

1(a) For what values of h will y be in $\text{span}\{v_1, v_2, v_3\}$ if
 $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$ $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ & $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$

$$y = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array} \right] \quad R_2 \rightarrow R_2 + R_1$$

$$R_3 \rightarrow R_3 + 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array} \right] \quad R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-8+3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array} \right] \quad h-5=0$$

$$h=5$$

Q:- Express the vector $b = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of the vectors $v_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Soln we need to find numbers x_1, x_2, x_3 satisfying

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$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

The vector equation is equivalent to matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} \quad X V = b$$

$$X [v_1 \ v_2 \ v_3] = b$$

Reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{bmatrix} \quad R_2 \rightarrow R_2 - 5R_1$$

$$R_3 \rightarrow R_3 + R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{bmatrix} \quad R_3 \leftrightarrow R_2 \quad R_3 \rightarrow \frac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -1 & 3 \end{bmatrix} \quad R_3 \rightarrow R_3 + 3R_1$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 15 \end{bmatrix}$$

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$$x_1 + x_2 + x_3 = 2$$

$$x_2 + x_3 = 4$$

$$5x_3 = 15 \Rightarrow x_3 = 3$$

$$x_2 + 3 = 4 \Rightarrow x_2 = 1$$

$$x_1 + 1 + 3 = 2 \Rightarrow x_1 = 2 - 4 = -2$$

$$x_1 = -1 \quad x_2 = 1 \quad x_3 = 3$$

$$b = v_1 - 2v_2 + 3v_3$$

$$\begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 3 \\ 5 - 4 + 12 \\ -1 + 2 + 9 \end{bmatrix}$$

Q) write $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Solu. x_1, x_2, x_3 are real numbers.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{R}_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{\text{R}_3 - 2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \xrightarrow{\text{R}_3 \cdot (-\frac{1}{4})} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 = 1$$

$$x_1 + x_2 = 0$$

$$-4x_3 = 1 \Rightarrow x_3 = -\frac{1}{4}$$

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$$x_1 + 4(-\frac{1}{4}) = 1$$

$$x_1 = 1 - 1 = 0$$

$$x_1 + 2x_2 + 2x_3 = 1$$

$$0 + 2(-\frac{1}{4}) + 2(-\frac{1}{4}) = 1$$

$$x_1 + 4 - \frac{1}{4} = 1 \frac{3}{4}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

$$b = \frac{15}{4}v_1 - \frac{3}{2}v_2 + \frac{7}{8}v_3$$

Null Space

Definition: The null space of an m x n matrix A, written as Null A, is the set of all solutions to the homogeneous equation $Ax = 0$. An n-vector

$$\text{Null } A = \{x \mid Ax = 0 \text{ and } Ax = 0\}$$

Ex.: Let $A = \begin{bmatrix} 1 & -2 & -2 \\ -3 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Determine if $a = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$ belongs to the null space of A.

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Solu. a satisfies $Ab = 0$.

$$\begin{bmatrix} 1 & -2 & -2 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 10 + 4 \\ -15 + 0 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus a is in Null A.

The column space of a matrix

Definition: The column space of an m x n matrix A, written as Col A, is the set of all linear combinations of the columns.

Col A = $\{a_1, \dots, a_n\}$, $\text{Ran } A$.

Col A = Span $\{a_1, \dots, a_n\}$

The column space of an m x n matrix A is a

subspace of \mathbb{R}^m .

$$3(a) \text{ Let } v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \text{ & } w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(i) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

Ans No, 3 vectors.

(ii) How many vectors are in $\text{span}\{v_1, v_2, v_3\}$?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 4 : 3 \\ 0 & 1 & 2 : 1 \\ 0 & 5 & 7 : 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 : 3 \\ 0 & 1 & 2 : 1 \\ 0 & 5 & 10 : 5 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 5R_2} \begin{bmatrix} 1 & 2 & 4 : 3 \\ 0 & 1 & 2 : 1 \\ 0 & 0 & 0 : 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 4 : 3 \\ 0 & 1 & 2 : 1 \\ 0 & 0 & 0 : 0 \end{bmatrix}$$

$q_1 < n$ consistent.

and has infinitely many solution.

(iii) x_3 is a free variable. $x_3 = 1$ gives $x_2 = x_2 - x_3 = 1$
 $x_1 + 2x_2 + 4x_3 = 3 \Rightarrow x_1 + 2 + 4 = 3 \Rightarrow x_1 = 3 - 2 = 1$

$x_1 + 2x_2 + 4x_3 = 3 \Rightarrow x_1 + 2 + 4 = 3 \Rightarrow x_1 = 3 - 2 = 1$

Note:- The subspace of all linear combination of the set of given vector space is called the subspace generated by these vectors

- ② The subspace spanned by any non-zero vector a of a vector space V consists of all scalar multiples of a . Geometrically it represents a line through the origin and a .
- ③ The subspace spanned by any two non-zero vectors a & b , which are not multiples of each other represents a plane passing through the origin a & b .

w is a subspace

$$w = -1v_1 - v_2 + v_3$$

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-2+4 \\ 0-1+2 \\ -1+3+6 \end{bmatrix}$$

spanned by $\{v_1, v_2, v_3\}$

3(b) Show that w is in the subspace \mathbb{R}^7 spanned by v_1, v_2, v_3 where

$$\begin{pmatrix} v_1 & v_2 & v_3 & w \\ 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -1 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{pmatrix} \quad R_4 \rightarrow R_4 - \frac{9}{7}R_1$$

$$R_2 \rightarrow R_2 + \frac{4}{7}R_1$$

$$R_3 \rightarrow R_3 + \frac{4}{7}R_1$$

$$\begin{pmatrix} 7 & -4 & -9 & -9 \\ 0 & \frac{19}{7} & \frac{8}{7} & -\frac{4}{7} \\ 0 & -\frac{15}{7} & \frac{10}{7} & \frac{10}{7} \\ 0 & -\frac{13}{7} & \frac{22}{7} & \frac{13}{7} \end{pmatrix} \quad R_2 \rightarrow 7R_2 \quad R_3 \rightarrow \frac{1}{5}R_3$$

$$R_3 \rightarrow 7R_3 \quad R_2 \leftrightarrow R_3$$

$$R_4 \rightarrow 7R_4$$

$$\begin{pmatrix} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 19 & -8 & 13 \\ 0 & -13 & 32 & 137 \end{pmatrix} \quad R_3 \rightarrow R_3 + \frac{19}{3}R_2$$

$$R_4 \rightarrow R_4 - \frac{13}{3}R_2$$

$$\begin{pmatrix} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & \frac{4}{3} & \frac{7}{3} \\ 0 & 0 & \frac{70}{3} & \frac{385}{3} \end{pmatrix} \quad R_3 \rightarrow 3R_3$$

$$R_4 \rightarrow 3R_4$$

$$\begin{pmatrix} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 70 & 385 \end{pmatrix} \quad R_4 \rightarrow R_4 - \frac{70}{14}R_3$$

$$\begin{pmatrix} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{consistency}$$

$$7x_1 - 4x_2 - 9x_3 = -9$$

$$-3x_2 + 2x_3 = 2$$

$$14x_3 = 77 \quad x_3 = 5.5$$

$$-3x_2 + 2(5.5) = 2$$

$$-3x_2 = 2 - (5.5)2$$

$$-3x_2 = -9$$

$$x_2 = 3$$

$$7x_1 - 4(3) - 9(5.5) = -9$$

$$7x_1 = -9 + 12 + 49.5$$

$$x_1 = \frac{52.5}{7} = 7.5$$

4(b) Find a spanning set for the null space of

$$\text{matrix } A = \begin{bmatrix} -3 & 6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solve

$$AX = 0$$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Aug. } [A:0] \quad \begin{bmatrix} -3 & 6 & -1 & 1 & 7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + 3R_1} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2}$$

$$R_1 \rightarrow R_1 - 2R_2 \quad \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{n=2} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{n=5}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad x_1 + 2x_2 + x_4 + 3x_5 = 0 \\ x_1 + 2x_2 + x_4 + 3x_5 = 0 \\ 2x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0.$$

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x_2, x_4, x_5 are free variables.

$$x_1 = 2x_2 + x_4 + 3x_5 \quad x_3 = -2x_4 + 2x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 + 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

$$5(a) \text{ Let } A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \text{ let } u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

(a) Determine if u is in $\text{Nul } A$. Could u be in $\text{Col } A$?

(b) Determine if v is in $\text{Col } A$ could v be in $\text{Nul } A$?

$$\text{Solve } \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u is not a solution of $Ax=0$, so u is not in $\text{Nul } A$.

Also with four entries u could not possibly be in $\text{Col } A$.
since $\text{Col } A$ is a subspace of \mathbb{R}^3

(b) $[A \ v]$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1 \\ \quad R_3 \rightarrow 2R_3 - 3R_1$$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 2 & 10 & 9 & -3 \end{bmatrix} \quad R_3 \rightarrow R_3 + 2R_2$$

$$= \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & 4 & 2 \\ 0 & 0 & 0 & 1 & -7 \end{bmatrix}$$

It is clear that the equation $Ax=v$ is consistent so v is in $\text{Col } A$ with only three entries, v could not possibly be in $\text{Nul } A$ since $\text{Nul } A$ is subspace of \mathbb{R}^4

5(b) Let $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix}$ For each of the following vectors determine whether the vectors are in the null space $N(A)$

① $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ② $\begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ ③ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ④ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, Then describe the null space $N(A)$ of the matrix A.

Solu Null space is $AX = 0$

$$\textcircled{a} \quad \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+0+3+0 \\ 0+0+1+0 \\ -3+0+4+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not a Null space.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+0+6-2 \\ 0+3+2+1 \\ -4-3+8-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is a } N(A)$$

$$\textcircled{c} \quad \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is a } N(A)$$

$\textcircled{d} \quad \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ * The size of matrix A is 3×4
 A \times is 3×1 matrix multiplication
 is not possible $\therefore \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ vector is not in $N(A)$.

- ④ determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent express one vector in the set as a linear combination of the other.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution Consider the linear combination:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 0 \quad \textcircled{1}$$

with variables x_1, x_2, x_3, x_4, x_5

we determine whether there is $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$ satisfying the above linear combination $\textcircled{1}$.

The linear combination $\textcircled{1}$ is written as the matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 2 & -2 & 2 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

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To find the solutions of the equations we apply the [A|0] can be reduced by elementary row operations.

$$\text{[A|0]} : \begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 2 & -2 & 2 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0 \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \quad R_2 \rightarrow \frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & \frac{1}{4} \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix} \quad R_3 \rightarrow 2R_3 - R_2$$

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$$\begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 2 & \frac{1}{2} \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix}$$

The solution is given by

$$x_1 = x_5 - \frac{1}{2}x_4, \quad x_2 = -x_4, \quad x_3 = 2x_4, \quad \text{where } x_4 \text{ is a free variable}$$

If we take $x_4=1$ then we have a unique solution

$$x_1 = x_5 - 2, \quad x_2 = -1, \quad x_3 = 2, \quad x_4 = 1$$

Thus the set is linearly dependent.

Substituting the values into $\textcircled{1}$ we have

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

Solving for the last vector we obtain the linear

combination

$$\begin{bmatrix} -2 \\ -2 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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Q7 Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 4 \\ b \end{bmatrix}$ be vectors in \mathbb{R}^3 . Determine a condition on the scalars a, b so that the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent.

Sol Consider the equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$
 where 0 is the three dimensional zero vector.
 Our goal is to find a condition on a, b so that
 the above equation has a non-trivial solution x_1, x_2, x_3 .
 This equation is equivalent to the 3×3 homogeneous system
 of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & a & 4 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & a-2 & 4 \\ 0 & 0 & b \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 4R_1}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & a-2 & 4 \\ 0 & 0 & b-4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & a-2 & 0 \\ 0 & 0 & b-4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - (a-2)R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & a-2 & 0 \\ 0 & 0 & b-4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & a-2 & 0 & 0 \\ 0 & 0 & b-4 & 0 \end{bmatrix}$$

Case (i)

If $b(a-2) = 0$ Then we obtain matrix in echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Case (ii)
 If $b(a-2) \neq 0$ Then we obtain matrix in echelon form
 The implies x_3 is a free variable hence homogeneous
 system has a non-zero solution x_1, x_2, x_3
 Hence the set $\{v_1, v_2, v_3\}$ is linearly dependent.

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Case (iii)
 If $b(a-2) \neq 0$
 Then divide the third row by this number and obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To obtain a solution $x_1 = x_2 = x_3 = 0$
 Thus in this case the set $\{v_1, v_2, v_3\}$ is linearly independent.

Thus we conclude the set $\{v_1, v_2, v_3\}$ is linearly dependent if $b(a-2) \neq 0$.

Thus the condition on a, b is $b(a-2) = 0$

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Find a basis for $\text{span}(S)$ where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

Solve $x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 6 & 1 & 0 \\ 1 & -1 & -2 & 3 & 0 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -4 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \\ R_1 \rightarrow R_1 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad R_2 \rightarrow \frac{1}{2}R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Since the above matrix has leading 1's in first and third columns we can conclude the first and third vectors of S form a basis of $\text{span}(S)$.
 $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \right\}$ is a basis for $\text{span}(S)$.

Q2 Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where
 $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 5 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 2 \\ 7 \\ 0 \\ 2 \end{bmatrix}$

Find a basis for the span(S)

Solve $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 1 & 7 \\ 2 & 1 & -1 & 4 & 0 \\ -1 & 1 & 5 & -1 & 2 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$R_4 \rightarrow R_4 + R_1$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & -1 & -3 & 2 & -4 \\ 0 & 2 & 6 & 0 & 4 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$R_3 \rightarrow R_3 + R_2$$

$$R_4 \rightarrow R_4 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 2R_3$$

$$R_2 \rightarrow R_2 + R_3$$

$$R_4 \rightarrow R_4 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the 1st, 2nd & 4th column vectors of the matrix contain the leading 1 entries. Hence the 1st, 2nd & 4th column vectors of A form a basis of span(S).

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for span(S)

that if a vector space V has a basis $B = \{b_1, b_2, \dots, b_n\}$ then any set in V containing more than n vectors must be linearly dependent.

Sol: Let $\{u_1, \dots, u_p\}$ be a set in V with more than n vectors. The coordinate vectors $[u_1]_B, \dots, [u_p]_B$ form a linearly dependent set in \mathbb{R}^n because there are more vectors (p) than entries (n) in each vector, so \exists scalars c_1, \dots, c_p not all zero such that

$$c_1[u_1]_B + \dots + c_p[u_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

since the coordinate mapping is a linear combination

$$[c_1u_1 + \dots + c_pu_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector on the right contains the n weights needed to build the vector $c_1u_1 + \dots + c_pu_p$ from the basis vectors in B , i.e. $c_1u_1 + \dots + c_pu_p = 0 \cdot b_1 + \dots + 0 \cdot b_n = 0$.

Since the c_i are not all zero, $\{u_1, u_2, \dots, u_p\}$ is linearly dependent.

This implies that if a vector space V has a basis

$B = \{b_1, \dots, b_n\}$, then each linearly independent set in V has no more than n vectors.

Implementation Let $u = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

9(b) Find the dimension of the subspace spanned by the given vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}$, $v_4 = \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

Solu

$$\left[\begin{array}{cccc} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{array} \right] R_3 \rightarrow R_3 - 2R_1$$

$$\left[\begin{array}{cccc} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 5 & -20 & -15 \end{array} \right] R_3 \rightarrow \frac{1}{5}R_3$$

$$\left[\begin{array}{cccc} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 1 & -4 & -3 \end{array} \right] R_1 \rightarrow R_1 - 3R_2$$

$$\left[\begin{array}{cccc} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -8 & 0 \end{array} \right]$$

1st & 2nd column are basis.

$$\text{Span} \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 2 \end{array} \right], \left[\begin{array}{c} 3 \\ 1 \\ 1 \end{array} \right] \right\} \quad \dim V = 2$$

no more than n vectors.

10(a) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation. Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ be 2 dimensional vectors. Suppose that

$$T(u) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \text{ and } T(v) = T\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \text{ let } w = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\in \mathbb{R}^2$ Find the formula for $T(w)$ in terms of x & y .

Solve $w = au + bv$

$$\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+3b \\ 2a+5b \end{bmatrix}$$

$$x = a+3b \quad x_2 \Rightarrow 2x = 2a+6b$$

$$y = 2a+5b \quad -y = -2a-5b$$

$$2x-y = b$$

$$x = a+3(2x-y)$$

$$x = a+6x-3y \Rightarrow a = 3y-5x$$

$$w = au + bv \Rightarrow T(w) = T(au+bv)$$

$$T(w) = aT(u) + bT(v)$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = aT\begin{bmatrix} 1 \\ 2 \end{bmatrix} + bT\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} -3 \\ 5 \end{bmatrix} + b\begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a+7b \\ 5a+b \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3(3y-5x)+7(2x-y) \\ 5(3y-5x)+(2x-y) \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9y+15x+14x-7y \\ 15y-25x+2x-y \end{bmatrix} = \begin{bmatrix} 29x-16y \\ -23x+14y \end{bmatrix}$$

- (b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be linear transformation such that
 $T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $T\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$ find the matrix representation
of T (with respect to the standard basis for \mathbb{R}^3)

$$e_1 = au + bv$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 3 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - \frac{2}{3}R_1$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \quad 3 - \frac{2}{3}(4) = \frac{9-8}{3}$$

$$3a + 4b = 1 \quad 3a + 4(-\frac{2}{3}) = 1$$

$$b\frac{1}{3} = -\frac{2}{3} \quad 3a = 1 + \frac{8}{3}$$

$$b = -2 \quad 3a = \frac{11}{3}$$

$$a = \frac{11}{9}$$

$$a = 3$$

$$e_2 = au + bv$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} 3 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad T(e_2) = aT(u) + bT(v)$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 - \frac{2}{3}R_1$$

$$3 - \frac{2}{3}(4) = \frac{9-8}{3}$$

$$\begin{bmatrix} 3 & 4 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \quad 1 - \frac{2}{3}(0)$$

$$3a + 4b = 0 \quad b = 3$$

$$\frac{1}{3}b = 1$$

$$b = 3$$

$$3a + 4(3) = 0$$

$$3a = -12$$

$$a = -4$$

$$A = \{T(e_1), T(e_2)\}$$

$$A = \left\{ 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}, -4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 3-0 & -4+0 \\ 6+10 & -8-15 \\ 9-2 & -12+3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 7 & -9 \end{bmatrix}$$

Q) Find modulus & amplitude of $\frac{(1+i)^2}{3+i}$

$$z = x+iy$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - amplitude

$$|z| = \sqrt{x^2+y^2} \rightarrow \text{modulus}$$

∴ So, first we need to transform $\frac{(1+i)^2}{3+i}$ to $z = x+iy$ form.

$$\Rightarrow \frac{1+i^2+2i}{3+i}$$

$$= \frac{1-1+2i}{3+i}$$

$$= \frac{2i}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{6i-2i^2}{3^2-i^2}$$

$$= \frac{6i+2}{10}$$

$$= \frac{6i+2}{10} \Rightarrow \frac{2}{10} + \frac{6i}{10}$$

$$= \left(\frac{1}{5}\right) + \left(\frac{3}{5}\right)i$$

$$x + iy$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3/5}{1/5}\right) = \tan^{-1}(3)$$

$$= \underline{\underline{71.5}}$$

$$|z| = \sqrt{x^2+y^2}$$

$$= \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2} \Rightarrow \sqrt{\frac{1}{25} + \frac{9}{25}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{5} //$$

1. b) Prove that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Now put $x = iy$

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \\ &= 1 + iy - \frac{y^2}{2!} + \frac{-iy^3}{3!} + \frac{iy^4}{4!} \end{aligned}$$

$$e = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Now put $x = iy$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} \dots$$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} \dots$$

$$\frac{e^{iy}}{e^{-iy}} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots \right) + i \left(y - \frac{y^3}{3!} + \dots \right)$$

$$\boxed{e^{iy} = \cos y + i \sin y} \rightarrow \textcircled{1}$$

If $x = -iy$

$$\boxed{e^{-iy} = \cos y - i \sin y} \rightarrow \textcircled{2}$$

$\textcircled{1} + \textcircled{2}$

$$e^{iy} + e^{-iy}$$

$$= 2 \cos y$$

$$\boxed{\cos y = \frac{e^{iy} + e^{-iy}}{2}}$$

$\textcircled{1} - \textcircled{2}$

$$e^{iy} - e^{-iy}$$

$$= 2i \sin y$$

$$\boxed{\sin y = \frac{e^{iy} - e^{-iy}}{2i}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x =$$

Q) b) Define the following.

i) Limit of a complex function

A complex valued function $f(z)$ defined on a neighbourhood of a point z_0 is said to have a limit l as $z \rightarrow z_0$, if for every $\epsilon > 0$, however small there exists a positive real number δ such that $|f(z) - l| < \epsilon$ when $|z - z_0| < \delta$. We write $\lim_{z \rightarrow z_0} f(z) = l$.

ii) Continuity of a complex function

A complex value function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exists and $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

iii) Differentiability of a complex function

A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique. This limit when exist is called derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$. Suppose we write $z - z_0 = \delta z$, then $z \rightarrow z_0 \rightarrow z$ $\delta z \rightarrow 0$. Hence $f'(z_0) = \lim_{\delta z \rightarrow 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$

3. a) If $f(z) = u + iv$ is analytic, then prove that Cauchy Riemann equations $u_x = v_y$, $v_x = -u_y$ are true.

Proof: given $f(z) = u + iv$

$$z = x + iy$$

$$\therefore f(x+iy) = u + iv \rightarrow ①$$

Differentiating ① w.r.t x

$$f'(x+iy) = u_x + iv_x \rightarrow ②$$

Differentiating ① w.r.t partially w.r.t y ,

$$f'(x+iy)i = u_y + iv_y$$

$$f'(x+iy) = \frac{1}{i} [u_y + iv_y]$$

$$= \frac{u_y}{i} + v_y \quad \left[\frac{1}{i} = -i \right]$$

$$\Rightarrow -i u_y + v_y \quad \longrightarrow (3)$$

Equating LHS RHS of (2) & (3),

$$u_x + iv_x = -iu_y + v_y$$

$$\Rightarrow \underline{u_x = v_y} \quad \text{and} \quad \underline{v_x = -u_y}$$

b) If $f'(z) = u(r, \theta) + iv(r, \theta)$ is analytic, then prove that Cauchy Riemann equation in polar form $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ are true. i.e., $u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$ are true.

Proof:- Given $f(z) = u+iv$.

$$z = re^{i\theta}$$

$$\Rightarrow f(re^{i\theta}) = u+iv \quad \longrightarrow (1)$$

Differentiate (1) partially w.r.t r !

$$f'(re^{i\theta})(e^{i\theta}) = u_r + iv_r \quad \longrightarrow (2)$$

Differentiate (1) partially w.r.t θ .

$$f'(re^{i\theta})(ire^{i\theta}) = u_\theta + iv_\theta \quad \text{or}$$

$$\Rightarrow f'(re^{i\theta})(e^{i\theta}) = \frac{1}{ir} [u_\theta + iv_\theta]$$

$$= -\frac{i u_\theta}{r} + \frac{v_\theta}{r} \quad \longrightarrow (3)$$

Equating real & imaginary parts of (2) & (3)

$$u_r + iv_r = -\frac{i u_\theta}{r} + \frac{v_\theta}{r}$$

$$\therefore u_r = \frac{v_\theta}{r} \quad \text{and} \quad v_r = -\frac{u_\theta}{r}$$

$$\Rightarrow \underline{u_r = \frac{1}{r} v_\theta} \quad \text{and} \quad \underline{v_r = -\frac{1}{r} u_\theta}$$

4 a) If $u = x^2 - y^2$ and $v = x^3 - 3xy^2$, show that u and v are harmonic functions but $f(z) = u + iv$ is not analytic.

u and v are said to be harmonic, if it satisfies Laplace equation.

$$U_{xx} + U_{yy} = 0$$

$$V_{xx} + V_{yy} = 0$$

$$u = x^2 - y^2$$

$$u_x = 2x$$

$$u_{xx} = 2$$

$$u_y = -2y$$

$$u_{yy} = -2$$

$$v = x^3 - 3xy^2$$

$$v_x = 3x^2 - 3y^2$$

$$v_{xx} = 6x$$

$$v_y = -6xy$$

$$v_{yy} = -6x$$

$$u_{xx} + u_{yy} = 2 - 2 = 0 \quad \therefore u \text{ is harmonic}$$

$$v_{xx} + v_{yy} = 6x - 6x = 0 \quad \therefore v \text{ is harmonic}$$

To know functions are analytic, it must satisfy C-R equations.

$$u_x = -v_y \quad \text{and} \quad v_x = -u_y$$

$$u_x = 2x \quad \therefore -v_y = 6xy$$

$$u_x \neq -v_y$$

$$v_x = 3x^2 - 3y^2 \quad -uy = 2y$$

$$\therefore v_x \neq -uy \quad \therefore u + iv \text{ is not analytic}$$

4 b) If $u = \frac{x^2}{y}$, $y \neq 0$ and $v = x^2 + 2y^2$, then show that curves $u = \text{constant}$ and $v = \text{constant}$ are orthogonal but $f(z) = u + iv$ is not an analytic function.

$$u = \frac{x^2}{y} = C_1$$

$$v = x^2 + 2y^2 = C_2$$

Diffr w.r.t x treating y as a function of x .

$$\underline{y^2(3x) - x^2 \frac{dy}{dx}} = 0$$

$$y^2$$

$$2xy - x^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2xy}{x^2}$$

$$\frac{dy}{dx} = \frac{2y}{x} = m_1$$

$$u = x^2/y$$

$$u_x = \frac{2y}{y} = 2$$

$$u_y = -\frac{x^2}{y^2}$$

$$\underline{2x + 4y - C_2} \quad 2x + 4y \cdot \frac{dy}{dx} = 0$$

$$x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{2y}$$

$$m_2 = -\frac{x}{2y}$$

$$m_1 \times m_2 = \frac{2y}{x} \times -\frac{x}{2y} = -1$$

\therefore orthogonal.

$$v = x^2 + 2y^2$$

$$v_x = 2x$$

$$v_y = 4y$$

$$u_x \neq v_y \quad \text{and} \quad v_x \neq -u_y$$

(5) a) Show that $f(z) = z^n$ is analytic. Hence find its derivative.

$$\text{given } f(z) = z^n \quad [z = re^{i\theta}]$$

$$f(re^{i\theta}) = (re^{i\theta})^n$$

$$u + iv = (re^{i\theta})^n$$

$$\therefore r^n e^{in\theta} \rightarrow r^n [\cos n\theta + i \sin n\theta]$$

$$\therefore u + iv = r^n \cos n\theta + i r^n \sin n\theta$$

$$u = r^n \cos n\theta$$

$$v = r^n \sin n\theta$$

$$u_r = nr^{n-1} \cos n\theta$$

$$v_r = nr^{n-1} \sin n\theta$$

$$u_\theta = -r^n \sin n\theta$$

$$v_\theta = +r^n \cos n\theta$$

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta \text{ is satisfied } \therefore \text{analytic}$$

Also we have in the polar form, $f'(z) = e^{i\theta} [u_r + iv_r]$

$$\Rightarrow e^{-i\theta} [nr^{n-1} \cos n\theta + i nr^{n-1} \sin n\theta]$$

$$\Rightarrow e^{-i\theta} \cdot nr^{n-1} (\cos n\theta + i \sin n\theta)$$

$$= e^{-i\theta} \cdot nr^{n-1} \cdot e^{in\theta}$$

$$= e^{i(n-1)\theta} \cdot nr^{n-1}$$

$$= n [re^{i\theta}]^{n-1}$$

$$\Rightarrow \underline{n z^{n-1}}$$

5 b) Show that function $f(z) = \log z$ is analytic & hence find its derivative.

$$\text{given } f(z) = \log z \quad \text{taking } z = re^{i\theta}$$

$$u + iv = \log(re^{i\theta})$$

$$= \log r + i\theta e^{i\theta}$$

$$= \log r + i\theta \cdot \log e \quad [\log e = 1]$$

$$= \log r + i\theta$$

$$u = \log r \quad v = \theta$$

$$u_r = \frac{1}{r}$$

$$v_r = 0$$

$$u_\theta = 0$$

$$v_\theta = 1$$

$$u_r = \frac{1}{r} v_\theta$$

$$v_\theta = -\frac{1}{r} u_r$$

Satisfied \therefore analytic.

In polar form, $f'(z) = e^{i\theta} [u_r + iv_r]$

$$= e^{-i\theta} \left[\frac{1}{r} + i0 \right]$$

$$= e^{-i\theta} \left[\frac{1}{r} \right]$$

$$= \frac{1}{r e^{i\theta}}$$

$$= \frac{1}{z}$$

$\therefore \frac{1}{z}$ is derivative

⑥ a) Show that the function $f(z) = \cosh z$ is analytic & hence find its derivative.

$$f(z) = \cosh z$$

$$z = x + iy$$

$$\rightarrow \cosh(x+iy) = \cos i(x+iy) = \cos(ix-y)$$

~~$\Rightarrow \cosh x \cosh iy + \sinh x \sin iy = \cos a \cos b + \sin a \sin b.$~~

$$\Rightarrow \cos ix \cdot \cos y + \sin ix \cdot \sin y.$$

$$\Rightarrow \cosh x \cdot \cos y + i \sinh x \cdot \sin y.$$

$$u = \cosh x \cdot \cos y$$

$$v = \sinh x \cdot \sin y.$$

$$u_x = \sinh x \cdot \cos y$$

$$v_x = \cosh x \cdot \sin y$$

$$u_y = -\cosh x \cdot \sin y$$

$$v_y = \cosh x \cdot \cos y$$

$$u_x = v_y \text{ and } v_x = -u_y \quad \therefore \text{analytic}$$

In cartesian form,

$$f'(z) = u_x + iv_x$$

$$= \sinh x \cdot \cos y + i \cosh x \cdot \sin y.$$

$$= i \sin ix \cdot \cos y + i \cos ix \cdot \sin y.$$

$$= i [\sin ix \cdot \cancel{\cos y} + \cos ix \cdot \cancel{\sin y}]$$

$$= i [\sin(ix + iy)] \quad i [\sin(ix + iy)]$$

$$\downarrow = i [\sin(ix - iy)] \Rightarrow \underline{\sinh x \cdot \cos y + i \cosh x \cdot \sin y}$$

$$\sinh x \cdot \cos y + i \cosh x \cdot \sin y$$

Multiply & divide i in RHS.

$$\Rightarrow \underline{\sinh z}$$

$$f'(z) = \frac{i}{i} [i \sin ix \cdot \cos y + i \cos ix \cdot \sin y]$$

$$= \frac{1}{i} [\sinh x \cdot \cos y + i \cosh x \cdot \sin y]$$

$$= \frac{1}{i} [i \sinh x \cdot \cos y - \cosh x \cdot \sin y]$$

$$= \frac{1}{i} [\sin ix \cdot \cos y - \cosh x \cdot \sin y]$$

$$= \frac{1}{i} [\sin(ix - y)] \quad \text{By } \sin(a-b) \text{ formula}$$

$$= \frac{1}{i} [\sin i(x+iy)]$$

$$= \frac{1}{i} \sin i z$$

$$= \underline{\sinh z}$$

$$\boxed{\begin{aligned} \sin ix &= i \sinh x \\ \sinh x &= -i \sin ix \\ &= \frac{1}{i} \sin i x \end{aligned}}$$

6. b) Show that $w = z + e^z$ is analytic & hence find its derivative

$$f(z) = z + e^z$$

$$u+iv = (x+iy) + e^{(x+iy)}$$

$$= x+iy + e^x \cdot e^{iy}$$

$$= x+iy + e^x [\cos y + i \sin y]$$

$$= x+iy + e^x \cos y + i e^x \sin y$$

$$u = x + e^x \cos y$$

$$u_x = 1 + e^x \cos y$$

$$u_y = -e^x \sin y$$

$$v = y + e^x \sin y$$

$$v_x = e^x \sin y$$

$$v_y = 1 + e^x \cos y$$

$$\cos y \quad \cos y$$

$$\sin y \quad \sin y$$

$u_x = v_y$ and $v_x = -u_y$ \therefore CR eqn satisfied, \therefore analytic.

In cartesian form, $f'(z) = u_x + iv_x$.

$$\Rightarrow 1 + e^x \cos y + i[e^x \sin y]$$

$$= 1 + e^x [\cos y + i \sin y]$$

$$1 + e^x e^{iy}$$

$$= 1 + e^{x+iy}$$

Final answer $= \underline{1 + e^z}$

7 a) Show that the function $f(z) = \sin 2z$ is analytic & hence find

$$\text{given } f(z) = \sin 2z \quad \text{taking } z = x+iy$$

$$u+iv = \sin(2x+iy)$$

$$= \sin(2x + iy)$$

$$= \sin 2x \cdot \cos iy + \cos 2x \cdot \sin iy$$

$$= \sin 2x \cdot \cosh iy + i(\cos 2x \cdot \sinh iy)$$

$$u = \sin 2x \cdot \cosh iy$$

$$v = \cos 2x \cdot \sinh iy$$

$$u_x = 2\cos 2x \cdot \cosh iy$$

$$v_x = -2\sin 2x \cdot \sinh iy$$

$$u_y = 2\sin 2x \cdot \sinh iy$$

$$v_y = 2\cos 2x \cdot \cosh iy$$

$$u_x = v_y \quad \& \quad v_x = -u_y \quad \text{Hence CR eqn satisfied} \quad \therefore \text{analytic.}$$

In cartesian form, $f'(z) = u_x + iv_x$

$$\Rightarrow 2\cos 2x \cdot \cosh iy + i(-2\sin 2x \cdot \sinh iy)$$

$$= 2\cos 2x \cdot \cosh iy - 2\sin 2x \cdot \sinh iy$$

$$= 2[\cosh iy \cdot \cos 2x - \sinh iy \cdot \sin 2x]$$

$$= 2 \cos(\alpha z + i\beta y)$$

$$= 2 \cos \alpha (x+iy)$$

$$= \underline{2 \cos \alpha z}$$

7 b) Find the analytic function $f(z) = u+i\nu$, where

$$u = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$u = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$u_x = 2x + \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}$$

$$= 2x + \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \Rightarrow u_x = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_y = -2y + x \left(\frac{-2y}{(x^2 + y^2)^2} \right)$$

$$= -\left[2y + \frac{2xy}{(x^2 + y^2)^2} \right]$$

$$\text{Consider } f'(z) = u_x + i\nu_x \\ = u_x - iu_y \quad [\because \nu_x = -u_y]$$

$$= 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \left[2y + \frac{2xy}{(x^2 + y^2)^2} \right]$$

Putting $x=z$ and $y=0$ to obtain $f'(z)$ as function of z

$$= 2z + \frac{(-z^2)}{z^4} + i \left[\frac{0}{z^4} \right]$$

$$f'(z) = 2z - \frac{1}{z^2}$$

On integrating we get

$$f(z) = \int \left(2z - \frac{1}{z^2} \right) dz + C$$

$$= \frac{2z^2}{2} + \frac{1}{z} + C$$

$$\therefore f(z) = z^2 + \frac{1}{z} + C$$

8 a) Find the analytic function whose real part is

$$\log \sqrt{x^2 + y^2}$$

given $u = \log \sqrt{x^2 + y^2} \Rightarrow u = \log(x^2 + y^2)^{1/2} \Rightarrow u = \frac{1}{2} \log(x^2 + y^2)$

$$u_x = \frac{1}{2} x \cdot \frac{\partial x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{1}{2} x \cdot \frac{\partial y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

$$f'(z) = u_x + i v_x$$

$$= u_x - i u_y \quad [v_x = -u_y]$$

$$\Rightarrow \frac{u}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

Put $x=z$ and $y=0$ to find $f'(z)$ as function of z .

$$\Rightarrow \frac{z}{z^2} - i(0)$$

$$\underline{f'(z) = \frac{1}{z}}$$

Integrate,

$$\int f'(z) dz = \int \frac{1}{z} dz$$

$$\underline{f(z) = \log z + C}$$

8 b) Find analytic function $f(z) = u + iv$, where $v = \frac{y}{x^2 + y^2}$

$$v = \frac{y}{x^2 + y^2}$$

$$v_x = \frac{-(2x)y}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$v_y = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$f'(z) = \cancel{u_x + iv_x} \quad \cancel{v_x + iv_y} \quad u_x + iv_x$$

$$\cancel{u_x - i v_y} \quad \cancel{v_x + i v_y} \quad v_y + i v_x$$

$$f'(z) = \frac{x^2 - y^2}{(x^2 + y^2)^2} + i \left[\frac{-2xy}{(x^2 + y^2)^2} \right]$$

Putting $x=z, y=0$ we have,

$$f'(z) = \frac{z^2}{z^4} = \frac{1}{z^2}$$

$$f(z) = \int \frac{1}{z^2} dz + C$$

$$f(z) = -\frac{1}{z} + C$$

Q. a) Find analytic function $f(z) = u + iv$, where $v = e^x(x \sin y + y \cos y)$
using milne thompson method.

$$v = e^x(x \sin y + y \cos y)$$

$$v_x = e^x(\sin y) + (x \sin y + y \cos y)e^x$$

$$v_x = e^x[\sin y + x \sin y + y \cos y]$$

$$v_y = e^x(x \cos y + [y \sin y + \cos y])$$

$$= e^x(x \cos y - y \sin y + \cos y)$$

$$f'(z) = \cancel{v_x + iv_y} \quad u_x + iv_x \\ = \cancel{v_x + iv_y} \quad v_y + iv_x$$

$$= e^x(\sin y + x \sin y + y \cos y) + i[e^x(x \cos y - y \sin y + \cos y)]$$

Put $x=z$ and $y=0$ to get $f'(z)$ in function of z .

$$\Rightarrow e^z(z) + i[e^z(z+1)]$$

$$\Rightarrow e^z(z \cos y - y \sin y + \cos y) + i[e^z(\sin y + x \sin y + y \cos y)]$$

Put $x=z$ and $y=0$ to get $f'(z)$ in function of z

$$= e^z(z+1)$$

$$\therefore f'(z) = e^z(z+1)$$

Integrate

$$f(z) = \int e^z(z+1)$$

$$= \int (z+1)e^z$$

$$= (z+1)\int e^z - \int [se^z] \cdot \frac{d(z+1)}{dz}$$

$$= (z+1)e^z - \int e^z \cdot 1$$

$$= (z+1)e^z - e^z$$

$$= e^z(z+1-1)$$

$$f(z) = \boxed{e^z \cdot z}$$

$$\int uv dz = u \int v dz - \int \left(\frac{du}{dz} v dz \right)$$

$$\int u (dv) = uv - \int v du$$

$$\int u v = u \int v - \int v \cdot \frac{du}{dz}$$

$$= x \sin x \int \cancel{\cos x}$$

$$x \cos x$$

$$x \sin x - \int \sin x \cdot 1$$

Sol:

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9.b) Find the analytic function $f(z) = u + iv$ where $u = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$ using milne thompson method.

$$u = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$$

$$u_2 = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \} + e^{-x} \{ 2y \cos y + 2x \sin y \}$$

$$\underline{u_2 = e^{-x} \{ -2y \cos y + 2x \cos y + 2x \sin y \}}$$

$$u_3 = e^{-x} \{ -2y \cos y + (x^2 - y^2)(-\sin y) + 2xy \cos y + 2x \sin y \}$$

$$= e^{-x} \{ -2y \cos y - (x^2 - y^2)(\sin y) + 2xy \cos y + 2x \sin y \}$$

$$f'(z) = u_2 + iv_2$$

$$f'(z) = u_2 - iu_3$$

$$= e^{-x} [2x \cos y + 2y \sin y - (x^2 - y^2) \cos y - 2xy \sin y] - ie^{-x} [2y \cos y - (x^2 - y^2) \sin y + 2xy \cos y + 2x \sin y]$$

$$x = z, y = 0$$

~~$$f'(z) = e^{-x} f(z) = e^{-x} [2z - z^2]$$~~

~~$$f(z) = \{ (2z - z^2) e^{-x}$$~~

$$= (2z - z^2) (-e^{-x}) - \{ -e^{-x} (2 - 2z)$$

$$\begin{aligned}
 &= 2z - z^2(-e^{-z}) - (2-z)e^{-z} + (z)e^{-z} + c \\
 &= -2ze^{-z} + z^2e^{-z} - 2e^{-z} + 2ze^{-z} + ze^{-z} + c \\
 f(z) &= \underline{z^2 e^{-z}} + c
 \end{aligned}$$

10 a) Find the analytic function $f(z) = u(r, \theta) + iv(r, \theta)$, where $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$ using Milne Thompson Method.

$$V = r^2 \cos 2\theta - r \cos \theta + 2$$

$$V_r = 2r \cos 2\theta - \cos \theta$$

$$\begin{aligned}
 V_\theta &= r^2(-2 \sin 2\theta) + r \sin \theta \\
 &= -2r^2 \sin 2\theta + r \sin \theta
 \end{aligned}$$

$$f'(z) = e^{-i\theta} (u_r + i v_r)$$

$$= e^{-i\theta} \left(\frac{1}{r} V_\theta + i V_r \right)$$

$$= e^{-i\theta} \left[\frac{1}{r} (-2r^2 \sin 2\theta + r \sin \theta) + i(2r \cos 2\theta - \cos \theta) \right] \quad V_\theta = r u_r$$

$$u_r = \frac{1}{r} V_\theta$$

$$u_\theta = -r V_r$$

$$v_r = -\frac{1}{r} u_\theta$$

Putting $r = z$, $\theta = 0$,

$$f'(z) = i(2z - 1)$$

on integrating

$$\begin{aligned}
 f(z) &= i \int (2z - 1) dz + c \\
 &= i \left(\frac{2z^2}{2} - z \right) + c \\
 &= \boxed{i(z^2 - z) + c}
 \end{aligned}$$

10 b) Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$

where $u(r, \theta) = \frac{\cos 2\theta}{r^2}$ using milne thompson method

$$u = \frac{\cos 2\theta}{r^2}$$

$$u_r = \frac{-2 \cos 2\theta}{r^3}$$

$$u_\theta = -\frac{2 \sin 2\theta}{r^2}$$

$$f'(z) = e^{-i\theta} (u_r + i v_r)$$

$$= e^{-i\theta} (u_r + i v_r) \quad \Rightarrow \quad e^{-i\theta} (u_r + i v_r)$$

$$= e^{-i\theta} \left(u_r + \left(\frac{i}{r} v_r \right) \right)$$

$$= e^{-i\theta} \left[-\frac{2 \cos 2\theta}{r^3} + i \frac{2 \sin 2\theta}{r^3} \right]$$

$$u_r = \frac{r^2}{2} v_\theta$$

$$v_r = -\frac{1}{r} u_\theta$$

$$u_\theta = -r v_r$$

$$r=2, \theta=0$$

$$\Rightarrow f'(z) = -\frac{2}{z^3}$$

$$f(z) = \int -\frac{2}{z^3} dz$$

$$= -2 \int \frac{1}{z^3} dz + C \quad \Rightarrow \quad -2 \left[-\frac{1}{2z^2} \right] + C$$

$$= \underline{\underline{\frac{1}{z^2} + C}}$$

11 a) Find the analytic function $f(z) = u + iv$, where $v = -\frac{\sin \theta}{r}$ using Milne thompson method.

$$V = -\frac{\sin \theta}{r} \quad \cancel{V_r = -\frac{r^{-1}}{r^2}}$$

$$V_r = \frac{\sin \theta}{r^2} \quad V_\theta = -\frac{\cos \theta}{r}$$

$$\begin{aligned} \therefore f'(z) &= e^{-i\theta} [u_r + i v_r] \\ &= e^{-i\theta} \left[\frac{1}{r} v_\theta + i v_r \right] \\ &= e^{-i\theta} \left[\frac{\cos \theta}{r^2} + i \frac{\sin \theta}{r^2} \right] \end{aligned}$$

$$\text{Put } r=2, \theta=0$$

$$\Rightarrow \cancel{\frac{-1}{z^2}}$$

$$f'(z) = -\frac{1}{z^2}$$

$$f(z) = \int \frac{-1}{z^2} dz$$

$$\Rightarrow \boxed{\underline{\underline{\frac{1}{z} + C}}}$$

11. b) Find the analytic function $f(z) = u + iv$, where $u = r^2 \cos \theta$.

$$u = r^2 \cos \theta$$

$$u_r = 2r \cos \theta$$

$$u_\theta = -r^2 - 2r^2 \sin \theta$$

$$\begin{aligned} f'(z) &= e^{-i\theta} [u_r + i v_r] \\ &= e^{-i\theta} [u_r + i \frac{1}{r} u_\theta] \\ &= e^{-i\theta} [2r \cos \theta + i 2r^2 \sin \theta] \end{aligned}$$

Put $r = x$ and $\theta = 0$.

$$\Rightarrow f'(z) = 2z$$

$$\begin{aligned} f(z) &= \int (2z) dz + c \\ &= \underline{\underline{2z}} = \frac{2z^2}{2} \end{aligned}$$

$$\boxed{f(z) = \underline{\underline{z^2}} + c}$$

12. a) Show that $u = e^x (x \cos y - y \sin y)$ is harmonic and find its harmonic conjugate.

$$u = e^x (x \cos y - y \sin y) = e^x x \cos y$$

$$u_x = e^x (\cos y) + (x \cos y - y \sin y) e^x$$

$$\Rightarrow u_x = e^x [\cos y + x \cos y - y \sin y]$$

$$u_{xx} = e^x \cdot \cos y + [\cos y + x \cos y - y \sin y] e^x$$

$$\Rightarrow e^x [2 \cos y + x \cos y - y \sin y] \longrightarrow ①$$

$$\text{Also, } u_y = e^x [-x \sin y - y \cos y - \sin y]$$

$$u_y = -e^x [x \sin y + y \cos y + \sin y]$$

$$u_{yy} = -e^x [x \cos y + (-y \sin y + \cos y) + \cos y]$$

$$= -e^x [x \cos y - y \sin y + 2 \cos y] \longrightarrow ②$$

$$\text{Now, } u_{xx} + u_{yy} = 0$$

$\therefore u$ is harmonic.

Now C-R equation $U_x = V_y$ and $V_x = -U_y$

$$V_y = e^x [\cos y + x \cos y - y \sin y] \quad \text{--- (3)}$$

$$V_x = e^x [x \sin y + y \cos y + \sin y] \quad \text{--- (4)}$$

From (3),

$$V = e^x \left[\int x \cos y dy + x \int \cos y dy - \int y \sin y dy \right] + f(x)$$

$$= e^x \left[\sin y + x \sin y - \left[y(-\cos y) + \int \cos y dy \right] + f(x) \right]$$

$$= e^x \left[\sin y + x \sin y + y \cos y + \sin y \right] + f(x)$$

$$V = e^x [x \sin y + y \cos y] + f(x) \quad \text{--- (5)}$$

From (4)

$$V = x \sin y \int x e^x dx + y \cos y \int e^x dx + \sin y \int e^x dx$$

$$= \sin y (x e^x - e^x) + y \cos y \cdot e^x + \sin y e^x + g(y)$$

$$= x \cdot e^x \sin y - e^x \sin y + y \cos y \cdot e^x + \sin y e^x + g(y)$$

$$= x e^x \sin y + y \cos y \cdot e^x$$

$$= e^x [x \sin y + y \cos y] \rightarrow (6)$$

Comparing (5) & (6)

$$f(x) = 0 \text{ and } g(y) = 0$$

$$\text{Now } f(z) = u + iv$$

$$= e^x [x \cos y - y \sin y] + i e^x [x \sin y + y \cos y]$$

Putting $x=z$ and $y=0$

$$\underline{f(z) = e^z \cdot z}$$

13. b) Show that $u = e^x \cos y + xy$ is harmonic & find its harmonic conjugate and also find corresponding analytic function.

$$u = e^x \cos y + xy$$

$$u_x = e^x \cos y + y$$

$$= e^x \cos y + y$$

$$u_{xx} = e^x \cos y \rightarrow ①$$

$$u_y = e^x (-\sin y) + x$$

$$u_{yy} = -e^x \cos y \rightarrow ②$$

$$\therefore u_{xx} + u_{yy} = 0$$

$\therefore u$ is harmonic.

Now C-R eqn, $u_x = v_y$ and $v_x = -u_y$

$$\therefore v_y = e^x \cos y + y \rightarrow ③$$

$$v_x = e^x \sin y - x \rightarrow ④$$

From ③,

$$v = e^x \int e^y dy + \int y dy + f(x)$$

$$= e^x \sin y + \frac{y^2}{2} + f(x) \rightarrow ⑤$$

From ④

$$v = \sin y \int e^x dx - \int x dx + g(y)$$

$$= e^x \sin y - \frac{x^2}{2} \rightarrow ⑥$$

Comparing ⑤ & ⑥ and by choosing $f(x) = -\frac{x^2}{2}$ & $g(y) = y^2$

$$v = e^x \sin y + \frac{y^2}{2} - \frac{x^2}{2}$$

$$f(z) = u + iv$$

$$= e^x \cos y + xy + i [e^x \sin y + \frac{y^2}{2} - \frac{x^2}{2}]$$

Put $x = z$ and $y = 0$.

$$e^z + i \left(\frac{-z^2}{2} \right)$$

$$\therefore \boxed{f(z) = e^z + i \left[\frac{-z^2}{2} \right]} //$$

13 a) Show that $V = \cos x \sinhy$ is harmonic and find its harmonic conjugate.

$$V = \cos x \sinhy$$

$$V_x = -\sin x - \sin x \cdot \sinhy$$

$$V_{xx} = -\cos x \cdot \sinhy \rightarrow ①$$

$$V_y = \cos x \cosh y$$

$$V_{yy} = \cos x \cdot \sinhy \rightarrow ②$$

$$\text{from } ① \text{ & } ② \quad V_{xx} + V_{yy} = 0 \quad \therefore V \text{ is harmonic.}$$

$$\text{C-R eqn } u_x = V_y \text{ and } v_x = -V_y$$

$$u_x = \cos x \cosh y \rightarrow ③$$

$$v_y = \sin x \sinhy \rightarrow ④$$

from ③,

$$\begin{aligned} u &= \int \cos x \cosh y (\cos x \, dx + g(y)) \\ &= \sin x \cosh hy \quad - ⑤ \end{aligned}$$

from ④

$$\begin{aligned} u &= \sin x \int \sinhy \, dy + f(x) \\ &= \sin x \cosh hy + f(x) \quad - ⑥ \end{aligned}$$

Comparing ⑤ & ⑥ $f(x) \& g(y) = 0$.

$$\therefore u = \underline{\underline{\sin x \cosh hy}}$$

$$\text{Then } f(z) = u + iv$$

$$= \sin x \cosh hy + i [\cos x \sinhy]$$

Put $x=z$ and $y=0$.

$$f(z) = \underline{\underline{\sin z}}$$

13 b) Show that $V = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1$ is harmonic and find its harmonic conjugate.

$$V = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1$$

$$V_x = 3x^2 - 3y^2 - 6x \quad \text{--- (1)}$$

$$V_{xx} = 6x - 6 \quad \text{--- (1)}$$

$$V_y = -6xy + 6y$$

$$V_{yy} = -6x + 6 \quad \text{--- (2)}$$

$$V_{xx} + V_{yy} = 0 \quad \therefore V \text{ is harmonic}$$

$$\text{By C-R eqn, } u_x = V_y \quad V_x = -6y$$

$$\therefore u_x = -6xy + 6y \quad \text{--- (3)}$$

$$u_y = -3x^2 + 3y^2 + 6x \quad \text{--- (4)}$$

from (3),

$$u_x = -6xy + 6y$$

$$\cancel{u = \int (6xy - 6y) dx + g(y)}$$

$$= \frac{6x^2y}{2} - 6xy + g(y)$$

$$= 3x^2y - 6xy + g(y)$$

from (4),

$$u_y = -3x^2 + 3y^2 + 6x$$

$$u = \int (-3x^2 + 3y^2 + 6x) dy + f(x)$$

$$= -3x^2y + \frac{3y^3}{3} + 6xy + f(x)$$

$$= -3x^2y + y^3 + 6xy + f(x)$$

14 a) Show that $u = \left(r + \frac{1}{r}\right) \cos\theta$ is harmonic and find its harmonic conjugate.

$$u = \left(r + \frac{1}{r}\right) \cos\theta$$

$$u_r = \left(1 - \frac{1}{r^2}\right) \cos\theta ; u_\theta = \left(r + \frac{1}{r}\right) (-\sin\theta)$$

$$u_{rr} = \left(\frac{2}{r^3}\right) \cos\theta ; u_{\theta\theta} = -\cos\theta \left(\frac{1}{r^2}\right)$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$\Rightarrow \frac{2}{r^3} \cos\theta + \frac{1}{r} \left[1 - \frac{1}{r^2}\right] \cos\theta - \frac{1}{r^2} \left[r + \frac{1}{r}\right] \cos\theta$$

$$\Rightarrow \frac{2}{r^3} \cos\theta + \left[\frac{1}{r} - \frac{1}{r^3}\right] \cos\theta - \left[\frac{1}{r} + \frac{1}{r^3}\right] \cos\theta$$

$$= \frac{2}{r^3} \cos\theta + \frac{1}{r} \cos\theta - \frac{1}{r^3} \cos\theta - \frac{1}{r} \cos\theta + \frac{1}{r^3} \cos\theta$$

$$= 0,$$

$\therefore u$ is harmonic.

$$u_r = \frac{1}{r} v_\theta \quad \cancel{v_\theta = -\frac{1}{r} u_r} \quad v_r = -\frac{1}{r} u_\theta$$

$$\therefore v_r = -\frac{1}{r} \left(r + \frac{1}{r}\right) (-\sin\theta)$$

$$v_r = \left(1 + \frac{1}{r^2}\right) \sin\theta \quad \text{--- (3)}$$

$$v_\theta = r u_r$$

$$= r \left[1 - \frac{1}{r^2}\right] \cos\theta$$

$$= \left(r - \frac{1}{r}\right) \cos\theta \quad \text{--- (4)}$$

from (3) & (4)

$$v_r = \left(1 + \frac{1}{r^2}\right) \sin\theta$$

$$v_\theta = \left(r - \frac{1}{r}\right) \cos\theta$$

$$v = \sin\theta \left(1 + \frac{1}{r^2}\right) dr + g(\theta)$$

$$v = \left(r - \frac{1}{r}\right) \int \cos\theta d\theta + f(r)$$

$$= \sin\theta \left(r\theta - \frac{1}{r}\right) + g(\theta)$$

$$= \left(r - \frac{1}{r}\right) \sin\theta + f(r)$$

$\therefore f(r)$ and $g(\theta) = 0$.

$$\therefore v = \sin\theta \left(r - \frac{1}{r}\right)$$

Put $r = 2$ and $\theta = 0$

$$\boxed{f(x) = x + \frac{1}{x}}$$

14b) Show that $V = \left(r - \frac{1}{r}\right) \sin\theta$ is harmonic & find its harmonic conjugate.

$$V = \left(r - \frac{1}{r}\right) \sin\theta$$

$$V_r = \left(\frac{1}{r^2}\right) \sin\theta$$

$$V_\theta = \left(1 - \frac{1}{r}\right) \cos\theta$$

$$V_{rr} = \left(-\frac{2}{r^3}\right) \sin\theta$$

$$V_{\theta\theta} = \left(1 - \frac{1}{r}\right) (-\sin\theta)$$

Now, $V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta}$ should be 0.

$$\Rightarrow \left(-\frac{2}{r^3}\right) \sin\theta + \frac{1}{r^3} \sin\theta + \frac{1}{r^2} \left[1 - \frac{1}{r}\right] (-\sin\theta)$$

$$= \left(-\frac{2}{r^3}\right) \sin\theta + \frac{1}{r^3} \sin\theta + \left(\frac{1}{r^2} - \frac{1}{r^3}\right) (-\sin\theta)$$

$$= -\frac{2}{r^3} \sin\theta + \frac{1}{r^3} \sin\theta - \frac{1}{r^2} \sin\theta + \frac{1}{r^3} \sin\theta$$

$$V = \left(r - \frac{1}{r}\right) \sin\theta$$

$$V_r = \left(1 + \frac{1}{r^2}\right) \sin\theta \quad V_\theta = \left(r - \frac{1}{r}\right) \cos\theta$$

$$V_{rr} = -\frac{2}{r^3} \sin\theta \quad V_{\theta\theta} = \left(r - \frac{1}{r}\right) (-\sin\theta)$$

$$V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = -\frac{2}{r^3} \sin\theta + \frac{1}{r} \left[1 + \frac{1}{r^2}\right] \sin\theta + \frac{1}{r^2} \left[r - \frac{1}{r}\right] (-\sin\theta)$$

$$= -\frac{2}{r^3} \sin\theta + \frac{1}{r} \sin\theta + \frac{1}{r^3} \sin\theta + \frac{1}{r} \sin\theta + \frac{1}{r^3} \sin\theta$$

$$= 0 / \therefore V \text{ is harmonic.}$$

By CR eqn, $U_r = \frac{1}{r} V_\theta$ and $V_r = -\frac{1}{r} U_\theta$

$$\therefore U_r = \frac{1}{r} \left(r - \frac{1}{r}\right) \cos\theta$$

$$= \left(1 - \frac{1}{r^2}\right) \cos\theta$$

$$U_\theta = ?$$

$$= r \left(1 + \frac{1}{r^2}\right) \sin\theta$$

$$u_r = \left(1 - \frac{1}{r^2}\right) \cos \theta$$

~~$$u_{\theta} = \left(1 + \frac{1}{r^2}\right) \sin \theta$$~~

$$u_{\theta} = -r \left(1 + \frac{1}{r^2}\right) \sin \theta$$

$$= \left(-r - \frac{1}{r^2}\right) \sin \theta$$

$$u = \cos \theta \left(1 - \frac{1}{r^2}\right) + g(\theta) \quad u = -(r + \frac{1}{r}) \int \sin \theta d\theta + f(r)$$

$$= \cos \theta \left(r + \frac{1}{r}\right)$$

$$= \left(r + \frac{1}{r}\right) \cos \theta + f(r)$$

$$\therefore f(r) \text{ and } g(\theta) = 0$$

$$\text{and } u = \cos \theta \left(r + \frac{1}{r}\right)$$

$$f(z) = u + iv$$

$$= \cos \theta \left(r + \frac{1}{r}\right) + i \left(r - \frac{1}{r}\right) \sin \theta$$

$$\text{Put } r = z \text{ and } \theta = 0.$$

$$\Rightarrow \underline{\underline{z + \frac{1}{z}}}$$

$$\therefore \boxed{f(z) = z + \frac{1}{z}}$$

15 a) Show that $v = rs \sin \theta + \frac{\cos \theta}{r}$ is harmonic and find its harmonic conjugate, and also find corresponding analytic function.

$$V = rs \sin \theta + \frac{\cos \theta}{r}$$

$$V_r = \sin \theta + \left(-\frac{1}{r^2}\right) \cos \theta$$

$$V_\theta = \cancel{-r \cos \theta} \quad r \cos \theta - \frac{\sin \theta}{r}$$

$$V_{rr} = \sin \theta - \frac{1}{r^2} \cos \theta$$

$$V_{\theta\theta} = -r \sin \theta - \frac{\cos \theta}{r}$$

$$V_{rrr} = \frac{2}{r^3} \cos \theta$$

$$V_{rrr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = \frac{2}{r^3} \cos \theta + \frac{1}{r} \left[\sin \theta - \frac{1}{r^2} \cos \theta \right] + \cancel{\frac{1}{r^2} \left[rs \sin \theta - \frac{\cos \theta}{r} \right]}$$

$$= \frac{2}{r^3} \cos \theta + \frac{1}{r} \sin \theta - \frac{1}{r^3} \cos \theta - \frac{1}{r} \sin \theta - \frac{\cos \theta}{r^3}$$

$$= 0$$

$\therefore V$ is harmonic

By C-R equation,

$$U_r = \frac{1}{r} V_\theta \text{ and } V_r = -\frac{1}{r} U_\theta$$

$$\therefore U_r = \frac{1}{r} \left[r \cos \theta - \frac{\sin \theta}{r} \right] \quad U_\theta = -r V_r$$

$$= \cos \theta - \frac{\sin \theta}{r^2} \rightarrow (3) \quad = -r \left[\sin \theta - \frac{1}{r^2} \cos \theta \right]$$

$$= -r \sin \theta + \frac{1}{r} \cos \theta \rightarrow (4)$$

By (3),

$$U_r = \cos \theta - \frac{\sin \theta}{r^2}$$

$$U = \int \cos \theta \, dr - \int \frac{\sin \theta}{r^2} \, dr + g(\theta)$$

$$= r \cos \theta + \frac{1}{r} \sin \theta + g(\theta) \rightarrow (5)$$

By (4),

$$U_\theta = -r \sin \theta + \frac{1}{r} \cos \theta$$

$$U = -r \left(\sin \theta d\theta + \frac{1}{r} \int \cos \theta \, d\theta \right) + f(r)$$

$$= r \cos \theta + \frac{1}{r} \sin \theta + f(r) \rightarrow (6)$$

∴ from (5) and (6), $f(r) = g(\theta) = 0$

$$\therefore \boxed{U = r \cos \theta + \frac{1}{r} \sin \theta}$$

$$f(z) = U + iV$$

$$= r \cos \theta + \frac{1}{r} \sin \theta + i \left(\cancel{r \cos \theta} + \frac{\sin \theta}{r} \right) i \left(r \sin \theta + \frac{\cos \theta}{r} \right)$$

but $r=x$ and $\theta=0$

$$z + \frac{i}{x}$$

$$\therefore \boxed{f(z) = z + \frac{i}{z}} //$$

15.b) Show that $U = \frac{\cos \theta}{r}$ is harmonic and find its harmonic conjugate and also find its corresponding analytic function.

$$U = \frac{\cos \theta}{r}$$

$$U_r = -\frac{\cos \theta}{r^2} \quad U_\theta = \frac{-\sin \theta}{r}$$

$$U_{rr} = \frac{2}{r^3} \cos \theta \quad U_{\theta\theta} = -\frac{\cos \theta}{r}$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{2}{r^3} \cos \theta + \frac{1}{r} \left[-\frac{\cos \theta}{r^2} \right] + \frac{1}{r^2} \left[-\frac{\sin \theta}{r} \right]$$

$$\Rightarrow \frac{2}{r^3} \cos \theta = -\frac{\cos \theta}{r^3} - \frac{\cos \theta}{r^3}$$

$$\therefore \frac{\cos \theta}{r^3} - \frac{\sin \theta}{r^3} = 0$$

$\therefore u$ is harmonic

By C-R eqn, $u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$

$$\therefore v_r = -\frac{1}{r} u_\theta$$

$$v_\theta = r u_r$$

$$= -\frac{1}{r} \left(-\frac{\sin \theta}{r} \right)$$

$$= r \left(-\frac{\cos \theta}{r^2} \right)$$

$$v_r = \frac{\sin \theta}{r^2} \rightarrow (3)$$

$$v_\theta = -\frac{\cos \theta}{r} \quad (4)$$

from (3),

$$v_r = \frac{\sin \theta}{r^2}$$

from (4)

$$v_\theta = -\frac{\cos \theta}{r}$$

$$v = \int \frac{\sin \theta}{r^2} dr + g(\theta)$$

$$v = \int -\frac{\cos \theta}{r} \Rightarrow -\frac{1}{r} \int \cos \theta d\theta + f(r)$$

$$= -\frac{1}{r} \sin \theta + g(\theta)$$

$$= -\frac{1}{r} \sin \theta + f(r)$$

$$\therefore g(\theta) = f(r) = 0$$

$$v = -\frac{1}{r} \sin \theta$$

$$f(z) = u + iv$$

Put $r = z$ and $\theta = 0$

$$\Rightarrow \frac{1}{r} \cos \theta + i \left(-\frac{1}{r} \sin \theta \right)$$

$$= \frac{1}{z} + i \left(-\frac{1}{z} (0) \right)$$

$$= \frac{1}{z}$$

$$\therefore \boxed{f(z) = \frac{1}{z}}$$

16. a) Show that $u = x^2 + 4x - y^2 + 2y$ is harmonic and find its harmonic conjugate & also find corresponding analytic function.

$$u = x^2 + 4x - y^2 + 2y$$

$$u_x = 2x + 4$$

$$u_{xx} = 2 \rightarrow ①$$

$$u_y = -2y + 2$$

$$u_{yy} = -2 \rightarrow ②$$

$$u_{xx} + u_{yy} = 0 \quad \therefore u \text{ is harmonic.}$$

By C-R eqn, $u_x = v_y$ and $v_y = -u_y$.

$$\Rightarrow v_x = -u_y$$

$$v_x = 2y - 2 \rightarrow ③$$

$$v_y = u_x$$

$$v_y = 2x + 4 \rightarrow ④$$

from ③ & ④

$$v_x = 2y - 2$$

$$v = \int (2y - 2) dx + g(y)$$

$$= 2xy - 2x + g(y) \rightarrow ⑤$$

$$v_y = 2x + 4$$

$$v = \int (2x + 4) dy + f(x)$$

$$= 2xy + 4y + f(x) \rightarrow ⑥$$

From ⑤ and ⑥

$$f(x) = -2x$$

$$\text{and } g(y) = 4y$$

$$\therefore v = 2xy - 2x + 4y$$

$$f(z) = u + iv$$

$$\Rightarrow x^2 + 4x - y^2 + 2y + \cancel{2xy} - \cancel{i[2xy]} = i[2xy - 2x + 4y]$$

Put $x = z$ and $y = 0$

$$z^2 + 4z + i[-2z]$$

$$= z^2 + 4z - 2iz$$

$$f(z) = \underline{\underline{z^2 + 4z - 2zi}}$$

16 b) Show that $V = 2xy - 2x + 4y$ is harmonic & find its harmonic conjugate & also find the corresponding analytic function.

$$V = 2xy - 2x + 4y$$

$$V_x = 2y - 2 \quad V_y = 2x + 4$$

$$V_{xx} = -2 \quad V_{yy} = 2$$

$$V_{xx} + V_{yy} = 0$$

$\therefore V$ is harmonic

From C-R eqn,

$$U_x = V_y \text{ and } V_x = -U_y$$

$$\therefore U_x = 2x + 4 \rightarrow (1)$$

$$U_y = -2y + 2 \rightarrow (2)$$

from (1)

$$U_x = 2x + 4$$

$$U = \int (2x + 4) dx + g(y)$$

$$= \frac{2x^2}{2} + 4x + g(y)$$

$$= x^2 + 4x + g(y) \rightarrow (3)$$

$$U_y = -2y + 2$$

$$U = \int (-2y + 2) dy + f(x)$$

$$= -\frac{2y^2}{2} + 2y$$

$$= 2y - y^2 \rightarrow (4)$$

Comparing (3) and (4)

$$U = x^2 + 4x - y^2 + 2y$$

$$f(z) = U + iV$$

$$= x^2 + 4x - y^2 + 2y + i[2xy - 2x + 4y]$$

Put $x = z$ and $y = 0$

$$= z^2 + 4z + i(2z)$$

$$f(z) = z^2 + 4z + 2zi$$

6
anal

17(a) Find the analytic function $f(z) = u+iv$ if $u+v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$ ($r \neq 0$)

given $u+iv = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta) \rightarrow ①$

diff ① partially w.r.t r .

$$u_r + v_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$= -\frac{2}{r^3} \cos 2\theta + \frac{2}{r^3} \sin 2\theta \rightarrow ②$$

diff ① partially w.r.t θ

$$u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$$

$$\Rightarrow -rv_r + rur = -\frac{2}{r^2} \sin 2\theta - \frac{2}{r^2} \cos 2\theta$$

$$\begin{cases} u_\theta = -rv_r \\ v_\theta = \cancel{r}v_r \end{cases}$$

$$-v_r + ur = -\frac{2}{r^3} \sin 2\theta - \frac{2}{r^3} \cos 2\theta \rightarrow ③$$

$$(2) + (3) \Rightarrow 2ur = -\frac{4}{r^3} \cos 2\theta$$

$$ur = -\frac{2}{r^3} \cos 2\theta.$$

$$(2) - (3) \Rightarrow 2v_r = \frac{4}{r^3} \sin 2\theta$$

$$v_r = \frac{2}{r^3} \sin 2\theta.$$

$$\therefore u_r = -\frac{2}{r^3} \cos 2\theta \quad v_r = \frac{2}{r^3} \sin 2\theta.$$

we have, $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= e^{-i\theta} \left(-\frac{2}{r^3} \cos 2\theta + i \frac{2}{r^3} \sin 2\theta \right)$$

Putting $r=2$ and $\theta=0$

$$\Rightarrow -\frac{2}{2^3} / \quad f'(z) = -\frac{2}{z^3}$$

$$f(z) = \int -\frac{2}{z^3}$$

$$= -\frac{2}{2z^2} \Rightarrow -\frac{1}{z^2} + C_1$$

$$\boxed{\therefore f(z) = \frac{1}{z^2} + C}$$

17 b) find the analytic function $f(z) = u+iv$ if $u+v = r(\cos\theta + \sin\theta)$

$$+ \frac{1}{r}(\cos\theta - \sin\theta)$$

$$u+v = r(\cos\theta + \sin\theta) + \frac{1}{r}(\cos\theta - \sin\theta) \rightarrow ①$$

diff ① partially w.r.t r

$$u_r + v_r = \cos\theta + \sin\theta - \frac{1}{r^2}(\cos\theta - \sin\theta) \rightarrow ②$$

diff ① partially w.r.t θ

$$u_\theta + v_\theta = r(\cos\theta - \sin\theta) + \frac{1}{r}(-\sin\theta - \cos\theta)$$

$$-r(\cos\theta - \sin\theta)$$

$$-rv_r + r u_r = r(-\sin\theta + \cos\theta) + \frac{1}{r}(-\sin\theta - \cos\theta)$$

$$\Rightarrow -v_r + u_r = (-\sin\theta + \cos\theta) + \frac{1}{r^2}(-\sin\theta - \cos\theta) \rightarrow ③$$

$$(2) + (3) \quad 2u_r = \underline{\cos\theta + \sin\theta} - \frac{1}{r^2}\cos\theta + \frac{1}{r^2}\sin\theta - \underline{\sin\theta + \cos\theta} - \frac{1}{r^2}\sin\theta$$

$$2u_r = 2\cos\theta - \frac{2}{r^2}\cos\theta$$

$$u_r = \cos\theta - \frac{1}{r^2}\cos\theta \rightarrow ④$$

(2) - (3)

$$2v_r = \underline{\cos\theta + \sin\theta} - \frac{1}{r^2}\cos\theta + \frac{1}{r^2}\sin\theta + \sin\theta - \underline{\cos\theta} + \frac{1}{r^2}\sin\theta + \frac{1}{r^2}\cos\theta$$

$$2v_r = 2\sin\theta + \frac{2}{r^2}\sin\theta$$

$$v_r = \sin\theta + \frac{1}{r^2}\sin\theta$$

$$v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta \rightarrow ⑤$$

$$f'(z) = e^{i\theta}(u_r + iv_r)$$

$$= e^{i\theta} \left[\cancel{\cos\theta} - \left(1 - \frac{1}{r^2}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)i\sin\theta \right]$$

Put $r=x$ and $\theta=0$

$$\Rightarrow \cancel{\cos\theta} \left[1 - \frac{1}{x^2} \right]$$

$$= \therefore \boxed{f'(z) = 1 - \frac{1}{z^2}}$$

$$f(z) = \boxed{(1 - \frac{1}{z^2})}$$

$$\boxed{f(z) = z + \frac{1}{z} + c}$$

(e) 18 a) Find the analytic function $f(z) = u + iv$ of $u + iv = (x+y) + e^x(\cos y + \sin y)$

$$u + v = (x+y) + e^x(\cos y + \sin y) \rightarrow ①$$

diff ① partially w.r.t. x

$$u_x + v_x = 1 + e^x(\cos y + \sin y) \rightarrow ②$$

diff ① partially w.r.t. y .

$$u_y + v_y = 1 + e^x(-\sin y + \cos y) \rightarrow ③$$

$$-v_x + u_x = 1 + e^x(-\sin y + \cos y) \rightarrow ④$$

$$(2) + (3)$$

$$\begin{aligned} 2u_x &= 1 + e^x(\cos y + \sin y) + 1 + e^x(-\sin y + \cos y) \\ &= 1 + e^x \cos y + e^x \sin y + 1 - e^x \sin y + e^x \cos y \\ &= 2 + 2e^x \cos y \end{aligned}$$

$$\underline{u_x = 1 + e^x \cos y} \quad \rightarrow ⑤$$

$$(2) - (3)$$

$$2v_x = 1 + e^x(\cos y + \sin y) - 1 - e^x(-\sin y + \cos y)$$

$$= 1 + e^x \cos y + e^x \sin y - 1 + e^x \sin y - e^x \cos y$$

$$2v_x = 2e^x \sin y$$

$$\underline{v_x = e^x \sin y} \quad \rightarrow ⑥$$

$$f'(z) = u_x + i v_x$$

$$\Rightarrow 1 + e^x \cos y + i e^x \sin y.$$

Put $x = z$ and $y = 0$.

$$1 + e^x \cos(0) \neq$$

$$= \underline{\underline{1 + e^x}}$$

$$f(z) = \int (1 + e^x)$$

$$f(z) = \underline{\underline{z + e^x}} + C$$

18 b) Find the analytic function $f(z)$ as a function of z given sum of its real and imaginary part is $x^3y^3 + 3xy(x-y)$.

$$u+v = x^3y^3 + 3xy(x-y) \rightarrow ①$$

$$\text{diff } ① \text{ w.r.t } x. \quad u+v = x^3y^3 + 3x^2y - 3xy^2$$

$$\cancel{u_x+v_x} = \cancel{3x^2} + 3y \quad u_x+v_x = 3x^2 + 6xy - 3y^2 \rightarrow ②$$

diff ① w.r.t y .

$$u_y+v_y = -3y^2 + 3x^2 - 6xy$$

$$-v_x+u_x = -3y^2 + 3x^2 - 6xy \rightarrow ③$$

$$(2) + (3)$$

$$2u_x = 3x^2 + 6xy - 3y^2 - 3y^2 + 3x^2 - 6xy$$

$$2u_x = 6x^2 - 6y^2$$

$$u_x = 3(x^2 - y^2)$$

$$(2) - (3)$$

$$2v_x = 3x^2 + 6xy - 3y^2 + 3y^2 - 3x^2 + 6xy$$

$$2v_x = 12xy$$

$$v_x = 6xy$$

$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= 3(x^2 - y^2) + i(6xy) \end{aligned}$$

$$\text{Put } x=z \text{ and } y=0$$

$$f'(z) = \underline{\underline{3z^2}}$$

$$p(z) = \int (3z^2) dz + c$$

$$\boxed{f(z) = \underline{\underline{z^3 + c}}}$$

19. a) Find the analytic function $f(z) = u + iv$ if $u - v = (x-y)(x^2 + 4xy + y^2)$

$$u - v = (x-y)(x^2 + 4xy + y^2)$$

$$\begin{aligned} u - v &= x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3 \\ &= x^3 + 3x^2y - 3xy^2 - y^3 \rightarrow ① \end{aligned}$$

Diff ① partially w.r.t x

$$u_x - v_x = 3x^2 + 6xy - 3y^2 \rightarrow ②$$

Diff ① partially w.r.t y .

$$uy - Vy = 3x^2 - 6xy - 3y^2$$

$$-v_x - u_x = 3x^2 - 6xy - 3y^2$$

$$v_x + u_x = -3x^2 + 6xy + 3y^2 \rightarrow ③$$

(2) + (3)

$$\Rightarrow 2u_x = 12xy$$

$$u_x = 6xy \rightarrow ④$$

$$(2) - (3) = 6x^2 - 6y^2$$

$$-2v_x = (x^2 - y^2)$$

$$-v_x = 3(y^2 - x^2) \rightarrow ⑤$$

Put $x = 7, y = 0$

$$f'(z) = u_x + iv_x$$

$$= 6xy + i(3(y^2 - x^2))$$

Put $x = 7, y = 0,$

$$= 6 \cdot 7(0) + i[3(-7^2)]$$

$$= -3i7^2$$

$$f(z) = \int (-3iz^2) dz + C$$

$$= -3i \int z^2 dz + C$$

$$+ \frac{z^3}{3} k - 3i$$

$$\boxed{f(z) = -iz^3 + C}$$

19.b) Find the analytic function $f(z) = u+iv$ if $u-v = e^x(\cos y - \sin y)$

$$u-v = e^x(\cos y - \sin y)$$

$$u-v = e^x \cos y - e^x \sin y \rightarrow ①$$

diff. ① w.r.t. x

$$u_x - v_x = e^x \cos y - e^x \sin y \rightarrow ②$$

diff ① w.r.t. y ,

$-\sin y - \cos y$

$$u_y - v_y = -e^x \sin y - e^x \cos y$$

$$-v_x - u_x = e^x(-\sin y - \cos y)$$

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}$$

$$v_x + u_x = -e^x(-\sin y - \cos y) \rightarrow ③$$

② + ③

$$2u_x = e^x \cos y - e^x \sin y + e^x \sin y + e^x \cos y$$

$$u_x - v_x - v_x$$

$$2u_x = 2e^x \cos y$$

$$u_x = e^x \cos y$$

② - ③

$$-2v_x = e^x \cos y - e^x \sin y - e^x \sin y - e^x \cos y$$

$$-2v_x = -2e^x \sin y$$

$$v_x = e^x \sin y$$

$$f'(z) = u_x + iv_x$$

$$\Rightarrow e^x \cos y + i e^x \sin y.$$

Put $x=z$ and $y=0$.

$$f'(z) = \underline{e^z}$$

$$f(z) = \int e^z dz + C$$

$$\boxed{f(z) = e^z + C}$$

Q) a) An electrostatic field in x-y plane is given by potential function $\phi = 3x^2y - y^3$. Find stream function.

$$\text{Given: } \phi = 3x^2y - y^3$$

$$\phi_x = 6xy$$

$$\begin{aligned}\phi_y &= -3y^2 + 3x^2 \\ &= 3x^2 - 3y^2\end{aligned}$$

$$\text{By C-R eqn: } \phi_x = \psi_y, \psi_x = -\phi_y$$

$$\psi_y = \phi_x$$

$$= 6xy \rightarrow ①$$

$$\psi_x = -\phi_y$$

$$= -3x^2 + 3y^2 \rightarrow ②$$

Integrating ①

$$\psi = \int 6xy dy + f(x)$$

$$= 6x \frac{y^2}{2} + f(x)$$

$$= 3xy^2 + f(x) \rightarrow ③$$

Integrating ②

$$\psi = \int (-3x^2 + 3y^2) dx + g(y)$$

$$= -\frac{3x^3}{3} + \frac{3y^2 x}{2} + g(y)$$

$$= -x^3 + 3xy^2 + g(y) \rightarrow ④$$

From ③ & ④

$$f(x) = -x^3$$

$\therefore \psi = \underline{3xy^2 - x^3}$ is the required stream function.

Q b)

$$w = a \log z + c$$

MOD-1
Sol

MOD-2

Sol
