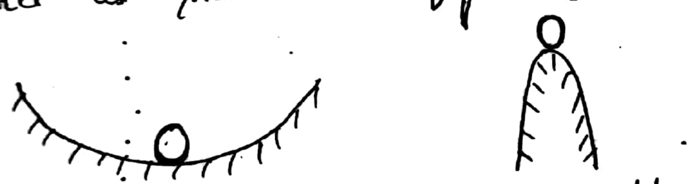
Unit 4: Stability Analysis

For a controll system designer main concern is whether designed system is stable or unstable because, stability is a very important characteristic of any controll system. Almost every working system is designed to be stable. Thus, study of whether the system is stable or unstable is known as stability analysis.

"A system is said to be stable if its output is bounded for any bounded input". [BIBO]

→ **Concepts of stability**

Consider circular object (ball) ~~resting~~ resting on different types of surfaces under the presence of gravitational force as shown in fig.



Stable unstable

For analysis and design purpose, stability of the system can be classified as:

(1) **Absolute stability**

Absolute stability refers to the condition of whether the system is stable or unstable; it is a YES or NO answer.

(2) **Relative stability**

Relative stability is the quantitative measure of how stable is the system and this is the degree of stability.

Terms used

(1) **stable system [BIBO]**

A system is said to be stable if the output is bounded for any bounded Input. thus a system is said to be in equilibrium if the output eventually comes back to its equilibrium state when the system is subjected to a disturbance

(2) **Unstable System**

A system is said to be unstable if the output is unbounded for a bounded Input. It means with Zero Input the output may Increase Indefinetly.

(3) **Absolute stable**

A system is said to be absolutely stable w.r.t a parameter of the system if its stable for all Value of there parameters.

(4) **Conditionally stable**

A system is said to be Conditionally stable with respect to a parameters of the System if it is stable only for a Certain bounded range of Values of there parameters.

(5) **Marginally stable (Critically stable)**

A system is said to be marginally stable if the system is just on the verge of belonging unstable.

(6) **Relative stability**

It is quantitative measure of the degree of stability, determines how stable is the system and how close it to being unstable.

→ Time Response from pole-zero plots

for all the examples shown below, the input to the system is assumed to be unit impulse function.

ex: 1 T.F $\dfrac{C(s)}{R(s)} = \dfrac{1}{s+1}$

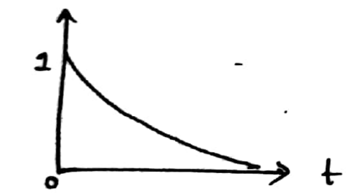
$\therefore$ o/p $C(s) = \dfrac{1}{s+1} R(s)$

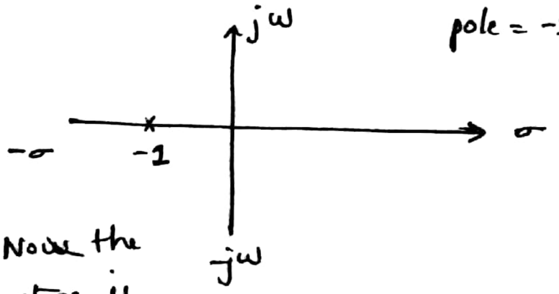
for unit impulse input. $R(s) = 1$.

$\therefore$ $C(s) = \dfrac{1}{s+1}$

Taking Inverse L.T. $\quad L^{-1}\left[\dfrac{1}{s+a}\right] = e^{-at}$

$c(t) = e^{-t}$





pole = -1

Now the system is stable.

s-plane

×
o

ex: 2 T.F $\dfrac{C(s)}{R(s)} = \dfrac{s+2}{(s+1)(s+3)}$ → zero (numerator), → pole (denominator)

o/p. $C(s) = \dfrac{(s+2)}{(s+1)(s+3)} \cdot R(s)$.

$s=-2$

$(s+1)(s+3)$.

$s=-1 \quad s=-3$

for unit ~~step~~ impulse input $R(s) = 1$ $\quad r(t) = \delta(t)$, $R(s) = 1$

$\therefore$ $C(s) = \dfrac{s+2}{(s+1)(s+3)}$

using partial fractions.

$$C(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$s+2 = A(s+3) + B(s+1)$$

$s=-3$

$$-1 = B(-2)$$

$$B = 1/2$$

$s=-1$

$$1 = 2A + 0$$

$$A = 1/2$$

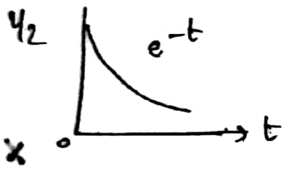
$\therefore A = 1/2, \quad B = 1/2$

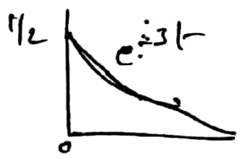
$s = \sigma + j\omega$

$$C(s) = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)}$$

Taking Inverse laplace transform

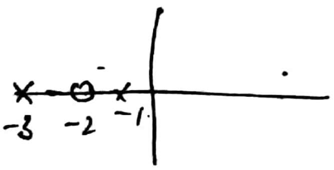
$$c(t) = \frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t}.$$





faster decaying than e^{-t}

x
o



It is stable system.

pole zero plot.

ex: 3.

$$T.F = \frac{C(s)}{R(s)} = \frac{1}{s-1}$$

o/p $\quad C(s) = \frac{1}{s-1} R(s) \qquad R(s) = 1$

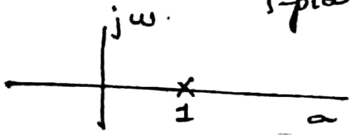
$$\therefore C(s) = \frac{1}{s-1}$$

$s-1=0$
$s=1$

$s+1=0$
$s=-1$

taking Inverse L.T.

$$c(t) = e^{t}.$$



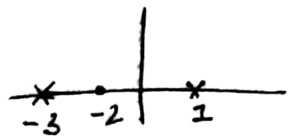
system is unstable

ex: 4

T.F $\frac{C(s)}{R(s)} = \frac{s+2}{(s-1)(s+3)}$

$R(s) = 1$.

$C(s) = \frac{s+2}{(s-1)(s+3)}$.



$\frac{s+2}{} = \frac{A}{s-1} + \frac{B}{s+3}$.

$s+2 = A(s+3) + B(s-1)$.

$s=1$ | $3 = A(4) + 0$ | $A = 3/4$

$s=-3$ | $-1 = 0 + (-4B)$ | $B = 1/4$

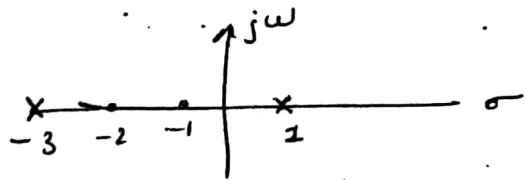
$C(s) = \frac{3}{4(s-1)} + \frac{1}{4(s+3)}$

Inverse L.T.

$c(t) = 3/4\, e^{t} + 1/4\, e^{-3t}$.

faster decaying.

$\therefore$ The system is unstable.



s-plane.

Note: (1) If all the poles of a system lie on the left of the s-plane it is a stable system

(2) If any one pole lie on the right of s plane then it is a unstable system.

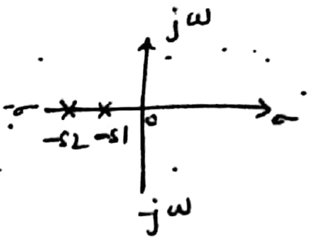
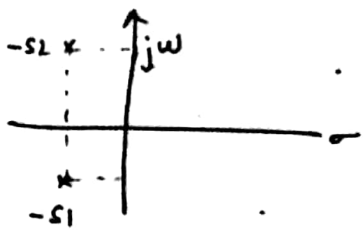
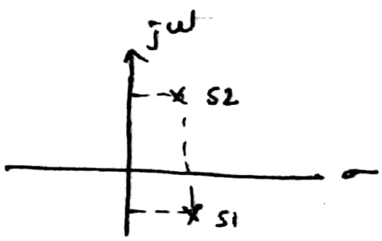
(3) Converging systems are stable

(4) Diverging systems are unstable

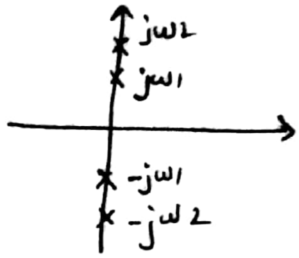
(5) The characteristics of a given system depends on the location of the poles.

(6) The location of zero's decides the starting magnitude of the signal.

Closed loop poles = Roots of characteristics equation

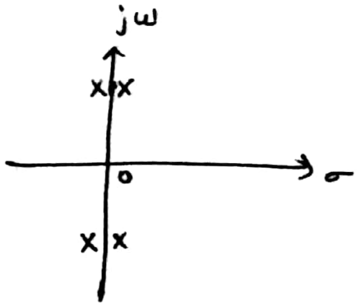
Nature of Roots	S-plane	~~Explanation~~	Stability Condition
(1) Real, negative i.e. all the roots are in left half of s-plane			Absolutely stable
(2) Complex conjugate with negative real part i.e. L.H.S of s-plane.			Absolutely stable.
(3) Real, positive ie R.H.S of s-plane (Any one closed loop pole in right half Irrespective of number of poles in left half of s-plane)			unstable.
(4) Complex conjugate of positive real part			unstable

(5) Non-repeated pair on Imaginary axis without any poles in R.H.S of s-plane



Marginally stable.

(6) Repeated pair of on Imaginary axis without any pole in R.H.S of s-plane



unstable.

● Necessary condition for system to be stable

(1) If all the roots of the characteristic equation have a negative real part then system is <u>stable</u>

(2) If any roots of the characteristic equation has a positive real part or if there is any repeated roots on the Imaginary axis of s-plane then the system is said to be <u>unstable</u>.

(3) If some of the roots of the characteristic equation have a negative real parts and remaining roots are on the Imaginary axis then the system is said to be <u>marginally stable</u>

⇒ Routh's Hurwitz Criterion (RH Criterion)

- It is an algebraic method of determining the stability of linear time Invariant system.
- This Criterion provides Information about whether system is stable or not depending upon the position of the roots of the characteristic equation whether they lies on left half of s-plane or right half of s-plane.
- It deals with absolute stability.

Limitations

Using RH Criterion even though it is possible to obtain the Information regarding the stability of feedback controll system, it is not possible to obtain complete Information regarding the relative stability of system.

- Relative stability (graphical method)
  (1) Root locus technique
  (2) Bode plot
  (3) Nyquist plot.
- Graphical methods will give better Information regarding relative stability
- All the above methods are used to find the stability of closed loop system without actually solving for the roots of characteristic equation.

RH Criterian

The characteristic equation of the system is given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

a_0, a_1, a_2 are real co-efficients

Necessary condition

(1) All the coefficients of the polynomial have the same sign

(2) There should not be any missing coefficients in the given characteristic equation.

Hurwitz - Criterion

The sufficient condition for having all the roots of characteristic equation in left half of s-plane is given by hurwitz. It is refered as hurwitz criterion

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots & a_{2n-5} \\ \vdots & & \vdots & & \\ 0 & & & \cdots & a_n \end{vmatrix}$$

$$D_1 = |a_1|, \quad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \quad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

for the system to be stable, all the above determinants must be positive.

ex: $s^3+s^2+s^1+4=0$ determine stability of the give characteristic equation by Hurwitz method.

$a_0=1,\ a_1=1,\ a_2=1,\ a_3=4,$ $n=3$

$$H=\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$D_1 = |1| = 1,$ $D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 1-4 = -3$

$D_K = |H|$

$$\begin{array}{cccc} & + & - & + \\ D_3 = & \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} \end{array} = 4-16 = -12$$

$4-4\times 4 = -12$

As D2, & D3 are negative, given system is unstable

ex: 2 $s^3+s+4=0 \rightarrow$ missing term s^2.
system is unstable.

<u>disadvantage of Hurwitz method</u>.

(1) For higher order of system, to solve the determinants of higher order is very complicated and time consuming.

(2) Difficult to predict marginal stability of the system.

(3) Number of roots located in right half of s-plane for unstable system cannot be judged by this method.

Due to this limitation, a new method is suggested by Scientist Routh called "Routh method." It is also called "Routh Hurwitz method."

→ **Routh's stability Criterion**

It is also called Routh's array method or R H method

Routh suggested a method of tabulating the coefficients of characteristic equation in a particular way. Tabulation of coefficients gives an array called "Routh's array"

Array

$$
\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdot \\
 & \vdots & & & \\
s^0 & a_n & & &
\end{array}
$$

Note

The coefficients of any row are corresponding to alternate power of 's' starting from the power indicated against it.

→ Coefficients for first 2 rows are written directly from characteristic equation.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

→ from these 2 rows, next rows can be obtained as follows.

$$b_1 = \frac{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_0 a_5}{d_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

from 2nd & 3rd row, 4th row can be obtained by

$$c_1 = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n.

→ **RH Criterion**

The necessary and sufficient condition for system to be stable is " All the terms in the first column of Routh's Array must have same sign. There shld not be any sign change in first column of Routh's Array.

If there is any sign changes existing then,

(a) System is unstable

(b) The number of sign changes equals the number of roots lying in right half of the s-plane.

ex: (1) $s^3 + 6s^2 + 11s + 6 = 0$

$a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6$ $\quad n = 3$

$$\begin{array}{c|cc} s^3 & 1 & 11 \\ s^2 & 6 & 6 \\ s^1 & \frac{11\times6-6}{6} = 10 & 0 \\ s^0 & 6 & \end{array}$$

$$\begin{array}{cc} 6 & 6 \\ 10 & 0 \end{array}$$

$$\frac{10\times6-0}{10}$$

$$\frac{60}{10} = 6$$

As there is no sign change in first column, system is stable

ex (2) $s^3 + 4s^2 + s + 16 = 0$

$a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 4 & 16 \\ s^1 & \frac{4-16}{4} = -3 & 0 \\ s^0 & \frac{-3\times16-0}{-3} = \frac{\cancel{-}48}{\cancel{-3}} = +16 & 0 \end{array}$$

As there are 2 sign changes, system is unstable.
Number of roots located in right half of s-plane
= Number of sign changes = 2.

→ Special cases of Rouths Criterion

Special Case 1

First element of any of the rows of Routh's array is zero and the same remaining row contains at least one non-zero element.

effect: The terms in the new row become infinite and Routh test fails.

ex: $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

$a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_4 = 6$, $a_5 = 2$, $a_6 = 1$, $n = 5$

$1/0 = \infty$

s^5	1	3	2
s^4	2	6	1
s^3	$\boxed{0}$	1.5	0
s^2	∞	. . .	. . .

← Routh's array failed.

$b_2 = \frac{2\times2-1}{2} = \frac{4-1}{2} = 3/2 = 1.5$

I method

Substitute a small positive number 'ε' in place of a zero occurred as a first element in a row. complete the array with this number 'ε'. Then examine the sign change by taking $\lim_{\varepsilon \to 0}$.

Consider above example,

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon-3}{\varepsilon}$	1	0
s^1	$\frac{1.5\frac{(6\varepsilon-3)}{\varepsilon}-\varepsilon}{\frac{6\varepsilon-3}{\varepsilon}}$	0	0
s^0	1	0	0

$$\frac{\frac{1.5(6\varepsilon-3)}{\varepsilon}-\varepsilon}{\frac{6\varepsilon-3}{\varepsilon}} = \frac{\frac{9\varepsilon-4.5-\varepsilon^2}{\varepsilon}}{\frac{6\varepsilon-3}{\varepsilon}} = \frac{9\varepsilon-4.5}{6\varepsilon-3} = \frac{1.5(6\varepsilon-3)}{6\varepsilon-3} = 1.5$$

neglected

To examine sign change,

$$\lim_{\varepsilon \to 0} \left[\frac{6\varepsilon - 3}{\varepsilon} \right] = 6 - \lim_{\varepsilon \to 0} \frac{3}{\varepsilon}$$

$$= 6 - \infty = -\infty \quad \text{sign is negative.}$$

(change in sign + to -ve)

$$\lim_{\varepsilon \to 0} \frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \to 0} \frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$$= +1.5 \quad \text{sign is positive}$$

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$-\infty$	1	0
s^1	+1.5	0	0
s^0	1	0	0

As there are 2 sign changes, system is unstable.

II Method

To solve the above difficulty one more method can be used. In this replace, 's' by $1/z$ in original equation. Taking LCM, rearrange characteristic equation in descending powers of 'z'. Then complete the Routh's array with this new equation in 'z' & examine the stability with this array.

Consider $F(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

$$s = 1/z$$

$$\frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

$$z^5 + 2z^4 + 6z^3 + 3z^2 + 2z + 1 = 0$$

× throughout by z^5 and writing in reverse order

z^5	1	6	2
z^4	2	3	1
z^3	$\frac{6\times2-3}{2} = 4.5$	$\frac{2\times2-1}{2} = 1.5$	0
z^2	$\frac{4.5\times3-2\times1.5}{4.5} = 2.33$	$\frac{4.5}{4.5} = 1$	0
z^1	$\frac{2.33\times1.5-4.5}{2.33} = -0.429$	0	
z^0	1		

$$\frac{-0.429(1) - 2.33(0)}{-0.429} = 1$$

As there are 2 sign changes, system is unstable. The result is same.

→ Special Case 2 :-

All the elements of a row in a Routh's array are zero.

effect: The terms of the next row cannot be determined and Routh's test fails.

ex:

s^5	a	b	c
s^4	d	e	f
s^3	0	0	0

← Rows of zero's.

∴ This indicates Non-availability of coefficient in that row procedure to eliminate this difficulty

(1) Form an equation by using the coefficients of a row which is just above the row of zero's. Such an equation is called Auxiliary equation denoted as A(s).

$$A(s) = ds^4 + es^2 + f$$

(2) Take the derivative of an auxiliary equation w.r.t 's'

$$\frac{dA(s)}{ds} = 4d\,s^3 + 2e\,s$$

(3) Replace row of zero's by the coefficients of $\frac{dA(s)}{ds}$

s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0.

(4) complete the array interms of these new coefficients.

→ Advantages of Routh's Criterion

(1) Stability of the system can be judged without actually solving characteristic equation

(2) No evaluation of determinants, which saves calculation time.

(3) For unstable system it gives number of roots of characteristic equation having positive real part.

(4) By using this criterion, critical value of system gain can be determined hence frequency of sustained oscillations can be determined.

(5) It helps in finding out ranges of values of K for system stability

(6) It helps in finding out intersection points of root locus with imaginary axis.

→ <u>Limitation of Routh's criterion</u>

(1) It is valid only for real coefficients of the characteristic equation

(2) Applicable only for linear system

(3) It does not suggest methods of stabilizing an unstable system.

(4) It does not provide exact locations of the closed loop poles in left or right half of s-plane

<u>problems</u>

(1) $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$

$$
\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 15 \\
s^3 & \boxed{0} & \frac{3-15}{1} = -12 & 0
\end{array}
$$

Replace 0 by small positive number ε

$$
\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 15 \\
s^3 & \varepsilon & -12 & 0 \\
s^2 & \frac{2\varepsilon + 12}{\varepsilon} & \frac{15\cancel{\varepsilon} - 0}{\cancel{\varepsilon}} = 15 & 0
\end{array}
$$

$$\begin{array}{c|ccc} s^1 & \dfrac{\left(\frac{2\varepsilon+12}{\varepsilon}\right)(-12) - [15\varepsilon]}{\left(\frac{2\varepsilon+12}{\varepsilon}\right)} & 0 & 0 \\ s^0 & 15 & & \end{array}$$

$1/0 = \infty$

$$\lim_{\varepsilon \to 0} \frac{2\varepsilon+12}{\varepsilon} = 2 + \frac{12}{\varepsilon} = +\infty$$

$$\lim_{\varepsilon \to 0} \frac{\left(\frac{2\varepsilon+12}{\varepsilon}\right)(-12) - 15\varepsilon}{\frac{2\varepsilon+12}{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{-24\varepsilon - 144 - 15\varepsilon^2}{2\varepsilon+12}$$

$$= \frac{0 - 144 - 0}{12} = -12$$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & \varepsilon & -12 & 0 \\ s^2 & +\infty & 15 & 0 \\ s^1 & -12 & 0 & \\ s^0 & 15 & & \end{array}$$

There are 2 sign changes, so system is unstable.

(2) $s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$

find the number of roots of this equation with positive real part, zero real part, & negative real part.

$a_0 = 1 \quad a_2 = 3 \quad a_4 = -16 \quad a_6 = -48.$

$a_1 = 4 \quad a_3 = 0 \quad a_5 = -64$

s^6	1	3	-16	-48
s^5	4	0	-64	0
s^4	$12/4 = 3$	0	$-48 = \frac{4\times(-48)-0}{4} = -48$	
s^3	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	

$\leftarrow$ special case 2.

$A(s) = 3s^4 - 48 = 0.$

$\frac{dA(s)}{ds} = 12s^3.$

s^6	1	3	-16	-48	
s^5	4	0	-64	0	
s^4	3	0	-48	0	
s^3	$\boxed{12}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	
s^2	$0[\varepsilon]$	$\frac{\cancel{12}(-48)}{\cancel{12}} = -48$	0	0	
s^1	$\frac{-(-48)(12)}{\varepsilon} = \frac{576}{\varepsilon}$	0	0		
s^0	$\frac{-48\left(\frac{\cancel{576}}{\cancel{\varepsilon}}\right)}{\frac{\cancel{576}}{\cancel{\varepsilon}}} = -48.$				

$\swarrow$ Auxiliary eq^n.

$$\therefore \lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

one sign change and system is unstable

Thus there is one root in RHS of s-plane i.e with positive real part.

Now solve $A(s) = 0$ for dominant roots

$$A(s) = 3s^4 - 48 = 0.$$

put $s^2 = y$.

$$3y^2 = 48 \qquad \sqrt{16} = 4$$

$$y^2 = \pm 16. \qquad y = \pm\sqrt{16} = \pm 4.$$

$$y = \sqrt{16} = 4$$

$$\therefore s^2 = \pm 4. \qquad y^2 = 16$$

$$s^2 = -4$$

$$s = \pm 2. \qquad s = \pm 2j$$

So $s = \pm 2j$ are the 2 roots on Imaginary axis i.e with zero real part. Root in RHS indicated by a sign change $s = +2$ as obtained by solving $A(s) = 0$.

$\therefore$ Total there are 6 roots as $n = 6$

Roots with positive real part = 1 $\quad (s-2) = 0 \rightarrow s = +2$

Roots with zero real part = 2 $\quad (0+2j)\ (0-2j)$

Roots with negative real part = $6 - 2 - 1 = 3$ $\quad (s+2) = 0 \quad s \to -2$ $\quad (0+2j) \quad s \to -2j$

→ <u>Marginal K and frequency of sustained oscillations</u>

Marginal value of 'K' is that value of 'K' for which system becomes marginally stable. For a marginally stable system there must be a row of zero's occuring in Routh's array.

(3) For unity feedback system $G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$, find range of values of K, marginal value of K and frequency of sustained oscillations.

→ Sol^n

Char^c equ: $1 + G(s)\,H(s) = 0$ & $H(s) = 1$

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)}(1) = 0 \quad ; K>0$$

$$s\left[1+0.65s + 0.1s^2\right] + K = 0$$

$$0.1s^3 + 0.65s^2 + s + K = 0$$

s^3	0.1	1
s^2	0.65	K
s^1	$\frac{0.65-0.1K}{0.65}$	0
s^0	K	

For closed loop CS to be stable, the following conditions must be true.

from s^0 (1) $K>0$

s^1 (2) $0.65K - 0.1K > 0$

$$0.65K > 0.1K.$$

$\times^{ly}$ 10) $\quad 6.5 > K.$

∴ Range of values of K, $\underline{0<K<6.5}$

Marginally value of, $K = K_{mar}$

$$0.65 - 0.1K_{mar} = 0$$

$$\boxed{K_{mar} = 6.5}$$

To find freq^c, find out roots of auxiliary equ at marginal value of K

$A(s) = 0.65s^2 + K = $ $\quad s^2 = -10$ $\quad$ ie $s = \pm j\,3.162$.

$s = \pm j\omega \qquad \omega = 3.162$ rad/sec

So value of K which makes any row of Routh array as row of zero's is called marginal value of K.

→ Now K = 0, makes row s^0 as row of zeros but K = 0 cannot be marginal value, because for K = 0, constant term in the charac equation becomes zero, i.e 1 coefficient for s^0 vanishes, which makes system unstable instead of marginally stable.

"Hence marginal value of K is a value which makes any row other than s^0 as row of zero's."

→ To obtain the frequency of sustained oscillations, solve the auxiliary equation $A(s) = 0$ for $K = K_{mar}$.
The magnitude of imaginary roots of $A(s) = 1$, obtained for marginal value of K (K_{mar}) indicates the frequency of sustained oscillations, which system is going to produce.

(4) for system, $s^4 + 22s^3 + 10s^2 + s + K = 0$ find K_{mar}, ω at K_{mar}

s^4	1	10	K
s^3	22	1	0
s^2	9.95	$\frac{22K}{22}$	0
s^1	$\frac{9.95 - 22K}{9.95}$	0	
s^0	K		

$$\frac{22 \times 10 - 1}{22} = \frac{219}{22} = 9.95$$

$$\frac{22K}{22}$$

Marginal value of 'K' which makes row s^1 as row of zero's

$$9.95 - 22K_{mar} = 0$$

$$K_{mar} = 0.4524$$

$$A(s) = 9.95s^2 + K = 0.$$

$$9.95s^2 + 0.4524 = 0$$

$$s^2 = -0.04546$$

$$s = \pm j\, 0.2132$$

Hence the freq^c of osc $\omega = 0.2132$ rad/sec.

(5) For a system with char^c equation, examine stability

$$f(s) = s^6 + 3s^5 + 4s^4 + 6s^3 + 5s^2 + 3s + 2 = 0.$$

$a_0 = 1$ $a_2 = 4$ $a_4 = 5$ $a_6 = 2$, $n = 6$

$a_1 = 3$ $a_3 = 6$ $a_5 = 3$ $a_7 = 0$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	0	0	0	0

← special case 2

Row of zero's

$$A(s) = 2s^4 + 4s^2 + 2 = 0$$

$\div 2$ $\quad = s^4 + 2s^2 + 1 = 0$

$$\frac{dA(s)}{ds} = 4s^3 + 4s$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	0	0	0	0
s^0				

← special case 2

$$A'(s) = 2s^2 + 2$$

$$\frac{dA'(s)}{s} = 4s$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	4	0	0	0
s^0	2	0	0	0

$S^4 + 2S^2 + 1 = 0$

$(s^2)^2 + 2(s^2) + 1 = 0$

No sign change, hence no root in RHS of s-plane

As row of zero's occur system may be marginally stable or unstable

→ To examine this find the roots of auxiliary equation

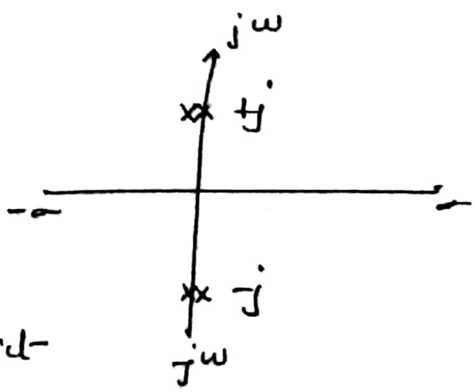
$$A(s) = s^4 + 2s^2 + 1 = 0.$$

$n = 4$

$$s^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$s^2 = -1$ | $s^2 = -1$

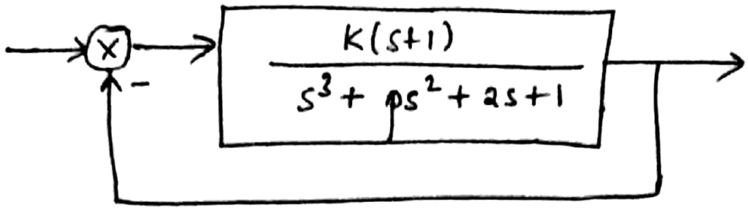
$s_{1,2} = \pm j$ | $s_{3,4} = \pm j$



The roots of $A'(s) = 0$ are the roots of $A(s) = 0$, so donot solve II auxiliary eq^u. predict the stability from the nature of roots of I auxiliary equation

As there are repeated roots on Imaginary axis, system is unstable

(6) A given system oscillates with freq^c 2 rad/sec find values of 'Kmax' and 'p'. No poles are in RHS



Sol^n: As system oscillates, it is marginally stable and value of 'K' at this situation is marginal value of 'K'. As system is marginally stable there must be row of zero's occurring in Routh's array.

char^c eq^n : $1 + G(s)\,H(s) = 0$. $\quad H(s) = 1$

$$1 + \frac{K(s+1)}{s^3 + ps^2 + 2s + 1}(1) = 0$$

$$s^3 + ps^2 + \underline{2s} + 1 + \underline{Ks} + K = 0.$$

$$s^3 + ps^2 + (2+K)s + (1+K) = 0$$

$$\begin{array}{c|cc}
s^3 & 1 & 2+K \\
s^2 & p & 1+K \\
s^1 & \frac{p(2+K) - (1+K)}{p} & 0 \\
s^0 & 1+K &
\end{array}$$

At marginal value of 'K'

$$(2+K)p - (1+K) = 0.$$

$$(2+K)p = 1+K.$$

$$p = \frac{K+1}{K+2}.$$

Now at this value,

$$A(s) = ps^2 + K + 1 = 0$$

$$s^2 = -\frac{(K+1)}{p}$$

$$s = \pm j\sqrt{\frac{K+1}{p}}$$

Compare with $s = \pm j\omega$, $\omega = 2$.

$$\sqrt{\frac{K+1}{p}} = 2$$

Taking squares,

$$\frac{K+1}{p} = 4$$

$$p = \frac{K+1}{4}$$

$$\frac{K+\cancel{1}}{K+2} = \frac{K+\cancel{1}}{4}$$

$$K+2 = 4$$

$$\boxed{K = 2}$$

$$K = K_{max} = 2$$

$$p = \frac{K_{max}+1}{2+2} = \frac{3}{4} = 0.75$$

(7) A unity feedback cs has $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$ using Routh's Criterion calculate the range of K for which the system is (i) stable, (ii) has its closed loop poles more negative than -1

Soln: charc equ. $1 + G(s) \cdot H(s) = 0$ $H(s) = 1$

$$1 + \frac{K(s+13)}{s(s+3)(s+7)}(1) = 0$$

$$s^3 + 10s^2 + 21s + Ks + 13K = 0$$

$$s^3 + 10s^2 + s(K+21) + 13K = 0$$

Routh's array.

$$\begin{array}{c|cc} s^3 & 1 & K+21 \\ s^2 & 10 & 13K \\ s^1 & \frac{210-3K}{10} & 0 \\ s^0 & 13K & \end{array}$$

(i) for stability

$(a-b)^2 =$

from s^0, $13K > 0$, $\therefore K > 0$.

a^3

from s^1, $210 - 3K = 0$ $\quad 210 > 3K$.

$\therefore \quad K < 70$.

$\therefore$ range of K $\quad 0 < K < 70$.

(ii) for the closed loop poles more negative than -1

$$s = s' - 1$$

$$(s'-1)^3 + 10(s'-1)^2 + \underbrace{(s'-1)(K+21)} + 13K = 0$$

$$(s')^3 - 3(s')^2 + 3s' - 1 + 10(s')^2 - 20s' + 10 + Ks' - K + 21s'$$
$$- 21 + 13K = 0.$$

$$\therefore (s')^3 + 7(s')^2 + s'[4+K] + [12K - 12] = 0.$$

$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

Routh's array

$$\begin{array}{c|cc} (s')^3 & 1 & 4+K \\ (s')^2 & 7 & 12K-12 \\ (s')^1 & \frac{28+7K-12K+12}{7} & 0 \\ (s')^0 & 12K-12 & \end{array}$$

from (s^0) $\quad 12K - 12 > 0 \quad , \quad K > 1$

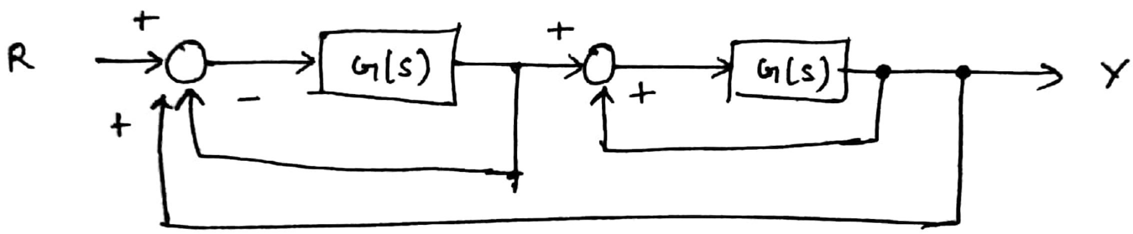
for (s^1) $\quad -5K + 40 > 0 \quad\quad 40 > 5K$

$K < 8$

range of K, $\quad 1 < K < 8$.

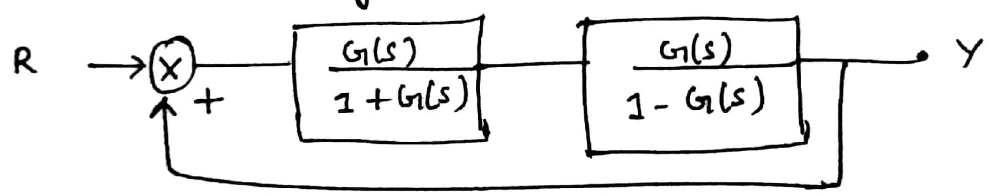
This is range of K to get closed loop poles more negative than -1.

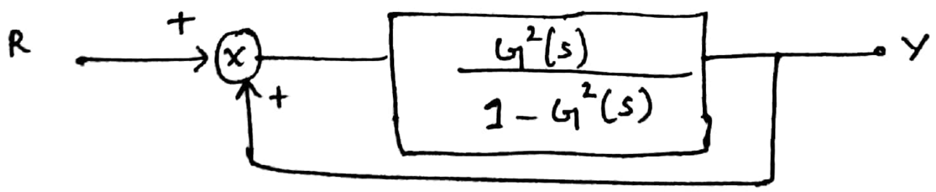
(8) The block diagram of a feedback c.s is shown in fig. Apply RH Criterion to determine the range of K for stability if $G(s) = \dfrac{K}{(s+4)(s+5)}$



Soln: Eliminating minor feedback loops

series





$$\frac{Y(s)}{R(s)} = \frac{\dfrac{G^2(s)}{1-G^2(s)}}{1 - \dfrac{G^2(s)}{1-G^2(s)}\,(1)} = \frac{G^2(s)}{1-2G^2(s)}$$

Substituting the value of $G(s) = \frac{K}{(s+4)(s+5)}$

$$\frac{G(s)}{1+G(s)H(s)}$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{K^2}{(s+4)^2(s+5)^2}}{1 - \frac{2K^2}{(s+4)^2(s+5)^2}}$$

$$\frac{Y(s)}{R(s)} = \frac{K^2}{(s+4)^2(s+5)^2 - 2K^2}$$

$$\frac{Y(s)}{R(s)} = \frac{K^2}{\underbrace{s^4 + 18s^3 + 121s^2 + 360s + 400 - 2K^2}} = 0.$$

$\therefore$ characteristic equation $1 + G(s)H(s) = 0$.

$$s^4 + 18s^3 + 121s^2 + 360s + \underbrace{400 - 2K^2} = 0$$

Routh's array is,

s^4	1	121	$400-2K^2$
s^3	18	360	0
s^2	101	$\frac{\cancel{18}(400-2K)^2}{\cancel{18}}$	0
s^1	$\frac{360 \cdot 360 - 18(400-2K)^2}{101}$	0	
s^0	$400-2K^2$.		

for row s^0; $400 - 2K^2 > 0$

$400 > 2K^2$

$200 > K^2$

$K < \sqrt{200}$

$K < 14.1421$.

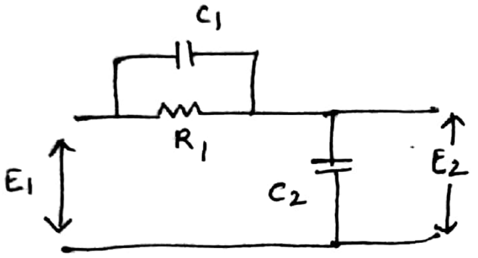
$$\frac{18 \times 121 - (1)(360)}{18} = 101$$

$0 < K < 14.14$

for any value of K between $0 < K < \infty$, the term s^1 is always positive. Thus range of K for stability

$0 < K < 14.1421$.

(8) The open loop T.F of a unity feedback C.S ($H(s) = 1$) is given by $G(s) = \dfrac{50}{s(1+0.05s)(1+0.2s)}$. Apply RH Criterion show that the system is unstable. Confirm that the Introduction of the 2 terminal pair n/w connected in cascade with G(s) make the system stable.



$C_1 = 0.5 \mu F$

$C_2 = 10 \mu F$

$R_1 = 1 M\Omega$

Sol^n:

$$G(s) = \frac{50}{s(1+0.05s)(1+0.2s)} \qquad H(s) = 1$$

$$F(s) = 1 + G(s) H(s) = 0$$

$$= 1 + \frac{50}{s(1+0.05s)(1+0.2s)} =$$

$s(1+0.05s)(1+0.2s) + 50 = 0$

$s[1 + 0.2s + 0.05s + 0.01s^2] + 50 = 0$

$s + 0.2s^2 + 0.05s^2 + 0.01s^3 + 50 = 0$

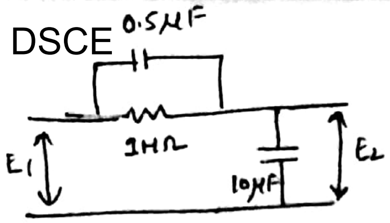
$0.01s^3 + 0.25s^2 + s + 50 = 0$

i.e $\quad 0.01s^3 + 0.25s^2 + s + 50 = 0$

Routh's array

s^3	0.01	1
s^2	0.25	50
s	−1	0
s^0	50	

There are 2 sign changes in I column of routh's array hence the system is unstable



$$Z_1(s) = R_1 \| C_1$$

$$= \frac{R_1 \times 1/sC_1}{R_1 + 1/sC_1} = \frac{R_1}{1 + sR_1C_1}$$

$$= \frac{1\times10^6}{1 + s\times1\times\cancel{10^6}\times0.5\times\cancel{10^{-6}}}$$

$$Z_2(s) = \frac{1}{10\times10^{-6}s}$$

$$Z_2(s) = \frac{10^5}{s}$$

$$Z_1(s) = \frac{10^6}{1+s0.5}$$

By Voltage division rule,

$$\frac{E_2(s)}{E_1(s)} = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} = \frac{1\times10^5/s}{\frac{10^6}{1+0.5s} + \frac{1\times10^5}{s}} = \frac{1\times10^5\,(1+0.5s)}{10^6 s + 1\times10^5(1+0.5s)}$$

$$\frac{E_2(s)}{E_1(s)} = \frac{1+0.5s}{1+10.5s}$$

$$\therefore G'(s) = \frac{50(1+0.5s)}{s(1+0.05s)(1+0.2s)(1+10.5s)}, \quad H(s)=1.$$

$$F(s) = G'(s)H(s) + 1 = 0$$

$$0.105s^4 + 2.635s^3 + 10.75s^2 + 26s + 50 = 0$$ check ..

→ New Routh's array

s^4	0.105	10.75	50
s^3	2.635	26	0
s^2	0.190	50	0
s^1	12.4369	0	
s^0	50		

$$G(s) = \frac{50}{s(1+0.05s)(1+0.2s)}$$

$$G'(s) = G(s)\cdot\frac{E_2(s)}{E_1(s)}$$

As there are no sign changes in I column of Routh's array, the system is stable. This shows that n/w connected in cascade with G(s), makes the system stable.