DAYANANDA SAGAR COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS

Staff Incharge: D. R. Sasi Rekha

COMPLEX VARIABLES

Basic Concepts:

1. The number of the form $z = x + iy$ where x and y are real numbers and $i = \sqrt{-1}$ or $i^2 = -1$ is called **Complex number (Cartesian form)** whose real part is x and the imaginary part is y.
2. If $z = x + iy$ is a complex number, then $\bar{z} = x - iy$ is called the **Complex Conjugate** of z.
3. $\boldsymbol{cosx + isinx = e^{ix}}$ **;** $\boldsymbol{cosx - isinx = e^{-ix}}$
4. $\boldsymbol{cosx = \frac{e^{ix}+e^{-ix}}{2}}$; $\boldsymbol{sinx = \frac{e^{ix}-e^{-ix}}{2i}}$
5. $\boldsymbol{cos(ix) = coshx}$; $\boldsymbol{sin(ix) = isinhx}$.
6. $\boldsymbol{(cos\theta + isin\theta)^n = cos\, n\theta + i\, sin\, n\theta}$, where n is a real number is called **De-Moivre's Theorem**
7. If $x = rcos\theta$, $y = rsin\theta$ then
   (i) $r = \sqrt{x^2 + y^2}$ is called the **modulus of Z** (or) $|\boldsymbol{Z}|$.
   (ii) $\theta = tan^{-1}\left(\frac{y}{x}\right)$ is called the **amplitude of Z** (or) $\boldsymbol{arg\ Z}$
8. If $z = x + iy = r(cos\theta + isin\theta) = re^{i\theta}$, is called **Complex number** (**polar form**) of Z.

Neighbourhood:

A neighbourhood of a point z_0 in the complex plane is the set of all points z such that $|z - z_0| < \delta$ where δ is a small positive real number. Geometrically this represents a circle with centre (x_0, y_0) and radius δ.

Function of a Complex Variable:

$w = f(z)$ is called a function of the complex variable z. w is said to be single valued or many valued function of z according as for a given value of z there corresponds one or more than one value of w. Thus $w = f(z) = u + iv$, is expressed in in two forms as:

(i) $w = f(z) = u(x, y) + i\, v(x, y)$ [**Cartesian Form**]

(ii) $w = f(z) = u(r, \theta) + i\, v(r, \theta)$ [**Polar Form**]

Example:

Let us consider $\boldsymbol{w = f(z) = z^2}$

Cartesian form

$$w = u + iv = z^2 = (x + iy)^2 = (x^2 - y^2) + i\,(2xy)$$

where $u = x^2 - y^2$ and $v = 2xy$.

Polar form

$$w = u + iv = z^2 = \left(re^{i\theta}\right)^2 = r^2 e^{2i\theta} = r^2(cos2\theta + isin2\theta) = r^2cos2\theta + ir^2sin2\theta$$

where $u = r^2cos2\theta$ and $v = r^2sin2\theta$.

Limit, Continuity, Differentiability of a Complex Valued Function:

Limit:

A complex valued function $f(z)$ defined in a neighbourhood of a point z_0 is said to have a limit l as $z \to z_0$, if for every $\epsilon > 0$, however small there exists a positive real number δ suchthat $|f(z) - l| < \epsilon$ when $|z - z_0| < \delta$. We write $lim_{z \to z_0} f(z) = l$.

Continuity:

A Complex Valued Function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exists and $lim_{z \to z_0} f(z) = f(z_0)$.

Differentiability:

A complex valued function f(z) is said to be differentiable at $z = z_0$ if $lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and is unique. This limit when exist is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$. Suppose we write $z - z_0 = \delta z$, then $z \to z_0$ ➔ $\delta z \to 0$. Hence $f'(z_0) = lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$.

Analytic Function:

A complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if $\frac{dw}{dz} = f'(z_0) = lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$ exist and is unique at z_0 and in the neighbourhood of z_0. Further $f(z)$ is said to be analytic in a region if it is analytic at every point of the region.

Note:

1. A Complex Valued Function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and also at all points in the neighbourhood of z_0.
2. Analytic function is also called a **regular function** or **holomorphic function**.

Theorem – 1:

Cauchy – Riemann Equations in Cartesian Form (C – R Equations in Cartesian Form)

<u>Statement:</u> If $f(z) = u + iv$ is analytic, then prove that Cauchy – Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ie., $u_x = v_y$, $v_x = -u_y$ are true.

<u>Proof:</u>

Given $\quad f(z) = u + iv$, where $z = x + iy$

$\therefore \quad f(x+iy) = u + iv$ ------------------------(1)

Differentiate (1) partially .w.r.t x

$f'(x+iy) = u_x + iv_x$ -------------------(2)

Differentiate (1) partially .w.r.t y

$f'(x+iy)i = u_y + iv_y$

$f'(x+iy) = \frac{1}{i}(u_y + iv_y)$

$= \frac{u_y}{i} + v_y$

$f'(x+iy) = -iu_y + v_y$ --------(3) $\quad \left(\because \ \frac{1}{i} = -i\right)$

Equating RHS of (2) and (3) we get

$u_x + iv_x = -iu_y + v_y$

Equating real and imaginary parts we get

$\boldsymbol{u_x = v_y} \ , \quad \boldsymbol{v_x = -u_y}$ These are C R Equation in Cartesian form

Theorem – 2:

Cauchy – Riemann Equations in Polar Form (C – R Equations in Polar Form)

<u>Statement:</u> If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic, then prove that Cauchy – Riemann equations in polar form $\frac{\partial u}{\partial r} = \frac{1}{r}\ \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\ \frac{\partial u}{\partial \theta}$ ie., $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$ are true.

<u>Proof:</u>

Given $\quad f(z) = u + iv$, where $z = re^{i\theta}$

$\therefore \quad f(re^{i\theta}) = u + iv$ ------------------(1)

Differentiate (1) partially .w.r.t r

$f'(re^{i\theta})\ (e^{i\theta}) \ = u_r + iv_r$ ------------------(2)

Differentiate (1) partially .w.r.t θ

$f'(re^{i\theta})\,(ire^{i\theta}) = u_\theta + iv_\theta$

$f'(re^{i\theta})\,(e^{i\theta}) = \frac{1}{ir}(u_\theta + iv_\theta)$

$= \frac{u_\theta}{ir} + \frac{v_\theta}{r}$

$= -i\frac{u_\theta}{r} + \frac{v_\theta}{r}$ ------------------(3) $\left(\because \ \frac{1}{i} = -i\right)$

From equation (2) and (3), we get

$u_r + iv_r = -i\frac{u_\theta}{r} + \frac{v_\theta}{r}$

Equating real and imaginary parts we ge

$\boldsymbol{u_r = \frac{1}{r} v_\theta}$, $\boldsymbol{v_r = -\frac{1}{r} u_\theta}$ These are C R Equation in Polar form

Harmonic Function:

A function $\emptyset$ is said to be harmonic if it satisfies Laplace's equation $\nabla^2\emptyset = 0$.

In Cartesian form the equation is $\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0.$

In the polar form the equation is $\frac{\partial^2\emptyset}{\partial r^2} + \frac{1}{r}\frac{\partial\emptyset}{\partial r} + \frac{1}{r^2}\frac{\partial^2\emptyset}{\partial \theta^2} = 0$

Properties of Analytic Functions:

- **Harmonic Property:** The real and imaginary parts of an analytic function are harmonic.

- **Orthogonal Property:** If $f(z) = u + iv$ is analytic then the family of curves $u(x, y) = c_1$, $v(x, y) = c_2$ where c_1 and c_2 are constants, intersect each other orthogonally.

Note:

- **A function is said to be analytic if it satisfy C – R Equations.**
- **A function is said to be harmonic if it satisfy Laplace Equations.**

Note:

The converse of the above properties is not true.

- **The real and imaginary part u and v are harmonic but they are not analytic.**

Example:

If $u = x^2 - y^2$ and $v = x^3 - 3xy^2$ then show that u and v are harmonic but $f(z) = u + iv$ not an analytic function

Solution:

Consider $u = x^2 - y^2$; $v = x^3 - 3xy^2$

$u_x = 2x$; $v_x = 3x^2 - 3y^2$

$u_{xx} = 2$; $v_{xx} = 6x$

$u_y = -2y$; $v_y = -6xy$

$u_{yy} = -2$; $v_{yy} = -6x$

$\therefore \quad u_{xx} + u_{yy} = 2 - 2 = 0$ ➔ u is harmonic

and $v_{xx} + v_{yy} = 6x - 6x = 0$ ➔ v is harmonic

Therefore u and v are harmonic

To verify C R Equation

$u_x = 2x \neq v_y = -6xy$ and $v_x = 3x^2 - 3y^2 \neq -u_y = 2y$

C R Equation not satisfied so they are not analytic

Hence u and v are harmonic but not satisfying C-R equations. i.e., the function $u + iv$ is not analytic.

- **The real and imaginary part u and v are intersect each other orthogonally but they are not analytic.**

Example:

If $u = \frac{x^2}{y}; y \neq 0$ and $v = x^2 + 2y^2$ then show that curve u = constant and v = constant are orthogonal but $f(z) = u + iv$ is not an analytic function

Solution:

Given $u = \text{constant}$; $v = \text{constant}$

$$\frac{x^2}{y} = c_1 \quad ; \quad x^2 + 2y^2 = c_2$$

D. w.r.t. x, treating y as a function of x

$$\frac{y(2x) - x^2\frac{dy}{dx}}{y^2} = 0 \quad ; \quad 2x + 4y\frac{dy}{dx} = 0$$

$$2xy - x^2\frac{dy}{dx} = 0 \quad ; \quad x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2xy}{x^2} = \frac{2y}{x} \quad ; \quad \frac{dy}{dx} = -\frac{x}{2y}$$

$$m_1 = \frac{2y}{x} \quad ; \quad m_2 = -\frac{x}{2y}$$

Therefore $m_1 \,.\, m_2 = \left(\frac{2y}{x}\right)\left(-\frac{x}{2y}\right) = -1$

Hence the curve u = constant and v = constant are intersect each other orthogonally

To verify C R Equation

Consider $u = \frac{x^2}{y}$; $v = x^2 + 2y^2$

$$u_x = \frac{2x}{y} \quad ; \quad v_x = 2x$$

$$u_y = -\frac{x^2}{y^2} \quad ; \quad v_y = 4y$$

$u_x = \frac{2x}{y} \neq v_y = 4y$ and $v_x = 2x \neq -u_y = \frac{x^2}{y^2}$

C R Equation not satisfied so they are not analytic

Hence, the real and imaginary part u and v are intersect each other orthogonally but they are not analytic.

Type 1: Finding the derivative of an analytic function

Working procedure:

Cartesian form

- Given $w = f(z)$, we substitute $\boldsymbol{z = x + iy}$, to find the real and imaginary part u and v as a function of x, y.
- Find the first order partial derivative $\boldsymbol{u_x}$, $\boldsymbol{u_y}$, $\boldsymbol{v_x}$, $\boldsymbol{v_y}$.
- Verify C R equation $\boldsymbol{u_x = v_y}$, $\boldsymbol{v_x = -u_y}$ to conclude the function is analytic.
- To find the derivative of $f(z)$, use $\boldsymbol{f'(z) = u_x + iv_x}$
- We substitute for the partial derivatives and rearrange as a function of $x + iy$ which is z, with the result $f'(z)$ is obtained as a function of z.

Polar form

- Given $w = f(z)$, we substitute $\boldsymbol{z = re^{i\theta}}$ to find the real and imaginary part u and v as a function of r, θ.
- Find the first order partial derivative $\boldsymbol{u_r}$, $\boldsymbol{u_\theta}$, $\boldsymbol{v_r}$, $\boldsymbol{v_\theta}$.
- Verify C R equation $\boldsymbol{u_r = \frac{1}{r} v_\theta}$, $\boldsymbol{v_r = -\frac{1}{r} u_\theta}$ to conclude the function is analytic.
- To find the derivative of $f(z)$, use $\boldsymbol{f'(z) = e^{-i\theta}(u_r + iv_r)}$
- We substitute for the partial derivatives and rearrange as a function of $re^{i\theta}$ which is z, with the result $f'(z)$ is obtained as a function of z.

Problems

1. Show that $f(z) = z + e^z$ is analytic and hence $\frac{dw}{dz}$.

<u>Solution:</u>

Given $f(z) = z + e^z$, taking $z = x + iy$

$$u + iv = (x + iy) + e^{x+iy}$$
$$= (x + iy) + e^x e^{iy}$$
$$= (x + iy) + e^x(cosy + isiny)$$
$$u + iv = x + iy + e^x cosy + ie^x siny$$

$\therefore \quad u = x + e^x cosy \quad ; \quad v = y + e^x siny$

$u_x = 1 + e^x cosy \quad ; \quad v_x = e^x siny$

$u_y = -e^x sin\, y \quad ; \quad v_y = 1 + e^x cosy$

Cauchy Riemann equations $u_x = v_y$ and $v_x = -u_y$ are satisfied

Thus $f(z) = z + e^z$ is analytic.

Also we have $f'(z) = u_x + iv_x$

i.e., $f'(z) = 1 + e^x cosy + ie^x siny$

$$= 1 + e^x(cosy + isiny)$$
$$= 1 + e^x e^{iy}$$
$$= 1 + e^{x+iy}$$

$\boldsymbol{f'(z) = 1 + e^z} \qquad (\because \quad z = x + iy\ )$

2. Show that $f(z) = sin\, 2z$ is analytic and hence $f'(z)$.

Solution:

Solution:

Gn. $f(z) = sin\, 2z$, taking $z = x + iy$

$$u + iv = sin\, 2(x + iy) = \sin(2x + i2y)$$

$$= sin\, 2x\, cos(i\, 2y) + cos\, 2x\, sin(\, i\, 2y) \quad (\because\ sin(A + B) = sin\, A\, cos\, B + cos\, A\, sin\, B)$$

$$u + iv = sin\, 2x\, cos\, h\, 2y + i\, cos\, 2x\, sin\, h\, 2y \quad (\because\ cos\, h\, y = cos\, i\, y\ ;\ i\, sin\, h\, y = sin\, iy)$$

$$\therefore \quad u = sin\, 2xcos\, h\, 2y \quad ; \quad v = cos2x\, sin\, h\, 2y$$

$$u_x = 2cos\, 2x\, cos\, h\, 2y \quad ; \quad v_x = -\, 2sin\, 2x\, sin\, h\, 2y$$

$$u_y = 2sin\, 2x\, sin\, h\, 2y \quad ; \quad v_y = 2cos\, 2x\, cos\, h\, 2y$$

Cauchy Riemann equations $u_x = v_y$ and $v_x = -u_y$ are satisfied

Thus $f(z) = sin2\, z$ is analytic.

Also we have $f'(z) = u_x + iv_x$

i.e., $f'(z) = 2cos\, 2x\, cosh\, 2y + i\, (-2\, sin\, 2x\, sin\, h\, 2y)$

$$= 2(cos\, 2x\, cos(i\, 2y) - sin\, 2x\, sin(i2y))$$

$$= 2cos(2x + i2y) \quad (\because\ cos\, A\, cos\, B - sin\, A\, sin\, B = cos(A + B)\,)$$

$$= 2cos\, 2(x + iy)$$

$$\boldsymbol{f'(z) = 2cos\, 2z} \quad (\because\ z = x + iy\ )$$

3. Show that $w = \log z$, $z \neq 0$ is analytic and hence find $\frac{dw}{dz}$

Solution:

Given $w = f(z) = \log z$, taking $z = re^{i\theta}$ we have,

$$u + iv = \log\left(re^{i\theta}\right)$$
$$= \log r + \log\left(e^{i\theta}\right)$$
$$= \log r + i\theta \log_e e$$
$$u + iv = \log r + i\,\theta \qquad (\because \log_e e = 1)$$

$$\therefore \quad u = \log r \quad ; \quad v = \theta,$$
$$u_r = \frac{1}{r} \quad ; \quad v_r = 0$$
$$u_\theta = 0 \quad ; \quad v_\theta = 1$$

C – R equation in the polar form : r $u_r = v_\theta$ *and* $r\, v_r = -u_\theta$ are satisfied.

Thus $w = \log z$ is analytic.

Also we have in the polar form,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

i.e., $$f'(z) = e^{-i\theta}\left[\ \frac{1}{r} + i.0\ \right]$$
$$= \frac{1}{re^{i\theta}}$$
$$\boldsymbol{f'(z) = \frac{1}{z}} \qquad (\because \quad z = re^{i\theta})$$

4. Show that $f(z) = z^n$, where n is a positive integer is analytic and hence find its derivative.

Solution:

Given $f(z) = z^n$ taking $z = re^{i\theta}$ we have,

$$u + iv = \left(re^{i\theta}\right)^n$$
$$= r^n e^{in\theta}$$
$$= r^n(cosn\theta + i\, sinn\theta)$$
$$u + iv = r^n cosn\theta + i\, r^n sinn\theta$$

$$\therefore \quad u = r^n cosn\theta \quad ; \quad v = r^n sinn\theta,$$
$$u_r = nr^{n-1} cosn\theta \quad ; \quad v_r = nr^{n-1} sinn\theta$$
$$u_\theta = -n\, r^n sinn\theta \quad ; \quad v_\theta = n\, r^n cosn\theta$$

C – R equation in the polar form : r $u_r = v_\theta$ *and* $r\, v_r = -u_\theta$ are satisfied.

Thus $f(z) = z^n$ is analytic.

Also we have in the polar form, $f'(z) = e^{-i\theta}(u_r + iv_r)$

i.e.,
$$f'(z) = e^{-i\theta}\left[\, nr^{n-1} cosn\theta + inr^{n-1} sinn\theta\right]$$
$$= nr^{n-1}e^{-i\theta}\left[\, cosn\theta + i\, sinn\theta\right]$$
$$= nr^{n-1}e^{-i\theta}\, e^{in\theta}$$
$$= nr^{n-1}e^{i(n-1)\theta}$$
$$= n(re^{i\theta})^{n-1}$$
$$\boldsymbol{f'(z) = nz^{n-1}} \qquad (\because \quad z = re^{i\theta})$$

Exercise:

1. Show that $f(z) = coshz$ is analytic and find its derivative
2. Show that $f(z) = e^{2z}$ is analytic and find its derivative
3. Show that $f(z) = cosz$ is analytic and find its derivative
4. Show that $f(z) = sinhz$ is analytic and find its derivative
5. Show that $f(z) = z^2 + 2z$ is analytic and find its derivative
6. Show that $f(z) = sinz$ is analytic and find its derivative

Answers:

1. $\boldsymbol{sinhz}$ **2.** $\boldsymbol{2e^{2z}}$ **3.** $\boldsymbol{-sinz}$ **4.** $\boldsymbol{coshz}$ **5.** $\boldsymbol{2z+2}$ **6.** $\boldsymbol{cosz}$

Type 2: Construction of analytic function $f(z)$ given its real or imaginary part. (Milne – Thomson method)

Working procedure:

Cartesian form

- Given the real or imaginary part, ie., u or v as a function of x, y
- Find the first order partial derivative $\boldsymbol{u_x}$, $\boldsymbol{u_y}$ or $\boldsymbol{v_x}$, $\boldsymbol{v_y}$.
- Consider $\boldsymbol{f'(z) = u_x + iv_x}$ and C R equation $\boldsymbol{u_x = v_y}$, $\boldsymbol{v_x = -u_y}$
- If u is given, $f'(z)$ becomes $\boldsymbol{f'(z) = u_x - iu_y}$ or if v is given $f'(z)$ becomes $\boldsymbol{f'(z) = v_y + iv_x}$
- We substitute the first order partial derivative in the RHS of $f'(z)$
- Put $\boldsymbol{x = z}$, $\boldsymbol{y = 0}$ to obtain $f'(z)$ as a function of z.
- Integrate with respect to z, we get $f(z)$

Polar form

- Given the real or imaginary part, ie., u or v as a function of r, θ
- Find the first order partial derivative $\boldsymbol{u_r}$, $\boldsymbol{u_\theta}$ or $\boldsymbol{v_r}$, $\boldsymbol{v_\theta}$.
- Consider $\boldsymbol{f'(z) = e^{-i\theta}(u_r + iv_r)}$ and C R equation $\boldsymbol{u_r = \frac{1}{r}v_\theta}$, $\boldsymbol{v_r = -\frac{1}{r}u_\theta}$
- If u is given, $f'(z)$ becomes $\boldsymbol{f'(z) = e^{-i\theta}\left(u_r - i\left(\frac{1}{r}u_\theta\right)\right)}$ or if v is given $f'(z)$ becomes $\boldsymbol{f'(z) = e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right)}$
- We substitute the first order partial derivative in the RHS of $f'(z)$
- Put $\boldsymbol{r = z}$, $\boldsymbol{\theta = 0}$ to obtain $f'(z)$ as a function of z.
- Integrate with respect to z, we get $f(z)$

Problems

5. Determine the analytic function $f(z) = u + iv$ given that the real part $u = x^2 - y^2 + \frac{x}{x^2+y^2}$

Solution:

Given, $u = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$u_x = 2x + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = -2y + x\left(\frac{-2y}{(x^2+y^2)^2}\right) = -\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)$$

Consider

$f'(z) = u_x + i\, v_x$

$= u_x - i\, u_y$ (by C R Eqn $v_x = -u_y$)

$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2} + i\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)$

Putting $x = z, y = 0$ we have ,

$$f'(z) = 2z + \frac{-z^2}{(z^2)^2} = 2z - \frac{1}{z^2}$$

On integrating we get

$$f(z) = \int \left(2z - \frac{1}{z^2}\right) dz + c$$

$$= \frac{2z^2}{2} + \frac{1}{z} + c$$

$$\boldsymbol{f(z) = z^2 + \frac{1}{z} + c}$$

6. Determine the analytic function $f(z) = u + iv$ given that the imaginary part $v = \frac{y}{x^2+y^2}$

Solution:

Given, $v = \frac{y}{x^2+y^2}$

$$v_x = y\left(\frac{-2x}{(x^2+y^2)^2}\right)$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)\,(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Consider, $f'(z) = u_x + i\,v_x$

$$= v_y + i\,v_x \qquad \text{(by C R Eqn} \quad u_x = v_y)$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} + i\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

Putting $x = z\,, y = 0$ we have,

$$f'(z) = \frac{z^2}{(z^2)^2} = \frac{1}{z^2}$$

On integrating we get

$$f(z) = \int \frac{1}{z^2} dz +$$

$$\boldsymbol{f(z) = -\frac{1}{z} + c}$$

7. Determine the analytic function $f(z) = u + iv$ given that the imaginary part $v = e^x(x \sin y + y \cos y)$

Solution:

Given, $v = e^x\,(x \sin y + y \cos y)$

$$v_x = e^x(\sin y) + (x \sin y + y \cos y)e^x$$

$$= e^x(\sin y + x \sin y + y \cos y)$$

$$v_y = e^x\big(x \cos y - y \sin y + \cos y\big)$$

Consider, $f'(z) = u_x + i\,v_x$

$$= v_y + i\,v_x \qquad \text{(by C R Eqn } u_x = v_y)$$

$$= e^x\big(x \cos y - y \sin y + \cos y\big) + i\,e^x(\sin y + x \sin y + y \cos y)$$

Putting $x = z\,, y = 0$ we have,

$$f'(z) = e^z(z + 1)$$

On integrating we get

$$f(z) = \int (z + 1)\,e^z dz + c$$

$$= (z + 1)\,e^z - 1(e^z) + c$$

$$= (z + 1 - 1)\,e^z + c$$

$$\boldsymbol{f(z) = z\,e^z + c}$$

8. Determine the analytic function $f(z) = u + iv$ given that the real part $u = r^2 cos2\theta$

Solution:

Given, $u = r^2 cos2\theta$

$$u_r = 2r\ cos2\theta$$

$$u_\theta = r^2(-2sin2\theta)$$
$$= -2r^2 sin2\theta$$

Consider

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$
$$= e^{-i\theta}\left(u_r - i\left(\frac{1}{r}u_\theta\right)\right) \qquad \text{(by C R Eqn } v_r = -\frac{1}{r}u_\theta)$$
$$= e^{-i\theta}\left(2r\ cos2\theta + \frac{i}{r}2r^2 sin2\theta\right)$$

Putting $r = z, \theta = 0$ we have ,

$$f'(z) = 2z$$

On integrating we get

$$f(z) = \int (2z)dz + c$$
$$= \frac{2z^2}{2} + c$$
$$\boldsymbol{f(z) = z^2 + c}$$

9. Determine the analytic function $f(z) = u + iv$ given that the real part $u = \frac{\cos 2\theta}{r^2}$

Solution:

Given, $u = \frac{\cos 2\theta}{r^2}$

$$u_r = -\frac{2}{r^3}\cos 2\theta$$

$$u_\theta = \frac{1}{r^2}(-2\sin 2\theta)$$

$$= -\frac{2\sin 2\theta}{r^2}$$

Consider

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(u_r - i\left(\frac{1}{r}u_\theta\right)\right) \quad \text{(by C R Eqn } v_r = -\frac{1}{r}u_\theta)$$

$$= e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + \frac{i}{r}\frac{2\sin 2\theta}{r^2}\right)$$

Putting $r = z, \theta = 0$ we have ,

$$f'(z) = -\frac{2}{z^3}$$

On integrating we get

$$f(z) = -2\int \frac{1}{z^3}dz + c$$

$$= -2\left(\frac{1}{-2z^2}\right) + c$$

$$f(z) = \frac{1}{z^2} + c$$

10. Determine the analytic function $f(z) = u + iv$ given that the real part $v = r^2cos2\theta - rcos\theta + 2$

Solution:

Given, $v = r^2cos2\theta - rcos\theta + 2$

$$v_r = 2r\,cos2\theta - cos\theta$$

$$v_\theta = r^2(-2sin2\theta) + rsin\theta$$

$$= -2r^2sin2\theta + rsin\theta$$

Consider

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right) \qquad \text{(by C R Eqn } u_r = \frac{1}{r}v_\theta)$$

$$= e^{-i\theta}\left(\frac{1}{r}(-2r^2sin2\theta + rsin\theta) + i(2r\,cos2\theta - cos\theta)\right)$$

Putting $r = z, \theta = 0$ we have ,

$$f'(z) = i(2z - 1)$$

On integrating we get

$$f(z) = i\int(2z - 1)dz + c$$

$$= i\left(\frac{2z2}{2} - z\right) + c$$

$$\boldsymbol{f(z) = i(z^2 - z) + c}$$

Exercise:

1. Construct the analytic function whose real part is $u = log\sqrt{x^2 + y^2}$
2. Construct the analytic function whose real part is $u = e^{-x}\{(x^2 - y^2)cosy + 2xysiny\}$
3. Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{2x}(x\ cos\ 2y - y\ sin\ 2y)$
4. Construct the analytic function whose imaginary part is $\left(r - \frac{k^2}{r}\right) sin\theta;\ \ r \neq 0$
5. Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{x}(x\ cos\ y - y\ sin\ y)$
6. Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{-x}(x\ cos\ y + y\ sin\ y)$

Answers:

1. $\boldsymbol{logz + c}$ **2.** $\boldsymbol{z^2 e^{-z} + c}$ **3.** $\boldsymbol{ze^{2z} + c}$ **4.** $\boldsymbol{z + \frac{k^2}{z} + c}$ **5.** $\boldsymbol{ze^{z} + c}$ **6.** $\boldsymbol{ze^{-z} + c}$

Complex Variable

Type 3: Finding the conjugate harmonic function and the analytic function.

We know that, the real and imaginary parts of an analytic function $\boldsymbol{f}\,(\boldsymbol{z}) = \boldsymbol{u} + \boldsymbol{iv}$ are harmonic. $\boldsymbol{u}$ and $\boldsymbol{v}$ are called conjugate harmonic functions (harmonic conjugates) Given $\boldsymbol{u}$ we can find $\boldsymbol{v}$ and vice-versa.

Working procedure:

Cartesian form

- Given $\boldsymbol{u}$ *or* $\boldsymbol{v}$, we find $\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_{xx}}, \boldsymbol{u_{yy}}$ *or* $\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_{xx}}, \boldsymbol{v_{yy}}$
- If $\boldsymbol{u_{xx}} + \boldsymbol{u_{yy}} = \boldsymbol{0}$ then $\boldsymbol{u}$ is harmonic or $\boldsymbol{v_{xx}} + \boldsymbol{v_{yy}} = \boldsymbol{0}$ then $\boldsymbol{v}$ is harmonic
- Consider C-R equation $\boldsymbol{u_x} = \boldsymbol{v_y}$, $\boldsymbol{v_x} = -\boldsymbol{u_y}$
- Substituting for $\boldsymbol{u_x}$, $\boldsymbol{u_y}$ or $\boldsymbol{v_x}, \boldsymbol{v_y}$ we obtain a system of two non-homogenous PDE of the form $\boldsymbol{v_y} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ $\boldsymbol{v_x} = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$ **or** $\boldsymbol{u_x} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ $\boldsymbol{u_y} = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$
- These can be solved by direct integration to obtain the required $\boldsymbol{v}$ *or* $\boldsymbol{u}$**.**
- To find the analytic function $f(z)$, further $\boldsymbol{u} + \boldsymbol{iv}$ will give us as a function of $\boldsymbol{x}, \boldsymbol{y}$. Putting $\boldsymbol{x} = \boldsymbol{z}, \boldsymbol{y} = \boldsymbol{0}$ we can obtain $\boldsymbol{f}(\boldsymbol{z})$ as a function of $\boldsymbol{z}$

Polar form

- Given $\boldsymbol{u}$ $\boldsymbol{or}$ $\boldsymbol{v}$, we find $\boldsymbol{u_r}$, $\boldsymbol{u_\theta}$, $\boldsymbol{u_{rr}}$, $\boldsymbol{u_{\theta\theta}}$ $\boldsymbol{or}$ $\boldsymbol{v_r}$, $\boldsymbol{v_\theta}$, $\boldsymbol{v_{rr}}$, $\boldsymbol{v_{\theta\theta}}$
- If $\boldsymbol{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0}$, then $\boldsymbol{u}$ is harmonic and $\boldsymbol{v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0}$, then $\boldsymbol{v}$ is harmonic
- Consider C-R equation $\boldsymbol{u_r = \frac{1}{r}v_\theta}$, $\boldsymbol{v_r = -\frac{1}{r}u_\theta}$
- Substituting for $\boldsymbol{u_r}$, $\boldsymbol{u_\theta}$ or $\boldsymbol{v_r}$, $\boldsymbol{v_\theta}$, we obtain a system of two non-homogenous PDE of the form $\boldsymbol{v_\theta = f(r,\theta)}$ $\boldsymbol{v_r = g(r,\theta)}$ or $\boldsymbol{u_r = f(r,\theta)}$ $\boldsymbol{u_\theta = g(r,\theta)}$
- These can be solved by direct integration to obtain the required $\boldsymbol{v}$ $\boldsymbol{or}$ $\boldsymbol{u}$.
- To find the analytic function $f(z)$, further $\boldsymbol{u + iv}$ will give us as a function of $\boldsymbol{r}$, $\boldsymbol{\theta}$. Putting $\boldsymbol{r = z}$, $\boldsymbol{\theta = 0}$ we can obtain $f(z)$ as a function of z

Problems:

1. Show that $\boldsymbol{u = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1}$ is harmonic and find its harmonic conjugate and also find the corresponding analytic function

Solution:

Consider $u = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 - 6x \quad ; \quad u_{xx} = 6x - 6$$

$$u_y = -6xy + 6y \quad ; \quad u_{yy} = -6x + 6$$

$$\therefore \quad u_{xx} + u_{yy} = 6x - 6 - 6x + 6$$

$$\boldsymbol{u_{xx} + u_{yy} = 0}$$

Thus $\boldsymbol{u}$ is harmonic

Let us find the harmonic conjugate of u, ie., v

Consider C-R equation $\boldsymbol{u_x = v_y \; , v_x = -u_y}$

Substituting for $\boldsymbol{u_x \; , \; u_y}$ we have,

$\therefore \quad v_y = u_x \qquad ; \quad v_x = -u_y$

$v_y = 3x^2 - 3y^2 - 6x \qquad ; \quad v_x = -(-6xy + 6y)$

$v = \int (3x^2 - 3y^2 - 6x\,)dy + f(x) \qquad ; \quad v = \int (\,6xy - 6y\,)dx + g(y)$

$v = 3x^2y - \frac{3y^3}{3} - 6xy + f(x) \qquad ; \quad v = \frac{6x^2y}{2} - 6xy + g(y)$

$v = 3x^2y - y^3 - 6xy + f(x) \qquad ; \quad v = 3x^2y - 6xy + g(y)$

Now we have to properly choose $f(x)$ and $g(y)$ to obtain a unique expression for $\boldsymbol{v}$

Simple comparison yields $f(x) = 0\,,\; g(y) = -y^3$.

Thus the required harmonic conjugate is $\boldsymbol{v = 3x^2y - y^3 - 6xy}$

The analytic function is $f\,(\,z\,) \;=\; u \;+\; iv$

$= x^3 - 3xy^2 - 3x^2 + 3y^2 + 1 + i(3x^2y - y^3 - 6xy)$

Putting $\boldsymbol{x = z\,, y = 0}$, then

$$\boldsymbol{f(z) = z^3 - 3z^2 + 1}$$

2. Show that $u = x^2 + 4x - y^2 + 2y$ is harmonic and find its harmonic conjugate and also find the corresponding analytic function

Solution:

Consider $u = x^2 + 4x - y^2 + 2y$

$u_x = 2x + 4$; $u_{xx} = 2$

$u_y = -2y + 2$; $u_{yy} = -2$

$\therefore$ $u_{xx} + u_{yy} = 2 - 2$

$$\boldsymbol{u_{xx} + u_{yy} = 0}$$

Thus u is harmonic

Let us find the harmonic conjugate of u, ie., v

Consider C-R equation $\boldsymbol{u_x = v_y}$, $\boldsymbol{v_x = -u_y}$

Substituting for $\boldsymbol{u_x}$, $\boldsymbol{u_y}$ we have,

$\therefore$ $v_y = u_x$; $v_x = -u_y$

$v_y = 2x + 4$; $v_x = -(-2y + 2)$

$v = \int(2x + 4)dy + f(x)$; $v = \int(2y - 2)dx + g(y)$

$v = 2xy + 4y + f(x)$; $v = 2xy - 2x + g(y)$

Now we have to properly choose $f(x)$ and $g(y)$ to obtain a unique expression for $\boldsymbol{v}$

Simple comparison yields $f(x) = -2x$, $g(y) = 4y$.

Thus the required harmonic conjugate is $\boldsymbol{v = 2xy + 4y - 2x}$

The analytic function is $f\ (z\ ) \ = \ u \ + \ iv$

$$= x^2 + 4x - y^2 + 2y + i(2xy + 4y - 2x)$$

Putting $\boldsymbol{x} = \ \boldsymbol{z}\,, \boldsymbol{y} \ = \ \boldsymbol{0}\,,$ then

$$\boldsymbol{f(z) = z^2 + 4z - 2zi}$$

3. Show that the given function is harmonic and find the harmonic conjugate. Also find the analytic function , $\boldsymbol{v = rsin\theta + \frac{1}{r}cos\theta}$; $\boldsymbol{r \neq 0}$

Solution:

Given $\quad v = rsin\theta + \frac{1}{r}cos\theta$

$$v_r = sin\theta - \frac{1}{r^2}cos\theta \qquad ; \ v_{rr} = \frac{2}{r^3}cos\theta$$

$$v_\theta = rcos\theta - \frac{1}{r}sin\theta \qquad ; \ v_{\theta\theta} = -rsin\theta - \frac{1}{r}cos\theta$$

$$\therefore \quad v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = \frac{2}{r^3}cos\theta + \frac{1}{r}\left(sin\theta - \frac{1}{r^2}cos\theta\right) + \frac{1}{r^2}\left(-rsin\theta - \frac{1}{r}cos\theta\right)$$

$$= \frac{2}{r^3}cos\theta + \frac{1}{r}sin\theta - \frac{1}{r^3}cos\theta - \frac{1}{r}sin\theta - \frac{1}{r^3}cos\theta$$

$$\boldsymbol{v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0}$$

Thus v is harmonic

Let us find the harmonic conjugate of v, ie., u

Consider C-R equation $\boldsymbol{u_r = \frac{1}{r} v_\theta \;, v_r = -\frac{1}{r} u_\theta}$

Substituting for $\boldsymbol{v_r \;, \; v_\theta}$ we have,

$$\therefore \quad u_r = \frac{1}{r} v_\theta \quad ; \quad u_\theta = -r v_r$$

$$u_r = \frac{1}{r}\left(r\cos\theta - \frac{1}{r}\sin\theta\right) \quad ; \quad u_\theta = -r\left(\sin\theta - \frac{1}{r^2}\cos\theta\right)$$

$$u = \int \left(\cos\theta - \frac{1}{r^2}\sin\theta\right) dr + f(\theta) \quad ; \quad u = \int (-r\sin\theta + \frac{1}{r}\cos\theta)\, d\theta + g(r)$$

$$u = r\cos\theta + \frac{1}{r}\sin\theta + f(\theta) \quad ; \quad u = r\cos\theta + \frac{1}{r}\sin\theta + g(r)$$

Now we have to properly choose $g(r)$ and $f(\theta)$ to obtain a unique expression for u

Simple comparison yields $f(\theta) = 0 = g(r)$

Thus the required harmonic conjugate is $\boldsymbol{u = r\cos\theta + \frac{1}{r}\sin\theta}$

The analytic function is $f(z) = u + iv$

$$= r\cos\theta + \frac{1}{r}\sin\theta + i(r\sin\theta + \frac{1}{r}\cos\theta)$$

Putting ***r = z , θ = 0*** , then

$$\boldsymbol{f(z) = z + \frac{i}{z}}$$

4. Show that the given function is harmonic and find the harmonic conjugate. Also find the analytic function , $\boldsymbol{v = \left(r - \frac{1}{r}\right) sin\theta \,;\, r \neq 0}$

Solution:

Given $\quad v = \left(r - \frac{1}{r}\right) sin\theta$

$$v_r = \left(1 + \frac{1}{r^2}\right) sin\theta \quad ; \quad v_{rr} = \frac{-2}{r^3} sin\theta$$

$$v_\theta = \left(r - \frac{1}{r}\right) cos\theta \quad ; \quad v_{\theta\theta} = -\left(r - \frac{1}{r}\right) sin\theta$$

$$\therefore \quad v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = \frac{-2}{r^3} sin\theta + \frac{1}{r}\left(1 + \frac{1}{r^2}\right) sin\theta - \frac{1}{r^2}\left(r - \frac{1}{r}\right) sin\theta$$

$$= \frac{-2}{r^3} sin\theta + \frac{1}{r} sin\theta + \frac{1}{r^3} sin\theta - \frac{1}{r} sin\theta + \frac{1}{r^3} sin\theta$$

$$\boldsymbol{v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0}$$

Thus v is harmonic

Let us find the harmonic conjugate of v, ie., u

Consider C-R equation $\boldsymbol{u_r = \frac{1}{r} v_\theta \;,\; v_r = -\frac{1}{r} u_\theta}$

Substituting for $\boldsymbol{v_r \;,\; v_\theta}$ we have,

$$\therefore \quad u_r = \frac{1}{r} v_\theta \quad ; \quad u_\theta = -r v_r$$

$$u_r = \frac{1}{r}\left(r - \frac{1}{r}\right) cos\theta \quad ; \quad u_\theta = -r\left(1 + \frac{1}{r^2}\right) sin\theta$$

$$u = \int \left(1 - \frac{1}{r^2}\right) cos\theta \;\, dr + f(\theta) \quad ; \quad u = -\int \left(r + \frac{1}{r}\right) sin\theta \;\, d\theta + g(r)$$

$$u = \left(r + \frac{1}{r}\right) cos\theta + f(\theta) \quad ; \quad u = \left(r + \frac{1}{r}\right) cos\theta + g(r)$$

Now we have to properly choose $g(r)$ and $f(\theta)$ to obtain a unique expression for u

Simple comparison yields $g(r) = 0 = f(\theta)$

Thus the required harmonic conjugate is $\boldsymbol{u = \left(r + \frac{1}{r}\right) cos\theta}$

The analytic function is $f\ (z\ ) = u + iv$

$$= \left(r + \frac{1}{r}\right) cos\theta + i\left(r - \frac{1}{r}\right) sin\theta$$

Putting $\boldsymbol{r = z\ ,\ \theta = 0}$, then

$$\boldsymbol{f(z) = z + \frac{1}{z}}$$

5. Show that the given function is harmonic and find the harmonic conjugate. Also find the analytic function , $\boldsymbol{u = \left(r + \frac{1}{r}\right) cos\theta\ ;\ r \neq 0}$

<u>Solution:</u>

Given $\quad u = \left(r + \frac{1}{r}\right) cos\theta$

$$u_r = \left(1 - \frac{1}{r^2}\right) cos\theta \qquad ;\ u_{rr} = \frac{2}{r^3} cos\theta$$

$$u_\theta = -\left(r + \frac{1}{r}\right) sin\theta \qquad ;\ u_{\theta\theta} = -\left(r + \frac{1}{r}\right) cos\theta$$

$$\therefore \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{2}{r^3} cos\theta + \frac{1}{r}\left(1 - \frac{1}{r^2}\right) cos\theta + \frac{1}{r^2}\left(-\left(r + \frac{1}{r}\right) cos\theta\right)$$

$$= \frac{2}{r^3} cos\theta + \frac{1}{r} cos\theta - \frac{1}{r^3} cos\theta - \frac{1}{r} sin\theta - \frac{1}{r^3} sin\theta$$

$$\boldsymbol{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0}$$

Thus $\boldsymbol{u}$ is harmonic

Let us find the harmonic conjugate of $\boldsymbol{u}$, ie., $\boldsymbol{v}$

Consider C-R equation $\boldsymbol{u_r = \frac{1}{r} v_\theta \ , v_r = -\frac{1}{r} u_\theta}$

Substituting for $\boldsymbol{v_r \ , \ v_\theta}$ we have,

$$\therefore \quad v_\theta = r u_r \quad ; \quad v_r = -\frac{1}{r} u_\theta$$

$$v_\theta = r\left(1 - \frac{1}{r^2}\right) cos\theta \quad ; \quad v_r = \frac{1}{r}\left(r + \frac{1}{r}\right) sin\theta$$

$$v = \int \left(r - \frac{1}{r}\right) cos\theta \ \ d\theta + f(r) \quad ; \quad v = \int \left(1 + \frac{1}{r^2}\right) sin\theta \ \ dr + g(\theta)$$

$$u = \left(r - \frac{1}{r}\right) sin\theta + f(r) \quad ; \quad u = \left(r - \frac{1}{r}\right) sin\theta + g(\theta)$$

Now we have to properly choose $f(r)$ and $g(\theta)$ to obtain a unique expression for $\boldsymbol{v}$
Simple comparison yields $f(r) = 0 = g(\theta)$

Thus the required harmonic conjugate **is** $\boldsymbol{v = \left(r - \frac{1}{r}\right) sin\theta}$

The analytic function is $f\,(\,z\,) \; = \; u \; + \; iv$

$$= \left(r + \frac{1}{r}\right) cos\theta + i\left(r - \frac{1}{r}\right) sin\theta$$

Putting $\boldsymbol{r = z \ , \ \theta = 0}$, then

$$\boldsymbol{f(z) = z + \frac{1}{z}}$$

Exercise

Show that the given function is harmonic function, find its harmonic conjugate and also find the analytic function

1. $v = cosx\ sinhy$
2. $u = (x-1)^3 - 3xy^2 + 3y^2$
3. $v = e^{-2y} sin2x$
4. $u = e^x cosy + xy$
5. $v = 2xy - 2x + 4y$
6. $u = \frac{1}{r} cos\theta$

Answers

1. $u = sinx\ coshy$; $f(z) = sinz$
2. $v = 3x^2y - 6xy + 3y - y^3$; $f(z) = (z-1)^3$
3. $u = e^{-2y} cos2x$; $f(z) = e^{2iz}$
4. $v = e^x siny + \frac{y^2}{2} - \frac{x^2}{2}$; $f(z) = e^z - \frac{i}{2} z^2$
5. $u = x^2 + 4x - y^2 + 2y$; $f(z) = z^2 + 4z - 2iz$
6. $v = -\frac{1}{r} sin\theta$; $f(z) = \frac{1}{z}$

Type4: Miscellaneous problems

Problems:

6. Find the analytic function $f(z) = u + iv$ given $u - v = e^x(cosy - siny)$

Solution:

Given $\quad u - v = e^x(cosy - siny)$ --------(1)

Diff (1) .Partially w. r. t. x

$$u_x - v_x = e^x(cosy - siny)$$

$$u_x - v_x = e^x cosy - e^x siny \quad \text{----------(2)}$$

Diff (1) .Partially w. r. t. y

$$u_y - v_y = e^x(-siny - cosy)$$

$$-v_x - u_x = e^x(-siny - cosy) \qquad (\because \boldsymbol{v_y = u_x} \ , \ \boldsymbol{u_y = -v_x})$$

$$v_x + u_x = e^x siny + e^x cosy) \quad \text{---------- (3)}$$

(2) + (3) ➔ $\quad 2u_x = 2e^x cosy$

$$\boldsymbol{u_x = e^x cosy}$$

(2) - (3) ➔ $\quad -2v_x = -2e^x siny$

$$\boldsymbol{v_x = e^x siny}$$

We have $\quad f'(z) = u_x + iv_x$

$$= e^x cosy + ie^x siny$$

putting $\boldsymbol{x = z, y = 0}$

$$f'(z) = e^z$$

On integration we get

$$f(z) = \int e^z \, dz + c$$

$$\boldsymbol{f(z) = e^z + c}$$

7. Find the analytic function $\boldsymbol{f(z) = u + iv}$ given $\boldsymbol{u - v = (x - y)(x^2 + 4xy + y^2)}$

Solution:

Given $u - v = (x - y)(x^2 + 4xy + y^2)$

$$= x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$= x^3 + 3x^2y - 3xy^2 - y^3 \text{--------}(1)$$

Diff (1) Partially w. r. t. x

$$u_x - v_x = 3x^2 + 6xy - 3y^2 \text{----------}(2)$$

Diff (1) Partially w. r. t. y

$$u_y - v_y = 3x^2 - 6xy - 3y^2$$

$$-v_x - u_x = 3x^2 - 6xy - 3y^2 \qquad (\because \boldsymbol{v_y = u_x}\ ,\ \boldsymbol{u_y = -v_x})$$

$$v_x + u_x = -3x^2 + 6xy + 3y^2 \text{ ---------- }(3)$$

(2) + (3) ➔ $2u_x = 12xy$

$$\boldsymbol{u_x = 6xy}$$

(2) - (3) ➔ $-2v_x = 6x^2 - 6y^2$

$$\boldsymbol{v_x = 3(y^2 - x^2)}$$

We have $f'(z) = u_x + iv_x$

$$= 6xy + 3i(y^2 - x^2)$$

putting $\boldsymbol{x = z, y = 0}$

$$f'(z) = -3iz^2$$

On integration we get

$$f(z) = -3i\int z^2\, dz + c$$

$$= -3i\left(\frac{z^3}{3}\right) + c$$

$$\boldsymbol{f(z) = -iz^3 + c}$$

8. Find the analytic function $\boldsymbol{f(z) = u + iv}$ given $\boldsymbol{u + v = \frac{1}{r^2}(cos2\theta - sin2\theta)}$

<u>Solution</u>

Given $u + v = \frac{1}{r^2}(cos2\theta - sin2\theta)$ -------------(1)

Diff (1) Partially w. r. t. r

$$u_r + v_r = \frac{-2}{r^3}(cos2\theta - sin2\theta)$$

$$= -\frac{2}{r^3}cos2\theta + \frac{2}{r^3}sin2\theta \quad \text{-----------(2)}$$

Diff (1) .Partially w. r. t. θ

$$u_\theta + v_\theta = \frac{1}{r^2}(-2sin2\theta - 2cos2\theta)$$

$$-rv_r + ru_r = -\frac{2}{r^2}sin2\theta - \frac{2}{r^2}cos2\theta \qquad (\because \ \boldsymbol{v_\theta = ru_r \ , u_\theta = -rv_r})$$

$$-v_r + u_r = -\frac{2}{r^3}sin2\theta - \frac{2}{r^3}cos2\theta \qquad \text{------------(3)}$$

(2) + (3) ➔ $$2u_r = -\frac{4}{r^3}cos2\theta$$

$$\boldsymbol{u_r = -\frac{2}{r^3}cos2\theta}$$

(2) - (3) ➔ $$2v_r = \frac{4}{r^3}sin2\theta$$

$$\boldsymbol{v_r = \frac{2}{r^3}sin2\theta}$$

We have $$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(-\frac{2}{r^3}cos2\theta + i\frac{2}{r^3}isin2\theta\right)$$

putting $\boldsymbol{r = z, \theta = 0}$

$$f'(z) = -\frac{2}{z^3}$$

On integration we get

$$f(z) = -2\int\frac{1}{z^3}dz + c$$

$$= \boldsymbol{-2\left(\frac{1}{-2z^2}\right) + c}$$

$$\boldsymbol{f(z) = \frac{1}{z^2} + c}$$

9. Find the analytic function $f(z) = u + iv$ given $u + v = r(\cos\theta + \sin\theta) + \frac{1}{r}(\cos\theta - \sin\theta)$

Solution

Given $\quad u + v = r(\cos\theta + \sin\theta) + \frac{1}{r}(\cos\theta - \sin\theta)$ -------------(1)

Diff (1) Partially w. r. t. r

$$u_r + v_r = \cos\theta + \sin\theta - \frac{1}{r^2}(\cos\theta - \sin\theta)$$

$$= \cos\theta + \sin\theta - \frac{1}{r^2}\cos\theta + \frac{1}{r^2}\sin\theta \quad \text{-----------(2)}$$

Diff (1) .Partially w. r. t. θ

$$u_\theta + v_\theta = r(-\sin\theta + \cos\theta) + \frac{1}{r}(-\sin\theta - \cos\theta)$$

$$-rv_r + ru_r = r(-\sin\theta + \cos\theta) + \frac{1}{r}(-\sin\theta - \cos\theta) \qquad (\because \; \boldsymbol{v_\theta = ru_r \;, u_\theta = -rv_r})$$

$$-v_r + u_r = -\sin\theta + \cos\theta - \frac{1}{r^2}\sin\theta - \frac{1}{r^2}\cos\theta \quad \text{------------(3)}$$

(2) + (3) ➔ $\quad 2u_r = 2\cos\theta - \frac{2}{r^2}\cos\theta$

$$u_r = \cos\theta - \frac{1}{r^2}\cos\theta$$

$$\boldsymbol{u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta}$$

(2) - (3) ➔ $\quad 2v_r = 2\sin\theta + \frac{2}{r^2}\sin\theta$

$$v_r = \sin\theta + \frac{1}{r^2}\sin\theta$$

$$\boldsymbol{v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta}$$

We have $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= e^{-i\theta}\left(\left(1 - \frac{1}{r^2}\right)cos\theta + i\left(1 + \frac{1}{r^2}\right)sin\theta\right)$$

putting $\boldsymbol{r = z, \theta = 0}$

$$f'(z) = 1 - \frac{1}{z^2}$$

On integration we get

$$f(z) = \int\left(1 - \frac{1}{z^2}\right)dz + c$$

$$\boldsymbol{f(z) = z + \frac{1}{z} + c}$$

Exercise

1. **Find the analytic function $\boldsymbol{f(z) = u + iv}$ given $\boldsymbol{u + v = x^3 - y^3 + 3xy(x - y)}$**
2. **Find the analytic function $\boldsymbol{f(z) = u + iv}$ given $\boldsymbol{u + v = (x - y) + e^x(cosy + siny)}$**

Answers

1. $\boldsymbol{z^3 + c}$
2. $\boldsymbol{z + e^z + c}$

Applications

Velocity potential, Stream function and Complex potential

The flow pattern is represented by $\boldsymbol{w = f(z) = \varphi(x,y) + i\psi(x,y)}$ is called the **complex potential,** where $\boldsymbol{\varphi(x,y)}$ is the **velocity potential** and $\boldsymbol{\psi(x,y)}$ is a **stream function or flux function**

Note:

Two concentric circular cylinders of radii $r_1, r_2\,(r_1 < r_2)$ are kept at potentials $\emptyset_1$ and $\emptyset_2$ respectively then

- The potential difference $= \varphi_2 - \varphi_1$
- The total charge or flux $= \int_0^{2\pi} d\psi$
- The capacitance without dielectric $= \frac{\text{The total charge or flux}}{\text{The potential difference}}$

Problems

10. An electrostatic field in the xy-plane is given by the potential function $\varphi = 3x^2y - y^3$. Find the stream function.

Solution:

Given $\varphi = 3x^2y - y^3$

$$\varphi_x = 6xy$$

$$\varphi_y = 3x^2 - 3y^2$$

Consider C-R equation $\boldsymbol{\varphi_x = \psi_y \quad , \quad \psi_x = -\varphi_y}$

Substituting for $\boldsymbol{\varphi_x}$, $\boldsymbol{\varphi_y}$ we have,

$$\psi_y = \varphi_x \quad ; \quad \psi_x = -\varphi_y$$

$$\psi_y = 6xy \quad ; \quad \psi_x = 3y^2 - 3x^2$$

$$\psi = \int 6xy \; dy + f(x) \quad ; \quad \psi = \int (3y^2 - 3x^2) \; dx + g(y)$$

$$\psi = \frac{6xy^2}{2} + f(x) \quad ; \quad \psi = 3xy^2 - \frac{3x^3}{3} + g(y)$$

$$\psi = 3xy^2 + f(x) \quad ; \quad \psi = 3xy^2 - x^3 + g(y)$$

Now we have to properly choose $f(x)$ and $g(y)$ to obtain a unique expression for $\boldsymbol{\psi}$.

Simple comparison yields $f(x) = -x^3$, $g(y) = 0$

$\therefore$ $\boldsymbol{\psi = 3xy^2 - x^3}$ **is the required stream function.**

11. Two concentric circular cylinders of radii $r_1, r_2\,(r_1 < r_2)$ are kept at potentials $\emptyset_1$and $\emptyset_2$ respectively.Using complex function $\omega = alogz + c$, prove that the capacitance per unit length of the capacitor formed by them is $\frac{2\,\pi\,\lambda}{log\left(\frac{r_2}{r_1}\right)}$ where λ is the dielectric constant of the medium.

<u>Solution:</u>

Given $\omega = alogz + c$

$$\varphi + i\psi = alog\left(re^{i\theta}\right) + c$$
$$= a\left\{logr + log\left(e^{i\theta}\right)\right\} + c$$
$$= a\{logr + i\theta\, log\, e\} + c$$
$$= a\{logr + i\theta\} + c$$
$$= alogr + ia\theta + c$$
$$\varphi + i\psi = alogr + c + ia\theta$$

$$\therefore \quad \varphi = alogr + c \quad ; \quad \psi = a\theta$$

$\because$ Two concentric circular cylinders of radii $r_1, r_2\,;\,(r_1 < r_2)$

$$\varphi_1 = alogr_1 + c \quad ; \quad \varphi_2 = alogr_2 + c$$

$\therefore$ The potential difference $= \varphi_2 - \varphi_1$

$$= alogr_2 + c - (\,alogr_1 + c)$$
$$= alogr_2 + c - alogr_1 - c$$
$$= a[logr_2 - logr_1]$$
$$= alog\left(\frac{r_2}{r_1}\right)$$

The total charge or flux $= \int_0^{2\pi} d\psi$

$$= \int_0^{2\pi} d(a\theta)$$

$$= a\int_0^{2\pi} d(\theta)$$

$$= a[\theta]_0^{2\pi} = a(2\pi - 0) = 2\pi a$$

The capacitance being the charge required to maintain a unit potential difference.

The capacitance without dielectric $= \frac{\text{The total charge or flux}}{\text{The potential difference}}$

$$= \frac{2\pi a}{a\log\left(\frac{r_2}{r_1}\right)} = \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)}$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge.

Thus the capacitance with dielectric $= \frac{2\,\pi\,\lambda}{\log\left(\frac{r_2}{r_1}\right)}$

Cauchy-Riemann equations in the Cartesian form

The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. These are known as Cauchy-Riemann (C-R) equations:

$$u_x = v_y \text{ and } v_x = -u_y$$

Proof

$$w = f(z) = u(x,y) + iv(x,y)$$

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Given $f(z)$ is analytic

$\Rightarrow f'(z)$ exist and is unique.

$$f(z) = u(x,y) + iv(x,y)$$

Diff. w.r.t x.

$$f'(x+iy) \cdot 1 = u_x + iv_x$$

$$\Rightarrow \boxed{f'(z) = u_x + iv_x} \quad \text{--- (1)}$$

$z = x+iy$

$f(z) = f(x+iy)$

$f'(z)\Big]_{w.r.t\ x} = f'(x+iy) \cdot (1)$

Coeff of x

$= f'(z)$

Again $f(z) = u(x,y) + iv(x,y)$

Differentiating w.r.t y.

$$\boxed{f'(z) \cdot i = u_y + iv_y} \quad \text{--- (2)}$$

$u = x^2$

$\boxed{u_x = 2x}$

$$\Rightarrow f'(z) = \frac{1}{i} u_y + \frac{\cancel{i}}{\cancel{i}} v_y$$

$$\Rightarrow \boxed{f'(z) = \frac{1}{i} u_y + v_y} \quad \text{--- (3)}$$

But $f'(z)$ is unique. it can't have two value.

from Eq (1) & (3) we get

$$u_x + iv_x = \frac{1}{i} u_y + v_y$$
$$= -i\, u_y + v_y$$

$$\left[\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i\right]$$

Compare the real part & imaginary part on both sides.

$$\boxed{u_x = v_y \text{ and } u_y = -v_x}$$

Cauchy-Riemann equations in the polar form

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

i.e $\boxed{u_r = \frac{1}{r} v_\theta}$ and $\boxed{u_\theta = -r v_r}$

These are known as Cauchy-Riemann (C-R) equations in the polar form.

Proof $f(z)$ is analytic

$\Rightarrow f'(z)$ exist and is unique.

$\therefore f(z) = u(r,\theta) + iv(r,\theta)$

$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

Diff. partially w.r.t 'r'.

$f'(re^{i\theta})e^{i\theta} = u_r + iv_r$

$\Rightarrow \boxed{f'(z) = \frac{1}{e^{i\theta}}(u_r + iv_r)}$ —— (1)

Diff. w.r.t θ

$f'(re^{i\theta})\, ri e^{i\theta} = u_\theta + iv_\theta$

$\Rightarrow \boxed{\frac{1}{rie^{i\theta}}(u_\theta + iv_\theta) = f'(z)}$ —— (2)

$f'(z)$ is unique.

$$\frac{1}{\cancel{e^{i\theta}}}(u_r + iv_r) = \frac{1}{ri\cancel{e^{i\theta}}}(u_\theta + iv_\theta)$$

$$\Rightarrow u_r + iv_r = \frac{1}{r} \cdot \frac{1}{i}(u_\theta + iv_\theta)$$

$$\Rightarrow u_r + iv_r = \frac{1}{ri}u_\theta + \frac{\cancel{i}}{r\cancel{i}}v_\theta$$

$$= \frac{1}{ri}u_\theta + \frac{1}{r}v_\theta$$

$$\boxed{u_r + iv_r = \frac{-i}{r}u_\theta + \frac{1}{r}v_\theta}$$

Compairing the real part and Imaginary part.

$$\boxed{u_r = \frac{1}{r}v_\theta}$$

and $\boxed{v_r = \frac{-1}{r}u_\theta}$

Proved

~~Proved~~

Construction of Analytic Function :-

→ Suppose u (or) v given. By using this, we have to construct $f(z)$ which is analytic.

→ For this, one method is there is called **Milne Thomson's method.**

Working Procedure :-

- Suppose u is given.
- Find u_x, u_y. consider
  $f'(z) = u_x + i v_x$. (But we dont have v_x value).
- Since we are constructing the analytic function, we use CR eq^n.
  so, $v_x = -u_y$
  $\therefore f'(z) = u_x - i u_y$
- then put $x = z, y = 0$ to obtain $f'(z)$ as a function of z.
- Integrating w.r.t 'z' we get $f(z)$.

=
- Given v
- Find v_x, v_y. consider
  $f'(z) = u_x + i v_x$ (we don't have u_x value)
- By CR eq^n, $u_x = v_y$
  $f'(z) = v_y + i v_x$
- Put $x = z, y = 0$ to obtain $f'(z)$ as a function of z.
- Integrating w.r.t 'z' we get $f(z)$.

Polar

- Similarly in the case of Polar co-ordinates, r, θ we consider

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

and use CR eq^n in the RHS $v_r = -\frac{1}{r}u_\theta$ given u or $u_r = \frac{1}{r}v_\theta$ given v.

- We use the substitution $r = z$, $\theta = 0$ to obtain $f'(z)$ as a function of z.
- Integrating w.r.t z we get $f(z)$.