DIGITAL COMMUNICATION SYSTEM
18EC5DCDCS

COURSE OBJECTIVES:

- To provide knowledge about source coding techniques.
- Analyze the channel capacity using Shannon’s theorem.
- Introduce error control coding techniques.
- Get knowledge of theoretical aspects of detection of the transmitted data from the noisy received data.
- Analyze the generation and detection of basic digital modulation techniques and arrive at the respective probability of errors and PSD.
- Develop communication system for specific application by taking example of Spread spectrum communication

COURSE OUTCOMES

- Apply the concept of Probability to measure information and capacity of channel.
- Apply various channel coding techniques for error detection and correction.
- Recognize the theoretical models, ideal receivers for the detection of the message from noisy received signal.
- Analyze transmitter and receiver blocks for binary and quaternary modulation techniques to arrive at probability of error and Power Spectral Density.
- Interpret Spread spectrum concepts, its types and applications.
- Work in a team to simulate digital communication systems for different applications.

MODULE 1

INFORMATION THEORY:

- Digital Communication block diagram
- Information
- Entropy

SOURCE ENCODING:

- Shannon’s Encoding Algorithm
- Huffman coding

CHANNEL

- Discrete memoryless channels
- Binary Symmetric Channel
- Channel capacity
- Shannon Hartley Theorem and its implications.

Prerequisite

- Knowledge of subjects like :

  Basics of Probability Theory

  Signals and Systems

  Analog communication.

<u>Self Steady Components:</u>

Module 1 :	**Mutual Information and its Properties, extension of source, other discrete channels, Simulation**
Module 2 :	Types of Errors, Methods of Controlling Errors, Simulation.
Module 3 :	Matched filter for RF pulse, Simulation
Module 4 :	Simulation of digital modulation techniques: ASK, FSK, PSK, QPSK, DPSK
Module 5 :	Construction of state Diagrams, Counter Design and Maximum length sequence generation by simulation.

TEXT BOOKS :

	Simon Haykin, “Digital communication”, ISBN-9971-51-205-X, John Wiley & Sons (Asia), Pvt. Ltd, 2008
	K. Sam Shanmugam, “Digital and analog communication systems”, John Wiley India Pvt. Ltd, 1996.
	Shu Lin, Daniel J Costello Jr., “Error Control Coding”, Pearson Education Asia, Second Edition, 2011.
	John. G.Proakis, “Communication Systems Engineering”, 2nd Edition, Pearson.
	John. G. Proakis, Masoud Salehi, “Digital Communications”, Mac Graw Hill, 2008

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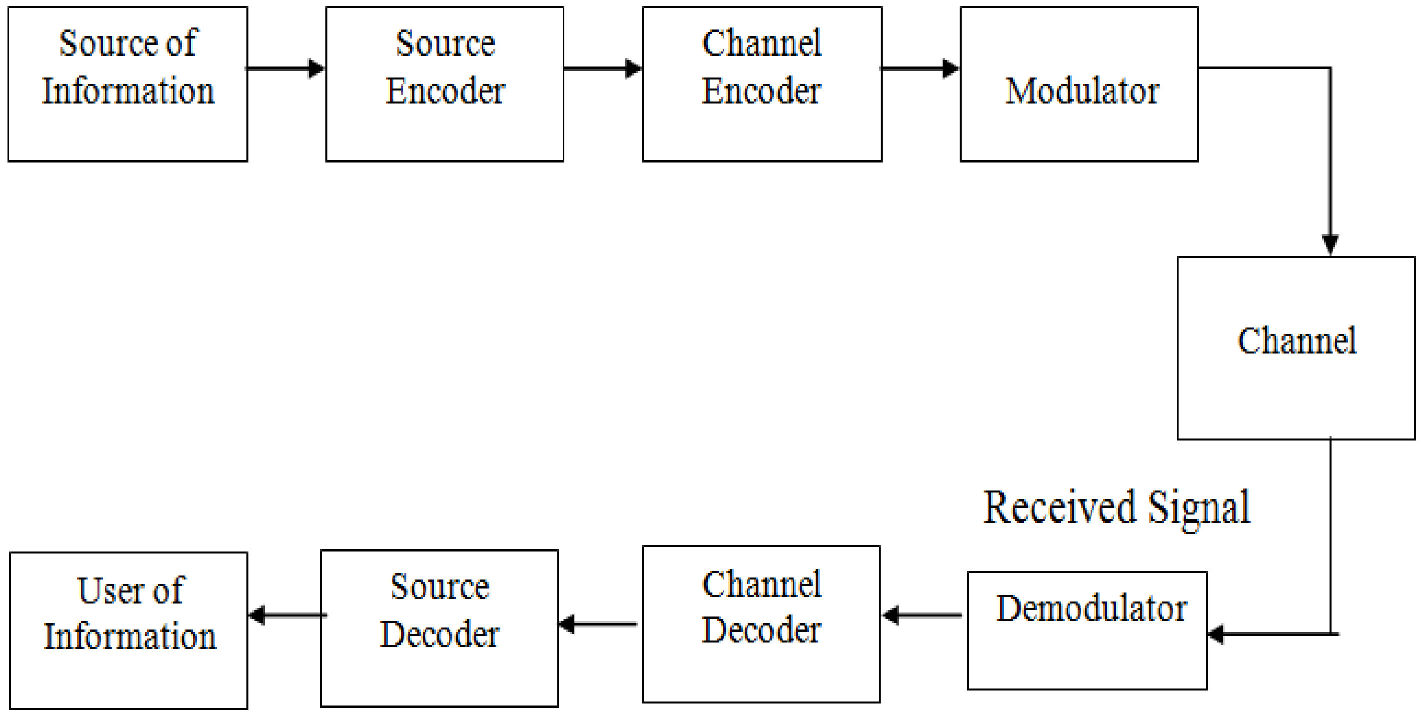
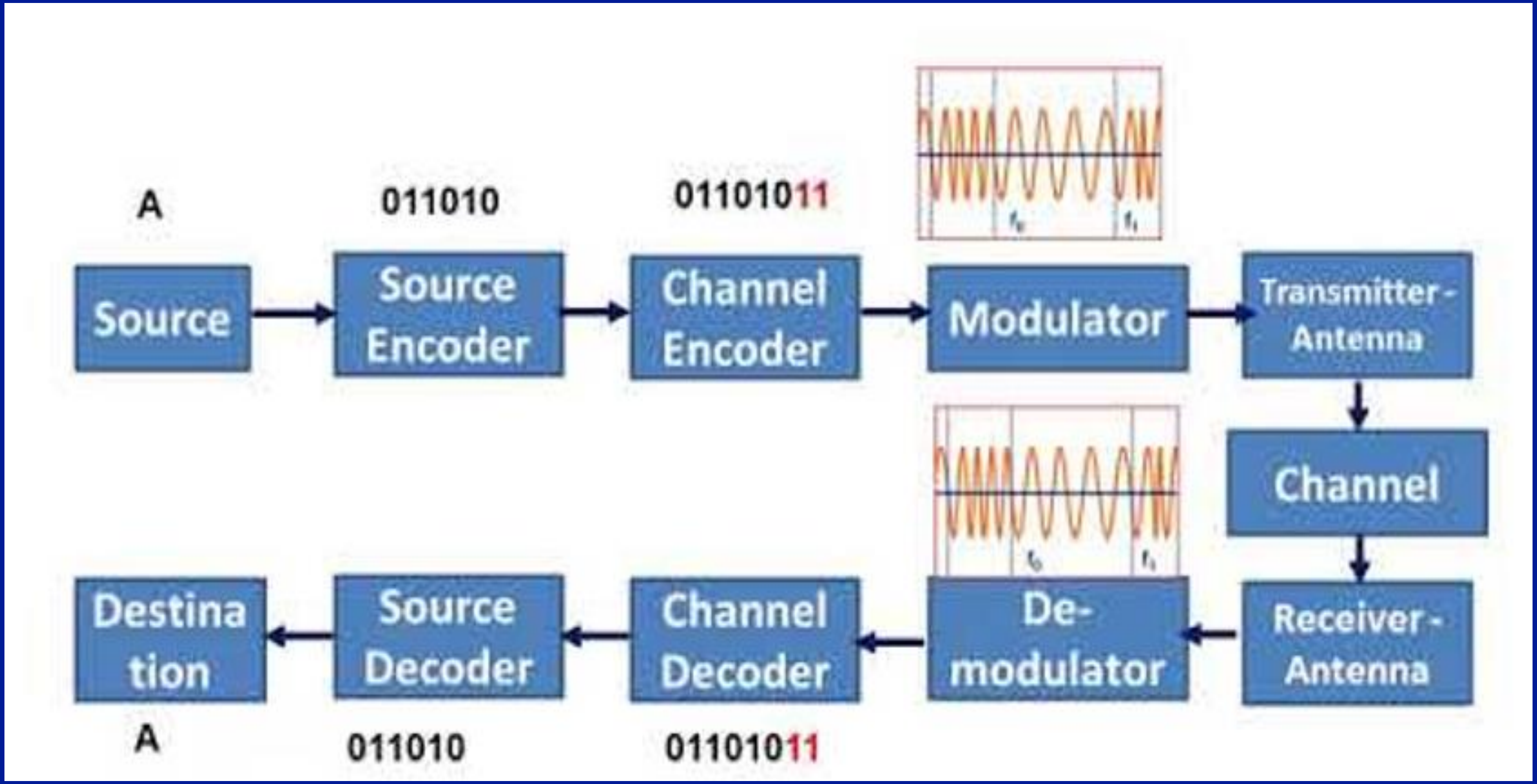
	Simon Haykin, “Digital and Analog Communication”, John Wiley, India Pvt. Ltd., 2008.
	Bernard Sklar, “Digital Communication”, Pearson Education, 2007.
	John G. Proakis , Masoud Salehi, Gerhard Bauch, “Contemporary Communication Systems Using MATLAB”, 3rd Edition.
	B.P. Lathi, “Modern digital and analog Communication systems”, Oxford University Press, 4thedn., 2010.
	Ranjan Bose, “ ITC and Cryptography”, TMH, 2nd edition, 2007

Digital communication system



Digital communication system



Information Source:

- The source of information can be analog or digital.
- Ex: Analog-Audio or video signal
- Video-Teletype signal.

Source Encoder :

- Converting the output of whether analog or digital source into a sequence of binary digits.
- The source decoder converts the binary output of the channel decoder into a symbol sequence.
- Aim : remove the redundancy in the transmitting information, so that bandwidth required for transmission is minimized.

Channel encoder:

- Introduce some redundancy in the binary information sequence.
- Ex: Let k be the information sequence and map that k bits to unique n bit sequence called code word.
- These extra bits do not convey any information but helps the receiver to detect and / or correct some of the errors in the information bearing bits.
- The Channel decoder recovers the information bearing bits from the coded binary stream
- Error detection and possible correction is also performed by the channel decoder.

Modulator Demodulator

- Efficient transmission of the signal.
- It operates by keying shifts in the amplitude, frequency or phase of a sinusoidal carrier wave to the channel encoder output.
- The digital modulation techniques are referred to as amplitude- shift keying, frequency- shift keying or phase-shift keying respectively.
- converts the input bit stream into an electrical waveform suitable for transmission over the communication channel.

- The detector performs demodulation, thereby producing a signal the follows the time variations in the channel encoder output.

Information

- Information theory, originally known as mathematical theory of communication, deals with mathematical modelling and analysis of a communication system rather than with physical sources and physical channels.

- The output of a discrete information source is a message that consists of a sequence of symbols.

- Information of an event depends only on its probability of occurrence and is not dependent on its content.

- The randomness of happening of an event and the probability of its prediction as a news is known as information.

INTRODUCTION

- The discrete information source consist of a discrete set of letters or symbol. A message emitted by a source consist of sequence of symbols.
- Every message coming out of the source contains some information.
- However the information contained by the message varies from one another. Some message convey more information than others.
- In order to identify the information content of the message and the average information content of symbols in messages 'measure of information' is necessary.

BASICS OF PROBABILITY:

* The probability of an event A is a non negative number between 0 and 1.

$$0 \leq P(A) \leq 1$$

* Probability of a sure event is equal to unity.

$$P(S) = 1 \,, \quad S \longrightarrow \text{sample space}$$

* Probability of a null event is equal to zero.

$$P(\phi) = 0.$$

* For any two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A + B) = P(A) + P(B) - P(AB)$$

* If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

* $P(A) + P(\bar{A}) = 1$

* $\sum_i P_i = P_1 + P_2 + \cdots + P_n = 1$

* Conditional Probability:

The probability of some event B occuring given that some other event A has already occured is given by

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P\left(\frac{B}{A}\right) P(A) = P\left(\frac{A}{B}\right) \cdot P(B)$$

Measure of Information

Let $S = \{S_1, S_2, S_3 \ldots S_q\}$ be the symbols with probabilities $P_1, P_2, P_3 \ldots P_q$ such that $P_1 + P_2 + P_3 + \ldots + P_q = 1$

i.e., $\sum_{i=1}^{q} P_i = 1$

Let S_K be the symbol chosen for transmission with a probability P_K. Suppose that a receiver correctly identifies the messages, then amount of information or self information is given by

$$I_K = \log_2 \left(\frac{1}{P_K} \right)$$

If the base of the log is 2, then unit is 'bits'. If the base is 10, unit is 'Hartleys' / 'Decits'. If base is 'e', unit is 'Nats'. (Bits → BInary uniTS)

Reason for using log

- The self information cannot be negative
- The lowest value for self information is zero, since the prob of the certain event is equal to one.
- More information is carried by the message which is less likely one or rare. The message associated with an event least likely to occur contains more information.
- The total amount of self information conveyed should be sum of individual information

BASIC LOGARITHMIC FORMULAE:

1. If 'n' and 'a' are positive real numbers, and $a \neq 1$, then if $a^x = n$, then $\log_a n = x$.
   (definition of logarithm)
2. log of 1 to any base is zero.
   i.e., $\log_b 1 = 0$
3. log of any number to the base as itself is 1.
   i.e., $\log_a a = 1$.
4. $\log_a (pq) = \log_a p + \log_a q$
5. $\log_a (p/q) = \log_a p - \log_a q$
6. $\log_a p^n = n \log_a p$
7. $a^{(\log_a p)} = p$
8. $\log_a n = \log_b n \times \log_a b$
9. $\log_a b = \dfrac{\log_n b}{\log_n a}$
10. $\dfrac{1}{\log_b a} = \log_a b$

Problem -1

The binary symbols 0's and 1's are transmitted with probabilities 1/4 and 3/4 respectively. Calculate the amount of information.

The binary symbols 0's and 1's are transmitted with probabilities 1/4 and 3/4 respectively. Calculate the amount of information.

$P(0) = P_0 = 1/4 \quad ; \quad P(1) = P_1 = 3/4$

$I_0 = \log_2 \left(\frac{1}{P_0}\right) = \log_2 \left(1/\tfrac{1}{4} \right) = \log_2 4 = 2$

$\therefore \underline{\underline{I_0 = 2 \text{ bits}}}$

$I_1 = \log_2 \left(\frac{1}{P_1}\right) = \log_2 \left(\frac{1}{3/4}\right) = \log_2 \left(\frac{4}{3}\right) = 0.415$

$\therefore I_1 = \underline{\underline{0.415 \text{ bits}}}$

Relation between Bits, Hartleys and Nats

Consider 3 equations given by

$$I = \log_{10}(1/p) \quad \text{Hartley} \quad \rightsquigarrow (1)$$

$$I = \log_{e}(1/p) \quad \text{Nats} \quad \rightsquigarrow (2)$$

$$I = \log_{2}(1/p) \quad \text{Bits} \quad \rightsquigarrow (3)$$

From eq^n (1) and (2),

$$\log_{10}(1/p) \cdot \text{Hartley} = \log_e(1/p) \cdot \text{Nats}$$

$$\therefore 1 \text{ Hartley} = \frac{\log_e(1/p)}{\log_{10}(1/p)} \cdot \text{Nats} = \frac{-\log_e p}{-\log_{10} p} = \frac{\log_p 10}{\log_p e}$$

$$1 \text{ Hartley} = \log_e 10 \text{ Nats} \quad \left| \because \frac{1}{\log_b a} = \log_a b \right.$$

$$\boxed{\begin{aligned} &1 \text{ Hartley} = 2.303 \text{ Nats} \\ &1 \text{ Nat} = \frac{1}{2.303} \text{ Hartley} = 0.434 \text{ Hartley} \end{aligned}}$$

From (1) and (3),

$$\log_{10}(1/p) \text{ Hartley} = \log_2(1/p) \text{ Bits}$$

$$\therefore \text{Hartley} = \frac{\log_2(1/p)}{\log_{20}(1/p)} \text{ Bits}$$

$$= \log_2 10$$

$$= 3.32 \text{ bits}$$

$$\boxed{\begin{aligned} 1 \text{ Hartley} &= 3.32 \text{ bits} \\ 1 \text{ bit} &= \frac{1}{3.32} \text{ Hartley} \\ 1 \text{ Bit} &= 0.301 \text{ Hartley} \end{aligned}}$$

From (2) and (3),

$$\log_e (1/p) \text{ Nats} = \log_2 (1/p) \text{ Bits}$$

$$1 \text{ Nat} = \frac{\log_2 (1/p)}{\log_e (1/p)} \text{ Bits}$$

$$\boxed{\begin{aligned} 1 \text{ Nat} &= \log_2 e = 1.443 \text{ bits} \\ 1 \text{ bit} &= 1/1.443 = 0.693 \text{ Nats} \end{aligned}}$$

Average Information content(Entropy) of symbols in long independent sequences

Consider message Length N

- p_1N number of messages of type s_1 contains $p_1 N \log_2\left(\frac{1}{p_1}\right)$bits
- p_2N number of messages of type s_2 contains $p_2 N \log_2\left(\frac{1}{p_2}\right)$bits.
- .
- .
- $I_{total} = p_1 N \log_2\left(\frac{1}{p_1}\right) + p_2 N \log_2\left(\frac{1}{p_2}\right) + \cdots \ldots + p_q N \log_2\left(\frac{1}{p_q}\right)$

$$\boldsymbol{I_{total} = N \sum_{i=1}^{q} p_i \, log_2\left(\frac{1}{p_i}\right)} \textbf{ bits}$$

- The average information *per* symbol is obtained by dividing the total information content of the message by the number of symbols in the message

$$Entropy = H(s) = \frac{I_{total}}{N} = \sum_{i=1}^{q} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits/symbol}$$

The average information content per symbol is called the source entropy

Average information rate:

If the symbols are emitted by source at a fixed time rate r_s,
then the average information rate R_s is given by

$$R_s = r_s * H(\mathrm{s})\ \text{bits/sec}$$

bits/sec Symbol/sec bits/symbol

Problem 2

Consider a source S = {S1, S2, S3} with probabilities {0.5, 0.25, 0.25}. Find

i) Self information of each message
ii) Entropy

Problem 2

Sol^n: i) $I_1 = \log_2\left(\frac{1}{P_1}\right) = \log_2\left(\frac{1}{0.5}\right) = \underline{\underline{1 \text{ bit}}}$

$I_2 = \log_2\left(\frac{1}{P_2}\right) = \log_2\left(\frac{1}{0.25}\right) = \underline{\underline{2 \text{ bits}}}$

$I_3 = \log_2\left(\frac{1}{P_3}\right) = \log_2\left(\frac{1}{0.25}\right) = \underline{\underline{2 \text{ bits}}}$

ii) Entropy $= H(s) = \sum_{i=1}^{q} P_i \log_2\left(\frac{1}{P_i}\right)$

$= 0.5 \log_2\left(\frac{1}{0.5}\right) + 0.25 \log_2\left(\frac{1}{0.25}\right) + 0.25 \log_2(4)$

$= 0.5 \times 1 + 0.25 \times 2 + 0.25 \times 2$

$H(s) = \underline{\underline{1.5}}$ bits/symbol

Problem 3

Q> The collector voltage of a circuit is to lie between -5 and -12. The voltage can take on only these values -5, -6, -7, -9, -10, -11, -12 volts with respective probabilities $\{1/6, 2/6, 1/12, 1/12, 1/12, 1/6, 1/12\}$. This voltage is recorded with a pen recorder. Determine the average self info associated with the record in terms of bits/level.

Solution 3

$$H(S) = \sum_{i=1}^{q} P_i \log_2\left(\frac{1}{P_i}\right) = \sum_{i=1}^{7} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$H(S) = \frac{1}{6}\log_2(6) + \frac{2}{6}\log_2\left(\frac{6}{2}\right) + \frac{1}{12}\log_2(12) + \frac{1}{12}\log_2(12) + \frac{1}{12}\log_2(12)$$

$$+ \frac{1}{6}\log_2(6) + \frac{1}{12}\log_2(12)$$

$$= \frac{2}{6} \times 2.585 + \frac{4}{12} \times 3.585 + \frac{2}{6} \times 1.585$$

$$= 2.585$$

$$H(S) = \underline{2.585} \text{ bits/level}$$

Problem 4

Q5> The output of an information source consists of 128 bits, 16 of which occur with probability 1/32 and remaining 112 occur with probability 1/224. The source emits 1000 symbols/sec. Assuming that symbols are chosen independently, find the average information rate of this source.

Solution 4

Sol^n :

$$H(S) = \sum_{i=1}^{128} P_i \log\left(\frac{1}{P_i}\right)$$

$$= 16 \times \frac{1}{32} \log_2(32) + 112 \times \frac{1}{224} \log_2(224)$$

$$= \frac{1}{2} \times 5 + \frac{1}{2} \times 7.807$$

$$H(S) = \underline{6.4035} \text{ bits/symbol}$$

Given $r_s = 1000$ symbols/sec

$$\therefore R_s = 1000 \times 6.4035$$

$$R_s = \underline{6403.5} \text{ bps}$$

A card is drawn from a deck of playing cards.

i) You are informed that the card you draw is spade. How much information did you receive in bits?

ii) How much information did you receive if you are told that the card you drew is an ace?

iii) How much information did you receive if you are told that the card you drew is an ace of spades?

iv) Is the information content of the message “ace of spades” the sum of the information contents of the messages ”spade” and “ace”?

Basic concept on drawing a card:

In a pack or deck of 52 playing cards, they are divided into 4 suits of 13 cards each i.e. spades ♠ hearts ♥, diamonds ♦, clubs ♣.

Cards of Spades and clubs are black cards.

Cards of hearts and diamonds are red cards.

The card in each suit, are ace, king, queen, jack or knaves, 10, 9, 8, 7, 6, 5, 4, 3 and 2.

King, Queen and Jack (or Knaves) are face cards. So, there are 12 face cards in the deck of 52 playing cards.

In a deck, there are 13 cards of spade and 4 aces.

In total, there are 52 cards.

Probability of getting a spade $= \frac{13}{52} = \frac{1}{4}$ (Let it be event A)

Probability of getting an ace $= \frac{4}{52} = \frac{1}{13}$ (Let it be event B)

Probability of getting an ace of spade $= \frac{1}{52}$ (event C)

Here event 'C' has the least probability of occurance followed by B and A. Let us find the information content of each event.

a> $I_A = \log_2 \left(\frac{1}{P_A}\right) = \log_2 (4) = 2 \text{ bits} \longrightarrow (1)$

b> $I_B = \log_2 \left(\frac{1}{P_B}\right) = \log_2 (13) = 3.7 \text{ bits} \longrightarrow (2)$

c> $I_c = \log_2 \left(\frac{1}{P_c}\right) = \log_2 (52) = 5.7 \text{ bits} \longrightarrow (3)$

From results (1), (2) and (3), we can infer that the message with least probability of occurance contains more information.

Also, d> $I_c = I_A + I_B$

$\therefore \quad I_{\text{Ace of spade}} = I_{\text{spade}} + I_{\text{ace}}$

Solution 5

Solⁿ: In a deck, there are 13 cards of spade and 4 aces.

In total, there are 52 cards.

$\therefore$ Probability of getting a spade $= \frac{13}{52} = \frac{1}{4}$ (Let it be event A)

Probability of getting an ace $= \frac{4}{52} = \frac{1}{13}$ (Let it be event B)

Probability of getting an ace of spade $= \frac{1}{52}$ (event C)

Here event 'C' has the least probability of occurrance followed by B and A. Let us find the information content of each event.

a) $I_A = \log_2 \left(\frac{1}{P_A}\right) = \log_2(4) = \underline{2 \text{ bits}} \longrightarrow (1)$

b) $I_B = \log_2 \left(\frac{1}{P_B}\right) = \log_2(13) = \underline{3.7 \text{ bits}} \longrightarrow (2)$

c) $I_C = \log_2 \left(\frac{1}{P_C}\right) = \log_2(52) = \underline{5.7 \text{ bits}} \longrightarrow (3)$

From results (1), (2) and (3), we can infer that the message with least probability of occurrance contains more information.

Also, d) $I_C = I_A + I_B$

$\therefore \quad I_{\text{Ace of spade}} = I_{\text{spade}} + I_{\text{ace}}$

Problem 6

6) The international Morse code uses a sequence of dot & dash to transmit the letters of english alphabet. The dash is represented by a current pulse that has a duration of 3 units and the dot has a duration of 1 unit. The probability of occurance of a dash is 1/3 of the probability of occurance of a dot.

a) Calculate information content of a dot and dash.

b) Calculate average information in dot-dash code.

c) Assume that the dot last for 10 msec which is the same time interval as the pause between the symbols. Find average rate of information in transmission

Sol^n: Wkt $P_{dot} + P_{dash} = 1 \longrightarrow (1)$

Given $\frac{1}{3} P_{dot} = P_{dash} \Rightarrow P_{dot} = 3 P_{dash}$

$\therefore P_{dot} + \frac{1}{3} P_{dot} = 1$

$P_{dot} = \frac{3}{4} = 0.75$ and $P_{dash} = 1 - 0.75 = 0.25$

a) $I_{dot} = \log_2\left(\frac{1}{P_{dot}}\right) = \log_2\left(\frac{4}{3}\right) = \underline{\underline{0.415}}$ bits

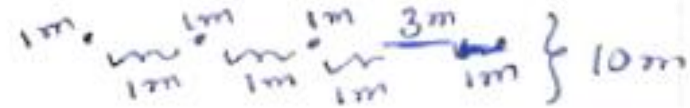
$I_{dash} = \log_2\left(\frac{1}{P_{dash}}\right) = \log_2(4) = \underline{\underline{2}}$ bits

b) $H(S) = \sum_{i=1}^{2} P_i \log_2\left(\frac{1}{P_i}\right)$

$= \frac{3}{4}\log_2\left(\frac{4}{3}\right) + \frac{1}{4}\log_2(4)$

$= \underline{\underline{0.81125}}$ bits/symbol

c) In every 4 symbols, 3 will be dots ($P_{dot} = 3/4$) and 1 will be dash ($P_{dash} = 1/4$)



$\therefore$ Totally 4 symbols will occur in 10 m sec

$$r_s = \frac{4}{10m} = 400 \text{ symbols/sec}$$

$$R_s = r_s H(s) = 400 \times 0.81125$$

$$R_s = \underline{\underline{324.5}} \text{ bits/sec}$$

A code is composed of dots and dashes. Assume that the dash is 3 times as long as the dot and has one-third the probability of occurrence. (i) Assess the information in dot and that in a dash; (ii) Estimate the average information in dot-dash code; and (iii) Assume that a dot lasts for 1 ms and this same time interval is allowed between symbols. Examine the average rate of information of transmission.

average rate of information in transmission.

Sol^n: Wkt $P_{dot} + P_{dash} = 1 \longrightarrow (1)$

Given $\frac{1}{3} P_{dot} = P_{dash} \implies P_{dot} = 3 P_{dash}$

$\therefore P_{dot} + \frac{1}{3} P_{dot} = 1$

$P_{dot} = \frac{3}{4} = 0.75$ and $P_{dash} = 1 - 0.75 = 0.25$

a) $I_{dot} = \log_2\left(\frac{1}{P_{dot}}\right) = \log_2\left(\frac{4}{3}\right) = \underline{\underline{0.415 \text{ bits}}}$

$I_{dash} = \log_2\left(\frac{1}{P_{dash}}\right) = \log_2(4) = \underline{\underline{2 \text{ bits}}}$

b) $H(S) = \sum_{i=1}^{2} P_i \log_2\left(\frac{1}{P_i}\right)$

$= \frac{3}{4} \log_2\left(\frac{4}{3}\right) + \frac{1}{4} \log_2(4)$

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$r_s = \frac{4}{10m} = 400$ symbols/sec

$R_s = r_s H(s) = 400 \times 0.81125$

$R_s = \underline{\underline{324.5}}$ bits/sec

PROPERTIES OF ENTROPY:

1. Entropy is average self information. It is given by

$$H(S) = \sum_{i=1}^{q} P_i \log(1/P_i)$$

$$H(S) = \sum_{i=1}^{q} P_i I_i \quad \text{bits/symbol}$$

where q represents number of symbols and also $\sum_{i=1}^{q} P_i = 1$.

2. For null event and sure event, the entropy vanishes.
3. The entropy is a symmetrical function of its arguments. The value of $H(S)$ remains the same irrespective of location of probabilities.

4. Extremal property: When all the source symbols become equiprobable, then the entropy attains the maximum value.

$$H(s)_{max} = \log_2 q.$$

5. When source symbols are not equiprobable, then entropy is less than maximum value.

6. The source efficiency, η_s is given by

$$\eta_s = \frac{H(S)}{H(S)_{max}}$$

7. The source redundancy R_{η_s} is given by

$$R_{\eta_s} = 1 - \eta_s$$

Usually efficiency and redundancy are represented in percentage.

PROPERTIES OF ENTROPY:

1. Entropy is average self information. It is given by

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$$R_{\eta_s} = 1 - \eta_s$$

Usually efficiency and redundancy are represented in percentage.

Q) A binary source is emitting an independent sequence of 0's and 1's with probabilities P and (1-P) respectively. Plot the entropy of the source versus P.

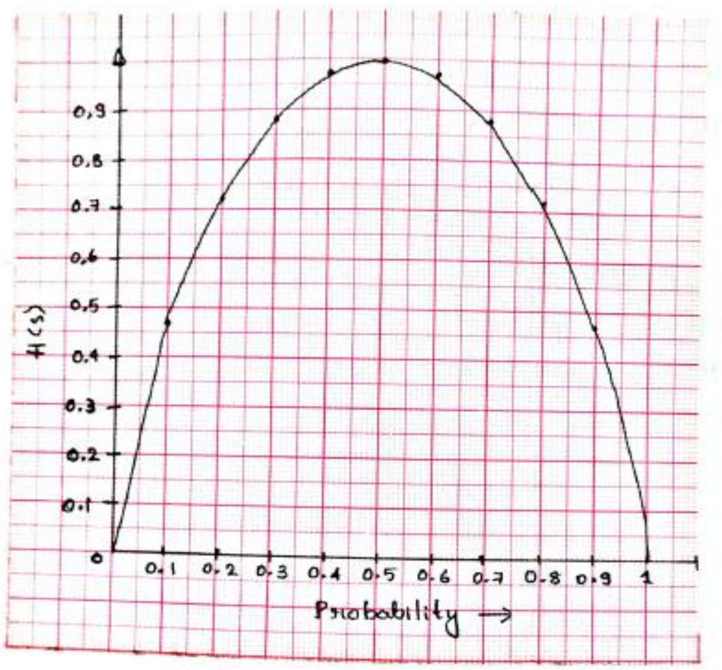
Sol^n: The entropy of the binary source is given by

$$H(S) = \sum_{i=1}^{2} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$H(S) = P \log_2\left(\frac{1}{P}\right) + (1-P)\log_2\left(\frac{1}{1-P}\right) \longrightarrow (1)$$

Type eq^n (1) in calculator and 'calculate' the values of H(s) for assumed values of P. P should vary from 0.1 to 1

P	H(s)
0.1	0.469
0.2	0.722
0.3	0.881
0.4	0.971
0.5	1
0.6	0.971
0.7	0.881
0.8	0.722
0.9	0.469
1.0	–



Problem

A discrete message source 's' emits two independent symbols x and y with probability 0.55 and 0.45 respectively. Calculate the efficiency of the source and its redundancy.

$\underline{Sol^n}$: $P(X) = P_x = 0.55$; $P(y) = P_y = 0.45$

$$H(S) = \sum_{i=1}^{2} P_i \log 1/P_i$$

$$= 0.55 \log \left(\frac{1}{0.55}\right) + 0.45 \log \left(\frac{1}{0.45}\right)$$

$$H(S) = 0.9928 \text{ bits/message symbol}$$

$$H(S)_{max} = \log_2 q \qquad | \text{ Here } q = 2$$

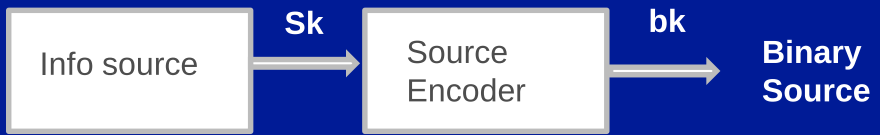
$$\therefore H(S)_{max} = \log_2 2 = 1$$

$$\eta_s = \frac{H(S)}{H(S)_{max}} = \frac{0.9928}{1} = \underline{\underline{0.9928}} \;/\; 99.28\%$$

$$R_{\eta_s} = 1 - \eta_s = \underline{\underline{0.0072}} \;/\; 0.72\%$$

Source Encoding

- Output of an information source is converted in to an r-array sequence.



- Coding: transformation of each of the source symbol S=$\{s_{1,}s_{2,}s_3 \ldots \ldots \ldots s_q\}$ where q-number of source
- using the symbols from the code alphabet
- X=$\{q_{1,}q_{2,}q_3 \ldots \ldots \ldots q_r)$ where r- number of coding symbols

- In binary X=(0,1).
- Ternary X=(0, 1 2)
- Qudranary X=(0, 1, 2 , 3)

- Transmission process is easier and Efficiency of the system can be increased.

Shannon's Encoding Algorithm

- Let the source symbols in the order of decreasing probabilities

  S={$s_{1,}s_{2,}s_3 \dots \dots ..s_q$**)**

  P={$p_{1,}p_{2,}p_3 \dots \dots ..p_q$**)**

  $p_1 \geq p_2 \geq p_3 \dots \dots . \geq p_q$

- Compute the sequence

  α_1**=0**

  α_2**=**$p_1 = p_{1+}\alpha_1$

  $\alpha_3 = p_2 + p_1 = p_2 + \alpha_2$

  $\alpha_4 = p_3 + p_2 + p_1 = p_3 + \alpha_3$

  $\alpha_{q+1} = p_q + \alpha_q = 1$

- Determine the smallest integer for l_i (length of code word) using the inequality

$$2^{l^i} \geq \frac{1}{p_i} \qquad \textbf{for all i=1 to q}$$

- Expand the decimal numbers α_i in binary form up to l_i places neglecting the expansion beyond l_i places.
- Remove the decimal point to get the desired code.

- **Code efficiency**
- The average length 'L' of any code is given by
- $\boldsymbol{L = \sum_{i=1}^{q} p_i l_i}$ binits/symbol
- where $l_i = l_1 l_2 l_3 l_4 \ldots \ldots . l_q$ is respective word length in expressed in binits
- Code efficiency, $\boldsymbol{\eta_c = \frac{H(S)}{L} * 100}$

- **Ex: 1**. Construct the Shannon's binary code for the following message symbols S=($s_{1,}s_{2,}s_3, s_4$) with probabilities P=(0.4, 0.1,0.2,0.3).
- Solution:
- ***0.4 > 0.3 > 0.2 > 0.1***

S_i	P_i	α_i	l_i	binary	Code.
S_1	0.4	0	2	$(0.00000\cdots)_2$	00
S_2	0.3	0.4	2	$(0.01100\cdots)_2$	01
S_3	0.2	0.7	3	$(0.101100\cdots)_2$	101
S_4	0.1	0.9	4.	$(0.11100\cdots)_2$	1110

$$2^{-l_1} \le 0.4 \to l_1 = 2$$
$$2^{-l_2} \le 0.3 \to l_2 = 2$$
$$2^{-l_3} \le 0.2 \to l_3 = 3$$
$$2^{-l_4} \le 0.1 \to l_4 = 4$$

0.4×2
$0.8\times2 \to 0$
$1.6 \to 1$
0.6×2
$1.2 \to 1$
0.2×2
$0.4 \to 0$
$\Rightarrow 0.0110\ldots$

0.7×2
$1.4 \to 1$
0.4×2
$0.8\times2 \to 0$
$1.6 \to 1$
$0.6\times2 \to 1$
1.2
0.2×2
$0.4\times2 \to 0$
$0.8 \to 0$
$\Rightarrow 0.101100\ldots$

0.9×2
$1.8 \to 1$
0.8×2
$1.6 \to 1$
$0.6\times2 \to 1$
1.2
$0.2\times2 \to 0$
0.4×2
$0.8\times2 \to 0$
$\Rightarrow 0.11100\ldots$

The codes are ***S1***: ***00, s2***: ***01, s3***: ***101, s4***: ***1110***

- The average length of this code is $L = \sum_{i=1}^{q} P_i l_i$

$$L = 2 \times 0.4 + 2 \times 0.3 + 3 \times 0.2 + 3 \times 0.1 = 2.4 \text{ Binits / message}$$

$$H(S) = 0.4 \log \frac{1}{0.4} + 0.3 \log \frac{1}{0.3} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} = 1.84644 \text{ bits / message;}$$

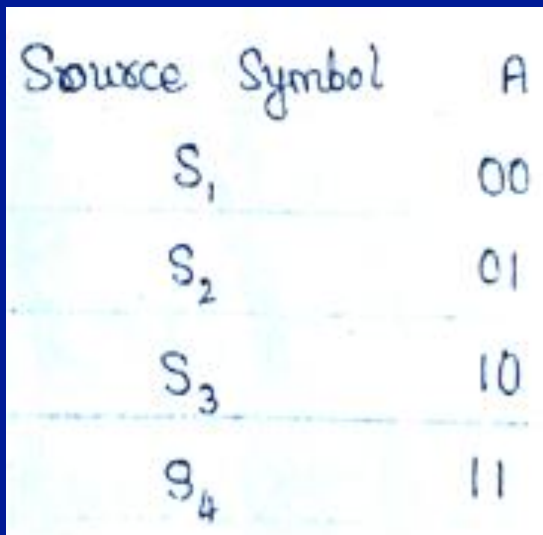
- $\%\eta_c = \frac{\mathrm{H(S)}}{\mathrm{L}} * 100 = \frac{1.8464}{2.4} * 100 = 76.93\%$

- Apply Shannon’s binary encoding procedure to the following set of messages and obtain code efficiency and redundancy. 1/8, 1/16, 3/16, 1/4, 3/8
- S={ S1, S2, S3, S4, S5}

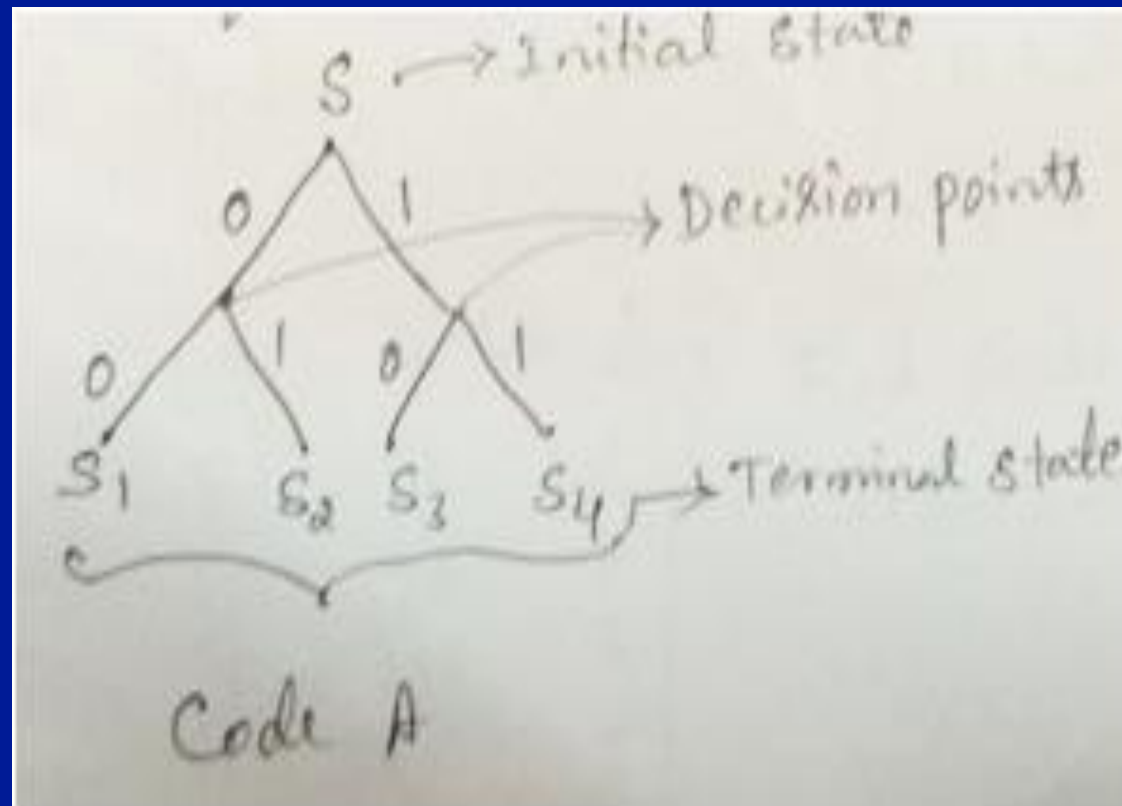
S_i	P_i	α_i	l_i	binary	Code
S_1	3/8	0	2	0.00	00
S_2	1/4	0.375	2	0.01	01
S_3	3/16	0.625	3	0.101	101
S_4	1/8	0.8125	3	0.110	110
S_5	1/16	0.9375	4	0.1111	1111

- $H(S) = \frac{1}{4}log_2 4 + \frac{3}{8}log_2\frac{8}{3} + \frac{1}{8}log_2 8 + \frac{3}{16}log_2\frac{16}{3} + \frac{1}{4}log_2 4 + \frac{1}{16}log_2 16$
- $H(S) = 2.1085\ bits/symbol$
- $L = \sum_{i=1}^{q} P_i l_i = \frac{1}{4}(2) + \frac{3}{8}(2) + \frac{1}{8}(3) + \frac{3}{16}(3) + \frac{1}{16}(4)$
- $L = 2.4375\ bits/symbol$
- $\%\eta = \frac{H(S)}{L} * 100 = 86.5\%$
- Redundancy=1- η=100-86.5=13.5%

Code Tree



Source Symbol	A
S_1	00
S_2	01
S_3	10
S_4	11



Consider a set of 5 codes as shown

Source Symbol	A	B	C	D	E
S_1	00	0	0	0	0
S_2	01	100	10	1000	10
S_3	10	110	100	1110	110
S_4	11	111	111	1111	11

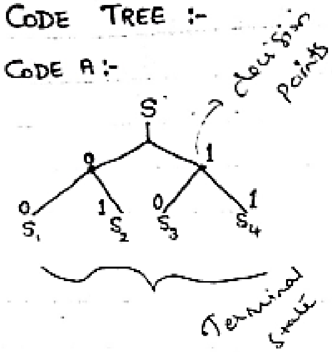
Code Tree

Consider a set of 5 codes as shown

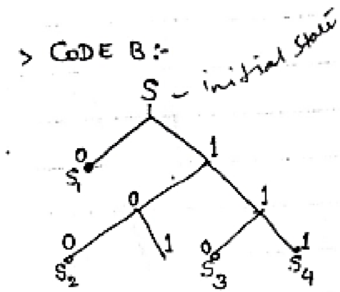
Source Symbol	A	B	C	D	E
S_1	00	0	0	0	0
S_2	01	100	10	1000	10
S_3	10	110	100	1110	110
S_4	11	111	111	1111	11

* CODE TREE :-

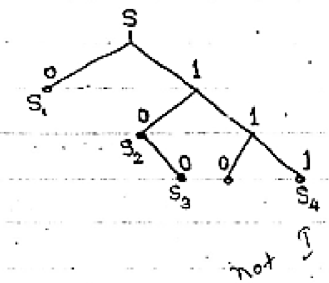
> CODE A :-



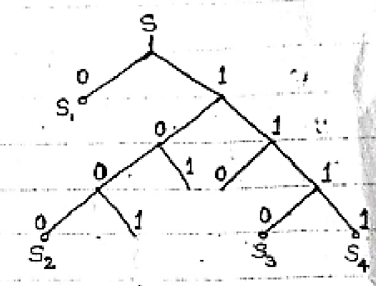
> CODE B :-



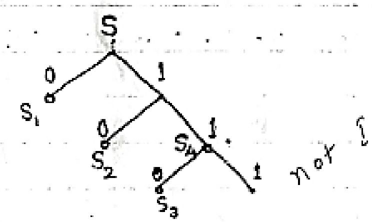
> CODE C :-



> CODE D :-



> CODE E :-



NOTE :- If a decision point represents a code, then that code is not instantaneous.

Huffman Coding

- The source symbols are listed in the decreasing order of probabilities.
- Check

  if $q = r + a(r-1)$ is satisfied and

  find the integer 'a',

  q is number of source symbols and

  r is number of symbols used in code alphabets.
- If 'a' is not integer, add dummy symbols of zero probability of occurrence.
- Combine the last 'r' symbols into a single composite symbol by adding their probability to get a reduced source.
- Repeat the above three steps, until in the final step exactly ***r***- symbols are left.

- The last source with ‘r’ symbols are encoded with ‘r’ different codes 0,1,2,3,….r-1

- In binary coding the last source are encoded with 0 and 1

- As we pass from source to source working backward, decomposition of one code word each time is done in order to form 2 new code words.

- This procedure is repeated till we assign the code words to all the source symbols of alphabet of source ‘s’ discarding the dummy symbols.

- Construct a Huffman Code for symbols having probabilities$\left\{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right\}$. Also find efficiency and redundancy.
- Solution:

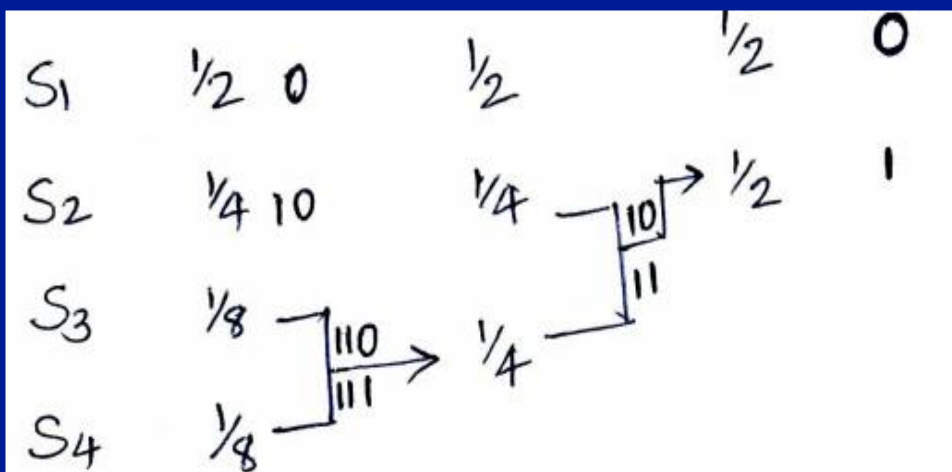
$$q = r + \boldsymbol{a}(r-1)$$

4=2+a(1) => a=2

- Construct a Huffman Code for symbols having probabilities$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$. Also find efficiency and redundancy.
- Solution:

$$q = r + \alpha(r - 1)$$

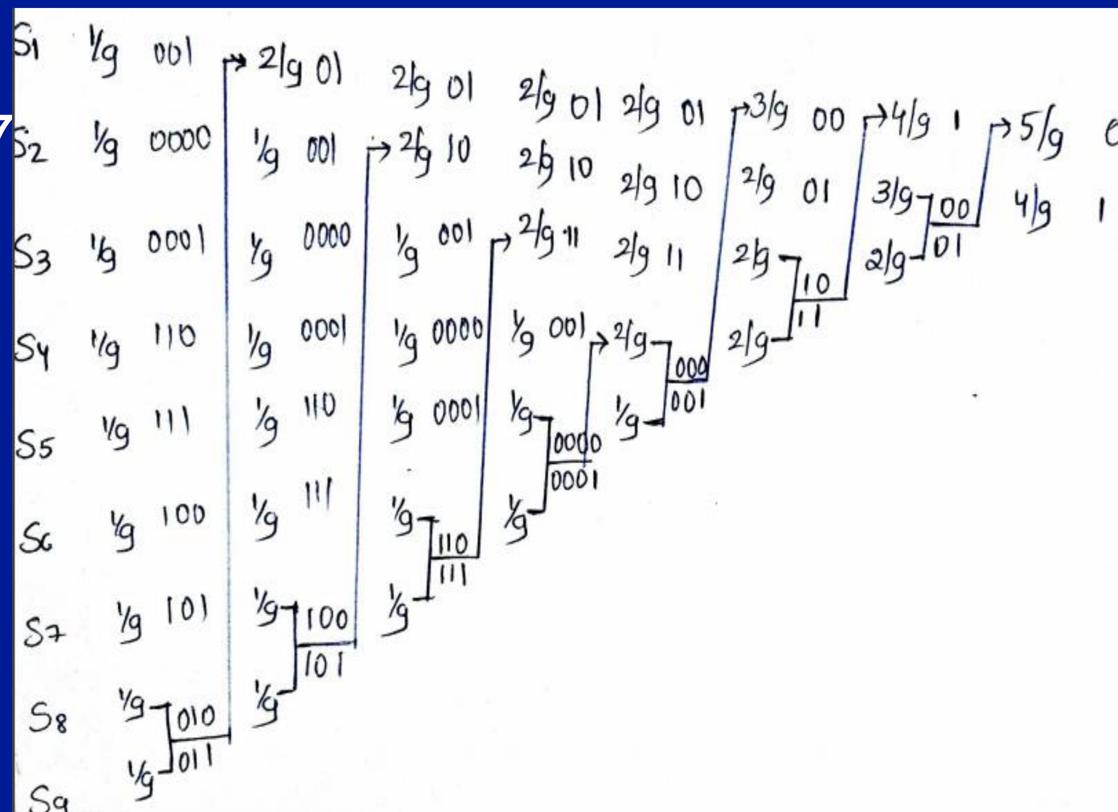
4=2+α(1) => α=2



Symbols	Codes	Probabilities	Length
S_1	0	½	1
S_2	10	¼	2
S_3	110	1/8	3
S_4	111	1/8	3

- $H(S) = \frac{1}{2} log_2 2 + \frac{2}{8} log_2 8 + \frac{1}{4} log_2 4$
- $H(S) = 1.75\ bits/symbol$
- $L = \sum_{i=1}^{4} P_i L_i = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3)$
- $L = 1.75\ bits/symbol$
- $\%\eta = \frac{H(S)}{L} * 100 = 100\%$
- Redundancy=0%

- Ex.2: A source has 9 symbols and each occur with a probability of 1/9. Construct a binary Huffman code. Find efficiency and redundancy of coding.
- Solution:
- $q = r + \alpha(r-1)$
- 9=2+α(1) => α=7



Symbols	Codes	Probabilities	Length
S_1	001	1/9	3
S_2	0000	1/9	4
S_3	0001	1/9	4
S_4	110	1/9	3
S_5	111	1/9	3
S_6	100	1/9	3
S_7	101	1/9	3
S_8	010	1/9	3
S_9	011	1/9	3

- $H(S) = \frac{9}{9} log_2 9$
- $H(S) = 3.17\ bits/symbol$
- $L = \sum_{i=1}^{9} P_i L_i = \frac{1}{9}$(3+4+4+3+3+3+3+3+3)
- $L = 3.22\ bits/symbol$
- $\%\eta = \frac{H(S)}{L} * 100 = 98.45\%$
- Redundancy=100-% η=1.55%

- Ternary $\%\eta = \frac{H3(S)}{L} * 100$
- $H3(S)$= H(s)/ log3

- Ex. 3:

Given the messages x_1, x_2, x_3, x_4, x_5 & x_6 with probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03. Construct binary and trinary code by applying Huffman encoding procedure. Also find efficiency and redundancy.

- Solution:
- Binary

$$q = r + \alpha(r-1)$$

$$6 = 2 + \alpha(1) \Rightarrow \alpha = 4$$

- Ex. 3: Given the messages x_1, x_2, x_3, x_4, x_5 & x_6 with probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03. Construct binary and trinary code by applying Huffman encoding procedure. Also find efficiency and redundancy.
- Solution:

$$q = r + \alpha(r-1)$$

$$6 = 2 + \alpha(1) \Rightarrow \alpha = 4 \in Z$$

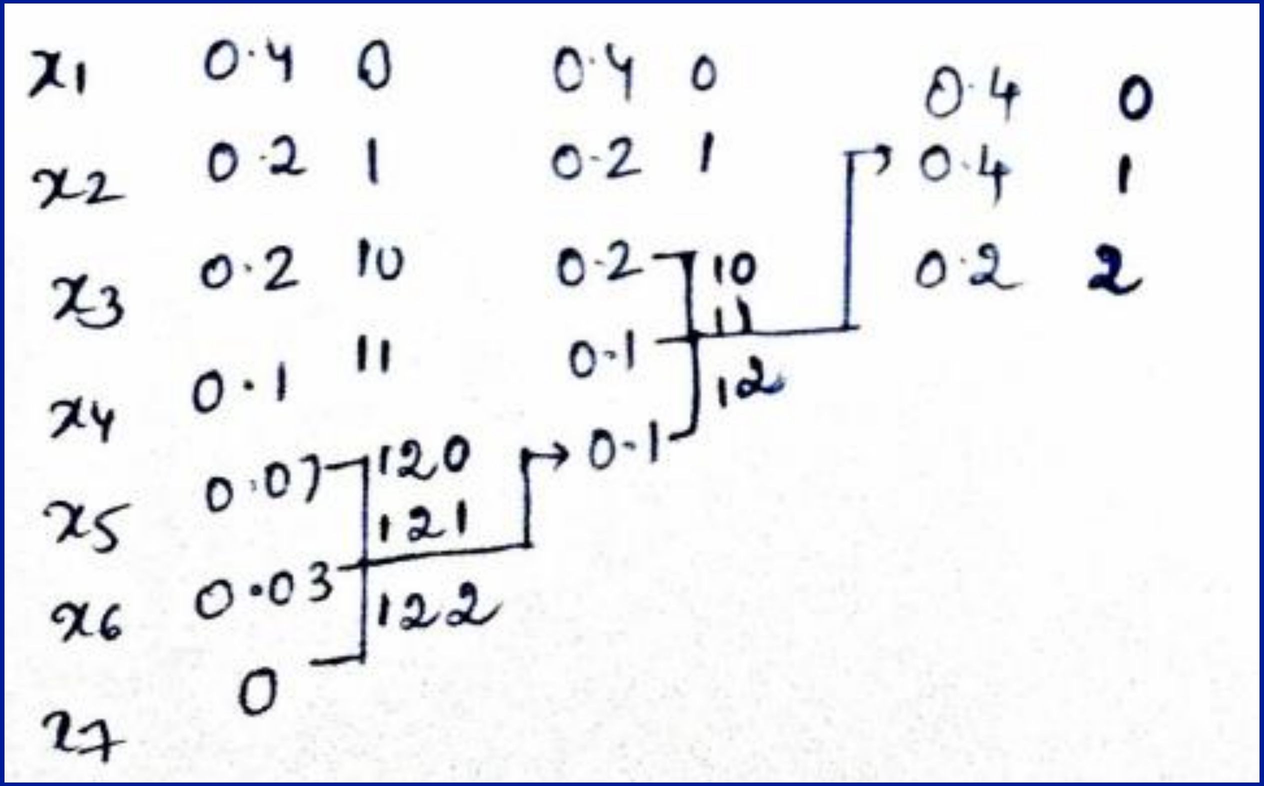
x_1	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
x_2	0.2	01	0.2	01	0.2	01	0.4	00	0.4	1
x_3	0.2	000	0.2	000	0.2	000	0.2	01		
x_4	0.1	0010	0.1	0010	0.2	001				
x_5	0.07	00110	0.1	0011						
x_6	0.03	00111								

Trinary: $q = r + \alpha(r-1)$

6=3+α(2) => α=3/2 ;

Let α=2, If α=2 => $q = 3 + 2(2)$= 7

Hence add a symbol x_7 with probability ‘0’.



Symbols	Codes	Probabilities	Len
x_1	0	0.4	1
x_2	1	0.2	1
x_3	10	0.2	2
x_4	11	0.1	2
x_5	120	0.07	3
x_6	121	0.03	3

Problem

Consider a Zero memory source with S=[S1,S2,S3,S4,S5,S6,S7] and Probabilities P=[0.4,0.2,0.1,0.1,0.1,0.05,0.05]

i. Construct a binary Huffman code by placing the composite symbol as low as possible.

ii. Repeat (i) by moving a composite symbol as high as possible.

iii. In each of the cases (i) and (ii) above,

- Compute the variances of the word lengths and comment on the result.
- Find Efficiency and Redundancy.

iv. Considering Case(ii) table,

- Write the code tree and decode the message 01110110011000100….
- Determine probabilities of 0's and 1's.

Tips: Variance= $\sum_{i=1}^{i=q} P_i\ (l_i - L)^2$

Probability of 0's : P(0)= $\frac{1}{L}\sum_{i=1}^{i=q}(No.\,of\,0'\,s\,in\,the\ \ code\ for\ Si)\ P_i$

As low as possible.

s_1	0.4	1	0.4 as low as possible
s_2	0.2	01	0.2 1
s_3	0.1	0010	0.1 0
s_4	0.1	0011	0.1 0
s_5	0.1	0000	0.1 1
s_6	0.05	00010	0.1 0
s_7	0.05	00011	0001

S_1	1
S_2	01
S_3	0010
S_4	0011
S_5	0000
S_6	00010
S_7	00011

$H(S) = 2.421$

$L = 2.5$

Variance = ~~0.45~~ 2.25

Efficiency $\eta = 96.84\%$

Redundancy $R_\eta = 3.16\%$

As high.

00 ↩ Ar k 1 ↗ 0·6 | 0

|

(ii) as high as possible

S_1	00
S_2	11
S_3	011
S_4	100
S_5	101
S_6	0100
S_7	0101

$H(S) = 2.421$

$L = 2.5$

Variance $= 0.45$

Probability of 0's $= 0.58$

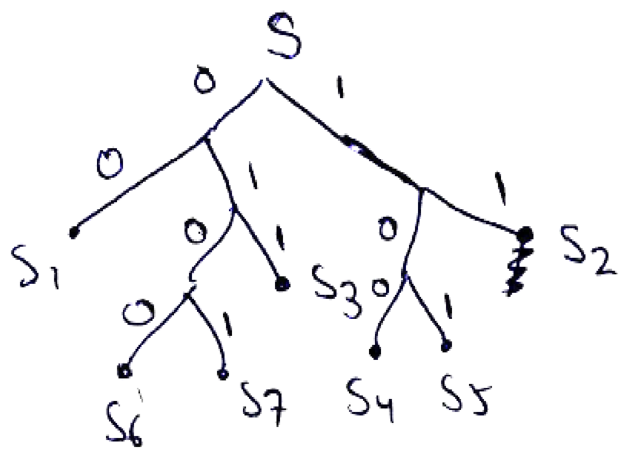
Probability of 1's $= 0.42$

Efficiency $\eta = 96.84\%$.

Redundancy $R_\eta = 3.16\%$.

0101

Code tree using code in case (ii)



Sequence ⇒

$$\underbrace{011}_{S_3}\underbrace{101}_{S_5}\underbrace{100}_{S_4}\underbrace{11}_{S_2}\underbrace{00}_{S_1}\underbrace{0100}_{S_6}$$

Consider a source with 8 alphabet A to H with respective probabilities of 0.22, 0.20, 0.18, 0.15, 0.10, 0.08, 0.05, 0.02.

i] Construct a binary compact code and determine the code efficiency.

ii] Construct a ternary compact code and determine efficiency of the code

iii] Construct a quaternary compact code and determine the code efficiency.

Compare and comment on the result. Draw code trees for all three cases.

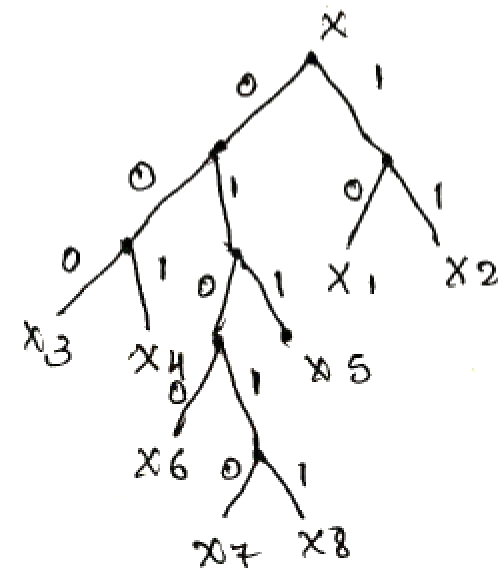
iv] Decode the messages using appropriate code trees

a) 0101001000001101011001…

b) 12111011020012002 ….

c) 03113202030010023 1 …..

ymbol	Prob													
x_1	0.22	10	0.22	10	0.22	10	0.25	01	0.33	00	0.42	1	0.58	0
x_2	0.20	11	0.20	11	0.20	11	0.22	10	0.25	01	0.33	00	0.42	1
x_3	0.18	000	0.18	000	0.18	000	0.20	11	0.22	10	0.25	01		
x_4	0.15	001	0.15	001	0.15	001	0.18	000	0.20	11				
x_5	0.10	011	0.10	011	0.18	010	0.15	001						
x_6	0.08	0100	0.08	0100	0.10	011								
x_7	0.05	01010	0.07	0101										
x_8	0.02	01011												



$$H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$$

$$H(S) = 2.75 \text{ bits / symbol}$$

$$L = \sum_{i=1}^{q} P_i l_i \qquad R\eta_c = 1.79\%$$

$$\eta_c = \frac{H(S)}{L}$$

$$L = 2.8$$

$$\eta_c = 98.25\%$$

iv a) $\underbrace{0101}\,\underbrace{00100}\,\underbrace{100}\,\underbrace{000}\,\underbrace{11}\,\underbrace{0101011}\,\underbrace{1001}$

$= X_7\ X_6\ X_3\ X_2\ X_8\ X_4$

ii) $q = 3 + \alpha(2)$

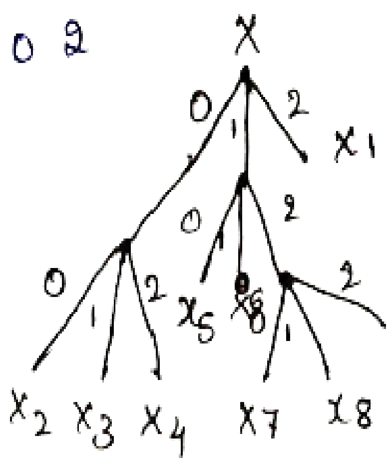
$\alpha = 2.5$

$\alpha = 3 \Rightarrow q = 9$

iv) 121 11 01 10 2 00 120 02

$x_8\ x_6\ x_3\ x_1\ x_2\ x_7\ x_4$

Symbol	Prob	Code		Code		Code		Code
x_1	0.22	2	0.22	2	0.25	1	0.53	0
x_2	0.20	00	0.20	00	0.22	2	0.25	1
x_3	0.18	01	0.18	01	0.20	00	0.22	2
x_4	0.15	02	0.15	02	0.18	01		
x_5	0.10	10	0.10	10	0.15	02		
x_6	0.08	11	0.08	11				
x_7	0.05	120	0.07	12				
x_8	0.02	121						
x_9	0	122						



$$H_r(S) = \frac{H(S)}{\log_2 r}$$

$$H_r(S) = \frac{H(S)}{\log_2(3)} = 1.735$$

$$L = \sum_{i=1}^{q} p_i l_i$$

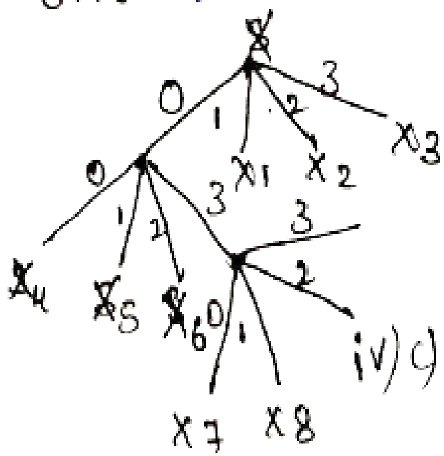
$$L = 1.85$$

$$\eta_c = \frac{H(S)}{L} = 93.73\%$$

$$R_{\eta_c} = 1 - \eta_c = 6.21\%$$

iii) $\alpha = 2 \quad q = 10$

Symbol	Prob					
x_1	0.22	1	0.22	1	0.4	0
x_2	0.20	2	0.20	2	0.22	1
x_3	0.18	3	0.18	3	0.20	2
x_4	0.15	00	0.15	00	0.18	3
x_5	0.10	01	0.10	01		
x_6	0.08	02	0.08	02		
x_7	0.05	030	0.07	03		
x_8	0.02	031				
x_9	0	032				
x_{10}	0	033				



$H(S) = 1.375$

$L = \sum_{i=1}^{q} p_i l_i$

$L = 1.47$

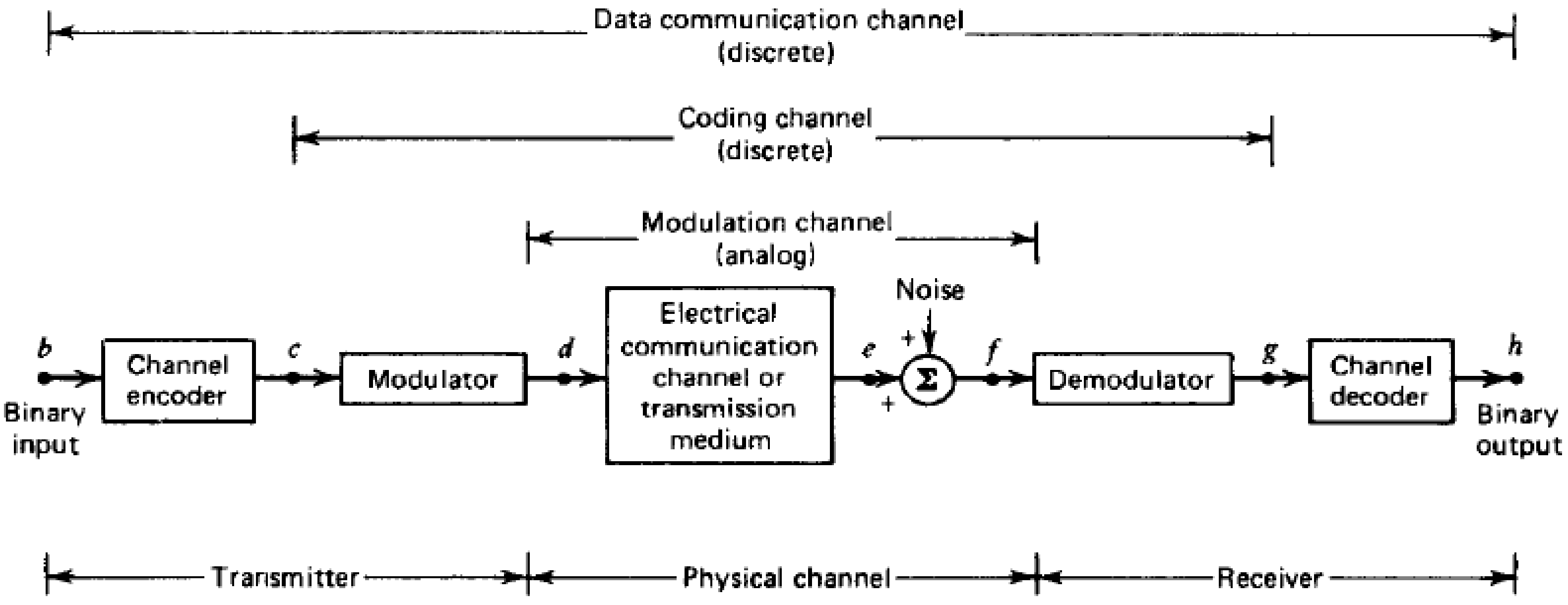
$\eta_c = \frac{H(S)}{L} = 93.53\%$

$R_{\eta_c} = 1 - \eta_c = 6.47\%$

iv) c) 0311320,203 001 002 3 1

$x_8 \; x_1 \; x_3 \; x_2 \; x_6 \; x_7 \; x_5 \; x_4 \; x_2 \; x_3 \; x_1$

Communication Channel



Characterization of a binary communication channel.

Discrete memoryless Channel:

- A channel is defined as the medium through which the coded signals which are generated by an information source are transmitted.
- In general, the input to the channel is a symbol belonging to an alphabet 'A' with 'r' symbols
- The output of a channel is a symbol belonging to an alphabet 'B' with 's' symbols
- Due to errors in the channel, the output symbols may differ from input symbols.

- **<u>Representation of a channel:</u>**
- A=$(a_{1,}a_{2,}a_3 \ldots \ldots \ldots a_r)$
- B=$(b_{1,}b_{2,}b_3 \ldots \ldots \ldots b_s)$
- set of conditional probability P$\left({}^{b_i}/_{a_i}\right)$ with i=1,2,….r and j=1,2,…….s

$$A \quad \left.\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{matrix}\right\} \rightarrow P\left({}^{b_i}/_{a_i}\right) \rightarrow \left\{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{matrix}\right. \quad B$$

- The conditional probabilities come in to the existence due to the presence of noise in the channel.

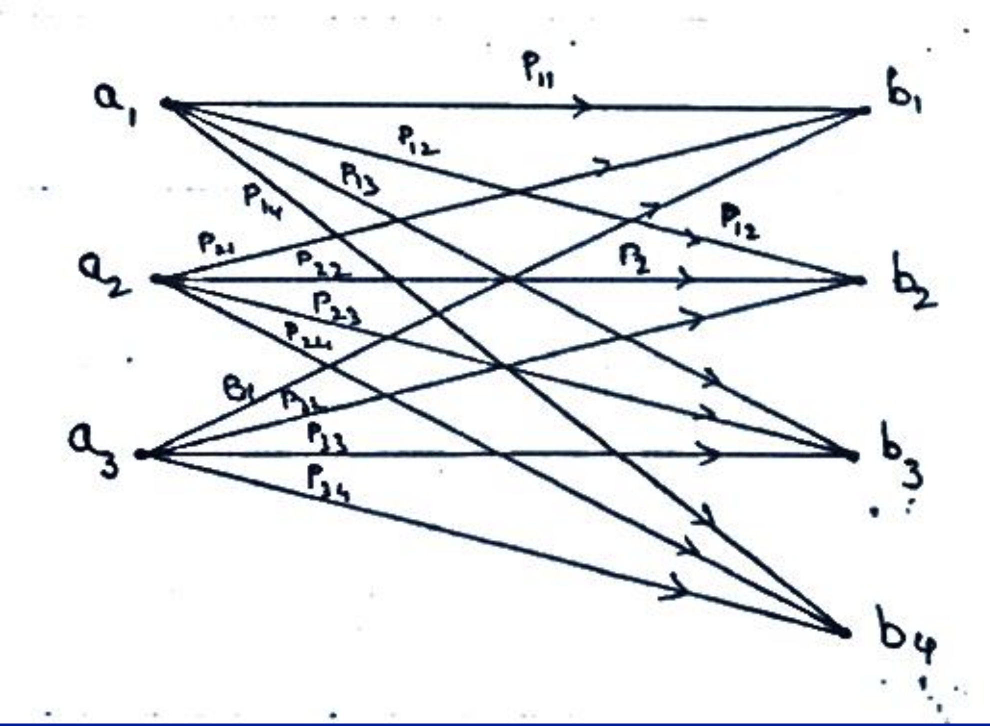
- **There are ‘s’ number of symbols at the receiver from ‘r’ symbols at transmitter.**

- **Totally there are r * s conditional probabilities represented in a form of matrix which is called as Channel Matrix or Noise Matrix.**

$$P(b_j/a_i) = \begin{array}{cc} & \begin{array}{ccccc} \quad b_1 \quad & \quad b_2 \quad & \quad b_3 \quad & & \quad b_s \quad \end{array} \\ \begin{array}{c} a_1 \\ \\ \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & P(b_3/a_2) & \cdots & P(b_s/a_2) \\ P(b_1/a_3) & P(b_2/a_3) & P(b_3/a_3) & \cdots & P(b_s/a_3) \\ P(b_1/a_4) & P(b_2/a_4) & P(b_3/a_4) & \cdots & P(b_s/a_4) \\ \vdots & & & & \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & \cdots & P(b_s/a_r) \end{bmatrix} \end{array}$$

Channel Diagram or Noise Diagram

Channel matrix can also be represented as channel diagram. For Example, consider a source A={a1, a2, a3} and output alphabet B={b1, b2, b3, b4} which can be represented in a channel diagram as shown below.



Channel Continued

- When a_1 is transmitted, it can be received as any one of the output symbols $(b_{1,} b_{2,} b_3 \ldots \ldots \ldots b_s)$
- $P_{11} + P_{12} + P_{13} + \cdots \ldots P_S$=1
- P$\left({}^{b_1}/{}_{a_1}\right)$+ P$\left({}^{b_2}/{}_{a_1}\right)$+ P$\left({}^{b_3}/{}_{a_1}\right)$+……+ P$\left({}^{b_s}/{}_{a_1}\right)$=1
- $\sum_{j=1}^{s} \mathrm{P}\left({}^{b_j}/{}_{a_i}\right) = 1$ for i= 1 to r
- Thus the sum of all the elements in any row of the channel matrix is equal to UNITY.

Joint Probability:

- $P(a_i \cap b_j) = \boldsymbol{P(a_i, b_j)} = \boldsymbol{P(A, B)} = P(^{b_j}/_{a_i})\boldsymbol{P(a_i)}$
- $\boldsymbol{P(a_i, b_j)} = P(^{a_i}/_{b_j})\boldsymbol{P(b_j)}$

Consider,

$$P(a_i, b_j) = P(b_j/a_i)\,P(a_i)$$

Multiply all the elements of the first row of channel matrix by $P(a_1)$ & 2^{nd} row by $P(a_2)$ & r^{th} row by $P(a_r)$

Then the matrix obtained is of the form:

$$
P(b_j/a_i)\,P(a_i) = P(a_i, b_j) =
\begin{array}{c}
\\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r
\end{array}
\begin{array}{c}
\begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_s \end{array} \\
\left[\begin{array}{ccccc}
P(b_1/a_1)P(a_1) & P(b_2/a_1)P(a_1) & P(b_3/a_1)P(a_1) & \cdots & P(b_s/a_1)P(a_1) \\
P(b_1/a_2)P(a_2) & P(b_2/a_2)P(a_2) & P(b_3/a_2)P(a_2) & \cdots & P(b_s/a_2)P(a_2) \\
P(b_1/a_3)P(a_3) & P(b_2/a_3)P(a_3) & P(b_3/a_3)P(a_3) & \cdots & P(b_s/a_3)P(a_3) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
P(b_1/a_r)P(a_r) & P(b_2/a_r)P(a_r) & P(b_3/a_r)P(a_r) & & P(b_s/a_r)P(a_r)
\end{array}\right]
\end{array}
$$

$$
P(a_i, b_j) =
\begin{array}{c}
\\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r
\end{array}
\begin{array}{c}
\begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_s \end{array} \\
\left[\begin{array}{ccccc}
P(a_1, b_1) & P(a_1, b_2) & P(a_1, b_3) & \cdots & P(a_1, b_s) \\
P(a_2, b_1) & P(a_2, b_2) & P(a_2, b_3) & \cdots & P(a_2, b_s) \\
P(a_3, b_1) & P(a_3, b_2) & P(a_3, b_3) & \cdots & P(a_3, b_s) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
P(a_r, b_1) & P(a_r, b_2) & P(a_r, b_3) & \cdots & P(a_r, b_s)
\end{array}\right]
\end{array}
$$

- The above matrix whose elements are various joint probabilities between input and output symbols is called joint probability matrix (JPM)

  The important properties of JPM are:

- The probabilities of input symbols can be obtained by adding the elements of JPM row wise
- The probabilities of output symbols can be obtained by adding the elements of JPM column wise
- The sum of all the elements of JPM is always equal to unity.

Properties:

- Consider the source alphabet A=$(a_{1,} a_{2,} a_3 \dots \dots .. a_r)$ and output alphabet B=$(b_{1,} b_{2,} b_3 \dots \dots .. b_s)$
- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$
- The entropy of the output is given by

$$H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$$

- If all the symbols are equi-probable, then maximum source entropy is $H(A)_{max} = log_2 r$

- The entropy of input symbols $a_{1,} a_{2,} a_3 \dots\dots\dots a_r$ after the transmission and reception of particular output symbol b_j is defined as <u>conditional entropy</u>, denoted by $H(A/b_j)$
- $H\left(A/b_j\right) = \sum_{i=1}^{r} \mathrm{P}\left(a_i/b_j\right) log_2 \frac{1}{\mathrm{P}\left(a_i/b_j\right)}$.
- If the average value of all the conditional probability is taken as j varies from 1 to s denoted by

$$H(A/B) = \sum_{j=1}^{s} P(b_j)\ H\left(A/b_j\right)$$

- $$H(A/B) = \sum_{j=1}^{s} \sum_{i=1}^{r} P(\,b_j)\ \mathrm{P}\left(a_i/b_j\right) log_2 \frac{1}{\mathrm{P}\left(a_i/b_j\right)}$$
- $H(A/B) = \sum_{j=1}^{s} \sum_{i=1}^{r} P(\,a_i, b_j)\ log_2 \frac{1}{P(a_i/b_j)}$ => Conditional entropy

of a transmitter

Properties

- Consider the source alphabet A=($a_{1,}a_{2,}a_3 \dots \dots . . a_r$) and

  output alphabet B=($b_{1,}b_{2,}b_3 \dots \dots . . b_s$)

- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$

- The entropy of the output is given by $H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$

- If all the symbols are equi-probable,

  then maximum source entropy is $H(A)_{max} = log_2 r$

- Conditional entropy of a transmitter: $H(^A/_B) = \sum_{j=1}^{s} \sum_{i=1}^{r} P(a_i, b_j)\, log_2 \frac{1}{P(^{a_i}/_{b_j})}$

- Conditional entropy of a Receiver: $H(^B/_A) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P(^{b_j}/_{a_i})}$

- Joint Conditional Entropy: $H(A, B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P(a_i, b_j)}$

Theorems

- $H(X,Y) = H(X) + H(Y/X)$
- $H(X,Y) = H(Y) + H(X/Y)$
- $H(A,B) = H(A) + H(B/A)$
- $H(A,B) = H(B) + H(A/B)$

Properties

Consider the source alphabet A=$(a_{1,}a_{2,}a_3 \dots\dots..a_r)$ & output alphabet B=$(b_{1,}b_{2,}b_3 \dots\dots..b_s)$

- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$
- The entropy of the output is given by $H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$
- If all the symbols are equi-probable, then maximum source entropy is $H(A)_{max} = log_2 r$

Equivocation:

- Conditional entropy of a transmitter: $H({}^{A}/_{B}) = \sum_{j=1}^{s}\sum_{i=1}^{r} P(a_i, b_j) \, log_2 \frac{1}{P({}^{a_i}/_{b_j})}$
- Conditional entropy of a Receiver: $H({}^{B}/_{A}) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \, log_2 \frac{1}{P({}^{b_j}/_{a_i})}$
- Joint Conditional Entropy: $H(A, B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \, log_2 \frac{1}{P(a_i, b_j)}$
- Theorems: $H(A,B) = H(A) + H(B/A)$

  $H(A,B) = H(B) + H(A/B)$

Mutual Information:

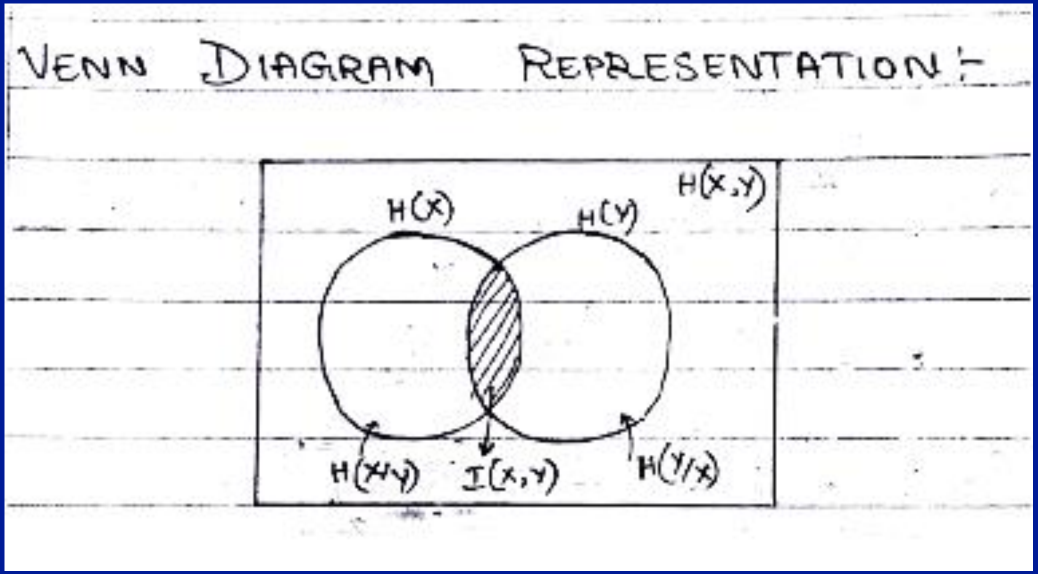
When an average amount of information H(x) is transmitted over a noisy channel, then an amount of information $H(^X/_Y)$ is lost in the channel. The balance of the information at the receiver is defined as Mutual Information I(X , Y)

I (X, Y)= H(X)- $H(^X/_Y)$

= H(Y)- $H(^Y/_X)$

Properties

- The Mutual Information is symmetric. $I(X,Y)=I(Y,X)$
- The mutual information is always non-negative.
- $I(X,Y) = H(X) + H(Y) - H(X,Y)$
- $I(X,Y) = H(X) - H(X/Y)$
- $I(X,Y) = H(Y) - H(Y/X)$



A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P(^B/_A)$, Calculate H(B) and H(A,B)

The entropy of the output is given by

$$H(B) = \sum_{j=1}^{s} P_{b_j} log_2 (\frac{1}{P_{b_j}})$$

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- We know that, $P(a_i, b_j)=P(a_i)P(b_j/a_i)$

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P(a_i,b_j)}$$

A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P(^B/_A)$, Calculate H(B) and H(A,B)

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$$H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$$

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- We know that, $P(a_i, b_j)=P(a_i)P(b_j/a_i)$

$$H(A, B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P(a_i,b_j)}$$

- Adding the element of each column, we get

$$P(b_1) = \frac{1}{5} + \frac{3}{40} = \frac{11}{40}$$

$$P(b_2) = \frac{9}{40} + \frac{1}{15} = \frac{7}{24}$$

$$P(b_3) = \frac{2}{15} + \frac{1}{30} + \frac{1}{5} = \frac{11}{30}$$

$$P(b_4) = \frac{1}{15}$$

$$H(B) = \sum_{j=1}^{4} P(b_j) \log \frac{1}{P(b_j)}$$

$$= \frac{11}{40} \log \frac{40}{11} + \frac{7}{24} \log \frac{24}{7} + \frac{11}{30} \log \frac{30}{11} + \frac{1}{15} \log 15$$

$$H(B) = 1.822 \text{ bits/message-symbol}$$

The JPM may now be constructed by multiplying 1st row elements by $P(a_1) = 0.2 = \frac{1}{5}$, 2nd row by $P(a_2) = 0.3 = \frac{3}{10}$, 3rd row by $P(a_3) = 0.2 = \frac{1}{5}$, 4th row by $P(a_4) = 0.1 = \frac{1}{10}$ and 5th row by $P(a_5) = 0.2 = \frac{1}{5}$.

$$P(A, B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \end{array}$$

The JPM may now be constructed by multiplying 1st row elements by $P(a_1) = 0.2 = \frac{1}{5}$, 2nd row by $P(a_2) = 0.3 = \frac{3}{10}$, 3rd row by $P(a_3) = 0.2 = \frac{1}{5}$, 4th row by $P(a_4) = 0.1 = \frac{1}{10}$ and 5th row by $P(a_5) = 0.2 = \frac{1}{5}$.

$$P(A,B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \end{array}$$

$$H(A,B) = \sum_{i=1}^{5}\sum_{j=1}^{4} P(a_i,b_j)\log\frac{1}{P(a_i,b_j)}$$

$$= \frac{1}{5}\log 5 + \frac{3}{40}\log\frac{40}{3} + \frac{9}{40}\log\frac{40}{9} + \frac{1}{15}\log 15 + \frac{2}{15}\log\frac{15}{2} + \frac{1}{30}\log 30 + \frac{1}{5}\log 5 + \frac{1}{15}\log 15$$

$H(A, B)$ = 2.7653 bits/message-symbol

A transmitter has an alphabet containing of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and the receiver has an alphabet of four letters$\{b_1, b_2, b_3, b_4\}$. The joint probabilities of the system are given below. Compute different entropies of this channel.
H(A), H(B), H(B/A), H(A/B), I(A,B)

$$P(A, B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.10 & 0.30 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0 \\ 0 & 0 & 0.05 & 0.1 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \end{array}$$

Channel Capacity

- It is known that average information content of the source is $H(X) = \sum_{i=1}^{M} p(x_i)\log_2\left(\frac{1}{p(x_i)}\right)$
- Average information per symbol going in to the channel is $R_{in} = r_s * H(X)$. Due to the error, it is not possible to reconstruct the input symbol sequence with certainty on the recovered sequence. Therefore source information is lost due to the errors.
- Therefore average rate of information transmission is given by

$$R_t = I(X,Y)\,.\,r_s.\ \text{Bits/sec.}$$

- The capacity of a discrete memoryless noisy channel is defined as maximum possible rate of information transmission over the channel. The maximum rate of transmission occurs when the source is matched to the channel.

$$C = Max(R_t)$$

$$= \mathrm{Max}[I(X,Y)\,.\,r_s]$$

$$C = \mathrm{Max}\{[H(X) - H(X/Y)]\,r_s\}$$

Channel Efficiency

- $\%\eta_{ch} = \frac{R_t}{C} * 100$

  $$= \frac{I(X,Y)\,.\,r_s}{\text{Max}\,[\{\,I(X,Y)\,.\,r_s\,\}]} * 100$$

- $\%\eta_{ch} = \frac{H(X) - H\left({}^{X}\!/_{Y}\right)}{Max\left[H(X) - H\left({}^{X}\!/_{Y}\right)\right]} * 100$

- **Redundancy =1-η_{ch}**

Shannon's Theorem on channel capacity

Shannon's second theorem is stated in 2 ways:

1) POSITIVE STATEMENT : It states that when the rate of information transmission $R_t \leq C$, there exists a coding technique which enables the transmission over channel with smaller probability of error.

2) NEGATIVE STATEMENT : It states that if $R_t > C$, then the reliable transmission of the information is not possible without errors i.e., errors cannot be controlled by any coding technique.

Symmetry Channel or Uniform Channel

- Symmetry channel is defined as the channel, in which the channel matrix has 2nd and subsequent rows, the same elements as the first row, but in different order.
- $\therefore H(^{Y}/_{X}) = h$, where →entropy of any single row. The channel capacity with r_s=1 sybols/sec is given by,

In general, symmetric channel can be represented as

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots & y_s \end{array} \\ \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_s \\ P_3 & P_2 & P_1 & \cdots & P_5 \\ P_9 & P_2 & P_5 & \cdots & P_9 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_5 & P_2 & P_s & \cdots & P_1 \end{bmatrix} \end{array}$$

$P_1, P_2, P_3 \ldots P_s$ are conditional probabilities whose permutation & combinations are present in other rows.

Symmetry Channel or Uniform Channel

- $C = Max(R_t)$

  $= \boldsymbol{Max}[I(X,Y)]\, r_s$

  $= \boldsymbol{Max}[I(X,Y)]$

  $= \boldsymbol{Max}\left(H(Y) - H(^{Y}/_{X})\right)$

  $= \boldsymbol{Max}[H(Y)] - \boldsymbol{Max}[H(^{Y}/_{X})]$

  $= \boldsymbol{Max}[H(Y)]$ - h

  $C = \text{Max}[H(Y)] - h$

In general, symmetric channel can be represented as

$$
\begin{array}{c c}
 & \begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots\cdots & y_s \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{array} &
\begin{bmatrix} P_1 & P_2 & P_3 & \cdots\cdots & P_s \\ P_3 & P_2 & P_1 & \cdots\cdots & P_5 \\ P_9 & P_2 & P_5 & \cdots\cdots & P_9 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_5 & P_2 & P_s & \cdots\cdots & P_1 \end{bmatrix}
\end{array}
$$

$P_1, P_2, P_3 \ldots P_s$ are conditional probabilities whose permutation & combinations are present in other rows.

- H(Y) is the entropy of symbol which becomes maximum if and only if all the receive symbols become Equi-probable. Since there are ‘s’ output symbols
- $\text{Max}[H(Y)] = log_2 s$
- $\therefore$ $\boldsymbol{C = log_2 s - h}$

Symmetry Channel or Uniform Channel

- Symmetry channel is defined as the channel, in which the channel matrix has 2nd and subsequent rows, the same elements as the first row, but in different order.
- $\therefore H(Y/X) = h$, where →entropy of any single row. The channel capacity with r_s=1 sybols/sec is given by,
- $C = Max(R_t)$

  $= Max[I(X,Y)]\, r_s$

  $= Max[I(X,Y)]$

  $= Max\left(H(Y) - H(Y/X)\right)$

  $= Max[H(Y)] - Max[H(Y/X)]$

  $= Max[H(Y)] - h$

  $C = \text{Max}[H(Y)] - h$

In general, symmetric channel can be represented as

$$\begin{array}{c|ccccc} & y_1 & y_2 & y_3 & \cdots & y_s \\ \hline x_1 & P_1 & P_2 & P_3 & \cdots & P_s \\ x_2 & P_3 & P_2 & P_1 & \cdots & P_5 \\ x_3 & P_9 & P_2 & P_5 & \cdots & P_9 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_r & P_5 & P_2 & P_s & \cdots & P_1 \end{array}$$

$P_1, P_2, P_3 \ldots P_s$ are conditional probabilities whose permutation & combinations are present in other rows.

- H(Y) is the entropy of symbol which becomes maximum if and only if all the receive symbols become Equi-probable. Since there are ‘s’ output symbols
- $\text{Max}[H(Y)] = log_2 s$
- $\therefore\ \boldsymbol{C = log_2 s - h}$

Problem

For a channel matrix shown below, find the channel capacity:

$$
P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\
\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}
\end{array}
$$

Problem

For a channel matrix shown below, find the channel capacity:

$$P(Y/X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ x_2 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ x_3 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{array}$$

It is a symmetric or uniform channel.

$\therefore$ Channel capacity, $C = \log_2 s - h$

Let $r_s = 1$ bit/sec

$$h = \sum_{j=1}^{3} P_j \log_2\left(\frac{1}{P_j}\right) = \frac{1}{2}\log_2 2 + \frac{1}{3}\log_2 3 + \frac{1}{6}\log_2 6$$

$$= 1.459 \text{ bits/symbol}$$

Problem

For a channel matrix shown below, find the channel capacity:

$$P(Y/X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ x_2 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ x_3 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{array}$$

It is a symmetric or uniform channel.

$\therefore$ Channel capacity, $C = \log_2 s - h$

Let $r_s = 1$ bit/sec

$$h = \sum_{j=1}^{3} P_j \log_2 \left(\frac{1}{P_j}\right) = \frac{1}{2}\log_2 2 + \frac{1}{3}\log_2 3 + \frac{1}{6}\log_2 6$$

$$= 1.459 \text{ bits/symbol}$$

$$C = \log_2 s - h$$

$$= \log_2 3 - 1.459$$

$$C = 0.126 \text{ bits/sec}$$

Repeat the above problem for channel matrix given below with $r_s = 1000$ symbols/sec.

$$P(Y/X) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0.3 & 0.2 & 0.1 \\ x_2 & 0.4 & 0.1 & 0.3 & 0.2 \\ x_3 & 0.1 & 0.2 & 0.4 & 0.3 \end{array}$$

Repeat the above problem for channel matrix given below with $r_s = 1000$ ~~bits~~ symbols/sec.

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

$$h = 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$$= 1.846 \text{ bits/symbol}$$

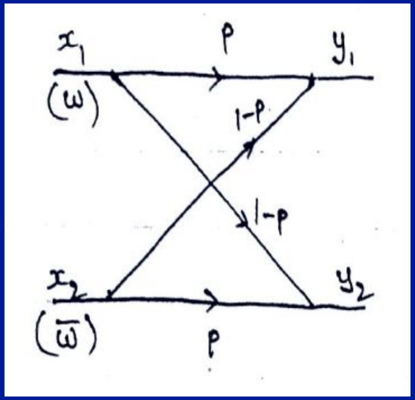
$$C = \left[\log_2 S - h\right] r_s$$

$$= \left[\log_2 4 - 1.846\right] 1000$$

$$\boxed{C = 154 \text{ bits/sec}}$$

Binary Symmetric Channel

- The binary symmetric channel is one of the most commonly and widely used channel whose channel diagram is given below



- From the above diagram, channel matrix can be written as
- $P(^{Y}/_{X}) = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix} = \begin{bmatrix} P & \bar{P} \\ \bar{P} & P \end{bmatrix}$
- The matrix is a symmetric matrix. Hence the channel is binary symmetric channel.

Channel Capacity

- It is known that C = Max$\{[H(Y) - H(^Y/_X)]\, r_s\}$.
- For symmetry channel, $H(^Y/_X) = h = P\log\frac{1}{P} + \bar{P}\log\frac{1}{\bar{P}}$
- Since it is a binary symmetric channel,

$$H(Y)_{max} = log_2 s = log_2 2 = 1$$

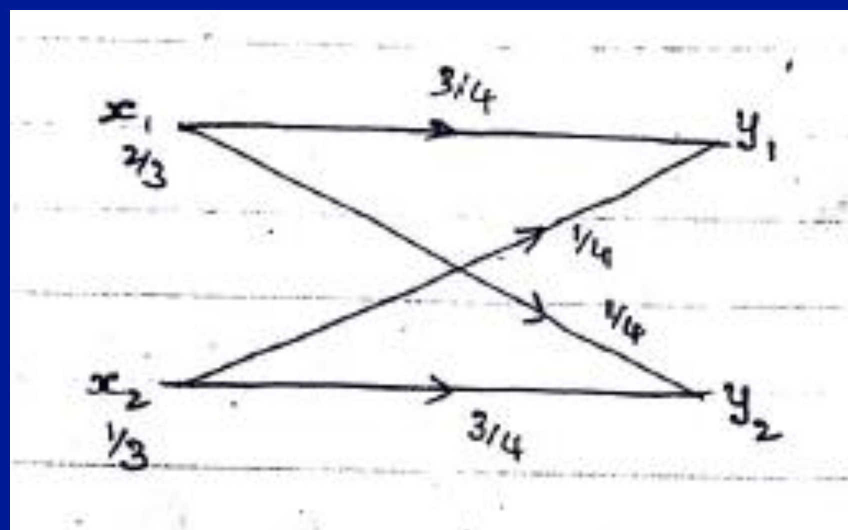
- $\therefore C = (1 - h)$ rs bits/sec.

A binary symmetric channel has the following matrix with source probabilities

$$P(x_1) = 2/3 \quad ; \quad P(x_2) = 1/3$$

$$P(Y/X) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Find , $H(x)$, $H(y)$, $I(x,y)$, C , η_{ch}



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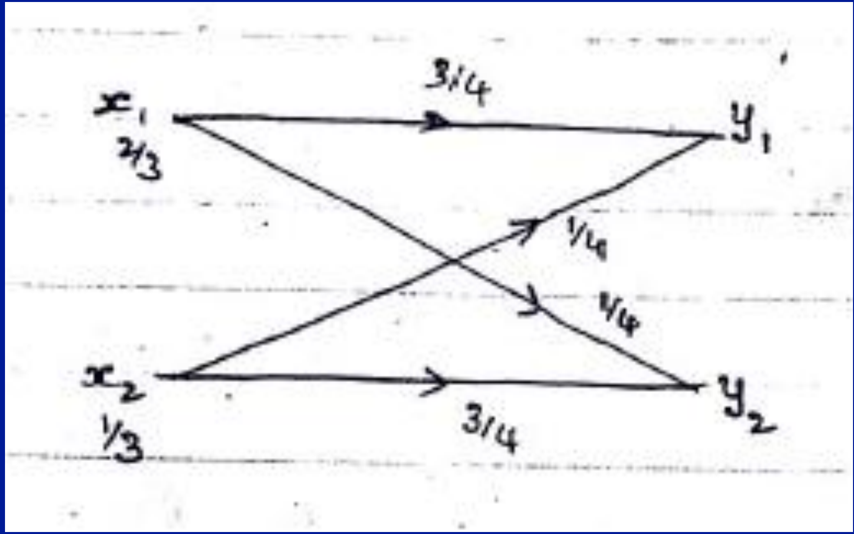
Find , $H(x)$, $H(y)$, $I(x,y)$, C, η_{ch}



$$\text{JPM} \Rightarrow \begin{bmatrix} 2/4 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$$

$$P(y_1) = 7/12 \quad ; \quad P(y_2) = 5/12$$

$$H(X) = \sum_{i=1}^{2} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= \frac{2}{3} \log_2 \left(\frac{3}{2}\right) + \left(\frac{1}{3}\right) \log_2 (3)$$

$$H(X) = 0.9183 \text{ bits/symbol}$$

$$H(Y) = \sum_{j=1}^{2} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$$= \frac{7}{12} \log_2 \left(\frac{12}{7}\right) + \frac{5}{12} \log_2 \left(\frac{12}{5}\right)$$

$$H(Y) = 0.9799 \text{ bits/symbol}$$

$$I(x,y) = H(Y) - H(Y/X)$$

$$H(Y/X) = h = p \log \frac{1}{p} + \bar{p} \log \frac{1}{\bar{p}}$$

$$= \frac{3}{4} \log_2 \left(\frac{4}{3}\right) + \frac{1}{4} \log_2 4$$

$$= \underline{0.8113} \text{ bits/symbol}$$

$$I(x,y) = H(Y) - H(Y/X)$$

$$= 0.9799 - 0.8113 = \underline{0.1686} \text{ bits/symbol}$$

$$\eta_{ch} = \frac{R_t}{C} \times 100 = \frac{I(x,y)\, r_s}{1-h} \times 100$$

$$\eta_{ch} = \frac{R_t}{C} \times 100 = \frac{I(x,y)\, r_s}{1-h} \times 100$$

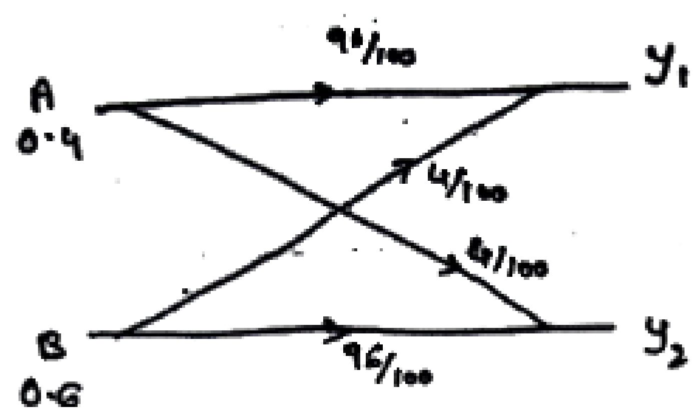
Let $r_s = 1$ bit/symbol

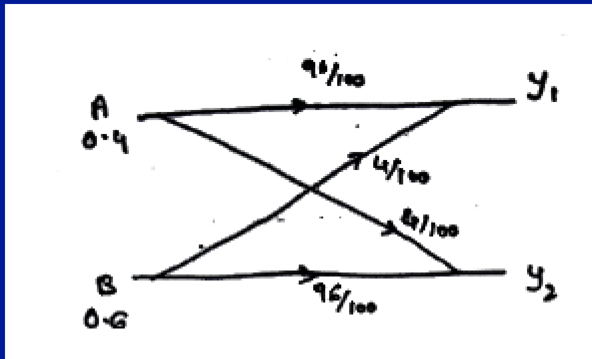
$$\therefore \eta_{ch} = \frac{0.1686}{1-0.8113} \times 100 = \frac{0.1686}{0.1887} \times 100$$

$$\boxed{\eta_{ch} = 89.35\ \%}$$

A message source produces 2 independent symbols A & B, with probabilities $P(A) = 0.4$ & $P(B) = 0.6$. Calculate efficiency of source & its redundancy if its symbols are received in an avg with 4 in every 100 symbols in error.

Calculate the transmission rate of the system.





$$\text{Channel matrix} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 96/100 & 4/100 \\ 4/100 & 96/100 \end{bmatrix} \end{array} = \begin{bmatrix} 0.96 & 0.04 \\ 0.04 & 0.96 \end{bmatrix}$$

$$JPM = \begin{bmatrix} 0.384 & 0.016 \\ 0.024 & 0.576 \end{bmatrix}$$

$P(y_1) = 0.408$; $P(y_2) = 0.592$

$$H(Y) = 0.408 \log_2\left(\frac{1}{0.408}\right) + 0.592 \log_2\left(\frac{1}{0.592}\right)$$

$$H(Y) = 0.9754 \text{ bits/symbol}$$

$$H(Y/X) = h = 0.96 \log_2\left(\frac{1}{0.96}\right) + 0.04 \log_2\left(\frac{1}{0.04}\right)$$

$$= 0.2423 \text{ bits/symbol}$$

$$I(x,y) = H(Y) - H(Y/X)$$

$$= 0.7331 \text{ bits/symbol}$$

$$R_t = I(x,y)\, r_s$$

Let $r_s = 100$ symbols/sec

$\therefore R_t = 73.31$ bits/sec

$$H(Y) = 0.408 \log_2\left(\frac{1}{0.408}\right) + 0.592 \log_2\left(\frac{1}{0.592}\right)$$

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$$R_t = I(x,y)\, r_s$$

Let $r_s = 100$ symbols/sec

$$\therefore R_t = \underline{73.31} \text{ bits/sec}$$

Shannon Hartley Theorem and its Implications

Statement of Shannon-Hartley Law :

Shannon-Hartley law also called Shannon's third theorem, states that the capacity of a band-limited Gaussian channel with AWGN is given by

$$C = B \log\left(1+\frac{S}{N}\right) \text{ bits/sec}$$

Where B = Channel bandwidth in Hz

S = Signal power in watts

N = Noise power in watts $= \eta B$

where the two sided power spectral density of noise is $(\eta/2)$ watts/Hz.

Noise: The unpredictable voltage waveform is a random process called noise

White noise: The noise introduced due to thermal motion of electrons is called **White Noise** and the noise resulting due to the flow of electrons across semiconductor junction is called **Shot Noise**.

When the noise adds to the signal is called **Additive Noise** and if it multiplies then it is called **fading.**

In channels, the noise is almost white and it has a distribution which resembles Gaussian distribution with zero mean and some variance. Hence this noise is also called " **Additive White Gaussian Noise (AWGN**)."

1st implication

1st Implication :

From Shannon-Hartley law, we have

$$C = B \log_2\left(1+\frac{S}{N}\right) \text{ bits/sec} \qquad \ldots\ldots (4.107)$$

It looks, from the above equation that when B is increased, channel capacity C also increases and since $R_{max} = C$, the maximum rate of information transmission can be enhanced to any large value as we please. However, the channel capacity does not become infinite when the bandwidth is made infinite. This is because, as B increases, the noise power N which is dependent on B, also increases thereby reducing $\left(\frac{S}{N}\right)$. Thus the product of B and $\log_2\left(1+\frac{S}{N}\right)$ will increase only upto a certain value and becomes constant with increasing B. This value is denoted by C_∞. Let us calculate that value.

Substituting $N = \eta B$ in equation (4.107), we get

$$C = B \log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\left(\frac{\eta B}{S}\right)\log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{(\eta B/S)}$$

Let $x = \frac{S}{\eta B}$

$$\text{Then} \quad C = \frac{S}{\eta}\log_2 (1+x)^{(1/x)}$$

Accordingly when $B \to \infty$, $x \to 0$

$$\therefore \quad \lim_{\substack{B\to\infty \\ x\to 0}} C = \lim_{\substack{B\to\infty \\ x\to 0}} \frac{S}{\eta}\log_2 (1+x)^{(1/x)}$$

$$\therefore \quad C_\infty = \frac{S}{\eta}\log_2 \lim_{x\to 0}\left[(1+x)^{(1/x)}\right]$$

$$\therefore \quad C_\infty = \frac{S}{\eta}\log_2 e$$

$$\text{or} \quad C_\infty = B\frac{S}{N}\log_2 e \text{ bits/sec}$$

SHANNON'S LIMIT :

We define an ***"ideal system"*** as one that transmits data at a bit rate R_t equal to the channel capacity C. We may then express the average transmitted power as

$$S = E_b C$$

Where. E_b = transmitted energy per bit in joules.

Using $N = \eta B$ and $S = E_b C$ in equation (4.107), we get for an ideal system

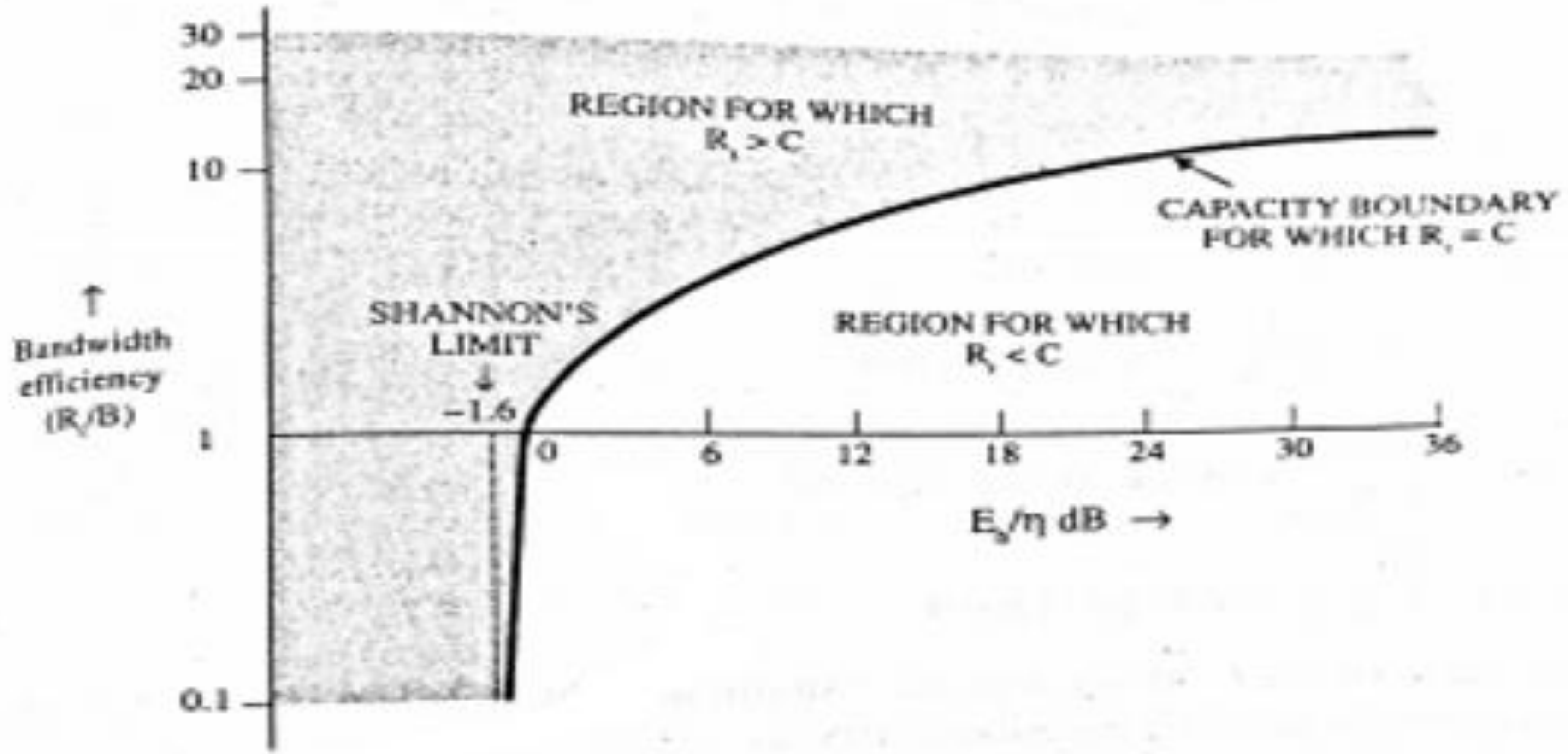
$$C = B \log_2 \left(1 + \frac{E_b}{\eta} \frac{C}{B}\right)$$

$$\text{or} \quad \frac{C}{B} = \log_2 \left(1 + \frac{E_b}{\eta} \frac{C}{B}\right)$$

The quantity $\left(\frac{C}{B}\right)$ is called ***"Bandwidth-efficiency"*** and the quantity (E_b/η) is given by

$$\frac{E_b}{\eta} = \frac{2^{C/B} - 1}{(C/B)}$$

When (R_t/B) is plotted as a function of (E_b/η), we get the bandwidth-efficiency diagram which is shown in figure 4.3. The resulting curve represents the capacity boundary for which



Illustrating Bandwidth-efficiency diagram

1. For infinite bandwidth, the signal energy-to-noise ratio E_b/η approaches the limiting value.

$$\left(\frac{E_b}{\eta}\right)_\infty = \lim_{B\to\infty}\left(\frac{E_b}{\eta}\right) = \lim_{B\to\infty}\left[\frac{2^{C/B}-1}{(C/B)}\right]$$

Let $\frac{C}{B} = x$. As $B \to \infty$, $x \to 0$

$$\therefore \quad \left(\frac{E_b}{\eta}\right)_\infty = \lim_{x\to 0}\left[\frac{2^x-1}{x}\right] \qquad \text{...... (4.112)}$$

Using L'Hospital Rule, the above limit can be evaluated as below:

Let $y = 2^x$

Taking ln on both sides

$$ln\, y = x\, ln2$$

Differentiating, $\frac{1}{y}\, dy = (ln2)\, dx$ ...... (4.113)

$$\therefore \quad \frac{dy}{dx} = y\,(ln2) = 2^x\,(ln2)$$

Differentiating both numerator and denominator of the RHS of equation (4.112) with respect to 'x', we get

$$\left(\frac{E_b}{\eta}\right)_{\infty} = \lim_{x\to 0}\left[\frac{\frac{d}{dx}(2^x-1)}{\frac{d}{dx}(x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2^x(\ln 2)}{1}\right] \text{ by using equation (4.113)}$$

$$= 2^0 \ln 2$$

$$\therefore \quad \left(\frac{E_b}{\eta}\right)_{\infty} = \ln 2 = 0.693$$

$$\text{or} \quad \left(\frac{E_b}{\eta}\right)_{\infty} \text{in dB} = 10 \log_{10}(0.693)$$

$$\therefore \quad \left(\frac{E_b}{\eta}\right)_{\infty} \text{in dB} \cong -1.6 \text{ dB} \qquad \text{...... (4.114)}$$

This value of – 1.6 dB is called the **"Shannon's Limit"**. The corresponding value of channel capacity is given by equation (4.108) as

$$C_{\infty} = B\frac{S}{N}\log_2 e \text{ bits/sec}$$

2. The capacity boundary, defined by the curve for critical bit rate $R_t = C$, separates combinations of system parameters that have the potential for supporting error free transmission ($R_t < C$) from those for which error-free transmission is not possible ($R_t > C$). The latter region is shown using dots in figure 4.3.
3. The diagram of figure 4.3 highlights trade-offs between (E_b/η) and (R_t/B). This is discussed in a difficult aspect in the 2nd implication of Shannon-Hartley law.

2nd Implication

2nd Implication :

Bandwidth - (S/N) Trade Off :

An important implication of Shannon-Hartley law is the exchange of bandwidth with signal to noise power ratio and vice-versa as given below :

$$\text{Suppose} \left(\frac{S_1}{N_1}\right) = 7 \text{ and } B_1 = 4 \text{ KHz.}$$

$$\therefore \text{ Channel capacity } C_1 = B_1 \log\left(1+\frac{S_1}{N_1}\right)$$

$$= 4 \times 10^3 \log(1+7)$$

$$= 12 \times 10^3 \text{ bits/sec.}$$

Keeping the channel capacity C_2 same as C_1 and if signal-to-noise ratio is increased to 15, then

$$C_2 = C_1 = 12 \times 10^3 = B_2 \log\left(1 + \frac{S_2}{N_2}\right)$$
$$= B_2 \log(1 + 15)$$
$$\therefore \quad \mathbf{B_2 = 3\ KHz}$$

Since the noise power $N = \eta B$, as the bandwidth gets reduced from 4 to 3 KHz, the noise also decreases indicating an increase in signal power as shown below.

We have $N_1 = \eta B_1 = (\eta)(4\ \text{KHz})$

and $N_2 = \eta B_2 = (\eta)(3\ \text{KHz})$

$$\text{Consider } \frac{(S_2/N_2)}{(S_1/N_1)} = \frac{15}{7}$$

$$\therefore \quad \frac{S_2}{S_1} = \frac{15N_2}{7N_1} = \frac{(15)(\eta)(3\text{KHz})}{(7)(\eta)(4\text{KHz})} = \frac{45}{28} = 1.6$$

Thus a 25% reduction in bandwidth from 4 KHz to 3 KHz requires a 60% approximate increase in signal power for maintaining the same channel capacity. Let us look into the exact significance by drawing the ***"trade-off curve"***.

From Shannon-Hartley law

$$\frac{B}{C} = \frac{1}{\log_2\left(1+\frac{S}{N}\right)} \quad \text{...... (4}$$

The values of (B/C) for different values of (S/N) are listed in table 4.1 below :

$\frac{S}{N}$	0.5	1	2	5	10	15	20	30
$\frac{B}{C}$	1.71	1	0.63	0.37	0.289	0.25	0.23	0.2

Table 4.1 : Table of values of (B/C) for different values of (S/N)

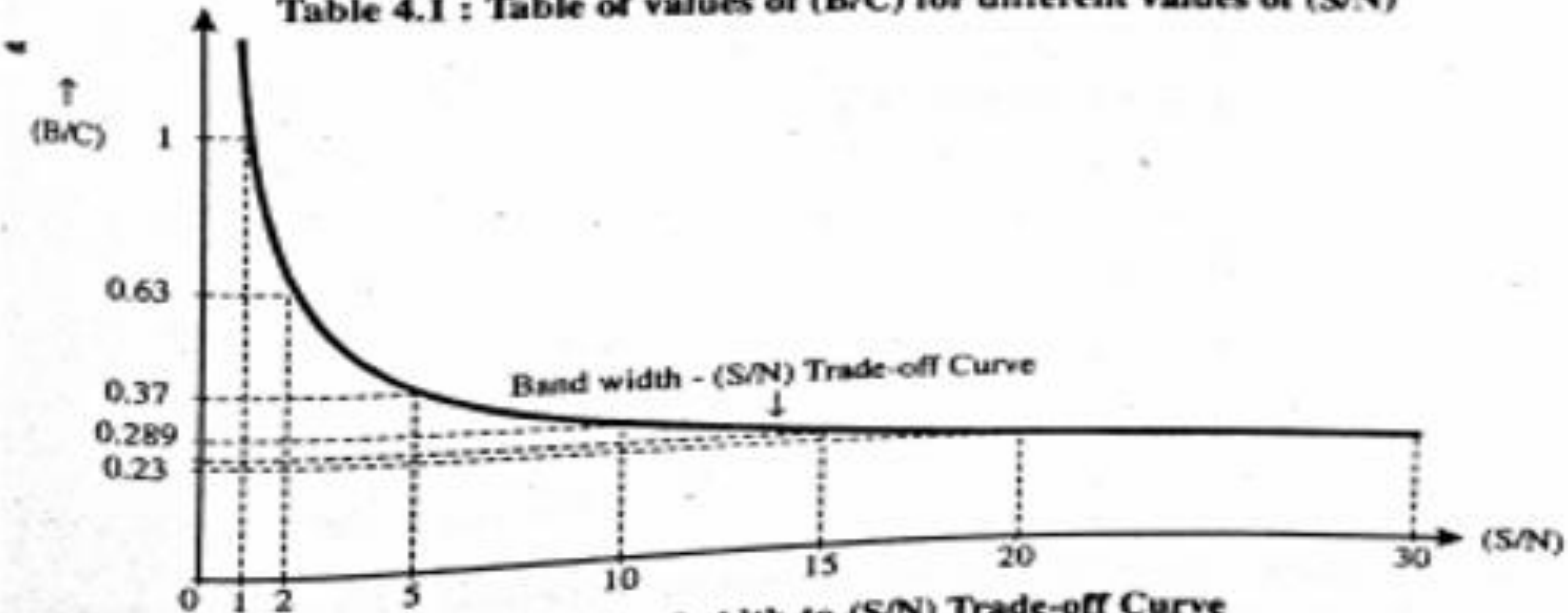


Fig. 4.4 : Bandwidth-to-(S/N) Trade-off Curve