M2177.003100
Deep Learning

[11: Generative Models (Part 1)]

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(last compiled at 16:01:00 on 2018/11/04)

Outline

References

- *Deep Learning* by Goodfellow, Bengio and Courville ▸ Link
  - Chapter 19: Approximate Inference
  - Chapter 20: Deep Generative Models
- *Pattern Recognition and Machine Learning* by Bishop
  - Chapter 10: Approximate Inference
- online resources:
  - *Stanford CS231n: CNN for Visual Recognition* ▸ Link
  - *CVPR 2018 GAN Tutorial* ▸ Link
  - *NIPS 2016 Variational Inference Tutorial* ▸ Link
  - *NIPS 2016 GAN Tutorial* ▸ Link

Outline

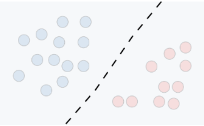
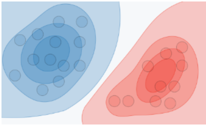
Supervised vs unsupervised learning

	supervised	**unsupervised**
data	$(\boldsymbol{x}, y)$ $\boldsymbol{x}$: data, y: label	$\boldsymbol{x}$ just data, no _____
goal	learn a *function* to map $\boldsymbol{x} \mapsto y$	learn inherent *structure* of the data
examples	classification regression object detection semantic segmentation	clustering dimensionality reduction representation learning density estimation

(source: cs231n)

Discriminative vs generative models

- assume supervised learning
  - goal: learn a *function* f to map $\boldsymbol{x} \mapsto y$

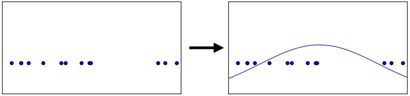
	discriminative	**generative**
goal	directly estimate $p(y \mid \boldsymbol{x})$	estimate $p(\boldsymbol{x} \mid y)$; then deduce[1] $p(y \mid \boldsymbol{x})$
should evaluate	$f(\boldsymbol{x}) = \underset{y}{\operatorname{argmax}}\, p(y \mid \boldsymbol{x})$	$f(\boldsymbol{x}) = \underset{y}{\operatorname{argmax}}\, p(\boldsymbol{x} \mid y) p(y)$
what's learned	decision boundary	probability __________ of data
		
examples	SVM, neural nets	Gaussian mixture, Bayes nets

(source: [Ng and Jordan, 2002], stackoverflow)

[1] *e.g.* using Bayes rule: $p(y \mid \boldsymbol{x}) = \frac{p(\boldsymbol{x} \mid y) p(y)}{p(\boldsymbol{x})}$

Generative modeling in unsupervised learning

- density estimation
  - goal: learn $p(\boldsymbol{x})$

training data  density function

- ______ generation
  - given training data, generate new samples from the same distribution

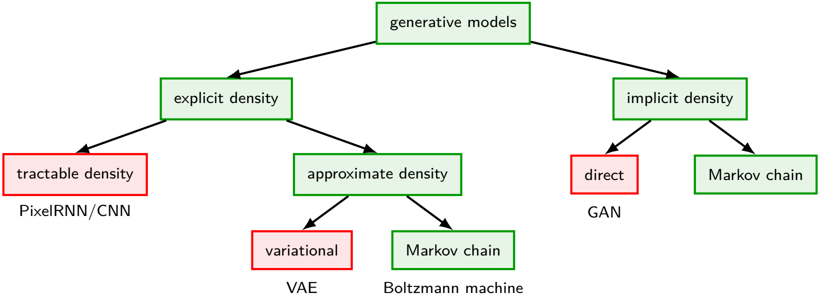
training data $\sim p_{\text{data}}(\boldsymbol{x})$

generated samples $\sim p_{\text{model}}(\boldsymbol{x})$

  - want to learn $p_{\text{model}}(\boldsymbol{x})$ similar to $p_{\text{data}}(\boldsymbol{x})$

(source: Goodfellow (2018), Karras (2017))

Taxonomy of generative models



(source: Goodfellow, 2017)

1. explicit density estimation
   - explicitly define and solve for $p_{\text{model}}(\boldsymbol{x})$
2. implicit density estimation
   - learn a model that can sample from $p_{\text{model}}(\boldsymbol{x})$ w/o explicitly defining it

Trends in generative models

- conventional GM approaches have one of three drawbacks [Doersch, 2016]:
  1. require strong assumptions about structure in data
  2. make approximations $\Rightarrow$ suboptimal results
  3. rely on computationally expensive inference procedures (*e.g.* MCMC)

- recent advances
  - train neural nets as powerful function approximators through backprop

  $\Rightarrow$ gives framework for backprop-based function approximators to build GMs

e.g. variational autoencoder: one of the most popular such frameworks
- assumptions of this model: weak
- training: fast via ________
- make an approximation but error is small given high-capacity models

- generative models based on neural nets

1. autoregressive models (or fully visible belief nets)
   - model $p(\boldsymbol{x})$ as $\prod_{i=1}^{n} p(x_i \mid x_1, \ldots, x_{i-1})$

   e.g. PixelRNN, PixelCNN, WaveNet

2. Helmholtz machines
   - model $p(\boldsymbol{x})$ as $\int p(\boldsymbol{z})p(\boldsymbol{x} \mid \boldsymbol{z})d\boldsymbol{z}$
   - two components: __________ net (encoder) + generative net (decoder)
   - use variational inference to maximize ELBO

   e.g. variational autoencoder (VAE)

3. generative adversarial network (GAN)
   - no explicit density modeling
   - two components: generator + discriminator
   - train models by solving minimax problem

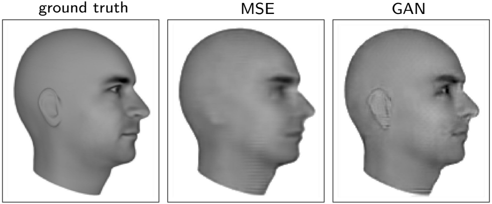
Why study generative models?

- excellent test of our ability
  - to represent/manipulate high-dim/complicated probability distributions
- can be incorporated into reinforcement learning (RL)
  - simulate possible futures for planning
- can be trained with missing data
  - can provide predictions on inputs that are missing data
  - ⇒ facilitate ____-supervised learning

(source: Chen+, 2018)

- enable machine learning to work with multi-modal outputs

  e.g. a single input → many different correction answers



← video frame prediction

- MSE: averaging

⇒ blurry

- GAN: sharp

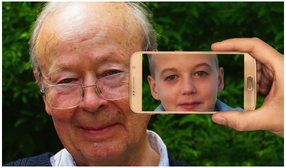
(source: Lotter+, 2015)

- enable inference of _____ representations

  ⇒ can be useful as general features

  e.g. identity-preserving latent representations →

(source: Antipov+, 2017)



age conditional GAN

- many tasks require realistic generation of samples

  e.g. single image super-resolution, art creation, image-to-image translation

first AI art sold at Christie ($432,000)

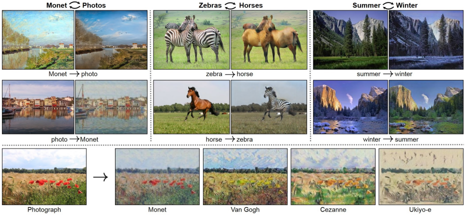
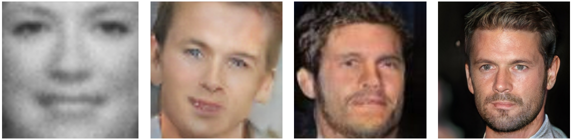


image-to-image translation



2014 2015 2016 2017

3.5 years of progress on faces

(source: Christie's, Zhu+ (2017), Brundage+ (2018))

Outline

Fully visible belief network (FVBN)

- explicit density model
- how it works:
  1. decompose likelihood of each input $\boldsymbol{x}$ into a product of 1D distributions

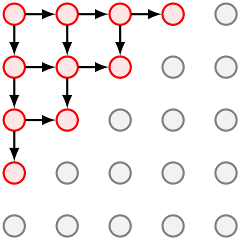
$$\underbrace{p(\boldsymbol{x})}_{\text{likelihood of input } \boldsymbol{x}} = \underbrace{\prod_{i=1}^{n} p(x_i \mid x_1, \ldots, x_{i-1})}_{\text{probability of } i\text{-th feature given all previous features}}$$

  2. maximize likelihood of training data

- complex distribution over feature values
  - we express it using a neural net
- main issue:
  - need to define ______ of "previous features"

PixelRNN

- context: image (ICML 2016 best paper[2])
- idea: the same as ________ modeling applied to image
  - generate image pixels starting from corner
  - model dependency on previous pixels using 2D LSTM
- limitation
  - sequential generation ⇒ slow



(source: [van den Oord et al., 2016c])

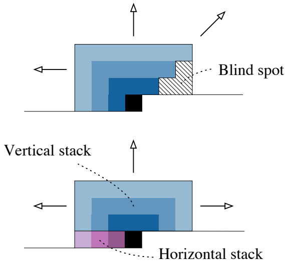
[2] this paper also proposed a simple PixelCNN, which was revised later in their NIPS paper (next page)

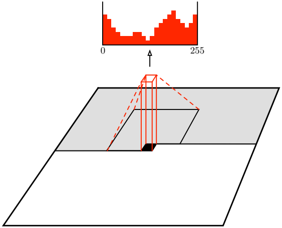
PixelCNN

- still generate image pixels starting from corner
- what's new
  - use CNN (not LSTM) to model dependency on previous pixels
  - two conv stacks to remove _____ spots
- training
  - maximize likelihood of training images
  - loss: softmax loss at each pixel
  - faster[3] than PixelRNN
- image generation
  - must still proceed sequentially

  ⇒ still slow

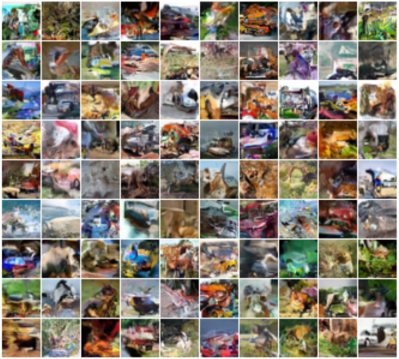
(source: [van den Oord et al., 2016b])



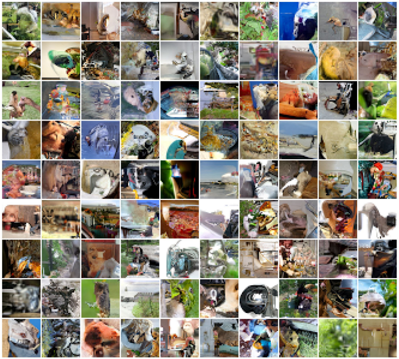


[3] we can parallelize convolutions since context region values are known from training images

Examples of generated samples



32×32 CIFAR-10



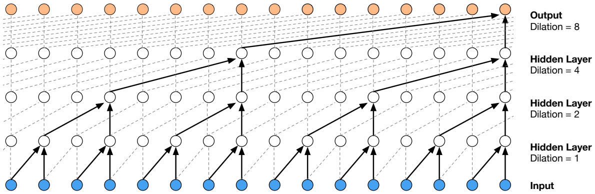
32×32 ImageNet

(source: [van den Oord et al., 2016c])

WaveNet

- can generate raw audios (trained on raw audio waveforms)
  - similar to PixelCNN in structure, but much more successful
  - amazing quality but slow generation (2 min to synthesize 1 sec audio)

- architecture
  - use dilated convolution to vastly _______ receptive field



(source: [van den Oord et al., 2016a])

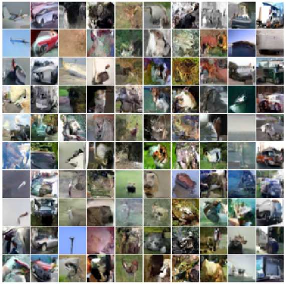
  - stack many layers like the above together
  - use classification (not regression) to generate next sample point
  - convert continuous audio → 256-level quantized classes

Remarks

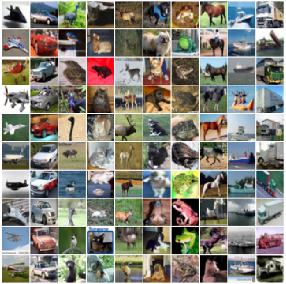
- autoregressive models: pros and cons
  - can explicitly compute $p(\boldsymbol{x})$
  - good samples
  - sequential generation: ___
- to improve performance
  - gated convolution layers
  - short-cut connections
  - discretized logistic loss
  - multi-scale
  - parallelization
  - and many more!
- more information: Link

(source: cs231n, [Salimans et al., 2017])

PixelCNN++ samples from CIFAR-10



real CIFAR-10 images



Outline

Probabilistic machine learning

- a probabilistic model
  - a joint distribution of hidden variables $\mathbf{z}$ and observed variables $\mathbf{x}$

$$p(\boldsymbol{z}, \boldsymbol{x})$$

  - describes how (a portion of) the world works

- inference about the unknowns:
  - through **posterior** ______

i.e. conditional distribution of the hidden variables given the observations

$$p(\boldsymbol{z} \mid \boldsymbol{x}) = \frac{p(\boldsymbol{z}, \boldsymbol{x})}{p(\boldsymbol{x})} \begin{array}{l} \leftarrow \text{model} \\ \leftarrow \text{data} \end{array}$$

  - the posterior links data and model $\Rightarrow$ used in all downstream analyses

- posterior inference:
  - therefore a central task in probabilistic models

Posterior inference

- refers to
  (1) computing posterior distribution $p(\boldsymbol{z} \mid \boldsymbol{x})$ or
  (2) taking expectations computed wrt this distribution

e.g. expectation-maximization (EM) algorithm

- evaluates expectation of complete-data log likelihood
  wrt posterior distribution of latent variables:

$$\mathcal{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}^{\mathrm{old}}) \triangleq \sum_{\mathbf{z}} p(\boldsymbol{z} \mid \boldsymbol{x}, \boldsymbol{\theta}^{\mathrm{old}}) \log p(\boldsymbol{x}, \boldsymbol{z} \mid \boldsymbol{\theta})$$

- for many models of practical interest
  - it is intractable to do (1)/(2) $\Rightarrow$ called “challenge of inference”

- challenge of inference
  - makes it difficult to ____ probabilistic models

Challenge of inference

- general reasons
  - dimensionality of latent space: too high to work with directly
  - posterior: highly complex $\Rightarrow$ expectations are not analytically tractable
- reasons in deep learning
  - interactions between ______ variables (*e.g.* connections between layers)
  - most neural nets with multiple layers of hidden variables
    - have intractable posterior distributions
- a solution: approximate posterior inference

Bayesian view

- we setup the general problem
  - consider a joint density of latent variables $\mathbf{z}$ and observations $\mathbf{x}$

$$p(\boldsymbol{z}, \boldsymbol{x}) = p(\boldsymbol{z}) p(\boldsymbol{x} \mid \boldsymbol{z})$$

  - latent variables help govern distribution of data in Bayesian models

- a Bayesian model
  - draws _____ variables from a prior $p(\boldsymbol{z})$, and then
  - relates them to observations through likelihood $p(\boldsymbol{x} \mid \boldsymbol{z})$

- inference in a Bayesian model
  - conditions on data and $\underbrace{\text{computes posterior } p(\boldsymbol{z} \mid \boldsymbol{x})}_{\uparrow}$

this often requires approximate inference

Approximate posterior inference

1. stochastic (*e.g.* Markov chain Monte Carlo: MCMC)
   - given infinite computational resource, can generate exact results
   - approximation arises from using a finite amount of processor time

   ☹ sampling methods can be computationally demanding

   $\Rightarrow$ their use is often limited to small-scale problems

   ☹ difficult to know whether __________ samples are being generated

2. deterministic (*e.g.* variational inference: VI)
   - some of which scale well to large applications
   - based on analytical approximations to posterior $p(\boldsymbol{z} \mid \boldsymbol{x})$

   e.g. assume it factorizes or has a __________ form (like Gaussian)

   $\Rightarrow$ never generate exact results

Markov chain Monte Carlo (MCMC)

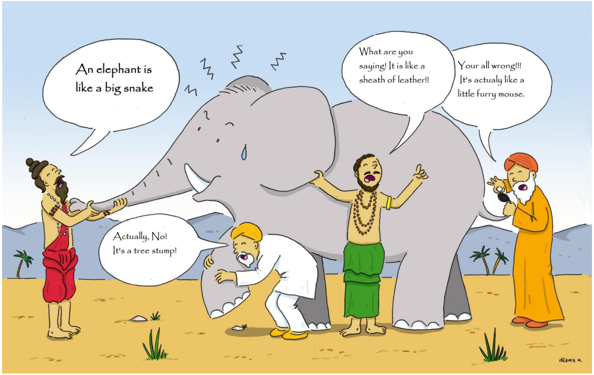
- dominant paradigm for decades
  1. construct an ergodic[4] $\underbrace{\text{chain}}_{\substack{\uparrow \\ \text{its stationary distribution } = \text{ posterior } p(\boldsymbol{z}\,|\,\boldsymbol{x})}}$ on $\mathbf{z}$
  2. sample from the chain to collect samples from the stationary distribution
  3. approximate the posterior with an $\underbrace{\text{empirical estimate}}_{\substack{\uparrow \\ \text{constructed from (a subset of) the collected samples}}}$

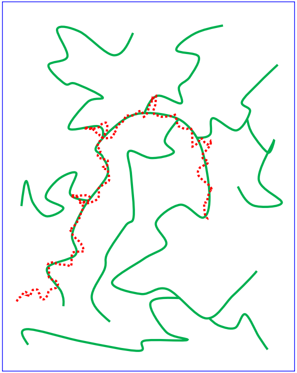
- widely studied, extended, and applied

  e.g. Metropolis-Hastings algorithm, Gibbs sampling

- too slow for large data sets/complex models
  - a good alternative in these settings: variational inference

[4] an ergodic system is a dynamical system with the following property: no matter where you start, you run the system and keep running it, then any other points in the space you will eventually reach





- goal: sample from a small subset of high-dim region where sampling is ____
  - often p has some (connected) structure
  - start somewhere, and start moving around (red dots)
  - try to move toward a region of high probability, and then get there
  - try to stay in the regions of high probability (green regions), doing random walks
  - ⁎ MCMC: do this by forming a Markov chain

Variational inference (VI)

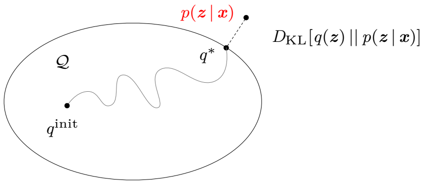
- main idea: turn inference into optimization (rather than sampling)

1. posit $\mathcal{Q}$, a *family* of approximate _______ over $\mathbf{z}$
   - each $q(\boldsymbol{z}) \in \mathcal{Q}$: a candidate approximation to the exact posterior

2. find the member of $\mathcal{Q}$ that minimizes D_{KL} to the exact posterior

$$q^*(\boldsymbol{z}) = \underset{q(\boldsymbol{z}) \in \mathcal{Q}}{\operatorname{argmin}} D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})]$$

3. approximate the posterior with q^*
   - use $q^*(\boldsymbol{z})$ in place of $p(\boldsymbol{z} \,|\, \boldsymbol{x})$ in downstream modeling



(source: [Blei et al., 2017])

Comparison

- different approaches to solving the same problem

	Markov chain Monte Carlo	variational inference
what it does	samples a Markov chain	solves an __________ problem
approximates posterior with	samples from the chain	optimization results
a tool for	simulating (sampling) from densities	approximating densities

when to use MCMC and when to use VI?

- MCMC
  - more computationally intensive than VI, but
  - guarantees producing (asymptotically) ____ samples from target density

  ⇒ suited to smaller data sets/scenarios
    - where we happily pay a heavier cost for more precise samples

- VI
  - has no such guarantees (only finds a density close to the target)
  - but faster than MCMC (also can use stochastic/distributed optimization)

  ⇒ suited to large data sets/scenarios
    - where we want to quickly explore many models

Outline

The phrase "variational" [Wainwright et al., 2008]

- an umbrella term that refers to mathematical tools for
  - optimization-based formulations of problems
  - associated techniques for their solution
- origin: *calculus of variations* (optimization over a space of functions)
- general idea:
  - express a quantity of interest as the solution of an optimization problem
- the optimization problem can then be "______"
  - either by approximating
    - the function to be optimized or
    - the set over which the optimization takes place
  - such relaxations allow us to
    - approximate the original quantity of interest

Variational inference (variational Bayes)

- inference = solving the following optimization:

$$q^*(\boldsymbol{z}) = \underset{q(\boldsymbol{z}) \in \mathcal{Q}}{\operatorname{argmin}} D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})] \tag{1}$$

  - complexity of $\mathcal{Q}$ determines complexity of this optimization

- choosing $\mathcal{Q}$ needs a good balance
  1. restrict $\mathcal{Q}$ sufficiently [for efficiency]

     $\Rightarrow$ it comprises only _______ distributions
  2. allow $\mathcal{Q}$ to be sufficiently rich/flexible [for accuracy]

     $\Rightarrow$ it can provide a good approximation to the true posterior

Optimization challenge

- the objective (1): not computable

$$q^*(\boldsymbol{z}) = \underset{q(\boldsymbol{z}) \in \mathcal{Q}}{\operatorname{argmin}} D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})] \tag{1}$$

- reason:

$$\begin{aligned} D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})] &= \mathbb{E}_{\mathbf{z} \sim q(\boldsymbol{z})}[\log q(\boldsymbol{z})] - \mathbb{E}_{\mathbf{z} \sim q(\boldsymbol{z})}[\log p(\boldsymbol{z} \,|\, \boldsymbol{x})] \\ &= \mathbb{E}_{\mathbf{z}}[\log q(\boldsymbol{z})] - \mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{z}, \boldsymbol{x})] + \underbrace{\log p(\boldsymbol{x})}_{\text{intractable}} \end{aligned} \tag{2}$$

- solution:
  - optimize an alternative objective called $\underbrace{\text{evidence lower bound (}\qquad\text{)}}_{\uparrow}$

also known as negative variational free energy

Evidence lower bound (ELBO)

- rearranging (2) gives

$$\underbrace{\log p(\boldsymbol{x})}_{\substack{\text{a constant}\\ \text{wrt } q(z)}} - D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \mid \boldsymbol{x})] = \underbrace{\mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{z}, \boldsymbol{x})] - \mathbb{E}_{\mathbf{z}}[\log q(\boldsymbol{z})]}_{\triangleq \mathcal{L}(q) \quad \text{“ELBO”}} \tag{3}$$

  - __________ ELBO $=$ minimizing the D_{KL}

- examining ELBO gives intuitions about optimal variational density

$$\begin{aligned}
\mathcal{L}(q) &= \mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{z}, \boldsymbol{x})] - \mathbb{E}_{\mathbf{z}}[\log q(\boldsymbol{z})] \\
&= \mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{z})] + \mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{x} \mid \boldsymbol{z})] - \mathbb{E}_{\mathbf{z}}[\log q(\boldsymbol{z})] \\
&= \underbrace{\mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{x} \mid \boldsymbol{z})]}_{\substack{\text{expected log likelihood}\\ \text{of the data}}} - \underbrace{D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z})]}_{\substack{D_{\mathrm{KL}} \text{ between}\\ \text{prior } p(z) \text{ and } q(z)}}
\end{aligned}$$

ELBO ("variational objective"): $\mathcal{L}(q) = \mathbb{E}_{\mathbf{z}\sim q(z)}[\log p(\boldsymbol{x} \mid \boldsymbol{z})] - D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z})]$

which values of z will this objective encourage $q(z)$ to place its mass on?

- first term: $\mathbb{E}_{\mathbf{z}}[\log p(\boldsymbol{x} \mid \boldsymbol{z})]$

  i.e. an expected likelihood

  - encourages densities that explain ________ data[5]

- second term: $-D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z})]$

  i.e. negative divergence between variational density and prior

  $\Rightarrow$ encourages densities close to prior

- thus the variational objective mirrors

  - usual balance between likelihood and prior

[5] *i.e.* densities that place their mass on configurations of the latent variables that explain the observed data

Why called ELBO?

- ELBO $\mathcal{L}(q)$ lower-bounds the (log) evidence $p(\boldsymbol{x})$

  i.e. for any $q(\boldsymbol{z})$

$$\log p(\boldsymbol{x}) \geq \mathcal{L}(q)$$

- proof:
  - notice that (3) gives

$$\log p(\boldsymbol{x}) = \mathcal{L}(q) + \underbrace{D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})]}_{\geq 0}$$

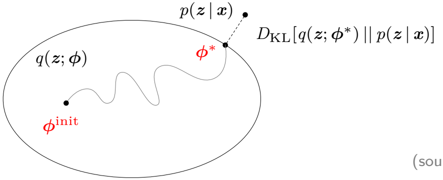
  - can also be derived through Jensen's inequality

Common restrictions on $\mathcal{Q}$

1. parameterization
   - use a parametric distribution governed by parameters $\boldsymbol{\phi}$

$$q(\boldsymbol{z}; \boldsymbol{\phi}) = q_{\boldsymbol{\phi}}(\boldsymbol{z})$$

   - $\boldsymbol{\phi}$: variational parameters



(source: [Blei et al., 2017])

   - ELBO $\mathcal{L}$ now becomes a function of $\boldsymbol{\phi}$: $\mathcal{L}(q, \boldsymbol{\phi})$
   - ________ $=$ finding $\boldsymbol{\phi}^*$
   - spoiler: VAE uses a neural net to model $q_{\boldsymbol{\phi}}(\boldsymbol{z})$ and sgd to train it

2. factorization

- assume q factorizes (called "mean-field approach")

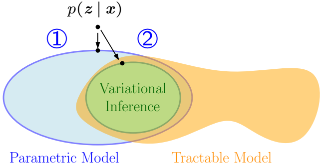
$$q(\boldsymbol{z}) = \prod_{j=1}^{m} q_j(z_j)$$

- each latent variable z_j :
  - governed by its own variational factor, the density $q_j(z_j)$
  - can take on any form appropriate to the corresponding random var

e.g. a continuous variable: a ________ factor

a categorical variable: a categorical factor

core idea behind VI: maximize $\mathcal{L}$ over a restricted family of distributions q



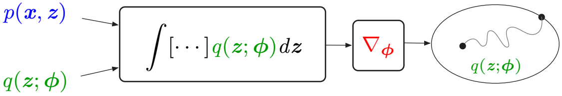
① $q(\boldsymbol{z}; \boldsymbol{\phi}_{\text{MLE}})$

② $q(\boldsymbol{z}; \boldsymbol{\phi}_{\text{ELBO}})$

(source: [Chen et al., 2018])

Variational inference recipe

1. start with a model: $p(\boldsymbol{x}, \boldsymbol{z})$
2. choose a variational approximation: $q(\boldsymbol{z}; \boldsymbol{\phi}) = q_{\boldsymbol{\phi}}(\boldsymbol{z})$
3. write down ELBO: $\mathcal{L}(\boldsymbol{\phi}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z}; \boldsymbol{\phi})}[\log p(\boldsymbol{x}, \boldsymbol{z}) - q(\boldsymbol{z}; \boldsymbol{\phi})]$
4. compute the expectation/integral: *e.g.* $\mathcal{L}(\boldsymbol{\phi}) = \boldsymbol{x}\boldsymbol{\phi}^2 + \log \boldsymbol{\phi}$
5. take derivatives: *e.g.* $\nabla_{\boldsymbol{\phi}} \mathcal{L}(\boldsymbol{\phi}) = 2\boldsymbol{x}\boldsymbol{\phi} + \frac{1}{\boldsymbol{\phi}}$
6. optimize: $\boldsymbol{\phi}_{t+1} = \boldsymbol{\phi}_t + \underline{\qquad\qquad}$



(source: [Blei et al., 2017])

Outline

Summary

- generative models: density estimation and sample generation
  - explicit density: PixelRNN/CNN, variational autoencoder (VAE)
  - implicit density: generative adversarial network (GAN)
- generative models are versatile and useful for many tasks
  - representation/manipulation of high-dim distributions
  - reinforcement learning, semi-supervised learning
  - multi-modal outputs, inference of latent representations
- approximate inference schemes fall into two classes: stochastic or deterministic
  - their strengths and weaknesses are complementary to each other
- variational inference: inference $\rightarrow$ optimization
  - goal: approximate posterior $p(\boldsymbol{z} \,|\, \boldsymbol{x}) \approx q(\boldsymbol{z}) \in \mathcal{Q}$ [$\mathbf{z}$: latent, $\mathbf{x}$: observed]
  - $\mathcal{Q}$: a family of restricted (parameterized/factorized) distributions over $\mathbf{z}$
  - $q^*(\boldsymbol{z}) = \operatorname{argmin}_{q(\boldsymbol{z}) \in \mathcal{Q}} D_{\mathrm{KL}}[q(\boldsymbol{z}) \,||\, p(\boldsymbol{z} \,|\, \boldsymbol{x})]$
  - VAE: we parameterize q using a neural net and optimize by backprop

Bibliography

Blei, D. M., Kucukelbir, A., and McAuliffe, J. D. (2017). Variational inference: A review for statisticians. *Journal of the American Statistical Association*, 112(518):859–877.

Chen, Y.-C., Wang, Y. S., Erosheva, E. A., et al. (2018). On the use of bootstrap with variational inference: Theory, interpretation, and a two-sample test example. *The Annals of Applied Statistics*, 12(2):846–876.

Doersch, C. (2016). Tutorial on variational autoencoders. *arXiv preprint arXiv:1606.05908*.

Ng, A. Y. and Jordan, M. I. (2002). On discriminative vs. generative classifiers: A comparison of logistic regression and naive bayes. In *Advances in neural information processing systems*, pages 841–848.

Salimans, T., Karpathy, A., Chen, X., and Kingma, D. P. (2017). Pixelcnn++: Improving the pixelcnn with discretized logistic mixture likelihood and other modifications. *arXiv preprint arXiv:1701.05517*.

van den Oord, A., Dieleman, S., Zen, H., Simonyan, K., Vinyals, O., Graves, A., Kalchbrenner, N., Senior, A. W., and Kavukcuoglu, K. (2016a). Wavenet: A generative model for raw audio. In *SSW*, page 125.

van den Oord, A., Kalchbrenner, N., Espeholt, L., Vinyals, O., Graves, A., et al. (2016b). Conditional image generation with pixelcnn decoders. In *Advances in Neural Information Processing Systems*, pages 4790–4798.

van den Oord, A., Kalchbrenner, N., and Kavukcuoglu, K. (2016c). Pixel recurrent neural networks. *arXiv preprint arXiv:1601.06759*.

Wainwright, M. J., Jordan, M. I., et al. (2008). Graphical models, exponential families, and variational inference. *Foundations and Trends® in Machine Learning*, 1(1–2):1–305.