DEPARTMENT
OF
ELECTRONICS & COMMUNICATION ENGINEERING

Digital Communication System

(Theory Notes)

Autonomous Course

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Module – 4 Contents
Digital Modulation Techniques: Digital Modulation formats, Coherent binary modulation techniques, Probability of error derivation of PSK and FSK, M-ary modulations-QPSK, QAM, PSD for different digital modulation techniques, Non-coherent binary modulation techniques –DPSK

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DIGITAL MODULATION TECHNIQUES

Modulation is defined Modulation is defined as the process by which some characteristics of a carrier is varied in accordance with a modulating wave. In digital communications, the modulating wave consists of binary data or an M-ary encoded version of it and the carrier is sinusoidal wave. In M-ary signalling, the modulator produces one of an available set of $M=2^m$ distinct signals in response to m bits of source data at a time. Binary modulation is a special case of M-ary modulation with M=2.

There are 3 basic modulation techniques for the transmission of digital data.

- Amplitude Shift Keying
- Frequency Shift Keying
- Phase Shift Keying

Fig 1 shows different modulations.

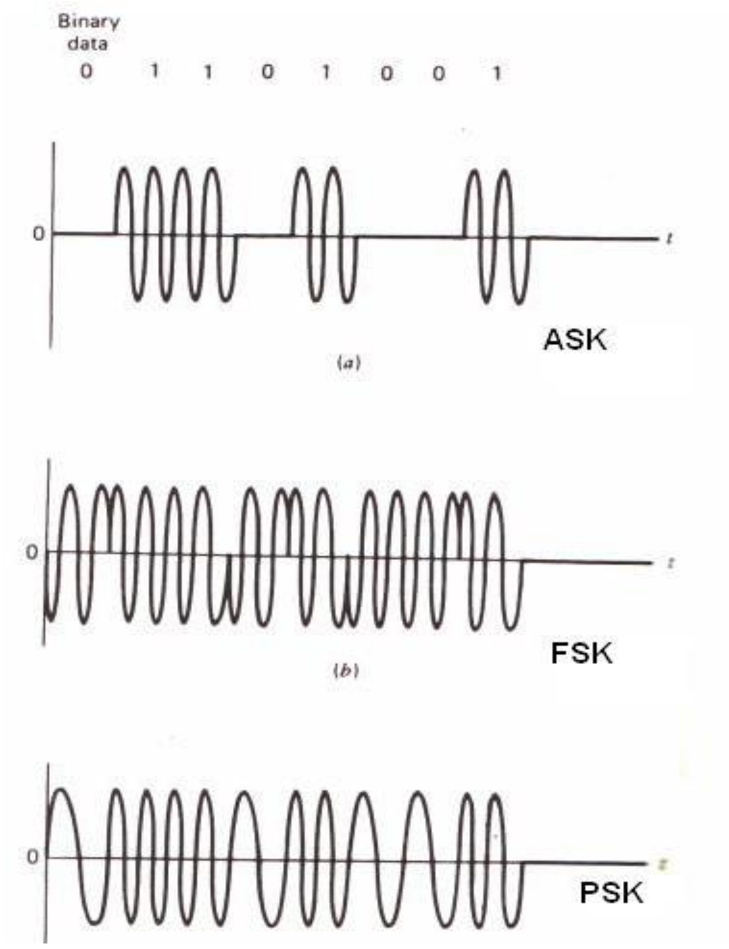


Fig 1

Demodulation at the receiver can be either

a)Coherent or

b) Noncoherent detection.

a) Coherent:

-Exact replicas of possible arriving signals are available at the receiver.

-Receiver is phase-locked to the transmitter.

- It is performed by cross-correlating the received signal with each one of the replicas, and then making a decision based on comparisions with preselected thresholds.

b) Non-coherent:

-Knowledge of the carrier wave's phase is not required.

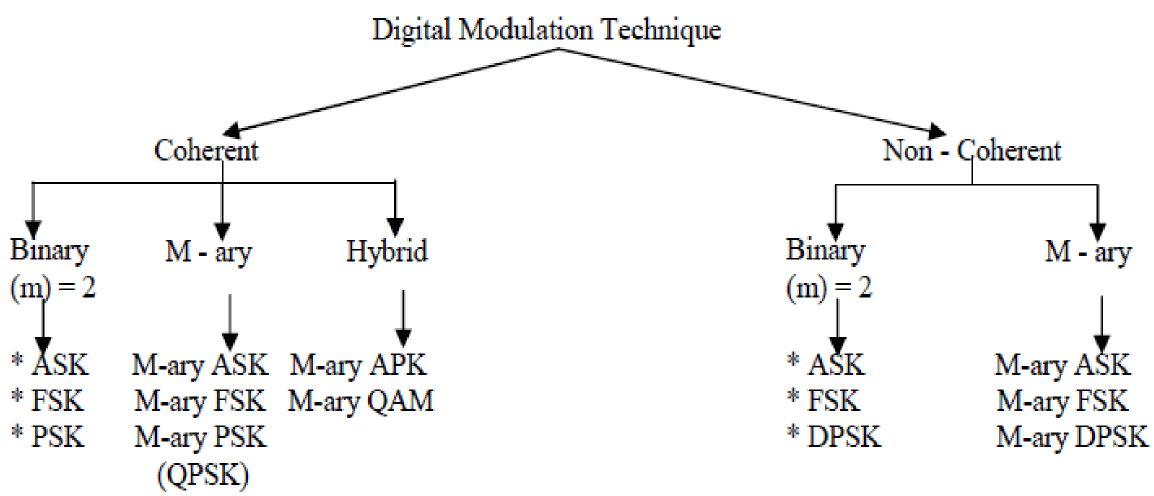
-Complexity of the receiver is reduced.

-It exhibits an inferior error performance, compared to a coherent system.

The choice among different scheme is made based on attaining as many of the following design goals.

1. Maximum data rate.
2. Minimum probability of symbol error.
3. Minimum transmitted power.
4. Minimum channel bandwidth
5. Maximum resistance to interfering signals
6. Minimum circuit complexity.

Hierarchy of digital modulation technique:



Amplitude Shift Keying

A binary ASK wave can be defined as

$S_t = A_c\, m(t)\ \cos 2\pi f_c t\ ,\ 0 \le t \le T_b$

where A_c is amplitude of carrier, m(t) is digital information signal.

f_c is carrier frequency, T_b is bit duration.

In a binary ASK , symbol '1' and '0' are transmitted as

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\ \cos\,(2\pi f_1\, t) \quad \text{for symbol 1}$$

$$s_2(t) = 0 \qquad\qquad \text{for symbol 0}$$

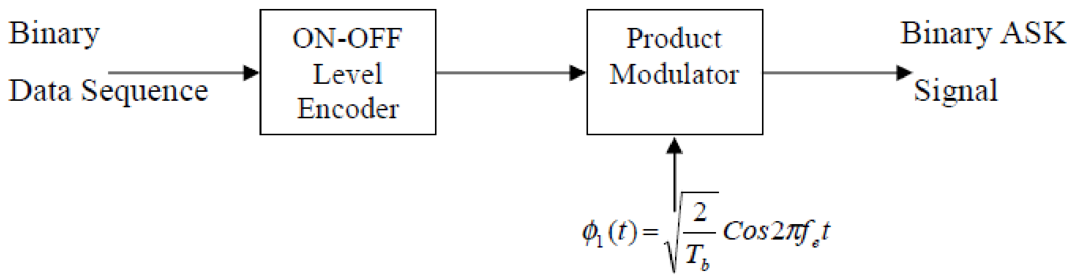


Fig 2: Block diagram of ASK Transmitter

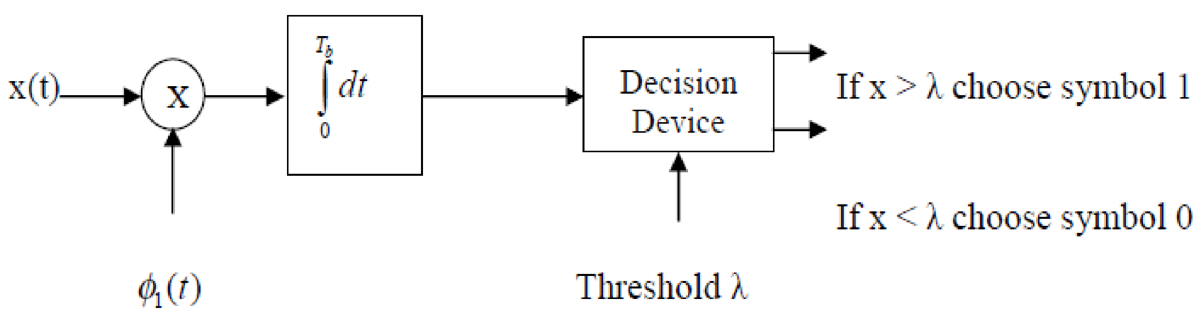


Fig 3:Block diagram of ASK Receiver

Coherent Binary PSK

In a coherent binary PSK system, the pair of signals, $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0 respectively.

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\ \cos\,(2\pi f_c\, t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\ \cos(2\pi f_c\, t + \pi) = -\sqrt{\frac{2E_b}{T_b}}\ \cos(2\pi f_c\, t) \quad \ldots\ldots\ldots\ldots\ldots.. \ (2)$$

where $0 \le t \le T_b$ and E_b is the transmitted signal energy per bit.

In binary PSK, it is clear that there is only one basis function of unit energy

$\Phi_1\ (t) = \sqrt{\frac{2_b}{T_b}}\ \ \cos\ (2\pi f_c\ t)$ 0≤t≤ T_b

Therefore transmitted signals

$s_1\ (t) = \sqrt{E_b}\ \ \Phi_1\ (t)$ 0≤t≤ T_b

$s_2\ t) = -\sqrt{E_b}\ \ \Phi_1(\mathrm{t})$ 0≤t≤ T_b

Here, Signal space is one-dimensional (N=1) and with two message points (M=2)

The coordinates of the message points equal

$$s_{11}\ \ = \int_0^{T_b} s_1\ (t)\Phi_1\ (t)\, dt =\ +\sqrt{E_b}$$

$$s_{21}\ \ = \int_0^{T_b} s_2\ (t)\Phi_1\ (t)\, dt =\ -\sqrt{E_b}$$

Signal space diagram for coherent binary PSK system is shown in fig. 4. Z_1 and Z_2 are decision regions.

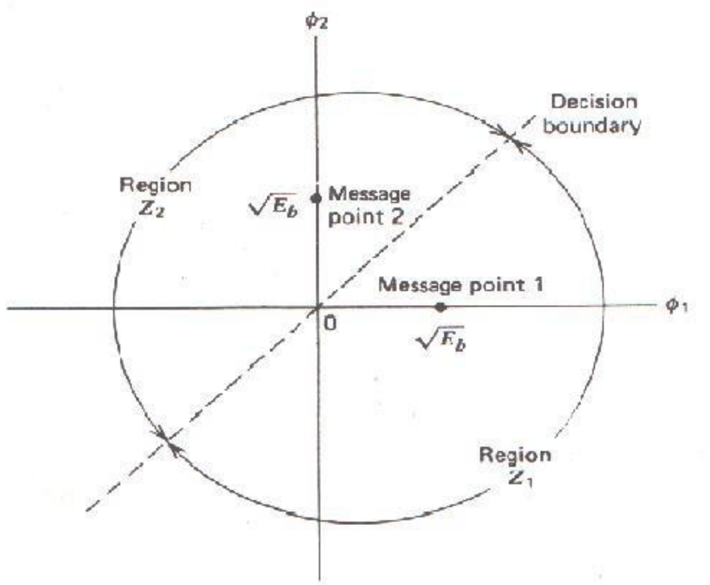


Fig 4. Signal space diagram for coherent binary PSK

<u>Binary PSK Transmitter and Receiver:</u>

To generate a binary PSK wave, input binary sequences are represented in polar form with symbols 1 and 0 by constant amplitude levels of $+\sqrt{E_b}$ and $-\sqrt{E_b}$ respectively. This binary wave and a sinusoidal carrier wave $\Phi_1\ (t)$ whose frequency f_c=n_c /T_b for some fixed integer n_c are applied to a product modulator as shown in fig 4a. The carrier and the timing pulses used to generate the binary wave are usually extracted from a common master clock. The desired PSK wave is obtained at the modulator output.

To detect the original binary sequence of 1s and 0s, noisy PSK wave x(t) (at the channel output) is applied to a correlator, which is also supplied with a locally generated coherent reference signal $\Phi_1\ (t)$ as shown in fig 5b. The correlator output x_i, is compared with a threshold of zero volts. If x_1>0, the receiver decides in favor of symbol1. If x_1<0, it decides in favor of symbol 0.

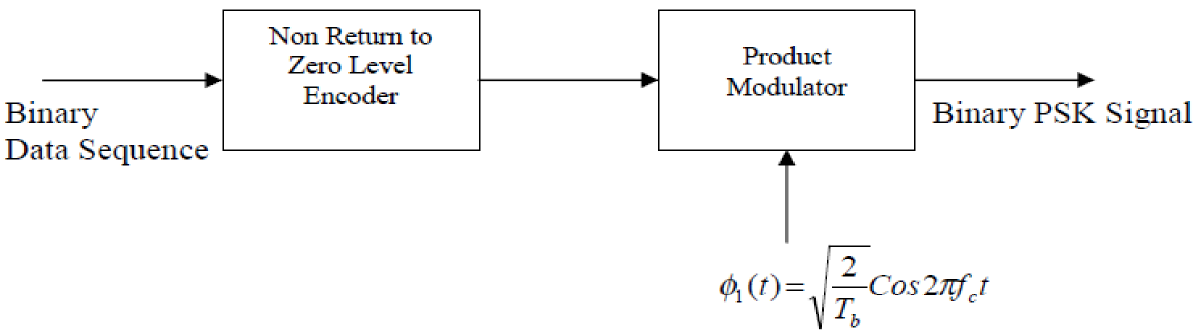


Fig 5a. Block diagram of BPSK Transmitter

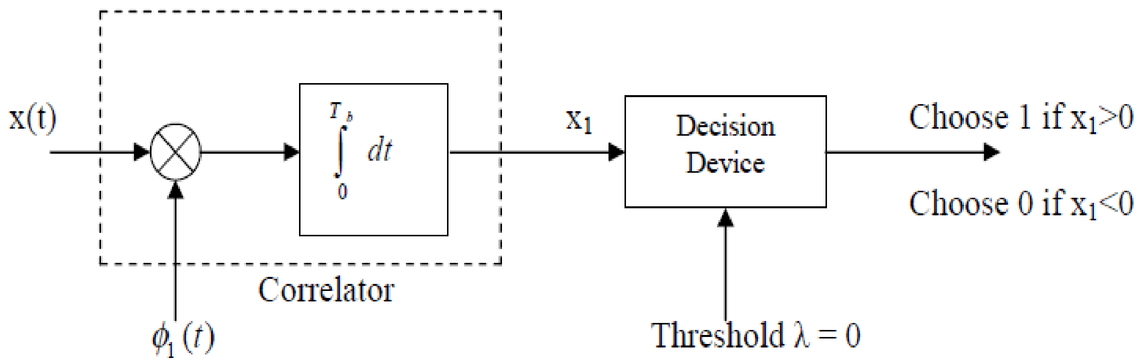


Fig 5b. Block diagram of BPSK Receiver.

Coherent Binary FSK

In a coherent binary FSK system, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency by a fixed amount.

Pair of sinusoidal waves is described by

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \; \cos(2\pi f_i t) & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

Where i=1, 2, and E_b is the transmitted signal energy per bit.

The transmitted frequency equals $f_i = \frac{n_c + i}{T_b}$ for some fixed integer n_c and i=1,2.

Thus symbol 1 is represented by $s_1(t)$ and symbol 0 by $s_2(t)$.

And $$\Phi_1(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \; \cos(2\pi f_i t) & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

where i=1,2.

For i=1,2 and j=1,2;

$$s_{ij} = \int_0^{T_b} s_i(t)\Phi_j(t)\,dt$$

$$= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}}\ \mathrm{COS}(2\pi f_i\, t) \sqrt{\frac{2}{T_b}}\ \mathrm{COS}\,(2\pi f_i\, t)dt$$

$$= \begin{cases} \sqrt{E_b} & i=j \\ 0 & i\neq j \end{cases}$$

Coherent binary FSK system is characterized by having a signal space that is two-dimensional (N=2) with two message points(M=2).

The two message points are defined by the signal vectors

$$S_1 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}$$

The distance between the two message points is equal to $\sqrt{2E_b}$.

The observation vector x has two elements x_1 and x_2, that are defined by respectively

$$x_1 = \int_0^{T_b} x(t)\Phi_1(t)\,dt$$

$$x_2 = \int_0^{T_b} x(t)\Phi_2(t)\,dt$$

Where x(t) is the received signal., the form of which depends on which symbol was transmitted.

If symbol 1 was transmitted, $x(t)=s_1(t)+w(t)$

If symbol 0 was transmitted, $x(t)=s_2(t)+w(t)$

Where w(t) is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$.

Here observation space is partitioned into two decision regions Z_1 and Z_2. The receiver decides in favor of symbol 1 if the received point represented by the observation vector x falls inside region Z_1. This occurs when $x_1>x_2$.

If $x_1<x_2$, the received signal point falls inside region Z_2. And receiver decides in favor of symbol 0.

The decision boundary, separating region Z_1 from region Z_2 is defined by $x_1=x_2$

Signal space diagram for coherent binary FSK system :

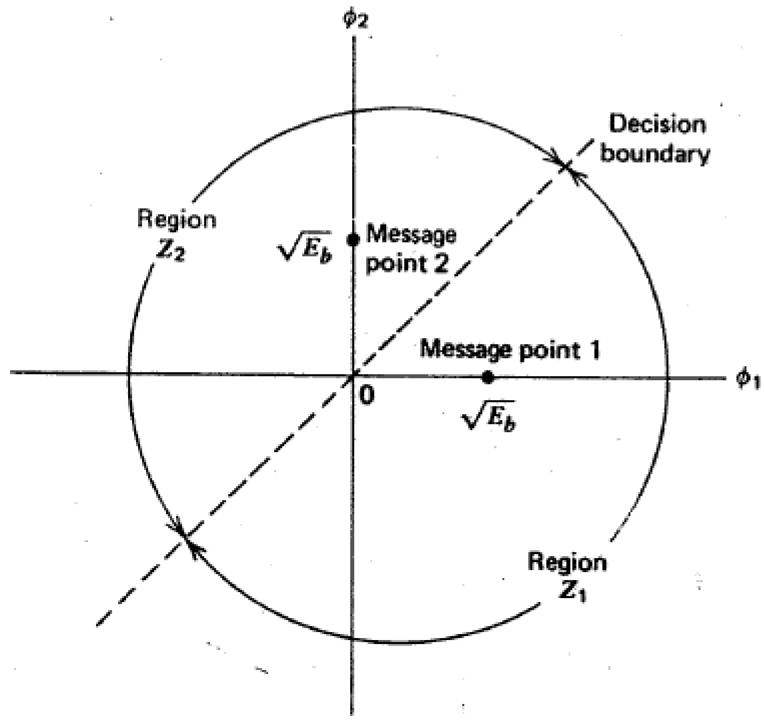


Fig 6. Signal space diagram for coherent binary FSK.

Binary FSK Transmitter and Receiver:

The input binary sequence is represented in its on-off form, with symbol 1 represented by a constant amplitude of $\sqrt{E_b}$ volts and symbol 0 represented by 0 volts. By using an inverter in the lower channel , with symbol 1 at its input, the oscillator with frequency f_1 in the upper channel is switched on while the oscillator with frequency f_2 in lower channel is switched off, hence frequency f_1 is transmitted.

Conversly, with symbol 0 at its input, the oscillator in the upper channel is switched off while the oscillator in lower channel is switched on, hence frequency f_2 is transmitted. The two frequencies f_1 and f_2 are chosen to equal integer multiples of the bit rate $1/T_b$.

In the transmitter, assume that the two oscillators are synchronized, so that the their outputs satisfy the requirements of the two orthonormal basis functions $\Phi_1\,(t)$ and $\Phi_2\,(t)$

The receiver consists of two correlators with a common input, which are supplied with locally generated coherent reference signals $\Phi_1\,(t)$ and $\Phi_2\,(t)$. The correlators outputs are then subtracted, one from the other, and the resulting difference "ℓ" is compared with a threshold of zero volts.

If $\ell > 0$, receiver decides in favor of 1. If $\ell < 0$, decides in favor of 0.

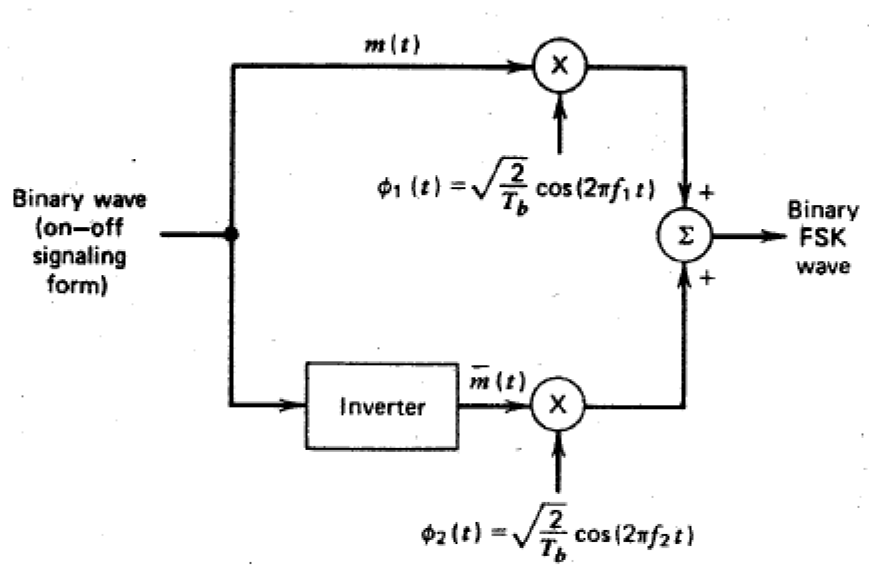


Fig 7a. Block diagram of Binary FSK transmitter

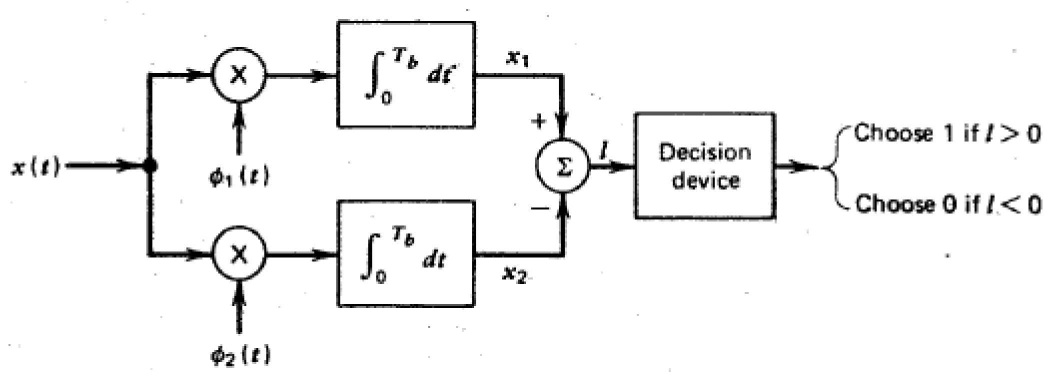


Fig 7b. Block diagram of Binary FSK receiver

Quadrature Phase Shift Keying

The QPSK is characterised by the fact that the information carried by the transmitted wave is contained in the phase. The phase of the carrier takes on one of four equally spaced values, such as π/4, 3π/4, 5π/4, 7π/4.Each possible value of the phase corresponds to a unique pair of bits called a dibit. (ex; set of phase values may represent the gray encoded set of dibits: 10,00,01 and 11)

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_C t + (2i-1)\frac{\pi}{4}\right] & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$$

Where i=1,2,3,4

E is transmitted signal energy per symbol. T is the symbol duration and carrier frequency f_c equals n_c/T for some fixed integer n_c.

$$
S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[(2i-1)\frac{\pi}{4}\right] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin\left[(2i-1)\frac{\pi}{4}\right] \sin(2\pi f_c t) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}
$$

where i=1,2,3,4

- There are only two orthonormal basis functions $\Phi_1(t)$ and $\Phi_2(t)$ contained in the expansion of $S_i(t)$

$$
\Phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_C t) \qquad 0 \le t \le T
$$

$$
\Phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_C t) \qquad 0 \le t \le T
$$

- There are four message points, and the associated signal vectors are defined by

$$
S_i = \begin{pmatrix} \sqrt{E} \cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E} \cos\left((2i-1)\frac{\pi}{4}\right) \end{pmatrix} \qquad i=1,2,3,4
$$

Table 1. Signal-space characterization of QPSK.

Input dibit $0 \leq t \leq T$	Phase of QPSK signal (*radians*)	Coordinates of message points	
		s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

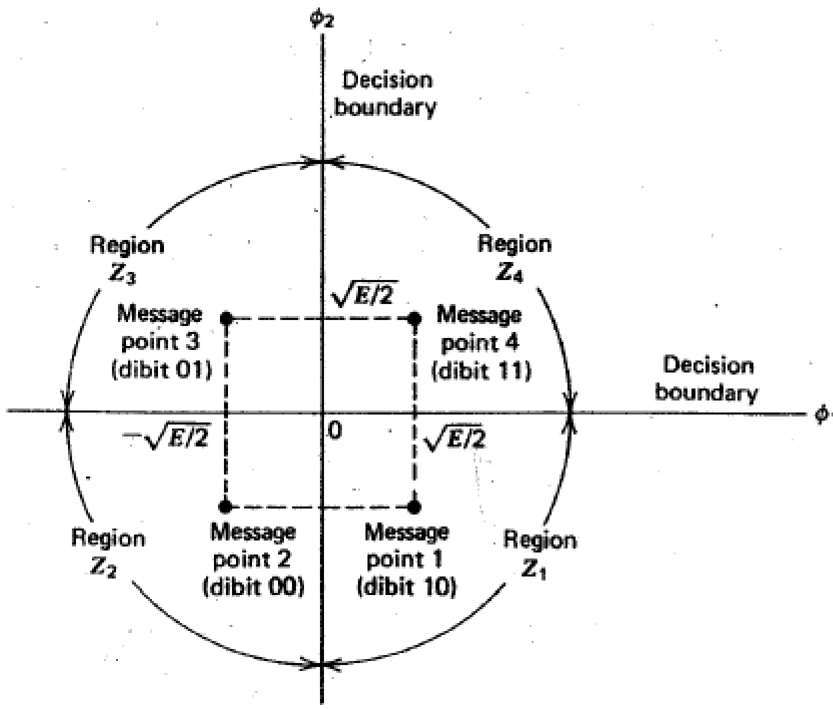


Fig8. Signal-space diagram

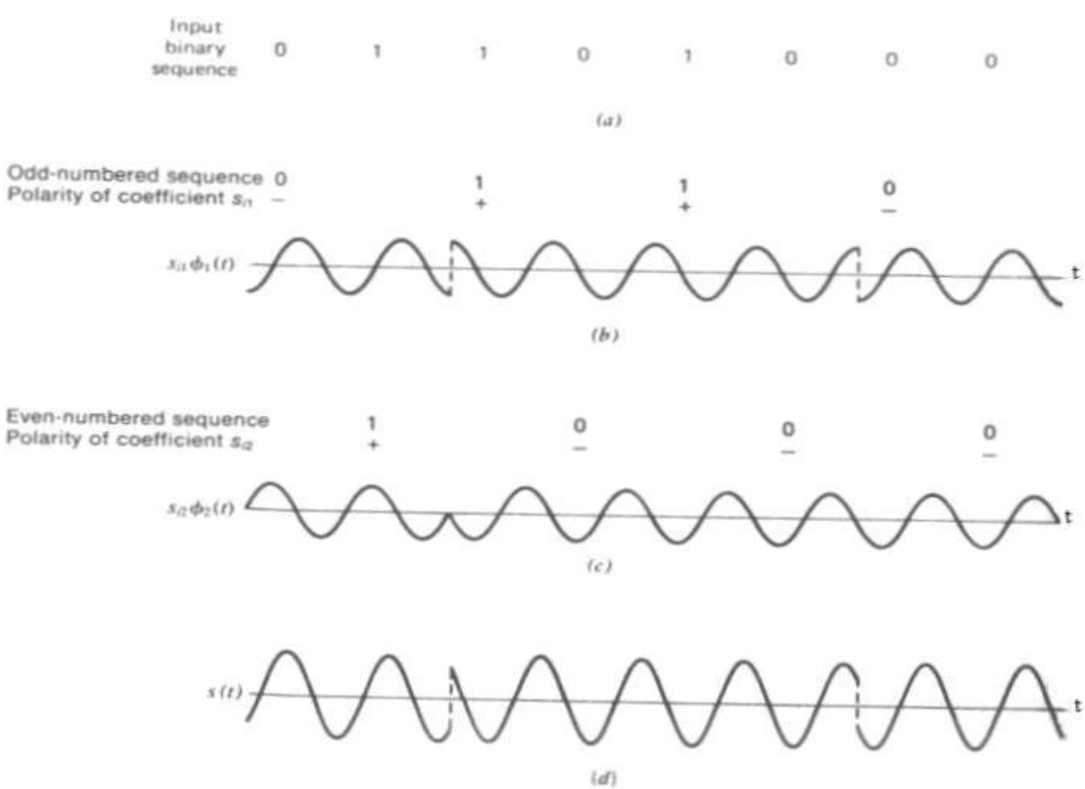


Fig 9. (d) QPSK WAVEFORM

QPSK transmitter:

The fig 10a shows block diagram of QPSK transmitter.

The input binary sequence b(t) is represented in polar form., with symbols 1 and 0 represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$ respectively. The binary wave is divided by means of a demultiplexer into two separate binary waves consisting of the odd-and even numbered input bits. These two binary waves are denoted by $b_1(t)$ and $b_2(t)$. In any signaling interval, the amplitudes of $b_1(t)$ and $b_2(t)$ equals s_{i1} and s_{i2} respectively, depending on the particular dibit that is being transmitted. The two binary waves are denoted by $b_1(t)$ and $b_2(t)$ are used to modulate a pair of quadrature carriers or orthonormal basis functions: $\Phi_1\ (t)$ and $\Phi_2\ (t)$. The result is a pair of binary PSK waves, which may be detected independently due to the orthogonality of $\Phi_1\ (t)$ and $\Phi_2\ (t)$. Finally the two binary PSK waves are added to produce the desired QPSK wave. Symbol duration T of a QPSK wave is twice as long as the bit duration $T_{b,}$ of the input binary wave. For a given bit rate 1/ $T_{b,}$ a QPSK wave requires half the transmission bandwidth of the corresponding binary PSK wave.

The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated pair of coherent reference signals $\Phi_1\ (t)$ and $\Phi_2\ (t)$. The correlators output x_1 and x_2 are each compared with a threshold of zero volts.

If $x_1>0$, a decision is made in favor of symbol 1 for the upper or in-phase channel output, but if $x_1<0$, a decision is made in favor of symbol 0. If $x_2>0$, decision is made in favor of symbol 1 for the lower or quadrature channel output, but if $x_2<0$, a decision is made in favor of symbol 0.

Finally, these two binary sequences at the in-phase and quadrature channel outputs are combined in a multiplexer to reproduce the original binary sequence at the transmitter input with the minimum probobality of symbol error.

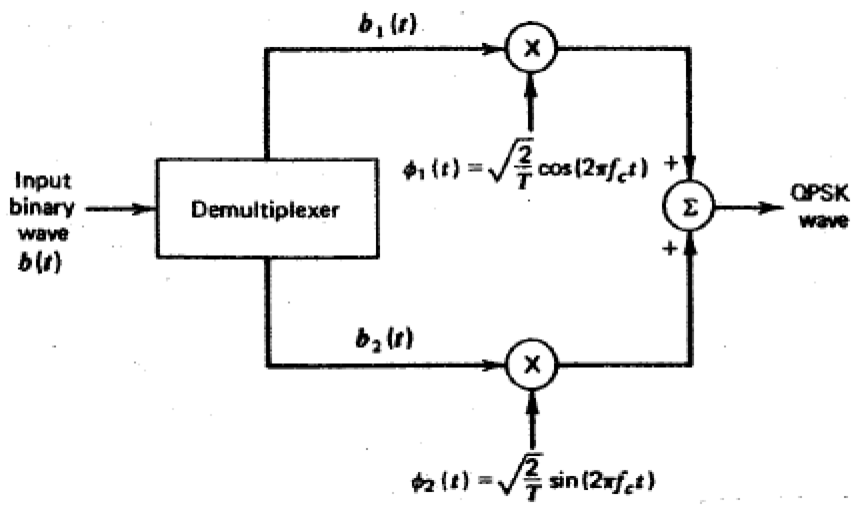


Fig 10a. Block diagram of QPSK transmitter

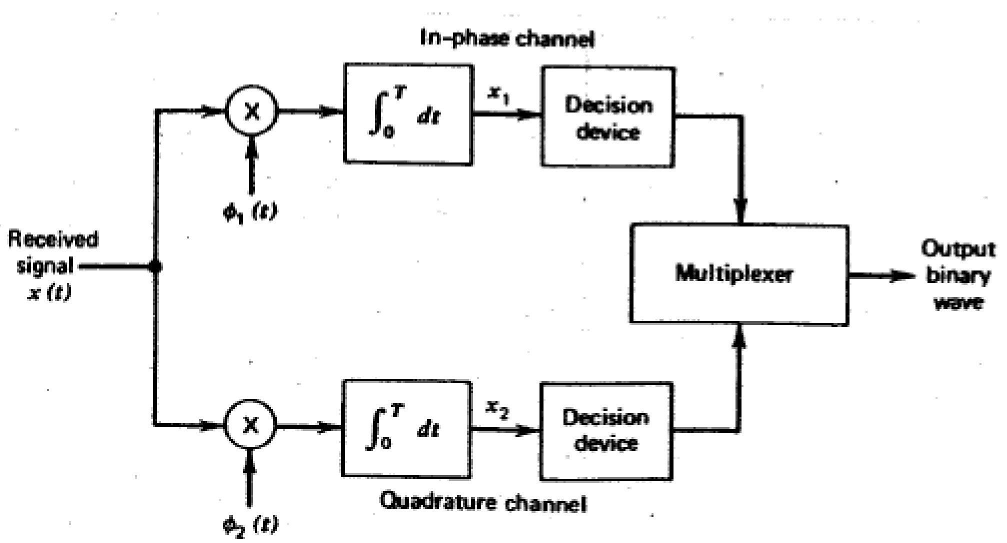


Fig 10b. Block diagram of QPSK Receiver

Differential Phase Shift Keying:

Differential phase –shift keying is the non-coherent version of the PSK. It eliminates the need for a coherent reference signal at the receiver by combining two basic operations at the transmitter.

1) Differential encoding of the input binary wave.
2) Phase-shift keying

   Hence, the name differential phase shift keying.

To send symbol 0, phase advance the current signal waveform by 180^0. To send symbol 1, leave the phase of the current signal waveform unchanged. The receiver is equipped with a storage capability, so that it can measure the relative phase difference between the waveforms received during two successive bit intervals. The unknown phase θ contained in the received wave varies slowly, the phase difference between waveforms received in two successive bit intervals will be independent of θ.

$$s_1(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t) & 0 \le t \le T_b \\ \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t) & T_b \le t \le 2T_b \end{cases}$$

Let $s_2(t)$ denote the transmitted DPSK signal for $0 \le t \le 2T_b$, for the case when there is binary symbol 0 at the transmitter input for $T_b \le t \le 2T_b$. The transmission of 0 advances the carrier phase by 180^0

$$s_1(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t) & 0 \le t \le T_b \\ \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t + \Pi) & T_b \le t \le 2T_b \end{cases}$$

- DPSK is a special case of noncoherent orthogonal modulation with $T=2T_b$.
  And $E=2E_b$
- Average probability of error of DPSK is given by

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_O}\right)$$

The differential encoding process at the transmitter input starts with an arbitrary first bit, serving as reference, and thereafter the differentially encoded sequence $\{d_k\}$ is generated by using the logical equation

$$\mathbf{d_k = d_{k-1}\, b_k + \bar{d}_{k-1}\, \bar{b}_k} \qquad \text{modulo-2}$$

where b_k is the input binary digit at time kT_b, and d_{k-1} is the previous value of the differentially encoded digit. The use of an overbar denotes logical inversion.

Table 2 illustrates the logical operations involved . Assuming that the reference bit added to the differentially encoded sequence $\{d_k\}$is a 1. The differentially encoded sequence $\{d_k\}$thus generated is used to phase-shift key a carrier with the phase angles 0 and π radians.

Table 2:Illustrating the generation of DPSK signal

$\{b_k\}$		1	0	0	1	0	0	1	1
$\{\bar{b}_k\}$		0	1	1	0	1	1	0	0
$\{d_{k-1}\}$		1	1	0	1	1	0	1	1
$\{\bar{d}_{k-1}\}$		0	0	1	0	0	1	0	0
$\{b_k d_{k-1}\}$		1	0	0	1	0	0	1	1
$\{\bar{b}_k \bar{d}_{k-1}\}$		0	0	1	0	0	1	0	0
Differentially encoded sequence $\{d_k\}$	1	1	0	1	1	0	1	1	1
Transmitted phase (*radians*)	0	0	π	0	0	π	0	0	0

DPSK Transmitter and Receiver

The block diagram of a DPSK transmitter is shown in fig 11a. It consists of a logic network and a one-bit delay element interconnected so as to convert an input binary sequence $\{b_k\}$into a differentially encoded sequence $\{d_k\}$. This sequence is amplitude-level shifted and then used to modulate a carrier wave of frequency f_c, thereby producing the desired DPSK wave.

A method of demodulating the DPSK signal is shown in fig 11b. At the receiver input, the received DPSK signal plus noise is passed through a bandpass filter centered at the carrier frequency f_c, to limit the noise power. The filter output and the delayed version of it, with the delay equal to the bit duration T_b, are applied to a correlator. The resulting correlator output is proportional to the cosine of the difference between the carrier phase angles in the two correlator inputs. Finally the correlator output is compared with a threshold of zero volts, and a decision is therby made in favor of symbol 1 or symbol 0. If the correlator output is positive, the phase difference between the waveforms received during the pertinent pair of intervals lies inside the range –π/2 to π/2, receiver decides in favor of symbol 1. If, correlator output is negative, the phase difference lies outside the range –π/2 to π/2, modulo -2 π, receiver decides in favor of symbol 0.

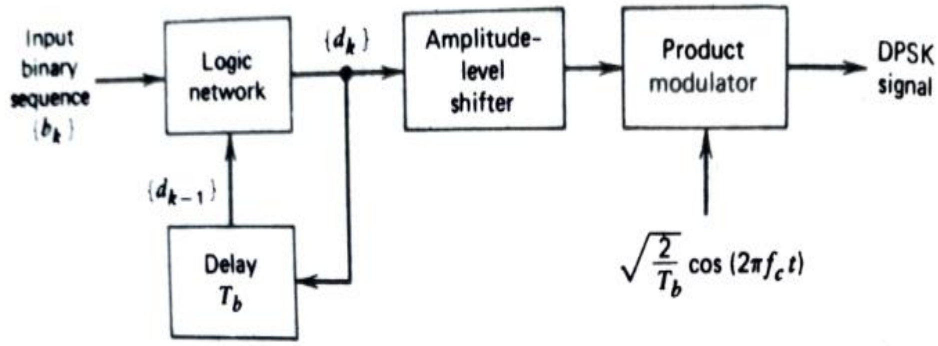


Fig11a: Block diagram of DPSK Transmitter

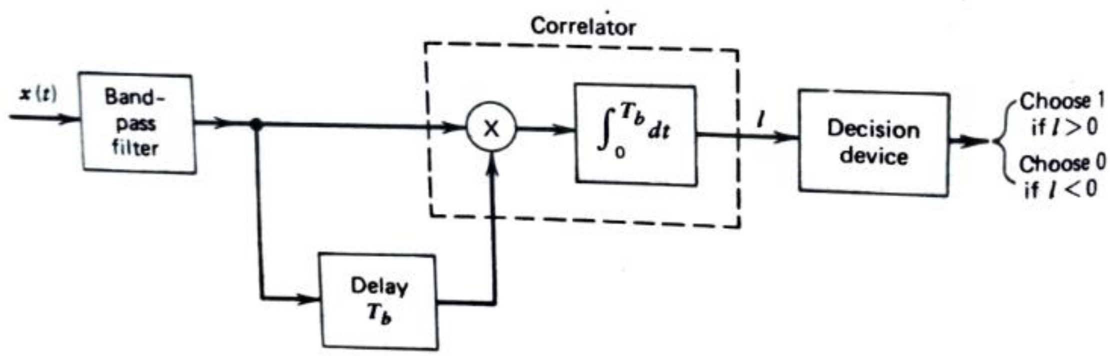


Fig11b: Block diagram of DPSK receiver

M-ary QAM

In M-ary PSK system, in-phase and quadrature components of the modulated signal are interrelated in such a way that the envelope is constrained to remain constant. This constraint manifests itself in a circular constellation for the message points. If this constraint is removed, and therby in-phase and quadrature components are permitted to be independent, the new obtained modulation scheme is called M-ary quadrature amplitude modulation (QAM).Here, carrier experiences amplitude as well as phase modulation.

The signal constellation for M-ary QAM consists of a square lattice of message points as shown in fig 12 for M=16. The corresponding signal constellations for the in- phase and quadrature components of the amplitude –phase modulated wave or shown in figs 13a and 13b respectively and the basic format of the signal constellations shown in fig is recognized to be that of a polar L-ary ASK signal with L=4, thus , in general, an M-ary QAM scheme enables the transmission of $M = L^2$ independent symbols over the same channel bandwidth as that required for one polar L-ary ASK scheme.

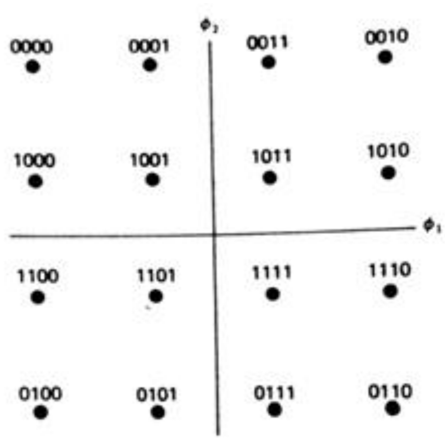


Fig 12: Signal constellation of M-ary QAM for M=16.

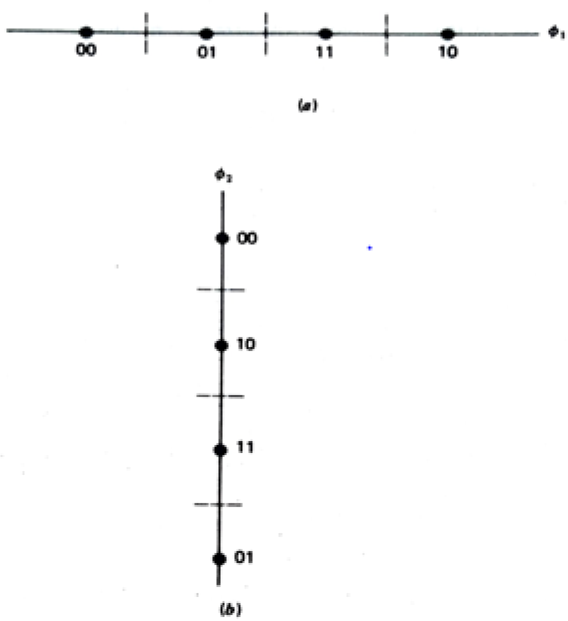


Fig 13: Decomposition of signal constellation of M-ary QAM(for M=16) into two signal-space diagrams for (a) in-phase component $\Phi_1\ (t)$, and (b) quadrature component $\Phi_2\ (t)$.

$s_i(t) = \sqrt{\frac{2E_0}{T}}\ a_i \cos\ (2\pi f_c\ t)\ +\ \sqrt{\frac{2E_0}{T}}\ b_i \sin\ (2\pi f_c\ t)$ $\qquad 0 \le t \le T$

Where E_0 is the energy of the signal with the lowest amplitude, and a_i and b_i are a pair of independent integers chosen in accordance with the location of the pertinent message point. The signal $s_i(t)$ consists of two phase-quadrature carriers, each of which is modulated by a set of discrete amplitudes:, hence the name ‘quadrature amplitude modulation’

The signal $s_i(t)$ can be expanded in terms of a pair of basis functions:

$\Phi_1\ (t) = \sqrt{\frac{2}{T}}\ \cos\ (2\pi f_C\ t)$ $\qquad 0 \le t \le T$

$\Phi_2\ (t) = \sqrt{\frac{2}{T}}\ \sin\ (2\pi f_C\ t)$ $\qquad 0 \le t \le T$

The coordinates of the ith message point are $a_i\sqrt{E}$ and $b_i\ \sqrt{E_0}$, where (a_i, b_i) is an element of the L-by – L matrix.

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \dots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \dots & (L-1, L-3) \\ \vdots & \vdots & & \vdots \\ (-L+1, -L+1) & (-L+3, -L+1) & \dots & (L-1, -L+1) \end{bmatrix}$$

Where L= $\sqrt{M}$

For example, for the 16- QAM, (Here L=4), the matrix is given by

$$\{a_i, b_i\} = \begin{bmatrix} (-3,3) & (-1,3) & (1,3) & (3,3) \\ (-3,1) & (-1,1) & (1,1) & (3,1) \\ (-3,-1) & (-1,-1) & (1,-1) & (3,-1) \\ (-3,-3) & (-1,-3) & (1,-3) & (3,-3) \end{bmatrix}$$

Signal constellation for special case of M-ary QAM for M=4 is given below.

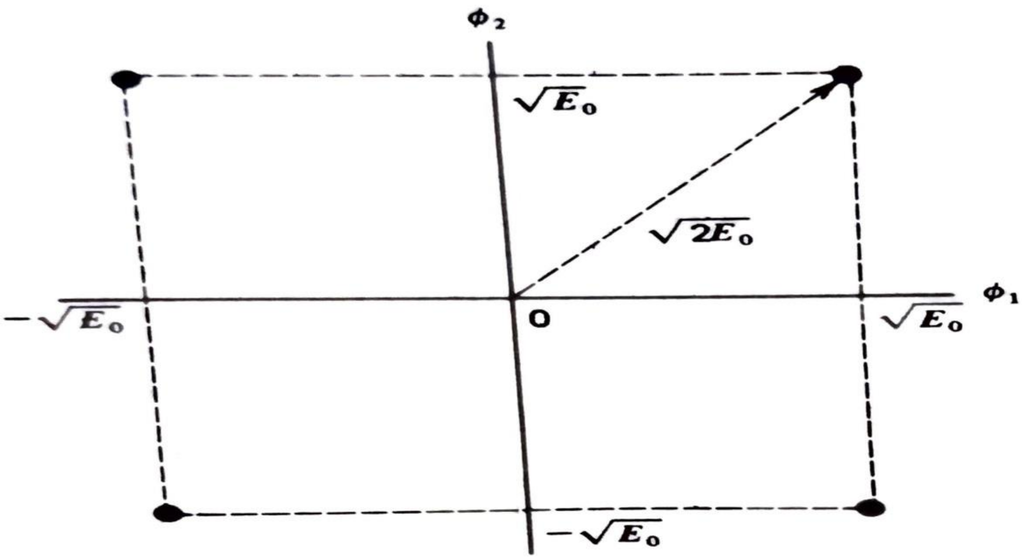


Fig 14: Signal constellation for special case of M-ary QAM for M=4

M-ary QAM Transmitter:

The serial-to-parallel converter accepts a binary sequence at a bit rate $R_b=1/T_b$ and produces two parallel binary sequences whose bit rates are $R_b/2$ each. The 2-to- L level converters, where L= $\sqrt{M}$, generate polar L-level signals in response to the respective in-phase and quadrature channel inputs. Quadrature-carrier multiplexing of the two polar L-level signals so generated produces the desired M-ary QAM signals.

Fig 15b shows block diagram of receiver. Decoding of each baseband channel is accomplished at the output of the pertinent decision circuit, which is designed to compare the L-level signals against L-1 decision thresholds. The two binary sequences so detected are then combined in the parallel –to-serial converter to reproduce the original binary sequence.

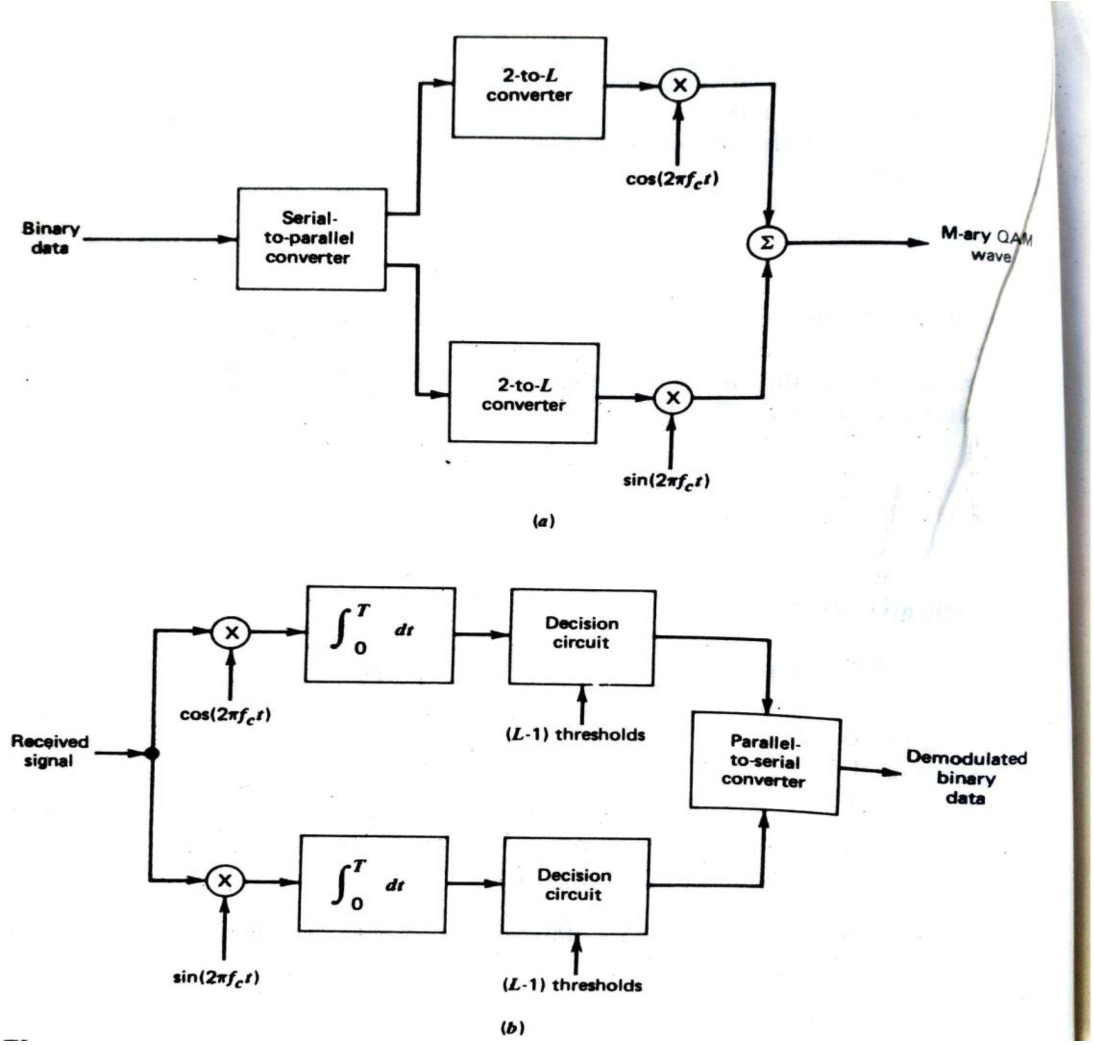


Fig 15. Block diagrams of M-ary QAM system (a) Transmitter (b) Receiver

Probability of error derivation of PSK:

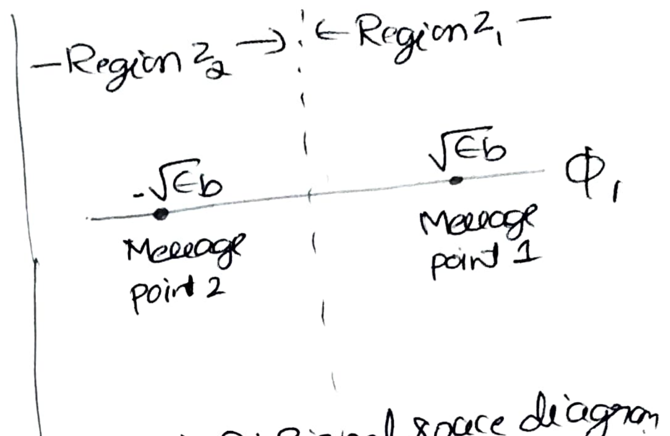


fig ①: Signal space diagram

From fig ①: decision region associated with symbol 1 (or) $S_1(t)$ is given by

$$Z_1 : 0 < x_1 < 1$$

where x_1 is observation scalar

$$x_1 = \int_0^{T_b} x(t)\,\phi_1(t)\,dt \rightarrow ①$$

$x(t)$ is received signal

⇒ WKT, likelihood functions of an AWGN channel are defined by

$$f_X(x|m_i) = (\pi N_0)^{-N/2} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right] \quad i=1,2,..M \rightarrow ②$$

⇒ when symbol 0 (or) signal $S_2(t)$ is transmitted, eqn ② becomes

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right] \quad \Bigg| \quad N=1,\ j=1$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] \rightarrow ③$$

⇒ The conditional probability of receiver deciding in favour of symbol 1, when symbol '0' was transmitted is given by

$$P_e(0) = \int_0^{\infty} f_{X_1}(x_1|0)\,dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\left(\frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}\right)^2\right] dx_1$$

Replace $\left(\frac{(x_1+\sqrt{E_b})}{\sqrt{N_0}}\right)$ by z;

$$z = \frac{x_1+\sqrt{E_b}}{\sqrt{N_0}} = \frac{1}{\sqrt{N_0}}(x_1+\sqrt{E_b})$$

$$\Rightarrow dz = \frac{1}{\sqrt{N_0}}(dx_1)$$

$$\Rightarrow dx_1 = \sqrt{N_0}\, dz$$

when $x=0$; $z=\sqrt{E_b/N_0}$

$x=\infty$; $z=\infty$

$\therefore$

$$Pe(0) = \frac{1}{\sqrt{\pi N_0}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2)\, dz . \sqrt{N_0} \longrightarrow (4)$$

But complementary error function

$$\mathrm{erfc}\left(\sqrt{E_b/N_0}\right) = \frac{2}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2)\, dz \rightarrow (5)$$

$\therefore$ eqn (4) ~~becomes~~ can be rewritten as

$$Pe(0) = \frac{1}{\cancel{\sqrt{N_0}}.\sqrt{\pi}} \cdot \frac{2}{2} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2)\, dz . \cancel{\sqrt{N_0}}$$

$$\therefore Pe(0) = \frac{1}{2}\, \mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

$\Rightarrow$ |||^{ly} $Pe(1) = \frac{1}{2}\, \mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$

Averaging $Pe(0)$ and $Pe(1)$; average probabality of symbol error is given by

$$\boxed{Pe = \frac{1}{2}\, \mathrm{erfc}\left(\sqrt{E_b/N_0}\right)}$$

where 'Pe' is probobality of symbol error for coherent binary PSK

Probability of error derivation of FSK:

⇒ For coherent binary FSK system;

$N = 2$; $M = 2$; (refer fig. 1)

⇒ Observation vector x has 2 elements x_1 and x_2;

$$x_1 = \int_0^{T_b} x(t)\,\phi_1(t)\,dt \rightarrow (1)$$

$$x_2 = \int_0^{T_b} x(t)\,\phi_2(t)\,dt \rightarrow (2)$$

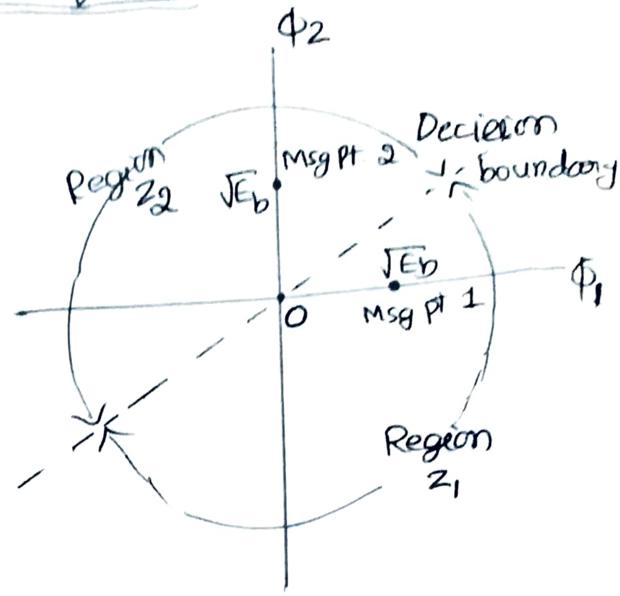


fig 1: Signal space diagram

$x(t)$ is received signal;

⇒ when symbol 1 was transmitted;

$$x(t) = s_1(t) + w(t)$$

⇒ when symbol '0' was transmitted

$$x(t) = s_2(t) + w(t)$$

$w(t)$ is sample function of white Gaussian noise process of zero mean & power spectral density $\frac{N_0}{2}$

⇒ Z_1 and Z_2 are the two decision regions;

⇒ Now, define a new Gaussian random variable 'L' whose sample value 'l' is equal to the difference between x_1 and x_2;

$$\therefore l = x_1 - x_2 \longrightarrow (3)$$

⇒ Mean value of 'L' depends on binary symbol transmitted

when symbol 1 was transmitted;

Gaussian random variables are X_1 and X_2

sample values are x_1 and x_2

mean equal to $\sqrt{E_b}$ and 0

∴ Conditional mean of random variable L, when symbol 1 was transmitted is given by

$$E[L|1] = E[X_1|1] - E[X_2|1] = +\sqrt{E_b} \rightarrow (4)$$

⇒ when symbol '0' was transmitted;

Gaussian random variables are X_1 and X_2

sample values are x_1 and x_2

mean equal to ~~...~~ 0 & $\sqrt{E_b}$

$$\therefore E[L|0] = E[X_1|0] - E[X_2|0] = -\sqrt{E_b} \rightarrow (5)$$

⇒ Variance of random variable L is independent of transmitted symbol.

Random variables X_1 & X_2 are statistically independent, each with a variance $= N_0/2$;

$$Var[L] = var[X_1] + var[X_2] = \frac{N_0}{2} + \frac{N_0}{2} = N_0 \rightarrow (6)$$

⇒ when symbol '0' was transmitted;

conditional probability density function of random variable L is;

$$f_L(l|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right] \rightarrow (7)$$

if $x_1 > x_2$ then $l > 0$; decision in favour of symbol 1;

$$\therefore P_e(0) = P(l > 0 \mid \text{symbol } 0 \text{ was sent})$$

$$= \int_0^\infty f_L(l|0)\,dl = \int_0^\infty \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right] dl$$

Replace $\frac{l+\sqrt{E_b}}{\sqrt{2N_0}} = z$

$$\therefore P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/2N_0}}^{x} \exp(-z^2)\,dz$$

$$P_e(0) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \rightarrow (8)$$

III'y $P_e(1) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$

Averaging $P_e(0)$ & $P_e(1)$; avg. probobality of symbol error is

$$\therefore \boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)}$$

⇑

"probobality of symbol error for coherent binary FSK"

x —— x —— x ——

POWER SPECTRA

→ Generally $s(t)$ is expressed in the form

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$
$$= Re[\tilde{s}(t)\exp(j2\pi f_c t)] \rightarrow (1)$$

where $Re[\cdot]$ is the real part of the expression contained inside the square brackets.

$$\tilde{s}(t) = s_I(t) + j s_Q(t) \rightarrow (2)$$

and

$$\exp(j2\pi f_c t) = \cos(2\pi f_c t) + j\sin(2\pi f_c t) \rightarrow (3)$$

→ The signal $\tilde{s}(t)$ is called the 'complex envelope' of the band-pass signal $s(t)$.

→ The components $s_I(t)$ and $s_Q(t)$ and $\tilde{s}(t)$ are all low-pass signals.

→ Let $S_B(f)$ denote the power spectral density density of the complex envelope $\tilde{s}(t)$.

→ $S_B(f)$ is also refered as baseband power spectral density.

→ $S_s(f)$ is the power spectral density of the original band-pass signal $s(t)$;

It is a frequency-shifted version of $S_B(f)$; (except for a scaling factor)

$$\therefore S_s(f) = \frac{1}{4}[S_B(f - f_c) + S_B(f + f_c)] \rightarrow (4)$$

∴ It is sufficient to evaluate $S_B(f)$.

1) Power Spectra of Binary PSK and FSK signals.

→ For PSK, it can be noted that, complex envelope of PSK consists of an in-phase component only.

→ Depending on whether there is a symbol '1' @ a symbol '0' at the modulator input during the signalling interval, $0 \leq t \leq T_b$; in-phase component equals $+g(t)$ @ $-g(t)$ respectively.

where $g(t)$ is the symbol shaping function;

$$g(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} & 0 \leq t \leq T_b \\ 0 & \text{elsewhere} \end{cases} \longrightarrow (5)$$

→ Assume input binary wave is random. Hence $S_B(f)$ of a binary PSK wave becomes

$$S_B(f) = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2}$$

$$= 2E_b \operatorname{sinc}^2(T_b f) \longrightarrow (6)$$

→ This power spectrum falls off as the inverse square of frequency.

⇒ For FSK wave;

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t \pm \frac{\pi t}{T_b}\right) \quad 0 \leq t \leq T_b$$

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left(\pm\frac{\pi t}{T_b}\right)\cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin\left(\pm\frac{\pi t}{T_b}\right)\sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{T_b}\right)\cos(2\pi f_c t) \mp \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{T_b}\right)\sin(2\pi f_c t) \longrightarrow (7)$$

In eq^n (7); plus sign corresponds to transmitting symbol 0. minus sign corresponds to transmitting symbol 1.

Here $g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$ → (8)

→ The energy spectral density of this symbol shaping function equals

$$\Psi_g(f) = \frac{8E_b T_b \cos^2(\pi T_b f)}{\pi^2(4T_b^2 f^2 - 1)^2} \longrightarrow (9)$$

→ power spectral density of quadrature component equals $\Psi_g(f)/T_b$.

$$\rightarrow S_B(f) = \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right] + \frac{8E_b \cos^2(\pi T_b f)}{\pi^2(4T_b^2 f^2 - 1)^2} \rightarrow (10)$$

substituting eq^n (10) in (4); it can be observed that the power spectrum of the binary FSK signal contains two discrete frequency components located at $(f_c + 1/2T_b) = f_1$ and $(f_c - 1/2T_b) = f_2$, with their average powers adding up to one-half the total power of the binary FSK signal.

→ Fig 7.29; shows the power spectra of binary PSK and FSK signals.

→ In both cases, $S_B(f)$ is shown normalized with respect to $2E_b$. and frequency is normalized w.r.t bit rate $R_b = 1/T_b$.

⇒ The differences in the rates of fall off of these spectra can be explained on the basis of the pulse shape $g(t)$.

⇒ The smoother the pulse, the faster is the drop of spectral tails to zero.

⇒ Thus, with binary FSK (continuous phase) having a smoother pulse shape, it has lower sidelobes than binary PSK.

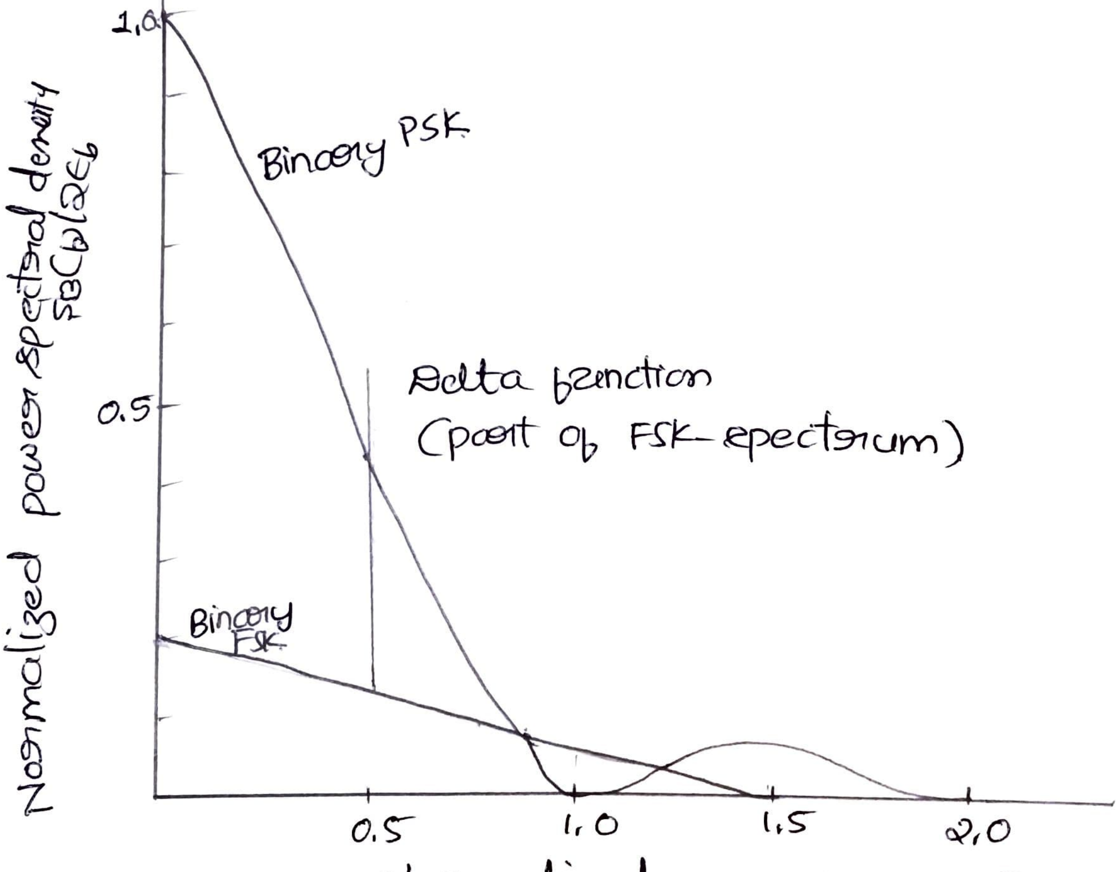


Figure 7.29: Power spectra of binary PSK and FSK signals.

2) Power Spectra of QPSK and MSK signals:

→ Depending on the dibit sent during the signalling interval $-T_b \le t \le T_b$, the in-phase component equals $+g(t)$ or $-g(t)$; and similarly for the quadrature component.

$$g(t) = \begin{cases} \sqrt{\frac{E}{T}} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \rightarrow (11)$$

$$\rightarrow S_B(f) = 2E \operatorname{sinc}^2(Tf)$$
$$= 4E_b \operatorname{sinc}^2(2T_b f) \rightarrow (12)$$

→ For MSK; Depending on the value of phase state $\theta(0)$; in-phase component equals $+g(t)$ or $-g(t)$, where

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) & -T_b \le t \le T_b \\ 0 & \text{elsewhere} \end{cases} \rightarrow (13)$$

$$\& \quad \Psi_g(f) = \frac{32 E_b T_b}{\pi^2} \left[\frac{\cos(2\pi T_b f)}{16 T_b^2 f^2 - 1}\right]^2 \rightarrow (14)$$

Hence, power spectral density of in-phase component equals $\Psi_g(f)/2T_b$.

→ For quadrature component,

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right) & 0 \le t \le 2T_b \\ 0 & \text{elsewhere} \end{cases} \rightarrow (15)$$

and $S_B(f) = 2\left[\frac{\Psi_g(f)}{2T_b}\right]$

$$= \frac{32E_b}{\pi^2}\left[\frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1}\right]^2 \rightarrow (16)$$

Fig 7.30 shows the power spectra of QPSK and MSK signals.

→ power spectral density is normalized w.r.to $4E_b$; frequency is normalized w.r.to bit rate $1/T_b$.

→ For $f \Rightarrow 1/T_b$; $S_B(f)$ of MSK signal falls off as the inverse fourth power of frequency;

In QPSK, it falls off as the inverse square of frequency.

→ MSK does not produce as much interference outside the signal band of interest as QPSK.

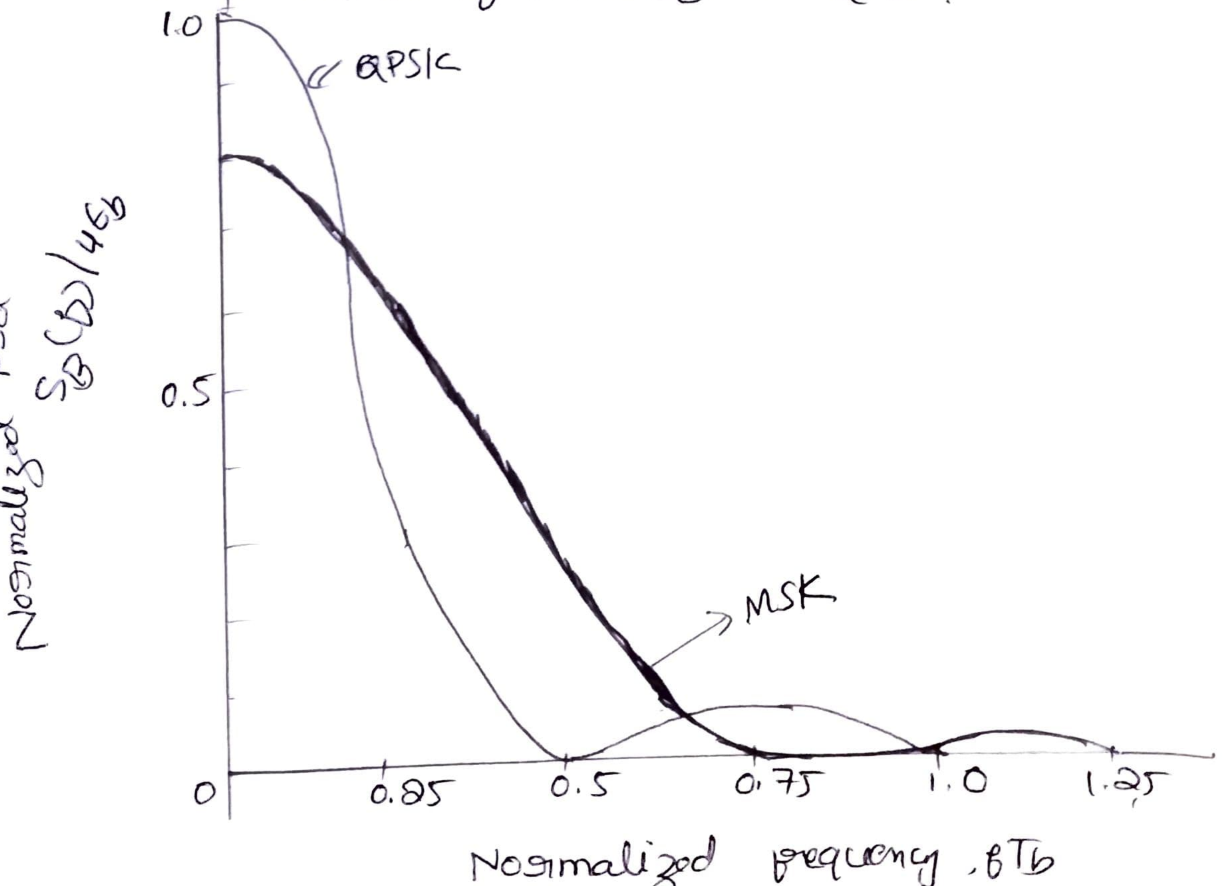


Fig 7.30: Power spectra of QPSK & MSK signals

3] Power Spectra of M-ary Signals

→ Binary PSK and QPSK are special cases of M-ary PSK signals.

→ The symbol duration of M-ary PSK is defined by

$$T = T_b \log_2 M \rightarrow (17)$$

where T_b is the bit duration

→ For M-ary PSK:

$$S_B(f) = 2E \,\text{sinc}^2 (Tf)$$

$$= 2E_b \log_2 M \,\text{sinc}^2 (T_b f \log_2 M) \rightarrow (18)$$

→ Fig 7.31 shows power spectra of M-ary PSK signals. Here power spectral density is normalized w.r.to $2E_b$; frequency is normalized w.r.to bit rate T_b;

→ Spectral analysis of M-ary FSK signal is much more complicated than that of M-ary PSK signals.

→ A case of particular interest occurs when the frequencies assigned to the multilevels make the frequency spacing uniform & freq. deviation $k=0.5$; i.e. M signal frequencies are separated by $1/2T$.

For $K=0.5$; the baseband power spectral density of M-ary FSK signals is defined by

$$S_B(f) = 4E_b\left[\frac{1}{2M}\sum_{i=1}^{M}\left(\frac{\sin\gamma_i}{\gamma_i}\right)^2 + \frac{1}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\cos(\gamma_i+\gamma_j)\left(\frac{\sin\gamma_i}{\gamma_i}\right)^2\left(\frac{\sin\gamma_j}{\gamma_j}\right)^2\right] \rightarrow (19)$$

where $\gamma_i = \left(fT_b - \frac{\alpha_i}{4}\right)\pi$

$\alpha_i = 2i - (M+1) \quad i = 1, 2, \ldots M$

Fig 7.32 is plotted for $M = 2, 4, 8$.

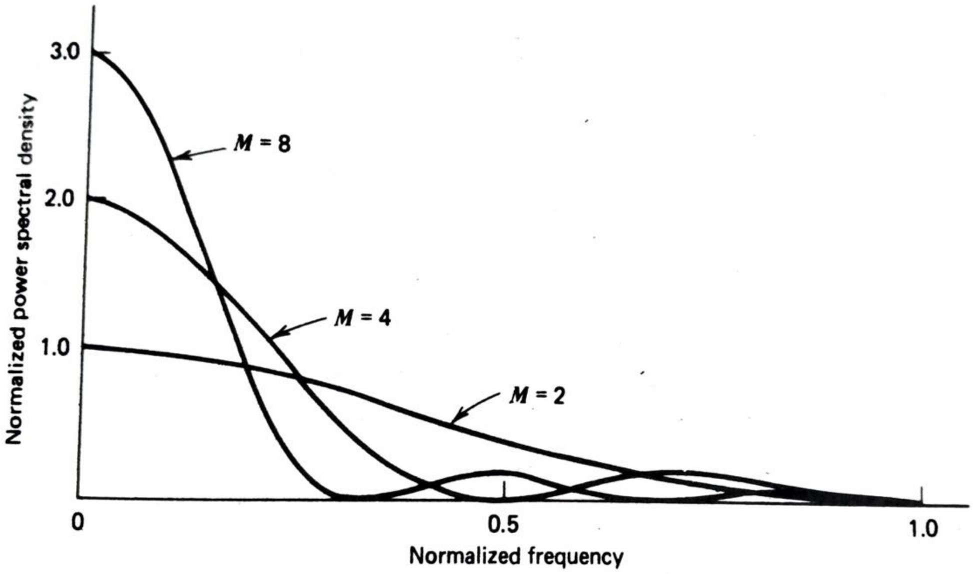


Figure 7.31 Power spectra of M-ary PSK signals for $M = 2, 4, 8$.

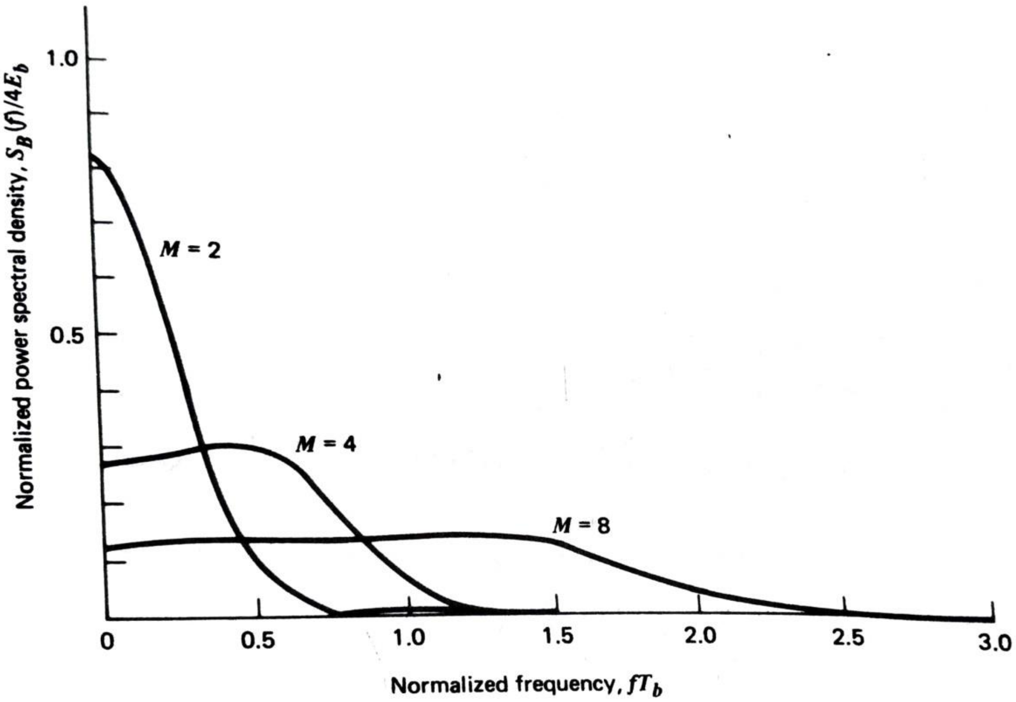


Figure 7.32 Power spectra of M-ary FSK signals for $M = 2, 4, 8$.