# Z-TRANSFORMS

$x(n) \xrightarrow{Z.T} X(z)$ — Z-trasform

$X(z) \xrightarrow{I.Z.T} x(n)$ — inverse Z-trasform.

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Defenition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

1) $x(n) = \delta(n)$

By defenition.

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Note:

$$\because \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

Here $x(n) = \delta(n)$

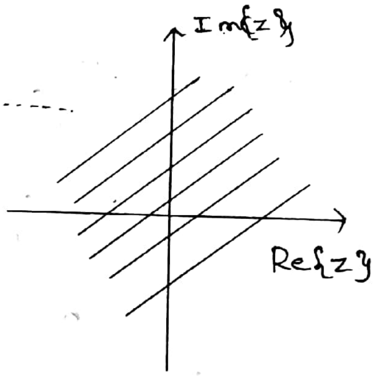
$$\therefore X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$\delta(n)$ exist only at $n=0$

$$= \ldots\ldots \delta(-1) z^{1} + \delta(0) z^{0} + \delta(1) z^{-1} + \ldots\ldots$$

$$= \ldots\ldots 0 + z^{0} + 0 + \ldots\ldots$$

$$\therefore \boxed{X(z) = 1}$$




Roc [Region of convergence]
is entere Z-plane.

2) $x(n) = \delta(n-k)$

W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\because \delta(n-k) = \begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases}$$

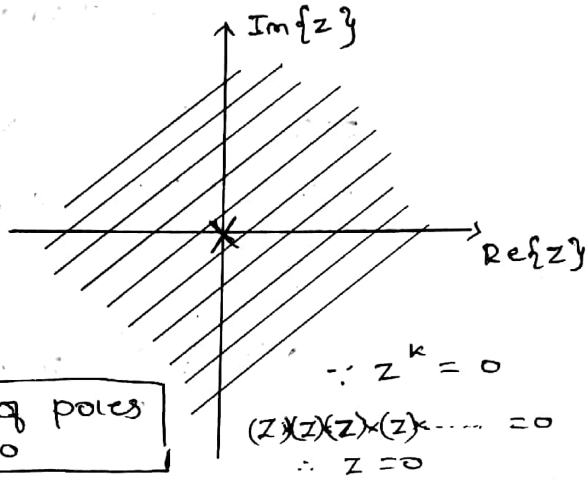
$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n-k) z^{-n}$$

$$= \ldots\ldots + \delta(-2-k) z^{2} + \delta(-1-k) z + \delta(0-k) z^{0} + \delta(1-k) z^{-1} + \ldots\ldots + \delta(k-k) z^{-k} + \ldots\ldots$$

$$= \delta(n-k) z^{-n} \Big|_{n=k} = \delta(k-k) z^{-k}$$

$= \delta(0) z^{-k} = Z^{-k}$

$$\therefore \boxed{X(Z) = Z^{-k} = \frac{1}{Z^k}}$$




Here Roc is $|Z| > 0$ i.e

Roc is entire Z-plane

Excluding $Z = 0$

$\therefore Z^k = 0$

$(Z)(Z)(Z)(Z)\ldots = 0$

$\therefore Z = 0$

$\boxed{\text{k number of poles at } Z=0}$

Poles are obtained by equating ~~equating~~ numerat-or of $X(z)$ to zero. indicated by $\times$

Zeroes are obtained by equating Denominator to zero. indicated by $O$

3) $x(n) = \delta(n+k)$

$$\delta(n+k) = \begin{cases} 1, & n = -k \\ 0, & n \neq -k \end{cases}$$

W.K.T, $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

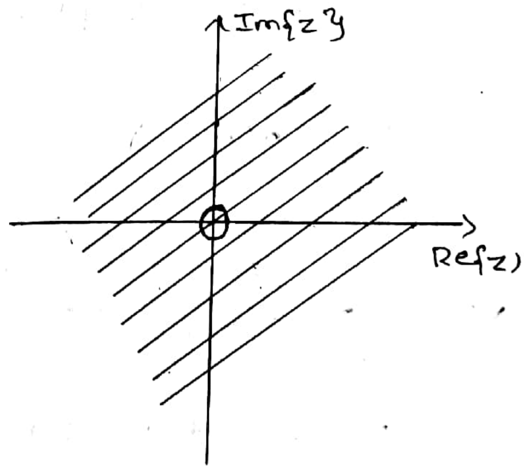
$$X(Z) = \sum_{n=-\infty}^{\infty} \delta(n+k) Z^{-n}$$

$$X(Z) = \ldots \delta(-k+k) Z^{k} + \ldots\ldots \delta(-1+k) Z^{1} + \delta(0+k) Z^{0} + \delta(1+k) Z^{-k} + \ldots\ldots$$

$$X(Z) = \delta(-k+k) Z^{k}$$

$$X(Z) = \delta(0) Z^{k}$$

$$X(Z) = Z^{k}$$




Roc is $|Z| < \infty$ i.e,

entire Z-plane Excluding

$Z = \infty$

'k' numbers of zeroes at $Z=0$

4. $x(n) = a^n u(n)$

Sol w.k.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Here $x(n) = a^n u(n)$

Substitute in above equation.

$\therefore X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$

$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$ $\because$ u(n) exists from 0 to $\infty$ Positive sequence.

$X(z) = \sum_{n=0}^{\infty} (a/z)^n$

we know that.

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \;;\; |a|<1$$

Hence, $X(z) = \sum_{n=0}^{\infty} (a/z)^n = \frac{1}{1-az^{-1}}$

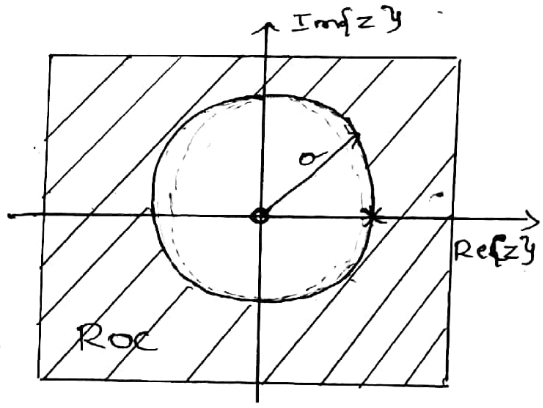
$$= \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$|a| < |z|$

$\Rightarrow |z| > |a|$

Hence Roc is $|z| > |a|$

Pole at $z = a$ and zeroes at the origin




5) $x(n) = -a^n u(-n-1)$ $\Rightarrow$ negative sequence.

w.k.t $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Here $x(n) = -a^n u(-n-1)$

$\therefore X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$

$X(z) = -\sum_{n=-\infty} a^n z^{-n}$ $\because$ u(-n-1) exists from $-\infty$ to $-1$

$$x(z) = -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

By substituting $n = -m$ we get. as $m = -n$

$$x(z) = -\sum_{m=\infty}^{1} \left(\frac{a}{z}\right)^{-m}$$

$n = -\infty \Rightarrow m = \infty$

$n = -1 \Rightarrow m = 1$

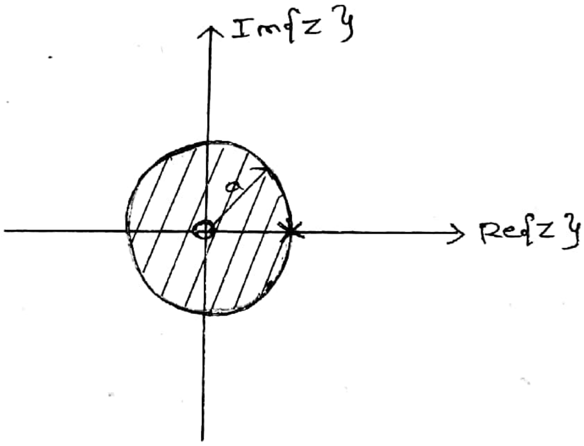
$$x(z) = -\sum_{m=\infty}^{1} \left[\left(\frac{z}{a}\right)^{-1}\right]^{-m} = -\sum_{m=\infty}^{1} \left(\frac{z}{a}\right)^m$$

$\left[ \text{WKT} \quad \sum_{n=1}^{\infty} p^n = \frac{p}{1-p} \; ; \; |p| < 1 \right]$

Hence $x(z) = -\left[\frac{z/a}{1 - z/a}\right] = -\left[\frac{z/a}{\frac{a-z}{a}}\right]$

$$x(z) = \frac{-z}{a-z} \quad \text{where } \left|\frac{z}{a}\right| < 1 \Rightarrow |z| < |a|$$

$$x(z) = \frac{z}{z-a}$$




ROC is $|z| < |a|$

Zeroes at $z = 0$.

Poles at $z = a$.

6) $x(n) = -u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$

Sol: WKT $x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Here $x(n) = -u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$

$$\therefore \; x(z) = \sum_{n=-\infty}^{\infty} \left[-u(-n-1) + \left(\frac{1}{2}\right)^n u(n)\right] z^{-n}$$

$$x(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n}$$

As $u(-n-1) = \begin{cases} 1, & n \le -1 \\ 0, & n > -1 \end{cases}$ and $u(n) = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$

Hence, $X(z) = -\sum_{n=-\infty}^{-1} u(-n-1) z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n}$

$$X(z) = -\sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n}$$

$$X(z) = -\sum_{n=\infty}^{1} z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n \qquad \left[\text{put } n=-m \text{ in } 1^{st} \text{ term}\right]$$

$$X(z) = \frac{-z}{1-z} + \frac{1}{1-\frac{1}{2z}}$$

$|z|<1$ and $\left|\frac{1}{2z}\right|<1 \Rightarrow \frac{1}{2}<|z|$

ROC is $\frac{1}{2}<|z|<1$

$$X(z) = \frac{z}{z-1} + \frac{2z}{2z-1}$$

$$X(z) = \frac{2z^2 - z + 2z^2 - 2z}{(z-1)(2z-1)}$$

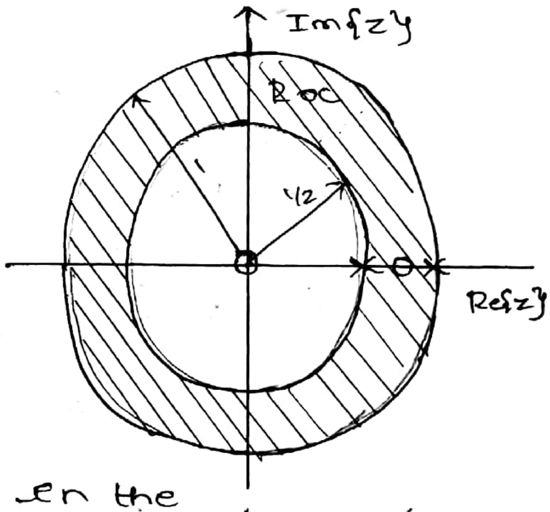
$$X(z) = \frac{4z^2 - 3z}{(z-1)(2z-1)}$$

zeroes are $4z^2 - 3z = 0$

$z(4z-3) = 0$

at $z=0$ and $z = 3/4$.

Poles at $z=1$ and $z=\frac{1}{2}$.




The transform here converges in the annular region and is known as double sided sequence.

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7) $x(n) = \left(\frac{1}{2}\right)^{|n|}$

Sol

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^{-n} & , n < 0 \\ \left(\frac{1}{2}\right)^{n} & , n > 0 \end{cases} \quad \therefore x(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1) + \left(\frac{1}{2}\right)^{n} u(n)$$

W.k.t $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-n} u(-n-1) z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n} u(n) z^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}.$$

$$X(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^{n}$$

$$X(z) = \sum_{n=1}^{\infty} \cancel{(2z)^n} \left(z/2\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^{n}$$

Series converges ~~at~~ if

Roc is $\left|z/2\right| < 1$ and $\left|\frac{1}{2z}\right| < 1$

$|z| < 2$ and $\frac{1}{2} < |z|$

Hence Roc is $1/2 < |z| < 2$

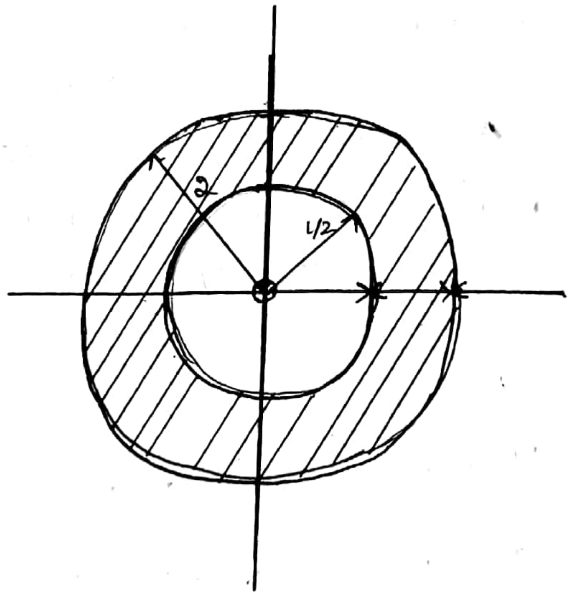
$$X(z) = \frac{\frac{z}{2}}{1 - z/2} + \frac{1}{1 - \frac{1}{2z}} \qquad \because \sum_{n=1}^{\infty} p^n = \frac{p}{1-p} \qquad \because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$X(z) = \frac{z}{2-z} + \frac{2z}{2z-1}$$

$$X(z) = \frac{\cancel{2z^2} - z + 4z - \cancel{2z^2}}{(2-z)(2z-1)}$$

$$x(z) = \frac{3z}{(2-z)(2z-1)}$$

Zeros at $z=0$ (origen) and
Poles at $z=2$ and $z=1/2$.




8) $x(n) = (-1/2)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

Sol W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Here $x(n) = (-1/2)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} \left[(-1/2)^n u(n) + \left(\tfrac{1}{4}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (-1/2)^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{4}\right)^n u(n) z^{-n}$$

Both are right sided sequence.

$$\therefore X(z) = \sum_{n=0}^{\infty} (-1/2)^n z^{-n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n z^{-n}$$

The above series converges if

$\left|\frac{1}{2z}\right| < 1$ and $\left|\frac{1}{4z}\right| < 1$

$\frac{1}{2} < |z|$ and $\left|\frac{1}{4}\right| < |z|$

Roc is $|z| > 1/2$ [taking common region]

$$X(z) = \frac{1}{1+1/2z} + \frac{1}{1-1/4z} = \frac{2z}{2z+1} + \frac{4z}{4z-1}$$

$$X(z) = \frac{8z^2 - 2z + 8z^2 + 4z}{(2z+1)(4z-1)}$$

$$X(z) = \frac{16z^2 + 2z}{(2z+1)(4z-1)}$$

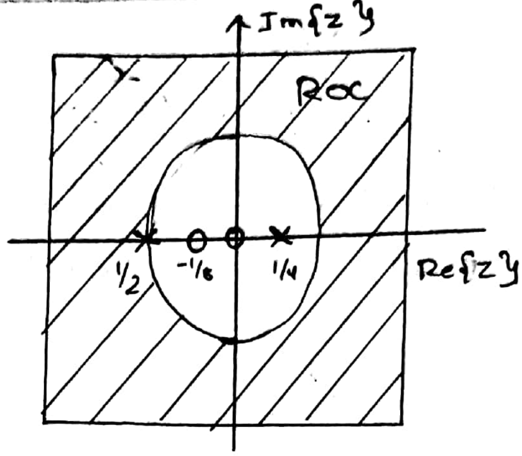
Zeros: $16z^2 + 2z = 0$

$z(16z + 2) = 0$

At $z = 0$ and $z = -1/8$

Poles: $2z + 1 = 0$ and $4z - 1 = 0$

$z = -1/2$ and $z = 1/4$.




a) $x(n) = (-1/2)^n u(-n) + (\frac{1}{4})^n u(-n)$

Sol: WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$x(n) = (-1/2)^n u(-n) + (1/4)^n u(-n)$

$\therefore X(z) = \sum_{n=-\infty}^{\infty} \left[(-1/2)^n u(-n) + (1/4)^n u(-n)\right] z^{-n}$

$= \sum_{n=-\infty}^{\infty} (-1/2)^n u(-n) z^{-n} + \sum_{n=-\infty}^{\infty} (1/4)^n u(-n) z^{-n}$.

Both are left sided sequence.

$\therefore X(z) = \sum_{n=0}^{-\infty} (-1/2)^n z^{-n} + \sum_{n=0}^{-\infty} (\frac{1}{4})^n z^{-n}$

$= \sum_{n=0}^{-\infty} \left(\frac{-1}{2z}\right)^n + \sum_{n=0}^{-\infty} \left(\frac{1}{4z}\right)^n$

$n \Rightarrow -n$.

$X(z) = \sum_{n=0}^{\infty} (-2z)^n + \sum_{n=0}^{\infty} (4z)^n$

The above series converges if.

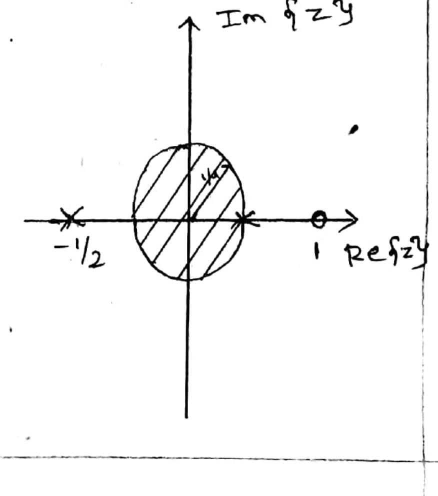
ROC is $|2z| < 1$ and $|4z| < 1$

$|z| < 1/2$ and $|z| < 1/4$.

Hence ROC is $|z| < 1/4$

$X(z) = \frac{1}{1+2z} + \frac{1}{1-4z} = \frac{1-4z+1+2z}{(1+2z)(1-4z)}$

$X(z) = \frac{2-2z}{(1+2z)(1-4z)}$




Zeros: At $z = 1$

Poles: At $z = -1/2$ and $z = 1/4$.

Determine the ZT of the following signal.

$x(n) = \alpha^{|n|}$ for i) $|\alpha| < 1$ ii) $|\alpha| > 1$.

$$x(n) = \begin{cases} \alpha^{-n}, & n < 0 \\ \alpha^{n}, & n > 0 \end{cases}$$

Sol: W.K.T $X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n}$$

$$X(z) = \sum_{-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^{n} z^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} \alpha^{n} z^{n} + \sum_{n=0}^{\infty} \alpha^{n} z^{-n} \quad [n \to -n \text{ in 1st term}]$$

$$X(z) = \sum_{n=1}^{\infty} (\alpha z)^{n} + \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^{n}$$

$$X(z) = \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\frac{\alpha}{z}} \qquad \because \sum_{n=1}^{\infty} p^n = \frac{p}{1-p}, \quad \sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$$

Series converges if.

$|\alpha z| < 1$ and $\left|\frac{\alpha}{z}\right| < 1 \Rightarrow |z| < \frac{1}{\alpha}$ and $\alpha < |z|$

ROC is $\alpha < |z| < \frac{1}{\alpha}$.

$$X(z) = \frac{\alpha z}{1-\alpha z} + \frac{z}{z-\alpha}$$

$$= \frac{\alpha z^2 - \alpha^2 z + z - \alpha z^2}{(1-\alpha z)(z-\alpha)}$$

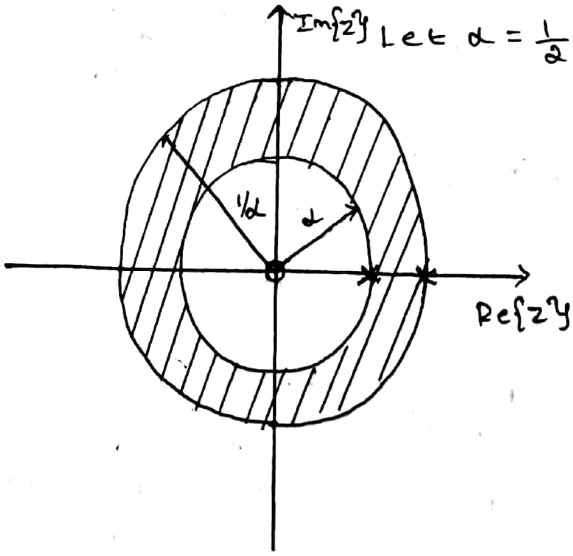
$$= \frac{z - \alpha^2 z}{(1-\alpha z)(z-\alpha)}$$

$$= \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}$$

i) When $|\alpha| < 1$ Roc is $\alpha < |z| < \frac{1}{\alpha}$.

Let $\alpha = \frac{1}{2}$ $\Rightarrow$ $\frac{1}{2} < |z| < 2$.

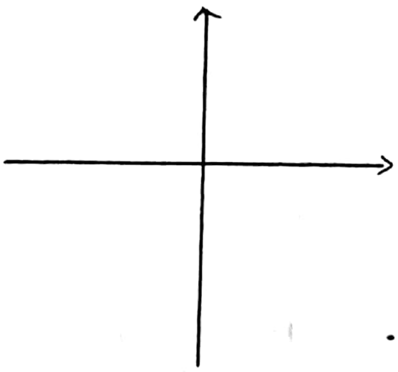
$\alpha < |z| < 1/\alpha$




Poles: $z = \frac{1}{\alpha}$ and $z = \alpha$

Poles $z = 0$

ii) When $|\alpha| > 1$.



Roc

$\left(|z| < \frac{1}{\alpha}\right) \cap \left(|z| > \alpha\right)$

Let $\alpha = 2$.

$\left(|z| < \frac{1}{2}\right) \cap \left(|z| > 2\right)$

$\therefore$ Roc does not exists or the series does not converge. (Roc is null)

# PROPERTIES OF Z-TRANSFORMS

## 1. Linearity

Given $\longrightarrow$

$x_1(n) \xrightarrow{Z.T} X_1(z)$ Roc : $R_1$

$x_2(n) \xrightarrow{Z.T} X_2(z)$ Roc : $R_2$

Prove that : $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z.T} a_1 X_1(z) + a_2 X_2(z)$

Roc : $R_1 \cap R_2$

**Proof:** $z\{a_1x_1(n)+a_2x_2(n)\} = \sum_{n=-\infty}^{\infty}[a_1x_1(n)+a_2x_2(n)]z^{-n}$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n}$$

$$= a_1\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

$$\therefore \quad = a_1X_1(z) + a_2X_2(z)$$

— Hence proved.

Roc : $\boxed{R_1 \cap R_2}$

2. Time Shifting property

Given $\longrightarrow$ $x(n) \xleftrightarrow{Z.T} X(z)$ Roc : R.

Prove that $\longrightarrow$ $x(n-n_0) \xleftrightarrow{Z} z^{-n_0}X(z)$

**Proof:** W.K.T $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

$$z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n}$$

Put $n-n_0=m \Rightarrow n=m+n_0$

if $n=-\infty \Rightarrow m=-\infty$

if $n=\infty \Rightarrow m=\infty$

$$\Rightarrow z\{x(n-n_0)\} = \sum_{m=-\infty}^{\infty} x(m)z^{-(m+n_0)}$$

$$z\{x(n-n_0)\} = \sum_{m=-\infty}^{\infty} x(m)z^{-m}.z^{-n_0}$$

$$= z^{-n_0}\sum_{m=-\infty}^{\infty} x(m)z^{-m}$$

$z\{x(n-n_0)\} = z^{-n_0}\,\underline{X(z)}$ Hence proved.

Roc is R Except for $z=0$ and at $z=\infty$

If $n_0 > 0$ poles introduced at origin $z=0$ which will overwrite the already existing zeros [due to $X(z)$] at origin.

If $n_0 < 0$ the multiplier introduces zeros at origin which may overwrite already existing poles due to $X(z)$

3. Multiplication by a complex exponential Sequence

Given $\rightarrow$ $x(n) \xleftrightarrow{z.t} X(z)$ ROC : R

Prove that $\rightarrow$ $a^n x(n) \xleftrightarrow{z.t} X(z/a)$ ROC : aR

Proof: $$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$\therefore Z\{a^n x(n)\} = X(z)\Big|_{z=z/a} = X(z/a)$$

$$\boxed{\text{ROC : aR}}$$

4. Time Reversal property

Given $x(n) \xleftrightarrow{z.T} X(z)$ ROC: R

prove that $x(-n) \xleftrightarrow{z} X(1/z)$ ROC : 1/R

Proof: $$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

put $n=-m$

As $m=\infty$ $\Rightarrow$ $m=\infty$ and

As $n=-\infty$ $\Rightarrow$ $m=-\infty$

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$$Z\{x(-n)\} = \sum_{m=\infty}^{-\infty} x(m)\, z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \left(\frac{1}{z}\right)^{-m}$$

$$Z\{x(-n)\} = X(1/z) \qquad ROC = 1/R$$

Hence proved.

5. Convolution

Given $\longrightarrow x(n) \xleftrightarrow{z} X(z) \quad ROC: R_1$

$h(n) \longleftrightarrow H(z) \quad ROC: R_2$

Prove that $\rightarrow x(n) * h(n) \xleftrightarrow{z} X(z)\, H(z) \quad ROC: R_1 \cap R_2$

Proof:

$$Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n}$$

$$Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} [x(n) * h(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(n-k)\, x(k) \cdot z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-n}$$ rearranging the terms.

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}\, H(z)$$

$$= H(z) \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$Z\{x(n) * h(n)\} = H(z)\, X(z)$$

$$\therefore Z\{x(n) * h(n)\} = X(z)\, H(z)$$

Hence proved.

ROC : $R_1 \cap R_2$

6. Multiplication by a Ramp / differentiation in Z-domain

Given $\rightarrow$ $x(n) \xleftrightarrow{Z} X(z)$ ROC : R.

Prove that $n x(n) \xleftrightarrow{Z} -z \cdot \frac{d}{dz} X(z)$

Proof: W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ — ①

differentiating w.r.t z, then.

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \left[-n . z^{-n-1}\right]$$

multiplying by $-z$,

$$-z \cdot \frac{d}{dz}\left[X(z)\right] = \sum_{n=-\infty}^{\infty} \{n x(n)\} \left[z^{-n-1} \, z\right]$$

$$-z \cdot \frac{d}{dz}\left[X(z)\right] = \sum_{n=-\infty}^{\infty} \{n x(n)\} z^{-n}$$

$$-z \cdot \frac{d}{dz}\left[X(z)\right] = Z\{n x(n)\}$$

$$\Rightarrow Z\{n x(n)\} = -z \cdot \frac{d}{dz}\left[X(z)\right]$$

ROC : R

Obtain the Z.T of the following:

1) $x(n) = \cos\Omega n \; u(n)$

w.k.t $\cos\Omega n = \dfrac{e^{j\Omega n} + e^{-j\Omega n}}{2} u(n)$

$\therefore x(n) = \dfrac{e^{j\Omega n} + e^{-j\Omega n}}{2} u(n)$

$x(n) = \frac{1}{2} e^{j\Omega n} u(n) + \frac{1}{2} e^{-j\Omega n} u(n)$

WKT $X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$X(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} e^{j\Omega n} u(n) + \frac{1}{2} e^{-j\Omega n} u(n) \right] z^{-n}$

$X(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{j\Omega n} z^{-n} u(n) + \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-j\Omega n} z^{-n} u(n)$

As $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$X(z) = \frac{1}{2} \sum_{n=0}^{\infty} (e^{j\Omega} z^{-1})^{n} + \frac{1}{2} \sum_{n=0}^{\infty} (e^{j\Omega} z)^{-n}$

Note: $\sum_{n=0}^{\infty} p^n = \dfrac{1}{1-p}$ ; $|P| < 1$

then series converges $|e^{j\Omega} z^{-1}| < 1$

$\Rightarrow |e^{j\Omega}| < |z| \Rightarrow 1 < |z| \Rightarrow |z| > 1$

$\because |z| = 1 = \sqrt{\cos^2\Omega + \sin^2\Omega}$

$X(z) = \frac{1}{2} \left[ \dfrac{1}{1 - \frac{e^{j\Omega}}{z}} \right] + \frac{1}{2} \left[ \dfrac{1}{1 - \frac{e^{-j\Omega}}{z}} \right]$

$= \frac{1}{2} \left[ \dfrac{z}{z - e^{j\Omega}} \right] + \frac{1}{2} \left[ \dfrac{z}{z - e^{-j\Omega}} \right]$

We could use property
$a^n u(n) \longleftrightarrow \dfrac{z}{z-a}$
ROC: $|z| > a$

$X(z) = \frac{1}{2} \left[ \dfrac{z^2 - ze^{-j\Omega} + z^2 - ze^{j\Omega}}{(z - e^{j\Omega})(z - e^{-j\Omega})} \right]$

$$x(z) = \frac{1}{2}\left[\frac{2z^2 - z(e^{j\Omega} + e^{-j\Omega})}{z^2 - ze^{j\Omega} - ze^{-j\Omega} + 1}\right]$$

$$x(z) = \frac{1}{2}\left[\frac{2z^2 - z(e^{j\Omega} + e^{-j\Omega})}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1}\right]$$

W.K.T $\quad e^{j\Omega} + e^{-j\Omega} = 2\cos\Omega.$

$$\therefore x(z) = \frac{1}{2}\left[\frac{2z^2 - 2z\cos\Omega}{z^2 - 2z\cos\Omega + 1}\right]$$

$$x(z) = \frac{z^2 - z\cos\Omega}{z^2 - 2z\cos\Omega + 1}$$

ROC is $|z| > 1$

2) $x(n) = \sin \Omega n \, u(n)$

Sol W.K.T $\quad \sin\Omega n = \frac{e^{+j\Omega n} - e^{-j\Omega n}}{2j} u(n)$

$$\therefore x(n) = \frac{1}{2j} e^{j\Omega n} u(n) - \frac{1}{2j} e^{-j\Omega n} u(n)$$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(z) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2j} e^{j\Omega n} u(n) - \frac{1}{2j} e^{-j\Omega n} u(n)\right] z^{-n}$$

$$x(z) = \frac{1}{2j}\sum_{n=0}^{\infty} e^{j\Omega n} z^{-n} - \frac{1}{2j}\sum_{n=0}^{\infty} e^{-j\Omega n} z^{-n}.$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$x(z) = \frac{1}{2j}\sum_{n=0}^{\infty} (e^{j\Omega} z^{-1})^n - \frac{1}{2j}\sum_{n=0}^{\infty} (e^{-j\Omega} z^{-1})^n$$

$$X(z) = \frac{1}{2j}\sum_{n=0}^{\infty}(e^{j\Omega}z^{-1})^n - \frac{1}{2j}\sum_{n=0}^{\infty}(e^{-j\Omega}z^{-1})^n$$

Roc is $|e^{j\Omega}z^{-1}|<1$ and $|e^{-j\Omega}z^{-1}|<1$

$|z^{-1}|<1$ $\quad$ $|z^{-1}|<1$

$\left|\frac{1}{z}\right|<1$ $\quad$ $\left|\frac{1}{z}\right|<1$

$|z|>1$ $\quad$ $|z|>1$

$$X(z) = \frac{1}{2j}\left[\frac{1}{1-\frac{e^{j\Omega}}{z}}\right] - \frac{1}{2j}\left[\frac{1}{1-\frac{e^{-j\Omega}}{z}}\right]$$

$$X(z) = \frac{1}{2j}\left[\frac{z}{z-e^{j\Omega}}\right] - \frac{1}{2j}\left[\frac{z}{z-e^{-j\Omega}}\right]$$

$$X(z) = \frac{1}{2j}\left[\frac{\cancel{z^2}-ze^{-j\Omega}-\cancel{z^2}+ze^{j\Omega}}{(z-e^{j\Omega})(z-e^{-j\Omega})}\right]$$

$$X(z) = \frac{1}{2j}\left[\frac{z(e^{j\Omega}-e^{-j\Omega})}{z^2-ze^{j\Omega}-ze^{-j\Omega}+1}\right]$$

$$X(z) = \frac{1}{2j}\left[\frac{z(e^{j\Omega}-e^{-j\Omega})}{z^2-z(e^{j\Omega}+e^{-j\Omega})+1}\right]$$

$$= \frac{1}{2j}\left[\frac{z.2j\sin\Omega}{z^2-z2(\cos\Omega)+1}\right]$$

$$= \frac{z\sin\Omega}{z^2-2z\cos\Omega+1}$$

Roc is $|z|>1$.

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3. $x(n) = a^n \cos(\Omega n)\, u(n)$

Sol : $w.k.t\ Z\{\cos(\Omega n)\, u(n)\} = \dfrac{z^2 - z\cos\Omega}{z^2 - 2z\cos\Omega + 1}$ ∴ ROC is $R_1: |z| > 1$

$Z\{x_1(n)\}$

Let $x_1(n) = \cos(\Omega n)\, u(n)$

$\therefore x(n) = a^n x_1(n)$

By multiplication property.

$$Z\{a^n x_1(n)\} = X_1\left(\frac{z}{a}\right). \quad \text{ROC}: aR_1$$

i.e. $|z| > a$

Here $x(n) = a^n \cos(\Omega n)\, u(n)$

$x(n) = a^n x_1(n)$

$w.k.t\ Z\{x_1(n)\} = \dfrac{z^2 - z\cos\Omega}{z^2 - 2z\cos\Omega + 1}.$

$$\Rightarrow Z\{x(n)\} = Z\{x_1(n)\}\Big|_{z \to z/a}$$

$$\therefore X(z) = \frac{z^2 - 2z\cos\Omega}{z^2 - 2z\cos\Omega + 1}\Bigg|_{z \to z/a}$$

$$= \frac{\frac{z^2}{a^2} - \frac{2z}{a}\cos\Omega}{\frac{z^2}{a^2} - \frac{2z}{a}\cos\Omega + 1.}$$

by taking $\frac{z^2}{a^2}$ common

$$X(z) = \frac{\cancel{\left(\frac{z}{a}\right)^2}\left[1 - az^{-1}\cos\Omega\right]}{\cancel{\left(\frac{z}{a}\right)^2}\left[1 - 2az^{-1}\cos\Omega + a^2 z^{-2}\right]}$$

$$X(z) = \frac{1 - az^{-1}\cos\Omega}{1 - 2az^{-1}\cos\Omega + a^2 z^{-2}.}$$

ROC $|z| > a.$

multiply and divide with $z^2$.

$$X(z) = \frac{z^2 - az\cos\Omega}{z^2 - 2az\cos\Omega + a^2}$$

4) $x(n) = a^n \sin(\Omega n) . u(n)$

Sol: w.k.T $Z\{\sin(\Omega n) u(n)\} = \dfrac{z \sin\Omega}{z^2 - 2z\cos\Omega + 1}$ : $|z| > 1$.

By multiplication property,

$$Z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$$

$$\therefore Z\{a^n \sin\Omega n \; u(n)\} = \frac{z/a \sin\Omega}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\Omega + 1}$$

$$= \frac{az \sin\Omega}{z^2 - 2az\cos\Omega + a^2}$$

5. $x(n) = \cosh(\Omega n) \cdot u(n)$

w.k.T $\cosh(\Omega n) = \dfrac{e^{\Omega n} + e^{-\Omega n}}{2}$

$\therefore x(n) = \frac{1}{2} e^{\Omega n} u(n) + \frac{1}{2} e^{-\Omega n} u(n)$

w.k.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} e^{\Omega n} u(n) + \frac{1}{2} e^{-\Omega n} u(n)\right] z^{-n}$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\therefore X(z) = \frac{1}{2}\sum_{n=0}^{\infty} e^{\Omega n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-\Omega n} z^{-n}$$

$$X(z) = \frac{1}{2}\sum_{n=0}^{\infty} (e^{\Omega} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (e^{-\Omega} z^{-1})^n$$

~~X(z)~~ =

ROC ~~con~~ $|e^{\Omega} z^{-1}| < 1$ $\quad$ $|e^{-\Omega} z^{-1}| < 1$

$|z| > e^{\Omega}$ $\quad$ $e^{-\Omega} < |z|$

$|z| > e^{-\Omega}$

$$X(z) = \frac{1}{2}\left[\frac{1}{1-\frac{e^{\Omega}}{z}}\right] + \frac{1}{2}\left[\frac{1}{1-\frac{e^{-\Omega}}{z}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\Omega}}\right] + \frac{1}{2}\left[\frac{z}{z-e^{-\Omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-\Omega} + z^2 - ze^{\Omega}}{(z-e^{\Omega})(z-e^{-\Omega})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{\Omega} + e^{-\Omega})}{z^2 - ze^{-\Omega} - ze^{\Omega} + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{\Omega} + e^{-\Omega})}{z^2 - z(e^{\Omega} + e^{-\Omega}) + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2z\cosh\Omega}{z^2 - z2\cosh\Omega + 1}\right]$$

$$= \frac{z^2 - z\cosh\Omega}{z^2 - 2z\cosh\Omega + 1}$$

6) $x(n) = na^n u(n)$

Sol Let $x_1(n) = a^n u(n)$

$\cancel{X_1(z) \xleftrightarrow{z} Z\{}$

$x_1(n) \xleftrightarrow{z} X_1(z)$

$x(n) = n \cdot x_1(n)$

$x(n) \xleftrightarrow{z} X(z)$

$\therefore \quad Z\{x_1(n)\} = X_1(z) = Z\{a^n u(n)\}$

$X_1(z) = \dfrac{z}{z-a} \quad ; \ |z| > a$

Now $Z\{x(n)\} = Z\{n \cdot x_1(n)\}$

$= -z\dfrac{d}{dz} X_1(z)$

$= -z \cdot \dfrac{d}{dz}\left[\dfrac{z}{z-a}\right]$

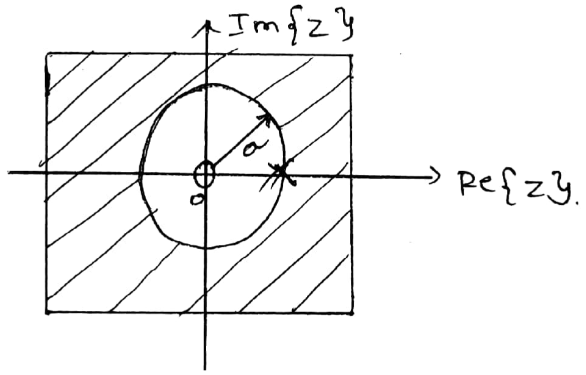
$= -z\left[\dfrac{1(z-a) - z(1)}{(z-a)^2}\right]$

$= \dfrac{-z[z-a-z]}{(z-a)^2}$

$= \dfrac{az}{(z-a)^2} \quad$ Roc $|z| > a$.

Note If $a = 1$ in above.

$nu(n) \xleftrightarrow{z} \dfrac{z}{(z-1)^2} \quad |z| > 1.$

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7) $x(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4} n\right) u(n)$

Sol : WKT $a^n \sin(\Omega n)\, u(n) \xrightarrow{Z} \dfrac{a z \sin\Omega}{z^2 - 2az\cos\Omega + a^2}$

$Z\{x(n)\} = Z\left\{\left(\frac{1}{3}\right)^n \sin(\pi/4)n \; u(n)\right\}$ Here, $a = 1/3$

$\Omega = \pi/4$

$$= \frac{z(1/3)\sin\pi/4}{(\cancel{1/3})^2 - 2(1/3)z\cos\pi/4 + (1/3)^2}$$

$$= \frac{1/3\, z \sin\pi/4}{z^2 - 2/3\, z(1/\sqrt{2}) + 1/9}$$

$$= \frac{\frac{z}{3\sqrt{2}}}{z^2 - \frac{2z}{3\sqrt{2}} + 1/9} \times \frac{9\sqrt{2}}{9\sqrt{2}}$$

$$= \frac{3z}{9\sqrt{2}z^2 - 6z + \sqrt{2}} \qquad |z| > 1/3$$

8) $x(n) = \sin\left[\frac{\pi}{8}n - \pi/4\right]u(n-2)$

Sol :

$$x(n) = \sin\left[\pi/8\,(n-2)\right]u(n-2)$$

Let $x_1(n) = \sin \pi/8\, n \;\; u(n)$

$$x(n) = x_1(n-2)$$

$x_1(n) \xleftrightarrow{z} X_1(z) \rightarrow$ ROC : R

$x(n) \xleftrightarrow{z} X(z)$

$x_1(n-2) \longleftrightarrow z^{-2} X_1(z)$ ROC : R except $z = 0, \infty$

$$Z\{x_1(n)\} = Z\{\sin(\pi/8\, n)\,[u(n)]\}$$

$$= \frac{z\sin(\pi/8)}{z^2 - 2z\cos\pi/8 + 1} \quad |z| > 1$$

$\because \; Z\{\sin \Omega n \;\; u(n)\} = \dfrac{z\sin\Omega}{z^2 - 2z\cos\Omega + 1} \quad |z| > 1$

$$Z\{x_1(n-2)\} = z^{-2} X_1(z)$$

$$= \frac{z \sin \pi/8}{(z^2 - 2z\cos\pi/8 + 1)} \times z^{-2}$$

$$= \frac{z^{-1}\sin\pi/8}{z^2 - 2z\cos\pi/8 + 1}$$

$$= \frac{\sin\pi/8}{z(z^2 - 2z\cos\pi/8 + 1)} \quad |z| > 1$$

ROC is R except at $z = 0$ and $z = \infty$

9) $x(n) = n \sin \pi/2 n \cdot u(-n)$

Let $x_1(n) = \sin \frac{\pi}{2} n \, u(n)$ . Put $n = -n$

$x_1(-n) = \sin \frac{\pi}{2}(-n) \, u(-n)$ $\therefore \sin(-n) = -\sin(n)$

$x_1(-n) = -\sin \frac{\pi}{2} n \, u(-n)$

$x(n) = -n \, x_1(-n)$

$Z\{x_1(n)\} = Z\{\sin \frac{\pi}{2} n \, u(n)\}$

$$X_1(z) = \frac{z \sin \pi/2}{z^2 - 2z \cos \pi/2 + 1} \quad |z| > 1$$

$$Z\{x_1(n)\} = \frac{z(1)}{z^2 - 2z(0) + 1} \quad |z| > 1$$

$$X_1(z) = \frac{z}{z^2+1}$$

Now.

$$Z\{x_1(-n)\} = X_1(z) \Big|_{z \to 1/z}$$

$$\Rightarrow Z\{x_1(-n)\} = \frac{z}{z^2+1} \Big|_{z \to 1/z}$$

$$= \frac{1/z}{1/z^2 + 1} = \frac{1/z}{\frac{1+z^2}{z^2}} = \frac{z^2/z}{1+z^2}$$

$$Z\{x_1(-n)\} = \frac{z}{1+z^2} \quad |z| < 1$$

$$\therefore Z\{n \, x_1(-n)\} = -z \frac{d}{dz} \{Z\{x_1(-n)\}\}$$

By multiplication of ramp property

$$Z\{nx(-n)\} = -z \cdot \frac{d}{dz}\left\{\frac{z}{1+z^2}\right\}$$

$$= -z \frac{(1+z^2) - z(2z)}{(1+z^2)^2}$$

$$= -z\left[\frac{1+z^2-2z^2}{(1+z^2)^2}\right]$$

$$= \frac{-z[1-z^2]}{(1+z^2)^2}$$

$$= \frac{z(z^2-1)}{(1+z^2)^2}$$

Zero: $z=0$ and $z^2=1 \Rightarrow z=+1,-1,0$

Poles: $(1+z^2)^2=0$

$1+z^2=0$

$z^2=-1$

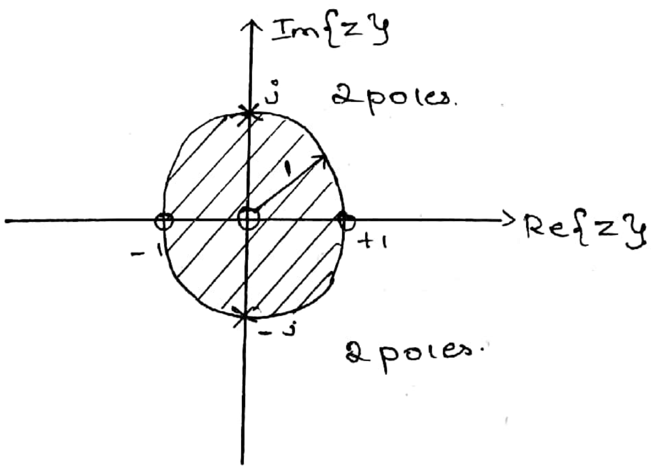
$z=\pm j$

$1+z^2=0$

$z^2=-1$

$z=\pm j$

$|z|<1$



9) $x(n) = 2\delta(n-3) - 2\delta(n+3)$

Sol:

$$x(n) = 2\delta(n-3) - 2\delta(n+3)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \cancel{2} \sum_{n=-\infty}^{\infty} \left[2\delta(n-3) - 2\delta(n+3)\right] z^{-n}$$

$$= 2\sum_{n=-\infty}^{\infty} \delta(n-3) z^{-n} - 2\sum_{n=-\infty}^{\infty} \delta(n+3) z^{-n}$$

$$= 2z^{-3} - 2z^{3}$$

$$X(z) = \frac{2}{z^3} - 2z^3$$

$$= \frac{2-2z^6}{z^3} = \frac{2(1-z^6)}{z^3}$$

ROC: Entire z-plane except $z=0$ and $\infty$

Zeros: 6 zeros at $z=1$

Poles: 3 poles at $z=0$

10) $x(n) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=0}^{N-1} a^n z^{-n}$$

W.K.T $\sum_{n=0}^{p} a^k = \frac{a^{p+1} - 1}{a-1}$

then $x(z) = \sum_{n=0}^{N-1} a^n z^{-n}$

$$= \sum_{n=0}^{N-1} (a z^{-1})^n \qquad |az^{-1}| < 1$$

$|a| < |z|$

$|z| > a.$

$$= \frac{(az^{-1})^{N-1+1} - 1}{az^{-1} - 1}$$

$$= \frac{a^N z^{-N} - 1}{\frac{a}{z} - 1}$$

$$= \frac{a^N z^{-N} \cdot z - z}{a - z}$$

$$= \frac{z - z a^N z^{-N}}{z - a}$$

(or)

$$= \frac{z\left[1 - \frac{a^N}{z^N}\right]}{z\left[1 - \frac{a}{z}\right]}$$

$$= \frac{1 - \frac{a^N}{z^N}}{1 - \frac{a}{z}}$$

$$= \frac{\frac{z^N - a^N}{z^N}}{\frac{z - a}{z}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z - a)}$$

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⇒ $x(n) = n\left(\frac{1}{2}\right)^n u(n)$. Find Z.T

Soln

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Let $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$

$\therefore x(n) = n\, x_1(n)$

$$X_1(z) = \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$

$$Z\{n\, x_1(n)\} = -z \frac{d X_1(z)}{dz} \quad |z| > \frac{1}{2}$$

$$= -z \frac{(z - \frac{1}{2}) - z}{(z - \frac{1}{2})^2}$$

$$= \frac{-z(-\frac{1}{2})}{(z - \frac{1}{2})^2}$$

$$X(z) = \left(\frac{1}{2}\right) \cdot \frac{z}{(z - \frac{1}{2})^2} \quad |z| > \frac{1}{2}$$

⇒ Find Z.T

$x(n) = n\, a^n u(n)$

Ans

$$X(z) = \frac{az}{(z-a)^2} \quad \boxed{\text{ROC: } |z| > a}$$

⇒ $x(n) = n\, a^{n-1} u(n)$

$$X(z) = \frac{z}{(z-a)^2} \quad \boxed{\text{ROC: } |z| > a.}$$

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① $x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$ - Find I Z.T

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1z^0 + 2z^{-1} + 2z^{-2} + 1z^{-3}$$

ROC: entire z plane except $z=0$ i.e.

ROC: $|z| > 0$ (for right sided sequence)

② $x(n) = (1 \ 1 \ 2 \ \underset{\uparrow}{2})$

$$X(z) = z^3 + z^2 + 2z + 2z^0$$

ROC entire z plane. except $z=\infty$

∴ ROC: $|z| < \infty$ (for left sided sequence)

③ $x(n) = (2, 1 \ \underset{\uparrow}{1}, 2)$

$$X(z) = 2z^2 + 1z^1 + 1z^0 + 2z^{-1}$$

ROC entire z plane except $z=0$ & $\infty$

i.e., ROC: $0 < |z| < \infty$ (for double sided sequence)

## Properties of ROC:

① ROC can't contain poles.

② For finite length causal sequence (right sided sequence), ROC is entire z plane except at $z=0$ i.e; ROC: $|z|>0$.

③ For finite length non causal sequence (left sided seq) ROC is entire z plane except at $z=\infty$

i.e., ROC : $|z| < \infty$

(4) For finite length double sided sequence, ROC is entire z plane except at $z = 0, \infty$ i.e.,

ROC $0 < |z| < \infty$

(5) For causal infinite length sequence. ROC is exterior to the circle with radius $r_{max}$. i.e., $|z| > r_{max}$.
(where $r_{max}$ : largest magnitude of any of the poles of $X(z)$) with possible exception of $z = \infty$.

6) For non causal infinite length sequence ROC is interior to the circle with radius $r_{min}$ i.e., $|z| < r_{min}$
( where $r_{min}$ : smallest magnitude of any of the poles of $X(z)$). with possible exception at $z = 0$.

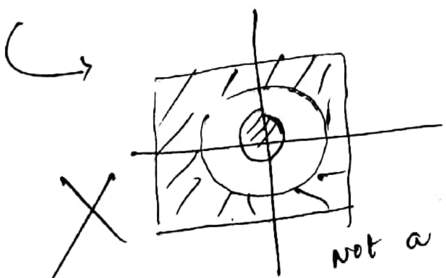
(7) For two sided sequence of infinite duration, the ROC is of the form

$$r_1 < |z| < r_2$$

where $r_1$ & $r_2$ are the magnitude of the poles of $X(z)$. Thus the ROC is an annular rings in the z-plane b/wn the circles $|z| = r_1$ & $|z| = r_2$ not containing any poles.

(8) The ROC of an LTI stable system contains the unit circle in the z plane.

(9) The ROC must be a connected region.



Not a connected region.

# Inverse z Transform: Partial Fraction Approach

1) Find the inverse z-transform of the sequence

$$x(z) = \frac{z}{3z^2 - 4z + 1}$$ for a) $|z| > 1$ (b) $|z| < 1/3$

(c) $1/3 < |z| < 1$

Soln

$3z^2 - 4z + 1 = 0$

$(z - 1/3)(z-1) = 0 \quad \times 3$

$(3z-1)(z-1) = 0$

$$x(z) = \frac{z}{(3z-1)(z-1)}$$

$$\frac{x(z)}{z} = \frac{1}{3(z-1/3)(z-1)}$$

$$\frac{x(z)}{z} = \frac{A}{(z-1/3)} + \frac{B}{z-1}$$

$$A = \frac{x(z)}{z}(z-1/3)\Big|_{z=1/3}$$

$$A = \frac{1}{3\cancel{(z-1/3)}(z-1)}\cancel{(z-1/3)}\Big|_{z=1/3}$$

$$A = \frac{1}{3(\frac{1}{3}-1)}$$

$$A = \frac{1}{3\frac{(-2)}{3}}$$

$$\boxed{A = -\frac{1}{2}}$$

$$B = \frac{x(z)}{z}(z-1)\Big|_{z=1}$$

$$B = \frac{1}{3(z-1/3)\cancel{(z-1)}}\cancel{(z-1)}\Big|_{z=1}$$

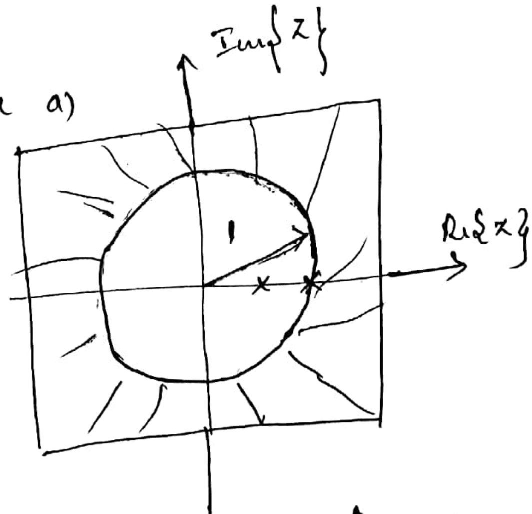
$$B = \frac{1}{3(1-1/3)}\Big|_{z=1}$$

$$B = \frac{1}{3(\frac{2}{3})}$$

$$\boxed{B = \frac{1}{2}}$$

$$\frac{x(z)}{z} = -\frac{1}{2}\frac{1}{(z-\frac{1}{3})} + \frac{1}{2}\frac{1}{(z-1)}$$

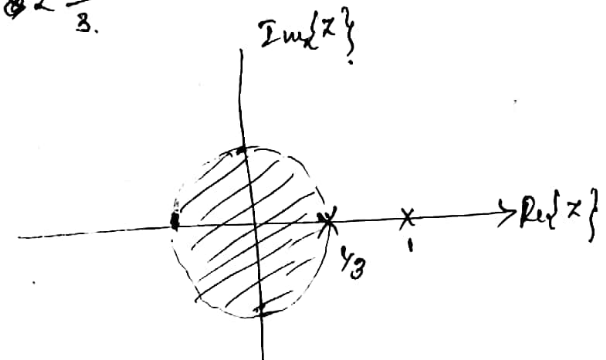
case a)




$|z| > 1 \implies$ it is right sided sequence. :.

$x(n) =$
$-\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) + \frac{1}{2} 1^n u(n)$

b) $|z| < \frac{1}{3}$

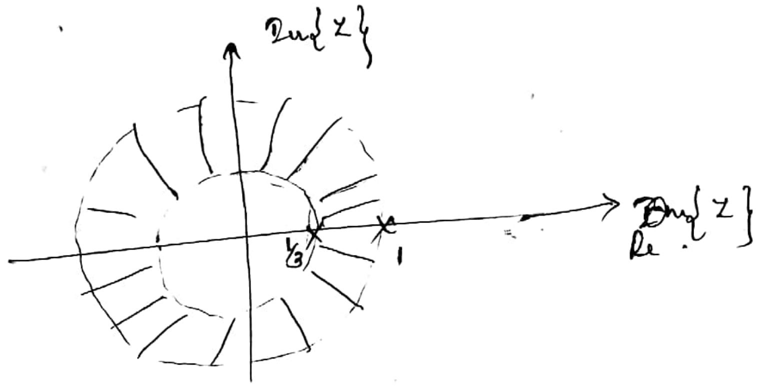



$-a^n u(-n-1) \xleftrightarrow{Z.T} \frac{z}{z-a} \quad |z| < a$

It is left sided sequence.

$\therefore x(n) = +\frac{1}{2}(1/3)^n u(-n-1) - \frac{1}{2} 1^n u(-n-1)$

c) $\frac{1}{3} < z < 1$




given $|z| > 1/3$ $\quad |z| < 1$

Ist term is for right sided sequence.

IInd term is for left sided sequence.

$x(n) = -\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) - 1/2 \,(1)^n u(-n-1)$

2 Q/n

$$X(z) = \frac{z+1}{3z^2-4z+1}$$

Find the ~~Roc~~ & inverse Z.T for the ROC $|z|>1$

soln

$$\frac{X(z)}{z} = \frac{z+1}{z(3z^2-4z+1)}$$

$$\frac{X(z)}{z} = \frac{z+1}{z(3z-1)(z-1)}$$

$$= \cancel{\frac{A}{z}} + \text{—}$$

$$\frac{X(z)}{z} = \frac{z+1}{3z(z-1/3)(z-1)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-1/3} + \frac{C}{z-1}$$

$$A = \frac{X(z)}{z}\, z \Big|_{z=0}$$

$$A = \frac{z+1}{3\cancel{z}(z-\frac{1}{3})(z-1)}\cancel{z}\Big|_{z=0}$$

$$A = \frac{1}{3(-1/3)(-1)}$$

$$\boxed{A = 1}$$

$$B = \frac{X(z)}{z}\left(z-\frac{1}{3}\right)\Big|_{z=1/3}$$

$$= \frac{z+1}{3z\cancel{(z-1/3)}(z-1)}\cancel{(z-1/3)}\Big|_{z=1/3}$$

$$= \frac{\frac{1}{3}+1}{3\,\frac{1}{3}\left(\frac{1}{3}-1\right)}$$

$$= \frac{\cancel{4}^{2}/\cancel{3}}{-\cancel{2}/\cancel{3}}$$

$$\boxed{B = -2}$$

$$C = \frac{X(z)}{z}(z-1)\Big|_{z=1}$$

$$= \frac{z+1}{3z(z-1/3)\cancel{(z-1)}}\cancel{(z-1)}\Big|_{z=1}$$

$$= \frac{2}{3(1)(1-1/3)} = \frac{\cancel{2}}{\cancel{3}\left(\frac{\cancel{2}}{\cancel{3}}\right)}$$

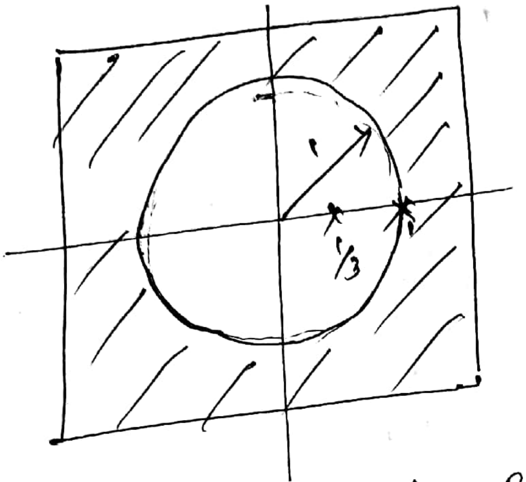
$$\boxed{C = 1}$$

$$\frac{X(z)}{z} = \frac{1}{z} + \frac{-2}{z-1/3} + \frac{1}{z-1}$$

$$X(z) = 1 - (2)\frac{z}{z-1/3} + \frac{z}{z-1}$$

with ROC $|z|>1$

(1) (c)




$|z| > 1$ ; it is for Right sided sequence.

$x(z) = \delta(n) - 2\left(\frac{1}{3}\right)^n u(n) + u(n)$

$$A = \frac{x(z)}{z}(z-2)\Big|_{z=2}$$

$$= \frac{1}{\cancel{(z-2)}(z+1)}\cancel{(z-2)}\Big|_{z=2}$$

$$\boxed{A = \frac{1}{3}}$$

$$B = \frac{x(z)}{z}(z+1)\Big|_{z=-1}$$

$$= \frac{1}{(z-2)\cancel{(z+1)}}\cancel{(z+1)}\Big|_{z=-1}$$

$$= \frac{1}{(-1-2)}$$

$$\boxed{B = -\frac{1}{3}}$$

(3) $x(z) = \dfrac{z^{-1}}{-2z^{-2} - z^{-1} + 1}$

ROC : $1 < |z| < 2$

Find inverse Z.T

soln

$$x(z) = \frac{z^{-1}}{-2z^{-2} - z^{-1} + 1} \times \frac{z^2}{z^2}$$

$$x(z) = \frac{z}{z^2 - z - 2}$$

$$\frac{x(z)}{z} = \frac{1}{(z-2)(z+1)}$$

$$\frac{x(z)}{z} = \frac{A}{z-2} + \frac{B}{z+1}$$

$\therefore$

$$\frac{x(z)}{z} = \frac{1}{3}\frac{1}{z-2} - \frac{1}{3}\frac{1}{z+1}$$

$$x(z) = \frac{1}{3}\frac{z}{z-2} - \frac{1}{3}\frac{z}{z-(-1)}$$

Given $|1| < |z| < |2|$ — Right sided sequence, left side sequence

~~$x(n) = \frac{1}{3}$~~

$$x(n) = -\frac{1}{3}\, 2^n u(-n-1) - \frac{1}{3}(-1)^n u(n)$$

Note: If $\frac{X(z)}{z}$ has the poles of multiplicity 'n' i.e.,

$$\frac{X(z)}{z} = \frac{\text{Numerator polynomial}}{(z-p)^n}$$

$$= \frac{A_1}{z-p} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_n}{(z-p)^n}.$$

where $A_1$ $A_2$ .. $A_n$ are given as.

$$A_k = \frac{1}{(n-k)!} \frac{d^{n-k}}{dz^{n-k}} \left\{ \frac{X(z)}{z} (z-p)^n \right\} \bigg|_{z=p}$$

$k = 1\ 2\ \ldots\ n.$

4

Qn Find the inverse Z.T.

$$X(z) = \frac{z(z+1)}{(z-1)^2 (z-\frac{1}{2})} \qquad \text{ROC} : |z|>1$$

$$\frac{X(z)}{z} = \frac{(z+1)}{(z-1)^2 (z-\frac{1}{2})}$$

$$\frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-1)^2} + \frac{B}{z-\frac{1}{2}}$$

$n = 2$

$$A_1 = \frac{1}{(2-1)!} \frac{d^1}{dz^1} \left\{ \frac{X(z)}{z} (z-1)^2 \right\} \bigg|_{z=1}$$

$$= \frac{1}{1} \frac{d}{dz} \left\{ \frac{z+1}{\cancel{(z-1)^2} (z-\frac{1}{2})} \cancel{(z-1)^2} \right\} \bigg|_{z=1}$$

$$= \frac{d}{dz} \left\{ \frac{z+1}{z-\frac{1}{2}} \right\} \bigg|_{z=1}$$

$$= \frac{(z-\frac{1}{2})(1) - (z+1)}{(z-\frac{1}{2})^2} \bigg|_{z=1}$$

$$= \frac{\cancel{z} - \frac{1}{2} - \cancel{z} - 1}{(z-\frac{1}{2})^2} \bigg|_{z=1}$$

$$= \frac{-\frac{3}{2}}{(1-\frac{1}{2})^2}$$

$$= \frac{-\frac{3}{2}}{\frac{1}{4}}$$

$$\boxed{A_1 = -6}$$

$$A_2 = \frac{1}{0!} \frac{X(z)}{z} (z-1)^2 \bigg|_{z=1}$$

$$A_2 = \frac{z+1}{\cancel{(z-1)^2}(z-\frac{1}{2})} \cancel{(z-1)^2} \bigg|_{z=1}$$

$$A_2 = \frac{2}{\frac{1}{2}} \qquad \boxed{A_2 = 4}.$$

$$B = \frac{X(z)}{z}\left(z-\tfrac{1}{2}\right)\Big|_{z=1/2}$$

$$= \frac{z+1}{(z-1)^2\cancel{\left(z-\tfrac{1}{2}\right)}}\cancel{\left(z-\tfrac{1}{2}\right)}\Big|_{z=1/2}$$

$$B = \frac{z+1}{(z-1)^2}\Big|_{z=1/2}$$

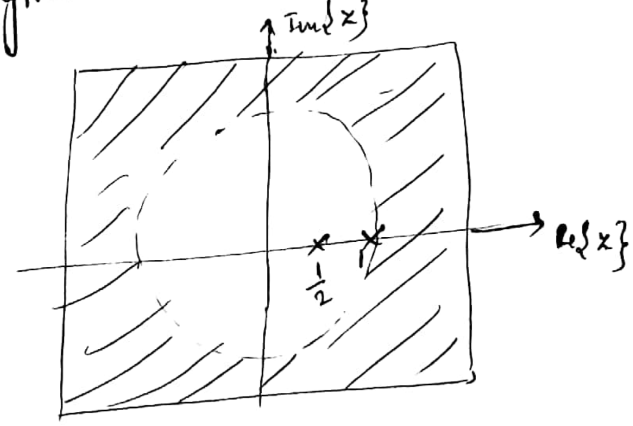
$$B = \frac{\frac{3}{2}}{\left(\frac{1}{2}-1\right)^2} = \frac{\frac{3}{\cancel{2}}}{\frac{1}{\cancel{4}\,2}}$$

$$\boxed{B = 6}$$

$$\frac{X(z)}{z} = -6\frac{1}{z-1} + 4\frac{1}{(z-1)^2} + \frac{6}{\left(z-\frac{1}{2}\right)}$$

$$X(z) = -6\frac{z}{z-1} + 4\frac{z}{(z-1)^2} + 6\frac{z}{z-\frac{1}{2}}$$

given ROC: $|z|>1$




$$x(n) = -6(1)^n u(n) + 4\,n\,u(n) + 6\left(\tfrac{1}{2}\right)^n u(n)$$

$$= 4n\,u(n) - 6u(n) + 6\left(\tfrac{1}{2}\right)^n u(n)$$

6 0

(5) Find the inverse z.T of

$$X(z) = \frac{z^4 + z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

ROC: $\frac{1}{2} < |z| < \infty$

soln

$$\frac{X(z)}{z} = \frac{z^3 + z}{z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}}$$

$$\frac{X(z)}{z} = \frac{z^3 + z}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

it is improper fraction so.

$$\begin{array}{r|l} & z + \frac{3}{4} \quad (Q) \\ z^2 - \frac{3}{4}z + \frac{1}{8} & \cancel{z^3} + z \\ (D_r) & \cancel{z^3} - \frac{3}{4}z^2 + \frac{1}{8}z \\ & (-) \quad (+) \quad (-) \\ \hline & +\cancel{\frac{3}{4}z^2} + \frac{7}{8}z \\ & \cancel{\frac{3}{4}z^2} - \frac{9}{16}z + \frac{3}{32} \\ & (-) \quad (+) \quad (-) \\ \hline & \frac{23}{16}z - \frac{3}{32} \quad (R) \end{array}$$

$$\frac{X(z)}{z} = Q + \frac{R}{D_r}$$

$$\frac{X(z)}{z} = z + \frac{3}{4} + \frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$\frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{\frac{23}{16}z - \frac{3}{32}}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\frac{\frac{23}{16}z - \frac{3}{32}}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{(z-\frac{1}{4})}$$

$$A = \frac{\frac{23}{16}z - \frac{3}{32}}{\cancel{(z-\frac{1}{2})}(z-\frac{1}{4})}\cancel{(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{2}}$$

$$\boxed{A = \frac{5}{2}}$$

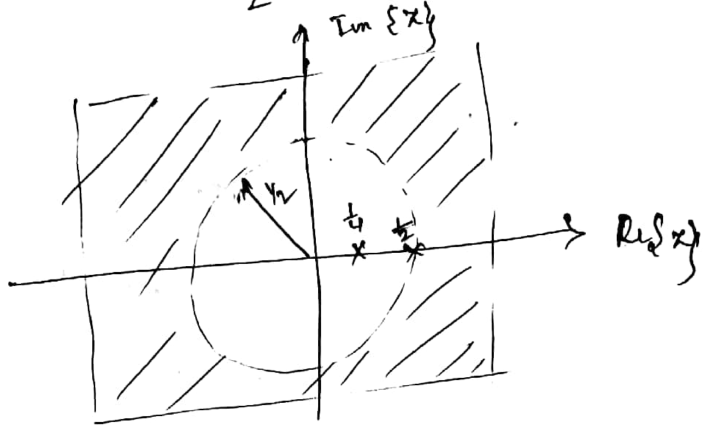
$$B = \frac{\frac{23}{16}z - \frac{3}{32}}{(z-\frac{1}{2})\cancel{(z-\frac{1}{4})}}\cancel{(z-\frac{1}{4})}\Bigg|_{z=\frac{1}{4}}$$

$$\boxed{B = -\frac{17}{16}}$$

$$\frac{X(z)}{z} = z + \frac{3}{4} + \frac{5}{2}\,\frac{1}{z-1/2} - \frac{17}{16}\,\frac{1}{z-\frac{1}{4}}$$

$$X(z) = z^2 + \frac{3}{4}z + \frac{5}{2}\,\frac{z}{z-1/2} - \frac{17}{16}\,\frac{z}{z-1/4}$$

given ROC : $\frac{1}{2} < |z| < \infty$




Taking I z.T

$$x(n) = \delta(n+2) + \frac{3}{4}\delta(n+1) + \frac{5}{2}\left(\tfrac{1}{2}\right)^n u(n) - \frac{17}{16}\left(\tfrac{1}{4}\right)^n u(n)$$

Note: Improper fraction: degree of Nr ≥ that of denominator.

LONG DIVISION METHOD / POWER SERIES EXPANSION TECHNIQUE

Qu ① Find the inverse z.T of the following

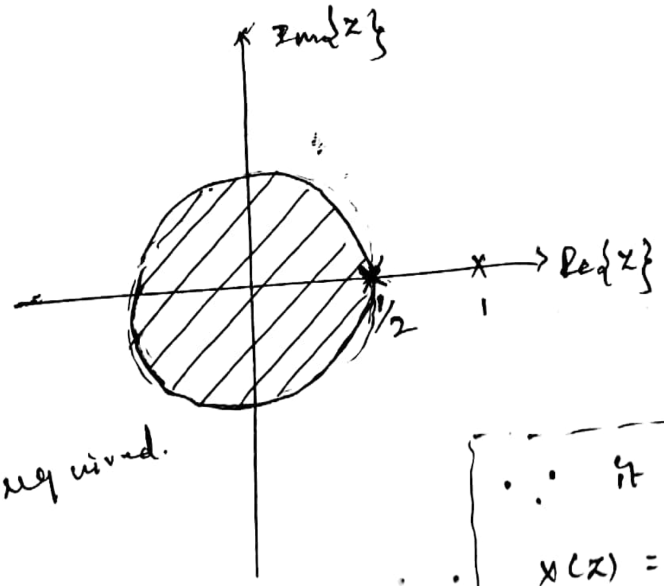
$$X(z) = \frac{z}{2z^2-3z+1} \quad ; \quad ROC: |z| < 1/2$$

soln To sketch ROC & locate poles

$$2z^2 - 3z + 1 = 0$$
$$2z^2 - 2z - z + 1 = 0$$
$$2z(z-1) - (z-1) = 0$$

$$(z-1)(2z-1) = 0$$
$$z = 1/2 \quad z = 1$$

Not required.




not required.

given ROC : $|z| < \frac{1}{2}$

$\therefore$ it is left sided sequence.

Thus quotient of long division must contain +ve powers of $z$.

$$\therefore X(z) = \frac{z}{1 - 3z + 2z^2}$$

$$
\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & z - 3z^2 + 2z^3 \\
 & (-)\ \ (+)\ \ \ (-) \\
\hline
 & 3z^2 - 2z^3 \\
 & 3z^2 - 9z^3 \pm 6z^4 \\
 & (-)\ \ (+)\ \ \ (-) \\
\hline
 & 7z^3 - 6z^4 \\
 & 7z^3 - 21z^4 \pm 14z^5 \\
 & (-)\ \ (+)\ \ \ (-) \\
\hline
 & 15z^4 - 14z^5 \\
 & \cdots
\end{array}
$$

$X(z)$ = Quotient of long division

$X(z) = \cdots + 15z^4 + 7z^3 + 3z^2 + z$

Taking $I\,ZT$

$$x(n) = (\cdots 15 \quad 7 \quad 3 \quad 1 \quad \underset{\uparrow}{0})$$

$\because$ it is compared with definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \cdots + x(-4) z^4 + x(-3) z^3 + x(-2) z^2 + x(-1) z^1$$

for left sided sequence only -ve values of $n$.

Note: The same problem solved using partial fraction approach we get

$$x(n) = -u(-n-1) + \left(\tfrac{1}{2}\right)^n u(-n-1)$$

$$x(n) = \left[\left(\tfrac{1}{2}\right)^n - 1\right] u(-n-1)$$

The same answer can be obtained by putting $n = -1, -2, \cdots$

(2) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$ ROC: $|z| > 1$

Soln Since ROC given indicates that it is a right sided sequence.

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \cdots \\
\hline
2z^2 - 3z + 1 & \cancel{z} \\
 & \cancel{z} - \frac{3}{2} + \frac{1}{2}z^{-1} \\
 & (-)\quad (+)\quad (-) \\
\hline
 & \cancel{\frac{3}{2}} - \frac{1}{2}z^{-1} \\
 & \cancel{\frac{3}{2}} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2} \\
 & (-)\quad (+)\quad (-) \\
\hline
 & \cancel{\frac{7}{4}z^{-1}} - \frac{3}{4}z^{-2} \\
 & \cancel{\frac{7}{4}z^{-1}} - \frac{21}{8}z^{-2} + \frac{7}{8}z^{-3} \\
 & (-)\quad (+)\quad (-) \\
\hline
 & \frac{15}{8}z^{-2} - \frac{7}{8}z^{-3}
\end{array}
$$

$X(z) =$ Quotient of long division

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \cdots$$

Taking I ZT

$$x(n) = \{ \underset{\uparrow}{0}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16} \cdots \}$$

<u>Note:</u> If partial fraction approach is used we get

$$x(n) = u(n) - \left(\tfrac{1}{2}\right)^n u(n)$$
$$= \left[1 - \left(\tfrac{1}{2}\right)^n\right] u(n)$$

③ Find $x(n)$ for $X(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$ $|z|>1$

Soln $|z|>1$, it is RSS & the powers of $z$ are $-$ve.

$$\begin{array}{r|l} & 1+1.5z^{-1}+1.75z^{-2}+1.875z^{-3} \\ \hline 1-1.5z^{-1}+0.5z^{-2} & 1 \\ & 1-1.5z^{-1}+0.5z^{-2} \\ & (-)\ (+)\ (-) \\ \hline & \cancel{1.5z^{-1}}-0.5z^{-2} \\ & \cancel{1.5z^{-1}}-2.25z^{-2}+0.75z^{-3} \\ & (-)\ (+)\ (-) \\ \hline & \cancel{1.75z^{-2}}-0.75z^{-3} \\ & \cancel{1.75z^{-2}}-2.625z^{-3}+0.75z^{-4} \\ & (-)\ (+)\ (-) \\ \hline & 1.875z^{-3}-0.875z^{-4} \\ & \vdots \end{array}$$

$x(z) = (1, 1.5, 1.75, 1.875 \ldots)$

④ $X(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$ $|z|<0.5$

soln: $|z|<0.5$ it is LSS; we need +ve powers of $z$

$\Rightarrow$ ~~x(z)~~

$$\begin{array}{r|l} & 2z^2+6z^3+14z^4+30z^5\ldots \\ \hline 0.5z^{-2}-1.5z^{-1}+1 & 1 \\ & \cancel{1}-3z+2z^2 \\ & (-)\ (+)\ (-) \\ \hline & \cancel{3z}-2z^2 \\ & \cancel{3z}-9z^2+6z^3 \\ & (-)\ (+)\ (-) \\ \hline & \cancel{7z^2}-6z^3 \\ & \cancel{7z^2}-21z^3+14z^4 \\ & (-)\ (+)\ (-) \\ \hline & 15z^3-14z^4 \\ & \ldots \end{array}$$

$X(z) = 2z^2+6z^3+14z^4+30z^5+\ldots$

$x(n) = (\ldots\ldots\ 30, 14, 6, 2, 0, \underset{\uparrow}{0})$

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(5) $X(z) = 1 + \dfrac{z^2 + z}{z^2 - 2z + 1} \quad |z| > 3$

Soln

$X(z) = 1 + X_1(z)$

Since $|z| > 3$, it is Right sided sequence (RSS) & it has $-$ve powers of $z$.

$$
\begin{array}{r|l}
 & 1 + 3z^{-1} + 5z^{-2} + 7z^{-3} + 9z^{-4} + \cdots \\
\hline
z^2 - 2z + 1 & \cancel{z^2} + z \\
 & \cancel{z^2} - 2z + 1 \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 3\cancel{z} - 1 \\
 & 3\cancel{z} - 6 + 3z^{-1} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & \cancel{5} - 3z^{-1} \\
 & \cancel{5} - 10z^{-1} + 5z^{-2} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 7\cancel{z^{-1}} - 5z^{-2} \\
 & 7\cancel{z^{-1}} - 14z^{-2} + 7z^{-3} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 9z^{-2} - 7z^{-3}
\end{array}
$$

$X_1(z) = 1 + 3z^{-1} + 5z^{-2} + 7z^{-3} + 9z^{-4} + \cdots$

$X(z) = 1 + X_1(z)$

$X(z) = 1 + 1 + 3z^{-1} + 5z^{-2} + \cdots$

$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$

$x(n) = \delta(n) + (\overset{x_1(0)}{1} + \overset{x(1)}{3} + 5 + 7 - \cdots)$

$x(n) = \delta(n) + (2n+1)\, u(n)$

# UNILATERAL OR SINGLE SIDED Z TRANSFORM

**Definition:**

The unilateral z transform $X_1(z)$ of a sequence $x(n)$ is defined as.

$$X_1(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

and differs from the double sided z.T because, the lower limit of the summation is ZERO. Thus the unilateral z.T of $x(n)$ can be thought of as the bilateral transform of $x(n)\,u(n)$.

Since, $x(n)u(n)$ is a causal sequence; the ROC of $X_1(z)$ is always the EXTERIOR to the circle in the z-plane.

Kumar. P
ECE dept

**Properties:**

Most of the properties of bilateral and unilateral z.T are identical, however, the following time shifting property is different for both the transforms.

If the unilateral z.T of $x(n)$ is

$z\{x(n)\} = X_1(z)$ then for $m \geq 0$ P.T

$$z\{x(n-m)\} = z^{-m} X_1(z) + x(-1)\, z^{-m+1} + x(-2)\, z^{-m+2} + \dots + x(-m).$$

**Proof:** By definition the unilateral z.T is given by

$$z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

but we need

$$z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m)\, z^{-n}$$

put $n - m = k$ $\quad n = k+m$

when $n=0$ $\quad k=-m$ $\quad$ when $n=\infty$ $\quad k=\infty$.

$$z\{x(k)\} = \sum_{k=-m}^{\infty} x(k)\, z^{-(k+m)}$$

$$= z^{-m} \sum_{k=-m}^{\infty} x(k)\, z^{-k}$$

$$= z^{-m}\left[ \sum_{k=-m}^{-1} x(k)\, z^{-k} + \sum_{k=0}^{\infty} x(k)\, z^{-k} \right]$$

$$= z^{-m}\left[ \sum_{k=-m}^{-1} x(k)\, z^{-k} + X_1(z) \right] \quad \text{where } X_1(z) = z\{x(k)\},\ k \geq 0$$

$$= \cancel{z^{-m}\{x(-m)\, z^{m} + x(-m+1)\, z^{-m1}}$$

$$= z^{-m}\left[ x(-1)\, z^{1} + x(-2)\, z^{2} + \cdots + x(-m)\, z^{m} + X_1(z) \right]$$

$$= z^{-m} X_1(z) + x(-1)\, z + x(-2)\, z^{2} + \cdots + x(-$$

**Note:** for double sided, z.T of sequence $x(n)$ has the following timing shifting property.

$$z\{x(n-m)\} = z^{-m} X(z).$$

corollary:

1) $z\{x(n-1)\} = x(-1) + z^{-1} X_1(z)$

2) $z\{x(n-2)\} = x(-2) + x(-1)\, z^{-1} + X_1(z)\, z^{-2}$

3) $z\{x(n-3)\} = x(-3) + x(-2)\, z^{-1} + x(-1)\, z^{-2} + X_1(z)\, z^{-3}$

Find the unilateral z transform of the following x(n)

1) $x(n) = 2^n u(n)$

2) $x(n) = (1/2)^{n+1} u(n+1)$

soln ①

Since, x(n) is a causal sequence, that is $x(n)=0$ for $n<0$, it implies $X(z) = X_1(z)$

$$X_1(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} (2z^{-1})^n$$

$$= \frac{1}{1-2z^{-1}} \; ; \quad ROC \; |2z^{-1}| < 1 \text{ or } |z| > 2.$$

$$= \frac{z}{z-2} \qquad |z| > 2.$$

② In this case, x(n) is not a causal sequence.

Hence $X_1(z) \neq X(z)$

$$X_1(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/2)^{n+1} u(n+1) z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/2)^{n+1} u(n) z^{-n} = 1/2 \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n \qquad ROC: \; |1/2 \, z^{-1}| < 1, \text{ OR } |z| > 1/2$$

$$= \frac{1}{2} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \boxed{X_1(z) = \frac{1}{2}\left[\frac{z}{z-1/2}\right]} \quad |z| > 1/2$$

# TRANSFORM ANALYSIS of LTI SYSTEM

Transfer function (System function) & impulse response.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$y(n) = x(n) * h(n)$

Taking Z.T on both side

$$Y(z) = X(z)\,H(z) \longrightarrow (1)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}} \longrightarrow (2)$$

In Eqn (2) $H(z)$ is called as system function or TRANSFER FUNCTION.

Causality & stability (of LTI systems)

The system is said to be causal if $h(n) = 0$ for $n < 0$ i.e, it must be RSS alternatively ROC must be outside the outer most pole.

The system is said to be stable if its impulse response is absolutely summable, alternatively the ROC must include unit circle.

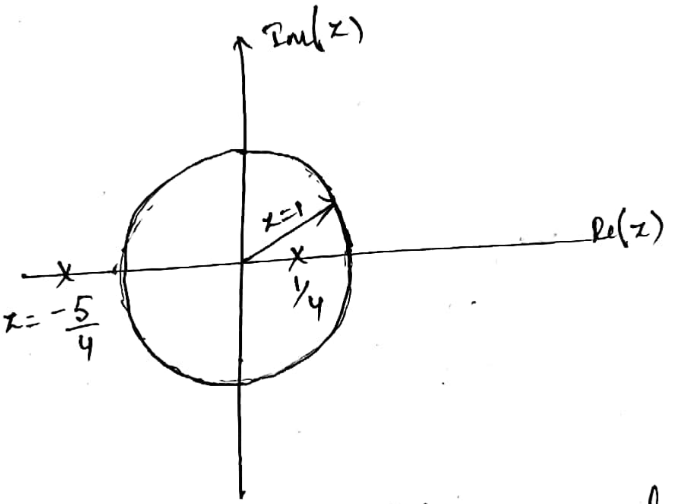
Generally a causal system to be stable, all the poles should lie within the unit circle in z plane.

Determine whether the system is both causal and stable.

① $H(z) = \dfrac{2z+1}{z^2 + z - \frac{5}{16}}$  ② $H(z) = \dfrac{z^2+2z}{z^2 + \frac{14}{8}z + \frac{49}{64}}$

$\downarrow$

$$H(z) = \frac{2z+1}{\left(z+\frac{5}{4}\right)\left(z-\frac{1}{4}\right)}$$




The system would be causal if $|z| > \frac{5}{4}$; but the system is not stable $\because$ all the poles are not lying inside the unit circle.

(3) A discrete LTI system is given by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

(a) Specify the ROC of H(z) and determine h(n) for the following:

i) when the sym is stable.
ii) when the system is causal.

$$H(z) = \frac{3 - 4z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}$$

soln: multiply & divide by $z^2$ on RHS

$$\frac{H(z)}{z} = \frac{3z - 4}{(z - 1/2)(z - 3)}$$

$$\frac{H(z)}{z} = \frac{A}{(z - 1/2)} + \frac{B}{z - 3}$$

$$A = \frac{3z - 4}{\cancel{(z - 1/2)}(z - 3)} \cancel{(z - 1/2)} \Big|_{z = 1/2}$$

$$A = \frac{\frac{3}{2} - 4}{(\frac{1}{2} - 3)} = \frac{\frac{-5}{2}}{\frac{-5}{2}}$$

$$\boxed{A = 1}$$

$$B = \frac{3z - 4}{(z - 1/2)\cancel{(z - 3)}} \cancel{(z - 3)} \Big|_{z = 3}$$

$$B = \frac{9 - 4}{3 - 1/2}$$

$$= \frac{5}{\frac{5}{2}}$$

$$\boxed{B = 2}$$

$$H(z) = 1 \frac{z}{z - 1/2} + 2 \frac{z}{z - 3}$$

i) when the sym is stable.

ROC: is given as

$\frac{1}{2} < |z| < 3$ ($\because$ we have to include unit circle)

$\left.\begin{matrix} |z| > 3 \\ |z| < 1/2 \end{matrix}\right\}$ does'nt include unit cle.

$\therefore$ it is bilateral sequence.

Apply I Z.T

$$h(n) = (1/2)^n u(n) - 2(3)^n u(-n-1)$$

ii) when the sym is causal

For causal sym, ROC must be outside the outermost pole. i.e. $|z| > 3$ i.e. RSS

$$h(n) = (1/2)^n u(n) + 2(3)^n u(n)$$

iii) when the sysm is anti causal.

$(|z| < 1/2) \cap (|z| < 3)$

$= |z| < 1/2$

$\therefore$ it is LSS

taking IZT

$h(n) = -(1/2)^n u(-n-1) - 2(3)^n u(-n-1)$

(4) A causal LTI system is described by the difference eqn $y(n) = y(n-1) + y(n-2) + x(n-1)$ i) find the sysm function ii) plot the poles & zeros. iii) indicate ROC. (iv) find the unit sample response $h(n)$. v) find a stable non causal unit sample response.

solⁿ given $y(n) = y(n-1) + y(n-2) + x(n-1)$

i) Taking ZT on both sides.

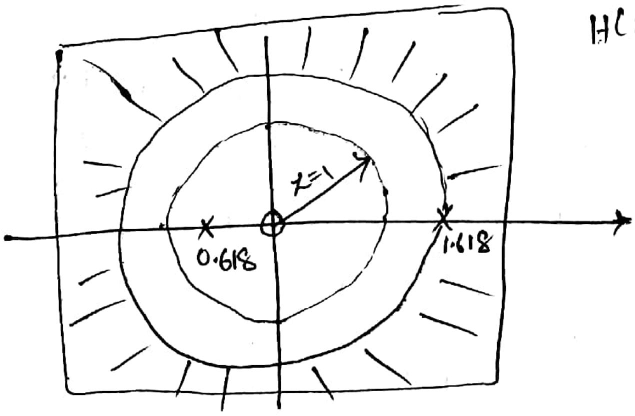
$$Y(z) = z^{-1} Y(z) + z^{-2} Y(z) + z^{-1} X(z)$$

$$Y(z)\left[1 - z^{-1} - z^{-2}\right] = z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

(OR) $$H(z) = \frac{z}{z^2 - z - 1}$$

(ii)




$$H(z) = \frac{z}{z^2 - z - 1}$$

$$= \frac{z}{(z+0.618)(z-1.618)}$$

iii) Since it is given that the system is causal ∴ the ROC must be outside of outer most pole. i.e., $|z| > 1.618$ (RSS)

iv) $$\frac{H(z)}{z} = \frac{1}{(z+0.618)(z-1.618)}$$

$$= \frac{A}{z+0.618} + \frac{B}{z-1.618}$$

$$A = \frac{H(z)}{z} \cdot (z+0.618)\Big|_{z=-0.618}$$

$$= \frac{1}{\cancel{(z+0.618)}(z-1.618)} \cancel{(z+0.618)}\Big|_{z=-0.618}$$

$$A = \frac{1}{z-1.618}$$

$$A = \frac{1}{-0.618-1.618}$$

$$A = -0.447$$

$$B = \frac{1}{(z+0.618)\cancel{(z-1.618)}} \cancel{(z-1.618)}\Big|_{z=1.618}$$

$$B = \frac{1}{1.618+0.618}$$

$$B = 0.447$$

$$H(z) = -0.447\frac{z}{(z+0.618)} + 0.447\frac{z}{z-1.618}$$

Since the system is causal ROC must be outside of outer most pole. i.e., $|z| > 1.618$

∴ it is RSS (right sided sequence)

Taking IZT

$$h(n) = -0.447(-0.618)^n u(n) + 0.447(1.618)^n u(n).$$

(v) when it is stable it should include unit circle which is possible with ROC $0.618 < |z| < 1.618$

∴ it is bilateral.

Taking IZT

$$h(n) = -0.447(-0.618)^n u(n) - 0.447(1.618)^n u(-n-1)$$

Qn 5

A causal discrete LTI system is discribed by

$$y(n) - \tfrac{3}{4}y(n-1) + \tfrac{1}{8}y(n-2) = x(n)$$

i) Determine the system function $H(z)$

ii) Find the impulse response

iii) find step response of sym

iv) Find the BIBO stability

v) find the frequency response.

Soln $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n)$

(i) Apply Z.T on both side.

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z)$$

$$Y(z)\left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right] = X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} = \frac{z^2}{z^2 - \frac{3}{4} z + 1/8}$$

(ii)

$$H(z) = \frac{z^2}{z^2 - \frac{1}{4} z - \frac{1}{2} z + \frac{1}{8}} ; \quad \frac{H(z)}{z} = \frac{z}{(z - 1/2)(z - \frac{1}{4})}$$

$$\frac{H(z)}{z} = \frac{A}{(z - 1/2)} + \frac{B}{(z - 1/4)}$$

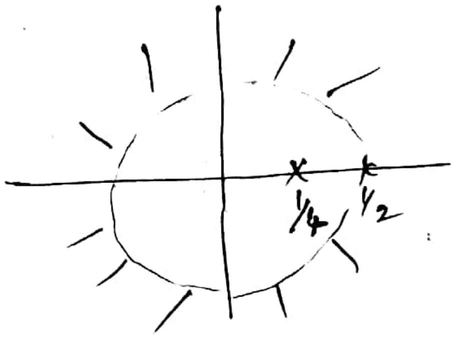
$$A = \frac{H(z)}{z}(z - 1/2)\Big|_{z=1/2} = \frac{z}{\cancel{(z-1/2)}(z-1/4)}\cancel{(z-1/2)}\Big|_{z=1/2}$$

$$A = \frac{\frac{1}{2}}{\left(\frac{1}{2} - 1/4\right)} \qquad \boxed{A = 2}$$

$$B = \frac{H(z)}{z} \cdot (z - 1/4)\Big|_{z=1/4} = \frac{z}{(z-1/2)\cancel{(z-1/4)}}\cancel{(z-1/4)}\Big|_{z=1/4}$$

$$B = \frac{\frac{1}{4}}{\left(\frac{1}{4} - 1/2\right)} \qquad \boxed{B = -1}$$

$$\frac{H(z)}{z} = 2\frac{1}{(z - 1/2)} - 1\frac{1}{(z - 1/4)}$$




$$H(z) = 2\frac{z}{(z - 1/2)} - \frac{z}{(z - 1/4)}$$

since system is causal $|z| > 1/2$ (outside of outermost pole). so RSS.

Apply IZT

$h(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$

(iii) To find step response.

$y(n) = x(n) * h(n)$

i/p is unit step signal, $x(n) = u(n)$

$y(n) = u(n) * h(n)$

$Y(z) = U(z)\ H(z)$

$$Y(z) = \frac{z}{z-1} \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

ROC: $(|z|>1) \cap (|z|>\frac{1}{2})$

i.e ROC: $|z|>1$

$\because a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$ $|z|>a$ (for above $a=1$)

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{(z-\frac{1}{2})} + \frac{C}{(z-\frac{1}{4})}$$

$$A = \frac{Y(z)}{z}(z-1)\Big|_{z=1} = \frac{z^2 \cancel{(z-1)}}{\cancel{(z-1)}(z-\frac{1}{2})(z-\frac{1}{4})}\Big|_{z=1} = \boxed{A = \frac{8}{3}}$$

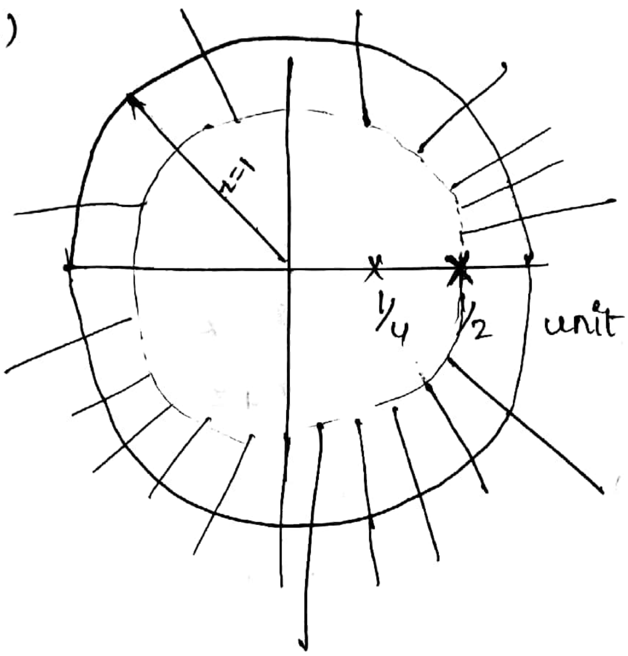
$$B = \frac{Y(z)}{z}(z-\frac{1}{2})\Big|_{z=\frac{1}{2}} = \frac{z^2 \cancel{(z-\frac{1}{2})}}{(z-1)\cancel{(z-\frac{1}{2})}(z-\frac{1}{4})}\Big|_{z=\frac{1}{2}} \quad \boxed{B = -2}$$

$$C = \frac{Y(z)}{z}(z-\frac{1}{4})\Big|_{z=\frac{1}{4}} = \frac{z^2 \cancel{(z-\frac{1}{4})}}{(z-1)(z-\frac{1}{2})\cancel{(z-\frac{1}{4})}}\Big|_{z=\frac{1}{4}} \quad \boxed{C = \frac{1}{3}}$$

$$Y(z) = \frac{8}{3}\frac{z}{z-1} - 2\frac{z}{z-\frac{1}{2}} + \frac{1}{3}\frac{z}{(z-\frac{1}{4})}$$ ROC: $|z|>1$

$$y(n) = \frac{8}{3}(1)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}\left(\frac{1}{4}\right)^n u(n)$$

v)




ROC : with respect to H(z)

$|z| > 1/2$

it is given that the sym is causal also all the poles are inside the unit circle; thus ROC includes unit circle. ∴ sym is both causal & stable.

v) To find frequency response.

$$H(z) = \frac{z^2}{(z-1/2)(z-1/4)}$$

for frequency response $z \longrightarrow e^{j\Omega}$

$$H(z) = \frac{e^{2j\Omega}}{(e^{j\Omega}-1/2)(e^{j\Omega}-1/4)}$$

Qn: 6 Given $y(n) - y(n-1) = x(n) + x(n-1)$

i) find the impulse response for causal sym (ii) Find the response of sym to i/p $x(n) = u(n)$ & $x(n) = 2^{-n}u(n)$, test its stability.

y(n) soln

i) $y(n) - y(n-1) = x(n) + x(n-1)$

Apply Z.T

$Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z) \longrightarrow (1).$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}} = \frac{z+1}{z-1}$

$H(z) = \frac{z}{z-1} + \frac{1}{z-1} = \frac{z}{z-1} + z^{-1}\frac{z}{z-1}$

Apply I Z.T (with sym causal ; i.e., RSS)

$h(n) = (1)^n u(n) + u(n-1) \quad |z|>1$

$h(n) = u(n) + u(n-1) \quad |z|>1$, Except $z=0, \infty$

ii) To find $y(n)$ with $x(n) = u(n)$ , $x(n) = 2^{-n}u(n)$

w.k.T. $y(n) = x(n) * h(n)$

$Y(z) = X(z)\,H(z)$

where $x(n) = u(n) \xleftrightarrow{Z.T} X(z) = \frac{z}{z-1} \quad |z|>1$

$H(z) = \frac{z+1}{z-1} \quad |z|>1$

$Y(z) = \frac{z}{z-1} \; \frac{z+1}{z-1} \quad (|z|>1) \cap (|z|>1)$

$Y(z) = \frac{z(z+1)}{(z-1)^2}; \quad \frac{Y(z)}{z} = \frac{z+1}{(z-1)^2}$

$= \frac{A_1}{(z-1)} + \frac{A_2}{(z-1)^2}$

$A_1 = \frac{1}{1!}\frac{d}{dz}\left[\frac{Y(z)}{z}\cdot(z-1)^2\right] = \frac{d}{dz}\left[z+1\right]\Big|_{z=1}$

$\boxed{A_1 = 1}$

$$A_2 = \frac{1}{0!}\left[\frac{Y(z)}{z}\cdot(z-1)^2\right]\Bigg|_{z=1} = \frac{z+1}{\cancel{(z-1)^2}}\,\cancel{(z-1)^2}\Bigg|_{z=1}$$

$$\boxed{A_2 = 2}$$

$$Y(z) = \frac{z}{z-1} + 2\,\frac{z}{(z-1)^2} \qquad |z|>1$$

Taking I.Z.T

$$y(n) = u(n) + 2n\,u(n)$$

When $x(n) = 2^{-n}u(n) = \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \frac{z}{z-\frac{1}{2}} \quad |z|>\frac{1}{2}$

$$Y(z) = X(z)\,H(z) \qquad \text{ROC}: \left(|z|>\tfrac{1}{2}\right)\cap |z|>1$$

$$= \frac{z}{z-\frac{1}{2}}\cdot\frac{z+1}{(z-1)} \;; \quad \frac{Y(z)}{z} = \frac{z+1}{\left(z-\frac{1}{2}\right)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-1} \qquad \text{ROC}: |z|>1$$

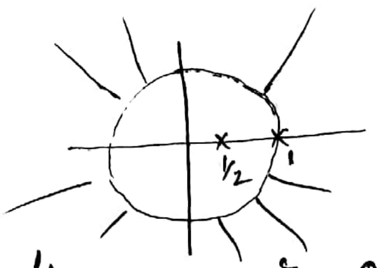
$$A = \frac{Y(z)}{z}\left(z-\tfrac{1}{2}\right)\Bigg|_{z=\frac{1}{2}} = \frac{z+1}{\cancel{\left(z-\frac{1}{2}\right)}(z-1)}\cancel{\left(z-\tfrac{1}{2}\right)}\Bigg|_{z=\frac{1}{2}}$$

$$A = \frac{3/2}{-1/2} \qquad \boxed{A=-3}$$

$$B = \frac{z+1}{\left(z-\frac{1}{2}\right)\cancel{(z-1)}}\cancel{(z-1)}\Bigg|_{z=1} = \frac{2}{1/2} = \boxed{B=4}$$

$$Y(z) = -3\,\frac{z}{z-\frac{1}{2}} + 4\,\frac{z}{z-1} \qquad |z|>1$$

$$= -3\left(\tfrac{1}{2}\right)^n u(n) + 4\,u(n)$$




As the pole $z=1$ lies on unit circle, the unit circle is not part of ROC ∴ system is unstable.

# Difference eqn analysis

① Solve the given difference qu.

$y(n) + 3y(n-1) = x(n)$ with $x(n) = u(n)$, $y(-1) = 1$

**Soln** Apply unilateral Z.T on both side.

$$Y(z) + 3\left[y(-1) + z^{-1}Y(z)\right] = X(z)$$

$$Y(z) + 3\left[1 + z^{-1}Y(z)\right] = \frac{z}{z-1}$$

$$Y(z)\left[1+3z^{-1}\right] + 3 = \frac{z}{z-1} \; ; \; Y(z)\left(1+3z^{-1}\right) = \frac{z}{z-1} - 3$$

$$Y(z) = \frac{z}{(z-1)(1+3z^{-1})} - \frac{3}{1+3z^{-1}}$$

$$Y(z) = \frac{z \cdot z}{(z-1)(z+3)} - \frac{3z}{z+3}$$

Let $Y_1(z) = \dfrac{z \cdot z}{(z-1)(z+3)}$

$$\frac{Y_1(z)}{z} = \frac{z}{(z-1)(z+3)}$$

$$\frac{Y_1(z)}{z} = \frac{A}{z-1} + \frac{B}{z+3}$$

$$A = \frac{Y_1(z)}{z}(z-1)\Big|_{z=1}$$

$$= \frac{z}{\cancel{(z-1)}(z+3)}\cancel{(z-1)}\Big|_{z=1}$$

$$\boxed{A = \frac{1}{4}}$$

$$B = \frac{Y_1(z)}{z}(z+3)$$

$$B = \frac{z}{(z-1)\cancel{(z+3)}}\cancel{(z+3)}\Big|_{z=-3}$$

$$= \frac{-3}{-4} \qquad \boxed{B = \frac{3}{4}}$$

$$\frac{Y_1(z)}{z} = \frac{1}{4}\frac{1}{z-1} + \frac{3}{4}\frac{1}{z+3}$$

$$Y_1(z) = \frac{1}{4}\frac{z}{z-1} + \frac{3}{4}\frac{z}{z+3}$$

$$\therefore Y(z) = \frac{1}{4}\frac{z}{z-1} + \frac{3}{4}\frac{z}{z+3} - \frac{3z}{z+3}$$

Apply Inverse Z.T $|z|>1$

$$y(n) = \frac{1}{4}u(n) + \frac{3}{4}(-3)^n - 3(-3)^n u(n)$$

$$y(n) = \frac{1}{4}u(n) - \frac{9}{4}(-3)^n u(n)$$

Qn.

② $y(n) - \frac{1}{9}y(n-2) = x(n-1)$, $y(-1) = 0$

$y(-2) = 1$ $\quad x(n) = 3u(n)$.