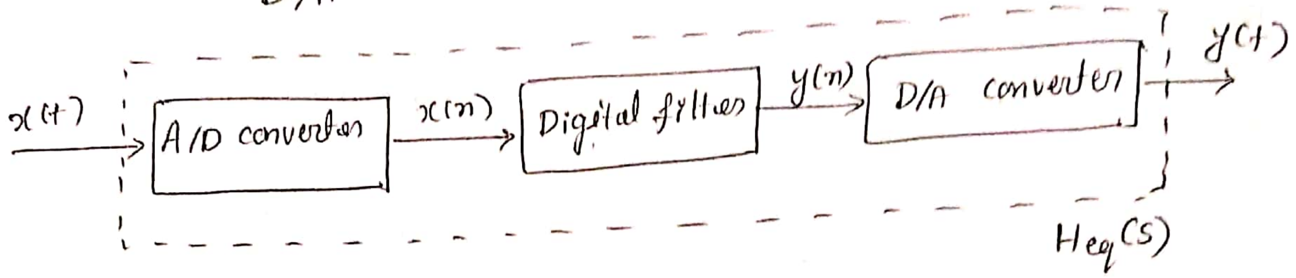
Digital filters

* IIR / recursive filter
* The Design of digital IIR filter is largely based on analog filter design techniques.
* To simulate an analog filter
  → The digital filter is used in A/D - Digital filter D/A



* {A/D - Digital filter - D/A} = $H_{eq}(S)$ → analog filter.
* Digital filter takes x(n) & O/P's y(n) i.e discrete-time sequence
  → Digital filter can be characterised by
  (i) unit sample response $h(n)$
  (ii) system function $H(z)$ (or)
  (iii) A difference equation realization.

Design procedure of a digital IIR filter

1) Selecting a method of transformation of a given analog filter to a digital filter having roughly the same freq response.
(2) Mapping the specifications of the digital IIR filter to Equivalent specifications of an an analog IIR filter.
(3) Designing the analog IIR filter according to the mapped specifications.
(4) Transforming the analog filter to an equivalent digital filter.

Mapping of an analog filter to a digital filter with approximately the same frequency response

(i) The backward difference method $s = \frac{1-z^{-1}}{T}$

(ii) The bilinear transformation.

(iii) The impulse invariant transformation

(iv) The matched - z transformation.

Backward difference Method :-

* $\boxed{A/D - H(z) - D/A}$ = Heq → analog filter in time domain

* Difference Eqn for same.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

taking laplace.

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$H_{eq}(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{M} a_k s^k} \longrightarrow (1)$$

1st order derivative.

$$\nabla^{(1)}\{y(n)\} = \frac{[y(n) - y(n-1)]}{T} \xleftrightarrow{ZT} \frac{(1-z^{-1})}{T} Y(z)$$

higher order = repetition of 1st order derivative.

$$\therefore \nabla^{(k)}\{y(n)\} \xleftrightarrow{ZT} \left[\frac{1-z^{-1}}{T}\right]^k Y(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\sum_{k=0}^{M} b_k \left[\frac{1-z^{-1}}{T}\right]^k}{\sum_{k=0}^{M} a_k \left[\frac{1-z^{-1}}{T}\right]^k} \longrightarrow (2)$$

by comparing ~~Eq~~ 1 & 2.

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}}$$

$$s = \frac{1-z^{-1}}{T}$$

$$sT = 1 - z^{-1}$$

$$z^{-1} = 1 - sT$$

$$z = \frac{1}{1-sT}$$

wkt $s = \sigma + j\Omega$.

$$\therefore z = \frac{1}{1-\sigma T - j\Omega T}$$

$$|z| = \frac{1}{\left[(1-\sigma T)^2 + (\Omega T)^2\right]^{1/2}}$$

OBSERVATION

(i) $\sigma < 0 \Rightarrow |z| < 1$

(ii) if Heq is stable $\Rightarrow$ All poles = Negative real parts

ie H(z) poles will have magnitudes less than 1

(ii) if $\sigma = 0$

$$z = \frac{1}{1-j\Omega T}$$

$$z - \tfrac{1}{2} = \frac{1}{1-j\Omega T} - \frac{1}{2} = \frac{(1+j\Omega T)}{2(1-j\Omega T)}$$

$$|z - \tfrac{1}{2}| = \tfrac{1}{2}$$

ie $j\Omega$ axis of the s-domain is a circle of radius 0.5 at $z = 0.5$ in the z domain

$\Rightarrow j\Omega \not\Rightarrow |z| = 1$

ie $H(\omega)$ will be considerably distorted wrt $H(j\Omega)$

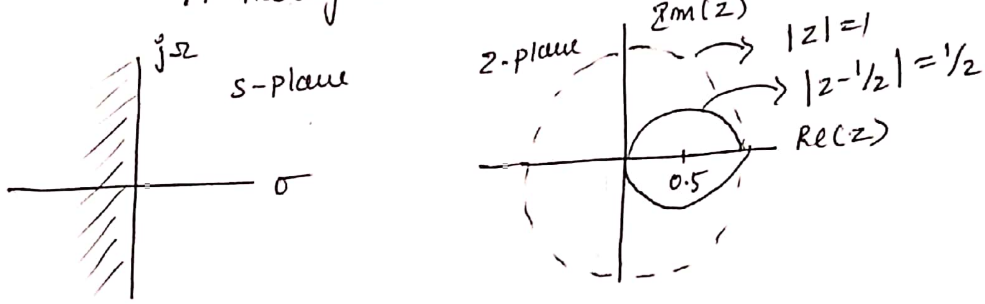
$\Rightarrow$ left-half of s-plane is mapped inside the circle. $|z - 0.5| = 0.5$ in the z-plane

ie Right-half of z-plane at $z = 0.5$

$\Rightarrow$ ie Analog Highpass filter cannot be mapped to Digital HPF $\because$ Poles of Digital filter cannot lie in the correct region.

$\Rightarrow$ Backward difference method does not preserve the shape of the frequency response ~~etc~~.

$\therefore$ Rarely used in real.



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2 problems

objective of IIT is to develop an IIR filter TF whose impulse response is sampled version of $H_a(s)$. It preserves characteristics of TS as small as possible & fs is high.

Impulse Invariant Transformation (IIT)

* Impulse response of an analog filter is $h_a(t)$
* The unit sample response discrete-time filter $H(z)$
* $\boxed{A/D - H(z) - D/A}$ → $H_{eq}(s)$ → sampled version of $h(t)$

$$h(n) = h_a(nT)$$
$$= h_a(t)\Big|_{t=nT}$$

$$H(z) = Z\{h(n)\}$$

laplace transform, Z-Transform.

LS & ZT

$\Rightarrow H(z) = H(s)\Big|_{s=z}$.

its incorrect form.

$\Rightarrow H(z)$ can be obtained directly from $H_a(s)$ without intervening steps $h_a(t)$ & $h_a(nT)$

* Analog transfer function with N different poles s-domain transfer function written in partial fraction expansion form.

$$H_a(s) = \sum_{i=1}^{N} \frac{C_i}{s-s_i} \longrightarrow (1)$$

* unit impulse response. $h_a(t) = \sum_{i=1}^{N} C_i e^{s_i t}$

Sampled response. $h_a(nT) = h(n) = \sum_{i=1}^{N} C_i e^{s_i nT}$

* $H(z) = Z\{h(n)\} = Z\Big\{\underbrace{\sum_{i=1}^{N} C_i e^{s_i nT}}_{h(n)}\Big\}$

$$= \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty}\sum_{i=1}^{N} C_i e^{s_i nT} z^{-n}$$

filter is assumed to be causal.

$$= \sum_{i=1}^{N} c_i \sum_{n=0}^{\infty} [e^{S_i T} z^{-1}]^n$$

$$= \sum_{i=1}^{N} c_i \frac{1}{1 - e^{S_i T} z^{-1}} \longrightarrow (2)$$

By comparing (1) & (2)

$$\frac{1}{S - S_i} \implies \frac{1}{1 - e^{S_i T} z^{-1}} \text{ (or) } \frac{z}{z - e^{S_i T}}$$

OBSERVATION : (i) Analog pole at $S = S_i$ is mapped to a digital pole at $z_i = e^{S_i T}$

* Transformed $H(z)$ properties.

(i) order is same as that of $H_a(s)$

(ii) poles are mapped according to

$$S_i \overset{IIT}{\Longrightarrow} z_i = e^{S_i T}, \quad 1 \le i \le N$$

ie $z = e^{sT}$ (digital & analog poles relation)

WKT $z = r e^{j\omega}$

$s = \sigma + j\Omega$

$$\therefore \quad r e^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

$$r = e^{\sigma T}$$

$$e^{j\omega} = e^{j\Omega T} \implies \boxed{\omega = \Omega T}$$

(a) $\sigma < 0 \implies 0 < r < 1$ LH poles mapped inside the unit circle

(b) $\sigma > 0 \implies r > 1$ RH poles mapped outside the unit circle.

(c) $\sigma = 0 \implies r = 1$ on unit circle. ($j\Omega$ axis = unit circle)

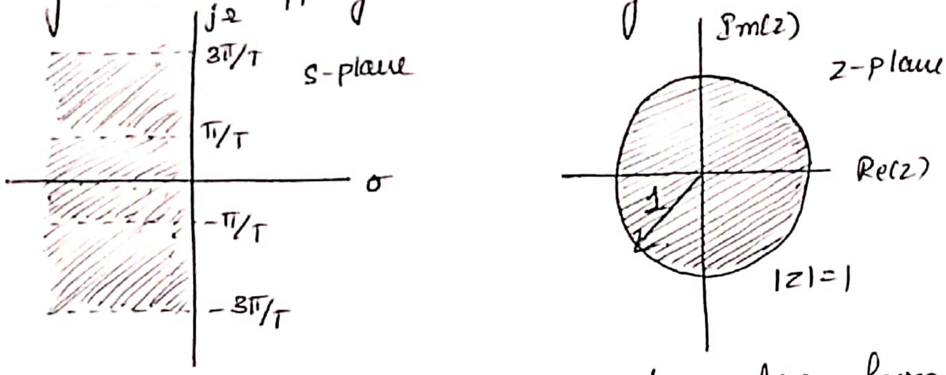
ie A stable analog filter $H_a(s)$ is transformed to a stable digital filter $H(z)$

$$\omega = \Omega T \implies \Omega = \frac{\omega}{T} \implies -\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \text{ (or) } (2q-1)\frac{\pi}{T} \le \Omega \le (2q+1)\frac{\pi}{T}$$

$$-\pi \le \omega \le \pi$$

$$\implies (2q-1)\frac{\pi}{T} \le \Omega \le (2q+1)\frac{\pi}{T} \implies q \text{ integer} \implies$$ Ω to ω is many to one.

⇒ Many to one mapping ie aliasing due to sampling.



The impulse invariant transformation from s-plane to the z-plane

→ Imaginary axis maps to the unit circle

→ Strip $\frac{2\pi}{T}$ maps the the disk.

(iii) $H(\omega)$ & $H_a(\Omega)$ are related by sampling theorem.

$H(\omega)$ ↓ freq response of digital filter

$H_a(\Omega)$ ↓ freq response of analog filter.

$$H(\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} H\left(\frac{\omega - 2\pi K}{T}\right)$$

Impulse Invariant Transformation :-

$$\frac{1}{(s+s_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds_i^{m-1}} \frac{1}{1-e^{-s_iT} z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

* The analog transfer function, $H(S) = \frac{2}{S^2+3S+2}$, determine $H(z)$ using impulse invariant transformation if (a) T = 1 seconds
(b) T = 0.1 seconds

Soln

$$H(S) = \frac{2}{S^2+3S+2} = \frac{2}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$A = \frac{2}{\cancel{(S+1)}(S+2)} \times \cancel{(S+1)} \Big|_{S=-1} = \frac{+2}{-1+2} = 2$$

$$B = \frac{2}{(S+1)(S+2)} \times (S+2) \Big|_{S=-2} = \frac{2}{-2+1} = -2$$

$$H(S) = \frac{2}{S+1} + \frac{-2}{S+2}$$

wkt Impulse invariant transformation $\frac{A_i}{S+P_i} \longrightarrow \frac{A_i}{1-e^{-P_iT}z^{-1}}$

by applying IIT on H(S), $P_1 = 1$, $P_2 = 2$

$$H(z) = \frac{2}{1-e^{-1T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}$$

(a) T = 1 seconds

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} + \frac{-2}{1-e^{-2}z^{-1}} = \frac{2}{1-0.3679z^{-1}} + \frac{-2}{1-0.1353z^{-1}}$$

$$= \frac{2(1-0.1353z^{-1}) - 2(1-0.3679z^{-1})}{(1-0.3679z^{-1})(1-0.1353z^{-1})}$$

$$= \frac{\cancel{2} - 0.2706z^{-1} - \cancel{2} + 0.7358z^{-1}}{1-0.1353z^{-1} - 0.3679z^{-1} + 0.0498z^{-2}} = \frac{0.4652z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

$$= \frac{0.4652z^{-1}}{z^{-2}(z^2-0.5032z+0.0498)}$$

$$= \frac{0.4652z}{z^2-0.5032z+0.0498}$$

(b) T = 0.1 seconds (T < 1)

$$H(z) = \frac{2}{1-e^{-0.1}z^{-1}} + \frac{-2}{1-e^{-0.2}z^{-1}} = \frac{2}{1-0.9048z^{-1}} + \frac{-2}{1-0.8187z^{-1}}$$

$$= \frac{2(1-0.8187z^{-1}) - 2(1-0.9048z^{-1})}{(1-0.9048z^{-1})(1-0.8187z^{-1})}$$

$$= \frac{\cancel{2} - 1.6374 z^{-1} - \cancel{2} + 1.8096 z^{-1}}{1 - 0.8187 z^{-1} - 0.9048 z^{-1} + 0.7408 z^{-2}}$$

$$= \frac{0.1722 z^{-1}}{1 - 1.7235 z^{-1} + 0.7408 z^{-2}}$$

$$= \frac{0.1722 z}{z^2 - 1.7235 z + 0.7408}$$

(b) is $T < 1$ i.e compute magnitude normalized transfer function $H_N(z)$.

$$H_N(z) = T \times H(z)$$
$$= 0.1 \times H(z) = 0.1 \times \frac{0.1722 z^{-1}}{1 - 1.7235 z^{-1} + 0.7408 z^{-2}}$$

$$= \frac{0.01722 z^{-1}}{1 - 1.7235 z^{-1} + 0.7408 z^{-2}}$$

$$= 0.1 \times H(z) = 0.1 \times \frac{0.1722 z}{z^2 - 1.7235 z + 0.7408}$$

$$= \frac{0.01722 z}{z^2 - 1.7235 z + 0.7408}$$

(*) Convert the analog filter with system transfer function, $H(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into a digital IIR filter by means of the IIM.

Soln

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{s+0.1}{(s+0.1)^2+3^2}$$

$$\boxed{\text{w.k.t } \frac{(s+a)}{(s+a)^2+b^2} \longrightarrow \frac{1 - e^{-aT}(\cos bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}}$$

$$H(z) = \frac{1 - e^{-0.1T}(\cos 3T) z^{-1}}{1 - 2e^{-0.1T}(\cos 3T) z^{-1} + e^{-2 \times 0.1T} z^{-2}}$$

$$T = 1 \Rightarrow H(z) = \frac{1 - e^{-0.1}(\cos 3) z^{-1}}{1 - 2e^{-0.1}(\cos 3) z^{-1} + e^{-0.2} z^{-2}}$$

$$= \frac{1 + 0.8958 z^{-1}}{1 + 1.7916 z^{-1} + 0.8187 z^{-2}}$$

$$H(z) = \frac{z^2 + 0.8958 z}{z^2 + 1.7916 z + 0.8187}$$

Bilinear Transformation

It is a conformal mapping that transforms img axis of S-plane into unit $\odot^{le}$ in z-plane only once, thus avoiding aliasing.

* used for transforming an analog filter to a digital filter
* It uses trapezoidal rule for integrating a continuous time function.
* It's defined by $S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

Let considering the derivative (1st order differential eqn of an analog sys)

$$\frac{d\,y(t)}{d(t} = x(t)$$

taking LT

$$S\,Y(S) = X(S) \longrightarrow (1)$$

* $y(t)$ with the limits $(n-1)T$ & nT
  T = Sampling period.

$$\therefore \int_{(n-1)T}^{nT} \frac{d\,y(t)}{dt}\,dt = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$y(nT) - y[(n-1)T] = \int_{(n-T)T}^{nT} x(t)\,dt$$

$$\int_{(n-1)T}^{nT} x(t)\,dt = \left[\frac{x(nT)+x[(n-1)T]}{2}\right]T$$

by trapezoidal rule. it states that if T is small the area (integral) can be approximated by the mean height of $x(t)$ b/w the 2 limits & then multiplying by the width.

$$y(nT) - y[(n-1)T] = \left[\frac{x(nT)+x[(n-1)T]}{2}\right]T$$

$$y(n) - y(n-1) = \left[\frac{x(n)+x(n-1)}{2}\right]T$$

Taking z-transform of the above eqn

$$Y(z) - z^{-1}Y(z) = \left[\frac{X(z) + z^{-1}X(z)}{2}\right]T$$

$$Y(z)\left(1-z^{-1}\right) = \left[\frac{1+z^{-1}}{2}\right]T\,X(z)$$

$$\frac{X(z)}{Y(z)} = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \longrightarrow (2)$$

by comparing eq^n 1 & 2

$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \longrightarrow 3$$

Illy transformation from z to s is

$$\frac{ST}{2} = \frac{1-z^{-1}}{1+z^{-1}}$$

$$\frac{ST}{2} + z^{-1}\frac{ST}{2} = 1 - z^{-1}$$

$$z^{-1} + z^{-1}\frac{ST}{2} = 1 - \frac{ST}{2}$$

$$z^{-1}\left(1+\frac{ST}{2}\right) = 1 - \frac{ST}{2}$$

$$z^{-1} = \frac{1-ST/2}{1+ST/2}$$

$$\boxed{Z = \frac{1+ST/2}{S-ST/2}} \longrightarrow 4$$

$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$S = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

Relation b/w Analog & digital filter poles in Bilinear transformation.

wkt $s = \sigma + j\Omega$

$z = re^{j\omega}$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega)+1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega - 1) + jr\sin\omega}{(r\cos\omega + 1) + j\sin\omega}\right]$$

$$= \frac{2}{T}\,\frac{(r\cos\omega-1)+jr\sin\omega}{(r\cos\omega+1)+jr\sin\omega} \times \frac{(r\cos\omega+1) - jr\sin\omega}{(r\cos\omega+1) - jr\sin\omega}$$

by Rationalizing.

$$= \frac{2}{T}\,\frac{r^2(\cancel{\cos^2\omega+\sin^2\omega})^{\to 1} - 1 + j2r\sin\omega}{r^2(\cancel{\cos^2\omega+\sin^2\omega})^{\to 1} + 1 + 2r\cos\omega}$$

$$= \frac{2}{T}\left[\frac{r^2-1+j2r\sin\omega}{r^2+1+2r\cos\omega}\right]$$

Separate real & imaginary.

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + \frac{j\,2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

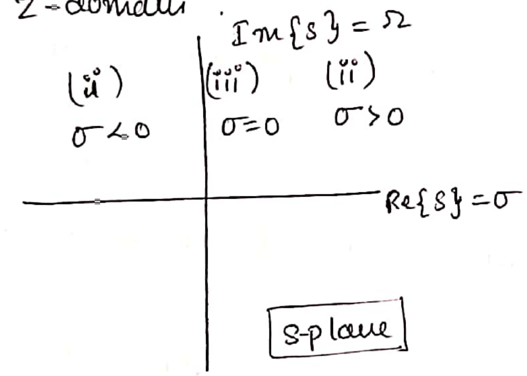
$$\therefore \quad \sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]$$

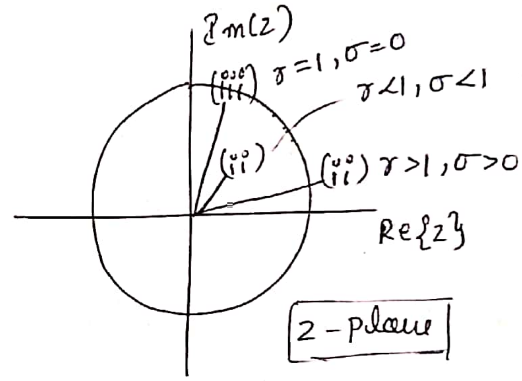
$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

(i) if $r<1 \Rightarrow \sigma<0$ ie LHS of the s-plane is mapped inside the circle $|z|=1$

(ii) if $r>1 \Rightarrow \sigma>0$ ie RHS of the s-plane is mapped outside the circle $|z|=1$

(iii) if $r=1$ we get $\sigma=0$ ie the imaginary axis in the s-domain is mapped to the circle of unit radius at $z=0$ in the z-domain





OBSERVATION :

$\Rightarrow$ Bilinear transformation preserves the stability of the transformed filter.

$\Rightarrow$ $r = 1 \Rightarrow \sigma = 0$

$$\Omega = \frac{2}{T}\left[\frac{2\sin\omega}{1+1+2\cos\omega}\right]$$

$$= \frac{2}{T}\left[\frac{\cancel{2}\sin\omega}{1+\cos\omega}\right] = \frac{2}{T}\left[\frac{\cancel{2}\sin\omega/2\,\cancel{\cos\omega/2}}{\cancel{2}\cos^2\omega/2}\right]$$

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$ analog frequency

$$\tan^{-1}\left(\frac{\Omega T}{2}\right) = \omega/2$$

$$\boxed{\therefore \quad \omega = 2\tan^{-1}\frac{\Omega T}{2}}$$ digital domain frequency.

* Digital-domain frequency ω is warped with respect to the analog frequency Ω

→ warping function ⇒ $2\tan^{-1}\left(\frac{\Omega T}{2}\right)$

→ analog frequencies $\Omega = \pm\infty$ are mapped to digital frequencies $\omega = \pm\pi$

∴ frequency mapping is not aliased.
ie Relationship b/w Ω & ω is one-to-one

∴ Bilinear transformation is adequate for all filter types.

* Frequency warping.

⇒ consider LPF having ripples in both the bands.

⇒ The frequency warping introduced by the bilinear transformation.

∴ necessary to prewarp the specifications of the analog filter.

* Design a LP Digital filter with passband & stopband. edge frequencies ω_p & ω_s.

(i) convert these frequencies to corresponding analog-domain band-edge frequencie Ω_p' & Ω_s'

$$\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

$$\Omega_s' = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

(ii) using Ω_p' & Ω_s' design the analog filter. $H_a(s)$

(iii) obtain the required Digital lowpass filter by using the bilinear transformation on the analog filter. $H_a(s)$ to $H(z)$ using $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ with $T = 1$ sec

NOTE : (i) The value of T used is immaterial as long as it is same in both (analog & digital) (T = 1 (or) 2)

(ii) Prewarping is done in the beginning of the design,
Bilinear transformation is performed in the end.

Warping & prewarping

→ Relation b/w analog frequency (Ω) & digital frequency (ω)

$$\Omega = \frac{2}{T} \tan \omega/2$$

⇒ for smaller values of ω, exist linear relationship b/w ω & Ω

⇒ for large values of ω, the relationship is non linear

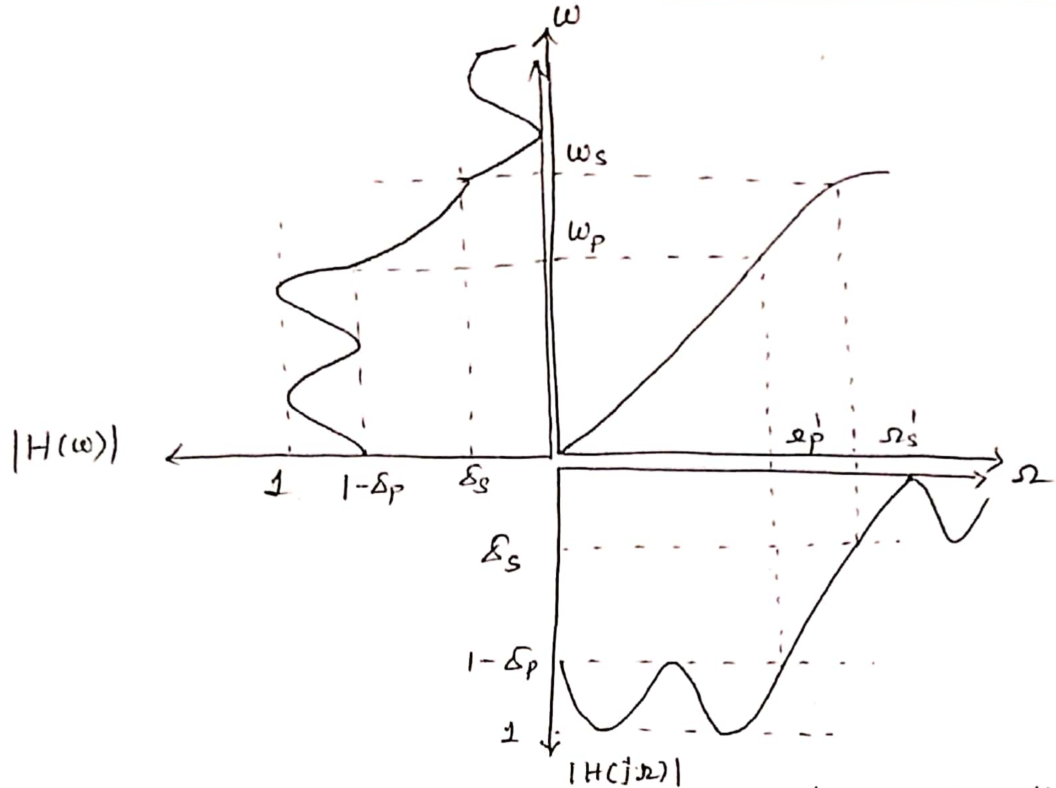
" this non-linearity introduces distortion in the frequency axis, this is known as warping effect"

⇒ This effect compresses the magnitude & phase response at high frequences.

→ Non-linear compression at high frequencies can be compensated

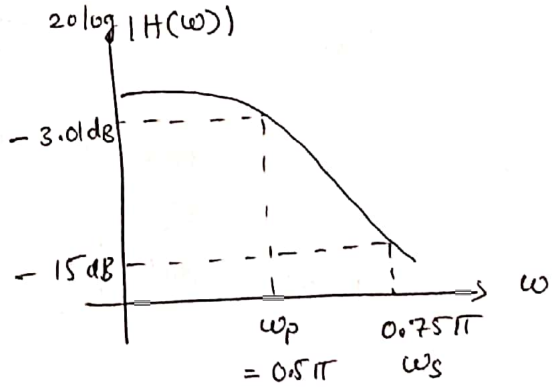
⇒ when the desired magnitude response is piece-wise constant over frequency, can be compensated by introducing a suitable prescaling (or) prewarping the critical frequencies by using the formula

$$\Omega = \frac{2}{T} \tan \omega/2$$



Mapping of an analog LPF to digital LPF via the Bilinear Transformation

* A digital LPF is required to meet the following specifications
(i) Monotonic passband & stopband.
(ii) -3.01 dB cutoff frequency of 0.5π rad.
(iii) Stopband attenuation of atleast 15 dB at 0.75π rad.
Find the system function $H(z)$ & the difference equation realization. Verify the design by checking for the passband & stopband specifications.



Sol^n

Step 1 : Prewarp

$\omega_p = 0.5\pi$ r $\omega_s = 0.75\pi$ r

$T = 1$ sec

$$\Omega'_p = \frac{2}{T}\tan(\omega_p/2) = 2\tan\left(\frac{0.5\pi}{2}\right) = 2 \text{ r/s}$$

$$\Omega'_s = \frac{2}{T}\tan(\omega_s/2) = 2\tan\left(\frac{0.75\pi}{2}\right) = 4.8282 \text{ r/s}$$

Step-2 : Analog filter design . [Butterworth]

$$N = \frac{\log\left[\frac{10^{-0.1 A_p \text{ in dB}} - 1}{10^{-0.1 A_s \text{ in dB}} - 1}\right]}{2 \log \frac{\Omega'_p}{\Omega'_s}}$$

$$= \frac{\log\left[\frac{10^{-0.1\times -3.01} - 1}{10^{-0.1\times -15} - 1}\right]}{2 \log\left[\frac{2}{4.8282}\right]} = \frac{\log\left[\frac{0.9998}{30.62}\right]}{2 \log\left(\frac{2}{4.8282}\right)}$$

$= 1.9413$

$\simeq 2$ $\boxed{\therefore N = 2}$

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$$

$$Ha(s) = H_2(s)\Big|_{s \to \frac{s}{\Omega_c}} = \frac{s}{2}$$

$\Omega_c = \Omega_{cn}\, \Omega'_p = 2.000 r/s$

(or)

$$\Omega_c = \frac{\Omega'_p}{\left[10^{-0.1 A_p \text{ in dB}} - 1\right]^{1/2N}}$$

$$= \frac{1}{\left(\frac{s}{2}\right)^2 + \sqrt{2}\left(\frac{s}{2}\right) + 1}$$

$$Ha(s) = \frac{4}{s^2 + 2\sqrt{2}\, s + 4}$$

Step-3 : obtain $H(z)$ by bilinear transformation $(T=1)$

$$H(z) = Ha(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)^2 + 2\sqrt{2}\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right) + 4}$$

$$= \frac{(1+z^{-1})^2}{3.4142 + 0.5858\, z^{-2}}$$

Verification : $z = re^{j\omega}$ $r = 1$

$$H(z) = H(e^{j\omega}) = H(\omega) = \frac{(1+e^{-j\omega})^2}{3.4142 + 0.5858\, e^{-j2\omega}}$$

$$= \frac{(1 + \cos\omega - j\sin\omega)^2}{3.4142 + 0.5858\cos 2\omega - j0.5858\sin 2\omega}$$

$$|H(z)| = \frac{(1+\cos\omega)^2 + \sin^2\omega}{\sqrt{(0.5858\cos 2\omega + 3.4142)^2 + (0.5858\sin 2\omega)^2}}$$

$\omega = 0.5\pi$

* Convert the analog filter with system function H(s) into digital filter using bilinear transformation $H(s) = \frac{s+0.3}{(s+0.3)^2+16}$, $T=0.5$

Sol:
$$H(s) = \frac{s+0.3}{(s+0.3)^2+16} = \frac{s+0.3}{s^2+0.6s+0.09+16}$$

$$= \frac{s+0.3}{s^2+0.6s+16.09}$$

Bilinear transformation to get H(z) from H(s)

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{2/T\,\frac{1-z^{-1}}{1+z^{-1}}+0.3}{\left(\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)^2+0.6\left(\frac{2(1-z^{-1})}{T(1+z^{-1})}\right)+16.09}$$

$$= \frac{2/T\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.3}{\frac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2}+\frac{1.2(1-z^{-1})}{T(1+z^{-1})}+16.09}$$

$$= \frac{\frac{2(1-z^{-1})+0.3T(1+z^{-1})}{\cancel{T(1+z^{-1})}}}{\frac{4(1-z^{-1})^2+1.2(1-z^{-1})T(1+z^{-1})+16.09T^2(1+z^{-1})}{T^{2}(1+z^{-1})^{2}}}$$

$$= \frac{\left[2(1-z^{-1})+0.3T(1+z^{-1})\right](T(1+z^{-1})}{4(1-z^{-1})^2+1.2T(1-z^{-2})+16.09T^2(1+z^{-1})}$$

$$\boxed{T=0.5}$$

$$H(z) = \frac{\left[2(1-z^{-1})+0.3\times0.5(1+z^{-1})\right](0.5(1+z^{-1})}{4(1-z^{-1})^2+1.2\times0.5(1-z^{-2})+16.09(0.5)^2(1-z^{-1})}$$

$$= \frac{1(1-z^{-1})(1+z^{-1})+0.075(1+z^{-1})(1+z^{-1})}{4(1-2z^{-1}+z^{-2})+0.6(1-z^{-2})+4.0225(1+2z^{-1}+z^{-2})}$$

$$= \frac{1-\cancel{z^{-1}}+\cancel{z^{-1}}-z^{-2}+0.075(1+2z^{-1}+z^{-2})}{4-8z^{-1}+4z^{-2}+0.6-0.6z^{-2}+4.0255+8.0450z^{-1}+4.0225z^{-2}}$$

$$= \frac{1.075 + 0.15z^{-1} - 0.925z^{-2}}{8.6225 + 0.045z^{-1} + 7.4225z^{-2}}$$

$$= \frac{\frac{1.075}{8.6225} + \frac{0.15}{8.6225}z^{-1} - \frac{0.925}{8.6225}z^{-2}}{1 + \frac{0.045}{8.6225}z^{-1} - \frac{7.4225}{8.6225}z^{-2}}$$

$$\boxed{H(z) = \frac{0.1247 + 0.0174z^{-1} - 0.1073z^{-2}}{1 + 0.0052z^{-1} + 0.8608z^{-2}}}$$

$$\boxed{H(z) = \frac{0.1247z^2 + 0.0174z - 0.1073}{z^2 + 0.0052z + 0.8608}}$$

(*) Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1 second, to satisfy the following specifications.

$$0.707 \leq |H(e^{j\omega})| \leq 1.0 \;; \text{ for } 0 \leq \omega \leq 0.3\pi$$

$$|H(e^{j\omega})| \leq 0.2 \;; \text{ for } 0.75\pi \leq \omega \leq \pi$$

Soln:

$\omega_p = 0.3\pi$ r/samples

$\omega_s = 0.75\pi$ r/samples

$A_p = 0.707$

$A_s = 0.2$

$T = 1$ second (Sampling time period)

for Impulse Invariant transformation.

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.3\pi}{1} = 0.9425 \text{ rad/second}$$

$$\Omega_s = \frac{\omega_s}{T} = \frac{0.75\pi}{1} = 2.3562 \text{ rad/second}.$$

$$N = \frac{\log\left[\frac{1/A_s^2 - 1}{1/A_p^2 - 1}\right]}{2\log(\Omega_s/\Omega_p)} = \frac{\log\left[\frac{1/0.2^2 - 1}{1/0.707^2 - 1}\right]}{2\log\left(\frac{0.9425}{2.3562}\right)} = 1.7339$$

ie $\boxed{N = 2}$

$K = \frac{N}{2} = \frac{2}{2} = 1$

$= \frac{4}{2} = 2.1$

WKT

$$H(s) = \frac{1}{s^2 + 1.4142s + 1}$$

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(*) Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.1 second, to satisfy the following specification.

Passband ripple ≤ 4.436 dB

Stop —"— attenuation ≥ 20dB

$\omega_p = 0.35\pi$ r/samples

$\omega_s = 0.7\pi$ r/sample

Sol^n for bilinear transformation

$$\Omega_p = \frac{2}{T} \tan \omega_p/2$$

$$= \frac{2}{0.1} \tan\left(\frac{0.35\pi}{2}\right) = \cancel{\frac{2}{0.1} \tan}$$

$$= 12.2560 \text{ r/seconds}$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) = \frac{2}{0.1} \tan\left(\frac{0.7\pi}{2}\right) = 39.2522 \text{ r/second.}$$

$$N = \frac{\log\left[\dfrac{10^{-0.1\,Ap\text{ in dB}} - 1}{10^{-0.1\,As\text{ in dB}} - 1}\right]}{2 \log \dfrac{\Omega_s}{\Omega_p}} = \frac{\log\left[\dfrac{10^{-0.1\times-4.436} - 1}{10^{-0.1\times-20} - 1}\right]}{2\log\left(\dfrac{39.2522}{12.2560}\right)}$$

$$= 1.7267 = 2$$

ie $\boxed{N=2}$

$$\therefore H(s) = \frac{1}{s^2 + 1.4142 s + 1}$$

$$H(s)\Big|_{s = s/\Omega_c} \Rightarrow \frac{\Omega_s}{\left[10^{-0.1\,As\text{ in dB}} - 1\right]^{1/2N}} \;\; \cancel{=\Omega_P} = \Omega_c = \frac{39.2522}{\left[10^{-0.1\times-20} - 1\right]^{1/4}} = 12.2560 \text{ r/s}$$

$$= 12.4439 \text{ rad/sec}$$

$= 10.6149$

$$H(s)\Big|_{s \to \frac{s}{12.4439}} = \frac{1}{\left(\dfrac{s}{12.4439}\right)^2 + 1.4142\left(\dfrac{s}{12.4439}\right) + 1}$$

$$= \frac{154.8506 \quad (112.6769)}{s^2 + 17.5982 s + 154.8506}$$

$s^2 + 15.0115 s + 112.6769$