Fourier representation of Signals

Introduction

Here we consider representing a signal as a weighted superposition of complex sinusoids. If such a signal is applied to a linear system, then the system output is weighted superposition of the system response to each complex sinusoid. Representation of signals as superposition of complex sinusoids not only leads to a useful expression for the system output but also provides a very insightful characterisation of signals and systems

The study of signals and system using sinusoidal representation is termed fourier analysis after Joseph fourier for his contributions to the theory of representing functions as weighted superpositions of sinusoids. The fourier representation for different classes of signals are given below;

i) Discrete-time periodic signals : Discrete time fourier series (DTFS)

ii) Continuous-time periodic signals : Fourier Series (FS)

iii) Discrete-time Non periodic signals : Discrete time fourier transform (DTFT)

iv) Continuous-time Non periodic signals : Fourier transform

1) Discrete-time periodic signals : The discrete time fourier series [DTFS]

The DTFS representation of a periodic sequence $x(n)$ is given by

$$x[n] = \sum_{K=\langle N\rangle} X[K]\, e^{j\Omega_0 n K} \longrightarrow (1)$$

$$X[K] = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jK\Omega_0 n} \text{ — } (2)$$

where $x[n]$ has fundamental period N and Ω_0 is equal to $\frac{2\pi}{N}$ (radians)

* In the above equation, $X[K]$ are known as discrete time fourier series coefficients. The co-efficients specify a decomposition of $x(n)$ into a sum of N harmonically related complex exponentials

* eqn (1) is known as synthesis equation and eqn (2) is known as analysis eqn ; we say that $x(n)$ and $X[K]$ are DTFS pair and is denoted by

$$x(n) \xleftrightarrow{DTFS : \Omega_0} X[K]$$

where $x(n)$ is time domain representation and $X[K]$ is frequency domain representation.

* The magnitude of $X[K]$ is $|X(K)|$ is known as magnetic spectrum of $x(n)$ and phase of $X[K]$ is $\arg\{X[K]\}$ is known as the phase spectrum of $x[n]$

* From the N values of $X[K]$ we may determine $x[n]$ using eqn (1) and from N values of $x[n]$ we may determine $X[K]$ using

eqn (2). Either $X[k]$ or $x[n]$ provides a complete description of the signal. In some problems it is advantageous to represent the signal using its time values $x[n]$, while in others the DTFS coefficients $X[k]$ after a more convinient description of the signal

Properties of DTFS,

The different properties of DTFS are

1) Linearity
2) Time shift
3) Freqency shift
4) Scaling
5) convolution
6) Modulation
7) Parseval's theorems
8) Duality
9) Symmetry

1) Linearity

If $x(n) \xleftrightarrow{DTFS\,:\,\Omega_0} X[k]$ &

$y(n) \xleftrightarrow{DTFS\,:\,\Omega_0} Y[k]$

Then $z(n) = ax(n) + b\,y(n) \xleftrightarrow{DTFS\,:\,\Omega_0} Z[k] = aX[k] + bY[k]$

In this case, both $x(n)$ and $y(n)$ are assumed to have the same fundamental period $N = \frac{2\pi}{\Omega_0}$

W.K.t, $X[k] = \frac{1}{N} \sum_{n=\langle N\rangle} x(n)\, e^{-jk\Omega_0 n}$

$Y[k] = 1/N \sum_{n=\langle N\rangle} y(n)\, e^{-jk\Omega_0 n}$

$Z[k] = \frac{1}{N} \sum_{n\langle N\rangle} z(n)\, e^{-jk\Omega_0 n}$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \left[a\, x[n] + b\, y[n]\right] e^{-jk\Omega_0 n}$$

$$= \frac{1}{N}\, a \sum_{n=\langle N \rangle} x[n]\, e^{-jk\Omega_0 n} + \frac{1}{N}(b) \sum_{n=\langle N \rangle} y[n]\, e^{-jk\Omega_0 n}$$

$$\therefore Z[k] = a\, X[k] + b\, Y[k]$$

Hence the proof

2) <u>Time shift</u>: if $x(n) \xleftrightarrow{DTFS:\Omega_0} X[k]$

Then $w(n) = x(n-n_0) \xleftrightarrow{DTFS:\Omega_0} W[k] = e^{-jk\Omega_0 n_0} X[k]$

Proof: we have $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\Omega_0 n}$

$$W[k] = \frac{1}{N} \sum_{n=\langle N \rangle} w[n]\, e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n-n_0)\, e^{-jk\Omega_0 n}$$

put $n - n_0 = m$, then

$$W[k] = \frac{1}{N} \sum_{m=\langle N \rangle} x[m]\, e^{-jk\Omega_0(m+n_0)}$$

$$= e^{-jk\Omega_0 n_0} \frac{1}{N} \sum_{m=\langle N \rangle} x[m]\, e^{-jk\Omega_0 m}$$

$$\therefore W[k] = e^{-jk\Omega_0 n_0} X[k]$$

Hence the proof

3) Frequency shift

if $x[n] \xleftrightarrow{DTFS:\Omega_o} X[K]$

then $g(n) = e^{jK_o\Omega_o n} x[n] \xleftrightarrow{DTFS:\Omega_o} G[K] = X[K-K_o]$

proof: we have $G(K) = \frac{1}{N} \sum_{n=\langle N \rangle} g(n) e^{-jK\Omega_o n}$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} e^{jK_o\Omega_o n} x[n] e^{-jK\Omega_o n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j(K-K_o)\Omega_o n}$$

$$G[K] = X[K-K_o]$$

Hence the proof

4) Scaling

Consider a periodic discrete time signal $x[n]$ with fundamental period N, such that $x[n] = 0$; Unless n/p is an integer

then $z[n] = x[pn]$ has fundamental period N/p

In this case, if $x[n] \xleftrightarrow{DTFS:\Omega_o} X[K]$

then $x[pn] \xleftrightarrow{DTFS:\Omega_o} z[K] = PX[K]$

where $p>0$

Then the scaling operation the harmonic spacing from Ω_o to $p\Omega_o$ and amplifies the DTFS

5) Convolution

if $x[n] \xleftrightarrow{DTFS:\Omega_o} X(K)$

& $y[n] \xleftrightarrow{DTFS:\Omega_o} Y[K]$

Then $z[n] = x[n] \circledast y[n] \xleftrightarrow{DTFS:\Omega_0} Z[K] = N X[K] Y[K]$

Here $\circledast$ denotes periodic convolution

Proof: We have $X[K] = \frac{1}{N} \sum'_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$

$$Y[K] = \frac{1}{N} \sum'_{n=\langle N \rangle} y[n] e^{-jk\Omega_0 n}$$

$$\therefore Z[K] = \frac{1}{N} \sum'_{n=\langle N \rangle} z(n) e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum'_{n=\langle N \rangle} [x[n] \circledast y[n]] e^{-jk\Omega_0 n}$$

using definition of periodic convolution, we get

$$Z[K] = \frac{1}{N} \sum_{n=\langle N \rangle} \left[\sum'_{l=\langle N \rangle} x[l]\, y(n-l) \right] e^{-jk\Omega_0 n}$$

changing the orders of summation, we get

$$Z[K] = \frac{1}{N} \left[\sum'_{l=\langle N \rangle} x(l) \sum'_{n=\langle N \rangle} y(n-l) e^{-jk\Omega_0 n} \right]$$

put $n-l = m$ then,

$$Z[K] = \frac{1}{N} \left[\sum'_{l=\langle N \rangle} x(l) \sum'_{m=\langle N \rangle} y(m) e^{-jk\Omega_0 (m+l)} \right]$$

$$= \frac{1}{N} \left[\sum'_{l=\langle N \rangle} x(l) \sum'_{m=\langle N \rangle} y(m) e^{-jk\Omega_0 m} e^{-jk\Omega_0 l} \right]$$

$$= \frac{1}{N} [N x[K] \cdot N y[K]]$$

$$\therefore Z[K] = N X[K] Y[K]$$

$\therefore$ Convolution in time domain is transformed to multiplication of DTFS

6) Modulation

if $x[n] \xleftrightarrow{DTFS:\Omega_0} X[k]$; $\Omega_0 = \frac{2\pi}{N}$

& $y[n] \xleftrightarrow{DTFS:\Omega_0} Y(k]$

then $z[n] = x[n] \cdot y[n] \xleftrightarrow{DTFS:\Omega_0} z[k] = X[k] \circledast Y(k]$

Proof: we have $z(k] = \frac{1}{N} \sum_{n=\langle N \rangle} z(n) \, e^{-jk\Omega_0 n}$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} x[n] \cdot y[n] \, e^{-jk\Omega_0 n} \quad \text{——} \quad ①$$

we have $x[n] = \sum_{l=\langle N\rangle} X(l] \cdot e^{+jl\Omega_0 n}$ —— ②

substituting eqn 2 in 1

$$z(k] = \frac{1}{N} \sum_{n=\langle N\rangle} \left[\sum_{l=\langle N\rangle} X(l) \, e^{jl\Omega_0 n} \right] y[n] \, e^{-jk\Omega_0 n}$$

changing the order of summation we get

$$z(k] = \frac{1}{N} \sum_{l=\langle N\rangle} x(l) \sum_{n=\langle N\rangle} y(n) e^{-j(k-l)\Omega_0 n}$$

$$= \sum_{l=\langle N\rangle} X(l) \, y(k-l)$$

$\therefore$ $z[k] = X[k] \circledast Y(k]$

7) Parseval's theorem:

If $x[n] \xleftrightarrow{DTFS:\Omega_0} X(k]$

then $\frac{1}{N} \sum_{n=\langle N\rangle} |x(n)|^2 = \sum_{k=\langle N\rangle} |X(k)|^2$ —— ①

Proof : The LHS of the equation ① is the average power of a periodic discrete time signal with fundamental period N

ie $$P = \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$$

The above eqn can be written as

$$P = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] \cdot x^*[n]$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} x[n] \left[\sum_{k=\langle N\rangle} X^*[k] \cdot e^{-ik\Omega_0 n} \right]$$

changing the order of summation we get,

$$P = \sum_{k=\langle N\rangle} X^*[k] \left[\frac{1}{N} \sum_{n=\langle N\rangle} x[n] \cdot e^{-ik\Omega_0 n} \right]$$

$$= \sum_{k=\langle N\rangle} X^*[k] \cdot X[k] = \sum_{n=\langle N\rangle} |X[k]|^2$$

$$\therefore \frac{1}{N} \sum_{n=\langle N\rangle} |x(n)|^2 = \sum_{k=\langle N\rangle} |X[k]|^2 \longrightarrow ②$$

In equation 2, the sequence $|X[k]|^2$ for $k = 0, 1, 2, \ldots N-1$ is the distribution of power as a function of frequency and is called power density spectrum of signal $x[n]$

8) <u>Duality</u> if $x[n] \xleftrightarrow{DTFS : \Omega_0} X[k]$

then $x[n] \xleftrightarrow{DTFS : \Omega_0} \frac{1}{N} X(-k) = X(k)$

Proof : we have $x[n] = \sum_{k=\langle N\rangle} X[k]\, e^{ik\Omega_0 n}$

Replacing n by $-n$, we get

$$x[-n] = \sum_{k=\langle N\rangle} X[k] \cdot e^{-jk\Omega_0 n}$$

Replacing n by k and k by n we get

$$X[-k] = \sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}$$

Xly both the sides by $1/N$, we get

$$\frac{1}{N} X(-k) = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}$$

Comparing with eqn $x(k) = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}$, we get

$$x[n] \xleftrightarrow{DTFS:\Omega_0} \frac{1}{N} X(-k)$$

9) <u>Symmetry</u>

if $x[n] \xleftrightarrow{DTFS:\Omega_0} X[k]$

then $x[n]$ real $\xleftrightarrow{DTFS:\Omega_0}$ $X^*[k] = X[-k]$

$x[n]$ Img $\xleftrightarrow{DTFS:\Omega_0}$ $X^*[k] = -X[-k]$

$x[n]$ real & even $\xleftrightarrow{DTFS:\Omega_0}$ $\text{Img}\{X[k]\} = 0$

$x[n]$ real and odd $\xleftrightarrow{DTFS:\Omega_0}$ $R_e\{X[k]\} = 0$

<u>Problems</u>

1) Evaluate the DTFS representation for the signal given by $x(n) = \cos\frac{\pi n}{3}$

Soln : w.k.t $x(n) = \cos \Omega_0 n$ is periodic if Ω_0 is integer x^{le} of $\frac{2\pi}{N}$ where N is the fundamental period ie $\Omega_0 = \frac{2\pi}{N} \cdot m$

By comparing $x(n) = \cos \frac{\pi}{3} n$ with $x(n) = \cos \Omega_0 n$

we get, $\Omega_0 = \frac{\pi}{3} = \frac{2\pi}{N} m$ (if $m = 1$ $N = 6$)

Fundamental period $N = 6$

We have $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$; $k = 0, 1, 2, \ldots 5$

But the given $x[n] = \cos \frac{\pi}{3} n$ can be expressed in the form of exponential so that we can obtain $X[k]$ directly using eqn

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_0 n} \quad — ①$$

$\therefore x(n) = \cos \frac{\pi}{3} n = \frac{e^{j\pi/3 n}}{2} + \frac{e^{-j\pi/3 n}}{2} = \frac{1}{2} e^{j(1)\pi/3 n} + \frac{1}{2} e^{j(-1)\pi/3 n}$

comparing with equation ① we get

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$X[1] = \frac{1}{2}$ & $X[-1] = \frac{1}{2}$

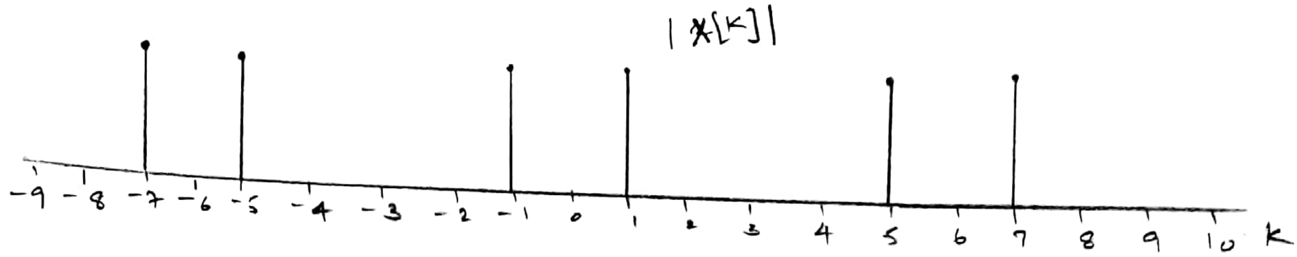
Since DTFS $X[k]$ forms a periodic sequence of period N, we can write

$\ldots = X[-11] = X[-5] = X[1] = X[7] = X[13] = \ldots = \frac{1}{2}$

$\ldots = X[-7] = X[-1] = X[5] = X[11] = X[17] = \ldots = \frac{1}{2}$

and other $X[k]$ are equal to zero

The spectra is shown in fig shown



2) Evaluate the DTFS representation for the signal $x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1 \ldots$; Sketch the magnitude & phase spectra

Soln: Given $x(n) = \sin\frac{4\pi}{21}n + \cos\frac{10\pi}{21}n + 1 \longrightarrow (1)$

here the 1st term has angular frequency $\Omega_{0_1} = \frac{4\pi}{21}$ and

II term has angular frequency $\Omega_{0_2} = \frac{10\pi}{21}$

So effectively the angular frequency of the summation Ω_0 = gcd of Ω_{0_1} & Ω_{0_2}

$\Omega_0 =$ gcd of $\frac{4\pi}{21}$ & $\frac{10\pi}{21}$

$\therefore \Omega_0 = \frac{2\pi}{21}$

Arranging eqn (1) as

$$x(n) = \frac{e^{j4\pi/21\,n} - e^{-j4\pi/21\,n}}{2j} + \frac{e^{j\frac{10\pi}{21}n} + e^{-j\frac{10\pi}{21}n}}{2} + 1$$

$$= \frac{1}{2j} e^{j(2)\left(\frac{2\pi}{21}n\right)} - \frac{1}{2j} e^{j(-2)\frac{2\pi}{21}n} + \frac{1}{2} e^{j5\left(\frac{2\pi}{21}n\right)} + \frac{1}{2} e^{-5j\left(\frac{2\pi}{21}n\right)} + 1 \quad \text{—(2)}$$

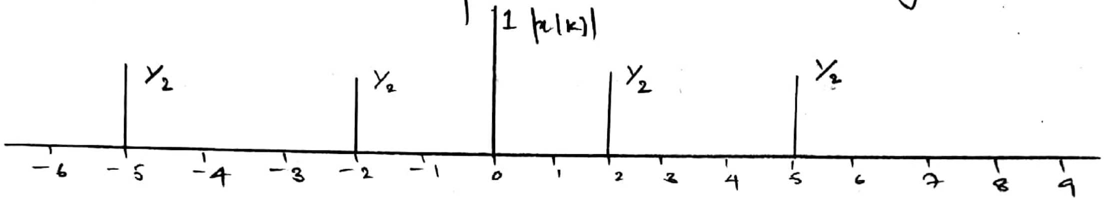
comparing eqn (2) with $x[n] = \sum_{k=\langle N \rangle} X[k]\, e^{jk\Omega_0 n}$ we get

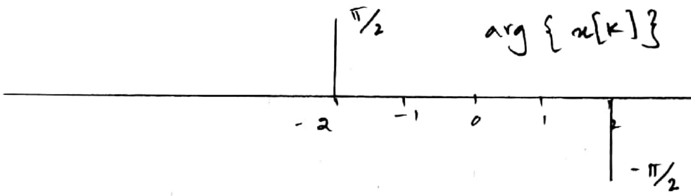
$X[0] = 1$

$X[-2] = -\frac{1}{2j}$; $X[2] = \frac{1}{2j}$

$X[-5] = \frac{1}{2}$; $X[5] = \frac{1}{2}$

The magnitude and phase spectrum is shown in fig below



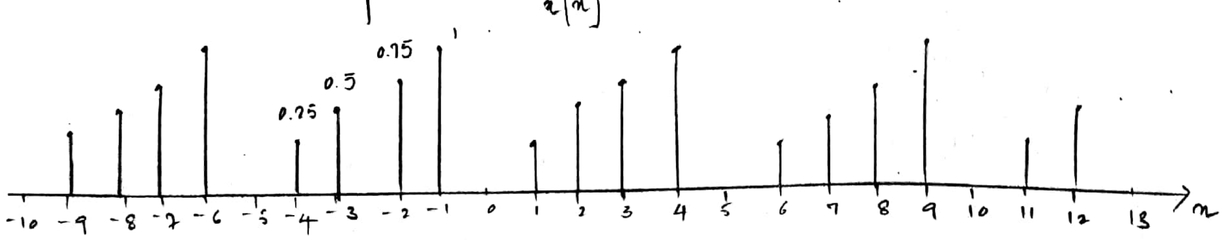


Here $N = 21$

N, $K \sum -N/2$ to $N/2$ ie $n\ k \sum -10, -9 \ldots\ldots 0, 1, 2, \ldots\ldots 10$

Also $X(k) = X[k+N] = X[k+21]$,

3) Evaluate the DTFS representation for the signal $x[n]$ shown in fig and sketch the spectra



By observation, we have the fundamental period of the given signal $x[n]$ is $N = 5$

$\therefore$ Angular frequency $\Omega_0 = 2\pi/5$

we have $X(k) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}$

$$\frac{1}{5}\left[0 \cdot e^{-jk\,2\pi/5 \cdot 0} + 0.25\, e^{-jk\,2\pi/5 (1)} + 0.5\, e^{-jk\,2\pi/5 \cdot 2} \right.$$

$$\left. + 0.75\, e^{-jk\,2\pi/5 (3)} + 1\, e^{-jk\,2\pi/5 \cdot 4} \right]$$

$$= \frac{1}{5}(0.25)\left[0 + e^{-jk\,2\pi/5 \cdot 1} + 2e^{-jk\,2\pi/5 \cdot 2} + 3e^{-j\,2\pi/5 \cdot 3} + 4e^{-jk\frac{2\pi}{5} \cdot 4} \right]$$

$$= \frac{1}{20}\left[e^{-jk2\pi/5} + 2e^{-j(4\pi/5)k} + 3e^{-j(6\pi/5)k} + 4e^{-j8\pi/5\, k} \right.$$

By taking one cycle is $k \in (0, 1 \ldots\ldots 4)$ we get,

$|X[0]| = 0.5 \qquad \arg\{X[0]\} = 0$

$|X[1]| = 0.21 \qquad \arg\{X[1]\} = 2.2$

$|X[2]| = 0.13 \qquad \arg\{X[2] = 2.8$

$|X[3]| = 0.13 \qquad \arg\{X[3]\} = -2.8$

$|X[4]| = 0.21 \qquad \arg\{X[4]\} = -2.2$

The magnitude & phase spectra are plotted in fig below

4) Consider the signal $x(n) = 2 + 2\cos \pi/_4 n + \cos \pi/_2 n + \frac{1}{2}\cos 3\pi/_4 n$

a) Determine and sketch its power density spectrum

b) Evaluate the power of the signal

given $x(n) = 2 + 2\cos \pi/_4 n + \cos \pi/_2 n + \frac{1}{2}\cos 3\pi/_4 n$ ..... (1)

Here 1 term is constant

Angular frequency of 2nd term $\Omega_{01} = \pi/_4$

Angular freq of 3rd term $\Omega_{02} = \pi/_2$

Angular freq of 4th term $\Omega_{03} = 3\pi/_4$

$\therefore$ Angular freq of summation ie of $x(n) = \Omega_0 =$ g.c.d of $(\Omega_0, \Omega_{02}, \Omega_{03})$

g.c.d of $(\pi/_4, \pi/_2, 3\pi/_4)$

$\boxed{\Omega_0 = \pi/_4} = 2\pi/_8 \downarrow 2\frac{\pi}{4}$

Fundamental period of $x(n)$ is $\boxed{N=8}$

The given $x(n)$ be written as,

$$x(n) = 2 + 2\left[\frac{e^{j\pi/_4 n} + e^{-j\pi/_4 n}}{2}\right] + \left[\frac{e^{j\pi/_2 n} + e^{-j\pi/_2 n}}{2}\right] + \frac{1}{2}\left[\frac{e^{\frac{j3\pi}{4}n} + e^{-\frac{j3\pi}{4}n}}{2}\right]$$

$$= 2 + e^{j(1)2\pi/_8 n} + e^{j(-1)(2\pi/_8)n} + \frac{1}{2}e^{j2\left(\frac{\pi}{8}(2)n\right)} + \frac{1}{2}e^{j(-2)(2\pi/_8 n)}$$

$$+ \frac{1}{4}e^{j(3)2\pi/_8 n} + \frac{1}{4}e^{j(-3)(2\pi/_8)n} \quad \text{—— (2)}$$

comparing the eqn with $x[n] = \sum_{k=\langle N \rangle} X[k]\, e^{jk\Omega_0 n}$ we get

$X[0] = 2$

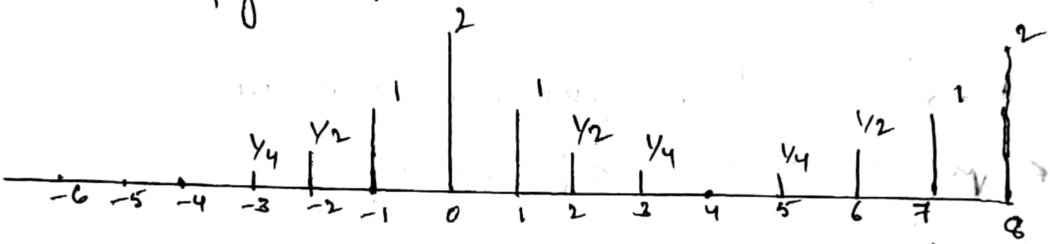
$X[-1] = X[1] = 1 \quad \therefore X[-1+8] = X[7] = 1$

$X[-2] = X[2] = \frac{1}{2} \quad \therefore X[-2+8] = X[6] = \frac{1}{2}$

$X[-3] = X[3] = \frac{1}{4} \quad \therefore X[-3+8] = X[5] = \frac{1}{4}$

$X[-4] = X[4] = 0 \quad \therefore X[-4+8] = X[4] = 0$

a) The power density spectrum $|X[k]|^2$ for one cycle ie $k \in (0 \ldots 7)$ is shown in fig below



b) The power of the signal (from parsoval's theorem)

$$P = \frac{1}{N} \sum_{n=<N>} |x(n)|^2 = \sum_{k=0}^{N-1} |X(k)|^2 \; ; k \in (0 \ldots 7)$$

$$= \frac{1}{8}\left[2^2 + 1^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2 + 0^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 1 \right.$$

$$P = \frac{1}{8}\left[4 + 1 + \frac{1}{4} + \frac{1}{16} + 0 + \frac{1}{16} + \frac{1}{4} + 1 \right]$$

$$\boxed{P = 0.828 \text{ Watt}}$$

5) Find ~~the~~ time domain signal corresponding to the DTFS coefficients

$$X[k] = \cos\left(\frac{6\pi k}{17}\right)$$

w.k.t DTFS coefficients $X[k]$ are always periodic

We have $X[k] = \cos \frac{6\pi}{17} k$

$\Omega_0 = \frac{6\pi}{17} = \frac{2\pi}{N} \times m$ where $m = 3$ $N = 17$

let us take $n, k \in \{-8, \ldots \ldots 8\}$

$$x(n) = \sum_{k=<N>} X[k] \, e^{jk\Omega_0 n}$$

$$= \sum \cos \frac{6\pi}{17} k \; e^{jk\frac{6\pi}{17}n}$$

$$= \sum_{k=-8}^{8} \frac{1}{2} \left[e^{\frac{j6\pi k}{17}} + e^{\frac{-j6\pi k}{17}} \right] e^{jk\frac{6\pi}{17}n}$$

$$x(n) = \frac{1}{2} \sum_{k=-8}^{8} \left\{ e^{jk\frac{2\pi}{17}(n+3)} + e^{jk\frac{2\pi}{17}(n-3)} \right\}$$

$$\therefore x(n) = \left.\begin{matrix} 0 & ; n \neq \pm 3 \\ = \frac{17}{2} & ; n = \pm 3 \end{matrix}\right\} n \in \{-8 \ldots 8\}$$

6) Find the ~~discrete-transform~~ time DT fourier series (DTFS) representation for the sequence $x[n] = \cos\left(\frac{\pi}{8} n + \phi\right)$

$$\Omega_0 = \frac{2\pi m}{8} = \frac{\pi}{8} \qquad N = 16 \quad \& \quad \Omega_0 = \frac{2\pi}{16}$$

$$\rightarrow x(n) = \frac{e^{j\left(\frac{\pi}{8}n+\phi\right)} + e^{-j\left[\frac{\pi}{8}n+\phi\right]}}{2} = \frac{1}{2} e^{-j\phi} e^{-j\frac{\pi}{8}n} + \frac{1}{2} e^{j\phi} e^{j\frac{\pi}{8}n} \quad \rightarrow (1)$$

Comparing above eqn with $x[n] = \sum_{k=\langle N \rangle} x[k] e^{jk\Omega_0 n}$ & taking starting index at $k = -7$

$$\therefore x[n] = \sum_{k=-7}^{8} X(k) \, e^{jk\pi/8 \, n} \quad \text{--- (2)}$$

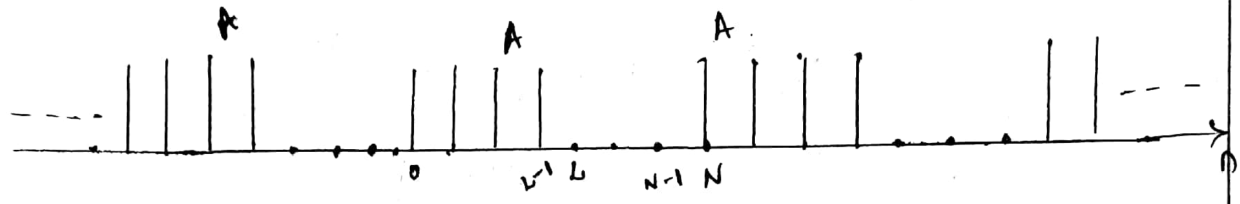
equating the terms in eqn 1 & 2, we get

$$x[n] \xleftrightarrow{DTFS, 2\pi/16} X[k] = \begin{cases} \frac{1}{2} e^{-j\phi} & k = -1 \\ \frac{1}{2} e^{j\phi} & k = 1 \\ 0 & -7 \le k \le 8 \;\&\; k \neq \pm 1 \end{cases}$$

since $x[k]$ has period $N = 16$, we have $X[15] = X[13] = \ldots \frac{1}{2} e^{-j\phi}$

& $X[17] = X[33] = \ldots = \frac{1}{2} e^{j\phi}$ with all other values of $X[k]$ equal to zero

7) Determine the fourier series coefficients of the periodic signal shown in figure.



we have,

$$X(K) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_0 K n} \qquad K = 0 \text{ to } N-1$$

$$\Omega_0 = 2\pi/N$$

$$\therefore X(K) = \frac{1}{N}\sum_{n=0}^{L-1} A\, e^{-j\frac{2\pi}{N} K n}$$

$$= \frac{A}{N}\sum_{n=0}^{L-1}\left[e^{-j\frac{2\pi K}{N}}\right]^n$$

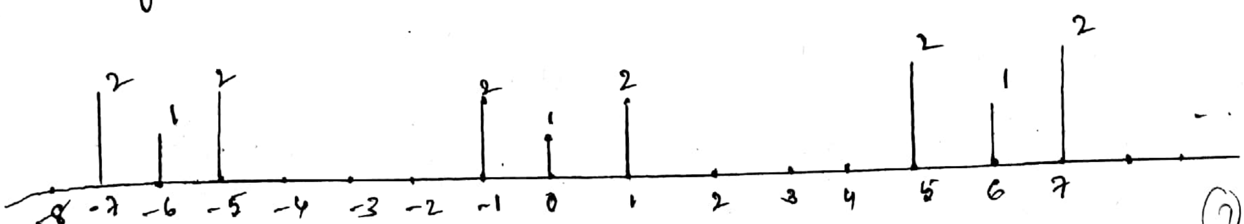
$$K = 0 \quad X(K) = AL/N$$

$$K \neq 0, \quad X(K) = A/N \left[\frac{1 - \left[e^{-j2\pi/N K}\right]}{1 - e^{-j2\pi/N K}}\right]$$

$$= \frac{A\, e^{-j\pi/N KL}}{N e^{-j\pi/N K}}\left[\frac{e^{j\pi/N KL} - e^{-j\frac{\pi}{N}KL}}{e^{j\frac{\pi}{N}K} - e^{-j\pi/N K}}\right]$$

$$X(k) = \frac{A}{N} e^{-j\frac{\pi K}{N}(L-1)} \frac{\sin(\pi/N\, K L)}{\sin(\pi/N\, K)}$$

8) Determine the DTFS coefficients for the periodic signal shown in figure



Here $N=6$, $\therefore \Omega_0 = 2\pi/6 = \pi/3$

$$X(k) = \frac{1}{6}\sum_{n=-1}^{4} x(n)\, e^{-jk\pi/3 n}$$

$$= \frac{1}{6}\left[x(-1)e^{jk\pi/3} + x(0) + x(1)e^{-jk\pi/3}\right]$$

$$= \frac{1}{6}\left[2e^{jk\pi/3} + 1 + 2e^{-jk\pi/3}\right]$$

$$X(k) = \frac{1}{6} + \frac{2}{3}\cos k\pi/3$$

*2) Continuous time periodic signals; The fourier series (FS)

The fourier series representation of periodic continuous time signal $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega_0 t} \quad \text{——— ①}$$

$$x(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt \quad \text{——— ②}$$

Where $x(t)$ has a fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$. Here $X[k]$ are known as fourier series coefficients of $x(t)$ eqn ① is known as synthesis equation and eqn ② is known as analysis equation, we say that $x(t)$ and $x[k]$ are FS pair and denote this relationship as

$$x(t) \xleftrightarrow{FS:\,\omega_0} X[k]$$

* Where $x(t)$ is time domain representation and $X(k)$ is frequency domain representation. In some problems it is

advantageous to represent the signal in the time domain as $x(t)$ while in others it is advantageous to represent the signal as $X[k]$

* The magnitude of $X[k]$ is $|X[k]|$ is known as magnitude spectrum of $x(t)$ and phase of $X[k]$ i.e $\arg\{X[k]\}$ is known as phase spectrum of $x(t)$

Properties of F.S

1) Linearity
2) Time shift
3) Frequency shift
4) Scaling
5) Time-differentiation
6) Convolution
7) Modulation
8) Parseval's theorem
9) Symmetry

1) Linearity

$$\text{if } x(t) \xleftrightarrow{FS:\,w_0} X[k]$$

$$\& \; y(t) \xleftrightarrow{FS:\,w_0} Y[k]$$

$$\text{then } z(t) = ax(t) + by(t) \xleftrightarrow{FS:\,w_0} Z[k] = aX[k] + bY[k]$$

In this case, both $x(t)$ & $y(t)$ are assumed to have same fundamental frequency.

Proof: We have, $X(k) = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$

$$Y(k) = \frac{1}{T}\int_{<T>} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$Z(k) = \frac{1}{T}\int_{<T>} z(t)\, e^{-jk\omega_0 t}\, dt$$

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$$= \frac{1}{T}\int_{<T>} [a x(t) + b y(t)]\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} a \int_{<T>} x(t)\, e^{-jk\omega_0 t} + \frac{1}{T} b \int_{<T>} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore Z(k) = aX(k) + bY(k)$$

Hence the proof

Time shift: if $x(t) \xleftrightarrow{FS:\omega_0} X(k)$

then $y(t) = x(t - t_0) \xleftrightarrow{FS:\omega_0} Y(k) = e^{-jk\omega_0 t_0} X(k)$

Proof: We have $X(k) = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$

$$\therefore Y(k) = \frac{1}{T}\int_{<T>} y(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{<T>} x(t-t_0)\, e^{-j\ \cdot\ t}\, dt$$

put $t - t_0 = m$, then

$$X(k) = \frac{1}{T}\int_{<T>} x(m)\, e^{-jk\omega_0 (m + t_0)}\, dm$$

$$= e^{-j\omega_0 k t_0}\, \frac{1}{T}\int_{<T>} x(m)\, e^{-jk\omega_0 m}\, dm = e^{-j\omega_0 k t_0} (k)$$

Hence the proof

3) Frequency shift

If $x(t) \xleftrightarrow{FS;\,\omega_0} X[k]$

then $y(t) = e^{jk_0\omega_0 t} x(t) \xleftrightarrow{FS;\,\omega_0} Y[k] = X[k-k_0]$

Proof: we have $X[k] = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t} dt$

$$\therefore\ Y[k] = \frac{1}{T}\int_{<T>} y(t) \cdot e^{-jk\omega_0 t} \cdot dt$$

$$= \frac{1}{T}\int_{<T>} e^{jk_0\omega_0 t} x(t) e^{-jk_0\omega_0 t} dt$$

$$= X[k-k_0]$$

Hence the proof.

4) Scaling

If $x(t) \xleftrightarrow{FS;\,\omega_0} X[k]$

then $z(t) = x(at) \xleftrightarrow{FS;\,a\omega_0} Z[k] = X[k]$; $a>0$

Proof:

If $x(t)$ is periodic, then $z(t) = x(at)$ is also periodic. If $x(t)$ has fundamental period T, then $z(t) = x(at)$ has fundamental period T/a.

Alternatively, if the fundamental frequency of $x(t)$ is ω_0, then the fundamental frequency of $z(t) = x(at)$ is $a\omega_0$.

we have, $x[k] = \frac{1}{T}\int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$\therefore z[k] = \frac{1}{(T/a)}\int_{\langle T/a \rangle} z(t)\, e^{-jka\omega_0 t}\, dt$$

$$= \frac{a}{T}\int_{\langle T/a \rangle} x(at)\, e^{-jka\omega_0 t}\, dt$$

$\therefore$ put $at = m$, then $dt = \frac{1}{a}\, dm$

$$z[k] = \frac{a}{T}\int_{\langle T/a \rangle} x(m)\, e^{-jk\omega_0 m}\, \frac{1}{a}\cdot dm$$

$$\therefore z[k] = x[k]$$

Therefore the FS coefficients of $x(t)$ and $x(at)$ are identical, but the harmonic changes from ω_0 to $a\omega_0$

5> Time differentiation

If $x(t) \xleftrightarrow{FS;\omega_0} x[k]$

then $\frac{dx(t)}{dt} \xleftrightarrow{FS;\omega_0} jk\omega_0 x[k]$

Proof:

we have,

$$x(t) = \sum_{k=-\infty}^{\infty} x[k] e^{jk\omega_0 t}$$

Differentiating both the sides w.r.t time t, we get

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} x[k] e^{jk\omega_0 t}\right]$$

Changing the order of differentiation and summation,

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{d}{dt} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot e^{jk\omega_0 t} \, jk\omega_0$$

$$= \sum_{k=-\infty}^{\infty} jk\omega_0 \, x[k] e^{jk\omega_0 t}$$

comparing with eq^n $x(t) = \sum_{k=-\infty}^{\infty} x[k] e^{jk\omega_0 t}$, we get

$$\frac{dx(t)}{dt} \xleftrightarrow{FS;\omega_0} jk\omega_0 x[k]$$

Hence the proof.

6) Convolution

If $x(t) \xleftrightarrow{FS;\,\omega_0} X[k]$

and $y(t) \xleftrightarrow{FS;\,\omega_0} Y[k]$; $\omega_0 = \frac{2\pi}{T}$

then $z(t) = x(t) \circledast y(t) \xleftrightarrow{FS;\,\omega_0} Z[k] = T X[k] \cdot Y[k]$

Here $\circledast$ denotes periodic convolution.

Proof: we have $X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t} \cdot dt$

$$Y[k] = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore Z[k] = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} [x(t) \circledast y(t)]\, e^{-jk\omega_0 t}\, dt$$

Using the definition of periodic convolution; we get

$$Z[k] = \frac{1}{T}\int_{t=\langle T\rangle} \left[\int_{L=\langle T\rangle} x(t)\, y(t-L)\, dL \right] e^{-jk\omega_0 t}\, dt$$

Changing the order of integration we get,

$$Z[k] = \frac{1}{T}\int_{l=\langle T\rangle} x(t) \int_{t=\langle T\rangle} y(t-L)\, e^{-jk\omega_0 t}\, dL\, dt.$$

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put $t-l=m$; $\therefore dt=dm$; then

$$z[k] = \frac{1}{T}\left[\int_{l=\langle T\rangle} x(l) \int_{m=\langle N\rangle} y(m)\, e^{-jk\omega_0(m+l)}\, dl\cdot dm\right.$$

$$= \frac{1}{T}\left[\int_{l=\langle T\rangle} x(l)\, e^{-jk\omega_0 l}\, dl \int_{m=\langle N\rangle} y(m)\, e^{-jk\omega_0 m}\, dm\right]$$

$$= \frac{1}{T}\left[T\,x[k]\cdot T\,y[k]\right]$$

$$\therefore z[k] = T\,x[k]\cdot y[k]$$

$\therefore$ Convolution in time domain is transformed to multiplication of FS co-efficients

7> Modulation

If $x(t) \xleftrightarrow{FS;\ \omega_0} X[k]$

and $y(t) \xleftrightarrow{FS;\ \omega_0} Y[k]$

then $z(t) = x(t)\cdot y(t) \xleftrightarrow{FS;\ \omega_0} z[k] = X[k] * Y[k]$

Proof: we have $z[k] = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\, y(t)\, e^{-jk\omega_0 t}\, dt \quad \text{—— ①}$$

we have; $x(t) = \sum_{l=-\infty}^{\infty} x[l]\, e^{jl\omega_0 t}$ —— ②

substituting ② in ①, we get;

$$z[k] = \frac{1}{T}\int_{\langle T\rangle}\left[\sum_{l=-\infty}^{\infty} x(l)\, e^{jl\omega_0 t}\right] y(t) \cdot e^{-jk\omega_0 t}\, dt$$

Changing the order of summation and integration, we get

$$z[k] = \frac{1}{T}\left[\sum_{l=-\infty}^{\infty} x(l) \cdot \int_{\langle T\rangle} y(t)\, e^{-j(k-l)\omega_0 t}\, dt\right]$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot y[k-l]$$

$$\therefore z[k] = x[k] * y[k]$$

87 Parseval's theorem:

If $x(t) \xleftrightarrow{FS;\,\omega_0} X[k]$

then $\frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt \longleftrightarrow = \sum_{k=-\infty}^{\infty} |X[k]|^2$

Proof: LHS of eq^n is the average power of a periodic continuous, time signal $x(t)$ with fundamental period T

i.e. $P = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt$

The above eq^n can be written as, $P = \frac{1}{T}\int_{\langle T\rangle} x(t)\, x^*(t)\, dt$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\left[\sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega_0 t}\right] dt$$

Changing the order of summation and integration we get,

$$P = \frac{1}{T} \sum_{k=-\infty}^{\infty} X[k] \cdot \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} X[k] \cdot X[k] = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

$$\therefore \frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2 \quad \text{——— (A)}$$

If eq^n (A) the sequence $|X[k]|^2$ for $k = 0, 1, 2 \ldots$ is the distribution of power as a fundamental of frequency and is called "power density spectrum" of the signal $|x(t)|$.

9> Symmetry

$$\text{If } x(t) \xleftrightarrow{FS;\omega_0} X[k]$$

$$\text{then } x(t)_{real} \xleftrightarrow{FS;\omega_0} X[k] = X[-k]$$

$$x(t)_{img} \xleftrightarrow{FS;\omega_0} X[k] = -X[-k]$$

$$x(t) \text{ real and even} \xleftrightarrow{FS;\omega_0} \text{Img}\{x[k]\} = 0$$

$$x(t) \text{ real and odd} \xleftrightarrow{FS;\omega_0} \text{Re}\{x[k]\} = 0$$

Problems:

1) For the signal $x(t) = \sin \omega_0 t$; find the fourier series and draw its spectrum.

Soln: Given, $x(t) = \sin \omega_0 t$

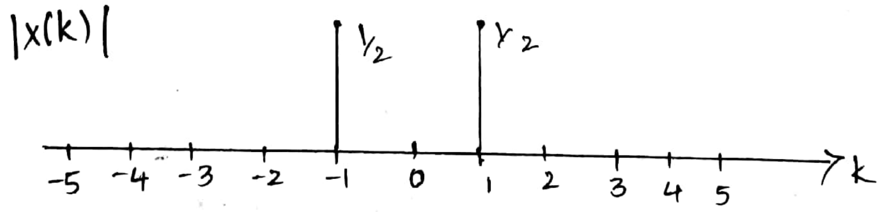
$$\therefore x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \quad \text{——} (1)$$

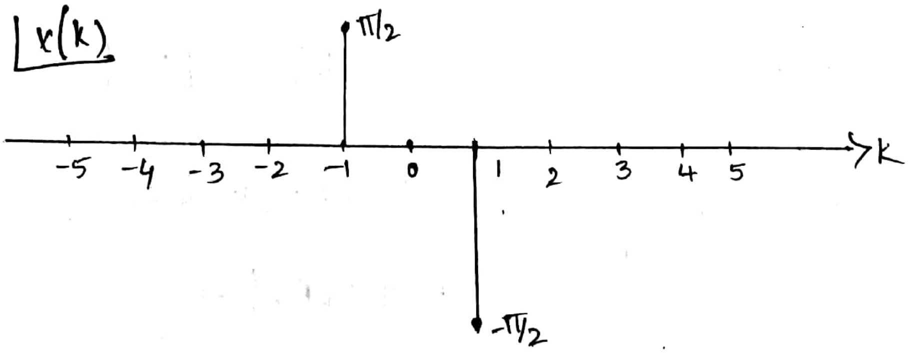
Comparing eqn (1) with eqn $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 t k}$ we have

$$X[1] = \frac{1}{2j}, \quad X[-1] = \frac{-1}{2j}$$

and $X[k] = 0$ for $k \neq \pm 1$

The magnitude and phase spectra is shown in figure below,





2> Evaluate the FS representation for the signal $x(t) = \sin(2\pi t) + \cos(3\pi t)$. Sketch the magnitude and phase spectra.

Soln: Given $x(t) = \sin(2\pi t) + \cos(3\pi t)$ ..... (1)

The first term has angular frequency $\omega_{0_1} = 2\pi$

The second term has angular frequency $\omega_{0_2} = 3\pi$

The angular frequency of the summation; of ω_{0_1} and ω_{0_2}

i.e. of $x(t)$ is $\omega_0 = g.c.d.(\omega_{0_1}, \omega_{0_2})$

$= g.c.d(2\pi, 3\pi)$

$\omega_0 = \pi$

eq^n (1) can be written as

$$x(t) = \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t} + \frac{1}{2} e^{j3\pi t} + \frac{1}{2} e^{-j3\pi t} \quad \ldots (2)$$

comparing eq^n (2) with eq^n $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$ we get;

$X[2] = \frac{1}{2j}$; $X[-2] = \frac{-1}{2j}$.

$X[3] = \frac{1}{2}$; $X[-3] = \frac{1}{2}$

and $X[k] = 0$ for $k \neq \pm 2, \pm 3$

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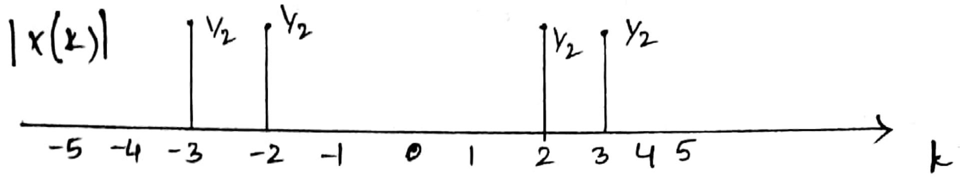
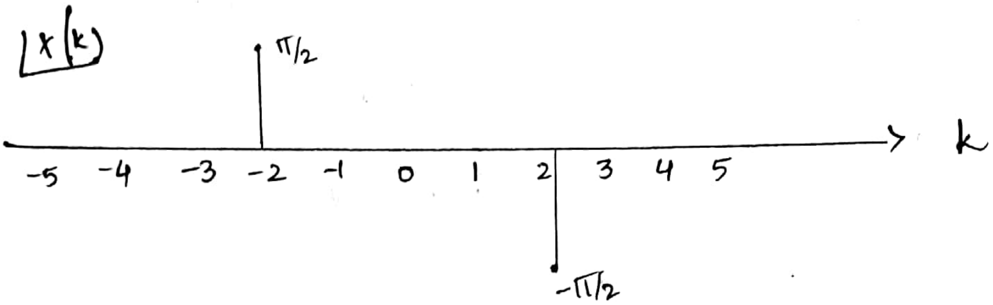
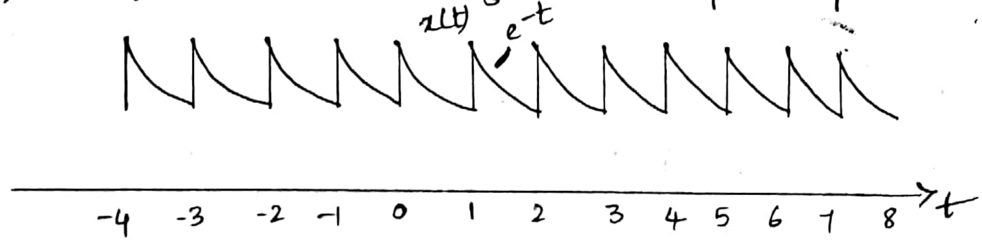
The magnitude and phase spectra is shown below;





3) For the signal $x(t)$ shown below in fig; find the FS representation and draw its magnitude and phase spectra.



Soln: By observing the above figure; we have $T = 1$, $\omega_0 = \frac{2\pi}{T} = 2\pi$

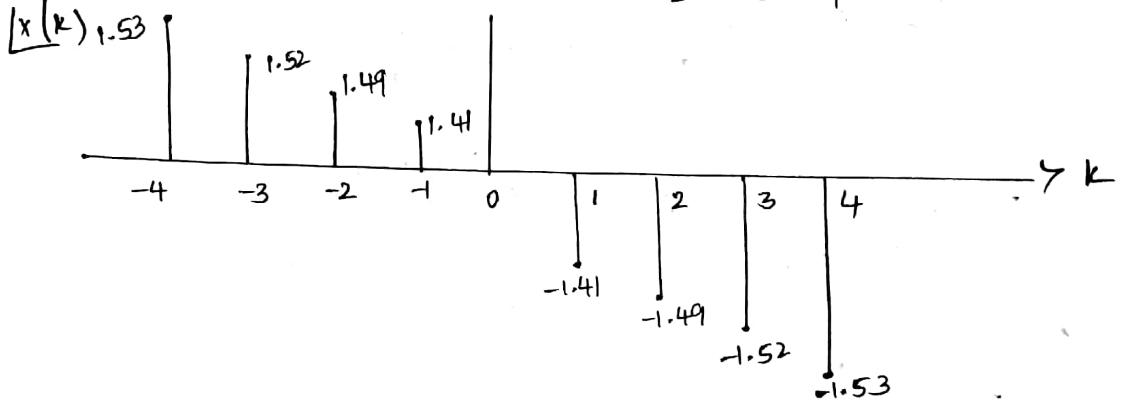
w.k.t. $x[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{1}\int_{t=0}^{1} e^{-t}\, e^{-jk(2\pi)t}\, dt$$

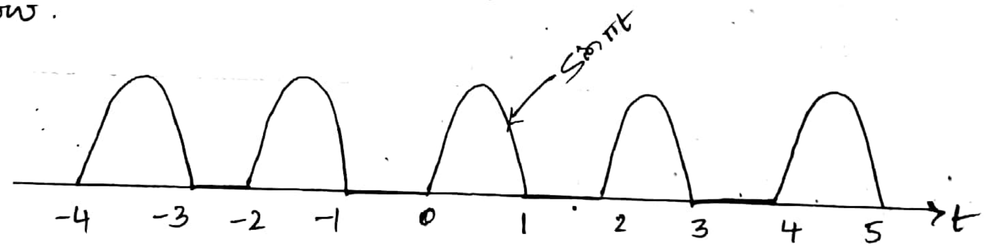
$$= \left.\frac{-e^{-(1+j2\pi k)t}}{(1+j2\pi k)}\right|_0^1 = \frac{1}{1+j2\pi k}\left[1 - e^{-(1+j2\pi k)}\right]$$

$$|x[k]| = \frac{1-e^{-1}}{1+j2\pi k}.$$

The magnitude and phase spectra is shown in figure below;



4) Find the FS co-efficient for signal $x(t)$ shown in figure below.



Soln: By observing the figure, we have, $T=2$ and $\omega_0 = \frac{2\pi}{T} = \pi$ //

From definition we have $x[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}dt$

$$x[k] = \frac{1}{2}\int_0^1 \sin\pi t\, e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_0^1 \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right] e^{-jk\pi t}dt$$

$$= \frac{1}{4j}\left[\int_0^1 e^{j(1-k)\pi t}\,dt - \int_0^1 e^{-j(1+k)\pi t}\,dt\right]$$

$$= \frac{1}{4j}\left[\left.\frac{e^{j(1-k)\pi t}}{j(1-k)\pi}\right|_0^1 + \left.\frac{e^{-j(1+k)\pi t}}{j(1+k)\pi}\right|_0^1\right]$$

$$= \frac{1}{4j}\left[\frac{e^{j(1-k)\pi} - e^0}{j(1-k)\pi} + \frac{e^{-j(1+k)\pi} - e^0}{j(1+k)\pi}\right]$$

$$= \frac{1}{4j}\left[\frac{(-1)^{k+1}-1}{j(1-k)\pi} + \frac{(-1)^{k+1}-1}{j(1+k)\pi}\right]$$

$$= \frac{-1}{4\pi}\left[(-1)^{k+1}-1\right]\cdot\left[\frac{2}{1-k^2}\right]$$

$$\therefore X[k] = \frac{1}{2\pi}\left[1-(-1)^{k+1}\right]\cdot\left[\frac{2}{1-k^2}\right]$$

5) Find the FS co-efficients for the periodic signal $x(t)$ with period 2 and $x(t) = e^{-t}$ for $-1 < t < 1$

Soln: Given $T = 2$; $\omega_0 = \frac{2\pi}{T} = \pi$

$$\therefore X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t}\, e^{-jk\pi t}\,dt$$

$$X[k] = \frac{(-1)^k\,[e - e^{-1}]}{2(1+jk\pi)} \text{ for all } k$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t}\, e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+jk\pi)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(1+jk\pi)t}}{-(1+jk\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{-2(1+jk\pi)}\left[e^{-(1+jk\pi)} - e^{(1+jk\pi)}\right]$$

$$x[k] = \frac{(-1)^k}{2(1+jk\pi)}\left[e - e^{-1}\right] \text{ for all } k$$

6> Determine the FS representation for the signal $x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$

Soln: The fundamental period of $x(t)$ is $T=4$.

Hence $w_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

we have $x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk(\pi/2)t}$

One approach to find $X[t]$ is used eq^n

$$X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk w_0 t}\, dt \quad \text{——— (A)}$$

However in the case $x(t)$ is expressed in terms of sinosoids so it is easier to obtain $X[k]$ by inspection.

$$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$

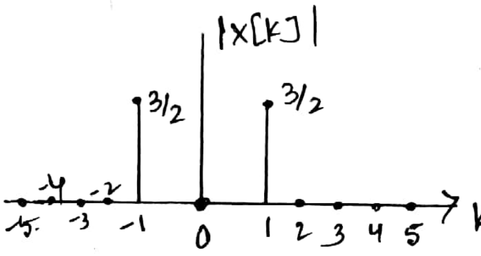
$$= 3\,\frac{e^{j(\pi/2)t + \pi/4} + e^{-j[(\pi/2)t + \pi/4]}}{2}$$

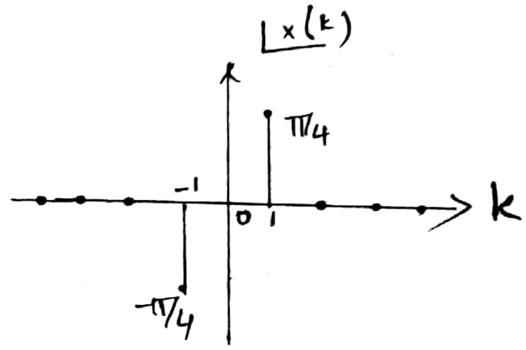
$$= \frac{3}{2}\, e^{-j\pi/4}\, e^{-j\pi/2 t} + \frac{3}{2}\, e^{j\pi/4}\, e^{j(\pi/2)t}$$

By, this last expression is in the form of fourier series.

$$X[k] = \begin{cases} \frac{3}{2} e^{-j\pi/4} & k = -1 \\ \frac{3}{2} e^{j\pi/4} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

The magnitude and phase of $X[K]$ are shown below.





4> Continuous Time Non periodic signals: The Fourier transform

A non-periodic continuous time signal $x(t)$ can be expressed as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \quad \text{——— (1)}$$

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{——— (2)}$$

$X(j\omega)$ is known as Fourier Transform (FT) of $x(t)$.

Alternatively we say that $X(j\omega)$ and $x(t)$ forms a FT pair, which can be expressed as

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

Here $X(j\omega)$ is the frequency domain representation of the time domain signal $x(t)$. $X(j\omega)$ is also known as spectrum of $x(t)$. eq^n (1) is known as synthesis eq^n and eq^n (2) is known as analysis equation.

The fourier transform $X(j\omega)$ for a continuous time signal $x(t)$ exists if the following conditions (referred to as Dirichlet conditions) are satisfied.

i) $x(t)$ is absolutely integrable i.e. $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

ii> $x(t)$ have a finite numbers of maxima and minima without any finite interval.

iii> $x(t)$ have a finite number of discontinuous within any finite interval.

Properties of FT:

1) Linearity
2) Time shift
3) Frequency shift
4) Scaling
5) Time differentiation
6) Integration
7) Convolution
8) Modulation
9) Parseval's theorem
10) Symmetry.
11) Frequency differentiation
12) Duality

1) Linearity

If $x(t) \xleftrightarrow{FT} X(j\omega)$

and $y(t) \xleftrightarrow{FT} Y(j\omega)$

then $z(t) = a\,x(t)$

$z(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$

Proof:

we have; $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$\therefore z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} dt$$

$$\therefore z(j\omega) = aX(j\omega) + bY(j\omega)$$

Hence the proof.

2) Time shift:

$$\text{If } x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{and } y(t) \xleftrightarrow{FT} Y(j\omega)$$

$$\text{then } y(t) = x(t-t_0) \xleftrightarrow{FT} Y(j\omega) = e^{-j\omega t_0} X(j\omega)$$

Proof: we have, $x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0) \, e^{-j\omega t} dt$$

put $t - t_0 = a$, then $dt = da$

$$y(j\omega) = \int_{-\infty}^{\infty} x(a)\, e^{-j\omega(a+t_0)}\, da$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(a)\, e^{-j\omega a}\, da$$

$$y(j\omega) = e^{-j\omega t_0}\, X(j\omega)$$

Hence the proof.

3) Frequency shift:

$$\text{If } x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then } y(t) \xleftrightarrow{FT}$$

Proof: we have;

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{j\beta t}\, \alpha(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\beta)t} \cdot dt$$

$\therefore Y(j\omega) = X(j(\omega-\beta))$ Hence the proof.

4) Scaling

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

then $y(t) = x(at) \xleftrightarrow{F.T.} Y(j\omega) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

Proof: we have

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\therefore Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

If a is +ve i.e. $a > 0$, put $at = \tau$

$$dt = \frac{1}{a} d\tau$$

$$\therefore Y(j\omega) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) \, e^{\frac{-j\omega\tau}{a}} . d\tau$$

$$= \frac{1}{a} X\left(\frac{j\omega}{a}\right) \text{ ——— (1)}$$

If a is -ve i.e. $a < 0$, put $-at = \tau$

$$dt = \frac{-1}{a} d\tau$$

$$\therefore Y(j\omega) = \frac{-1}{a} \int_{+\infty}^{-\infty} x(-\tau) e^{\frac{-j\omega}{a}(-\tau)} d\tau$$

$$= \frac{-1}{a}\int_{-\infty}^{\infty} x(-\tau)\, e^{-j(\omega/a)(-\tau)}\, d\tau = \frac{1}{a} X\left(\frac{-j\omega}{a}\right) \quad \text{--- (2)}$$

From eqn (1) and (2)

$$= \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

Hence the proof.

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5) Time Differentiation

$$\text{If } x(t) \xleftrightarrow{F.T.} X(j\omega).$$

$$\text{then } \frac{dx(t)}{dt} \xleftrightarrow{F.T.} j\omega X(j\omega)$$

Proof: we have $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$ --- (1)

differentiating both the sides w.r.to to "t" we get

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega\right]$$

changing the order of differentiation and integration, we get

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left(\frac{\partial}{\partial t} e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, j\omega \,.\, e^{j\omega t}\, d\omega$$

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$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega X(j\omega)]e^{j\omega t}d\omega \quad \text{——} ②$$

comparing eqn ② with eqn ①, we get,

$$\frac{dx(t)}{dt} \xleftrightarrow{F.T.} j\omega X(j\omega)$$

Hence the proof.

6) Frequency differentiation

$$\text{If } x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$\text{then } -jt\,x(t) \xleftrightarrow{F.T.} \frac{d}{d\omega}X(j\omega)$$

Proof: we have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{——} ①$$

diff both sides, we get

$$\frac{dX(j\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]$$

Interchanging the order of differentiation and integration,

we get, $$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)\left(\frac{\partial}{\partial\omega}e^{-j\omega t}\right)dt = \int_{-\infty}^{\infty} x(t)(-jte^{-j\omega t})dt$$

$$= \int_{-\infty}^{\infty}(-jt\,x(t))\,e^{-j\omega t}dt \quad \text{——} ②$$

comparing eq^n ① and eq^n ②

$$-jt\,x(t) \xleftrightarrow{F.T.} \frac{d}{d\omega} X(j\omega)$$

Hence the proof

7) Integration

$$\text{If } x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$\text{then } \int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T.} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$$

8) convolution

$$\text{If } x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$\text{and } y(t) \xleftrightarrow{F.T.} Y(j\omega)$$

$$\text{then } z(t) = x(t) * y(t) \xleftrightarrow{F.T.} Z(j\omega) = X(j\omega)\cdot Y(j\omega)$$

Proof:

we have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\, dt$$

$$\therefore Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)] \, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) \, y(t-\tau) \, d\tau \right] e^{-j\omega t} \cdot dt$$

Changing the order of differentiation and integration, we get

$$z(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(t-\tau) \, e^{-j\omega t} dt \right) d\tau$$

put $t - \tau = a$, then $dt = da$

$$z(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot \int_{-\infty}^{\infty} y(a) \, e^{-j\omega(a+\tau)} \, da \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \, e^{-j\omega\tau} \int_{-\infty}^{\infty} y(a) \, e^{-j\omega a} \cdot da$$

$$= X(j\omega) \cdot Y(j\omega)$$

Therefore convolution in time domain is equivalent to multiplication in frequency domain.

9) Modulation

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

and $y(t) \xleftrightarrow{F.T.} Y(j\omega)$

then $z(t) = x(t) \cdot y(t) \xleftrightarrow{F.T.} Z(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$

Proof : we have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d\omega$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} [x(t) \cdot y(t)] e^{-j\omega t} d\omega \quad \text{—— (1)}$$

we have, $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\beta) e^{j\beta t} d\beta$ —— (2)

subtract eq^n (2) in eq^n (1), we get,

$$Z(j\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\beta) \cdot e^{j\beta t} \cdot d\beta\right] y(t) \cdot e^{-j\omega t} dt$$

Changing the order of integrations, we get

$$Z(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\beta) \cdot \int_{-\infty}^{\infty} y(t) \cdot e^{-j(\omega-\beta)t} dt \cdot d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\beta)\, Y(j(\omega-\beta))\, d\beta$$

$$\therefore\ Z(j\omega) = \frac{1}{2\pi}\left[X(j\beta) * Y(j\beta)\right]$$

Multiplication in time domain is equivalent to convolution in frequency domain.

10> Parseval's Theorem:

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

then $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad \text{--- (1)}$$

Soln: eqn (1) $|X(j\omega)|^2$ is known as energy density spectrum of the signal $x(t)$. L.H.S of eqn (1) is the energy of the signal $x(t)$.

Proof:

we have $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

changing the order of integration, we get

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{*}(j\omega).\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{*}(j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Hence the proof.

11> Duality

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

then $X(jt) \xleftrightarrow{F.T.} 2\pi x(-\omega)$

Proof: we have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Interchanging t and ω, we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)\, e^{j\omega t} dt$$

$$\therefore 2\pi(-\omega)\, x = \int_{-\infty}^{\infty} X(jt)\, e^{-j\omega t} dt \quad \text{——} \; (1)$$

comparing eq^n ① with $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

we have $X(jt) \xleftrightarrow{F.T.} 2\pi x(-\omega)$

Hence the proof

12) Symmetry

If $x(t) \xleftrightarrow{F.T.} X(j\omega)$

then $x(t)$ real $\xleftrightarrow{F.T.} X^n(j\omega) = X(-j\omega)$

$x(t)$ img $\xleftrightarrow{F.T.} X^n(j\omega) = -X(-j\omega)$

If $x(t)$ real & even $\xleftrightarrow{F.T.}$ Img $\{X(j\omega)\} = 0$

$x(t)$ real & odd $\xleftrightarrow{F.T.}$ Re $\{X(j\omega)\} = 0$

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