1(a) For what values of h will y be in span $\{v_1, v_2, v_3\}$ if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$y = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right] \quad R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-8+3 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right] \quad \begin{array}{l} h-5=0 \\ h=5 \end{array}$$

Eg:- Express the vector $b = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of the vectors $v_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Solu: we need to find numbers x_1, x_2, x_3 satisfying

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b.$$

The vector equation is equivalent to matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} \qquad \begin{matrix} XV = b \\ X[v_1\ v_2\ v_3] = b \end{matrix}$$

Reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right] \qquad \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right] \qquad R_3 \leftrightarrow R_2 \qquad R_3 \to \tfrac{1}{2} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -1 & 3 \end{array}\right] \qquad R_3 \to R_3 + 3R_1$$

10/2
TC4

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

$$x_1 + x_2 + x_3 = 2$$

$$x_2 + x_3 = 4$$

$$5x_3 = 15 \Rightarrow x_3 = 3$$

$$x_2 + 3 = 4 \Rightarrow x_2 = 1$$

$$x_1 + 1 + 3 = 2 \Rightarrow x_1 = 2 - 4 = -2$$

$x_1 = 1 \quad x_2 = -2 \quad \& \quad x_3 = 3$

$$b = v_1 - 2v_2 + 3v_3$$

$$\begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 3 \\ 5 - 4 + 12 \\ -1 - 2 + 9 \end{bmatrix}$$

(5) Write the vector $\begin{bmatrix}1\\3\\-1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}2\\-2\\1\end{bmatrix}$ $\begin{bmatrix}2\\0\\4\end{bmatrix}$

Soln:-

x_1, x_2, x_3 are real numbers.

$$\begin{bmatrix}1\\3\\-1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-2\\1\end{bmatrix} + x_3\begin{bmatrix}2\\0\\4\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\0 & -2 & 0\\0 & 1 & 4\end{bmatrix} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\0 & -2 & 0 & 3\\0 & 1 & 4 & -1\end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\0 & 1 & 4 & -1\\0 & -2 & 0 & 2\end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c}1 & 2 & 2 & 1\\0 & 1 & 4 & -1\\0 & 0 & 8 & 1\end{array}\right]$$

$x_1 + 2x_2 + 2x_3 = 1$

$x_2 + 4x_3 = -1$

$8x_3 = 1 \Rightarrow x_3 = 1/8$

$x_2 + 4(1/8) = -1$

$x_2 = -1 - 1/2 = -3/2$

$x_1 + 2x_2 + 2x_3 = 1$

$x_1 + 2(-3/2) + 2(1/8) = 1$

$x_1 = 4 - 1/4 = 15/4$

$$\begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}15/4 + 2(-3/2) + 2(1/8)\\-2(-3/2)\\-3/2 + 4(1/8)\end{bmatrix}$$

$b = 15/4\, v_1 - 3/2\, v_2 + 1/8\, v_3$.

Null Space

Definition:- The null space of an $m \times n$ matrix A written as Null A is the set of all solutions to the homogeneous equation $Ax = 0$. In set notation

$$\text{Null } A = \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}$$

Ex:- Let $A = \begin{bmatrix}1 & -3 & -2\\-5 & 9 & 1\end{bmatrix}$ and let $u = \begin{bmatrix}5\\3\\-2\end{bmatrix}$. Determine if u belongs to the null space of A.

Soln: u satisfies $Au = 0$.

$$\begin{bmatrix}1 & -3 & -2\\-5 & 9 & 1\end{bmatrix}\begin{bmatrix}5\\3\\-2\end{bmatrix} = \begin{bmatrix}5-9+4\\-25+27-2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Thus u is in Null A.

The Column Space of a matrix

Definition: The column space of an $m \times n$ matrix A, written as Col A is the set of all linear combinations of the columns of A. If $A = [a_1 \ldots a_n]$ then

$$\text{Col } A = \text{Span}\{a_1, \ldots, a_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

3 (a) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ & $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

(i) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

Ans No, 3 vectors.

(ii) How many vectors are in Span $\{v_1, v_2, v_3\}$?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ -1 & 3 & 6 & : & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 5 & 10 & : & 5 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{5}R_3} \begin{bmatrix} 1 & 2 & 4 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 1 & 2 & : & 1 \end{bmatrix} \; R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$ $r = 2$ $n = 3$ $r < n$ Consistent.

and has infinitely many solution.

(iii) x_3 is a free variable. $x_3 = 1$

$x_2 + 2x_3 = 1$

$x_2 = 1 - 2 = -1$

$x_1 + 2x_2 + 4x_3 = 3 \Rightarrow x_1 - 2 + 4 = 3$ $x_1 = 3 - 2 = 1$

Note :- The subspace of all linear combination of the set of given vector space is called the subspace generated by these vectors or spanned by these vectors

(2) The subspace spanned by any non-zero vector α of a vector space V consists of all scalar multiples of α. Geometrically it represents a line through the origin and α.

(3) The subspace spanned by any two non zero vectors α & β which are not multiples of each other represents a plane passing through the origin α & β.

w is a subspace spanned by $\{v_1, v_2, v_3\}$

$w = 1v_1 - v_2 + v_3$

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-2+4 \\ 0-1+2 \\ -1-3+6 \end{bmatrix}$$

3(b) Show that w is in the subspace R^4 spanned by $v_1\ v_2\ v_3$ value.

(b)

$$\begin{array}{cccc} v_1 & v_2 & v_3 & w \end{array}$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -1 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + \frac{4}{7}R_1 \\ R_3 \to R_3 + \frac{2}{7}R_1 \\ R_4 \to R_4 - \frac{9}{7}R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & 19/7 & 8/7 & 13/7 \\ 0 & -15/7 & 10/7 & 10/7 \\ 0 & -13/7 & 32/7 & 137/7 \end{array}\right] \quad \begin{array}{ll} R_2 \to 7R_2 & R_3 \to \frac{1}{5}R_3 \\ R_3 \to 7R_3 & R_2 \leftrightarrow R_3 \\ R_4 \to 7R_4 & \end{array}$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 19 & -8 & 13 \\ 0 & -13 & 32 & 137 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + \frac{19}{3}R_2 \\ R_4 \to R_4 - \frac{13}{3}R_2 \end{array}$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14/3 & 77/3 \\ 0 & 0 & 70/3 & 385/3 \end{array}\right] \quad \begin{array}{l} R_3 \to 3R_3 \\ R_4 \to 3R_4 \end{array}$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 70 & 385 \end{array}\right] \quad R_4 \to R_4 - \frac{70}{14}R_3$$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{consistence.}$$

$$7x_1 - 4x_2 - 9x_3 = -9$$
$$-3x_2 + 2x_3 = 2$$
$$14x_3 = 77 \qquad x_3 = 5.5$$

$$-3x_2 + 2(5.5) = 2$$
$$-3x_2 = 2 - (5.5)2$$
$$-3x_2 = -9$$
$$x_2 = 3$$
$$7x_1 - 4(3) - 9(5.5) = -9.$$
$$7x_1 = -9 + 12 + 49.5$$
$$x_1 = \frac{52.5}{7} = 7.5$$

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4 (b) Find a spanning set for the null space of matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Soln

$$Ax = 0$$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$[A:0]$

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \quad R_2 \to \tfrac{1}{5} R_2$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r = 2 \\ n = 5 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 - x_4 + 3x_5 = 0$

$x_3 + 2x_4 - 2x_5 = 0$

$0 = 0$

x_2, x_4, x_5 are free variables.

$x_1 = 2x_2 + x_4 - 3x_5 \qquad x_3 = -2x_4 + 2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

5(a) Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ Let $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(a) Determine if u is in Nul A. Could u be in Col A ?

(b) Determine if v is in Col A could v be in Nul A ?

Solu

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u is not a solution of $Ax = 0$, so u is not a Nul A. u could not possibly be in Col A. Also with four entries u could not possibly be in Col A since Col A is a subspace of $\mathbb{R}^3$

(b) $[A \; v]$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ -2 & -5 & 7 & 3 & : & -1 \\ 3 & 7 & -8 & 6 & : & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to 2R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & -1 & 5 & 4 & : & 2 \\ 0 & 2 & 10 & 9 & : & -3 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & : & 3 \\ 0 & +1 & -5 & -4 & : & -2 \\ 0 & 0 & 0 & 1 & : & -7 \end{bmatrix}$$

It is clear that the equation $Ax = v$ is consistent so v is in Col A with only three entries, v could not possibly be in Nul A since Nul A is subspace of $\mathbb{R}^4$

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5(b) Let $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix}$ For each of the following vectors determine whether the vectors are in the null space $N(A)$

(a) $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then describe the null space $N(A)$ of the matrix A.

Soln: Null space is $AX = 0$

(a) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+0+3+0 \\ 0+0+1+0 \\ -3+0+4+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not a Null space.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+0+6-2 \\ 0-3+2+1 \\ -4-3+8-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ is a $N(A)$

(c) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ is a $N(A)$

(d) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ • the size of matrix A is 3×4 & X is 3×1 matrix multiplication is not possible $\therefore \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ vector is not in $N(A)$.

(5) Determine whether the following set of vectors linearly independent or linearly dependent. If the set is linearly dependent express one vector in the set as a linear combination of the other.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}$$

Soln: Consider the linear combination.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} = 0 \rightarrow (*)$$

with variables x_1, x_2, x_3, x_4

we determine whether there is $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$ satisfying the above linear combination (*)

The linear combination (*) is written as the matrix eq

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -2 \\ -1 & 3 & 0 & 7 \\ 0 & 4 & 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

To find the solutions of the equations we apply the $[A \mid 0]$ can be reduced by elementary row operation.

$$[A \mid 0] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ -1 & 3 & 0 & 7 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right] \quad \begin{matrix} R_3 + R_1 \\ \frac{1}{2}R_2 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 4 & -1 & 5 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right] \quad \begin{matrix} R_3 \rightarrow R_3 - 4R_2 \\ R_4 \rightarrow R_4 - 4R_2 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & 5 & 15 & 0 \end{array}\right] \quad \begin{matrix} R_3 \rightarrow \frac{1}{3}R_3 \\ R_4 \rightarrow \frac{1}{5}R_4 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \quad R_4 \rightarrow R_4 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

solution is given by

$x_1 = x_4 \quad x_2 = -2x_4 \quad x_3 = -3x_4$ where x_4 is a free variable

If we take $x_4 = 1$ then we have a nonzero solution

$x_1 = 1 \quad x_2 = -2 \quad x_3 = -3 \quad x_4 = 1$

Thus the set is linearly dependent.

Substituting the values into (*) we have

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} = 0$$

Solving for the last vector we obtain the linear combination

$$\begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2-3 \\ 0+4-6 \\ +1+6+0 \\ 0+8+3 \end{bmatrix}$$

(7) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ a \\ 5 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 4 \\ b \end{bmatrix}$ be vectors in R^3. Determine a condition on the scalars a, b so that the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent.

Sol: Consider the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ where 0 is the three dimensional zero vector. Our goal is to find a condition on a, b so that the above equation has a non trivial soln x_1, x_2, x_3.

This equation is equivalent to the 3×3 homogeneous system of linear equations.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & a & 4 \\ 0 & 5 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 2 & a & 4 & 0 \\ 0 & 5 & b & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & a-2 & 4 & 0 \\ 0 & 5 & b & 0 \end{array}\right] \quad \tfrac{1}{5} R_3$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & a-2 & 4 & 0 \\ 0 & 1 & b/5 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & a-2 & 4 & 0 \end{array}\right] \quad R_3 \to R_3 - (a-2)R_2$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & \frac{4 - b(a-2)}{5} & 0 \end{array}\right]$$

Case (i)

If $4 - \frac{b(a-2)}{5} = 0$ then we obtain matrix in echelon form

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This implies x_3 is a free variables hence homogeneous system has a non zero solution x_1, x_2, x_3. Hence in this case the set $\{v_1, v_2, v_3\}$ is linearly dependent.

Case (ii)

If $4 - \frac{b(a-2)}{5} \neq 0$

then divide the third row by this number and obtain

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

we obtain a solution $x_1 = x_2 = x_3 = 0$

Thus in this case the set $\{v_1, v_2, v_3\}$ is linearly independent.

Thus we conclude the set $\{v_1, v_2, v_3\}$ is linearly dependent iff $4 - \frac{b(a-2)}{5} = 0$.

Thus the condition on a, b is $b(a-2) = 20$

Find a basis for span (S) where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

Soln

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc:c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 6 & 1 & 0 \\ 1 & -1 & -2 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -4 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_1 \to R_1 - R_2 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to 1/2$$

$$\left[\begin{array}{cccc:c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the above matrix has leading 1's in first and third columns we can conclude the first and third vector of S form a basis of span (S).

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \right\}$$ is a basis for span (S).

(12) Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \quad v_5 = \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix}$$

Find a basis for the span (s)

Solu:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 1 & 7 \\ 2 & 1 & -1 & 4 & 0 \\ -1 & 1 & 5 & -1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & -1 & -3 & 2 & -4 \\ 0 & 2 & 6 & 0 & 4 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + R_3 \\ R_4 \to R_4 - 2R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that 1^{st} 2^{nd} & 4^{th} column vectors of R contain the leading 1 entries. Hence the 1^{st} 2^{nd} & 4^{th} column vectors of A form a basis of span(s)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$ is a basis for span(s)

that if a vector space V has a basis $B = \{b_1, b_2 \dots b_n\}$ then any set in V containing more than n vectors must be linearly dependent.

Soln: Let $\{u_1 \dots u_p\}$ be a set in V with more than n vectors. The coordinate vectors $[u_1]_B \dots [u_p]_B$ form a linearly dependent set in $\mathbb{R}^n$ because there are more vectors (p) than entries (n) in each vector. So $\exists$ scalars $c_1 \dots c_p$ not all zero such that

$$c_1[u_1]_B + \dots + c_p[u_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the coordinate mapping is a linear combination

$$[c_1u_1 + \dots + c_pu_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector on the right contains the n weights needed to build the vector $c_1u_1 + \dots + c_pu_p$ from the basis vectors in B. ie $c_1u_1 + \dots + c_pu_p = 0 \cdot b_1 + \dots + 0 \cdot b_n = 0$

Since the c_i are not all zero $\{u_1, u_2 \dots u_p\}$ is linearly dependent

This implies that if a vector space V has a basis $B = \{b_1 \dots b_n\}$ then each linearly independent set in V has no more than n vectors.

... transformation Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9(b) Find the dimension of the subspace spanned by the given vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}$ $v_4 = \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

Soln :-

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 5 & -20 & -15 \end{bmatrix} \quad R_3 \to \tfrac{1}{5} R_3$$

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 1 & -4 & -3 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -0 & 0 \end{bmatrix}$$

1^{st} & 2^{nd} column are basis.

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \text{Dim } V = 2$$

no more than n vectors.

10(a) Let $T: R^2 \to R^3$ be a linear transformation Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ be 2 dimensional vectors. Suppose that

$T(u) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $T(v) = T\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ Let $w = \begin{bmatrix} x \\ y \end{bmatrix}$

$\in R^2$ Find the formula for $T(w)$ in terms of x & y.

Solu

$w = au + bv$

$\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+3b \\ 2a+5b \end{bmatrix}$

$x = a + 3b \times 2 \Rightarrow 2x = \cancel{2a} + 6b$

$y = 2a + 5b \qquad - y = \cancel{2a} + 5b$

$2x - y = b$

$x = a + 3(2x - y)$

$x = a + 6x - 3y \Rightarrow a = 3y - 5x$

$w = au + bv \Rightarrow T(w) = T(au + bv)$

$T(w) = aT(u) + bT(v)$

$T\begin{bmatrix} x \\ y \end{bmatrix} = aT\begin{bmatrix} 1 \\ 2 \end{bmatrix} + bT\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$T\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} -3 \\ 5 \end{bmatrix} + b\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a+7b \\ 5a+b \end{bmatrix}$

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3(3y-5x) + 7(2x-y) \\ 5(3y-5x) + (2x-y) \end{bmatrix}$

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9y+15x+14x-7y \\ 15y-25x+2x-y \end{bmatrix} = \begin{bmatrix} 29x - 16y \\ -23x + 14y \end{bmatrix}$

(b) Let $T: R^2 \to R^3$ be linear transformation such that

$T\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $T\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}0\\-5\\1\end{bmatrix}$ Find the matrix representation of T (with respect to the standard basis for $\mathbb{R}^3$)

$e_1 = au + bv$

$\begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}3\\2\end{bmatrix} + b\begin{bmatrix}4\\3\end{bmatrix}$

$\left[\begin{array}{cc:c}3 & 4 & 1\\2 & 3 & 0\end{array}\right]$ $R_2 \to R_2 - \frac{2}{3}R_1$

$3 - \frac{2}{3}(4) = \frac{9-8}{3}$

$0 - \frac{2}{3}(1)$

$\left[\begin{array}{cc:c}3 & 4 & 1\\0 & \frac{1}{3} & -\frac{2}{3}\end{array}\right]$

$3a + 4b = 1$

$b\frac{1}{3} = -\frac{2}{3}$

$b = -2$

$3a + 4(-2) = 1$

$3a = 1 + 8$

$3a = 9$

$a = 3$

$e_2 = au + bv$

$\begin{bmatrix}0\\1\end{bmatrix} = a\begin{bmatrix}3\\2\end{bmatrix} + b\begin{bmatrix}4\\3\end{bmatrix}$

$T(e_1) = aT(u) + bT(v)$

$\begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\left[\begin{array}{cc:c}3 & 4 & 0\\2 & 3 & 1\end{array}\right]$ $R_2 \to R_2 - \frac{2}{3}R_1$

$3 - \frac{2}{3}(4) = \frac{9-8}{3}$

$1 - \frac{2}{3}(0)$

$\left[\begin{array}{cc:c}3 & 4 & 0\\0 & \frac{1}{3} & 1\end{array}\right]$

$3a + 4b = 0$

$\frac{1}{3}b = 1$ $b = 3$

$3a + 4(3) = 0$

$3a = -12$

$a = -4$

$A = \{T(e_1), T(e_2)\}$

$A = \left\{3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\-5\\1\end{bmatrix}, -4\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}0\\-5\\1\end{bmatrix}\right\}$

$A = \begin{bmatrix}3-0 & -4+0\\6+10 & -8-15\\9-2 & -12+3\end{bmatrix} = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}$