ERROR CONTROL CODING:

CHANNEL CODING:-

The main task required in digital communication is to construct cost effective system for transmitting information from end of system at a rate and level of reliability that are acceptable to the user [at the other end].

The coding techniques discussed previously deals with minimizing the average wordlength of the Codes with an objective of achieving higher efficiency.

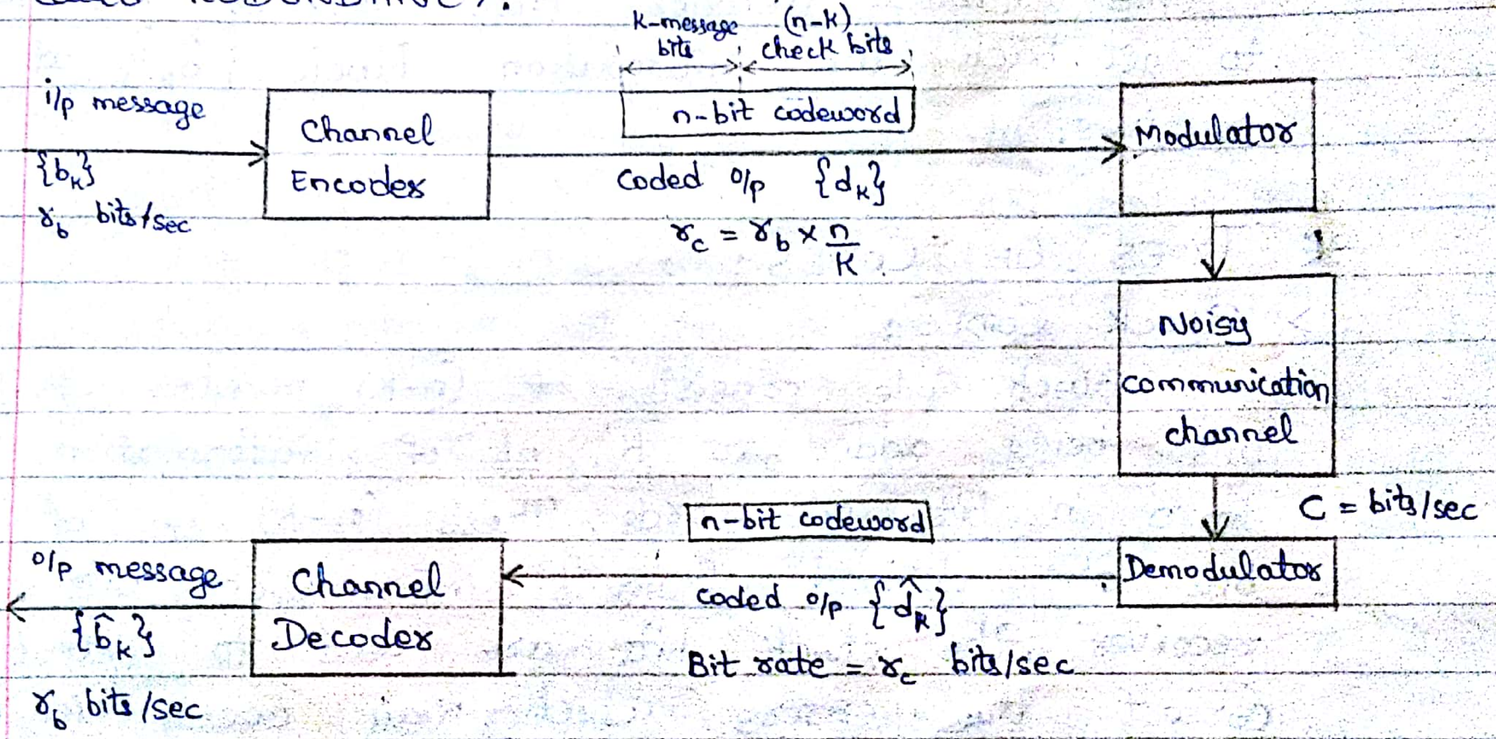
The main disadvantage with this type of coding is that they are variable length codes. Due to this, a single error which occurs due to noise present in the channel affects more than one block code word. Another disadvantage of variable length codes is that the time will fluctuate widely.

When fixed length codes are used, a single error will affect only that block which can be easily detected & corrected.

To detect & correct the errors, error control coding techniques are used that rely on the systematic addition of redundant symbols.

Error control coding is nothing but the calculated use of redundancy. The functional blocks that are used are channel encoder at the transmitter and channel decoder at the receiver.

The channel encoder at the transmitter systematically ~~adds~~ adds the digits to the transmitted message bits. These additional digits carry no information but makes it possible for the channel decoder to detect & correct the error in information bearing digits. This reduces the overall probability of error, thereby achieving the desired goal. These additional digits which carry no information are called REDUNDANT digits & the process of adding these digits is called REDUNDANCY.



Above figure shows block diagram of digital comm^n system employing error control coding. The main functional blocks are channel encoder, channel decoder, modulator, demodulator & noisy communication channel.

The source generates a message block $\{b_k\}$ at a rate of r_b bits/sec & feeds it to the channel encoder. The channel encoder then adds $(n-k)$ no of redundant bits to these k-bit messages to form n-bit code words. These $(n-k)$ no of additional bits are also called as CHECK BITS which donot carry any information but helps the channel decoder to detect & correct the errors. The bit rate of coded output block $\{d_k\}$ will be r_c bits/sec. This is the rate at which the MODEM operates to produce a message block $\{\hat{d}_k\}$ at the receiver.

$$r_c = r_b \times \frac{n}{k}$$

The channel decoder then decodes this message to get back the information block $\{\hat{b}_k\}$ at the receiver.

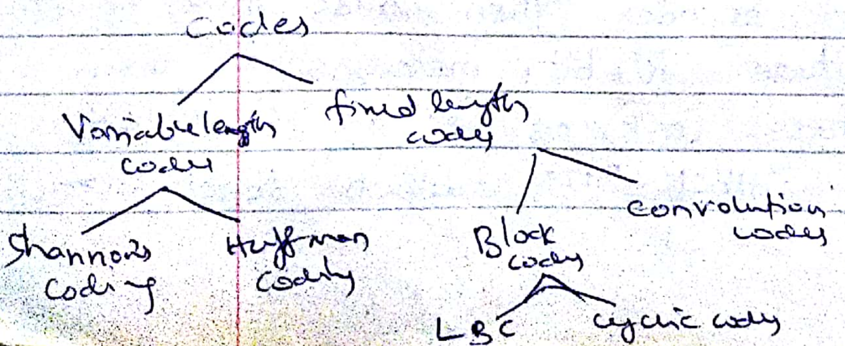
* TYPES OF CODES:

> BLOCK CODES:

Block Code consists of $(n-k)$ number of check bits being added to 'k' no of information bits to form 'n' bit code words. These $(n-k)$ no of check bits are derived from 'k' information bits. At the receiver, the check bits are used to detect & correct the errors which may occur in the entire n-bit code word.

> CONVOLUTION CODES:

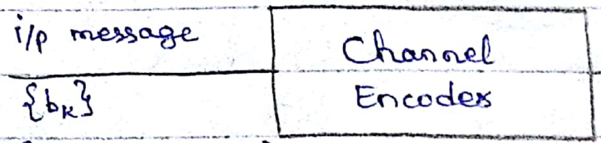
In convolution codes, the check bits are continuously interleaved with information bits. These check bits verify the information bits not only in the block immediately preceeding but in other blocks also.



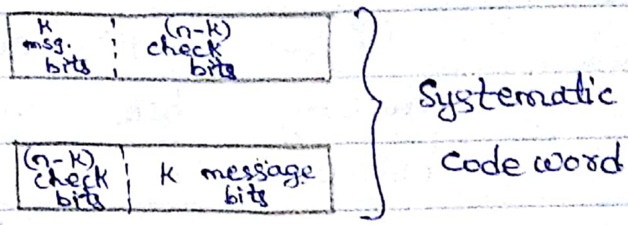
* LINEAR BLOCK CODES :-

$$C = \overbrace{\underbrace{c_1 c_2 \ldots c_k}_{\text{msg. bits}}\ \underbrace{c_{k+1} \ldots c_n}_{\text{check bits}}}^{}$$

$$C = c_1 c_2 c_3 \ldots\ldots c_n$$



$D = \{d_1, d_2, d_3 \ldots d_k\}$



A (n, k) block code is said to be (n, k) linear block code if it satisfies the condition given below.

Let C_a & C_b be any 2 code words [n-bits] belonging to a set of (n, k) block code. Then, if $C_a \oplus C_b$ [modulo-2 arithmetic] is also a n-bit code-word belonging to the same set of (n, k) block code, then, such a block code is called (n, k) LINEAR BLOCK CODE.

* MATRIX DESCRIPTION OF LINEAR BLOCK CODES:

Let the message block of k-bits be represented as a row vector called message vector given by

$$D = \{d_1 d_2 d_3 \ldots d_k\} \quad \text{—} (i)$$

where each message bit can be '0' or '1'. Thus, there are 2^k distinct message vectors.

Each message block is transformed to a codeword of length n-bits & there are 2^k code vectors.

$$C = \{c_1 c_2 c_3 \ldots c_n\} \quad \text{—} (ii)$$

In a systematic linear block code, the message bits appear at beginning of code vector &

remaining $(n-K)$ bits are check bits.

$$C = \{\underbrace{c_1\, c_2\, c_3 \cdots c_K}_{K \text{ message bits}}\ \underbrace{c_{K+1}\, c_{K+2} \cdots c_n}_{(n-K) \text{ check bits}}\}$$

These $(n-K)$ check bits are derived from the K message bits using a predefined rule as given below.

$$\left.\begin{aligned}
C_{K+1} &= P_{11}d_1 + P_{21}d_2 + P_{31}d_3 + P_{41}d_4 + \cdots + P_{K1}d_K \\
C_{K+2} &= P_{12}d_1 + P_{22}d_2 + P_{32}d_3 + P_{42}d_4 + \cdots + P_{K2}d_K \\
C_{K+3} &= P_{13}d_1 + P_{23}d_2 + P_{33}d_3 + P_{43}d_4 + \cdots + P_{K3}d_K \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\
C_n &= P_{1(n-K)}d_1 + P_{2(n-K)}d_2 + P_{3(n-K)}d_3 + \cdots + P_{K(n-K)}d_K
\end{aligned}\right\} \text{(iii)}$$

where $P_{11}, P_{21}, P_{12} \ldots$ are either '0' or '1' & the addition operation is modulo-2 arithmetic.

Combining above 3 equations, the equation result can be expressed in matrix form as.

$$[c_1\, c_2\, c_3 \cdots\cdots c_n] = [d_1\, d_2\, d_3 \cdots d_K] \left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & 0 \cdots & 0 & P_{11} & P_{12} & \cdots & P_{1(n-K)} \\
0 & 1 & 0 & 0 \cdots & 0 & P_{21} & P_{22} & \cdots & P_{2(n-K)} \\
0 & 0 & 1 & 0 \cdots & 0 & P_{31} & P_{32} & \cdots & P_{3(n-K)} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \cdots & 1 & P_{K1} & P_{K2} & \cdots & P_{K(n-K)}
\end{array}\right]$$

$$(1\times n) = (1\times K)\ \underbrace{(K\times K)\qquad (K\times (n-K))}_{K\times n}$$

$$[C] = [D]\,[G]$$

where $[G] \rightarrow$ generator matrix which consists of an identity matrix of order K & a parity matrix.

$$[G] = [I \,\vdots\, P_K]$$

$[G]$ is called Generator matrix of order $(k \times n)$ given by

$$[G] = [I_k \,\vdots\, P]_{k \times n}$$

where I_k is unit matrix of order k
P is an arbitrary matrix called parity matrix of the order $k \times (n-k)$

The Generator matrix can also be expressed as

$$[G] = [P \,\vdots\, I_k]_{k \times n}$$

In this case, the message bits, will be present at the end & the check bits at the beginning of code vector.

P) The Generator matrix for a (6,3) block code is given below. Find all the code vectors.

$$G = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

The message block size for this code is 3 & the length of code vector 'n' is 6. Since $k = 3$, there are $2^k = 2^3 = 8$ message vectors present.

The code vectors are found as follows.

$$[C] = [D]\,[G]$$

$$= [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = [d_1 \;\; d_2 \;\; d_3 \,\vdots\, (d_2+d_3) \;\; (d_1+d_3) \;\; (d_1+d_2)]$$

Message vector $d_1\ d_2\ d_3$	Code Vector $c_1\ c_2\ c_3\ c_4\ c_5\ c_6$
0 0 0	0 0 0 0 0 0
0 0 1	0 0 1 1 1 0
0 1 0	0 1 0 1 0 1
0 1 1	0 1 1 0 1 1
1 0 0	1 0 0 0 1 1
1 0 1	1 0 1 1 0 1
1 1 0	1 1 0 1 1 0
1 1 1	1 1 1 0 0 0

It can be verified that addition of any 2 code vector is a code vector belonging to the same (6,3) code.

Ex: Consider, $C_4 \oplus C_5$

$$\begin{array}{rl} C_4 = & 100011 \\ C_5 = & 101101 \\ \hline & 001110 \end{array} = C_1$$

is a code belonging to same (6,3) code.

$\therefore$ Above code is a linear block code.

P) Repeat the above problem for (7,4) block code generated by

$$G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$n = 7$

$k = 4$

$(n-k) = 3$

$2^k = 2^4 = 16$ message vectors will be present.

$$[C] = [D][G]$$

$$= [d_1 \ d_2 \ d_3 \ d_4] \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$[C] = [d_1, d_2, d_3, d_4, (d_1+d_2+d_3), (d_1+d_2+d_4), (d_1+d_3+d_4)]$$

Message vector $d_1 \ d_2 \ d_3 \ d_4$	Code vector $c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7$
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 1 1
0 0 1 0	0 0 1 0 1 0 1
0 0 1 1	0 0 1 1 1 1 0
0 1 0 0	0 1 0 0 1 1 0
0 1 0 1	0 1 0 1 1 0 1
0 1 1 0	0 1 1 0 0 1 1
0 1 1 1	0 1 1 1 0 0 0
1 0 0 0	1 0 0 0 1 1 1
1 0 0 1	1 0 0 1 1 0 0
1 0 1 0	1 0 1 0 0 1 0
1 0 1 1	1 0 1 1 0 0 1
1 1 0 0	1 1 0 0 0 0 1
1 1 0 1	1 1 0 1 0 1 0
1 1 1 0	1 1 1 0 1 0 0
1 1 1 1	1 1 1 1 1 1 1

P) For a systematic (6,3) linear block code, the parity matrix is given by

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find all the possible code vectors.

$n = 6$

$k = 3$

$2^3 = 8$ message vectors are present.

$$[G] = [I_k \vdots P]$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = [D][G]$$

$$= [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= [d_1, d_2, d_3, (d_1+d_3), (d_2+d_3), (d_1+d_2)]$$

Message ~~Code~~ vectors	Code vectors
0 0 0	0 0 0 0 0 0
0 0 1	0 0 1 1 1 0
0 1 0	0 1 0 0 1 1
0 1 1	0 1 1 1 0 1
1 0 0	1 0 0 1 0 1
1 0 1	1 0 1 0 1 1
1 1 0	1 1 0 1 1 0
1 1 1	1 1 1 0 0 0

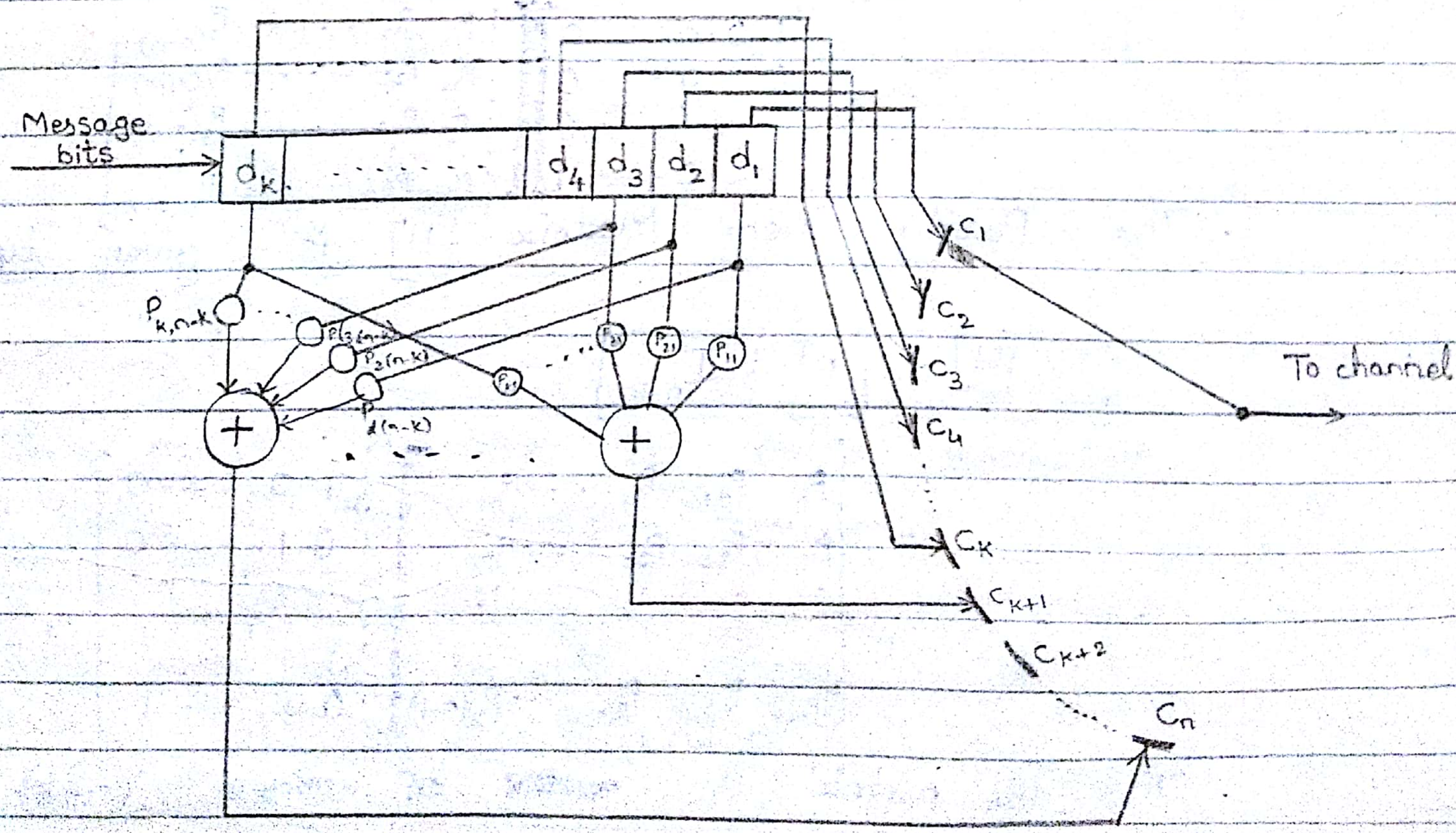
* ENCODING CIRCUIT FOR (n, K) LINEAR BLOCK CODE :

It is known that $[C] = [D][G]$.

Expanding above equation & equating the corresponding elements on both sides,

$$c_1 = d_1$$
$$c_2 = d_2$$
$$\vdots \qquad \vdots$$
$$c_K = d_K$$
$$c_{K+1} = P_{11}d_1 + P_{21}d_2 + P_{31}d_3 + \dots + P_{K1}d_K$$
$$c_{K+2} = P_{12}d_1 + P_{22}d_2 + P_{32}d_3 + \dots + P_{K2}d_K$$
$$\vdots \qquad \vdots$$
$$c_n = P_{1(n-K)}d_1 + P_{2(n-K)}d_2 + P_{3(n-K)}d_3 + \dots + P_{K(n-K)}d_K$$

The implimentation of above equation in a circuit results in the encoder for (n, K) linear block code. The realization of encoder circuit shown below consists of a k-bit shift register, n-segment commutator & (n-k) no of modulo-2 adders.



The entire data $d_1 d_2 d_3 \dots d_k$ is shifted into the k-bit shift register. The small circles $P_{11}, P_{21}, P_{31} \dots P_{k1}$ are either open circuit or short circuit depending either 0 or 1.

If $P_{11} = 0$, then there is no connection from d_1 to modulo-2 adder & if $P_{11} = 1$, then there is a connection.

When the message is shifted into the shift register, the modulo-2 adders generate the check bits which are fed into the commutator segment along with the message bits as shown in figure. When the commutator brush rotates & makes contact with the segment successively, the code vector bits will be transmitted through the channel.

* PARITY CHECK MATRIX:

The generator matrix is given by

$$[G] = [I_k \;\vdots\; P]$$

$$= \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & \cdots & 0 & P_{11} & P_{12} & \cdots & P_{1(n-k)} \\ 0 & 1 & 0 & \cdots & 0 & P_{21} & P_{22} & \cdots & P_{2(n-k)} \\ \vdots & \vdots & \vdots & \cdots & 0 & P_{31} & P_{32} & \cdots & P_{3(n-k)} \\ & & & & & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & P_{k1} & P_{k2} & \cdots & P_{k(n-k)} \end{array}\right]$$

The Parity Check Matrix $[H]$ is given by

$$[H] = [P^T \;\vdots\; I_{n-k}]$$

$$= \left[\begin{array}{cccc|cccc} P_{11} & P_{21} & P_{31} \cdots & P_{k1} & 1 & 0 & \cdots & 0 \\ P_{12} & P_{22} & P_{32} \cdots & P_{k2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ P_{1(n-k)} & P_{2(n-k)} & P_{3(n-k)} \cdots & P_{k(n-k)} & 0 & 0 & \cdots & 1 \end{array}\right]$$

The $[H]$ matrix is of order $(n-k) \times n$ and this matrix is used in error correction.

* To PROVE $CH^T = 0$:-

Consider the Generator [G] & [H] matrix as given above.

The i^{th} row of [G] matrix is given by

$$g_i = \underset{\substack{\uparrow\\ j^{th}\\ \text{element}}}{000\ldots1\ldots0}\ P_{i1}\ P_{i2}\ldots \underset{\substack{\uparrow\\ (k+j)^{th}\\ \text{element}}}{P_{ij}}\ldots P_{i(n-k)}$$

III^{ly}, consider j^{th} row of [H] matrix.

$$h_j = P_{1j}\ P_{2j}\ \ldots \underset{\substack{\uparrow\\ i^{th}\\ \text{element}}}{P_{ij}}\ldots P_{kj}\ \underset{\substack{\uparrow\\ (k+j)^{th}\\ \text{element}}}{000\ldots1\ldots0}$$

Consider $g_i h_j^T = [000\ldots1\ldots0\ P_{i1}\ P_{i2}\ldots P_{ij}\ldots P_{i(n-k)}] \begin{bmatrix} P_{ij} \\ P_{2j} \\ \vdots \\ P_{ij} \\ \vdots \\ P_{kj} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$= 0+0+\ldots+P_{ij}+\ldots\ldots 0+P_{ij}+0\ldots+0$

$= P_{ij}[1+1] = 0$

$[\because 1 \oplus_2 1 = 0$ modulo 2 addition$]$

$\therefore g_i h_j^T = 0$

This equation is true for every value of i & j & hence in the matrix form,

$$[G][H^T] = 0$$

$\otimes^{by}$ both sides by [D] message vector

$$\therefore [D][G][H^T] = [D]0 = 0$$

But $[D][G] = [C]$

$$\therefore \boxed{[C][H^T] = 0}$$

* ERROR CORRECTION AND SYNDROME :-

Let $[C] = [c_1 c_2 \ldots c_n]$ be a valid code vector transmitted over a noisy communication channel belonging to a (n,K) linear block code.

Let $[R] = [r_1 r_2 \ldots r_n]$ be a received vector. Due to the noise in the channel, $r_1, r_2 \ldots r_n$ may be different from $c_1, c_2, \ldots c_n$.

The error vector or error pattern E is defined as the difference b/w R & C.

$$\therefore \quad E = R - C$$

Therefore, the error vector can be represented by

$$E = [e_1 \, e_2 \ldots\ldots e_n]$$

From above equation, it is clear that E is also a vector where $e_i = 1$ if $R \neq C$
& $e_i = 0$ if $R = C$

The 1's present in error vector E represent the error's caused by the noise in the channel.

In the equation $E = R - C$, the receiver knows only R & it doesn't know C & E. In order to find E & then C, the receiver does the decoding operation by determining a $(n-k)$ vector S defined as

$$S = RH^T = [s_1 \, s_2 \, s_3 \ldots s_{n-k}]$$

This $(n-k)$ vector is called ERROR SYNDROME of R.

Consider $S = RH^T$

$$= [C+E][H^T]$$

$$= \cancel{CH^T}^{\,0} + EH^T = EH^T$$

$$\therefore \quad \boxed{S = EH^T}$$

The receiver finds E from the above equation as S & H^T are known. Then, from the equation $R = C + E$, the transmitted code vector C can be found out.

Note that the syndrome S of the received vector will be zero if R is a valid code vector. When $R \neq C$, then $S \neq 0$. The receiver then detects & corrects the error.

P> For a systematic (6,3) code, find all the transmitted code vector, draw the encoding circuit, if received vector $[R] = [110010]$, detect & correct the single error that has occured due to noise.

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

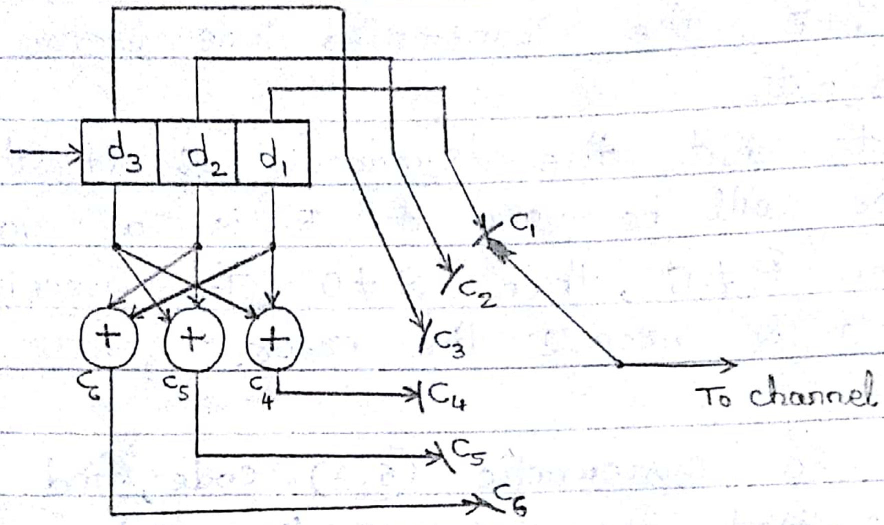
$$[C] = [D][G]$$

$$= [d_1 \; d_2 \; d_3] [\quad]$$

$$[C] = [d_1 \; d_2 \; d_3 \; (d_1 + d_3) \; (d_2 + d_3) \; (d_1 + d_2)]$$

Message vectors $d_1 \; d_2 \; d_3$	Code vectors
000	000 000
001	001 110
010	010 011
011	011 101
100	100 101
101	101 011
110	110 110
111	111 000

Encoding circuit:-



$$[H] = [P^T \vdots I_{n-k}]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$C = R - E \equiv R \oplus_2 E$

$1+0+0+0+0+0$

$$[S] = [R][H^T]$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$E = [0\,0\,0\,1\,0\,0]$

$C = R - E = [1\,1\,0\,0\,1\,0] - [0\,0\,0\,1\,0\,0]$

$\therefore C = [1\,1\,0\,1\,1\,0]$

Comparing the $[S] = [100]$ with the rows of $[H^T]$ matrix

The syndrome vector $[S] = [100]$ is present in the 4^{th} row of $[H^T]$ matrix, & hence the 4^{th} bit in the received vector counting from left is in error.

$\therefore$ The corrected code vector is $[110110]$ which is a valid transmitted code vector.

P) For a systematic (6,3) linear block code, the parity matrix is given by $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

a) Find all possible code vectors
b) Draw the encoding circuit
c) If the received code vector $[R] = [110011]$, find the syndrome, detect and correct the error.

$$[G] = [I_K \vdots P]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

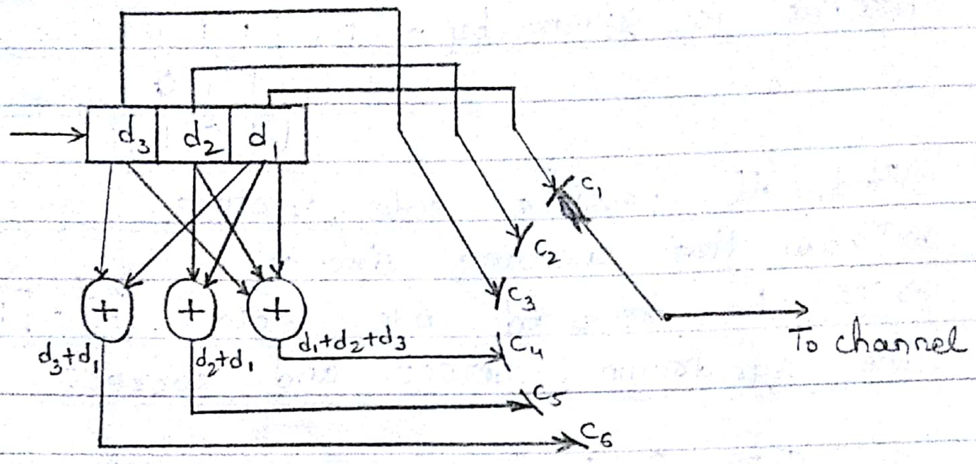
$$[C] = [D][G]$$

$$= [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$[C] = [d_1 \; d_2 \; d_3 \; (d_1+d_2+d_3) \; (d_1+d_2) \; (d_1+d_3)]$$

Message vectors	Code vectors
000	000000
001	001101
010	010110
011	011011
100	100111
101	101010
110	110001
111	111100

Encoding circuit



$$[H] = [P^T \vdots I_{n-k}]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$[S] = [R][H^T]$$

$$= [110011] \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$E = E - R - \{110011\}$
$= \{000010\}$
$\{000001\}$
$= \{11$

$$\boxed{[S] = [0\ 1\ 0]} \quad \therefore [E] = [0\,0\,0\,0\,1\,0]$$

The syndrome vector $[S] = [010]$ is present in 5^{th} row of $[H^T]$ matrix & hence 5^{th} bit in received vector from left is in error.

$\therefore$ The corrected code vector is $[110001]$ which is a valid transmitted code vector.

SYNDROME CALCULATION CIRCUIT :-

Let the received vector $R = [r_1\ r_2\ r_3 \ldots r_n]$. The syndrome vector $[S]$ is given by the equation

$$[S] = [S_1\ S_2\ S_3 \ldots S_{n-k}] = RH^T$$

Substituting each value,

$$[S_1\ S_2\ S_3 \ldots S_{n-k}] = [r_1\ r_2\ r_3 \ldots r_k\ r_{k+1} \ldots r_n] \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1(n-k)} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2(n-k)} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3(n-k)} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{k1} & P_{k2} & P_{k3} & \cdots & P_{k(n-k)} \\ \hline 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

multiplying both the equation using modulo 2 arithmetic, syndrome bits are given by

$$S_1 = P_{11}r_1 + P_{21}r_2 + P_{31}r_3 + \ldots + P_{k1}r_k + r_{k+1}$$

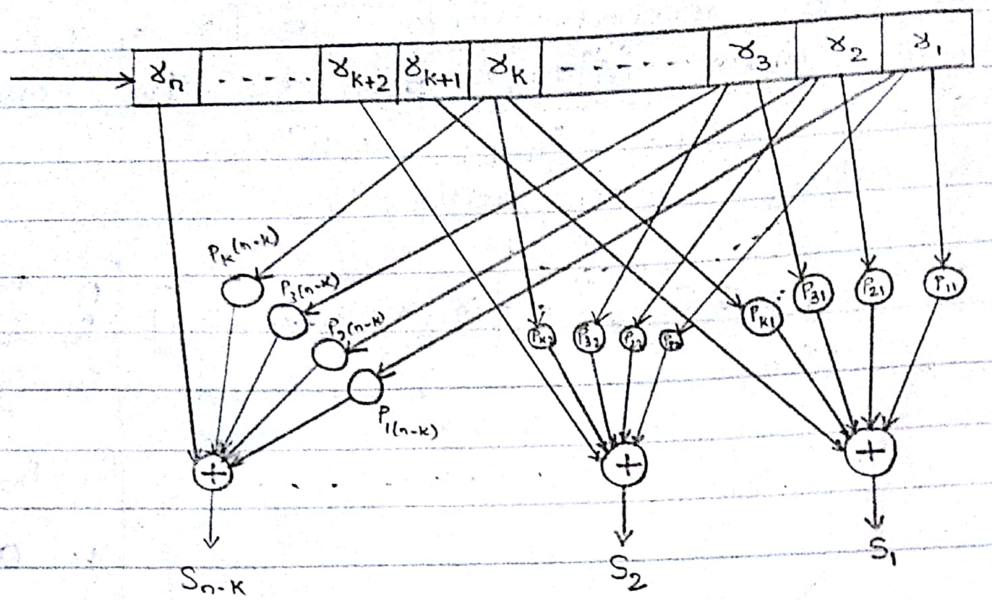
$$S_2 = P_{12}r_1 + P_{22}r_2 + P_{32}r_3 + \ldots + P_{k2}r_k + r_{k+2}$$

$$S_3 = P_{13}r_1 + P_{23}r_2 + P_{33}r_3 + \ldots + P_{k3}r_k + r_{k+3}$$

$$\vdots$$

$$S_{n-k} = P_{1(n-k)}r_1 + P_{2(n-k)}r_2 + P_{3(n-k)}r_3 + \ldots + P_{k(n-k)}r_k + r_n$$

The above equations can be realized using the circuit shown next which is called SYNDROME CALCULATION CIRCUIT:



P) For a systematic (6,3) code, the received vector $R = [r_1 \ r_2 \ r_3 \ r_4 \ r_5 \ r_6]$, construct the corresponding syndrome calculation circuit for $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$[H] = [P^T \,\vdots\, I_{n-K}]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[S] = [R][H^T]$$

$$= [r_1 \ r_2 \ r_3 \ r_4 \ r_5 \ r_6] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(r_1 + r_3 + r_4) \ \ (r_2 + r_3 + r_5) \ \ (r_1 + r_2 + r_6)]$$

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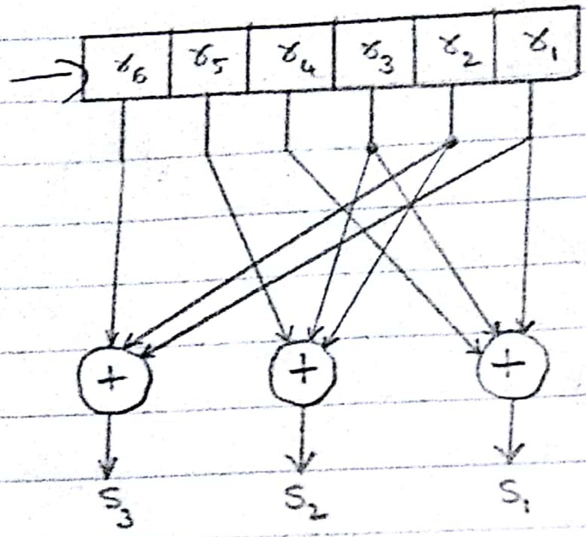

) For a systematic (7, 4) linear block code, the parity matrix is given by

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

a) Find all possible valid code vectors
b) Draw the corresponding encoding circuit
c) A single error has occurred in each of the received vectors. Detect & correct those errors

$$R_A = [0111110]$$
$$R_B = [1011100]$$
$$R_C = [1010000]$$

d) Draw the syndrome calculation circuit.

$$S = R\,H^T$$

$$H^T = \begin{bmatrix} P \\ - \\ I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

BINARY CYCLIC CODES :-

Binary Cyclic codes form the sub-class of linear block code.

ADVANTAGES :-

- Encoding & Syndrome calculation circuit can be easily implemented by simple shift register & feedback connection & by some basic gates.
- Cyclic codes have a mathematical structure that makes it possible to design the codes with useful error correcting property.

* ALGEBRAIC STRUCTURE OF CYCLIC CODES :-

"A (n, K) linear block code is said to be a cyclic code if every cyclic shift of the code is also a code vector."

Ex:- If $C_1 = 0111110$

$C_2 = 0011111$

$C_3 = 1001111$

$C_4 = 1100111$

$\vdots \qquad \vdots$

If $C_1, C_2, C_3 \ldots$ are also code vectors belonging to the same code, then the code is called CYCLIC CODE.

In general, let the n-bit vector be represented as

$V = (V_0\ V_1\ V_2 \ldots\ldots V_{n-1})$

$V(1) = (V_{n-1}\ V_0\ V_1\ V_2 \ldots\ldots V_{n-2})$

$V(2) = (V_{n-2}\ V_{n-1}\ V_0\ V_1 \ldots\ldots V_{n-3})$

$\vdots \qquad \vdots$

$V(i) = (V_{n-i}\ V_{n-i+1} \ldots\ldots V_0\ V_1\ V_2 \ldots\ldots V_{n-i-1})$

These equations which are obtained by shifting the 'V' vector cyclically successively are also the code vectors 'C'. This property of cyclic codes also allows to treat the elements of each code vector as a the co-efficients of polynomial of degree ~~(n-1)~~. n

$\therefore$ The equation will be

$V(x) = V_0 + V_1x + V_2x^2 + V_3x^3 + \ldots\ldots + V_{n-1}x^{n-1}$

$V'(x) = V_{n-1} + V_0x + V_1x^2 + V_2x^3 + \ldots\ldots + V_{n-2}x^{n-1}$

$V^2(x) = V_{n-2} + V_{n-1}x + V_0x^2 + V_1x^3 + \ldots\ldots + V_{n-3}x^{n-1}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$V^i(x) = V_{n-i} + V_{n-i+1}x + V_{n-i+2}x^2 + V_{n-i+3}x^3 + \ldots\ldots + V_{n-i-1}x^{n-1}$

* MODULO-2 ALGEBRA :-

P1) Find the product of polynomials $f_1(x) = x+1$ & $f_2(x) = x^3+x+1$ using modulo-2 algebra.

$$f_1(x) \cdot f_2(x) = (x+1)(x^3+x+1)$$
$$= x^4 + x^2 + x + x^3 + x + 1$$
$$= x^4 + x^2 + x^3 + x\overset{0}{(\cancel{1+1})} + 1$$
$$= \underline{x^4 + x^3 + x^2 + 1}$$

P2) Multiply $f_1(x) = 1+x+x^3$ and $f_2(x) = 1+x+x^2+x^4$

$$f_1(x) \cdot f_2(x) = (1+x+x^3)(1+x+x^2+x^4)$$
$$= 1+x+x^2+x^4 + x+x^2+x^3+x^5 + x^3+x^4+x^5+x^7$$
$$= 1 + x\overset{0}{(\cancel{1+1})} + x^2\overset{0}{(\cancel{1+1})} + x^3\overset{0}{(\cancel{1+1})} + x^4\overset{0}{(\cancel{1+1})} + x^5\overset{0}{(\cancel{1+1})} + x^7$$
$$= \underline{1+x^7}$$

P3) Divide $f_2(x) = x^6+x^5+x^2$ by $f_1(x) = x^3+x+1$

$x^3+x^2+x \longrightarrow$ Quotient polynomial

$$
\begin{array}{r|l}
 & x^3+x^2+x \\
\hline
x^3+x+1 & x^6+x^5+x^2 \\
 & \cancel{x^6}+x^4+x^3 \\
\hline
 & \cancel{x^5}+x^4+\cancel{x^3}+\cancel{x^2} \\
 & \cancel{x^5}+\quad \cancel{x^3}+\cancel{x^2} \\
\hline
 & \cancel{x^4} \\
 & \cancel{x^4}+x^2+x \\
\hline
 & x^2+x
\end{array}
$$

$x^2+x \longrightarrow$ Remainder polynomial

* PROPERTIES OF CYCLIC CODES :-

i) For a (n,k) cyclic code, there exists a generator polynomial of degree $(n-k)$ given by $g(x)$

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_{n-k-1} x^{n-k-1}$$

ii) The generator polynomial $g(x)$ of a (n, k) cyclic code is a factor of x^n+1

i.e., $x^n+1 = g(x)\,h(x)$

where, $h(x)$ is another polynomial of degree 'k' called PARITY-CHECK Polynomial.

iii) If $g(x)$ is a polynomial of degree $(n-k)$ & is a factor of x^n+1, then it generates the (n, k) cyclic code.

iv) The code vector polynomial can be found using

$$V(x) = D(x) \cdot g(x)$$

where $D(x) = d_0 + d_1 x + d_2 x^2 + \ldots\ldots + d_{k-1} x^{k-1}$ polynomial of degree 'k'. is the message vector

This method generates NON-SYSTEMATIC CYCLIC codes.

v) To generate a systematic cyclic code, the remainder polynomial $R(x)$ is obtained from the division of

$$\frac{x^{n-k} D(x)}{g(x)} = R(x)$$

The co-efficients of $R(x)$ are placed in beginning of code vector followed by co-efficients of message polynomial $D(x)$ to get the code vector.

n-bit code vector	
Co-efficients of $R(x)$	Co-efficients of $D(x)$

P) For (7,4) single error correcting cyclic code, $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$ and $x^n+1 = x^7+1 = (1+x+x^3)(1+x+x^2+x^4)$. Using the generator polynomial $g(x) = (1+x+x^3)$, find all 16 code vectors of cyclic code both in NON-SYSTEMATIC & SYSTEMATIC form.

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Non-systematic form:

$V(x) = D(x) g(x)$

Consider the message vector $D = [1011]$

The message vector polynomial $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$

$= 1 + 0.x + 1.x^2 + 1.x^3$

$= 1 + x^2 + x^3$

$V(x) = D(x).\ g(x)$

$$V(x) = (1+x^2+x^3)(1+x+x^3)$$

$$= 1+x+x^3+x^2+x^3+x^5+x^3+x^4+x^6$$

$$= 1+x+x^2+x^3(1+1+1)+x^4+x^5+x^6$$

$$= 1+x+x^2+x^3+x^4+x^5+x^6$$

$\therefore V = [1111111]$ is the code vector

Let $D = 1001$

$\therefore D(x) = 1+0.x+0.x^2+1.x^3$

$= 1+x^3$

$$V(x) = (1+x^3)(1+x+x^3)$$

$$= 1+x+x^3+x^3+x^4+x^6$$

$$= 1+x+x^4+x^6$$

$\therefore V = [1100101]$

Systematic form:

$$\frac{x^{n-k} d(x)}{g(x)} = R(x)$$

Let $D = 1011 \Rightarrow D(x) = 1+x^2+x^3$

$$\therefore \frac{x^3(1+x^2+x^3)}{1+x+x^3} = \frac{x^3+x^5+x^6}{1+x+x^3}$$

$$\begin{array}{r|l}
 & x^3+x^2+x+1 \\
\hline
x^3+x+1 & \cancel{x^6}+x^5+\cancel{x^3} \\
 & \cancel{x^6}+x^4+\cancel{x^3} \\
\hline
 & \cancel{x^5}+x^4 \\
 & \cancel{x^5}+x^3+x^2 \\
\hline
 & \cancel{x^4}+x^3+x^2 \\
 & \cancel{x^4}+x^2+x \\
\hline
 & \cancel{x^3}+\cancel{x} \\
 & \cancel{x^3}+\cancel{x}+1 \\
\hline
\end{array}$$

$$\therefore \quad R(x) = 1 = R_0 + R_1 x^1 + R_2 x^2$$

$$\therefore \quad R = [1\,0\,0]$$

$$D = [1\,0\,1\,1]$$

$$C = [R \,\vdots\, D] = \underline{[1\,0\,0\,1\,0\,1\,1]}$$

Let $D = 1001$

$$D(x) = 1 + x^3$$

$$\therefore \quad \frac{x^3(1+x^3)}{1+x+x^3} = \frac{x^3+x^6}{1+x+x^3}$$

$$\begin{array}{r|l}
 & x^3 + x \\
\hline
x^3+x+1 & \cancel{x^6} + \cancel{x^3} \\
 & \cancel{x^6} + x^4 + \cancel{x^3} \\
\hline
 & \quad \cancel{x^4} \\
 & \quad \cancel{x^4} + x^2 + x \\
\hline
 & \qquad x^2 + x
\end{array}$$

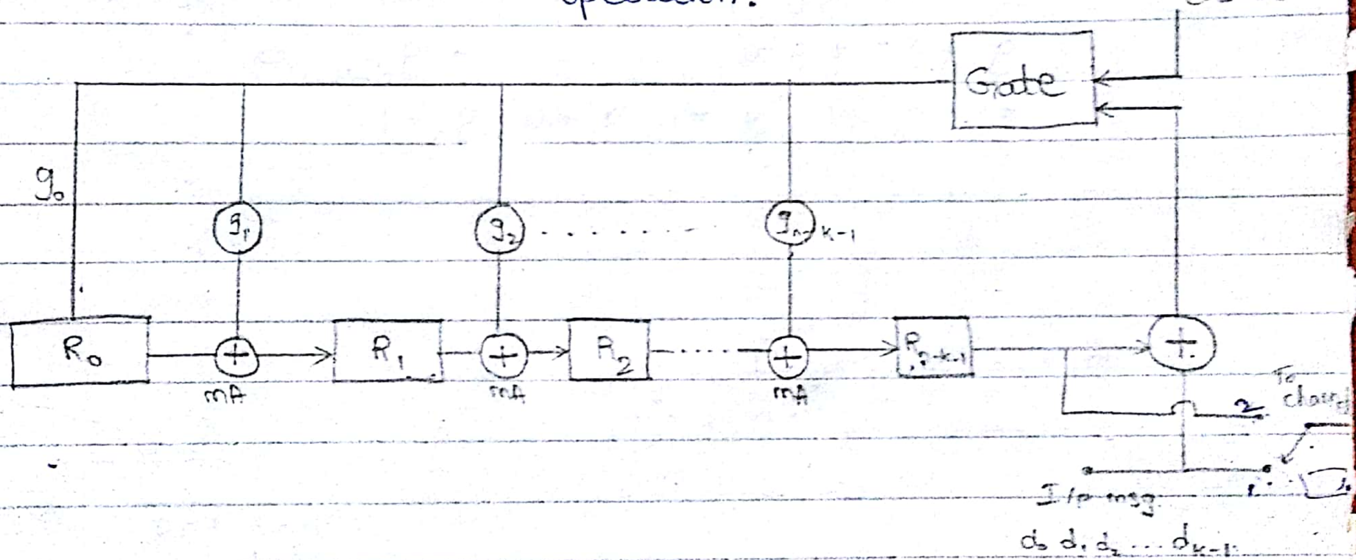
$$\therefore \quad R(x) = x^2 + x = R_0 + R_1 x^1 + R_2 x^2$$

$$R = [0\,1\,1]$$

$$\therefore \quad C = \underline{[0\,1\,1\,1\,0\,0\,1]}$$

* ENCODING USING (n-k) BIT SHIFT REGISTER :-

In order to obtain the remainder polynomial $R(x)$, the division of $x^{n-k} D(x)$ by generator polynomial $g(x)$ is done to calculate the parity check polynomial $V(x)$. The hardware required to implement the encoding system consists of
i) $(n-K)$ bit shift register
ii) $(n-K)$ modulo-2 adder
iii) AND gate
iv) Counter to keep track of shifting operation.



It is assumed that at the occurence of the clock pulse, the inputs are shifted into the register and appear at the output at the end of clock pulse.

Step 1 : With the gate turned on with the switch in position 1, with the information bits or digits $d_0 d_1 d_2 \ldots d_{k-1}$ are shifted into the register (with d_{k-1} first) & simultaneously into the communication channel. As soon as the k information digits have been shifted into the register, the register contains parity check bits $r_0, r_1, r_2 \ldots r_{n-k-1}$.

Step 2 : With gate turned off & swith in position 2, contents of shift register are shifted into the channel.

Thus, the code vector $R_0 R_1 \ldots R_{n-K-1} D_0 D_1 D_2 \ldots D_{K-1}$ is generated & sent over the channel.

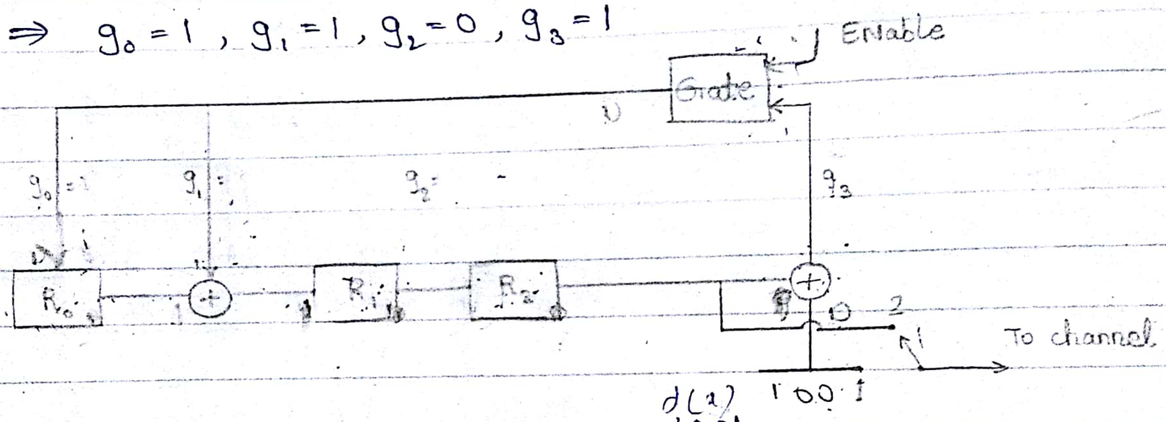
P) Design an encoder for (7,4) binary cyclic code generated by $g(x) = 1 + x + x^3$ and verify its operation using message vector (1001) and (1011)

$n-K = 7-4 = 3$ bit shift register

In general, the generator polynomial is given by

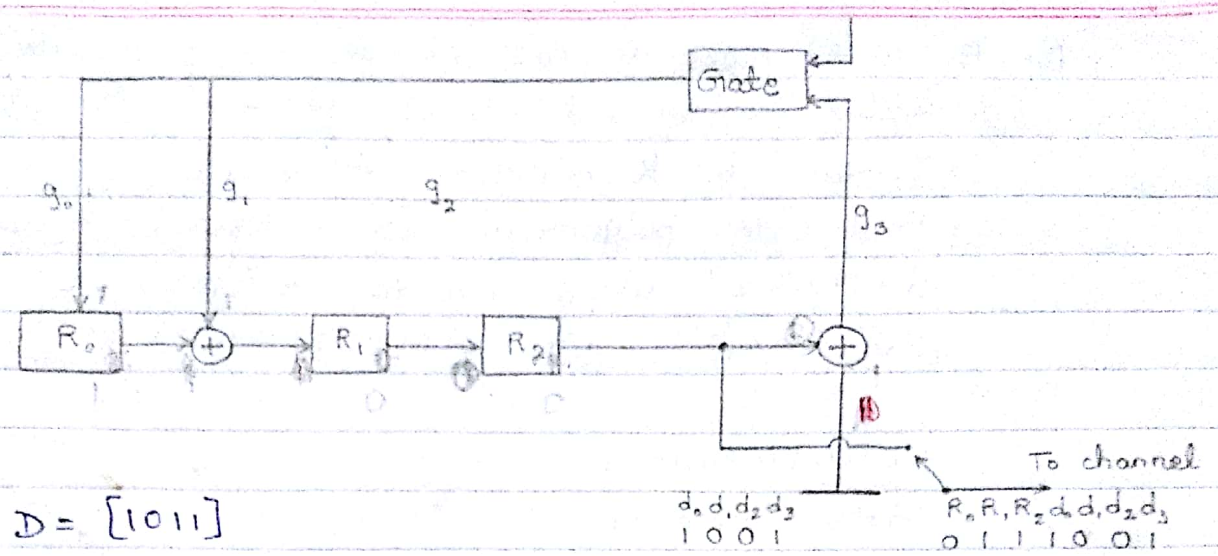
$g_0 + g_1 x + g_2 x^2 + \ldots + g_{n-K-1} x^{n-K-1}$

$\Rightarrow g_0 = 1,\ g_1 = 1,\ g_2 = 0,\ g_3 = 1$



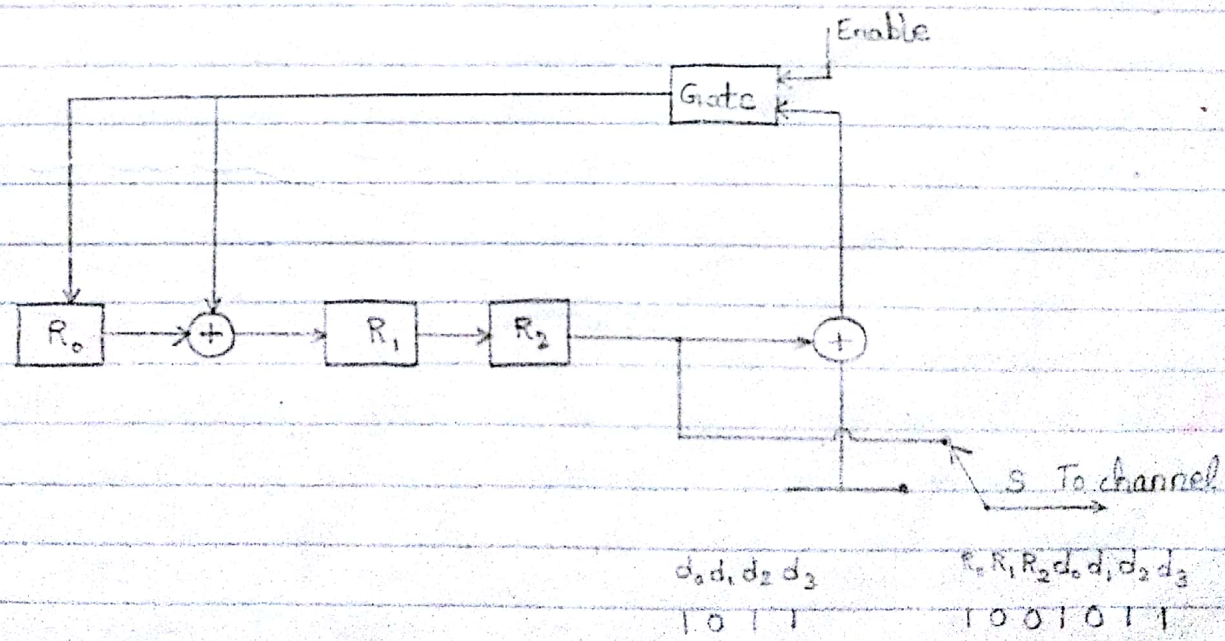
i) For the message vector (1001) the shift register contents are

	No of Shifts	I/P	R_0	R_1	R_2	Remainder
Switch S in position 1 & gate ON	1	1	0	0	0	—
	2	0	1	1	0	—
	3	0	0	1	1	—
	4	1	1	1	1	—
Switch S in position 2 & gate OFF	5	x	0	1	1	1
	6	x	0	0	1	1
	7	x	0	0	0	0
		X	0	0	0	



ii) $D = [1011]$

	No of Shifts	I/P	SR contents R_0	R_1	R_2	Remainder
Switch S in position 1 & gate ON	1	1	0	0	0	—
	2	1	1	1	0	—
	3	0	1	0	0	—
	4	1	1	0	0	—
Switch S in position 2 & gate OFF	5	X	0	1	0	0
	6	X	0	0	1	0
	7	X	0	0	0	1



A (15, 5) Algebraic code (cyclic) is generated using generator polynomial $g(x) = 1+x+x^2+x^4+x^5+x^8+x^{10}$

i) Draw block diagram of encoder

ii) Find code polynomial for message polynomial $d = 1+x^2+x^4$ using encoder diagram

$d = (10101)$ $g(x) = (11101100101)$

Switch : position 1

Gate : Turned on

No of Shifts	I/p	Shift register contents
		1110110010
1	1	
2	0	
3	1	
4	0	
5	1	

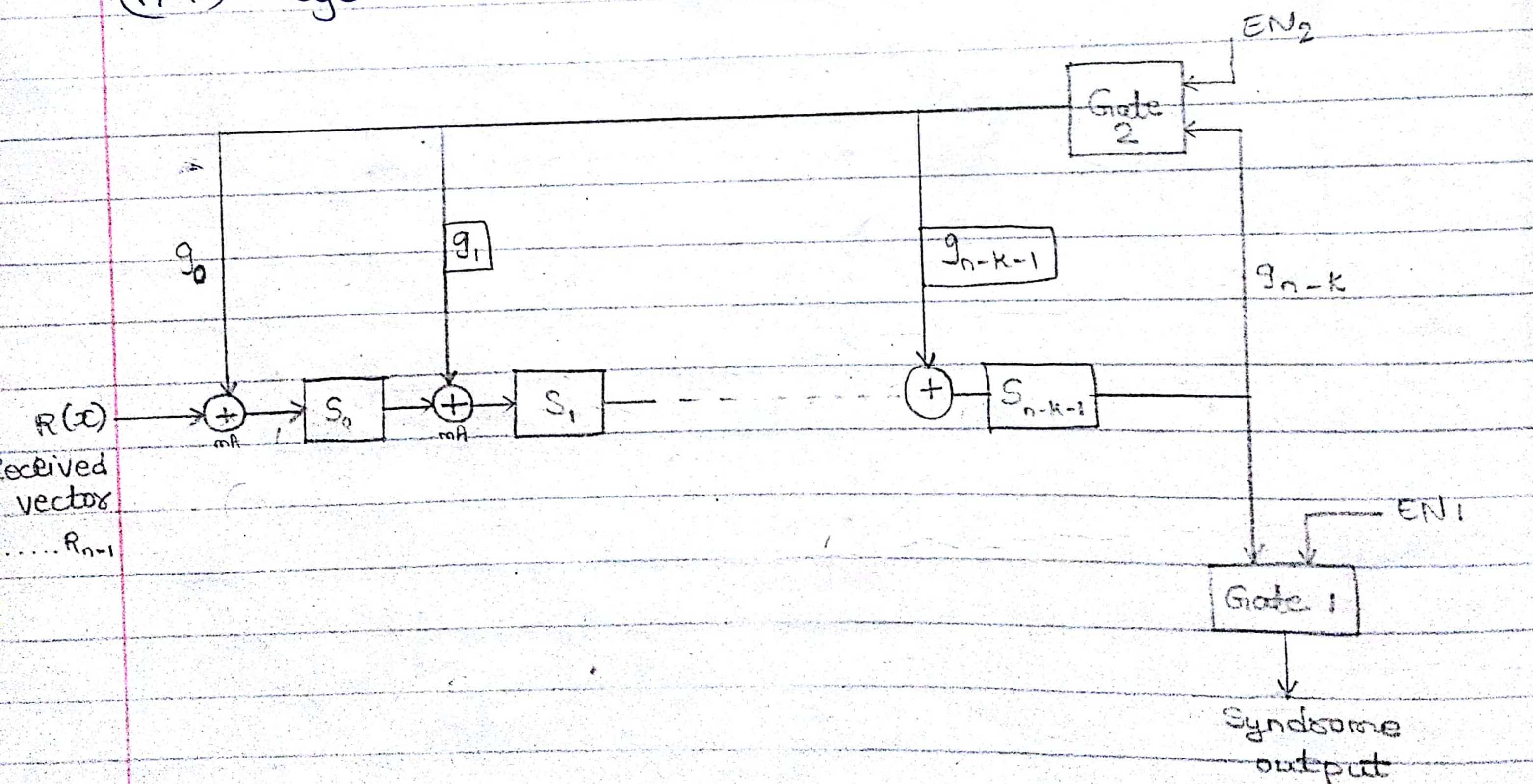
* SYNDROME CALCULATION CIRCUIT :-

If $V(x)$ is the transmitted code vector & $R(x)$ is received code vector & if $V(x) = R(x)$, then the Syndrome polynomial $S(x) = 0$

If $V(x) \neq R(x)$, then $S(x) \neq 0$

To calculate syndrome polynomial, the received code vector is divided by generator polynomial. If remainder of division is '0', then there is no error in received code vector. The remainder of division gives the error syndrome. The error polynomial depends upon syndrome polynomial. To Determine the co-efficients of syndrome polynomial, the dividing circuit for a $(n-k)$ cyclic code is shown below.

$\frac{R(x)}{g(x)}$?



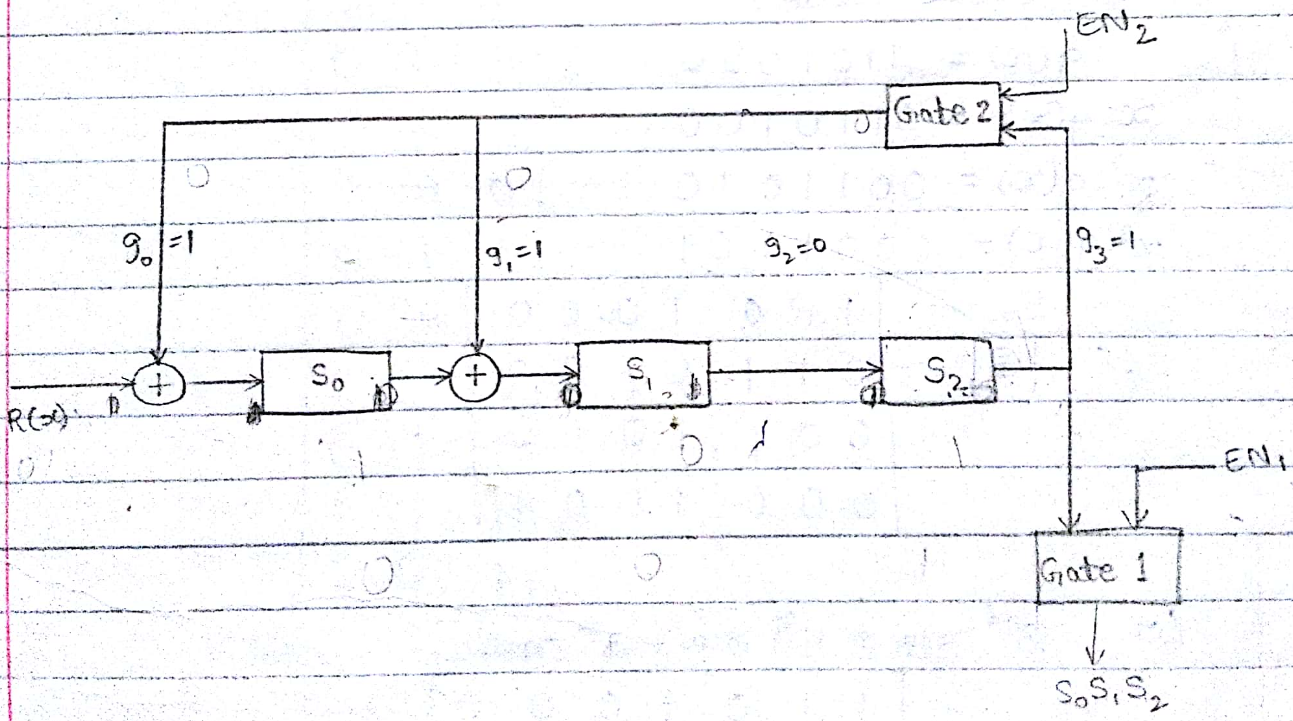
With Gate 1 turned OFF & Gate 2 turned ON, the received code vector is loaded into the shift register with (R_{n-1}) as first digit. At the end of 'n' clock pulses, the flip-flops will have the co-efficients of syndrome polynomial. After the message is loaded into the shift register, gate 2 is turned OFF & gate 1 is turned ON and the information present in syndrome calculating circuit is shifted to an error detection & correction circuit.

P) For a (7,4) cyclic code, the received vector is 1110101 and the generator polynomial $g(x) = 1+x+x^3$. Draw the syndrome calculation circuit & correct the single error in the received vector.

$n-k = 7-4 = 3$ bit shift register

$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3$

$g_0 = 1$; $g_1 = 1$; $g_2 = 0$; $g_3 = 1$



No of shifts	I/p	Shift register S_0	S_1	S_2	~~Remainder~~
	LSB	0	0	0	
1	1	1	0	0	
2	0	0	1	0	
3	1	1	0	1	
4	0	1	0	0	
5	1	1	1	0	
6	1	1	1	1	
7	1	0	0	1	

Initially
Gate-1 is OFF
Gate-2 is ON

Indicates error

Consider $g(x) = 1+x+x^3$

It is known that $g(x)$, $x\,g(x)$, $x^2 g(x)$ & $x^3 g(x)$ also represent the code vector polynomial of the same cyclic code.

$g(x) = 1101000$

$x.g(x) = 0110100$

$x^2.g(x) = 0011010$

$x^3.g(x) = 0001101$

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Let 3^{rd} row = 1^{st} row + 3^{rd} row

$$[G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Let 4^{th} row $= 1^{st}$ row $+ 2^{nd}$ row $+ 4^{th}$ row

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Generally

$G = [I_K : P]$

$H = [P^T : I_{n-k}]$

Now

$G = [P_{n-k} : I_K]$

$H = [I_{n-k} : P^T]$

$$H^T = \left[\begin{array}{c} I_{n-k} \\ \hline P \end{array}\right]$$

$[H] = [I_{n-k} : P^T]$

$$= \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$[H^T] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right]$$

$\leftarrow$ Syndrome present in 3^{rd} row

$\therefore$ Error pattern $= [0\,0\,1\,0\,0\,0\,0]$

$R + E = C$

$\therefore$ Corrected vector $= [1\,1\,0\,0\,1\,0\,1]$

11^{ly} do for $R(x) = \underline{0111011}$

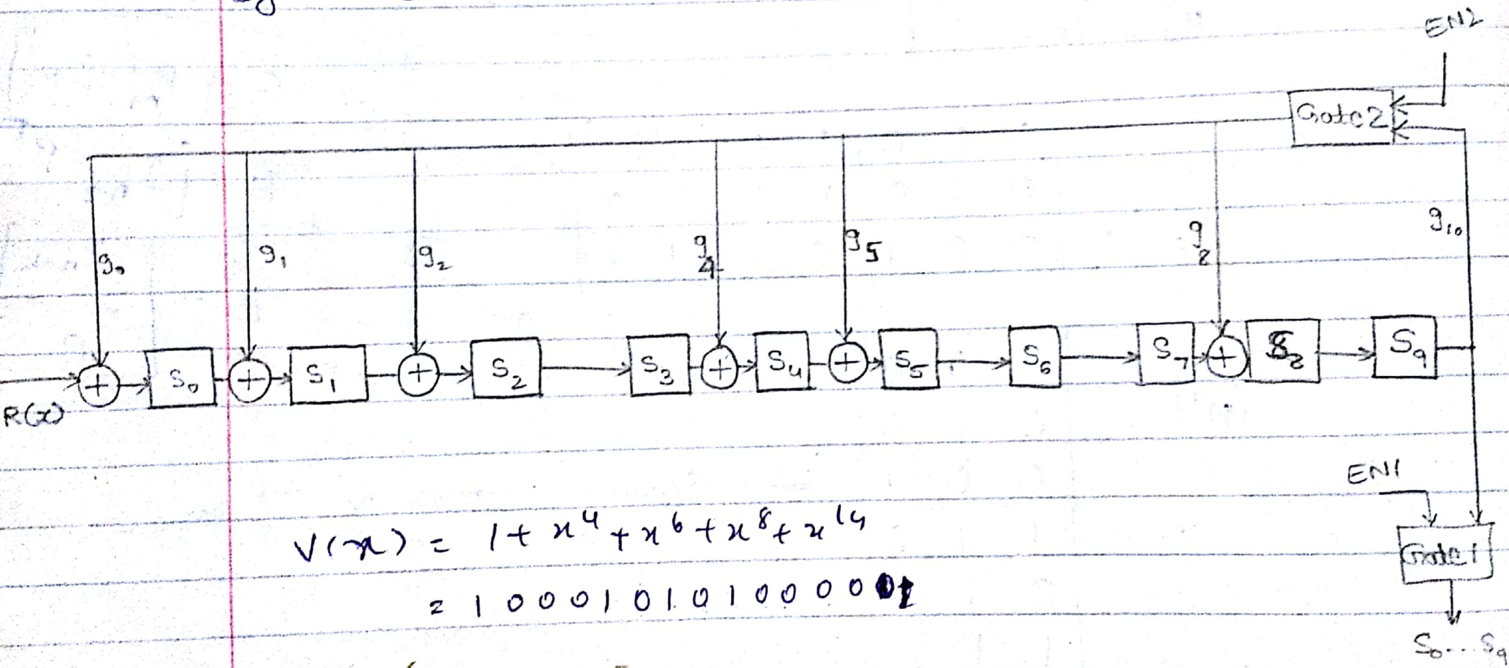
Syndrome $= 111$

P) Consider a (15,11) cyclic code generated by $g(x) = 1+x+x^4$.

a) Device a feedback register encoder for this code

b) Illustrate the encoding procedure with the message vector 11001101011 by listing the states of the registers.

P) A (15,5) linear cyclic code has a generator polynomial $g(x) = 1+x+x^2+x^4+x^5+x^8+x^{10}$. Draw the syndrome calculation circuit. Is $V(x) = 1+x^4+x^6+x^8+x^{14}$ a code polynomial? If not, find the syndrome for $V(x)$.



$V(x) = 1+x^4+x^6+x^8+x^{14}$

$= 1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,0\,0\,1$

If $V(x)$ is a code polynomial it should be perfectly divisible by $g(x)$ with remainder zero.

* CONVOLUTION CODING:-

In block codes, a block of 'n' digits generated by the encoder in a particular time unit depends only on one block of 'k' input message digits within that time unit.

A convolution encoder takes a sequence of message digits & generates a sequence of code digits. In any time unit, a message block consisting of 'k' digits is fed into the encoder & the encoder generates a code block consisting of 'n' code digits.

The 'n' digit code word depends not only on 'k' digit message block of the same time unit but also on the previous (m-1) message block.

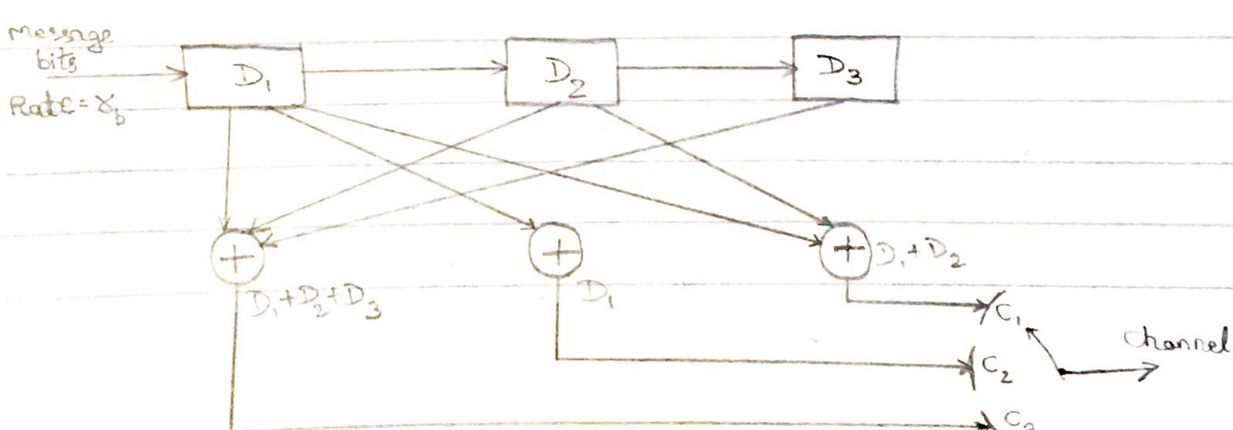
The code generated by the above encoder is called (n, k, m) convolution code of constrained length "nm" digits & rate efficiency "k/n" where n = no of outputs = no of modulo 2 adders

k = no of i/p bits entering at any time

m = no of stages of the flip-flop.

The block codes are better suited for error detection & convolution codes for error correction.

Ex:- Consider an encoder for (n, k, m) = (3, 1, 3) to generate a convolution code as shown below:



Let $d_K = 10110$
$\quad\quad\quad d_1 d_2 d_3 d_4 d_5$

	0 – T_b	T_b – $2T_b$	$2T_b$ – $3T_b$	$3T_b$ – $4T_b$	$4T_b$ – $5T_b$	$5T_b$ – $6T_b$	$6T_b$ – $7T_b$
	d_1	d_2	d_3	d_4	d_5		
Input →	1	0	1	1	0	0	0
Contents of SR →	100	010	101	110	011	001	000
output →	111	101	011	010	001	100	000

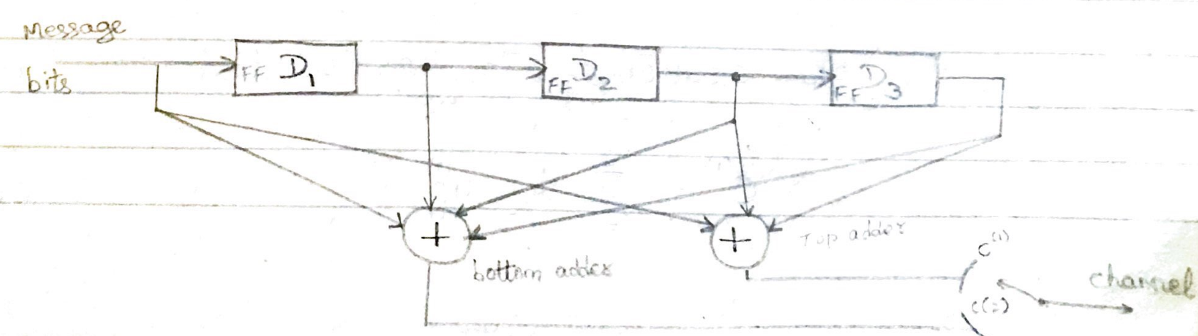
In convolution encoder, the message ~~tree~~ stream continuously runs through the encoder whereas in block coding schemes, the message stream is first divided into long blocks & then encoded.

In general, there are 2 methods of generating convolution codes.

i) Time domain approach
ii) Transfer domain approach

⇒ Encoding of convolution codes using time domain approach :-

P) Consider a $(n, K, m) = (2, 1, 3)$ convolution encoder as shown in fig. Determine the codes using time domain approach & transfer domain approach.



The time domain behaviour of a binary convolution encoder may be defined in terms of a set of 'n' impulse responses. Let the sequence $[g_1^{(j)}\ g_2^{(j)}\ g_3^{(j)} \ldots g_{m+1}^{(j)}]$ denote the impulse responses, also called GENERATOR sequences of the input / output path of 'n' no of modulo-2 adders.

In the encoder, there are 2 modulo-2 adders labelled top adder & bottom adder. Hence, there will be 2 generator sequences.

Let $d_1 d_2 d_3 \ldots d_L$ represent the input message sequence that enters into the encoder one bit at a time starting with d_1.

Then, the encoder generates 2 o/p sequences $c^{(1)}$ & $c^{(2)}$ defined by the discrete convolution sum given by

$$c^{(1)} = [d] * g^{(1)}$$
$$c^{(2)} = [d] * g^{(2)}$$

We have, $g^{(1)} = [1011]$; $g^{(2)} = [1111]$

From the definition of discrete convolution,

$$c_l^j = \sum_{i=0}^{m} d_{l-i}\, g_{i+1}^j$$

L = no of message sequence bits

where i varies from 0 to m = (0 to 3)

l varies from 1 to $(L+m)$ = (1 to 8)

$d_{l-i} = 0$, for $l \le i$

Let the message sequence be 10111
$d_1 d_2 d_3 d_4 d_5$

The o/p sequence is calculated as follows:

For $j = 1$,

$$c_l^{(1)} = \sum_{i=0}^{3} d_{l-i}\, g_{i+1}^{(1)}$$

Code word = L+m

L = no. of message bits

m = no. of FFs.

$$c_l^{(1)} = d_l g_1^{(1)} + d_{l-1} g_2^{(1)} + d_{l-2} g_3^{(1)} + d_{l-3} g_4^{(1)}$$

$l=1$ $\quad c_1^{(1)} = d_1 g_1^{(1)} + 0 + 0 + 0 = (1)(1) = 1$

$l=2$ $\quad c_2^{(1)} = d_2 g_1^{(1)} + d_1 g_2^{(1)} + 0 + 0 = (0)(1) + 1(0) = 0$

$l=3$ $\quad c_3^{(1)} = d_3 g_1^{(1)} + d_2 g_2^{(1)} + d_1 g_3^{(1)} + 0 = (1)(1) + (0)(0) + (1)(1) = 0$

$l=4$ $\quad c_4^{(1)} = d_4 g_1^{(1)} + d_3 g_2^{(1)} + d_2 g_3^{(1)} + d_1 g_4^{(1)} = (1)(1) + (1)(0) + (0)(1) + (1)(1) = 0$

$$c_5^{(1)} = d_5 g_1^{(1)} + d_4 g_2^{(1)} + d_3 g_3^{(1)} + d_2 g_4^{(1)} = (1)(1) + (0)(1) + (1)(1) + (1)(0) = 0$$

$$c_6^{(1)} = d_6 g_1^{(1)} + d_5 g_2^{(1)} + d_4 g_3^{(1)} + d_3 g_4^{(1)} = (0)(1) + (1)(0) + (1)(1) + (1)(1) = 0$$

$$c_7^{(1)} = d_7 g_1^{(1)} + d_6 g_2^{(1)} + d_5 g_3^{(1)} + d_4 g_4^{(1)} = (0) + 0 + (1)(1) + (1)(1) = 0$$

$$c_8^{(1)} = d_8 g_1^{(1)} + d_7 g_2^{(1)} + d_6 g_3^{(1)} + d_5 g_4^{(1)} = 0 + 0 + 0 + 1 = 1$$

$$\therefore \boxed{c^{(1)} = 10000001}$$

For $j = 2$

$$c_l^{(2)} = \sum_{i=0}^{3} d_{l-i}\, g_{i+1}^{(2)}$$

$$= d_l g_1^{(2)} + d_{l-1} g_2^{(2)} + d_{l-2} g_3^{(2)} + d_{l-3} g_4^{(2)}$$

But $g^{(2)} = [1111]$

$$\therefore c_l^{(2)} = d_l + d_{l-1} + d_{l-2} + d_{l-3}$$

$$c_1^{(2)} = d_1 + 0 + 0 + 0 = 1$$

$$c_2^{(2)} = d_2 + d_1 + 0 + 0 = 1$$

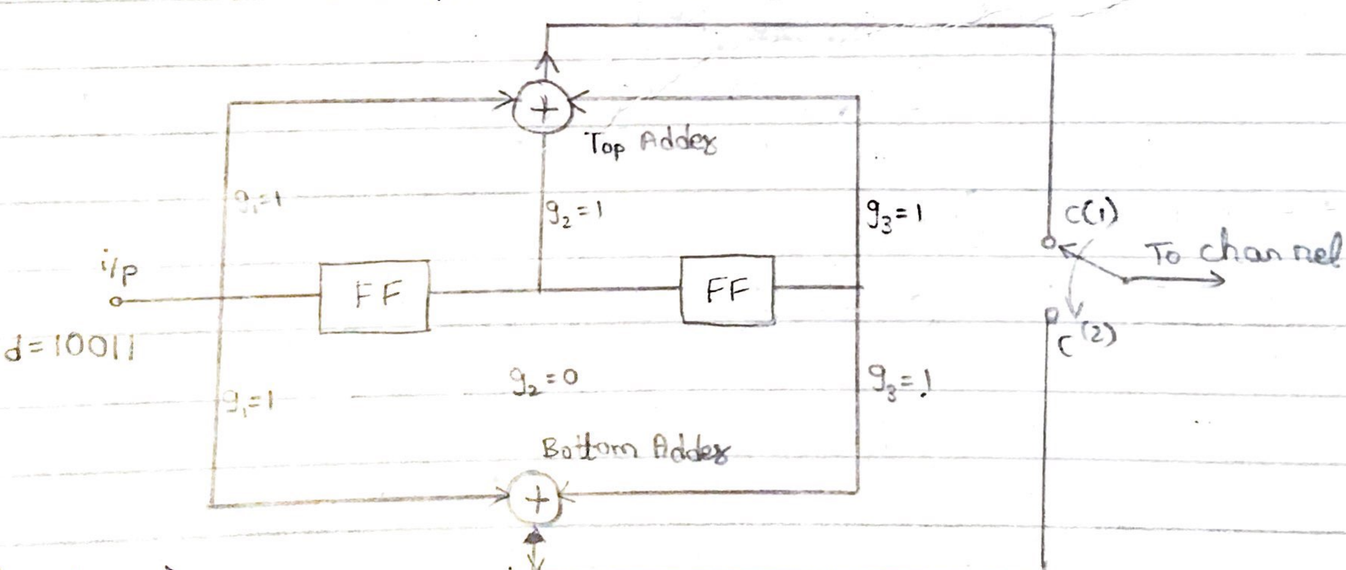
$c_3^{(2)} = d_3 + d_2 + d_1 + 0 = 0$
$c_4^{(2)} = d_4 + d_3 + d_2 + d_1 = 1$
$c_5^{(2)} = d_5 + d_4 + d_3 + d_2 = 1$
$c_6^{(2)} = 0 + d_5 + d_4 + d_3 = 1$
$c_7^{(2)} = 0 + 0 + d_5 + d_4 = 0$
$c_8^{(2)} = 0 + 0 + 0 + d_5 = 1$

$\therefore \boxed{c^{(2)} = [11011101]}$

After encoding, the 2 o/p sequences are multiplexing into a single sequence called code word for transmission over channel.
The code word at o/p of convolution encoder is given in general as
$[c_1^{(1)}\ c_1^{(2)}\ c_2^{(1)}\ c_2^{(2)}\ c_3^{(1)}\ c_3^{(2)} \ldots\ldots c_l^{(1)}\ c_l^{(2)}]$

$\therefore$ Code word = 1101 000101 01 0011

P> For the convolution encoder shown, $d = 10011$ ($d_1 d_2 d_3 d_4 d_5$). Find the output sequence using the following 2 approaches.
i) Time domain approach ii) Transfer domain approach.



$(n, k, m) = (2, 1, 2)$

$$g^{(1)} = g_1^{(1)} g_2^{(1)} g_3^{(1)} g_4^{(1)} \cdots\cdots g_{m+1}^{(1)}$$

$$g^{(2)} = g_1^{(2)} g_2^{(2)} g_3^{(2)} g_4^{(2)} \cdots\cdots g_{m+1}^{(2)}$$

$$\therefore \quad g^{(1)} = (111) \quad ; \quad g^{(2)} = (101)$$

$$L = 5 \quad ; \quad m = 2 \quad ; \quad l = L + m = 7$$

$$C_l^{(j)} = \sum_{i=0}^{m} d_{l-i}\, g_{i+1}^{j}$$

10011
$d_1 d_2 d_3 d_4 d_5$

$j=1$
$$C_l^{(1)} = \sum_{i=0}^{2} d_{l-i}\, g_{i+1}^{(1)}$$

$$C_l^{(1)} = d_l g_1^{(1)} + d_{l-1} g_2^{(1)} + d_{l-2} g_3^{(1)}$$

$$= d_l(1) + d_{l-1}(1) + d_{l-2}(1)$$

$$C_l^{(1)} = d_l + d_{l-1} + d_{l-2}$$

$l=1 \Rightarrow C_1^{(1)} = d_1 + 0 + 0 = 1$

$l=2 \Rightarrow C_2^{(1)} = d_2 + d_1 + 0 = 1$

$l=3 \Rightarrow C_3^{(1)} = d_3 + d_2 + d_1 = 1$

$l=4 \Rightarrow C_4^{(1)} = d_4 + d_3 + d_2 = 1$

$l=5 \Rightarrow C_5^{(1)} = d_5 + d_4 + d_3 = 0$

$l=6 \Rightarrow C_6^{(1)} = 0 + d_5 + d_4 = 0$

$l=7 \Rightarrow C_7^{(1)} = 0 + 0 + d_5 = 1$

$$\therefore \quad \boxed{C^{(1)} = 1110001}$$

$j=2$
$$C_l^{(2)} = \sum_{i=0}^{2} d_{l-i}\, g_{i+1}^{(2)}$$

$$= d_l g_1^{(2)} + d_{l-1} g_2^{(2)} + d_{l-2} g_3^{(2)}$$

10011
$d_1 d_2 d_3 d_4 d_5$

$$= d_l(1) + d_{l-1}(0) + d_{l-2}(1)$$

$$c_l^{(2)} = d_l + d_{l-2}$$

$l=1 \Rightarrow c_1^{(2)} = d_1 + 0 = 1$
$l=2 \Rightarrow c_2^{(2)} = d_2 + 0 = 0$
$l=3 \Rightarrow c_3^{(2)} = d_3 + d_1 = 1$
$l=4 \Rightarrow c_4^{(2)} = d_4 + d_2 = 1$
$l=5 \Rightarrow c_5^{(2)} = d_5 + d_3 = 1$
$l=6 \Rightarrow c_6^{(2)} = d_6 + d_4 = 0 + d_4 = 1$
$l=7 \Rightarrow c_7^{(2)} = d_7 + d_5 = 0 + d_5 = 1$

$$\therefore \boxed{c^{(2)} = 1011111}$$

$\therefore$ $c^{(1)} = 1111001$
$c^{(2)} = 1011111$

$\therefore$ Code word $= [11, 10, 11, 11, 01, 01, 11]$

* MATRIX METHOD :-

The generator sequence
$g_1^{(1)} g_2^{(1)} g_3^{(1)} \dots g_{m+1}^{(1)}$
for the top adder and
$g_1^{(2)} g_2^{(2)} g_3^{(2)} \dots g_{m+1}^{(2)}$
for the bottom adder can be interlaced & arranged in a matrix form with the no of rows equal to no of digits in the message sequence i.e, L rows & no of columns equal to $n(L+m)$. Such matrix of the order $\{L \times n(L+m)\}$ is called GENERATOR MATRIX of the

convolution encoder. In general, for 2 modulo-2 adders convolution encoder, the generator matrix is given by

$$G = \begin{bmatrix} g_1^{(1)} g_1^{(2)} & g_2^{(1)} g_2^{(2)} & g_3^{(1)} g_3^{(2)} & \cdots & g_{m+1}^{(1)} g_{m+1}^{(2)} & 0\,0 & \cdots & 0\,0 \\ 0\;0 & g_1^{(1)} g_1^{(2)} & g_2^{(1)} g_2^{(2)} & \cdots & g_m^{(1)} \; g_m^{(2)} & g_{m+1}^{(1)} g_{m+1}^{(2)} & \cdots & 0\,0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0\;0 & 0\;0 & \cdots & & & & & g_{m+1}^{(1)} g_{m+1}^{(2)} \end{bmatrix}$$

P) Previous same problem:

$d = 10111$

$g^{(1)} = 1011$

$g^{(2)} = 1111$

$L = 5$; $C = n(L+m) = 2(5+3) = 16$

$\therefore$ A matrix of (5×16)

$$G = \begin{bmatrix} 11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}$$

$C = [d][G]$

$= [10111][G]$

$$\boxed{C = 11, 01, 00, 01, 01, 01, 00, 11}$$

P) Same previous problem :

$d = 10011$

$g^{(1)} = 111$

$g^{(2)} = 101$

$R = L = 5$ rows

$C = n(L+m) = 2(5+2) = 14$ columns

$\therefore$ Matrix is of order (5×14)

$$G = \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}$$

$C = [d][G]$

$= [10011][G]$

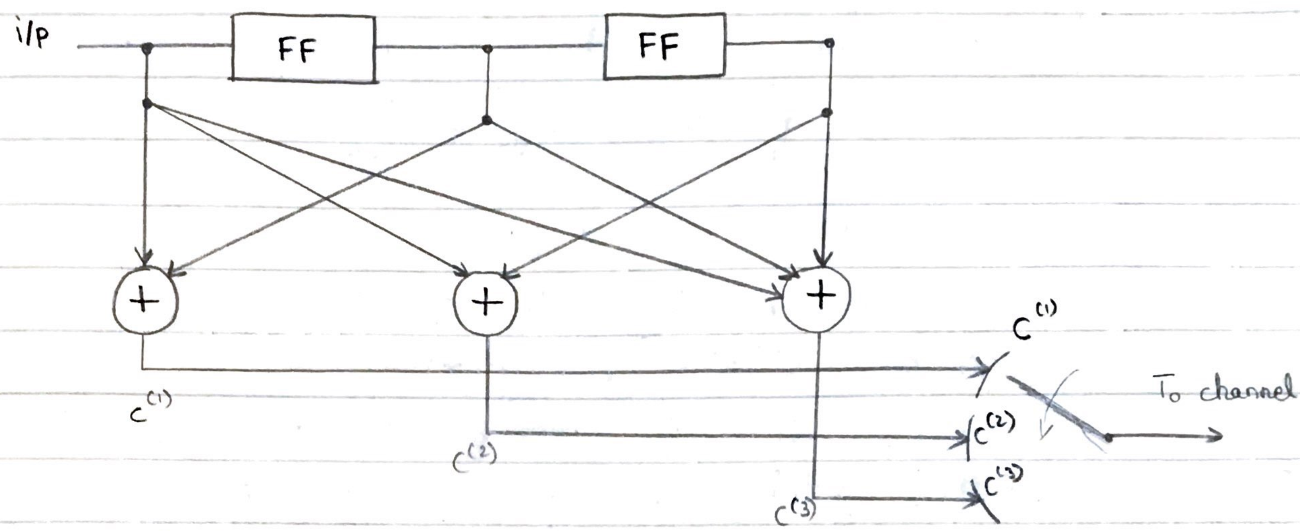
$$\boxed{C = [11; 10; 11; 11; 01; 01; 11]}$$

P) Consider a $(3,1,2)$ convolution code with $g^{(1)} = 110$, $g^{(2)} = 101$ and $g^{(3)} = 111$.

a) Draw the encoder block diagram

b) Find the generator matrix

c) Find the code word corresponding to $d = 11110$ using time domain approach.



Matrix will have $L = 6$ rows

& $n(L+m) = 3(6+2) = 24$ columns

$g^{(1)} = 110$

$g^{(2)} = 101$

$g^{(3)} = 111$

$$G = \begin{bmatrix} 111 & 101 & 011 & 000 & 000 & 000 & 000 & 000 \\ 000 & 111 & 101 & 011 & 000 & 000 & 000 & 000 \\ 000 & 000 & 111 & 101 & 011 & 000 & 000 & 000 \\ 000 & 000 & 000 & 111 & 101 & 011 & 000 & 000 \\ 000 & 000 & 000 & 000 & 111 & 101 & 011 & 000 \\ 000 & 000 & 000 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}$$

$d = [111101]$

$C = [d][G]$

$= [111, 010, 001, 001, 110, 100, 101, 011]$

* TRANSFORM DOMAIN METHOD :-

For j no of modulo-2 adders (where j varies from 1 to n), the Generator Polynomial is

$$g^{j}(x) = g_1^{j} + x g_2^{j} + x^2 g_3^{j} + x^3 g_4^{j} + \ldots\ldots + x^m g_{m+1}^{j}$$

where $j \rightarrow 1$ to n

The corresponding o/p of each of the adder is given by $C^{j}(x) = d(x)\, g^{j}(x)$

where $d(x)$ is message vector polynomial.

After getting the polynomials at the o/p of each of the adder, the final encoder o/p polynomial is obtained in the form

$$C(x) = c^{(1)}(x^n) + x\, C^{(2)}(x^n) + x^2 C^{(3)}(x^n) + \ldots + x^{n-1} c^{(n)}(x^n)$$

P)

$$g^{(1)} = 1011$$
$$g^{(2)} = 1111$$

$$g^{(1)}(x) = 1 + 0.x + 1.x^2 + 1.x^3 = 1 + x^2 + x^3$$
$$g^{(2)}(x) = 1 + 1.x + 1.x^2 + 1.x^3 = 1 + x + x^2 + x^3$$

$$d = 10111$$
$$d(x) = 1 + 0.x + 1.x^2 + 1.x^3 + 1.x^4 = 1 + x^2 + x^3 + x^4$$

$$c^{(1)}(x) = d(x)\, g^{(1)}(x)$$
$$= (1 + x^2 + x^3 + x^4)(1 + x^2 + x^3)$$
$$= 1 + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + \cancel{x^4} + \cancel{x^5} + \cancel{x^3} + \cancel{x^5} + \cancel{x^5} + \cancel{x^6} + \cancel{x^4} + \cancel{x^6} + \ldots$$
$$= 1 + x^7$$

$$c^{(1)}(x) = 1 + x^7$$

$c^{(2)}(x) = d(x)\, g^{(2)}(x)$

$= (1+x^2+x^3+x^4)(1+x+x^2+x^3)$

$= 1+x+\cancel{x^2}+\cancel{x^3}+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+\cancel{x^5}+x^3+\cancel{x^4}+\cancel{x^5}+\cancel{x^6}$
$+x^4+x^5+\cancel{x^6}+x^7$

$= 1+x+x^3+x^4+x^5+x^7$

$C(x) = c^{(1)} x^n + x\, c^{(2)} x^n$

$= c^{(1)}(x)^2 + x\, c^{(2)}(x)^2$

$= \cancel{(1+x^7)x^2 + x(1+x+x^3+x^4+x^5+x^7)x^2}$

$\cancel{= x^3 + x^9 +}$

$= (1+x^7)^2 + x\,(1+x+x^3+x^4+x^5+x^7)^2$

$= 1+x^{14} + x(1+x^2+x^6+x^8+x^{10}+x^{14})$

$= 1+x^{14}+x+x^3+x^7+x^9+x^{11}+x^{15}$

$= 1+x+x^3+x^7+x^9+x^{11}+x^{14}+x^{15}$

$\therefore\ C = [11,01,00,01,01,01,00,11]$

$[(1+x^7)^2 = 1+x^7+x^7+x^{14}$
$= 1+x^{14}$
$|||^{ly}$ for 2^{nd} term also$]$

p) $g^{(1)} = 111$

$g^{(2)} = 101$

$d = 10011$

$g^{(1)}(x) = 1+x+x^2$

$g^{(2)}(x) = 1+x^2 \quad ; \quad d(x) = 1+x^3+x^4$

$c^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1+x^3+x^4)(1+x+x^2)$

$= 1+x+x^2+x^3+\cancel{x^4}+\cancel{x^5}+\cancel{x^4}+\cancel{x^5}+x^6$

$c^{(1)}(x) = 1+x+x^2+x^3+x^6$

$c^{(2)}(x) = d(x)\, g^{(2)}(x) = (1+x^3+x^4)(1+x^2)$

$= 1+x^2+x^3+x^5+x^4+x^6$

$= 1+x^2+x^3+x^4+x^5+x^6$

$$C(x) = c^{(1)}(x)^n + x.c^{(2)}(x)^n$$
$$= c^{(1)}(x)^2 + x \cdot c^{(2)}(x)^2$$
$$= \{1+x+x^2+x^3+x^6\}^2 + x\{1+x^2+x^3+x^4+x^5+x^6\}^2$$
$$= 1+x^2+x^4+x^6+x^{12}+x+x^5+x^7+x^9+x^{11}+x^{13}$$
$$= 1+x+x^2+x^4+x^5+x^6+x^7+x^9+x^{11}+x^{12}+x^{13}$$
$$c = [11101111010111]$$

)

$g^{(1)} = 110$
$g^{(2)} = 101$
$g^{(3)} = 111$
$d = 111101$
$g^{(1)}(x) = 1+x$
$g^{(2)}(x) = 1+x^2$
$g^{(3)}(x) = 1+x+x^2$
$d(x) = 1+x+x^2+x^3+x^5$

$$c^{(1)}(x) = d(x) \cdot g^{(1)}(x)$$
$$= (1+x+x^2+x^3+x^5)(1+x)$$
$$= 1+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^5+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^4+x^6$$
$$= 1+x^4+x^5+x^6$$

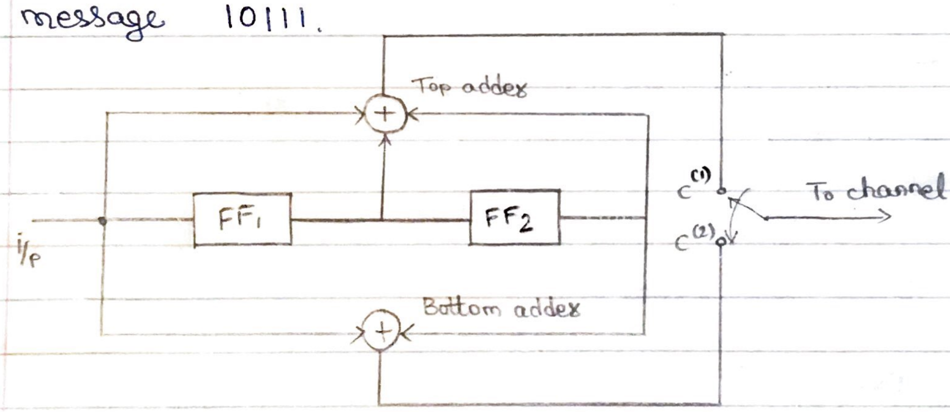
$$c^{(2)}(x) = d(x)\, g^{(2)}(x)$$
$$= (1+x+x^2+x^3+x^5)(1+x^2)$$
$$= 1+x+\cancel{x^2}+\cancel{x^3}+\cancel{x^5}+\cancel{x^2}+\cancel{x^3}+x^4+\cancel{x^5}+x^7$$
$$= 1+x+x^4+x^7$$

$$c^{(3)}(x) = d(x)\, g^{(3)}(x)$$
$$= (1+x+x^2+x^3+x^5)(1+x+x^2)$$
$$= 1+\cancel{x}+x^2+x^3+\cancel{x^5}+\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+x^6$$
$$+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+\cancel{x^5}+x^7$$
$$= 1+x^2+x^3+x^6+x^7$$

$$c(x) = c^{(1)}(x)^n + x c^{(2)}(x)^n + x^2 c^{(3)}(x)^n$$
$$= c^{(1)}(x^3) + x c^{(2)}(x^3) + x^2 c^{(3)}(x^3)$$
$$= 1+x^{12}+x^{15}+x^{18} + x\{1+x^3+x^{12}+x^{21}\}$$
$$+ x^2\{1+x^6+x^9+x^{18}+x^{21}\}$$
$$= 1+x^{12}+x^{15}+x^{18}+x+x^4+x^{13}+x^{22}+x^2+x^8+x^{11}+x^{20}+x^{23}$$
$$= 1+x+x^2+x^4+x^8+x^{11}+x^{12}+x^{13}+x^{15}+x^{18}+x^{20}+x^{22}+x^{23}$$
$$C = [111,010,001,001,110,100,101,011]$$

* STATE DIAGRAM & CODE TREE:-

P) Consider the binary convolution encoder shown in the figure. Draw the state table, state transition table, state diagram & corresponding code tree. Using the code tree, find the encoded sequence for the message 10111.



State table :-

$S_0 \leftarrow 00$

$S_1 \leftarrow 01$

$S_2 \leftarrow 10$

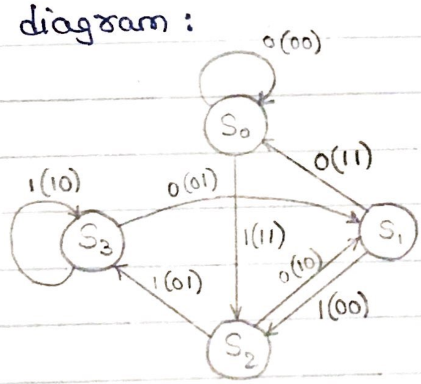
$S_3 \leftarrow 11$

State transition table :-

Present state	Binary	input	Next state	Binary description	$d_l d_{l-1} d_{l-2}$	$c^{(1)} c^{(2)}$
S_0	00	0	S_0	00	000	00
		1	S_2	10	100	11
S_1	01	0	S_0	00	001	11
		1	S_2	10	101	00
S_2	10	0	S_1	01	010	10
		1	S_3	11	110	01
S_3	11	0	S_1	01	011	01
		1	S_3	11	111	10

$$c^{(1)} = d_l + d_{l-1} + d_{l-2}$$
$$c^{(2)} = d_l + d_{l-2}$$

State diagram :



~~Code table~~

Code -

0 → Upper di

1 → Doubled d

$d_1 = 10111$

$c_1 = 111000$

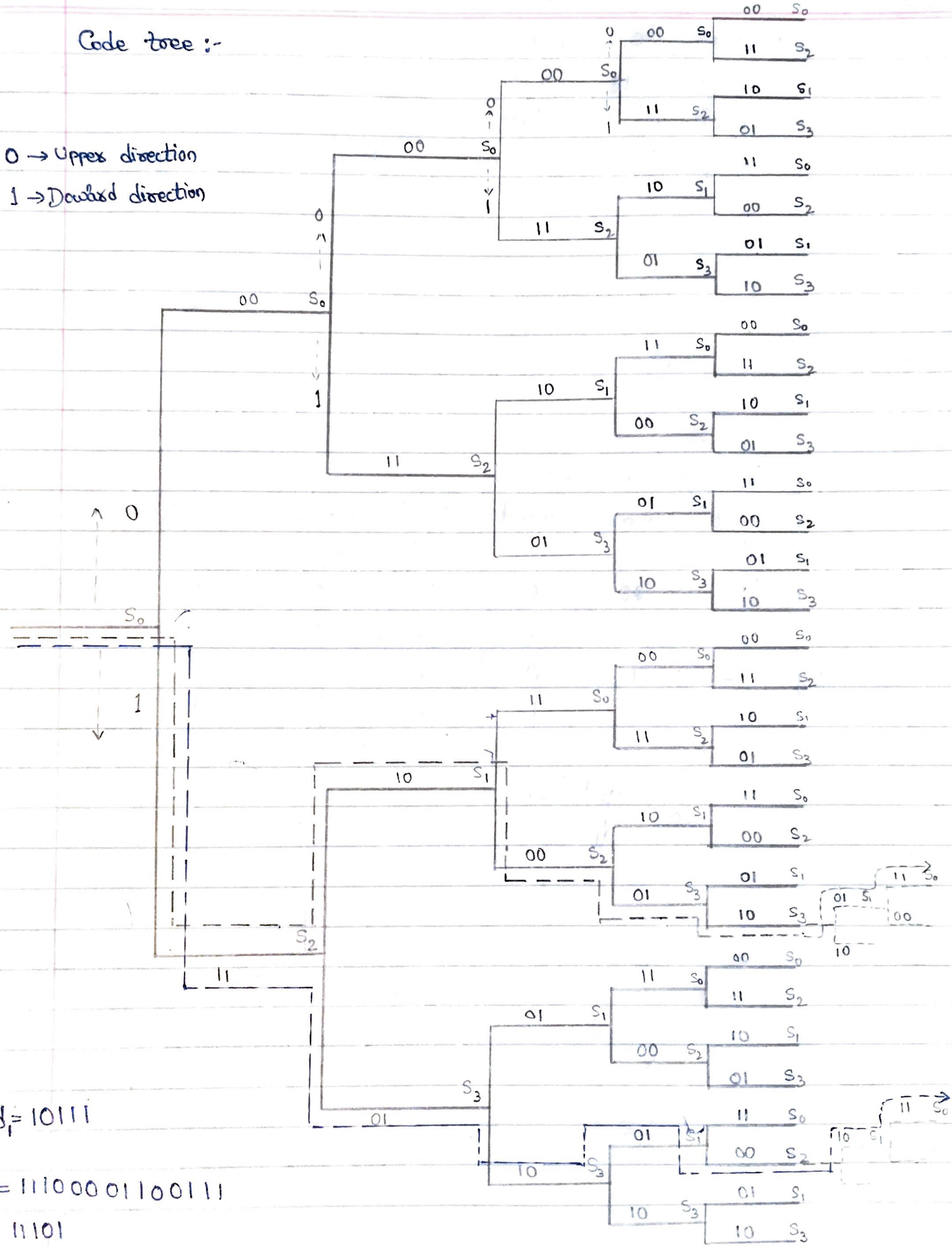
$d_2 = 11101$

$c_2 = 11011$

Code tree :-

0 → Upper direction

1 → Dowbsd direction



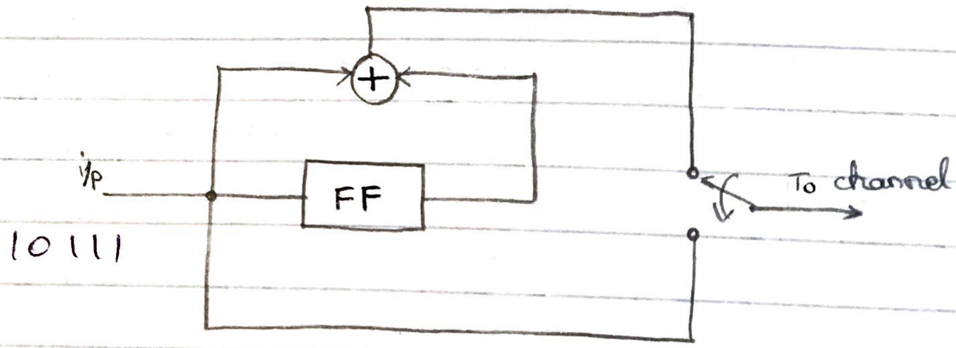
$d_1 = 10111$

$c_1 = 11100001100111$

$d_2 = 11101$

$c_2 = 11011001001011$

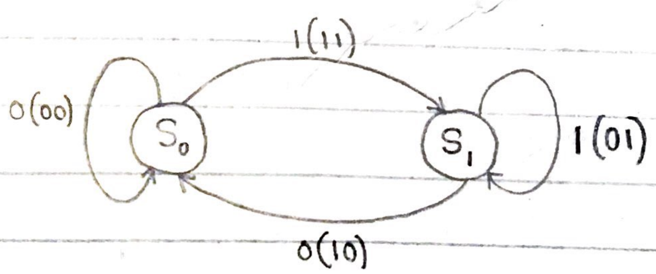
Consider the convolution encoder shown below. Draw the ⓐ state diagram ⓑ code tree ⓒ find the encoder o/p produced by the sequence 10111. Verify the o/p using time domain approach.

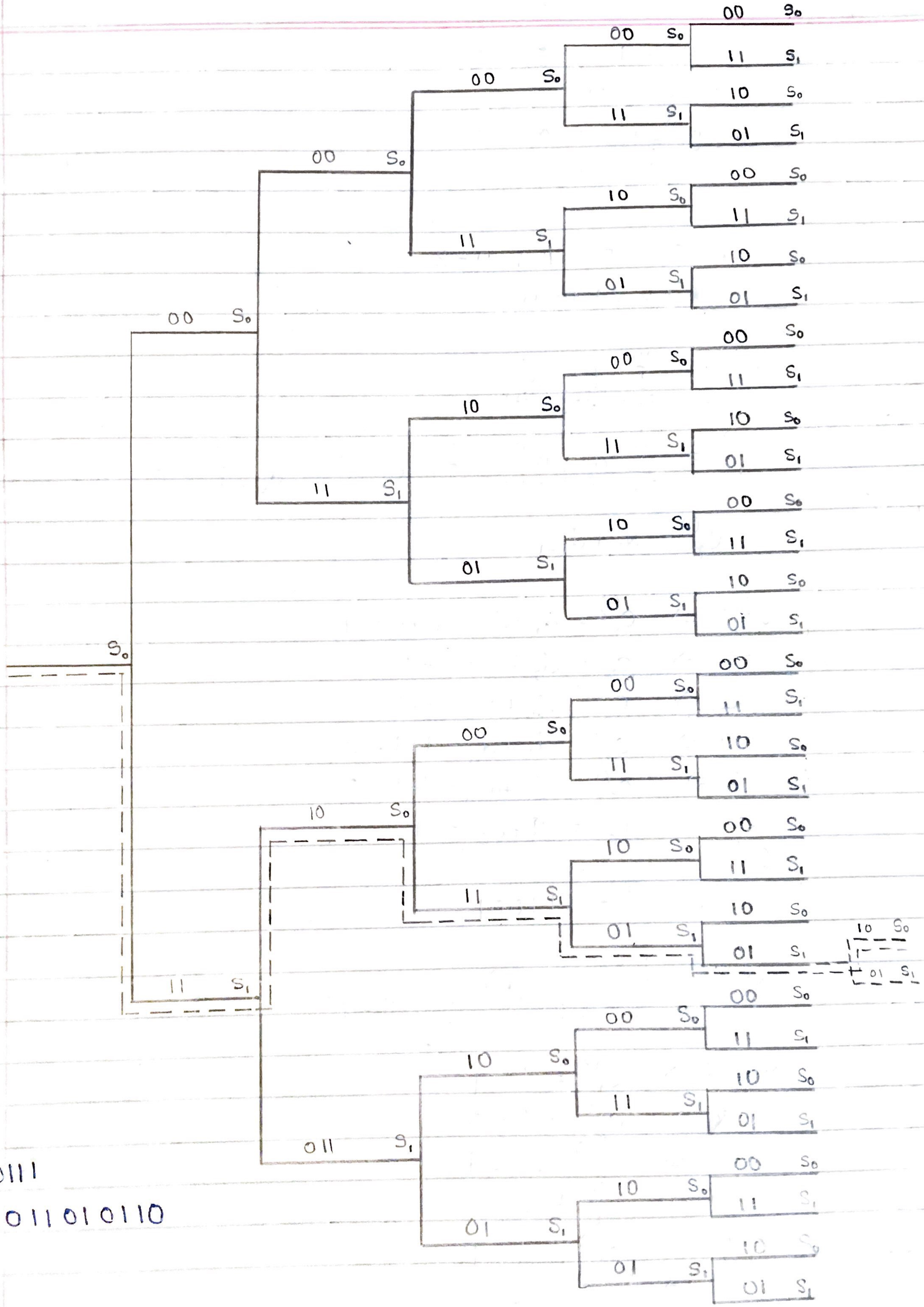


$S_0 \leftarrow 0$; $S_1 \leftarrow 1$

Present state	Binary	input	Next state	Binary description	$d_l d_{l-1}$	$c^{(1)} c^{(2)}$
S_0	0	0	S_0	0	00	00
		1	S_1	1	10	11
S_1	1	0	S_0	0	01	10
		1	S_1	1	11	01

$$c^{(1)} = d_l + d_{l-1}$$
$$c^{(2)} = d_l$$





d = 10111

C = 111011010110

$g^{(1)} = \{1\ 1\}$ $\quad g^{(2)} = \{1\ 0\}$

$g_1^{(1)} = 1 \;;\; g_2^{(1)} = 1$

$g^{(1)} = 11$

$g_1^{(2)} = 1 \;;\; g_2^{(2)} = 0$

$g^{(2)} = 10$

$L = 5 \;;\; m = 1 \;;\; n = 2$

$l = L + m = 6$

$\begin{matrix} 1 & 0 & 1 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix}$

$$C_l^{(j)} = \sum_{i=0}^{m} d_{l-i}\, g_{i+1}^{j}$$

$$C_l^{(1)} = \sum_{i=0}^{1} d_{l-i}\, g_{i+1}^{(1)} = d_l g_1^{(1)} + d_{l-1} g_2^{(1)} = d_l + d_{l-1}$$

$$C_l^{(1)} = d_l + d_{l-1}$$

$l = 1 \Rightarrow C_1^{(1)} = d_1 + d_0 = d_1 + 0 = 1$

$l = 2 \Rightarrow C_2^{(1)} = d_2 + d_1 = 1 + 0 = 1$

$l = 3 \Rightarrow C_3^{(1)} = d_3 + d_2 = 1 + 0 = 1$

$l = 4 \Rightarrow C_4^{(1)} = d_4 + d_3 = 1 + 1 = 0$

$l = 5 \Rightarrow C_5^{(1)} = d_5 + d_4 = 1 + 1 = 0$

$l = 6 \Rightarrow C_6^{(1)} = 0 + d_5 = 1$

$\therefore\ C^{(1)} = 111001$

$$C_l^{(2)} = \sum_{i=0}^{1} d_{l-i}\, g_{i+1}^{(2)} = d_l g_1^{(2)} + d_{l-1} g_2^{(2)} = d_l + 0 = d_l$$

$l = 1 \Rightarrow C_1^{(2)} = d_1 = 1$

$l = 2 \Rightarrow C_2^{(2)} = d_2 = 0$

$l = 3 \Rightarrow C_3^{(2)} = d_3 = 1$

$l = 4 \Rightarrow C_4^{(2)} = d_4 = 1$

$l = 5 \Rightarrow C_5^{(2)} = d_5 = 1$

$l = 6 \Rightarrow C_6^{(2)} = d_6 = 0$

$\therefore\ C^{(2)} = 101110$

$c^{(1)} = 111001$

$c^{(2)} = 101110$

$\therefore$ code word, $C = 111011010110$

matrix method:

$L = 5$ rows ; $g^{(1)} = 11$

$n(L+m) = 2(5+1) = 12$ columns ; $g^{(2)} = 10$

$\therefore$ Generator matrix is of order (5×12)

$$G = \begin{bmatrix} 11 & 10 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 \end{bmatrix}$$

$c = [d][G]$

$= [10111][G]$

$C = [11,10,11,01,01,10]$

Transform Domain

$g^{(1)}(x) = 1+x$ $g^{2}(x) = 1$

$d(x) = 1+x^2+x^3+x^4$

$c^{(1)}(x) = d(x) * g^1(x)$

$= (1+x^2+x^3+x^4)(1+x)$

$= 1+x^2+\cancel{x^3}+\cancel{x^4}+x+\cancel{x^3}+\cancel{x^4}+x^5$

$= 1+x+x^2+x^5$

$c^{(2)}(x) = d(x) * g^2(x)$

$= (1+x^2+x^3+x^4)(1)$

$$c(x) = \left[c_1^{(1)}(x)\right]^2 + x \cdot \left[c_2^{(2)}(x)\right]^2$$

$$= \left[1+x+x^2+x^5\right]^2 + x\left[1+x^2+x^3+x^4\right]^2$$

$$= 1+x^2+x^4+x^{10} + x\left[1+x^4+x^6+x^8\right]$$

$$= 1+x^2+x^4+x^{10}+x+x^5+x^7+x^9$$

$$= \cancel{1\,0\,1\,0\,1}$$

$$= \cancel{1+x+x^2+x^7+x^9+x^{10}}$$

$$= \cancel{1+x^2+}$$

$$= 1+x+x^2+x^4+x^5+x^7+x^9+x^{10}$$

$$= [1110 \quad 11 \quad 01 \quad 01 \quad 10]$$