

SYLLABUS**SUBJECT CODE : 10EC52****IA MARKS : 25****NO. OF LECTURE HRS/WEEK : 04****EXAM HOURS : 03****TOTAL NO. OF LECTURE HRS. : 52****EXAMMARKS : 100****UNIT - 1**

DISCRETE FOURIER TRANSFORMS (DFT): FREQUENCY DOMAIN SAMPLING AND RECONSTRUCTION OF DISCRETE TIME SIGNALS. DFT AS A LINEAR TRANSFORMATION, ITS RELATIONSHIP WITH OTHER TRANSFORMS.

6 HRS

UNIT - 2

PROPERTIES OF DFT, MULTIPLICATION OF TWO DFTs- THE CIRCULAR CONVOLUTION, ADDITIONAL DFT PROPERTIES.

6 HRS

UNIT - 3

USE OF DFT IN LINEAR FILTERING, OVERLAP-SAVE AND OVERLAP-ADD METHOD. DIRECT COMPUTATION OF DFT, NEED FOR EFFICIENT COMPUTATION OF THE DFT (FFT ALGORITHMS).

7 HRS

UNIT - 4

RADIX-2 FFT ALGORITHM FOR THE COMPUTATION OF DFT AND IDFT-DECIMATION-IN-TIME AND DECIMATION-IN-FREQUENCY ALGORITHMS. GOERTZEL ALGORITHM, AND CHIRP-Z TRANSFORM.

7 HRS

UNIT - 5

IIR FILTER DESIGN: CHARACTERISTICS OF COMMONLY USED ANALOG FILTERS – BUTTERWORTH AND CHEBYSHEVE FILTERS, ANALOG TO ANALOG FREQUENCY TRANSFORMATIONS.

6 HRS

UNIT - 6

IMPLEMENTATION OF DISCRETE-TIME SYSTEMS: STRUCTURES FOR IIR AND FIR SYSTEMS DIRECT FORM I AND DIRECT FORM II SYSTEMS, CASCADE, LATTICE AND PARALLEL REALIZATION.

7 HRS

UNIT - 7

FIR FILTER DESIGN: INTRODUCTION TO FIR FILTERS, DESIGN OF FIR FILTERS USING - RECTANGULAR, HAMMING, BARTLET AND KAISER WINDOWS, FIR FILTER DESIGN USING FREQUENCY SAMPLING TECHNIQUE.

6 HRS

UNIT - 8

DESIGN OF IIR FILTERS FROM ANALOG FILTERS (BUTTERWORTH AND CHEBYSHEV) - IMPULSE INVARIANCE METHOD. MAPPING OF TRANSFER FUNCTIONS: APPROXIMATION OF DERIVATIVE (BACKWARD DIFFERENCE AND BILINEAR TRANSFORMATION) METHOD, MATCHED Z TRANSFORMS, VERIFICATION FOR STABILITY AND LINEARITY DURING MAPPING

7 HRS

Digital Signal Processing

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TEXT BOOK:

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.

REFERENCE BOOKS:

1. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
2. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 3RD EDITION, 2010.
3. DIGITAL SIGNAL PROCESSING, LEE TAN: ELSIVIER PUBLICATIONS, 2007

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UNIT 1**DISCRETE FOURIER TRANSFORMS (DFT)****CONTENTS:-**

DISCRETE FOURIER TRANSFORMS (DFT): FREQUENCY DOMAIN SAMPLING AND RECONSTRUCTION OF DISCRETE TIME SIGNALS. DFT AS A LINEAR TRANSFORMATION, ITS RELATIONSHIP WITH OTHER TRANSFORMS. 6 HRS

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA Mc-GRAW HILL, 2ND EDITION, 2004.

UNIT 1

Discrete Fourier Transform

1.1 Introduction:

Before we introduce the DFT we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus we establish the relation between the sampled Fourier transform and the DFT. A discrete time system may be described by the convolution sum, the Fourier representation and the z transform as seen in the previous chapter. If the signal is periodic in the time domain DTFS representation can be used, in the frequency domain the spectrum is discrete and periodic. If the signal is non-periodic or of finite duration the frequency domain representation is periodic and continuous this is not convenient to implement on the computer. Exploiting the periodicity property of DTFS representation the finite duration sequence can also be represented in the frequency domain, which is referred to as Discrete Fourier Transform DFT.

DFT is an important mathematical tool which can be used for the software implementation of certain digital signal processing algorithms .DFT gives a method to transform a given sequence to frequency domain and to represent the spectrum of the sequence using only k frequency values, where k is an integer that takes N values, K=0, 1, 2,...,N-1.

The advantages of DFT are:

1. It is computationally convenient.
2. The DFT of a finite length sequence makes the frequency domain analysis much simpler than continuous Fourier transform technique.

1.2 FREQUENCY DOMAIN SAMPLING AND RECONSTRUCTION OF DISCRETE TIME SIGNALS:

Consider an aperiodic discrete time signal $x(n)$ with Fourier transform, an aperiodic finite energy signal has continuous spectra. For an aperiodic signal $x[n]$ the spectrum is:

$$X[w] = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \dots \dots \dots (1.1)$$

$$\left[\frac{2\pi k}{N} \right] = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x[n-lN] \right] e^{-j2\pi kn/N} \quad \text{for } k = 0, 1, 2, \dots, (N-1) \dots \dots \dots (1.3)$$

Denote the quantity inside the bracket as $x_p[n]$. This is the signal that is a repeating version of $x[n]$ every N samples. Since it is a periodic signal it can be represented by the Fourier series.

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad n = 0, 1, 2, \dots, (N-1)$$

With FS coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N} \quad k = 0, 1, 2, \dots, (N-1) \dots \dots \dots (1.4)$$

Comparing the expressions in equations (1.4) and (1.3) we conclude the following:

$$c_k = \frac{1}{N} X \left[\frac{2\pi}{N} k \right] \quad k = 0, 1, \dots, (N-1) \dots \dots \dots (1.5)$$

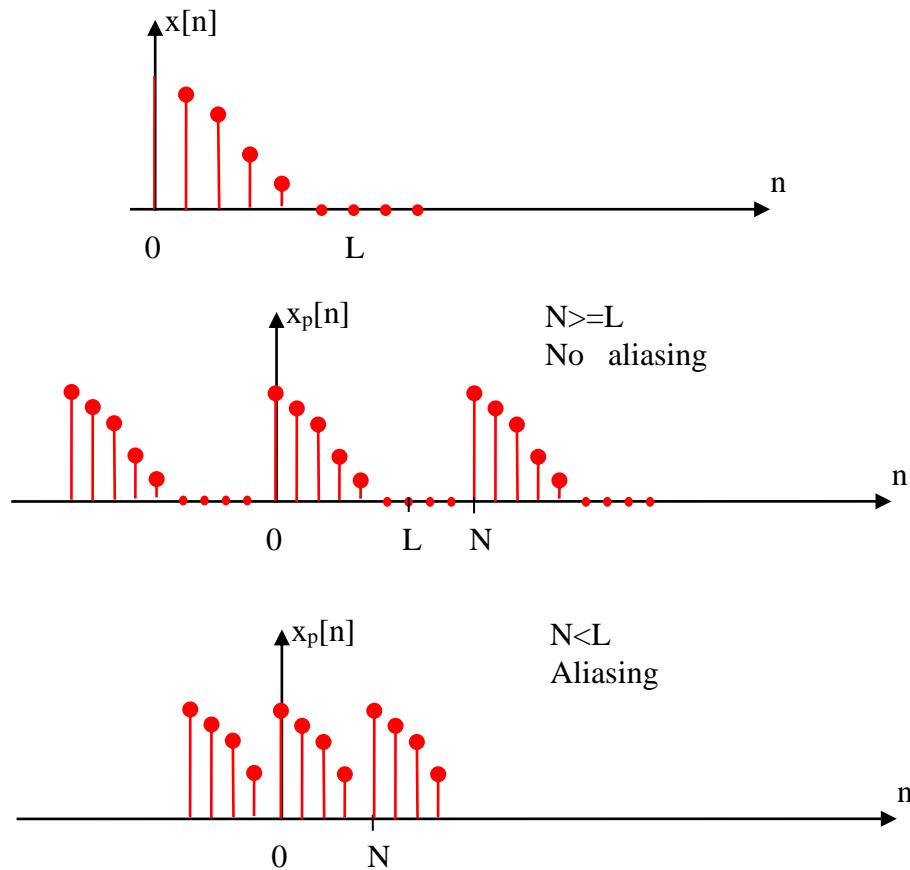
Therefore it is possible to write the expression $x_p[n]$ as below:

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X \left[\frac{2\pi}{N} k \right] e^{j2\pi kn/N} \quad n = 0, 1, \dots, (N-1) \dots \dots \dots (1.6)$$

The above formula shows the reconstruction of the periodic signal $x_p[n]$ from the samples of the spectrum $X[w]$. But it does not say if $X[w]$ or $x[n]$ can be recovered from the samples.

Let us have a look at that:

Since $x_p[n]$ is the periodic extension of $x[n]$ it is clear that $x[n]$ can be recovered from $x_p[n]$ if there is no aliasing in the time domain. That is if $x[n]$ is time-limited to less than the period N of $x_p[n]$. This is depicted in Fig. 1.2 below:

**Fig. 1.2 Signal Reconstruction**

Hence we conclude:

The spectrum of an aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $w_k = \frac{2\pi k}{N}$ if $N \geq L$.

We compute $x_p[n]$ for $n=0, 1, \dots, N-1$ using equation (1.6)
Then $X[w]$ can be computed using equation (1.1).

1.3 Discrete Fourier Transform:

The DTFT representation for a finite duration sequence is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(n) = \frac{1}{2\pi} \int X(j\omega) e^{j\omega n} d\omega, \quad \text{Where } \omega = 2\pi k/n$$

$$\frac{2\pi}{N}$$

Where $x(n)$ is a finite duration sequence, $X(j\omega)$ is periodic with period $2\pi/N$. It is convenient sample $X(j\omega)$ with a sampling frequency equal an integer multiple of its period $=m$ that is taking N uniformly spaced samples between 0 and 2π .

$$\text{Let } \omega_k = 2\pi k/N, 0 \leq k \leq N-1$$

$$\text{Therefore } X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}$$

Since $X(j\omega)$ is sampled for one period and there are N samples $X(j\omega)$ can be expressed as

$$X(k) = X(j\omega) \Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, 0 \leq k \leq N-1$$

1.4 Matrix relation of DFT

The DFT expression can be expressed as

$$[X] = [x(n)] [WN]$$

$$T$$

$$\text{Where } [X] = [X(0), X(1), \dots]$$

$[x]$ is the transpose of the input sequence. WN is a $N \times N$ matrix

$$\begin{aligned} WN = & \begin{matrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & wN1 & wN2 & wN3 & \dots & wN n-1 \\ 1 & wN2 & wN4 & wN6 & \dots & wN2(n-1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \dots & \dots & wN (N-1)(N-1) \end{matrix} \end{aligned}$$

ex;

4 pt DFT of the sequence 0,1,2,3

$$\begin{array}{lcl} X(0) & = & 1 & 1 & 1 & 1 \\ X(1) & = & 1 & -j & -1 & j \\ X(2) & = & 1 & -1 & 1 & -1 \\ X(3) & = & 1 & j & -1 & -j \end{array}$$

Solving the matrix $X(K) = 6, -2+2j, -2, -2-2j$

1.5 Relationship of Fourier Transforms with other transforms

1.5.1 Relationship of Fourier transform with continuous time signal:

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in Fourier series as

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

Where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain discrete time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 nT} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k n/N} \\ &= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi k n/N} \\ X(k) &\equiv N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N \tilde{c}_k \end{aligned}$$

Thus $\{\tilde{c}_k\}$ is the aliasing version of $\{c_k\}$

1.5.2 Relationship of Fourier transform with z-transform

Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

With ROC that includes unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $Z_k = e^{j2\pi k/N}$ for $K = 0, 1, 2, \dots, N-1$ we obtain

$$\begin{aligned} X(k) &\equiv X(z)|_{z=e^{j2\pi k/N}} \quad k = 0, 1, \dots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi n k/N} \end{aligned}$$

The above expression is identical to Fourier transform $X(\omega)$ evaluated at N equally spaced frequencies $\omega_k = 2\pi k/N$ for $K = 0, 1, 2, \dots, N-1$.

If the sequence $x(n)$ has a finite duration of length N or less. The sequence can be recovered from its N -point DFT. Consequently $X(z)$ can be expressed as a function of DFT as

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n)z^{-n} \\ X(z) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \right] z^{-n} \\ X(z) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} (e^{j2\pi k/N} z^{-1})^n \\ X(z) &= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}} \end{aligned}$$

Fourier transform of a continuous time signal can be obtained from DFT as

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}}$$

Recommended Questions with solutions

Question 1

The first five points of the 8-point DFT of a real valued sequence are $\{0.25, 0.125-j0.318, 0, 0.125-j0.0518, 0\}$. Determine the remaining three points

Ans: Since $x(n)$ is real, the real part of the DFT is even, imaginary part odd. Thus the remaining points are $\{0.125+j0.0518, 0, 0, 0.125+j0.318\}$.

Question 2

Compute the eight-point DFT circular convolution for the following sequences.

$$x_2(n) = \sin 3\pi n/8$$

Ans:

(a)

$$\begin{aligned}\bar{x}_2(l) &= x_2(l), \quad 0 \leq l \leq N-1 \\ &= x_2(l+N), \quad -(N-1) \leq l \leq -1 \\ \bar{x}_2(l) &= \sin\left(\frac{3\pi}{8}l\right), \quad 0 \leq l \leq 7 \\ &= \sin\left(\frac{3\pi}{8}(l+8)\right), \quad -7 \leq l \leq -1 \\ &= \sin\left(\frac{3\pi}{8}|l|\right), \quad |l| \leq 7\end{aligned}$$

$$\begin{aligned}\text{Therefore, } x_1(n) \circledast_8 x_2(n) &= \sum_{m=0}^3 \bar{x}_2(n-m) \\ &= \sin\left(\frac{3\pi}{8}|n|\right) + \sin\left(\frac{3\pi}{8}|n-1|\right) + \dots + \sin\left(\frac{3\pi}{8}|n-3|\right) \\ &= \{1.25, 2.55, 2.55, 1.25, 0.25, -1.06, -1.06, 0.25\}\end{aligned}$$

Question 3

Compute the eight-point DFT circular convolution for the following sequence

$$X_3(n) = \cos 3\pi n/8$$

$$\begin{aligned}\bar{x}_2(n) &= \cos\left(\frac{3\pi}{8}n\right), \quad 0 \leq l \leq 7 \\ &= -\cos\left(\frac{3\pi}{8}n\right), \quad -7 \leq l \leq -1 \\ &= [2u(n)-1]\cos\left(\frac{3\pi}{8}n\right), \quad |n| \leq 7\end{aligned}$$

$$\begin{aligned}\text{Therefore, } x_1(n) \circledast_8 x_2(n) &= \sum_{m=0}^3 \left(\frac{1}{4}\right)^m \bar{x}_2(n-m) \\ &= \{0.96, 0.62, -0.55, -1.06, -0.26, -0.86, 0.92, -0.15\}\end{aligned}$$

Question 4

Define DFT. Establish a relation between the Fourier series coefficients of a continuous time signal and DFT

Solution

The DTFT representation for a finite duration sequence is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega n} d\omega, \quad \text{Where } \omega = 2\pi k/n$$

Where $x(n)$ is a finite duration sequence, $X(j\omega)$ is periodic with period 2π . It is convenient sample $X(j\omega)$ with a sampling frequency equal an integer multiple of its period $= m$ that is taking N uniformly spaced samples between 0 and 2π .

$$\text{Let } \omega_k = 2\pi k/n, \quad 0 \leq k \leq N$$

$$\text{Therefore } X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}$$

Since $X(j\omega)$ is sampled for one period and there are N samples $X(j\omega)$ can be expressed as

$$X(k) = X(j\omega) \Big|_{\omega=2\pi kn/N} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad 0 \leq k \leq N-1$$

Question 5

5.7 If $X(k)$ is the DFT of the sequence $x(n)$, determine the N -point DFTs of the sequences

$$x_c(n) = x(n) \cos \frac{2\pi kn}{N} \quad 0 \leq n \leq N-1$$

and

$$x_s(n) = x(n) \sin \frac{2\pi kn}{N} \quad 0 \leq n \leq N-1$$

in terms of $X(k)$.

Solution:-

$$\begin{aligned} X_c(k) &= \sum_{n=0}^{N-1} \frac{1}{2} x(n) \left(e^{j \frac{2\pi k_0 n}{N}} + e^{-j \frac{2\pi k_0 n}{N}} \right) e^{-j \frac{2\pi k n}{N}} \\ &= \frac{1}{2} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi (k-k_0) n}{N}} + \frac{1}{2} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi (k+k_0) n}{N}} \\ &= \frac{1}{2} X(k - k_0)_{\text{mod}_N} + \frac{1}{2} X(k + k_0)_{\text{mod}_N} \end{aligned}$$

$$\text{similarly, } X_s(k) = \frac{1}{2j} X(k - k_0)_{\text{mod}_N} - \frac{1}{2j} X(k + k_0)_{\text{mod}_N}$$

Question 6

Find the 4-point DFT of sequence $x(n) = 6 + \sin(2\pi n/N)$, $n=0,1,\dots,N-1$

Solution :-

$$\text{Here } x(n) = 6 + \sin\left(\frac{2\pi n}{N}\right), \quad \text{with } N=4$$

$$x(n) = 6 + \sin\left(\frac{2\pi n}{4}\right), \quad n=0,1,2,3.$$

$$= 6 + \sin\left(\frac{\pi}{2}n\right), \quad n=0,1,2,3.$$

$$= \{6, 7, 6, 5\}.$$

The N-point DFT is given as,

$$\begin{aligned} X_N &= [W_N]x_N \\ \therefore X_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 6+7+6+5 \\ 6-j7-6+j5 \\ 6-7+6-5 \\ 6+j7-6-j5 \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ +j2 \end{bmatrix} \end{aligned}$$

Question 7

Determine the eight-point DFT of the signal

$$x(n) = [1, 1, 1, 1, 1, 1, 0, 0]$$

and sketch its magnitude and phase.

Solution

$$\begin{aligned}
 X(k) &= \sum_{n=0}^7 x(n)e^{-j\frac{2\pi}{7}kn} \\
 &= \{6, -0.7071 - j1.7071, 1 - j, 0.7071 + j0.2929, 0, 0.7071 - j0.2929, 1 + j, \\
 &\quad -0.7071 + j1.7071\} \\
 |X(k)| &= \{6, 1.8478, 1.4142, 0.7654, 0, 0.7654, 1.4142, 1.8478\} \\
 \angle X(k) &= \left\{0, -1.9635, \frac{-\pi}{4}, 0.3927, 0, -0.3927, \frac{\pi}{4}, 1.9635\right\}
 \end{aligned}$$

Question 8Compute the N -point DFTs of the signal

$$x(n) = \cos \frac{2\pi}{N} k_0 n \quad 0 \leq n \leq N-1$$

Solution

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk_0} e^{-j\frac{2\pi}{N}kn} \\
 &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} \\
 &= N\delta(k - k_0)
 \end{aligned}$$

$$x(n) = \frac{1}{2}e^{j\frac{2\pi}{N}nk_0} + \frac{1}{2}e^{-j\frac{2\pi}{N}nk_0}$$

$$\text{From (e) we obtain } X(k) = \frac{N}{2} [\delta(k - k_0) + \delta(k - N + k_0)]$$

UNIT 2**PROPERTIES OF DISCRETE FOURIER TRANSFORMS (DFT)****CONTENTS:-**

PROPERTIES OF DFT, MULTIPLICATION OF TWO DFTs- THE CIRCULAR CONVOLUTION,
ADDITIONAL DFT PROPERTIES. 6 HRS

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

Unit 2 Properties of DFT

2.1 Properties:-

The DFT and IDFT for an N-point sequence $x(n)$ are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0, 1, \dots, N-1$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad n = 0, 1, \dots, N-1$$

where W_N is defined as

$$W_N = e^{-j2\pi/N}$$

In this section we discuss about the important properties of the DFT. These properties are helpful in the application of the DFT to practical problems.

The notation used below to denote the N -point DFT pair $x(n)$ and $X(k)$ is

$$x(n) \xrightarrow[N]{\text{DFT}} X(k)$$

Periodicity:-

If $x(n)$ and $X(k)$ are an N -point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

2.1.2 Linearity: If

$$x_1(n) \xrightarrow[N]{\text{DFT}} X_1(k)$$

$$x_2(n) \xrightarrow[N]{\text{DFT}} X_2(k)$$

Then $A x_1(n) + b x_2(n) \longleftrightarrow a X_1(k) + b X_2(k)$

2.1.3 Circular shift:

In linear shift, when a sequence is shifted the sequence gets extended. In circular shift the number of elements in a sequence remains the same. Given a sequence $x(n)$ the shifted version $x(n-m)$ indicates a shift of m . With DFTs the sequences are defined for 0 to $N-1$.

If $x(n) = x(0), x(1), x(2), x(3)$

$$X(n-1) = x(3), x(0), x(1), x(2)$$

$$X(n-2) = x(2), x(3), x(0), x(1)$$

2.1.4 Time shift:

If $x(n) \longleftrightarrow X(k)$

\xrightarrow{mk}

Then $x(n-m) \longleftrightarrow W_N^m X(k)$

2.1.5 Frequency shift

If $x(n) \longleftrightarrow X(k)$

+no k

$W_N \quad x(n) \longleftrightarrow X(k+no)$

$N-1 \quad kn$

Consider $x(k) = \sum_{n=0}^{N-1} x(n) W_N^n$

n

$N-1$

$(k+no)n$

$$X(k+no) = \sum_{n=0}^{N-1} x(n) W_N^{(k+no)n}$$

$kn \quad non$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn} W_N^{non}$$

non

$$\therefore X(k+no) \longleftrightarrow x(n) W_N^{kn}$$

2.1.6 Symmetry:

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For a real sequence, if $x(n) \leftrightarrow X(k)$

$$X(N-K) = X^*(k)$$

For a complex sequence

$$\text{DFT}(x^*(n)) = X^*(N-K)$$

If $x(n)$ then $X(k)$

Real and even		real and even
Real and odd		imaginary and odd
Odd and imaginary		real odd
Even and imaginary		imaginary and even

2.2 Convolution theorem;

Circular convolution in time domain corresponds to multiplication of the DFTs

If $y(n) = x(n) \otimes h(n)$ then $Y(k) = X(k) H(k)$

Ex let $x(n) = 1,2,2,1$ and $h(n) = 1,2,2,1$

Then $y(n) = x(n) \otimes h(n)$

$$Y(n) = 9,10,9,8$$

N pt DFTs of 2 real sequences can be found using a single DFT

If $g(n)$ & $h(n)$ are two sequences then let $x(n) = g(n) + j h(n)$

$$G(k) = \frac{1}{2} (X(k) + X^*(k))$$

$$H(k) = \frac{1}{2j} (X(k) - X^*(k))$$

2N pt DFT of a real sequence using a single N pt DFT

Let $x(n)$ be a real sequence of length $2N$ with $y(n)$ and $g(n)$ denoting its N pt DFT

Let $y(n) = x(2n)$ and $g(2n+1)$

$$X(k) = Y(k) + WN G(k)$$

Using DFT to find IDFT

The DFT expression can be used to find IDFT

$$X(n) = 1/N [DFT(X^*(k))]^*$$

Recommended Questions with solutions

Question 1

State and Prove the Time shifting Property of DFT

Solution

The DFT and IDFT for an N-point sequence $x(n)$ are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0, 1, \dots, N-1$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad n = 0, 1, \dots, N-1$$

where W_N is defined as

$$W_N = e^{-j2\pi/N}$$

Time shift:

If $x(n) \longleftrightarrow X(k)$

$$mk$$

Then $x(n-m) \longleftrightarrow W_N^{-mk} X(k)$

Question 2

State and Prove the: (i) Circular convolution property of DFT; (ii) DFT of Real and even sequence.

Solution

(i) Convolution theorem

Circular convolution in time domain corresponds to multiplication of the DFTs

If $y(n) = x(n) \otimes h(n)$ then $Y(k) = X(k) H(k)$

Ex let $x(n) = 1, 2, 2, 1$ and $h(n) = 1, 2, 2, 1$ Then $y(n) = x(n) \otimes h(n)$

$$Y(n) = 9, 10, 9, 8$$

N pt DFTs of 2 real sequences can be found using a single DFT

Digital Signal Processing

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If $g(n)$ & $h(n)$ are two sequences then let $x(n) = g(n) + j h(n)$

$$G(k) = \frac{1}{2} (X(k) + X^*(k))$$

$$H(k) = \frac{1}{2j} (X(k) - X^*(k))$$

2N pt DFT of a real sequence using a single N pt DFT

Let $x(n)$ be a real sequence of length $2N$ with $y(n)$ and $g(n)$ denoting its N pt DFT

Let $y(n) = x(2n)$ and $g(2n+1)$

$$X(k) = Y(k) + W_N^{-k} G(k)$$

Using DFT to find IDFT

The DFT expression can be used to find IDFT

$$X(n) = \frac{1}{N} [DFT(X^*(k))]^*$$

(ii) DFT of Real and even sequence.

For a real sequence, if $x(n) \longleftrightarrow X(k)$

$$X(N-K) = X^*(k)$$

For a complex sequence

$$DFT(x^*(n)) = X^*(N-K)$$

If $x(n)$	then	$X(k)$
Real and even		real and even
Real and odd		imaginary and odd
Odd and imaginary		real odd
Even and imaginary		imaginary and even

Question 3

Distinguish between circular and linear convolution

Solution

- 1) Circular convolution is used for periodic and finite signals while linear convolution is used for aperiodic and infinite signals.
- 2) In linear convolution we convolved one signal with another signal where as in circular convolution the same convolution is done but in circular pattern depending upon the samples of the signal
- 3) Shifts are linear in linear convolution, whereas it is circular in circular convolution.

Question 4

For the sequences

$$x_1(n) = \cos \frac{2\pi}{N}n \quad x_2(n) = \sin \frac{2\pi}{N}n \quad 0 \leq n \leq N-1$$

determine the N -point:

- (a) Circular convolution $x_1(n) \textcircled{N} x_2(n)$
- (b) Circular correlation of $x_1(n)$ and $x_2(n)$
- (c) Circular autocorrelation of $x_1(n)$
- (d) Circular autocorrelation of $x_2(n)$

Solution(a)

$$\begin{aligned} x_1(n) &= \frac{1}{2} \left(e^{j \frac{2\pi}{N}n} + e^{-j \frac{2\pi}{N}n} \right) \\ X_1(k) &= \frac{N}{2} [\delta(k-1) + \delta(k+1)] \\ \text{also } X_2(k) &= \frac{N}{2j} [\delta(k-1) - \delta(k+1)] \\ \text{So } X_3(k) &= X_1(k)X_2(k) \\ &= \frac{N^2}{4j} [\delta(k-1) - \delta(k+1)] \\ \text{and } x_3(n) &= \frac{N}{2} \sin\left(\frac{2\pi}{N}n\right) \end{aligned}$$

Solution(b)

$$\begin{aligned} \tilde{R}_{xy}(k) &= X_1(k)X_2^*(k) \\ &= \frac{N^2}{4j} [\delta(k-1) - \delta(k+1)] \\ \Rightarrow \tilde{r}_{xy}(n) &= -\frac{N}{2} \sin\left(\frac{2\pi}{N}n\right) \end{aligned}$$

Solution(c)

$$\begin{aligned} \tilde{R}_{xx}(k) &= X_1(k)X_1^*(k) \\ &= \frac{N^2}{4} [\delta(k-1) + \delta(k+1)] \\ \Rightarrow \tilde{r}_{xx}(n) &= \frac{N}{2} \cos\left(\frac{2\pi}{N}n\right) \end{aligned}$$

Solution(d)

$$\begin{aligned}\bar{R}_{yy}(k) &= X_2(k)X_2^*(k) \\ &= \frac{N^2}{4} [\delta(k-1) + \delta(k+1)] \\ \Rightarrow \bar{r}_{yy}(n) &= \frac{N}{2} \cos\left(\frac{2\pi}{N} n\right)\end{aligned}$$

Question 5

Use the four-point DFT and IDFT to determine the sequence

$$x_3(n) = x_1(n) \textcircled{N} x_2(n)$$

where $x_1(n)$ and $x_2(n)$ are the sequence given

$$x_1(n) = \{1, 2, 3, 1\}$$

↑

$$x_2(n) = \{4, 3, 2, 2\}$$

↑

Solution

$$\begin{aligned}y(n) &= x_1(n) \textcircled{4} x_2(n) \\ &= \sum_{m=0}^3 x_1(m)_{\text{mod } 4} x_2(n-m)_{\text{mod } 4} \\ &= \{17, 19, 22, 19\}\end{aligned}$$

$$\begin{aligned}X_1(k) &= \{7, -2 - j, 1, -2 + j\} \\ X_2(k) &= \{11, 2 - j, 1, 2 + j\} \\ \Rightarrow X_3(k) &= X_1(k)X_2(k) \\ &= \{17, 19, 22, 19\}\end{aligned}$$

Question 6

A linear time-invariant system with frequency response $H(\omega)$ is excited with the periodic input

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n - kN)$$

Suppose that we compute the N -point DFT $Y(k)$ of the samples $y(n)$, $0 \leq n \leq N-1$ of the output sequence. How is $Y(k)$ related to $H(\omega)$?

Solution

$$\begin{aligned}
 x(n) &= \sum_{i=-\infty}^{\infty} \delta(n - iN) \\
 y(n) &= \sum_m h(m)x(n-m) \\
 &= \sum_m h(m) \left[\sum_i \delta(n-m-iN) \right] \\
 &= \sum_i h(n-iN)
 \end{aligned}$$

Therefore, $y(\cdot)$ is a periodic sequence with period N . So

$$\begin{aligned}
 Y(k) &= \sum_{n=0}^{N-1} y(n) W_N^{kn} \\
 &= H(w)|_{w=\frac{2\pi}{N}k}
 \end{aligned}$$

$$Y(k) = H\left(\frac{2\pi k}{N}\right) \quad k = 0, 1, \dots, N-1$$

UNIT 3

FAST FOURIER TRANSFORMS (FFT) ALGORITHMS

CONTENTS:-

USE OF DFT IN LINEAR FILTERING : OVERLAP-SAVE AND OVERLAP-ADD METHOD. DIRECT COMPUTATION OF DFT, NEED FOR EFFICIENT COMPUTATION OF THE DFT (FFT ALGORITHMS).

7 HRS

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

UNIT 3

FAST-FOURIER-TRANSFORM (FFT) ALGORITHMS

3.1 Digital filtering using DFT

In a LTI system the system response is got by convoluting the input with the impulse response. In the frequency domain their respective spectra are multiplied. These spectra are continuous and hence cannot be used for computations. The product of 2 DFT s is equivalent to the circular convolution of the corresponding time domain sequences. Circular convolution cannot be used to determine the output of a linear filter to a given input sequence. In this case a frequency domain methodology equivalent to linear convolution is required. Linear convolution can be implemented using circular convolution by taking the length of the convolution as $N \geq n_1+n_2-1$ where n_1 and n_2 are the lengths of the 2 sequences.

3.1.1 Overlap and add

In order to convolve a short duration sequence with a long duration sequence $x(n), x(n)$ is split into blocks of length N . $x(n)$ and $h(n)$ are zero padded to length $L+M-1$. circular convolution is performed to each block then the results are added. These data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0, 0, \dots, 0}_{M-1 \text{ zeros}}\}$$

$$x_2(n) = \{x(L), x(L+1), \dots, x(2L-1), \underbrace{0, 0, \dots, 0}_{M-1 \text{ zeros}}\}$$

$$x_3(n) = \{x(2L), \dots, x(3L-1), \underbrace{0, 0, \dots, 0}_{M-1 \text{ zeros}}\}$$

The two N -point DFTs are multiplied together to form

$$Y_m(k) = H(k)X_m(k) \quad k = 0, 1, \dots, N-1$$

The IDFT yields data blocks of length N that are free of aliasing since the size of the DFTs and IDFT is $N = L+M - 1$ and the sequences are increased to N -points by appending zeros to each block. Since each block is terminated with $M-1$ zeros, the last $M-1$ points from each output block must be overlapped and added to the first $M-1$ points of the succeeding

block. Hence this method is called the overlap method. This overlapping and adding yields the output sequences given below.

$$y(n) = \{y_1(0), y_1(1), \dots, y_1(L-1), y_1(L) + y_2(0), y_1(L+1) + \\ y_2(1), \dots, y_1(N-1) + y_2(M-1), y_2(M), \dots\}$$

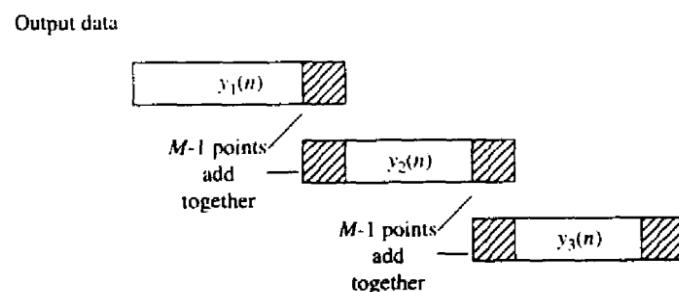
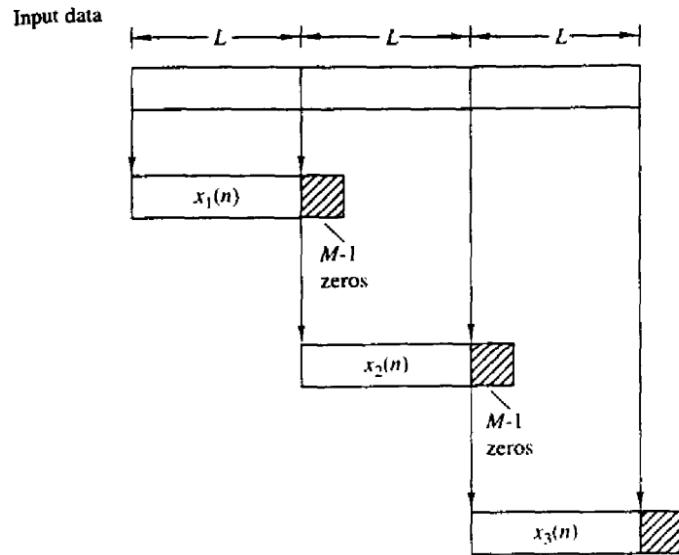


Figure 5.11 Linear FIR filtering by the overlap-add method.

2.1.2 Overlap and save method

In this method $x(n)$ is divided into blocks of length N with an overlap of $k-1$ samples. The first block is zero padded with $k-1$ zeros at the beginning. $H(n)$ is also zero padded to length N . Circular convolution of each block is performed using the N length DFT. The output signal is obtained after discarding the first $k-1$ samples the final result is obtained by adding the intermediate results.

In this method the size of the I/P data blocks is $N = L+M-1$ and the size of the DFts and IDFTs are of length N . Each data block consists of the last $M-1$ data points of the previous data block followed by L new data points to form a data sequence of length $N = L+M-1$. An N -point DFT is computed from each data block. The impulse response of the FIR filter is increased in length by appending $L-1$ zeros and an N -point DFT of the sequence is computed once and stored.

The multiplication of two N -point DFTs $\{H(k)\}$ and $\{X_m(k)\}$ for the m th block of data yields

$$\hat{Y}_m(k) = H(k)X_m(k) \quad k = 0, 1, \dots, N - 1$$

Then the N -point IDFT yields the result

$$\hat{y}_m(n) = \{\hat{y}_m(0), \hat{y}_m(1), \dots, \hat{y}_m(M-1), \hat{y}_m(M), \dots, \hat{y}_m(N-1)\}$$

Since the data record is of the length N , the first $M-1$ points of $Y_m(n)$ are corrupted by aliasing and must be discarded. The last L points of $Y_m(n)$ are exactly the same as the result from linear convolution and as a consequence we get

$$\hat{y}_m(n) = y_m(n), n = M, M + 1, \dots, N - 1$$

$$x_1(n) = \underbrace{\{0, 0, \dots, 0\}}_{M-1 \text{ points}}, \{x(0), x(1), \dots, x(L-1)\}$$

$$x_2(n) = \underbrace{\{x(L-M+1), \dots, x(L-1)\}}_{\substack{M-1 \text{ data points} \\ \text{from } x_1(n)}}, \underbrace{\{x(L), \dots, x(2L-1)\}}_{L \text{ new data points}}$$

$$x_3(n) = \underbrace{\{x(2L-M+1), \dots, x(2L-1)\}}_{\substack{M-1 \text{ data points} \\ \text{from } x_2(n)}}, \underbrace{\{x(2L), \dots, x(3L-1)\}}_{L \text{ new data points}}$$

and so forth. The resulting data sequences from the IDFT are given by (5.3.8), where the first $M-1$ points are discarded due to aliasing and the remaining L points constitute the desired result from linear convolution. This segmentation of the input data and the fitting of the output data blocks together to form the output sequence are graphically illustrated in Fig. 5.10.

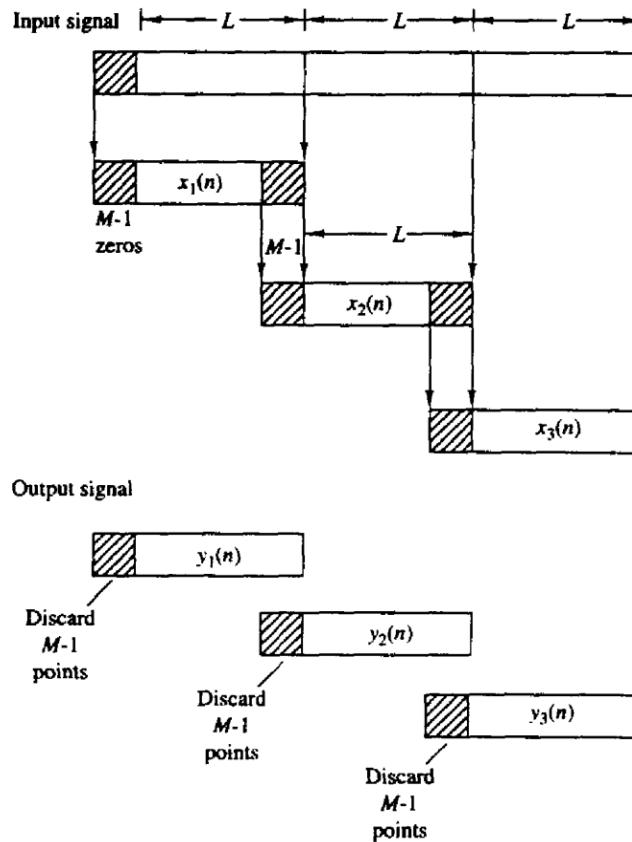


Figure 5.10 Linear FIR filtering by the overlap-save method.

3.2 Direct Computation of DFT

The problem:

Given signal samples: $x[0], \dots, x[N - 1]$ (some of which may be zero), develop a procedure to compute

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

for $k = 0, \dots, N - 1$ where

$$W_N = e^{-j\frac{2\pi}{N}}.$$

We would like the procedure to be fast, simple, and accurate. Fast is the most important, so we will sacrifice simplicity for speed, hopefully with minimal loss of accuracy

3.3 Need for efficient computation of DFT (FFT Algorithms)

Let us start with the simple way. Assume that W_N^{kn} has been precompiled and stored in a

table for the N of interest. How big should the table be? W_N^{mN} is periodic in m with period N , so we just need to tabulate the N values:

$$W_N^m = \cos\left(\frac{2\pi}{N}m\right) - j\sin\left(\frac{2\pi}{N}m\right)$$

(Possibly even less since Sin is just Cos shifted by a quarter periods, so we could save just Cos when N is a multiple of 4.)

Why tabulate? To avoid repeated function calls to Cos and sin when computing the DFT. Now we can compute each $X[k]$ directly from the formula as follows

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 + x[1] W_N^k + x[2] W_N^{2k} + \cdots + x[N-1] W_N^{(N-1)k}.$$

For each value of k, there are N complex multiplications, and (N-1) complex additions. There are N values of k, so the total number of complex operations is

$$N \cdot N + N(N - 1) = 2N^2 - N \equiv O(N^2).$$

Complex multiplies require 4 real multiplies and 2 real additions, whereas complex additions require just 2 real additions. N^2 complex multiplies are the primary concern.

N^2 increases *rapidly* with N, so how can we reduce the amount of computation? By exploiting the following properties of W:

- Symmetry property: $W_N^{k+N/2} = -W_N^k = e^{j\pi} W_N^k$
- Periodicity property: $W_N^{k+N} = W_N^k$
- Recursion property: $W_N^2 = W_{N/2}$

The first and third properties hold for even N, *i.e.*, when 2 is one of the prime factors of N. There are related properties for other prime factors of N.

Divide and conquer approach

We have seen in the preceding sections that the DFT is a very computationally intensive operation. In 1965, Cooley and Tukey published an algorithm that could be used to compute the DFT much more efficiently. Various forms of their algorithm, which came to be known as the Fast Fourier Transform (FFT), had actually been developed much earlier by other mathematicians (even dating back to Gauss). It was their paper, however, which stimulated a revolution in the field of signal processing.

It is important to keep in mind at the outset that the FFT is *not* a new transform. It is simply a very efficient way to compute an existing transform, namely the DFT. As we saw, a

straight forward implementation of the DFT can be computationally expensive because the number of multiplies grows as the square of the input length (i.e. N^2 for an N point DFT). The FFT reduces this computation using two simple but important concepts. The first concept, known as divide-and-conquer, splits the problem into two smaller problems. The second concept, known as recursion, applies this divide-and-conquer method repeatedly until the problem is solved.

Recommended Questions with solutions

Question1

A designer has available a number of eight-point FFT chips. Show explicitly how he should interconnect three such chips in order to compute a 24-point DFT.

Solution:-

Create three subsequences of 8-pts each

$$\begin{aligned}
 Y(k) &= \sum_{n=0,3,6,\dots}^{21} y(n)W_N^{kn} + \sum_{n=1,4,7,\dots}^{22} y(n)W_N^{kn} + \sum_{n=2,5,\dots}^{23} y(n)W_N^{kn} \\
 &= \sum_{i=0}^7 y(3i)W_N^{ki} + \sum_{i=0}^7 y(3i+1)W_N^{ki}W_N^k + \sum_{i=0}^7 y(3i+2)W_N^{ki}W_N^{2k} \\
 &\triangleq Y_1(k) + W_N^k Y_2(k) + W_N^{2k} Y_3(k)
 \end{aligned}$$

where Y_1, Y_2, Y_3 represent the 8-pt DFTs of the subsequences.

Question 2

Let $x(n)$ be a real-valued N -point ($N = 2^n$) sequence. Develop a method to compute an N -point DFT $X'(k)$, which contains only the odd harmonics [i.e., $X'(k) = 0$ if k is even] by using only a real $N/2$ -spoint DFT.

Solution:-

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad 0 \leq k \leq N-1 \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn} \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x(r + \frac{N}{2}) W_N^{(r+\frac{N}{2})k}
 \end{aligned}$$

$$\text{Let } X'(k') = X(2k+1), \quad 0 \leq k' \leq \frac{N}{2} - 1$$

$$\text{Then, } X'(k') = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^{(2k'+1)n} + x(n + \frac{N}{2}) W_N^{(n+\frac{N}{2})(2k'+1)} \right]$$

Using the fact that $W_N^{2k'n} = W_{\frac{N}{2}}^{k'n}$, $W_N^N = 1$

$$\begin{aligned}
 X'(k') &= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^n W_{\frac{N}{2}}^{k'n} + x(n + \frac{N}{2}) W_{\frac{N}{2}}^{k'n} W_N^n W_N^{\frac{N}{2}} \right] \\
 &= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x(n + \frac{N}{2}) \right] W_N^n W_{\frac{N}{2}}^{k'n}
 \end{aligned}$$

Question 3

The z -transform of the sequence $x(n) = u(n) - u(n-7)$ is sampled at five points on the unit circle as follows

$$x(k) = X(z)|_{z=e^{j2\pi k/5}} = e^{j2\pi k/5} \quad k = 0, 1, 2, 3, 4$$

Solution:-

$$\begin{aligned}
 X(z) &= 1 + z^{-1} + \dots + z^{-6} \\
 X(k) &= X(z)|_{z=e^{j2\pi k/5}} \\
 &= 1 + e^{-j\frac{2\pi}{5}} + e^{-j\frac{4\pi}{5}} + \dots + e^{-j\frac{12\pi}{5}} \\
 &= 2 + 2e^{-j\frac{2\pi}{5}} + e^{-j\frac{4\pi}{5}} + \dots + e^{-j\frac{12\pi}{5}} \\
 x'(n) &= \{2, 2, 1, 1, 1\} \\
 x'(n) &= \sum_m x(n+7m), \quad n = 0, 1, \dots, 4
 \end{aligned}$$

Temporal aliasing occurs in first two points of $x'(n)$ because $X(z)$ is not sampled at sufficiently small spacing on the unit circle.

Question 4

Consider a finite-duration sequence $x(n)$, $0 \leq n \leq 7$, with z-transform $X(z)$. We wish to compute $X(z)$ at the following set of values:

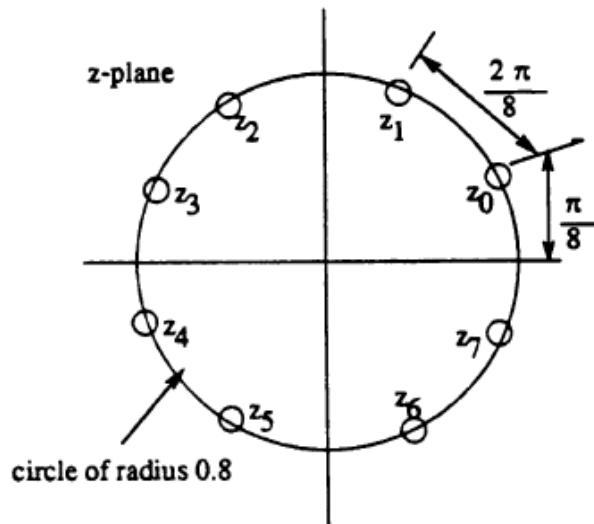
$$z_k = 0.8e^{j[(2\pi k/8) + (\pi/8)]} \quad 0 \leq k \leq 7$$

- (a) Sketch the points $\{z_k\}$ in the complex plane.
- (b) Determine a sequence $s(n)$ such that its DFT provides the desired samples of $X(z)$.

Solution:- (a)

$$Z_k = 0.8e^{j[\frac{2\pi k}{8} + \frac{\pi}{8}]}$$

(b)



$$\begin{aligned} X(k) &= X(z)|_{z=z_k} \\ &= \sum_{n=0}^7 x(n) \left[0.8e^{j[\frac{2\pi k}{8} + \frac{\pi}{8}]} \right]^{-n} \\ s(n) &= x(n) 0.8e^{-j\frac{\pi}{8}n} \end{aligned}$$

UNIT 4

FAST FOURIER TRANSFORMS (FFT) ALGORITHMS

CONTENTS:-

RADIX-2 FFT ALGORITHM FOR THE COMPUTATION OF DFT AND IDFT

DECIMATION IN- TIME AND DECIMATION-IN-FREQUENCY ALGORITHMS. GOERTZEL

ALGORITHM, AND CHIRP-Z TRANSFORM.

7 HRS

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

UNIT 4RADIX-2 FFT ALGORITHM FOR THE COMPUTATION OF DFT AND IDFT

4.1 Introduction:

Standard frequency analysis requires transforming time-domain signal to frequency domain and studying Spectrum of the signal. This is done through DFT computation. N-point DFT computation results in N frequency components. We know that DFT computation through FFT requires $N/2 \log_2 N$ complex multiplications and $N \log_2 N$ additions. In certain applications not all N frequency components need to be computed (an application will be discussed). If the desired number of values of the DFT is less than $2 \log_2 N$ than direct computation of the desired values is more efficient than FFT based computation.

4.2 Radix-2 FFT

Useful when N is a power of 2: $N = r^v$ for integers r and v. ‘r’ is called the **radix**, which comes from the Latin word meaning .a root, and has the same origins as the word radish.

When N is a power of $r = 2$, this is called **radix-2**, and the natural .divide and conquer approach. is to split the sequence into two sequences of length $N=2$. This is a very clever trick that goes back many years.

4.2.1 Decimation in time

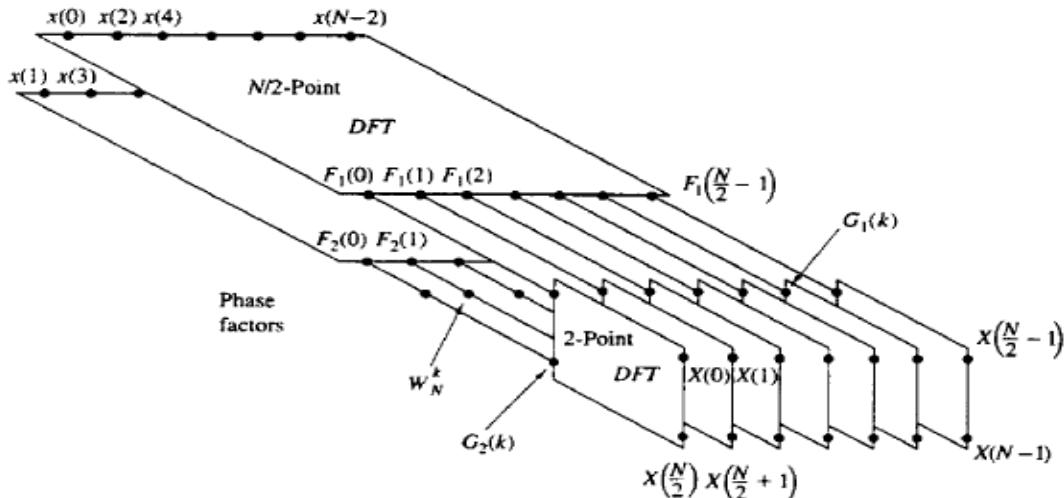
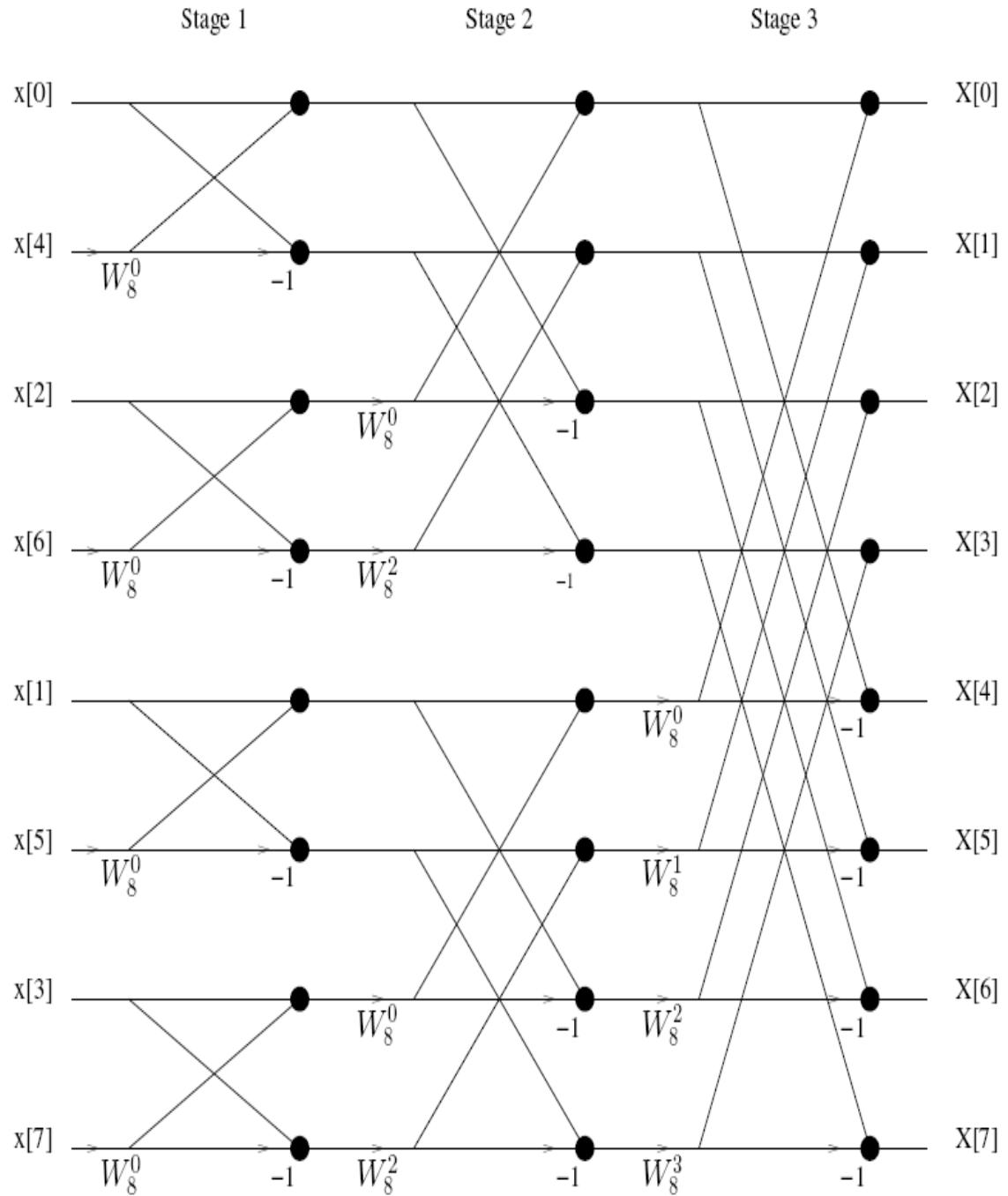


Fig 4.1 First step in Decimation-in-time domain Algorithm

$N = 8$ -point decimation-in-time FFT algorithm



$$W_N = e^{-j\frac{2\pi}{N}}$$

Each dot represents a complex addition.

Each arrow represents a complex multiplication.

4.2.2 Decimation-in-frequency Domain

Another important radix-2 FFT algorithm, called decimation-in-frequency algorithm is obtained by using divide-and-conquer approach with the choice of M=2 and L= N/2. This choice of data implies a column-wise storage of the input data sequence. To derive the algorithm, we begin by splitting the DFT formula into two summations, one of which involves the sum over the first N/2 data points and the second sum involves the last N/2 data points. Thus we obtain

$$\begin{aligned} X(k) &= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn} \\ &= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn} \end{aligned}$$

Since $W_N^{kN/2} = (-1)^k$, the expression (6.1.33) can be rewritten as

$$X(k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right) \right] W_N^{kn}$$

Now, let us split X(k) into the even and odd-numbered samples. Thus we obtain

$$X(2k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{kn} \quad k = 0, 1, \dots, \frac{N}{2} - 1 \quad (6.1.35)$$

and

$$X(2k+1) = \sum_{n=0}^{(N/2)-1} \left\{ \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^{kn} \right\} W_{N/2}^{kn} \quad k = 0, 1, \dots, \frac{N}{2} - 1 \quad (6.1.36)$$

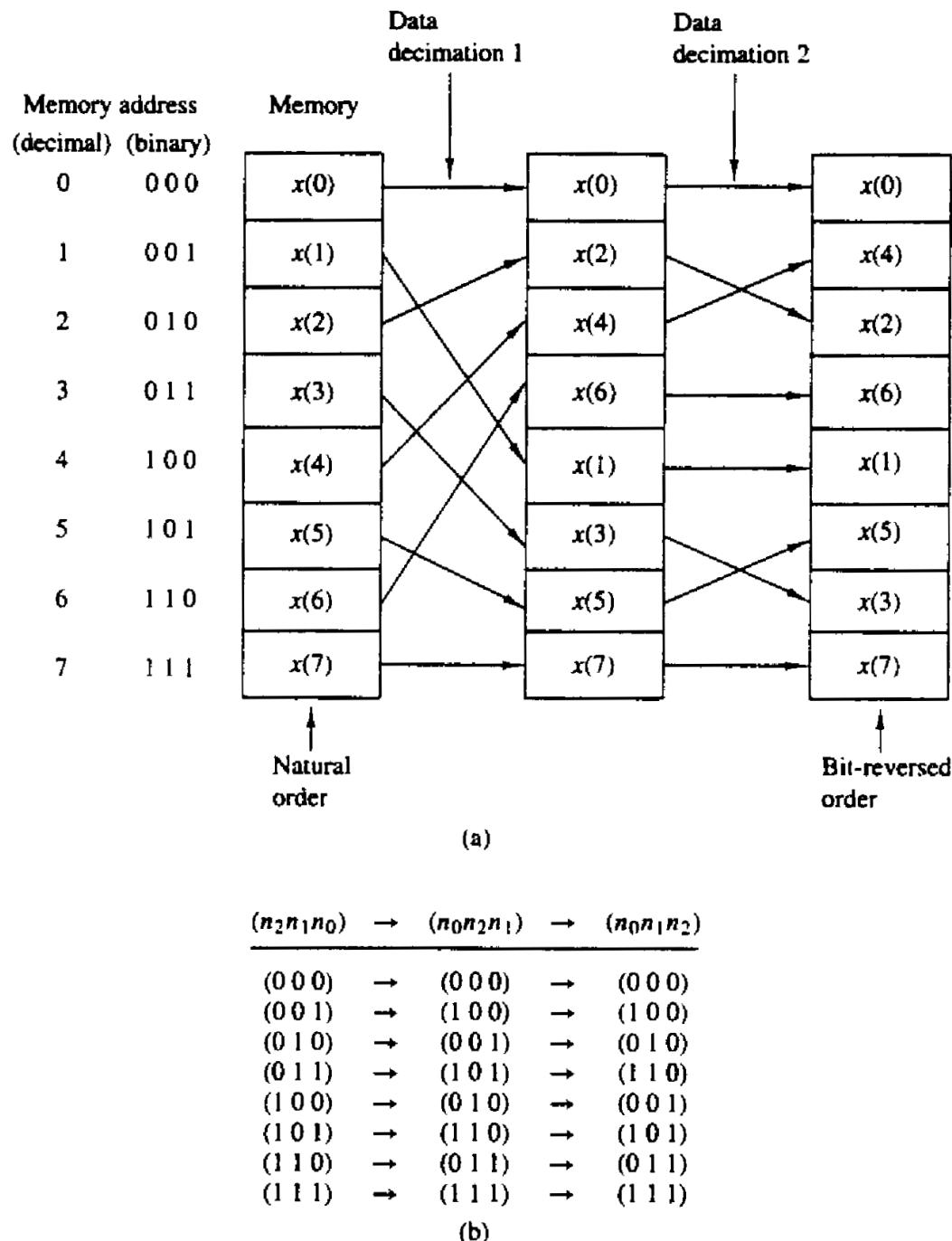
where we have used the fact that $W_N^2 = W_{N/2}$.

If we define the N/2-point sequences $g_1(n)$ and $g_2(n)$ as

$$\begin{aligned} g_1(n) &= x(n) + x\left(n + \frac{N}{2}\right) \\ g_2(n) &= \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \quad n = 0, 1, 2, \dots, \frac{N}{2} - 1 \end{aligned}$$

then

$$\begin{aligned} X(2k) &= \sum_{n=0}^{(N/2)-1} g_1(n) W_{N/2}^{kn} \\ X(2k+1) &= \sum_{n=0}^{(N/2)-1} g_2(n) W_{N/2}^{kn} \end{aligned}$$

**Fig 4.2 Shuffling of Data and Bit reversal**

The computation of the sequences $g_1(n)$ and $g_2(n)$ and subsequent use of these sequences to compute the $N/2$ -point DFTs depicted in fig we observe that the basic computation in this figure involves the butterfly operation.

The computation procedure can be repeated through decimation of the $N/2$ -point DFTs, $X(2k)$ and $X(2k+1)$. The entire process involves $v = \log_2 N$ of decimation, where each stage involves $N/2$ butterflies of the type shown in figure 4.3.

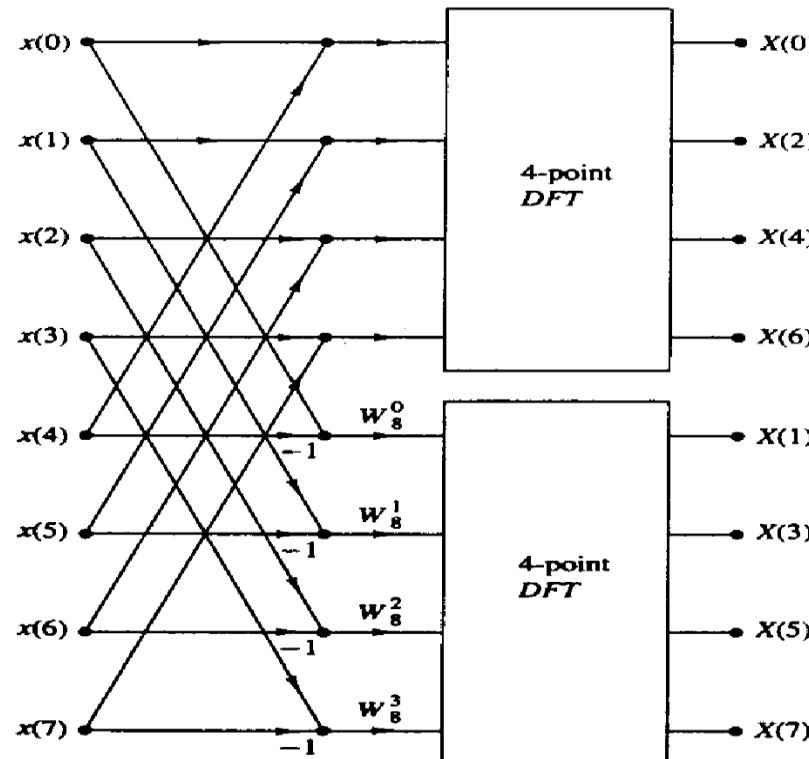


Fig 4.3 First step in Decimation-in-time domain Algorithm

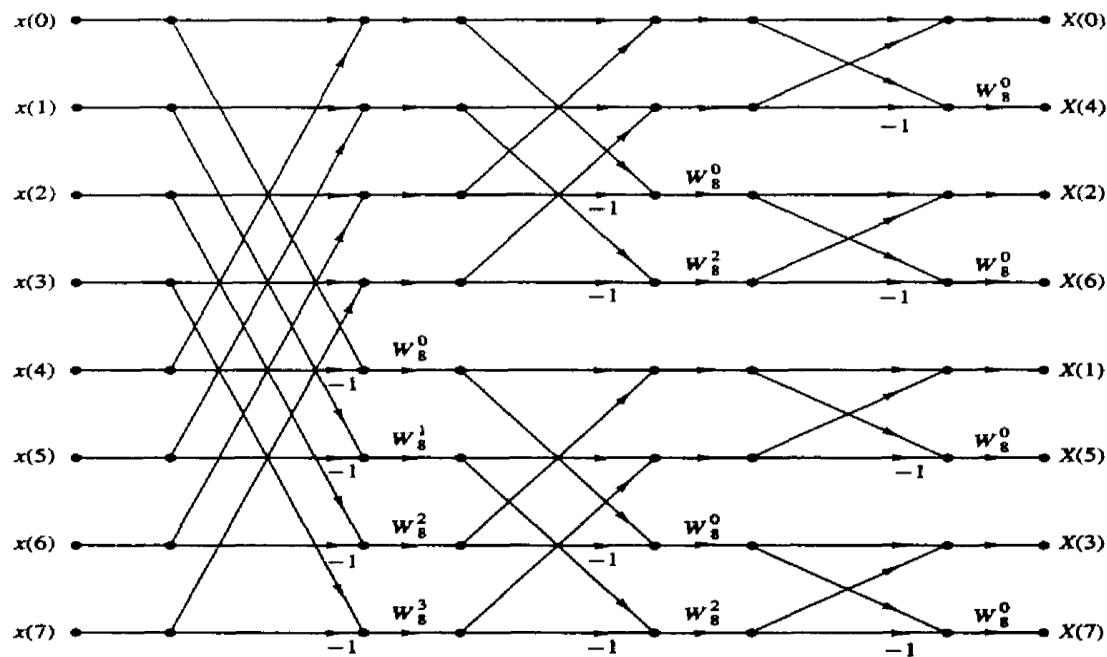
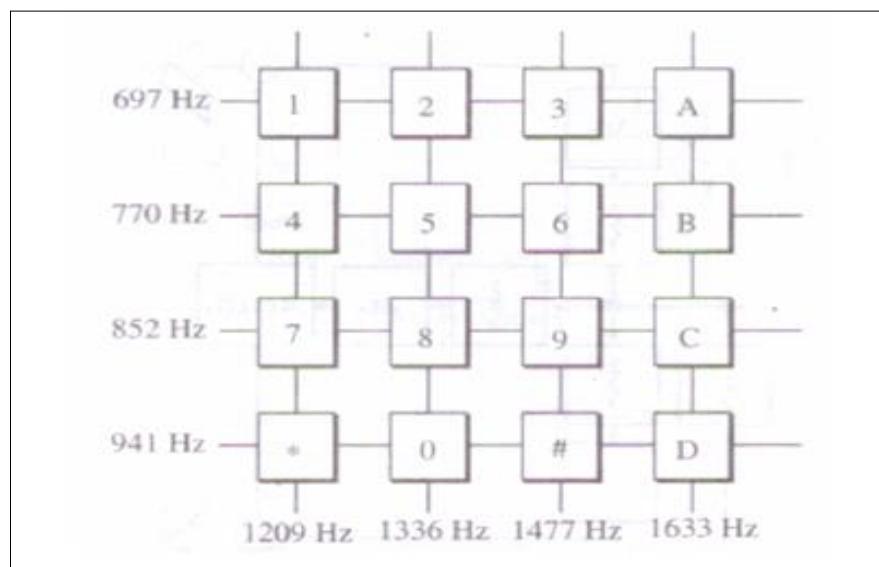


Fig 4.4 N=8 point Decimation-in-frequency domain Algorithm

4.2 Example: DTMF – Dual Tone Multi frequency

This is known as touch-tone/speed/electronic dialing, pressing of each button generates a unique set of two-tone signals, called DTMF signals. These signals are processed at exchange to identify the number pressed by determining the two associated tone frequencies. Seven frequencies are used to code the 10 decimal digits and two special characters (4x3 array)



In this application frequency analysis requires determination of possible seven (eight) DTMF fundamental tones and their respective second harmonics .For an 8 kHz sampling freq, the best value of the DFT length N to detect the eight fundamental DTMF tones has been found to be 205 .Not all 205 freq components are needed here, instead only those corresponding to key frequencies are required. FFT algorithm is not effective and efficient in this application. The direct computation of the DFT which is more effective in this application is formulated as a linear filtering operation on the input data sequence.

This algorithm is known as Goertzel Algorithm

This algorithm exploits periodicity property of the phase factor. Consider the DFT definition

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} \quad (1)$$

Since W_N^{-kN} is equal to 1, multiplying both sides of the equation by this results in;

$$X(k) = W_N^{-kN} \sum_{m=0}^{N-1} x(m)W_N^{mk} = \sum_{m=0}^{N-1} x(m)W_N^{-k(N-m)} \quad (2)$$

This is in the form of a convolution $y_k(n) = x(n) * h_k(n)$

$$y_k(n) = \sum_{m=0}^{N-1} x(m)W_N^{-k(n-m)} \quad (3)$$

$$h_k(n) = W_N^{-kn}u(n) \quad (4)$$

Where $y_k(n)$ is the out put of a filter which has impulse response of $h_k(n)$ and input $x(n)$.

The output of the filter at $n = N$ yields the value of the DFT at the freq $\omega_k = 2\pi k/N$

The filter has frequency response given by

$$H_k(z) = \frac{1}{1 - W_N^{-k}z^{-1}} \quad (6)$$

The above form of filter response shows it has a pole on the unit circle at the frequency $\omega_k = 2\pi k/N$.

Entire DFT can be computed by passing the block of input data into a parallel bank of N single-pole filters (resonators)

The above form of filter response shows it has a pole on the unit circle at the frequency $\omega_k = 2\pi k/N$.

Entire DFT can be computed by passing the block of input data into a parallel bank of N single-pole filters (resonators)

1.3 Difference Equation implementation of filter:

From the frequency response of the filter (eq 6) we can write the following difference equation relating input and output;

$$H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}$$

$$y_k(n) = W_N^{-k} y_k(n-1) + x(n) \quad y_k(-1) = 0 \quad (7)$$

The desired output is $X(k) = y_k(n)$ for $k = 0, 1, \dots, N-1$. The phase factor appearing in the difference equation can be computed once and stored.

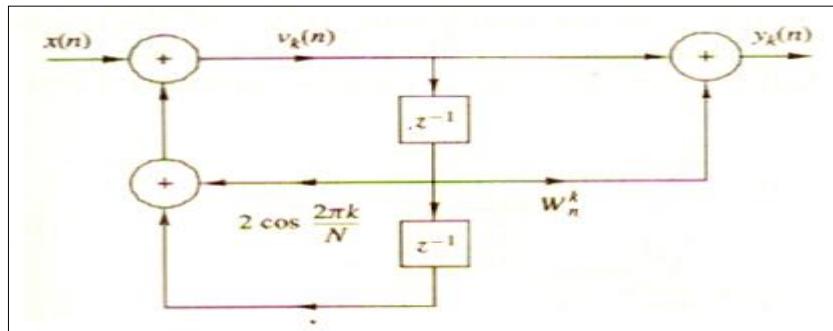
The form shown in eq (7) requires complex multiplications which can be avoided doing suitable modifications (divide and multiply by $1 - W_N^k z^{-1}$). Then frequency response of the filter can be alternatively expressed as

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{1 - 2 \cos(2\pi k / N) z^{-1} + z^{-2}} \quad (8)$$

This is second –order realization of the filter (observe the denominator now is a second-order expression). The direct form realization of the above is given by

$$v_k(n) = 2 \cos(2\pi k / N) v_k(n-1) - v_k(n-2) + x(n) \quad (9)$$

$$y_k(n) = v_k(n) - W_N^k v_k(n-1) \quad v_k(-1) = v_k(-2) = 0 \quad (10)$$



The recursive relation in (9) is iterated for $n = 0, 1, \dots, N$, but the equation in (10) is computed only once at time $n = N$. Each iteration requires one real multiplication and two additions. Thus, for a real input sequence $x(n)$ this algorithm requires $(N+1)$ real multiplications to yield $X(k)$ and $X(N-k)$ (this is due to symmetry). Going through the Goertzel algorithm it is clear that this algorithm is useful only when M out of N DFT values need to be computed where $M \leq 2\log_2 N$. Otherwise, the FFT algorithm is more efficient method. The utility of the algorithm completely depends on the application and number of frequency components we are looking for.

4.2. Chirp z- Transform

4.2.1 Introduction:

Computation of DFT is equivalent to samples of the z-transform of a finite-length sequence at equally spaced points around the unit circle. The spacing between the samples is given by $2\pi/N$. The efficient computation of DFT through FFT requires N to be a highly composite number which is a constraint. Many a times we may need samples of z-transform on contours other than unit circle or we may require dense set of frequency samples over a small region of unit circle. To understand these let us look in to the following situations:

1. Obtain samples of z-transform on a circle of radius ‘a’ which is concentric to unit circle
The possible solution is to multiply the input sequence by a^{-n}
2. 128 samples needed between frequencies $\omega = -\pi/8$ to $+\pi/8$ from a 128 point sequence

From the given specifications we see that the spacing between the frequency samples is $\pi/512$ or $2\pi/1024$. In order to achieve this freq resolution we take 1024- point FFT of the given 128-point seq by appending the sequence with 896 zeros. Since we need only 128 frequencies out of 1024 there will be big wastage of computations in this scheme.

For the above two problems Chirp z-transform is the alternative.

Chirp z- transform is defined as:

$$X(z_k) = \sum_{n=0}^{N-1} x(n)z_k^{-n} \quad k = 0, 1, \dots, L-1 \quad (11)$$

Where z_k is a generalized contour. z_k is the set of points in the z-plane falling on an arc which begins at some point z_0 and spirals either in toward the origin or out away from the origin such that the points $\{z_k\}$ are defined as,

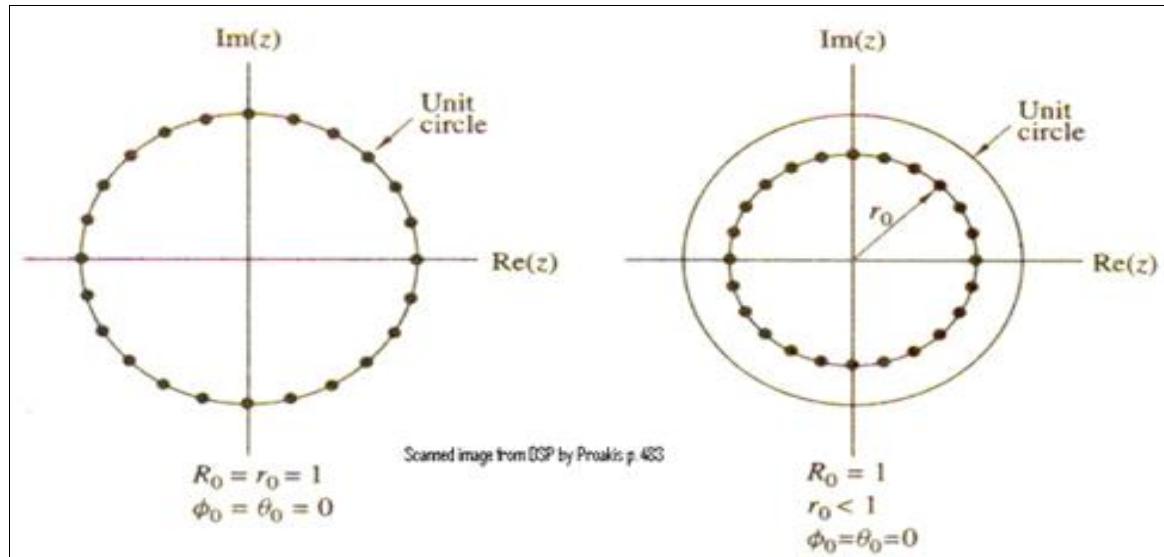
$$z_k = r_0 e^{j\theta_0} (R_0 e^{j\phi_0})^k \quad k = 0, 1, \dots, L-1 \quad (12)$$

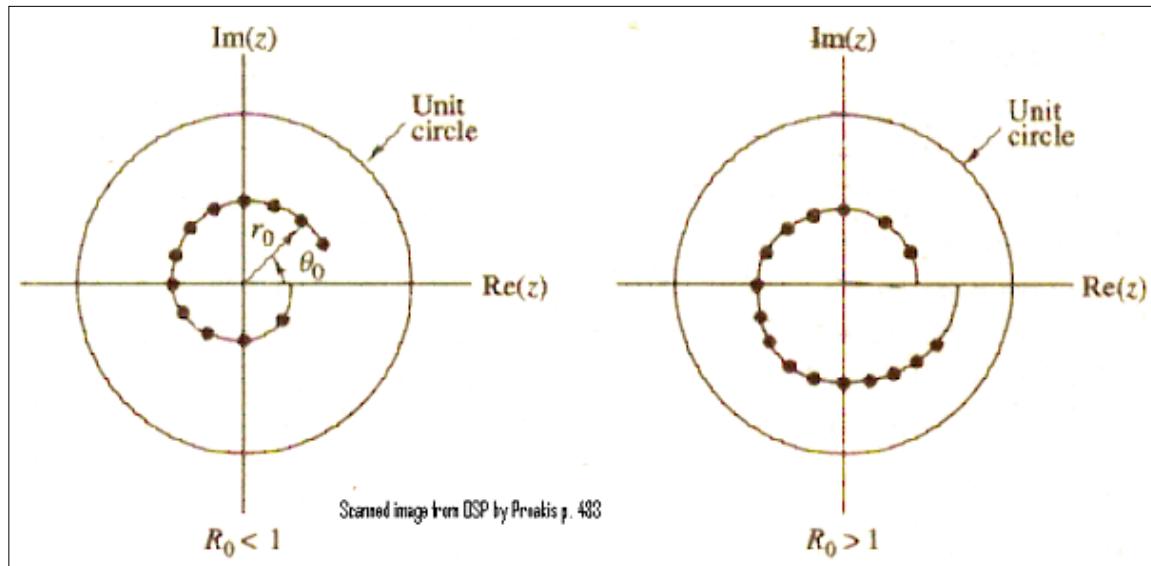
Note that,

- a. if $R_0 < 1$ the points fall on a contour that spirals toward the origin
- b. If $R_0 > 1$ the contour spirals away from the origin
- c. If $R_0 = 1$ the contour is a circular arc of radius
- d. If $r_0=1$ and $R_0=1$ the contour is an arc of the unit circle.

(Additionally this contour allows one to compute the freq content of the sequence $x(n)$ at dense set of L frequencies in the range covered by the arc without having to compute a large DFT (i.e., a DFT of the sequence $x(n)$ padded with many zeros to obtain the desired resolution in freq.))

- e. If $r_0 = R_0 = 1$ and $\theta_0 = 0$ $\Phi_0 = 2\pi/N$ and $L = N$ the contour is the entire unit circle similar to the standard DFT. These conditions are shown in the following diagram.





Substituting the value of z_k in the expression of $X(z_k)$

$$X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} = \sum_{n=0}^{N-1} x(n) (r_0 e^{j\theta_0})^{-n} W^{-nk} \quad (13)$$

where $W = R_0 e^{j\phi_0}$ (14)

4.2.2 Expressing computation of $X(z_k)$ as linear filtering operation:

By substitution of

$$nk = \frac{1}{2}(n^2 + k^2 - (k-n)^2) \quad (15)$$

we can express $X(z_k)$ as

$$X(z_k) = W^{-k^2/2} y(k) = y(k) / h(k) \quad k = 0, 1, \dots, L-1 \quad (16)$$

Where

$$h(n) = W^{n^2/2} \quad g(n) = x(n) (r_0 e^{j\theta_0})^{-n} W^{-n^2/2}$$

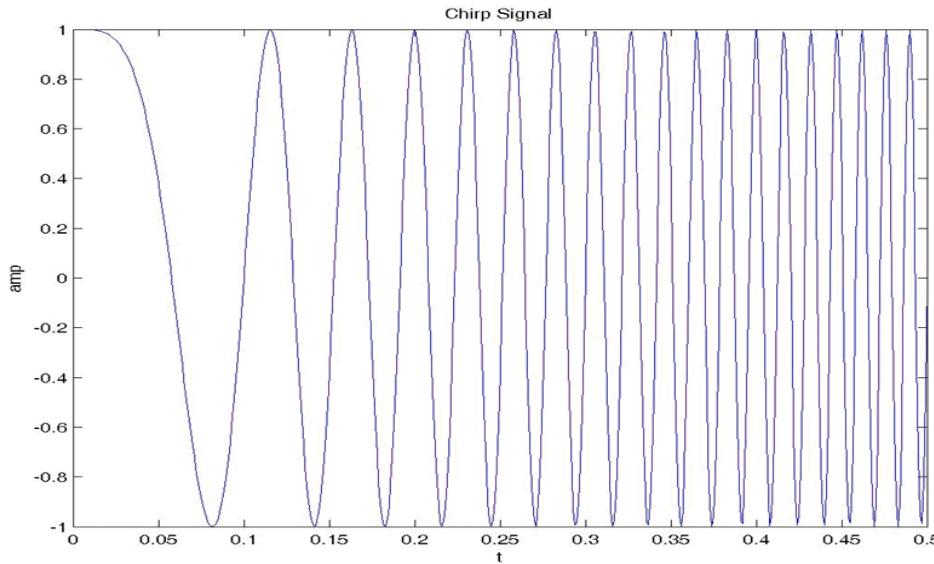
$$y(k) = \sum_{n=0}^{N-1} g(n) h(k-n) \quad (17)$$

both $g(n)$ and $h(n)$ are complex valued sequences

4.2.3 Why it is called Chirp z-transform?

If $R_0 = 1$, then sequence $h(n)$ has the form of complex exponential with argument $\omega n = n^2 \Phi_0/2 = (n \Phi_0/2) n$. The quantity $(n \Phi_0/2)$ represents the freq of the complex exponential

signal, which increases linearly with time. Such signals are used in radar systems are called chirp signals. Hence the name chirp z-transform.



4.2.4 How to Evaluate linear convolution of eq (17)

1. Can be done efficiently with FFT
2. The two sequences involved are $g(n)$ and $h(n)$. $g(n)$ is finite length seq of length N and $h(n)$ is of infinite duration, but fortunately only a portion of $h(n)$ is required to compute L values of $X(z)$, hence FFT could be still be used.
3. Since convolution is via FFT, it is circular convolution of the N -point seq $g(n)$ with an M - point section of $h(n)$ where $M > N$
4. The concepts used in overlap –save method can be used
5. While circular convolution is used to compute linear convolution of two sequences we know the initial $N-1$ points contain aliasing and the remaining points are identical to the result that would be obtained from a linear convolution of $h(n)$ and $g(n)$, In view of this the DFT size selected is $M = L+N-1$ which would yield L valid points and $N-1$ points corrupted by aliasing. The section of $h(n)$ considered is for $-(N-1) \leq n \leq (L-1)$ yielding total length M as defined
6. The portion of $h(n)$ can be defined in many ways, one such way is,

$$h_1(n) = h(n-N+1) \quad n = 0, 1, \dots, M-1$$

7. Compute $H_1(k)$ and $G(k)$ to obtain

$$Y_1(k) = G(k)H_1(k)$$

8. Application of IDFT will give $y_1(n)$, for

$n = 0, 1, \dots, M-1$. The starting $N-1$ are discarded and desired values are $y_1(n)$ for

$N-1 \leq n \leq M-1$ which corresponds to the range $0 \leq n \leq L-1$ i.e.,

$$y(n) = y_1(n+N-1) \quad n=0, 1, 2, \dots, L-1$$

9. Alternatively $h_2(n)$ can be defined as

$$\begin{aligned} h_2(n) &= h(n) & 0 \leq n \leq L-1 \\ &= h(n - (N + L - 1)) & L \leq n \leq M-1 \end{aligned}$$

10. Compute $Y_2(k) = G(k)H_2(k)$, The desired values of $y_2(n)$ are in the range

$0 \leq n \leq L-1$ i.e.,

$$y(n) = y_2(n) \quad n=0, 1, \dots, L-1$$

11. Finally, the complex values $X(z_k)$ are computed by dividing $y(k)$ by $h(k)$
For $k = 0, 1, \dots, L-1$

4.3 Computational complexity

In general the computational complexity of CZT is of the order of $M \log_2 M$ complex multiplications. This should be compared with $N \cdot L$ which is required for direct evaluation. If L is small direct evaluation is more efficient otherwise if L is large then CZT is more efficient.

4.3.1 Advantages of CZT

a. Not necessary to have $N = L$

b. Neither N or L need to be highly composite

c. The samples of Z transform are taken on a more general contour that includes the unit circle as a special case.

4.4 Example to understand utility of CZT algorithm in freq analysis

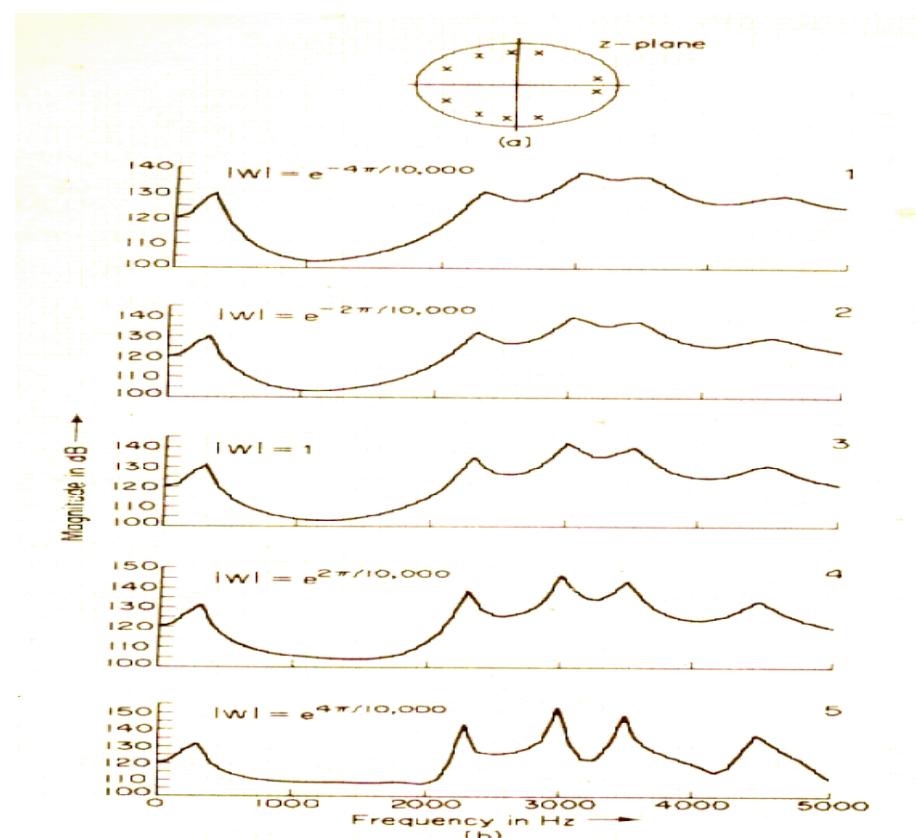
(ref: DSP by Oppenheim Schaffer)

CZT is used in this application to sharpen the resonances by evaluating the z-transform off the unit circle. Signal to be analyzed is a synthetic speech signal generated by exciting a

five-pole system with a periodic impulse train. The system was simulated to correspond to a sampling freq. of 10 kHz. The poles are located at center freqs of 270, 2290, 3010, 3500 & 4500 Hz with bandwidth of 30, 50, 60, 87 & 140 Hz respectively.

Solution: Observe the pole-zero plots and corresponding magnitude frequency response for different choices of $|w|$. The following observations are in order:

- The first two spectra correspond to spiral contours outside the unit circle with a resulting broadening of the resonance peaks
- $|w| = 1$ corresponds to evaluating z-transform on the unit circle
- The last two choices correspond to spiral contours which spirals inside the unit circle and close to the pole locations resulting in a sharpening of resonance peaks.

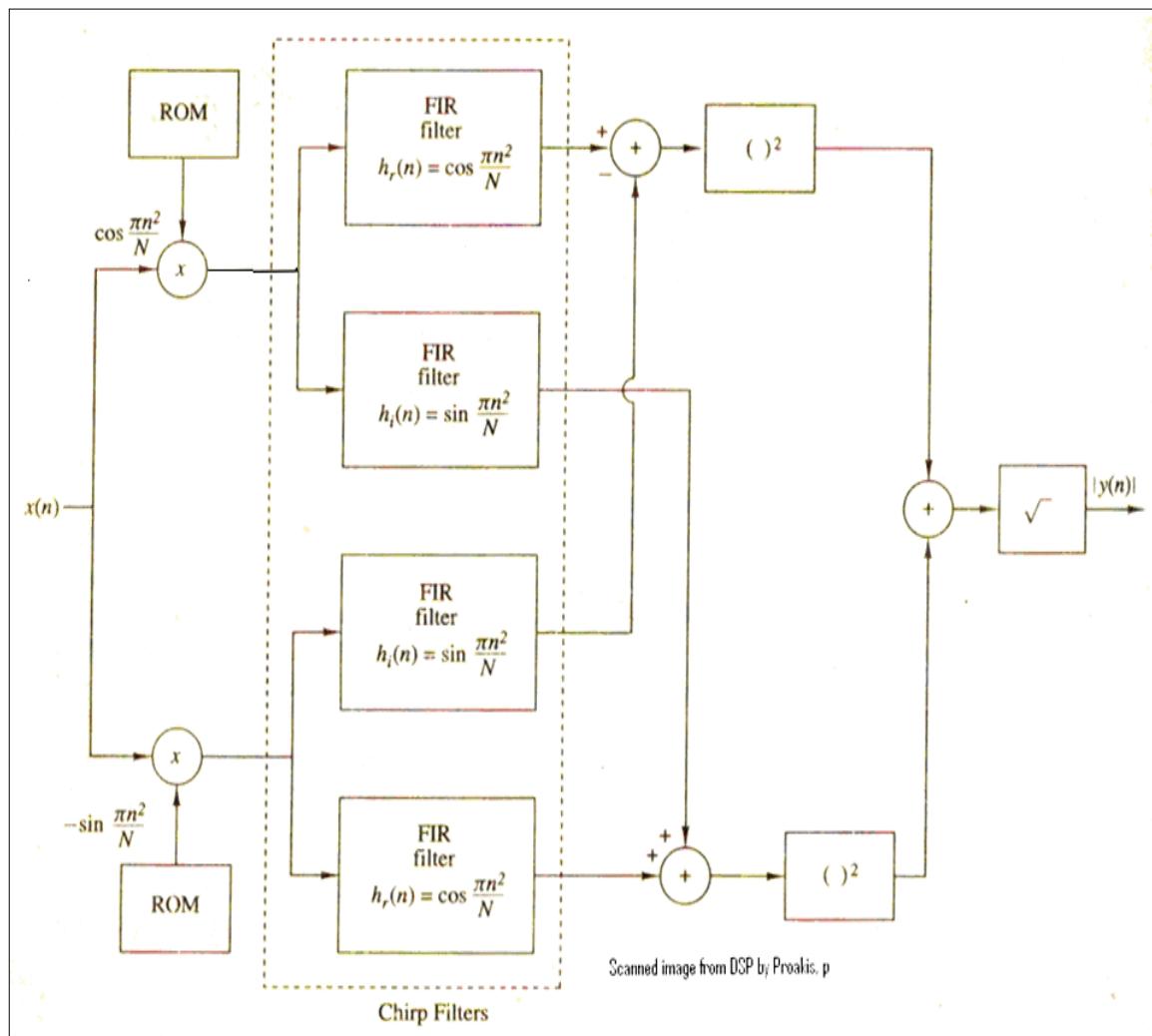


4.5 Implementation of CZT in hardware to compute the DFT signals

The block schematic of the CZT hardware is shown in down figure. DFT computation requires $r_0 = R_0 = 1$, $\theta_0 = 0$, $\Phi_0 = 2\pi/N$ and $L = N$.

The cosine and sine sequences in $h(n)$ needed for pre multiplication and post multiplication are usually stored in a ROM. If only magnitude of DFT is desired, the post multiplications are unnecessary,

In this case $|X(zk)| = |y(k)|$ $k = 0, 1, \dots, N-1$



Recommended Questions with solutions

Question 1

Compute the 16-point DFT of the sequence

$$x(n) = \cos \frac{\pi}{2}n \quad 0 \leq n \leq 15$$

using the radix-4 decimation-in-time algorithm.

Solution:-

$$\underline{A} \triangleq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\underline{x}_1 \triangleq [x(0) \ x(4) \ x(8) \ x(12)]^T$$

$$\underline{x}_2 \triangleq [x(1) \ x(5) \ x(9) \ x(13)]^T$$

$$\underline{x}_3 \triangleq [x(2) \ x(6) \ x(10) \ x(14)]^T$$

$$\underline{x}_4 \triangleq [x(3) \ x(7) \ x(11) \ x(15)]^T$$

$$\begin{bmatrix} F(0) \\ F(4) \\ F(8) \\ F(12) \end{bmatrix} = \underline{A}\underline{x}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F(1) \\ F(5) \\ F(9) \\ F(13) \end{bmatrix} = \underline{A}\underline{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F(2) \\ F(6) \\ F(10) \\ F(14) \end{bmatrix} = \underline{A}\underline{x}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F(3) \\ F(7) \\ F(11) \\ F(15) \end{bmatrix} = \underline{A}\underline{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As every $F(i) = 0$ except $F(0) = -F(2) = 4$,

$$\begin{bmatrix} x(0) \\ x(7) \\ x(8) \\ x(12) \end{bmatrix} = Ax_4 \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

which means that $X(4) = X(12) = 8$. $X(k) = 0$ for other K .

Question 2

Draw the flow graph for the decimation-in-frequency (DIF) SRFFT algorithm for $N = 16$. What is the number of nontrivial multiplications?

Solution :- There are 20 real , non trial multiplications

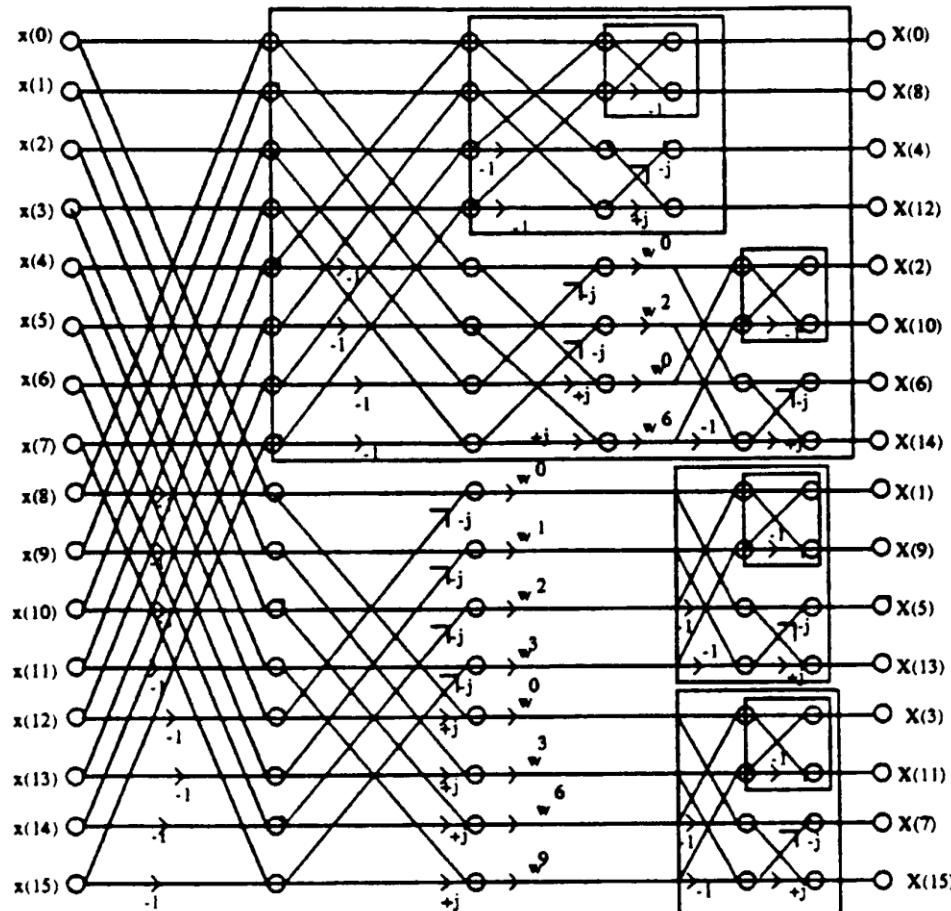


Figure 4.1 DIF Algorithm for $N=16$

Question 3

Explain how the DFT can be used to compute N equispaced samples of the z -transform, of an N -point sequence, on a circle of radius r .

Solution:-

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n}$$

$$\text{Hence, } X(z_k) = \sum_{n=0}^{N-1} x(n)r^{-n}e^{-j\frac{2\pi}{N}kn}$$

where $z_k = re^{-j\frac{2\pi}{N}k}, k = 0, 1, \dots, N - 1$ are the N sample points. It is clear that $X(z_k), k = 0, 1, \dots, N - 1$ is equivalent to the DFT (N -pt) of the sequence $x(n)r^{-n}, n \in [0, N - 1]$.

Question 4

Let $X(k)$ be the N -point DFT of the sequence $x(n), 0 \leq n \leq N - 1$. What is the N -point DFT of the sequence $s(n) = X(n), 0 \leq n \leq N - 1$?

Solution:-

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}$$

Let $F(t), t = 0, 1, \dots, N - 1$ be the DFT of the sequence on $k X(k)$.

$$\begin{aligned} F(t) &= \sum_{k=0}^{N-1} X(k)W_N^{tk} \\ &= \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x(n)W_N^{kn} \right] W_N^{tk} \\ &= \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} W_N^{k(n+t)} \right] \\ &= \sum_{n=0}^{N-1} x(n)\delta(n+t) \bmod N \\ &= \sum_{n=0}^{N-1} x(n)\delta(N-1-n-t) \quad t = 0, 1, \dots, N - 1 \\ &= \{x(N-1), x(N-2), \dots, x(1), x(0)\} \end{aligned}$$

Question 5

Develop a radix-3 decimation-in-time FFT algorithm for $N = 3^r$ and draw the corresponding flow graph for $N = 9$. What is the number of required complex multiplications? Can the operations be performed in place?

Solution:-

$$\begin{aligned}
 Y(k) &= \sum_{n=0}^8 y(n)W_9^{nk} \\
 &= \sum_{n=0,3,6} y(n)W_9^{nk} + \sum_{n=1,4,7} y(n)W_9^{nk} + \sum_{n=2,5,8} y(n)W_9^{nk} \\
 &= \sum_{m=0}^2 y(3m)W_9^{3km} + \sum_{m=0}^2 y(3m+1)W_9^{(3m+1)k} + \sum_{m=0}^2 y(3m+2)W_9^{(3m+2)k} \\
 &= \sum_{m=0}^2 y(3m)W_3^{km} + \sum_{m=0}^2 y(3m+1)W_3^{m,k}W_9^k + \sum_{m=0}^2 y(3m+2)W_3^{m,k}W_9^{2k}
 \end{aligned}$$

Question 6

Determine the system function $H(z)$ and the difference equation for the system that uses the Goertzel algorithm to compute the DFT value $X(N - k)$.

Solution:-

$$\begin{aligned}
 X(k) &= \sum_{m=0}^{N-1} x(m)W_N^{km} \\
 &= \sum_{m=0}^{N-1} x(m)W_N^{km}W_N^{-kN} \text{ since } W_N^{-kN} = 1 \\
 &= \sum_{m=0}^{N-1} x(m)W_N^{-k(N-m)}
 \end{aligned}$$

This can be viewed as the convolution of the N-length sequence $x(n)$ with impulse response of a linear filter

$$h_k(n) \stackrel{\Delta}{=} W_N^{kn}u(n), \text{ evaluated at time } N$$

$$H_k(z) = \sum_{n=0}^{\infty} W_N^{kn}z^{-n}$$

$$\begin{aligned} &= \frac{1}{1 - W_N^k z^{-1}} \\ &= \frac{Y_u(z)}{X(z)} \\ y_k(n) &= W_N^k y_k(n-1) + x(n), \quad y_k(-1) = 0 \\ y_k(N) &= X(k) \end{aligned}$$

UNIT 5**IIR FILTER DESIGN****CONTENTS:-**

IIR FILTER DESIGN: CHARACTERISTICS OF COMMONLY USED ANALOG FILTERS – BUTTERWORTH AND CHEBYSHEVE FILTERS, ANALOG TO ANALOG FREQUENCY TRANSFORMATIONS. **6 HRS**

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

Unit 5 Design of IIR Filters

5.1 Introduction

A digital filter is a linear shift-invariant discrete-time system that is realized using finite precision arithmetic. The design of digital filters involves three basic steps:

- ❖ The specification of the desired properties of the system.
- ❖ The approximation of these specifications using a causal discrete-time system.
- ❖ The realization of these specifications using finite precision arithmetic.

These three steps are independent; here we focus our attention on the second step. The desired digital filter is to be used to filter a digital signal that is derived from an analog signal by means of periodic sampling. The specifications for both analog and digital filters are often given in the frequency domain, as for example in the design of low pass, high pass, band pass and band elimination filters.

Given the sampling rate, it is straight forward to convert from frequency specifications on an analog filter to frequency specifications on the corresponding digital filter, the analog frequencies being in terms of Hertz and digital frequencies being in terms of radian frequency or angle around the unit circle with the point $Z=-1$ corresponding to half the sampling frequency. The least confusing point of view toward digital filter design is to consider the filter as being specified in terms of angle around the unit circle rather than in terms of analog frequencies.

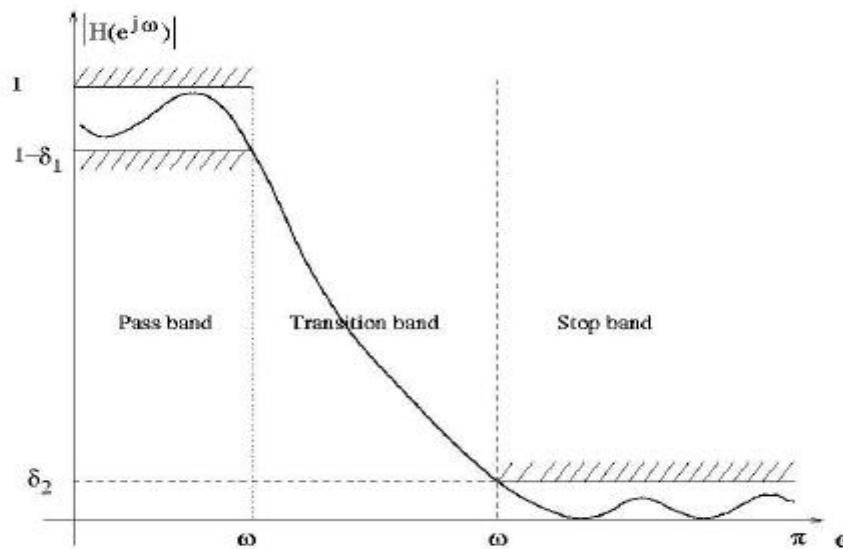


Figure 5.1: Tolerance limits for approximation of ideal low-pass filter

A separate problem is that of determining an appropriate set of specifications on the digital filter. In the case of a low pass filter, for example, the specifications often take the form of a tolerance scheme, as shown in Fig. 5.1.

$$1 - \delta_1 \leq |H(e^{j\omega})| \leq 1, \quad |\omega| \leq \omega_p$$

$$|H(e^{j\omega})| \leq \delta_2, \quad \omega_s \leq |\omega| \leq \pi$$

Many of the filters used in practice are specified by such a tolerance scheme, with no constraints on the phase response other than those imposed by stability and causality requirements; i.e., the poles of the system function must lie inside the unit circle. Given a set of specifications in the form of Fig. 5.1, the next step is to find a discrete time linear system whose frequency response falls within the prescribed tolerances. At this point the filter design problem becomes a problem in approximation. In the case of infinite impulse response (IIR) filters, we must approximate the desired frequency response by a rational function, while in the finite impulse response (FIR) filters case we are concerned with polynomial approximation.

5.1 Design of IIR Filters from Analog Filters:

The traditional approach to the design of IIR digital filters involves the transformation of an analog filter into a digital filter meeting prescribed specifications. This is a reasonable approach because:

- ❖ The art of analog filter design is highly advanced and since useful results can be achieved, it is advantageous to utilize the design procedures already developed for analog filters.
- ❖ Many useful analog design methods have relatively simple closed-form design formulas.

Therefore, digital filter design methods based on analog design formulas are rather simple to implement. An analog system can be described by the differential equation

$$\sum_{k=0}^N c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^M d_k \frac{d^k x_a(t)}{dt^k}$$

And the corresponding rational function is

$$H_a(s) = \frac{\sum_{k=0}^M d_k s^k}{\sum_{k=0}^N c_k s^k} = \frac{y_a(s)}{x_a(s)}$$

The corresponding description for digital filters has the form

$$\sum_{k=0}^N a_k y(n - k) = \sum_{k=0}^M b_k x(n - k)$$

and the rational function

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

In transforming an analog filter to a digital filter we must therefore obtain either $H(z)$ or $h(n)$ (inverse Z-transform of $H(z)$ i.e., impulse response) from the analog filter design. In such transformations, we want the imaginary axis of the S-plane to map into the unit circle of the Z-plane, a stable analog filter should be transformed to a stable digital filter. That is, if the analog filter has poles only in the left-half of S-plane, then the digital filter must have poles only inside the unit circle. These constraints are basic to all the techniques discussed here.

5.2 Characteristics of Commonly Used Analog Filters:

From the previous discussion it is clear that, IIT digital filters can be obtained by beginning with an analog filter. Thus the design of a digital filter is reduced to designing an appropriate analog filter and then performing the conversion from $H_a(s)$ to $H(z)$. Analog filter design is a well - developed field, many approximation techniques, viz., Butterworth, Chebyshev, Elliptic, etc., have been developed for the design of analog low pass filters. Our discussion is limited to low pass filters, since, frequency transformation can be applied to transform a designed low pass filter into a desired high pass, band pass and band stop filters.

5.2.1 Butterworth Filters:

Low pass Butterworth filters are all - pole filters with monotonic frequency response in both pass band and stop band, characterized by the magnitude - squared frequency response

$$|H_a(\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}} = \frac{1}{1 + \epsilon^2(\Omega/\Omega_p)^{2N}}$$

Where, N is the order of the filter, Ω_c is the -3dB frequency, i.e., cutoff frequency, Ω_p is the pass band edge frequency and $1 = (1 / 1 + \epsilon^2)$ is the band edge value of $|H_a(\Omega)|^2$. Since the product $H_a(s) H_a(-s)$ and evaluated at $s = j\Omega$ is simply equal to $|H_a(\Omega)|^2$, it follows that

$$H_a(s) H_a(-s) = \frac{1}{1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N}$$

The poles of $H_a(s) H_a(-s)$ occur on a circle of radius Ω_c at equally spaced points. From Eq. (5.29), we find the pole positions as the solution of

$$\frac{-s^2}{\Omega_c^2} = (-1)^{1/N} = e^{j(2k+1)\pi/N}, \quad k = 0, 1, \dots, N-1$$

And hence, the N poles in the left half of the s-plane are

$$s_k = \Omega_c e^{j\pi/2} e^{(2k+1)\pi/2N}, \quad k = 0, 1, \dots, N-1$$

$$= \sigma_k + j\Omega_k$$

Note that, there are no poles on the imaginary axis of s-plane, and for N odd there will be a pole on real axis of s-plane, for N even there are no poles even on real axis of s-plane. Also note that all the poles are having conjugate symmetry. Thus the design methodology to design a Butterworth low pass filter with δ_2 attenuation at a specified frequency Ω_s is Find N,

$$N = \frac{\log[(1/\delta_2^2) - 1]}{2 \log(\Omega_s/\Omega_c)} = \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)}$$

Where by definition, $\delta_2 = 1/\sqrt{1+\delta^2}$. Thus the Butterworth filter is completely characterized by the parameters N, δ_2 , ϵ and the ratio Ω_s/Ω_p or Ω_c . Then, from Eq. (5.31) find the pole positions s_k ; $k = 0, 1, 2, \dots, (N-1)$. Finally the analog filter is given by

$$H_a(s) = \prod_{k=1}^N \frac{1}{(s - s_k)}$$

5.2.2 Chebyshev Filters:

There are two types of Chebyshev filters. Type I Chebyshev filters are all-pole filters that exhibit equiripple behavior in the pass band and a monotonic characteristic in the stop band. On the other hand, type II Chebyshev filters contain both poles and zeros and exhibit a monotonic behavior in the pass band and an equiripple behavior in the stop band. The zeros of this class of filters lie on the imaginary axis in the s-plane. The magnitude squared of the frequency response characteristic of type I Chebyshev filter is given as

$$| H_a(\Omega) |^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)}$$

Where ϵ is a parameter of the filter related to the ripple in the pass band as shown in Fig. (5.7), and T_N is the Nth order Chebyshev polynomial defined as

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & |x| \leq 1 \\ \cosh(N \cosh^{-1} x), & |x| > 1 \end{cases}$$

The Chebyshev polynomials can be generated by the recursive equation

$$T_{N+1}(x) == 2xT_N(x) - T_{N-1}(x), \quad N = 1, 2, \dots$$

Where $T_0(x) = 1$ and $T_1(x) = x$.

At the band edge frequency $\Omega = \Omega_p$, we have

$$\frac{1}{\sqrt{1 + \epsilon^2}} = 1 - \delta_1$$

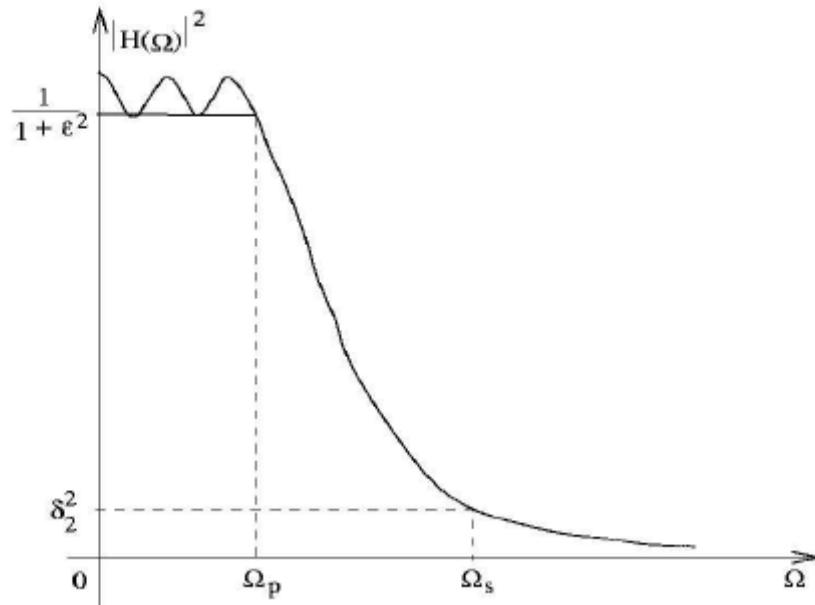


Figure 5.2: Type I Chebyshev filter characteristic

Or equivalently

$$\epsilon^2 = \frac{1}{(1 - \delta_1)^2} - 1$$

Where δ_1 is the value of the pass band ripple.

The poles of Type I Chebyshev filter lie on an ellipse in the s-plane with major axis

$$r_1 = \Omega_p \frac{\beta^2 + 1}{2\beta}$$

And minor axis

$$r_2 = \Omega_p \frac{\beta^2 - 1}{2\beta}$$

Where β is related to ϵ according to the equation

$$\beta = \left[\frac{\sqrt{1 + \epsilon^2} + 1}{\epsilon} \right]^{1/N}$$

The angular positions of the left half s-plane poles are given by

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k = 0, 1, \dots, N-1$$

Then the positions of the left half s-plane poles are given by

$$s_k = \sigma_k + j\Omega_k, \quad k = 0, 1, \dots, N-1$$

Where $\sigma_k = r_2 \cos \phi_k$ and $\Omega_k = r_1 \sin \phi_k$. The order of the filter is obtained from

$$\begin{aligned} N &= \frac{\log[(\sqrt{1 - \delta_2^2} + \sqrt{1 - \delta_2^2(1 + \epsilon^2)})/\epsilon\delta_2]}{\log[\frac{\Omega_s}{\Omega_p} + \sqrt{(\frac{\Omega_s}{\Omega_p})^2 - 1}]} \\ &= \frac{\cosh^{-1}(\frac{\delta}{\epsilon})}{\cosh^{-1}(\frac{\Omega_s}{\Omega_p})} \end{aligned}$$

Where, by definition $\delta_2 = 1/\sqrt{1+\delta^2}$.

Finally, the Type I Chebyshev filter is given by

$$H_a(s) = \prod_{k=1}^N \frac{1}{(s - s_k)}$$

A Type II Chebyshev filter contains zero as well as poles. The magnitude squared response is given as

$$|H_a(-\Omega)|^2 = \frac{1}{1 + \epsilon^2 [T_N^2(\frac{\Omega_s}{\Omega_p})/T_N^2(\frac{\Omega_s}{\Omega})]}$$

Where $T_N(x)$ is the N-order Chebyshev polynomial. The zeros are located on the imaginary axis at the points

$$z_k = j \frac{\Omega_s}{\sin \phi_k}, \quad k = 0, 1, \dots, N-1$$

and the left-half s-plane poles are given

$$s_k = \sigma_k + j\Omega_k, \quad k = 0, 1, \dots, N-1$$

Where

$$\sigma_k = \frac{\Omega_s r_2 \cos \phi_k}{\sqrt{r_2^2 \cos^2 \phi_k + r_1^2 \sin^2 \phi_k}}$$

and

$$\Omega_k = \frac{\Omega_s r_1 \sin \phi_k}{\sqrt{r_2^2 \cos^2 \phi_k + r_1^2 \sin^2 \phi_k}}$$

Finally, the Type II Chebyshev filter is given by

$$H_a(s) = \prod_{k=1}^N \frac{s - z_k}{s - s_k}$$

The other approximation techniques are elliptic (equiripple in both passband and stopband) and Bessel (monotonic in both passband and stopband).

5.3 Analog to Analog Frequency Transforms

Frequency transforms are used to transform lowpass prototype filter to other filters like highpass or bandpass or bandstop filters. One possibility is to perform frequency transform in the analog domain and then convert the analog filter into a corresponding digital filter by a mapping of the s-plane into z-plane. An alternative approach is to convert the analog lowpass filter into a lowpass digital filter and then to transform the lowpass digital filter into the desired digital filter by a digital transformation.

Suppose we have a lowpass filter with pass edge Ω_p and if we want convert that into another lowpass filter with pass band edge Ω'_p then the transformation used is

$$s \rightarrow \frac{\Omega_p}{\Omega'_p} s \quad (\text{lowpass to lowpass})$$

Thus we obtain a lowpass filter with system function $H_l(s) = H_p[(\Omega_p/\Omega'_p)s]$, where $H_p(s)$ is the system function of the prototype filter with passband edge frequency Ω_p .

To convert low pass filter into highpass filter the transformation used is

$$s \rightarrow \frac{\Omega_p \Omega'_p}{s} \quad (\text{lowpass to highpass})$$

The system function of the highpass filter is $H_h(s) = H_p(\Omega_p \Omega'_p / s)$.

The transformation for converting a lowpass analog filter with passband edge frequency Ω_p , into a band filter, having a lower band edge frequency Ω_l and an upper band edge frequency Ω_u , can be accomplished by first converting the lowpass

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \quad (\text{lowpass to bandpass})$$

Thus we obtain

$$H_b(s) = H_p \left(\Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \right)$$

Finally, if we wish to convert a lowpass analog filter with band edge frequency Ω_p , into a bandstop filter, the transformation is simply the inverse of (8.4.3) with the additional factor Ω_p serving to normalize for the band edge frequency of the lowpass filter. Thus the transformation is

$$s \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} \quad (\text{lowpass to bandstop})$$

The filter function is

$$H_{bs}(s) = H_p \left(\Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} \right)$$

Recommended Questions with answers

Question 1

I Design a digital filter to satisfy the following characteristics.

- ❖ -3dB cutoff frequency of 0.5 rad.
- ❖ Magnitude down at least 15dB at 0.75 rad.
- ❖ Monotonic stop band and pass band Using
- ❖ Impulse invariant technique
- ❖ Approximation of derivatives
- ❖ Bilinear transformation technique

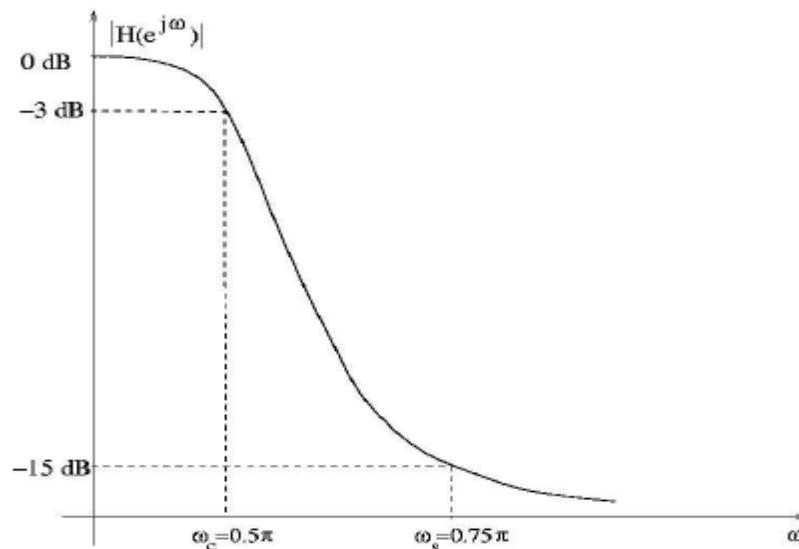


Figure 5.8: Frequency response plot of the example

Solution:-

a) Impulse Invariant Technique

From the given digital domain frequency, find the corresponding analog domain frequencies.

$$\Omega_c = \frac{\omega_c}{T} \text{ and } \Omega_s = \frac{\omega_s}{T}$$

Where T is the sampling period and 1/T is the sampling frequency and it always corresponds to 2π radians in the digital domain. In this problem, let us assume T = 1sec.

Then $\Omega_c = 0.5\pi$ and $\Omega_s = 0.75\pi$

Let us find the order of the desired filter using

$$N = \frac{\left(\frac{1}{\delta_2^2} - 1\right)}{2 \log\left(\frac{\Omega_s}{\Omega_c}\right)}$$

Where δ_2 is the gain at the stop band edge frequency ω_s .

$$-15 \text{ dB} = 20 \log \delta_2$$

$$\delta_2 = 10^{-\frac{15}{20}} = 0.1778$$

$$\delta_2 = 10^{-\frac{\pi}{20}} = 0.1778$$

$$N = \frac{\log\left(\frac{1}{(0.1778)^2} - 1\right)}{2 \log\left(\frac{0.75\pi}{0.5\pi}\right)} = 4.219 \simeq 5$$

Order of filter N = 5.

Then the 5 poles on the Butterworth circle of radius $\Omega_c = 0.5 \pi$ are given by

$$s_0 = 0.5\pi e^{j(\frac{\pi}{2} + \frac{\pi}{10})} = -0.485 + j1.493$$

$$s_1 = 0.5\pi e^{j(\frac{\pi}{2} + \frac{3\pi}{10})} = -1.27 + j0.923$$

$$s_2 = 0.5\pi e^{j(\frac{\pi}{2} + \frac{5\pi}{10})} = -1.57 + j0.0$$

$$s_3 = 0.5\pi e^{j(\frac{\pi}{2} + \frac{7\pi}{10})} = -1.27 - j0.923$$

$$s_5 = 0.5\pi e^{j(\frac{\pi}{2} + \frac{9\pi}{10})} = -0.485 - j1.493$$

Then the filter transfer function in the analog domain is

$$\begin{aligned} H_a(s) &= \frac{1}{(s + 0.485 - j1.493)(s + 1.27 - j0.923)(s + 1.57)(s + 1.27 + j0.923)(s + 0.485 + j0.923)} \\ &= \sum_{k=1}^5 \frac{A_k}{(s - s_k)} \end{aligned}$$

where A_k 's are partial fractions coefficients of $H_a(s)$.

Finally, the transfer function of the digital filter is

$$H(z) = \sum_{k=1}^5 \frac{A_k}{(1 - e^{s_k} z^{-1})}, \text{ where } s_k \text{'s are the poles on the Butterworth circle}$$

b)

$$H(z) = H_a(s) \Big|_{s=\frac{1-z^{-1}}{T} = 1-z^{-1}}$$

$$H(z) = \sum_{k=1}^5 \frac{1}{(1 - z^{-1} - s_k)}$$

c) For the bilinear transformation technique, we need to pre-warp the digital frequencies into corresponding analog frequencies.

i.e., $\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$

$$\Omega_c = 2 \tan\left(\frac{0.5\pi}{2}\right) = 2 \text{ rad.}$$

and

$$\Omega_s = 2 \tan\left(\frac{0.75\pi}{2}\right) = 4.828 \text{ rad.}$$

Then the order of the filter

$$N = \frac{\log\left(\frac{1}{(0.1778)^2} - 1\right)}{2 \log\left(\frac{4.828}{2}\right)}$$

The pole locations on the Butterworth circle with radius $\Omega_c = 2$ are

$$s_0 = 2e^{j(\frac{\pi}{2} + \frac{\pi}{4})} = -1.414 + j1.414$$

$$s_1 = 2e^{j(\frac{\pi}{2} + \frac{3\pi}{4})} = -1.414 - j1.414$$

Then the filter transfer function in the analog domain is

$$H_a(s) = \frac{1}{(s + 1.414 - j1.414)(s + 1.414 + j1.414)}$$

Finally, the transfer function of the digital filter is

$$H(z) = H_a(s) \Big|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}=2 \frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{1}{(2 \frac{1-z^{-1}}{1+z^{-1}} + 1.414 - j1.414)(2 \frac{1-z^{-1}}{1+z^{-1}} + 1.414 + j1.414)}$$

Question 2

Design a digital filter using impulse invariant technique to satisfy following characteristics

- (i) Equiripple in pass band and monotonic in stop band
- (ii) -3dB ripple with pass band edge frequency at 0.5π radians.
- (iii) Magnitude down at least 15dB at 0.75π radians.

Solution: Assuming $T=1$, $\Omega=0.5\pi$ and $s=0.75\pi$

The order of desired filter is

$$N = \frac{\log[(\sqrt{1-\delta_2^2}) + \sqrt{1-\delta_2^2(1-\epsilon^2)})/\epsilon\delta_2]}{\log[\frac{\Omega_s}{\Omega_p} + \sqrt{(\frac{\Omega_s}{\Omega_p})^2 - 1}]}$$

when

$$20 \log \frac{1}{\sqrt{1+\epsilon^2}} = -3 \text{ dB}$$

i.e.,

$$10 \log(1+\epsilon^2) = 3 \text{ dB}$$

$$\epsilon^2 = 10^{0.3} - 1 = 0.9952$$

$$\epsilon = 0.9976$$

and

$$20 \log \delta_2 = -15 \text{ dB}$$

$$\delta_2 = 10^{-0.75} = 0.1778$$

Hence

$$\begin{aligned} N &= \frac{[(\sqrt{1 - (0.1778)^2} + \sqrt{1 - (0.1778)^2(1 + 0.9952)})/0.9976 \times 0.1778]}{\log[\frac{0.75\pi}{0.5\pi} + \sqrt{(\frac{0.75\pi}{0.5\pi})^2 - 1}]} \\ &= 2.48 \\ &\simeq 3 \end{aligned}$$

The order of filter, $N = 3$.

The 3 poles on the ellipse are determined by

$$\beta = \left[\frac{\sqrt{1 + \epsilon^2} + 1}{\epsilon} \right]^{\frac{1}{N}} = \left[\frac{\sqrt{1 + 0.9976^2} + 1}{0.9976} \right]^{\frac{1}{3}} = 1.342$$

$$\begin{aligned} r_1 &= \Omega_p \frac{\beta^2 + 1}{2\beta} \\ &= 0.5\pi \times \frac{(1.341)^2 + 1}{2 \times 1.341} \\ &= 1.639 \end{aligned}$$

$$\begin{aligned} r_2 &= \Omega_p \frac{\beta^2 - 1}{2\beta} \\ &= 0.5\pi \times \frac{(1.341)^2 - 1}{2 \times 1.341} \\ &= 0.469 \end{aligned}$$

The angles,

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k = 0, 1, 2$$

The poles are at

$$s_k = r_2 \cos \phi_k + j r_1 \sin \phi_k$$

$$\begin{aligned}s_0 &= 0.469 \cos\left(\frac{4\pi}{6}\right) + j1.639 \sin\left(\frac{4\pi}{6}\right) \\ &= -0.2345 + j1.419\end{aligned}$$

$$\begin{aligned}s_1 &= 0.469 \cos(\pi) + j1.639 \sin(\pi) \\ &= -0.469 + j0.0\end{aligned}$$

$$\begin{aligned}s_2 &= 0.469 \cos\left(\frac{8\pi}{6}\right) + j1.6939 \sin\left(\frac{8\pi}{6}\right) \\ &= -0.2345 - j1.419\end{aligned}$$

The analog filter transfer function is given by

$$\begin{aligned}H_a(s) &= \frac{1}{s + 0.2345 - j1.419)(s + 0.469) + (s + 0.2345 + j1.419)} \\ &= \sum_{k=1}^3 \frac{A_k}{(s - s_k)}\end{aligned}$$

where A_k 's are the partial fraction coefficients.

Finally, the digital filter transfer function is given by

$$H(z) = \sum_{k=1}^3 \frac{A_k}{(1 - e^{s_k} z^{-1})}$$

Question 3

An IIR digital low-pass filter is required to meet the following specifications:

Passband ripple (or peak-to-peak ripple): ≤ 0.5 dB

Passband edge: 1.2 kHz

Stopband attenuation: ≥ 40 dB

Stopband edge: 2.0 kHz

Sample rate: 8.0 kHz

Use the design formulas in the book to determine the required filter order for

- (a) A digital Butterworth filter
- (b) A digital Chebyshev filter
- (c) A digital elliptic filter

Solution:-

For the design specifications we have

$$\epsilon = 0.349$$

$$\delta = 99.995$$

$$f_p = \frac{1.2}{8} = 0.15$$

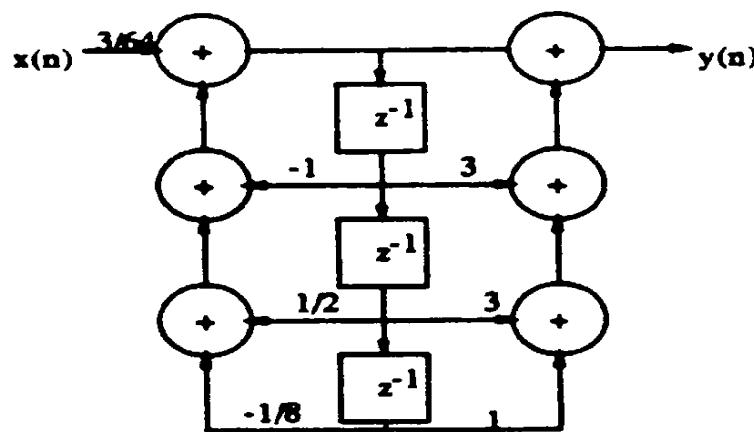
$$f_s = \frac{2}{8} = 0.25$$

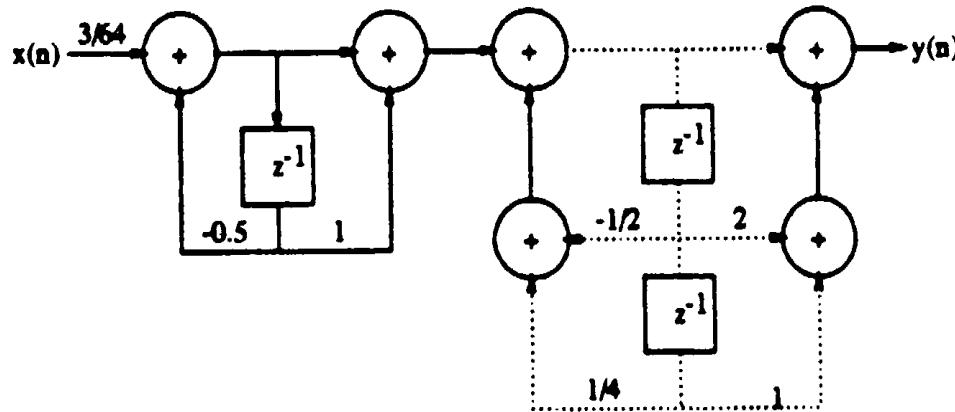
$$\Omega_p = 2 \tan \frac{\omega_p}{2} = 1.019$$

$$\Omega_s = 2 \tan \frac{\omega_s}{2} = 2$$

$$\eta = \frac{\delta}{\epsilon} = 286.5$$

$$k = \frac{\Omega_s}{\Omega_p} = 1.963$$





$$\text{Butterworth filter: } N_{\min} \geq \frac{\log \eta}{\log k} = 8.393 \Rightarrow N = 9$$

$$\text{Chebyshev filter: } N_{\min} \geq \frac{\cosh^{-1} \eta}{\cosh^{-1} k} = 4.90 \Rightarrow N = 5$$

$$\text{Elliptic filter: } N_{\min} \geq \frac{k(\frac{1}{k})}{k(\sqrt{1 - \frac{1}{k^2}})} \cdot \frac{k(\sqrt{1 - \frac{1}{\eta^2}})}{k(\frac{1}{\eta})} \Rightarrow N = 4$$

Question 4

Determine the system function $H(z)$ of the lowest-order Chebyshev digital filter that meets the following specifications:

- (a) $\frac{1}{2}$ -dB ripple in the passband $0 \leq |\omega| \leq 0.24\pi$.
- (b) At least 50-dB attenuation in the stopband $0.35\pi \leq |\omega| \leq \pi$. Use the bilinear transformation.

Solution:-

$$\text{Passband ripple} = 0.5 \text{dB} \Rightarrow \epsilon = 0.349$$

$$\text{Stopband attenuation} = 50 \text{dB}$$

$$w_p = 0.24\pi$$

$$w_s = 0.35\pi$$

$$\Omega_p = 2\tan \frac{w_p}{2} = 0.792$$

$$\Omega_s = 2\tan \frac{w_s}{2} = 1.226$$

$$\eta = \frac{\delta}{\epsilon} = 906.1$$

$$k = \frac{\Omega_s}{\Omega_p} = 1.547$$

$$N_{\min} \geq \frac{\cosh^{-1} \eta}{\cosh^{-1} k} = \frac{7.502}{1.003} = 7.48 \Rightarrow N = 8$$

UNIT 6**Implementation of Discrete time systems****CONTENTS:-**

**IMPLEMENTATION OF DISCRETE-TIME SYSTEMS: STRUCTURES FOR IIR AND FIR SYSTEMS
DIRECT FORM I AND DIRECT FORM II SYSTEMS, CASCADE, LATTICE AND PARALLEL
REALIZATION.** 7 HRS

RECOMMENDED READINGS:-

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

UNIT 6

Implementation of Discrete-Time Systems

6.1 Introduction

The two important forms of expressing system leading to different realizations of FIR & IIR filters are

- a) Difference equation form

$$y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{k=1}^M b_k x(n-k)$$

- b) Ration of polynomials

$$H(Z) = \frac{\sum_{k=0}^M b_k Z^{-k}}{1 + \sum_{k=1}^N a_k Z^{-k}}$$

The following factors influence choice of a specific realization,

- Computational complexity
- Memory requirements
- Finite-word-length
- Pipeline / parallel processing

6.1.1 Computation Complexity

This is do with number of arithmetic operations i.e. multiplication, addition & divisions. If the realization can have less of these then it will be less complex computationally.

In the recent processors the fetch time from memory & number of times a comparison between two numbers is performed per output sample is also considered and found to be important from the point of view of computational complexity.

6.1.2 Memory requirements

This is basically number of memory locations required to store the system parameters, past inputs, past outputs, and any intermediate computed values. Any realization requiring less of these is preferred.

6.1.3 Finite-word-length effects

These effects refer to the quantization effects that are inherent in any digital implementation of the system, either in hardware or in software. No computing system has infinite precision. With finite precision there is bound to be errors. These effects are basically to do with truncation & rounding-off of samples. The extent of this effect varies with type of arithmetic used(fixed or floating). The serious issue is that the effects have influence on system characteristics. A structure which is less sensitive to this effect need to be chosen.

6.1.4 Pipeline / Parallel Processing

This is to do with suitability of the structure for pipelining & parallel processing. The parallel processing can be in software or hardware. Longer pipelining make the system more efficient.

6.2 Structure for FIR Systems:

FIR system is described by,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

Or equivalently, the system function

$$H(Z) = \sum_{k=0}^{M-1} b_k Z^{-k}$$

Where we can identify $h(n) = \begin{cases} b_n & 0 \leq n \leq M-1 \\ 0 & otherwise \end{cases}$

Different FIR Structures used in practice are,

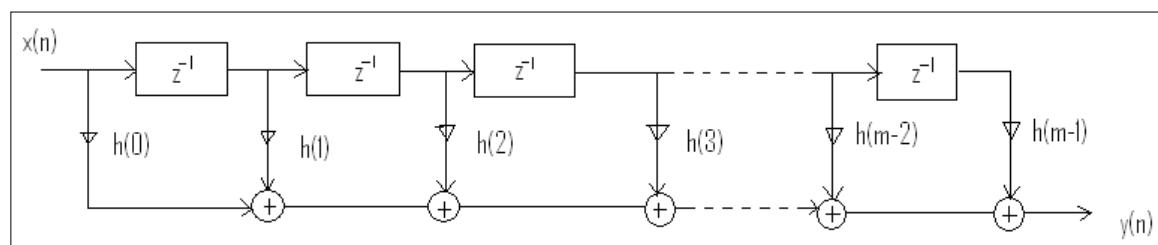
1. Direct form
2. Cascade form
3. Frequency-sampling realization
4. Lattice realization

6.2.1 Direct – Form Structure

Convolution formula is used to express FIR system given by,

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

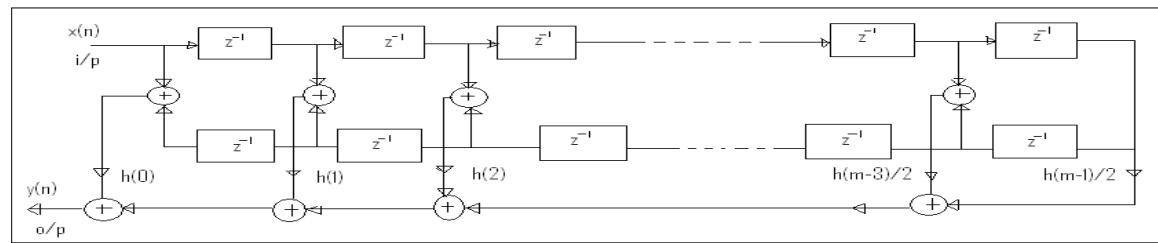
- It is Non recursive in structure



- As can be seen from the above implementation it requires M-1 memory locations for storing the M-1 previous inputs
- It requires computationally M multiplications and M-1 additions per output point
- It is more popularly referred to as tapped delay line or transversal system
- Efficient structure with linear phase characteristics are possible where $h(n) = \pm h(M-1-n)$

Digital Signal Processing

10EC52



Prob:

Realize the following system function using minimum number of multiplication

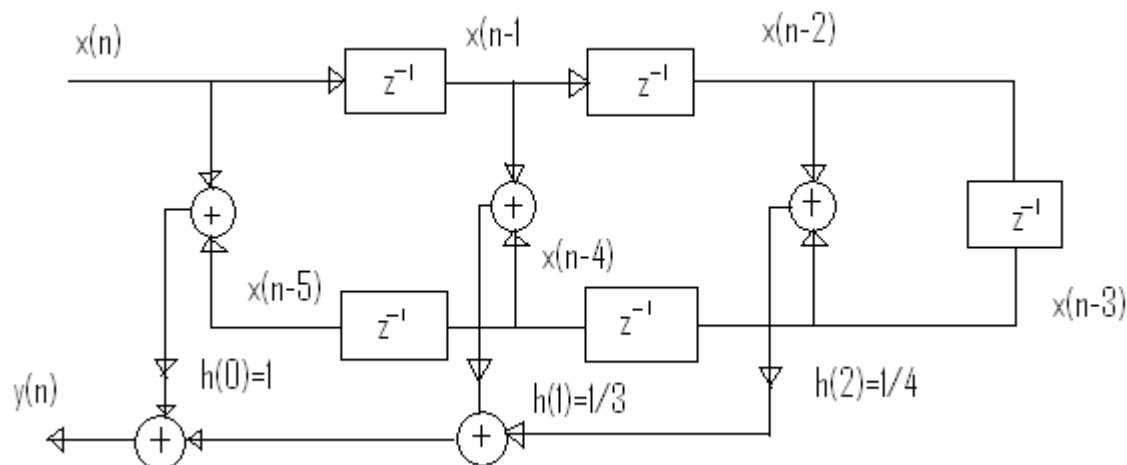
$$(1) H(Z) = 1 + \frac{1}{3}Z^{-1} + \frac{1}{4}Z^{-2} + \frac{1}{4}Z^{-3} + \frac{1}{3}Z^{-4} + Z^{-5}$$

We recognize $h(n) = \left[1, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, 1 \right]$

M is even = 6, and we observe $h(n) = h(M-1-n)$ $h(n) = h(5-n)$

i.e $h(0) = h(5)$ $h(1) = h(4)$ $h(2) = h(3)$

Direct form structure for Linear phase FIR can be realized



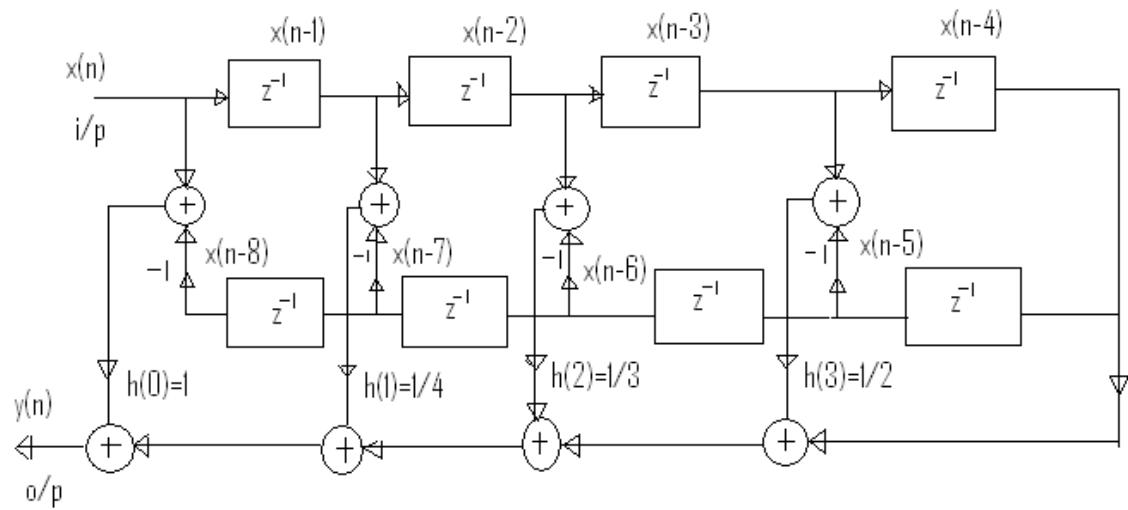
Exercise: Realize the following using system function using minimum number of multiplication.

$$H(Z) = 1 + \frac{1}{4}Z^{-1} + \frac{1}{3}Z^{-2} + \frac{1}{2}Z^{-3} - \frac{1}{2}Z^{-5} - \frac{1}{3}Z^{-6} - \frac{1}{4}Z^{-7} - Z^{-8}$$

$m=9$ $h(n) = \left[1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -1 \right]$

odd symmetry

$$\begin{aligned} h(n) &= -h(M-1-n); & h(n) &= -h(8-n); & h(m-1/2) &= h(4) = 0 \\ h(0) &= -h(8); & h(1) &= -h(7); & h(2) &= -h(6); & h(3) &= -h(5) \end{aligned}$$



6.2.2 Cascade – Form Structure

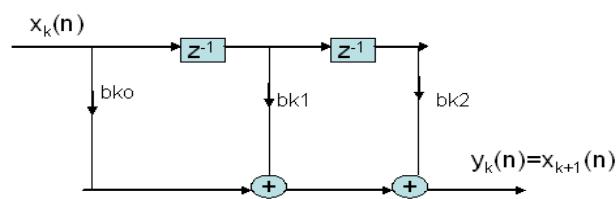
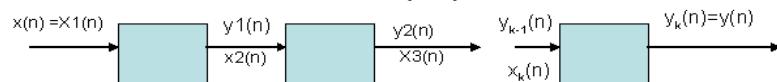
The system function $H(Z)$ is factored into product of second – order FIR system

$$H(Z) = \prod_{k=1}^K H_k(Z)$$

Where $H_k(Z) = b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}$ $k = 1, 2, \dots, K$

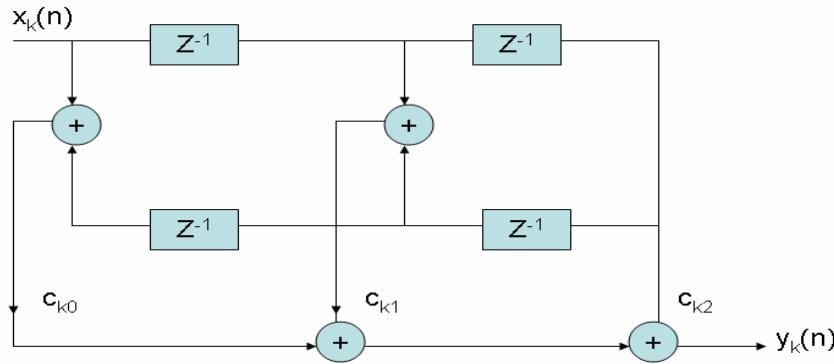
and $K = \text{integer part of } (M+1) / 2$

The filter parameter b_0 may be equally distributed among the K filter section, such that $b_0 = b_{10} b_{20} \dots b_{K0}$ or it may be assigned to a single filter section. The zeros of $H(z)$ are grouped in pairs to produce the second – order FIR system. Pairs of complex-conjugate roots are formed so that the coefficients $\{b_{ki}\}$ are real valued.



In case of linear -phase FIR filter, the symmetry in $h(n)$ implies that the zeros of $H(z)$ also exhibit a form of symmetry. If z_k and z_k^* are pair of complex – conjugate zeros then $1/z_k$ and $1/z_k^*$ are also a pair complex –conjugate zeros. Thus simplified fourth order sections are formed. This is shown below,

$$\begin{aligned} H_k(z) &= C_{k0}(1 - z_k z^{-1})(1 - z_k^* z^{-1})(1 - z^{-1}/z_k)(1 - z^{-1}/z_{k*}) \\ &= C_{k0} + C_{k1}z^{-1} + C_{k2}z^{-2} + C_{k3}z^{-3} + z^{-4} \end{aligned}$$

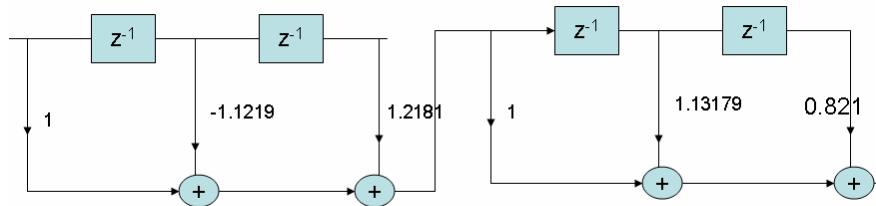


Problem: Realize the difference equation

$$y(n) = x(n) + 0.25x(n-1) + 0.5x(n-2) + 0.75x(n-3) + x(n-4)$$

in cascade form.

$$\begin{aligned} Y(z) &= X(z)\{1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4}\} \\ \text{Soln: } H(z) &= 1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4} \\ H(z) &= (1 - 1.1219z^{-1} + 1.2181z^{-2})(1 + 1.3719z^{-1} + 0.821z^{-2}) \\ H(z) &= H_1(z)H_2(z) \end{aligned}$$



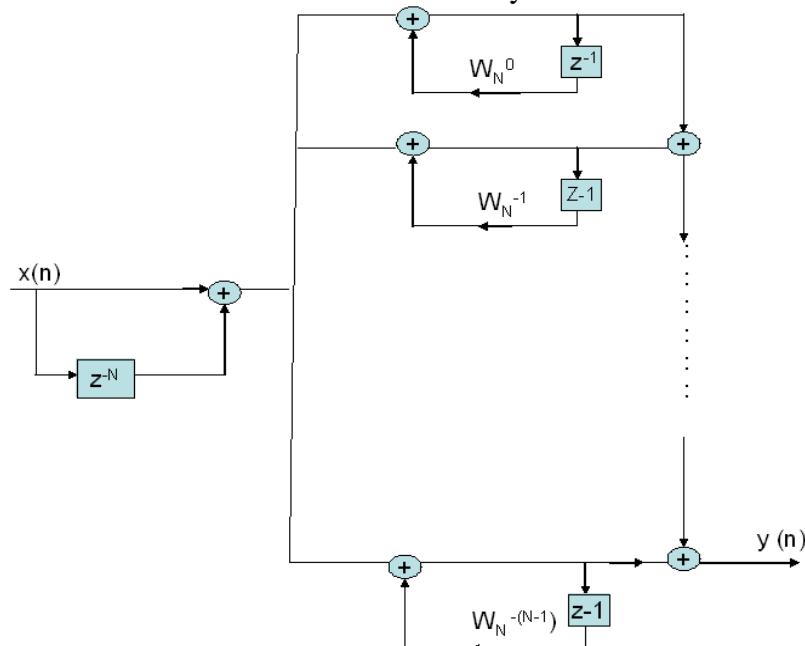
6.3 Frequency sampling realization:

We can express system function $H(z)$ in terms of DFT samples $H(k)$ which is given by

$$H(z) = (1 - z^{-N}) \frac{1}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - W_N^{-k} z^{-1}}$$

This form can be realized with cascade of FIR and IIR structures. The term $(1-z^{-N})$ is realized as FIR and the term $\frac{1}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-W_N^{-k} z^{-1}}$ as IIR structure.

The realization of the above freq sampling form shows necessity of complex arithmetic. Incorporating symmetry in $h(n)$ and symmetry properties of DFT of real sequences the realization can be modified to have only real coefficients.



6.4 Lattice structures

Lattice structures offer many interesting features:

1. Upgrading filter orders is simple. Only additional stages need to be added instead of redesigning the whole filter and recalculating the filter coefficients.
2. These filters are computationally very efficient than other filter structures in a filter bank applications (eg. Wavelet Transform)
3. Lattice filters are less sensitive to finite word length effects.

Consider

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \sum_{i=1}^m a_m(i)z^{-i}$$

m is the order of the FIR filter and $a_m(0)=1$

when $m = 1 \quad Y(z)/X(z) = 1 + a_1(1) z^{-1}$

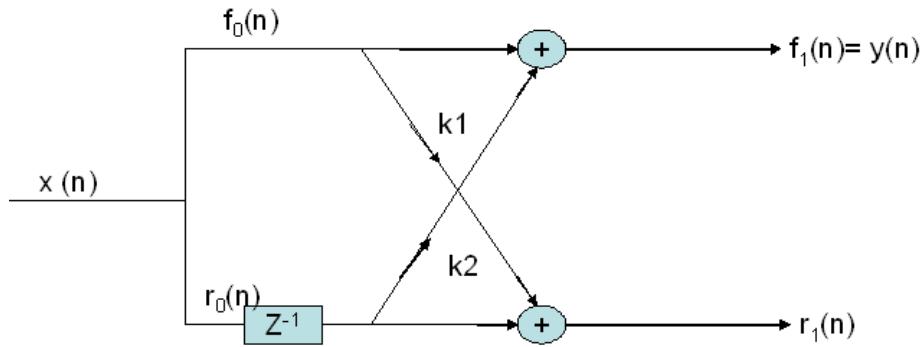
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$$y(n) = x(n) + a_1(1)x(n-1)$$

$f_1(n)$ is known as upper channel output and $r_1(n)$ as lower channel output.

$$f_0(n) = r_0(n) = x(n)$$



The outputs are

$$f_1(n) = f_0(n) + k_1 r_0(n-1) \quad 1a$$

$$r_1(n) = k_1 f_0(n) + r_0(n-1) \quad 1b$$

$$\text{if } k_1 = a_1(1), \text{then } f_1(n) = y(n)$$

If m=2

$$\frac{Y(z)}{X(z)} = 1 + a_2(1)z^{-1} + a_2(2)z^{-2}$$

$$y(n) = x(n) + a_2(1)x(n-1) + a_2(2)x(n-2)$$

$$y(n) = f_1(n) + k_2 r_1(n-1) \quad (2)$$

Substituting 1a and 1b in (2)

$$\begin{aligned} y(n) &= f_0(n) + k_1 r_0(n-1) + k_2 [k_1 f_0(n-1) + r_0(n-2)] \\ &= f_0(n) + k_1 r_0(n-1) + k_2 k_1 f_0(n-1) + k_2 r_0(n-2) \end{aligned}$$

$$\sin ce \quad f_0(n) = r_0(n) = x(n)$$

$$\begin{aligned} y(n) &= x(n) + k_1 x(n-1) + k_2 k_1 x(n-1) + k_2 x(n-2) \\ &= x(n) + (k_1 + k_1 k_2) x(n-1) + k_2 x(n-2) \end{aligned}$$

We recognize

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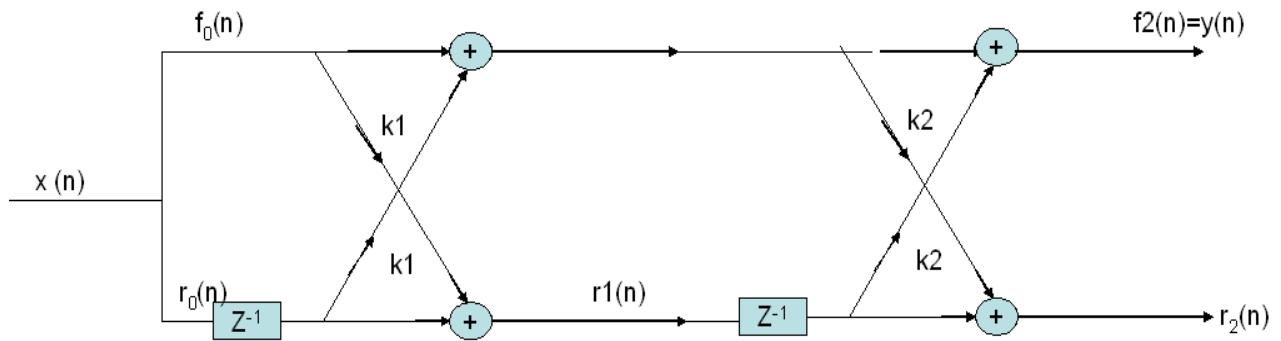
$$a_2(1) = k_1 + k_1 k_2$$

$$a_2(1) = k_2$$

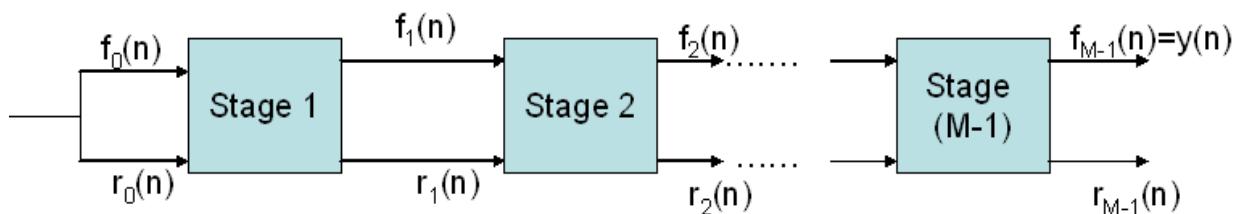
Solving the above equation we get

$$k_1 = \frac{a_2(1)}{1+a_2(2)} \quad \text{and} \quad k_2 = a_2(2) \quad (4)$$

Equation (3) means that, the lattice structure for a second-order filter is simply a cascade of two first-order filters with k_1 and k_2 as defined in eq (4)



Similar to above, an M th order FIR filter can be implemented by lattice structures with $M - 1$ stages



8.4.1 Direct Form –I to lattice structure

For $m = M, M-1, \dots, 2, 1$ do

$$k_m = a_m(m)$$

$$a_{m-1}(i) = \frac{a_m(i) - a_m(m)a_m(m-i)}{1 - k_m^2} \quad 1 \leq i \leq m-1$$

- The above expression fails if $k_m=1$. This is an indication that there is a zero on the unit circle. If $k_m=1$, factor out this root from $A(z)$ and the recursive formula can be applied for reduced order system.

for $m = 2$ and $m = 1$

$$k_2 = a_2(2) \quad \& \quad k_1 = a_1(1)$$

for $m = 2$ & $i = 1$

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - k_2^2} = \frac{a_2(1)[1 - a_2(2)]}{1 - a_2^2(2)} = \frac{a_2(1)}{1 + a_2(2)}$$

$$\text{Thus } k_1 = \frac{a_2(1)}{1 + a_2(2)}$$

8.4.2 Lattice to direct form -I

For $m = 1, 2, \dots, M-1$

$$a_m(0) = 1$$

$$a_m(m) = k_m$$

$$a_m(i) = a_{m-1}(i) + a_m(m)a_{m-1}(m-i) \quad 1 \leq i \leq m-1$$

Problem:

Given FIR filter $H(Z) = 1 + 2Z^{-1} + \frac{1}{3}Z^{-2}$ obtain lattice structure for the same

Given $a_1(1) = 2$, $a_2(2) = \frac{1}{3}$

Using the recursive equation for

$m = M, M-1, \dots, 2, 1$

here $M=2$ therefore $m = 2, 1$

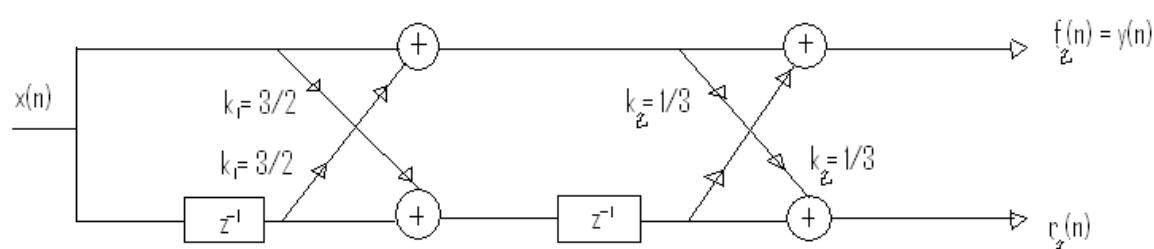
if $m=2$ $k_2 = a_2(2) = \frac{1}{3}$

if $m=1$ $k_1 = a_1(1)$

also, when $m=2$ and $i=1$

$$a_1(1) = \frac{a_2(1)}{1 + a_2(2)} = \frac{2}{1 + \frac{1}{3}} = \frac{3}{2}$$

Hence $k_1 = a_1(1) = \frac{3}{2}$



Recommended questions with solution

Problem:1

Consider an FIR lattice filter with co-efficients $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{4}$. Determine the FIR filter co-efficient for the direct form structure
 $(H(Z) = a_3(0) + a_3(1)Z^{-1} + a_3(2)Z^{-2} + a_3(3)Z^{-3})$

$$a_3(0) = 1 \quad a_3(3) = k_3 = \frac{1}{4} \quad a_2(2) = k_2 = \frac{1}{3} \\ a_1(1) = k_1 = \frac{1}{2}$$

for m=2, i=1

$$a_2(1) = a_1(1) + a_2(2)a_1(1) \\ = a_1(1)[1 + a_2(2)] = \frac{1}{2} \left[1 + \frac{1}{3} \right] \\ = \frac{4}{6} = \frac{2}{3}$$

for m=3, i=1

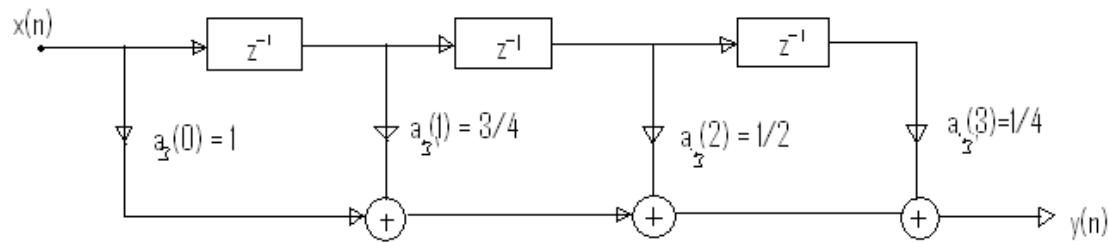
$$a_3(1) = a_2(1) + a_3(3)a_2(2) \\ = \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} \\ = \frac{2}{3} + \frac{1}{12} = \frac{8+1}{12} \\ = \frac{9}{12} = \frac{3}{4}$$

for m=3 & i=2

$$a_3(2) = a_2(2) + a_3(3)a_2(1) \\ = \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} \\ = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$a_3(0) = 1, \quad a_3(1) = \frac{3}{4}, \quad a_3(2) = \frac{1}{2}, \quad a_3(3) = \frac{1}{4}$$



6.5 Structures for IIR Filters

The IIR filters are represented by system function;

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

and corresponding difference equation given by,

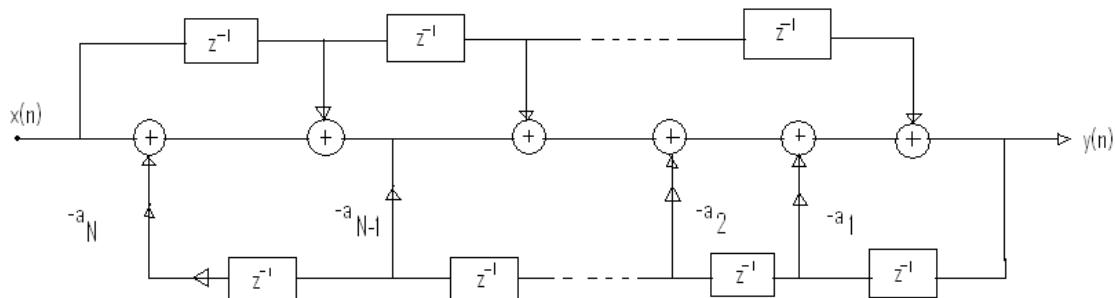
$$y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$$

Different realizations for IIR filters are,

1. Direct form-I
2. Direct form-II
3. Cascade form
4. Parallel form
5. Lattice form

6.5.1 Direct form-I

This is a straight forward implementation of difference equation which is very simple. Typical Direct form – I realization is shown below . The upper branch is forward path and lower branch is feedback path. The number of delays depends on presence of most previous input and output samples in the difference equation.



6.5.2 Direct form-II

The given transfer function $H(z)$ can be expressed as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} \cdot \frac{Y(z)}{V(z)}$$

where $V(z)$ is an intermediate term. We identify,

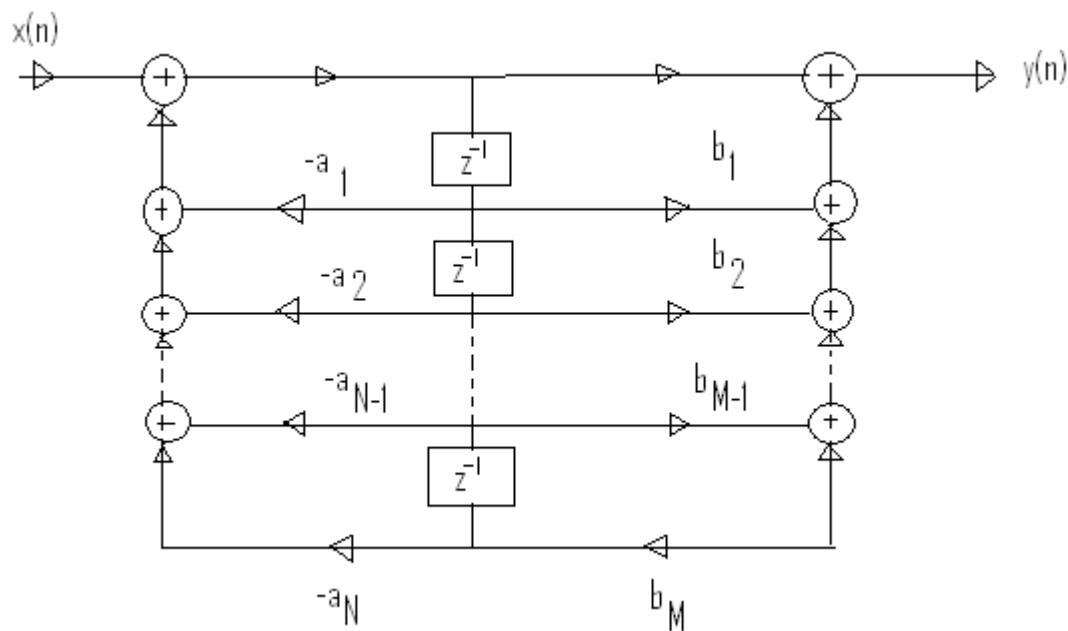
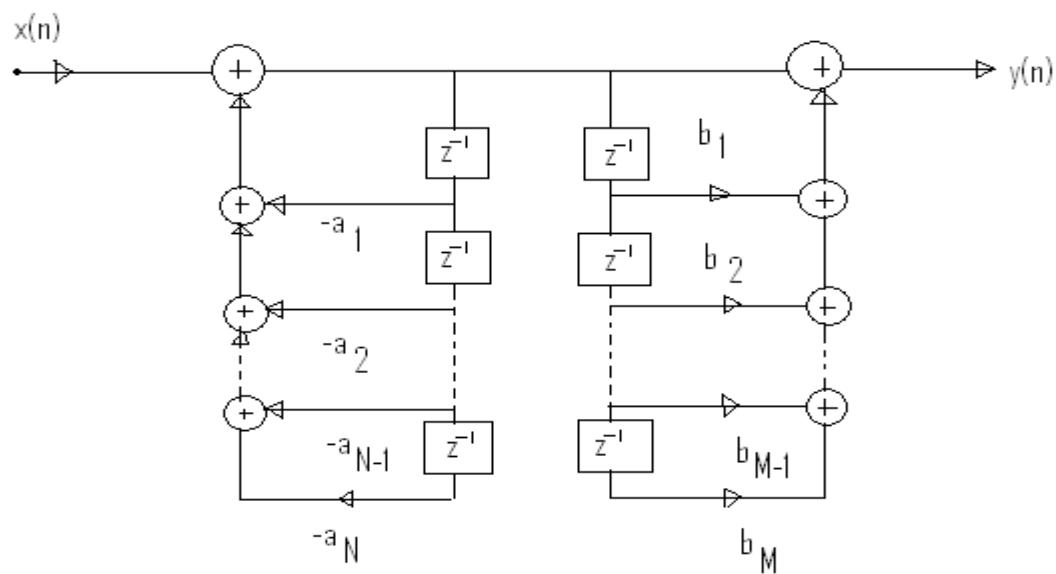
$$\frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}} \quad \text{----- all poles}$$

$$\frac{Y(z)}{V(z)} = \left(1 + \sum_{k=1}^M b_k z^{-k} \right) \quad \text{----- all zeros}$$

The corresponding difference equations are,

$$v(n) = x(n) - \sum_{k=1}^N a_k v(n-k)$$

$$y(n) = v(n) + \sum_{k=1}^M b_k v(n-1)$$



This realization requires $M+N+1$ multiplications, $M+N$ addition and the maximum of $\{M, N\}$ memory location

6.5.3 Cascade Form

The transfer function of a system can be expressed as,

$$H(z) = H_1(z)H_2(z)\dots H_k(z)$$

Where $H_k(Z)$ could be first order or second order section realized in Direct form – II form i.e.,

$$H_k(Z) = \frac{b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$$

where K is the integer part of $(N+1)/2$

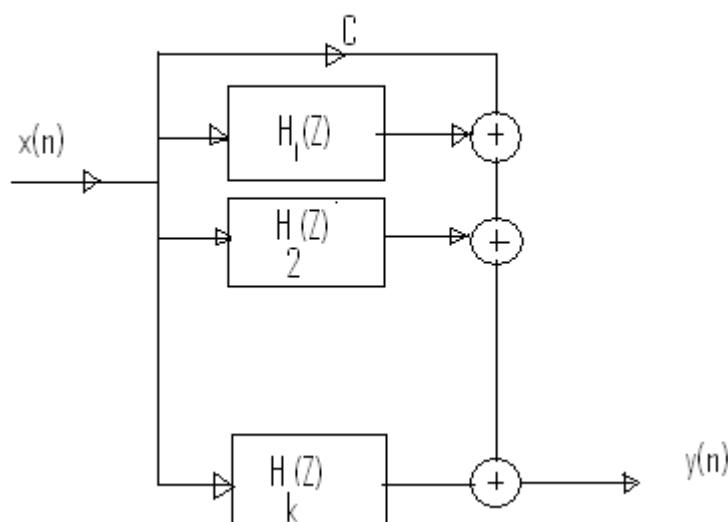
Similar to FIR cascade realization, the parameter b_0 can be distributed equally among the k filter section B_0 that $b_0 = b_{10}b_{20}\dots b_{K0}$. The second order sections are required to realize section which has complex-conjugate poles with real co-efficients. Pairing the two complex-conjugate poles with a pair of complex-conjugate zeros or real-valued zeros to form a subsystem of the type shown above is done arbitrarily. There is no specific rule used in the combination. Although all cascade realizations are equivalent for infinite precision arithmetic, the various realizations may differ significantly when implemented with finite precision arithmetic.

6.5.4 Parallel form structure

In the expression of transfer function, if $N \geq M$ we can express system function

$$H(Z) = C + \sum_{k=1}^N \frac{A_k}{1 - p_k Z^{-1}} = C + \sum_{k=1}^N H_k(Z)$$

Where $\{p_k\}$ are the poles, $\{A_k\}$ are the coefficients in the partial fraction expansion, and the constant C is defined as $C = b_N/a_N$, The system realization of above form is shown below.



$$\text{Where } H_k(Z) = \frac{b_{k0} + b_{k1}Z^{-1}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$$

Once again choice of using first-order or second-order sections depends on poles of the denominator polynomial. If there are complex set of poles which are conjugative in nature then a second order section is a must to have real coefficients.

Problem 2

Determine the

- (i) Direct form-I (ii) Direct form-II (iii) Cascade &
- (iv) Parallel form realization of the system function

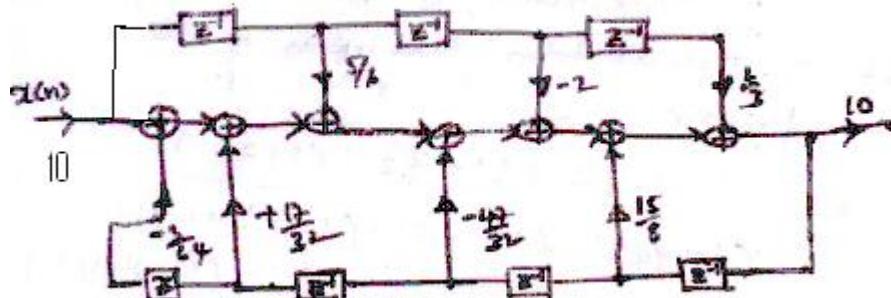
$$H(Z) = \frac{10\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 - \frac{2}{3}Z^{-1}\right)\left(1 + 2Z^{-1}\right)}{\left(1 - \frac{3}{4}Z^{-1}\right)\left(1 - \frac{1}{8}Z^{-1}\right)\left(1 - \left(\frac{1}{2} + j\frac{1}{2}\right)Z^{-1}\right)\left(1 - \left(\frac{1}{2} - j\frac{1}{2}\right)Z^{-1}\right)}$$

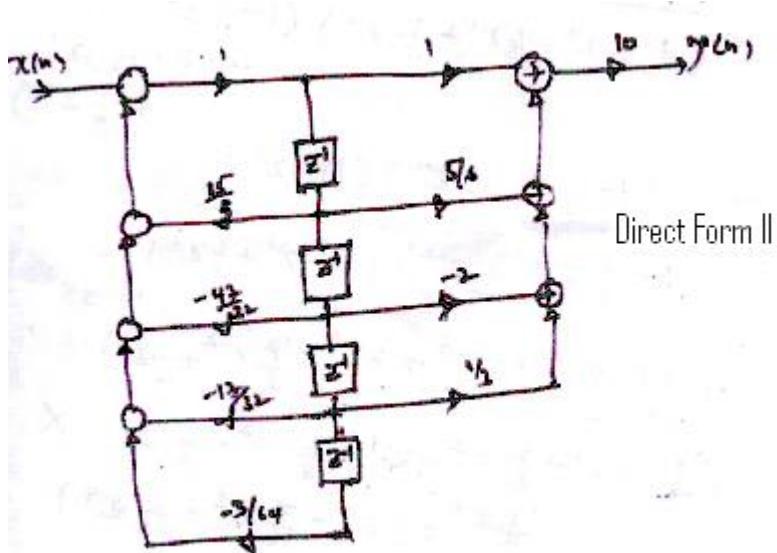
$$= \frac{10\left(1 - \frac{7}{6}Z^{-1} + \frac{1}{3}Z^{-2}\right)\left(1 + 2Z^{-1}\right)}{\left(1 + \frac{7}{8}Z^{-1} + \frac{3}{32}Z^{-2}\right)\left(1 - Z^{-1} + \frac{1}{2}Z^{-2}\right)}$$

$$H(Z) = \frac{10\left(1 + \frac{5}{6}Z^{-1} - 2Z^{-2} + \frac{2}{3}Z^{-3}\right)}{\left(1 - \frac{15}{8}Z^{-1} + \frac{47}{32}Z^{-2} - \frac{17}{32}Z^{-3} + \frac{3}{64}Z^{-4}\right)}$$

$$H(z) = \frac{(-14.75 - 12.90z^{-1})}{\left(1 + \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}\right)} + \frac{(24.50 + 26.82z^{-1})}{\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Direct Form I





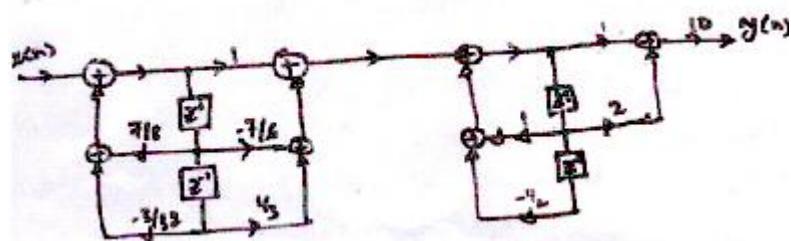
Cascade Form

$$H(z) = H_1(z) H_2(z)$$

Where

$$H_1(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}}$$

$$H_1(z) = \frac{10(1 + 2z^{-1})}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$



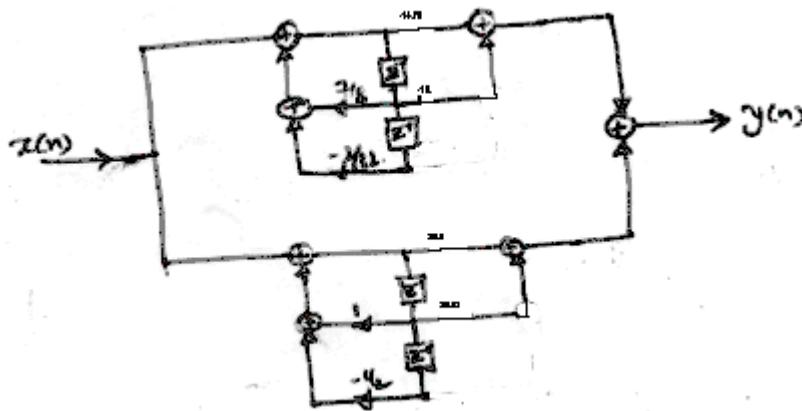
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Parallel Form

$$H(z) = H_1(z) + H_2(z)$$

$$H(z) = \frac{(-14.75 - 12.90z^{-1})}{(1 + \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2})} + \frac{(24.50 + 26.82z^{-1})}{(1 - z^{-1} + \frac{1}{2}z^{-2})}$$

**Problem: 3**

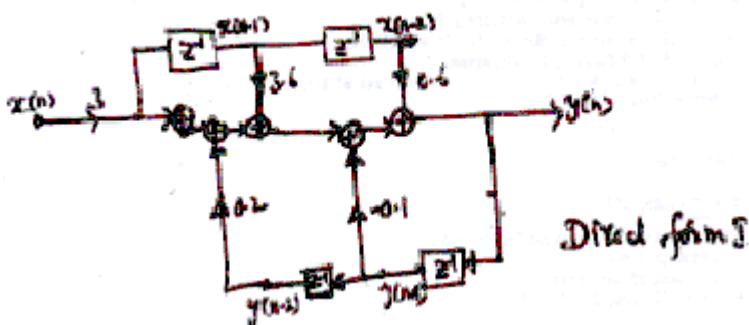
Obtain the direct form – I, direct form-II

Cascade and parallel form realization for the following system,

$$y(n) = -0.1 y(n-1) + 0.2 y(n-2) + 3x(n) + 3.6 x(n-1) + 0.6 x(n-2)$$

Solution:

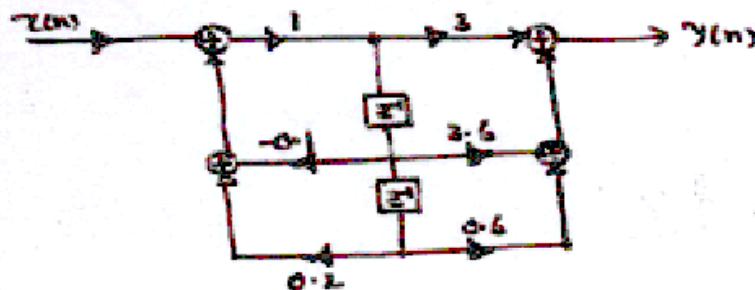
The Direct form realization is done directly from the given i/p – o/p equation, show in below diagram



Direct form –II realization

Taking ZT on both sides and finding H(z)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$



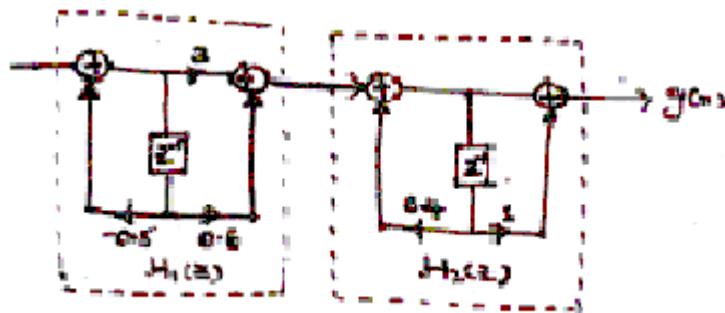
Cascade form realization

The transformer function can be expressed as:

$$H(z) = \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})}$$

which can be re written as

$$\text{where } H_1(z) = \frac{3 + 0.6z^{-1}}{1 + 0.5z^{-1}} \text{ and } H_2(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

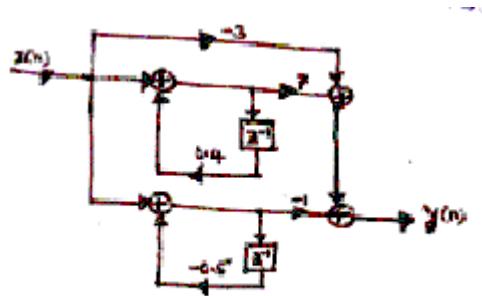


Parallel Form realization

The transfer function can be expressed as

$H(z) = C + H_1(z) + H_2(z)$ where $H_1(z)$ & $H_2(z)$ is given by,

$$H(z) = -3 + \frac{7}{1 - 0.4z^{-1}} - \frac{1}{1 + 0.5z^{-1}}$$



6.6 Lattice Structure for IIR System:

Consider an All-pole system with system function.

$$H(Z) = \frac{1}{1 + \sum_{k=1}^N a_N(k)Z^{-k}} = \frac{1}{A_N(Z)}$$

The corresponding difference equation for this IIR system is,

$$y(n) = -\sum_{k=1}^N a_N(k)y(n-k) + x(n)$$

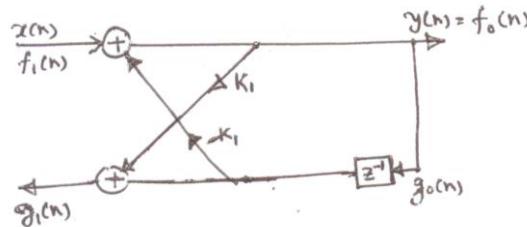
OR

$$x(n) = y(n) + \sum_{k=1}^N a_N(k)y(n-k)$$

For N=1

$$x(n) = y(n) + a_1(1)y(n-1)$$

Which can be realized as,



We observe

$$x(n) = f_1(n)$$

$$\begin{aligned} y(n) &= f_0(n) = f_1(n) - k_1 g_0(n-1) \\ &= x(n) - k_1 y(n-1) \end{aligned}$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1) = k_1 y(n) + y(n-1)$$

$$k_1 = a_1(1)$$

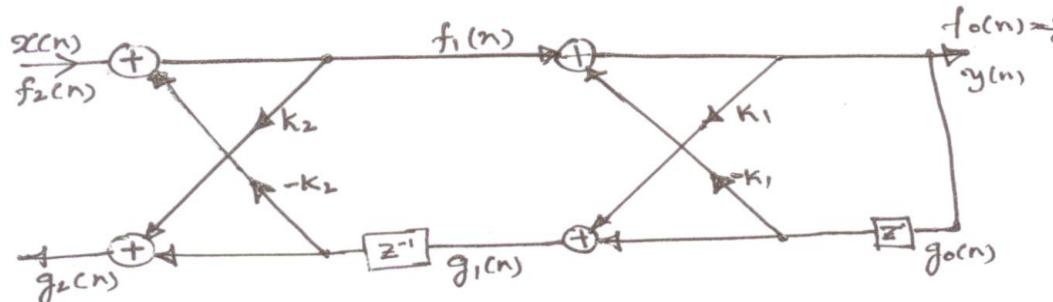
For N=2, then

$$y(n) = x(n) - a_2(1)y(n-1) - a_2(2)y(n-2)$$

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This output can be obtained from a two-stage lattice filter as shown in below fig



$$f_2(n) = x(n)$$

$$f_1(n) = f_2(n) - k_2 g_1(n-1)$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1)$$

$$f_0(n) = f_1(n) - k_1 g_0(n-1)$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1)$$

$$\begin{aligned} y(n) &= f_0(n) = g_0(n) = f_1(n) - k_1 g_0(n-1) \\ &= f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1) \\ &= f_2(n) - k_2 [k_1 f_0(n-1) + g_0(n-2)] - k_1 g_0(n-1) \\ &= x(n) - k_2 [k_1 y(n-1) + y(n-2)] - k_1 y(n-1) \\ &= x(n) - k_1(1+k_2)y(n-1) - k_2 y(n-2) \end{aligned}$$

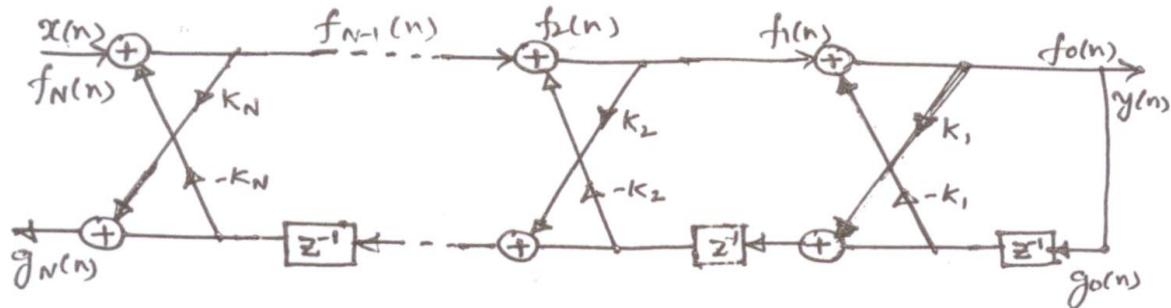
Similarly

$$g_2(n) = k_2 y(n) + k_1(1+k_2)y(n-1) + y(n-2)$$

We observe

$$a_2(0) = 1; a_2(1) = k_1(1+k_2); a_2(2) = k_2$$

N-stage IIR filter realized in lattice structure is,



$$f_N(n) = x(n)$$

$$f_{m-1}(n) = f_m(n) - k_m g_{m-1}(n-1) \quad m=N, N-1, \dots, 1$$

$$g_m(n) = k_m f_{m-1}(n) + g_{m-1}(n-1) \quad m=N, N-1, \dots, 1$$

$$y(n) = f_0(n) = g_0(n)$$

8.6.1 Conversion from lattice structure to direct form:

$$\begin{aligned} a_m(m) &= k_m; \quad a_m(0) = 1 \\ a_m(k) &= a_{m-1}(k) + a_m(m)a_{m-1}(m-k) \end{aligned}$$

Conversion from direct form to lattice structure

$$a_{m-1}(0) = 1 \quad k_m = a_m(m)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

6.6.2 Lattice – Ladder Structure:

A general IIR filter containing both poles and zeros can be realized using an all pole lattice as the basic building block.

If,

$$H(Z) = \frac{B_M(Z)}{A_N(Z)} = \frac{\sum_{k=0}^M b_M(k)Z^{-k}}{1 + \sum_{k=1}^N a_N(k)Z^{-k}}$$

Where $N \geq M$

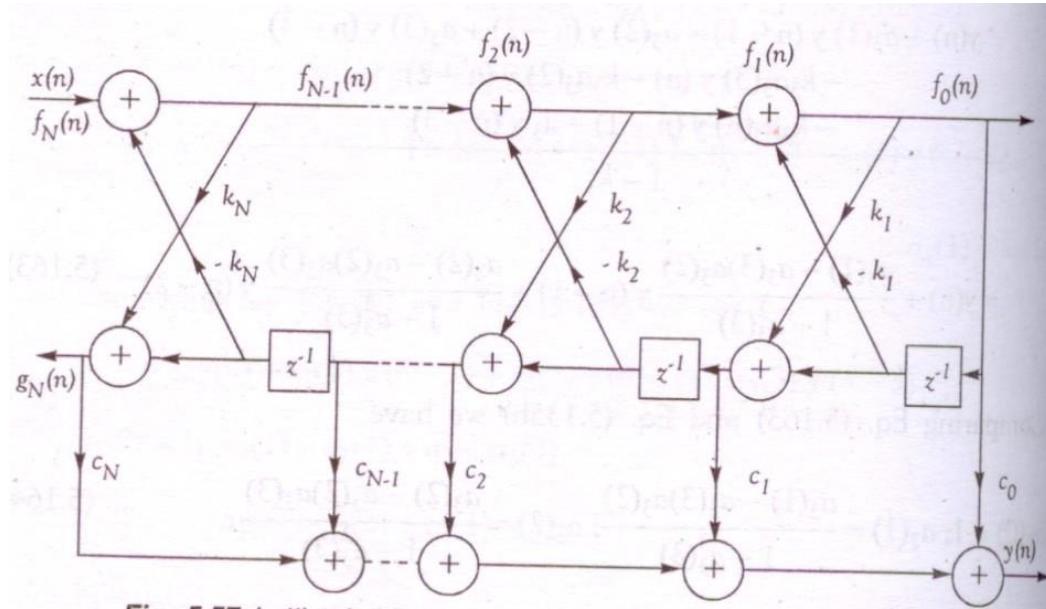
A lattice structure can be constructed by first realizing an all-pole lattice co-efficients k_m , $1 \leq m \leq N$ for the denominator $A_N(Z)$, and then adding a ladder part for $M=N$. The output of the ladder part can be expressed as a weighted linear combination of $\{g_m(n)\}$.

Now the output is given by

$$y(n) = \sum_{m=0}^M C_m g_m(n)$$

Where $\{C_m\}$ are called the ladder co-efficient and can be obtained using the recursive relation,

$$C_m = b_m - \sum_{i=m+1}^M C_i a_i(i-m); \quad m=M, M-1, \dots, 0$$

**Problem:4**

Convert the following pole-zero IIR filter into a lattice ladder structure,

$$H(Z) = \frac{1 + 2Z^{-1} + 2Z^{-2} + Z^{-3}}{1 + \frac{13}{24}Z^{-1} + \frac{5}{8}Z^{-2} + \frac{1}{3}Z^{-3}}$$

Solution:

$$\text{Given } b_M(Z) = 1 + 2Z^{-1} + 2Z^{-2} + Z^{-3}$$

$$\text{And } A_N(Z) = 1 + \frac{13}{24}Z^{-1} + \frac{5}{8}Z^{-2} + \frac{1}{3}Z^{-3}$$

$$a_3(0) = 1; \quad a_3(1) = \frac{13}{24}; \quad a_3(2) = \frac{5}{8}; \quad a_3(3) = \frac{1}{3}$$

$$k_3 = a_3(3) = \frac{1}{3}$$

Using the equation

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a^2 m(m)}$$

for m=3, k=1

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)} = \frac{\frac{13}{24} - \frac{1}{3} \cdot \frac{5}{8}}{1 - (\frac{1}{3})^2} = \frac{3}{8}$$

for m=3, & k=2

$$a_2(2) = k_2 = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$\frac{\frac{5}{8} - \frac{1}{3} \cdot \frac{13}{24}}{1 - \frac{1}{9}} = \frac{\frac{45-13}{72}}{\frac{8}{9}} = \frac{1}{2}$$

for m=2, & k=1

$$a_1(1) = k_1 = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

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$$\frac{\frac{3}{8} - \frac{1}{2} \cdot \frac{3}{8}}{1 - (\frac{1}{2})^2} = \frac{\frac{3}{8} - \frac{3}{16}}{1 - \frac{1}{4}} = \frac{1}{4}$$

for lattice structure $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{2}$, $k_3 = \frac{1}{3}$

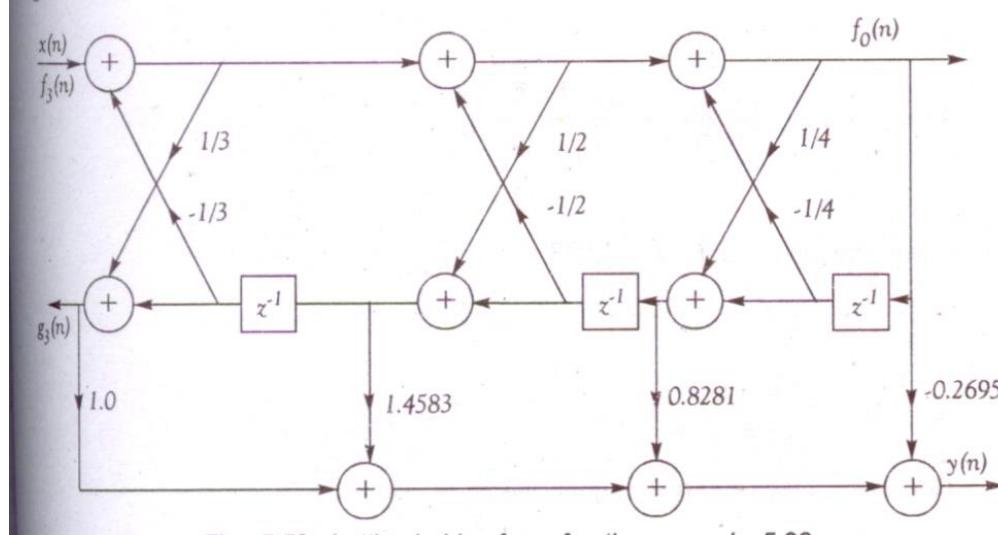
For ladder structure

$$C_m = b_m - \sum_{i=m+1}^M C_i a_i (1-m) \quad m=M, M-1, 1, 0$$

$M=3$ $C_3 = b_3 = 1; \quad C_2 = b_2 - C_3 a_3(1)$
 $= 2 - 1 \cdot (\frac{13}{24}) = 1.4583$

$$C_1 = b_1 - \sum_{i=2}^3 c_i a_i (i-m) \quad m=1$$
 $= b_1 - [c_2 a_2(1) + c_3 a_3(2)]$
 $= 2 - [(1.4583)(\frac{3}{8}) + \frac{5}{8}] = 0.8281$
 $c_0 = b_0 - \sum_{i=1}^3 c_i a_i (i-m)$
 $= b_0 - [c_1 a_1(1) + c_2 a_2(2) + c_3 a_3(3)]$
 $= 1 - [0.8281(\frac{1}{4}) + 1.4583(\frac{1}{2}) + \frac{1}{3}] = -0.2695$

To convert a lattice- ladder form into a direct form, we find an equation to obtain $a_N(k)$ from k_m ($m=1, 2, \dots, N$) then equation for c_m is recursively used to compute b_m ($m=0, 1, 2, \dots, M$).

**Problem 5**

A z-plane pole-zero plot for a certain digital filter is shown in figure 7. The filter has unity gain at DC. Determine the system function in the form

$$H(z) = A \left[\frac{(1 + a_1 z^{-1})(1 + b_1 z^{-2} + b_2 z^{-4})}{(1 + c_1 z^{-1})(1 + d_1 z^{-1} + d_2 z^{-2})} \right] \text{ giving the numerical values for parameters}$$

$A, a_1, b_1, b_2, c_1, d_1$ and d_2 . Sketch the direct form-II and cascade realizations of the system. [9]

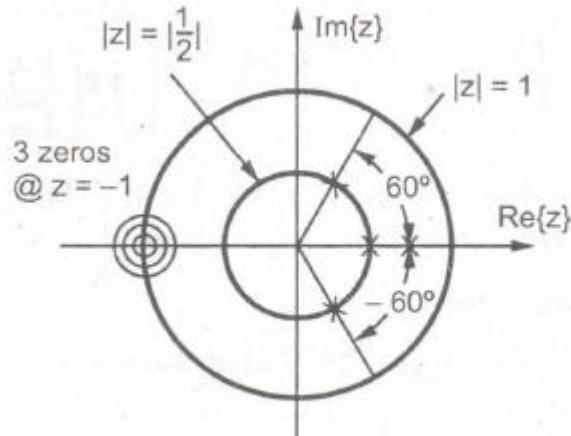


Fig. 7

Sol. :

$$H(z) = A \cdot \frac{(1 + z^{-1})^3}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}$$

$$H(z) = A \cdot \frac{(1 + z^{-1})(1 + 2z^{-1} + z^{-2})}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}$$

$$H(z)|_{z=1} = 1$$

$$\Rightarrow A = \frac{3}{64}, a_1 = 1, b_1 = 2, b_2 = 1, c_1 = -\frac{1}{2}, d_1 = -\frac{1}{2} \text{ and } d_2 = \frac{1}{4}$$

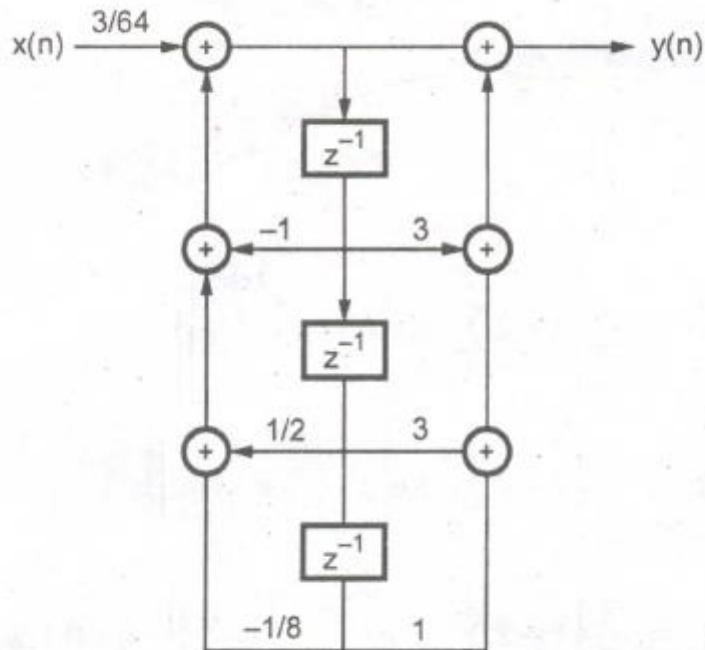


Fig. 8 (a)

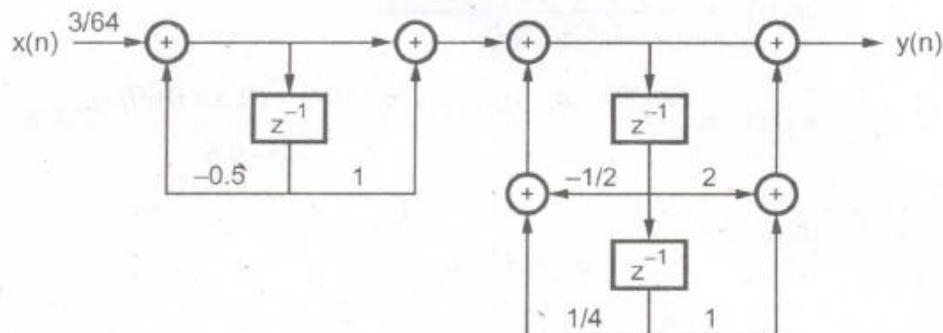


Fig. 8 (b)

Question 6

Consider a FIR filter with system function:

$H(z) = 1 + 2.82 Z^{-1} + 3.4048z^{-2} + 1.74z^{-3}$. Sketch the direct form and lattice realizations of the filter.

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$$\text{Sol. : } A_3(z) = H(z) = 1 + 2.82z^{-1} + 3.4048z^{-2} + 1.74z^{-3}$$

$$B_3(z) = 1.74 + 3.4048z^{-1} + 2.82z^{-2} + z^{-3}$$

$$\text{Hence } k_3 = 1.74$$

$$A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{1 - k_3^2}$$

$$= \frac{1 + 2.82z^{-1} + 3.4048z^{-2} + 1.74z^{-3} - 3.0276 - 5.9243z^{-1} - 4.9068z^{-2} - 1.74z^{-3}}{(-2.0276)}$$

$$= \frac{-2.0276 - 3.1043z^{-1} - 1.502z^{-2}}{(-2.0276)} = 1 + 1.531z^{-1} + 0.7407z^{-2}$$

$$B_2(z) = 0.7407 + 1.531z^{-1} + z^{-2}$$

$$k_2 = 0.7407$$

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - k_2^2}$$

$$= \frac{1 + 1.531z^{-1} + 0.7407z^{-2} - 0.5486 - 1.134z^{-1} - 0.7407z^{-2}}{0.4514}$$

$$= 1 + 0.8795z^{-1}$$

$$k_1 = 0.8795$$

Direct form realization :

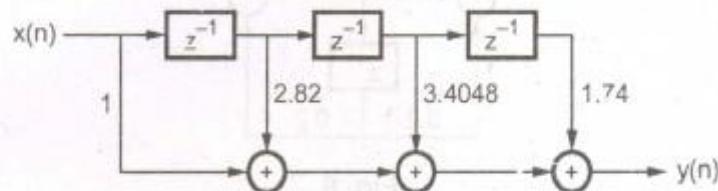


Fig. 5

Lattice realization :

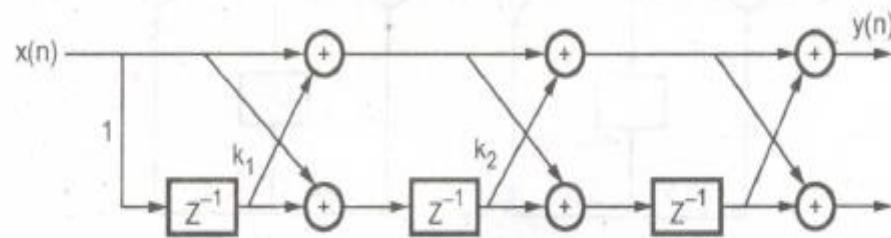


Fig. 6

UNIT 7**FIR FILTER DESIGN****CONTENTS:-**

FIR FILTER DESIGN: INTRODUCTION TO FIR FILTERS, DESIGN OF FIR FILTERS USING - RECTANGULAR, HAMMING, BARTLET AND KAISER WINDOWS, FIR FILTER DESIGN USING FREQUENCY SAMPLING TECHNIQUE. 6 HRS

RECOMMENDED READINGS

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

UNIT 7

Design of FIR Filters

7.1 Introduction:

Two important classes of digital filters based on impulse response type are

Finite Impulse Response (FIR)

Infinite Impulse Response (IIR)

The filter can be expressed in two important forms as:

1) System function representation;

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \quad (1)$$

2) Difference Equation representation;

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k) \quad (2)$$

Each of this form allows various methods of implementation. The eq (2) can be viewed as a computational procedure (an algorithm) for determining the output sequence $y(n)$ of the system from the input sequence $x(n)$. Different realizations are possible with different arrangements of eq (2)

The major issues considered while designing a digital filters are :

- Reliability (causal or non causal)
- Stability (filter output will not saturate)
- Sharp Cutoff Characteristics
- Order of the filter need to be minimum (this leads to less delay)
- Generalized procedure (having single procedure for all kinds of filters)
- Linear phase characteristics

The factors considered with filter implementation are ,

- a. It must be a simple design
- b. There must be modularity in the implementation so that any order filter can be obtained with lower order modules.
- c. Designs must be as general as possible. Having different design procedures for different types of filters(high pass, low pass,...) is cumbersome and complex.
- d. Cost of implementation must be as low as possible
- e. The choice of Software/Hardware realization

7.2 Features of IIR:

The important features of this class of filters can be listed as:

- Out put is a function of past o/p, present and past i/p's
- It is recursive in nature
- It has at least one Pole (in general poles and zeros)
- Sharp cutoff chas. is achievable with minimum order
- Difficult to have linear phase chas over full range of freq.
- Typical design procedure is analog design then conversion from analog to digital

7.3 Features of FIR : The main features of FIR filter are,

- They are inherently stable
- Filters with linear phase characteristics can be designed
- Simple implementation – both recursive and nonrecursive structures possible
- Free of limit cycle oscillations when implemented on a finite-word length digital system

7.3.1 Disadvantages:

- Sharp cutoff at the cost of higher order
- Higher order leading to more delay, more memory and higher cost of implementation

7.4 Importance of Linear Phase:

The group delay is defined as

$$\tau_g = -\frac{d\theta(\omega)}{d\omega}$$

which is negative differential of phase function.

Nonlinear phase results in different frequencies experiencing different delay and arriving at different time at the receiver. This creates problems with speech processing and data

communication applications. Having linear phase ensures constant group delay for all frequencies.

The further discussions are focused on FIR filter.

6.5 Examples of simple FIR filtering operations: **1. Unity Gain Filter**

$$y(n) = x(n)$$

2. Constant gain filter

$$y(n) = Kx(n)$$

3. Unit delay filter

$$y(n) = x(n-1)$$

4. Two - term Difference filter

$$y(n) = x(n) - x(n-1)$$

5. Two-term average filter

$$y(n) = 0.5(x(n) + x(n-1))$$

6. Three-term average filter (3-point moving average filter)

$$y(n) = 1/3[x(n) + x(n-1) + x(n-2)]$$

7. Central Difference filter

$$y(n) = 1/2[x(n) - x(n-2)]$$

When we say Order of the filter it is the number of previous inputs used to compute the current output and Filter coefficients are the numbers associated with each of the terms $x(n)$, $x(n-1)$,.. etc

The table below shows order and filter coefficients of above simple filter types:

Ex.	order	a0	a1	a2
1	0	1	-	-
2	0	K	-	-
3	1	0	1	-
4(HP)	1	1	-1	-
5(LP)	1	1/2	1/2	-
6(LP)	2	1/3	1/3	1/3
7(HP)	2	1/2	0	-1/2

7.6 Design of FIR filters:

The section to follow will discuss on design of FIR filter. Since linear phase can be achieved with FIR filter we will discuss the conditions required to achieve this.

7.6.1 Symmetric and Antisymmetric FIR filters giving out Linear Phase characteristics:

Symmetry in filter impulse response will ensure linear phase

An FIR filter of length M with i/p $x(n)$ & o/p $y(n)$ is described by the difference equation:

$$y(n) = b_0 x(n) + b_1 x(n-1) + \dots + b_{M-1} x(n-(M-1)) = \sum_{k=0}^{M-1} b_k x(n-k) \quad -(1)$$

Alternatively. it can be expressed in convolution form

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \quad -(2)$$

i.e $b_k = h(k)$, $k=0,1,\dots,M-1$

Filter is also characterized by

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$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$ -(3) polynomial of degree M-1 in the variable z^{-1} . The roots of this polynomial constitute zeros of the filter.

An FIR filter has linear phase if its unit sample response satisfies the condition
 $h(n) = \pm h(M-1-n)$ $n=0,1,\dots,M-1$ -(4)

Incorporating this symmetry & anti symmetry condition in eq 3 we can show linear phase chas of FIR filters

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

If M is odd

$$\begin{aligned} H(z) &= h(0) + h(1)z^{-1} + \dots + h\left(\frac{M-1}{2}\right)z^{-\left(\frac{M-1}{2}\right)} + h\left(\frac{M+1}{2}\right)z^{-\left(\frac{M+1}{2}\right)} + h\left(\frac{M+3}{2}\right)z^{-\left(\frac{M+3}{2}\right)} + \dots \\ &+ h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)} \end{aligned}$$

$$= z^{-\left(\frac{M-1}{2}\right)} \left[h(0)z^{\left(\frac{M-1}{2}\right)} + h(1)z^{\left(\frac{M-3}{2}\right)} + \dots + h\left(\frac{M-1}{2}\right) + h\left(\frac{M+1}{2}\right)z^{-1} + h\left(\frac{M+3}{2}\right)z^{-2} + \dots + h(M-1)z^{-\left(\frac{M-1}{2}\right)} \right]$$

Applying symmetry conditions for M odd

$$h(0) = \pm h(M-1)$$

$$h(1) = \pm h(M-2)$$

.

.

$$h\left(\frac{M-1}{2}\right) = \pm h\left(\frac{M-1}{2}\right)$$

$$h\left(\frac{M+1}{2}\right) = \pm h\left(\frac{M-3}{2}\right)$$

.

.

$$h(M-1) = \pm h(0)$$

$$H(z) = z^{-\left(\frac{M-1}{2}\right)} \left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n) \{ z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2} \} \right]$$

similarly for M even

$$H(z) = z^{-\left(\frac{M-1}{2}\right)} \left[\sum_{n=0}^{\frac{M-1}{2}} h(n) \{ z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2} \} \right]$$

7.6.2 Frequency response:

If the system impulse response has symmetry property (i.e., $h(n)=h(M-1-n)$) and M is odd
 $H(e^{j\omega}) = e^{j\theta(\omega)} |H_r(e^{j\omega})|$ where

$$H_r(e^{j\omega}) = \left[h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \cos \omega \left(\frac{M-1}{2} - n\right) \right]$$

$$\theta(\omega) = -\left(\frac{M-1}{2}\right)\omega \quad \text{if } |H_r(e^{j\omega})| \geq 0$$

$$= -\left(\frac{M-1}{2}\right)\omega + \pi \quad \text{if } |H_r(e^{j\omega})| \leq 0$$

In case of M even the phase response remains the same with magnitude response expressed as

$$H_r(e^{j\omega}) = \left[2 \sum_{n=0}^{\frac{M-1}{2}} h(n) \cos \omega \left(\frac{M-1}{2} - n\right) \right]$$

If the impulse response satisfies anti symmetry property (i.e., $h(n)=-h(M-1-n)$) then for M odd we will have

$$h\left(\frac{M-1}{2}\right) = -h\left(\frac{M-1}{2}\right) \text{ i.e., } h\left(\frac{M-1}{2}\right) = 0$$

$$H_r(e^{j\omega}) = \left[2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin \omega \left(\frac{M-1}{2} - n\right) \right]$$

If M is even then,

$$H_r(e^{j\omega}) = \left[2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin \omega \left(\frac{M-1}{2} - n \right) \right]$$

In both cases the phase response is given by

$$\begin{aligned}\theta(\omega) &= -\left(\frac{M-1}{2}\right)\omega + \pi/2 \quad \text{if } |H_r(e^{j\omega})| \geq 0 \\ &= -\left(\frac{M-1}{2}\right)\omega + 3\pi/2 \quad \text{if } |H_r(e^{j\omega})| \leq 0\end{aligned}$$

Which clearly shows presence of Linear Phase characteristics.

7.6.3 Comments on filter coefficients:

- The number of filter coefficients that specify the frequency response is $(M+1)/2$ when M is odd and $M/2$ when M is even in case of symmetric conditions
- In case of impulse response antisymmetric we have $h(M-1/2)=0$ so that there are $(M-1/2)$ filter coefficients when M is odd and $M/2$ coefficients when M is even

7.6.5 Choice of Symmetric and antisymmetric unit sample response

When we have a choice between different symmetric properties, the particular one is picked up based on application for which the filter is used. The following points give an insight to this issue.

- If $h(n)=-h(M-1-n)$ and M is odd, $H_r(w)$ implies that $H_r(0)=0$ & $H_r(\pi)=0$, consequently not suited for lowpass and highpass filter. This condition is suited in Band Pass filter design.
- Similarly if M is even $H_r(0)=0$ hence not used for low pass filter
- Symmetry condition $h(n)=h(M-1-n)$ yields a linear-phase FIR filter with non zero response at $w = 0$ if desired.

Looking at these points, antisymmetric properties are not generally preferred.

7.6.6 Zeros of Linear Phase FIR Filters:

Consider the filter system function

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

Expanding this equation

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

Since for Linear-phase we need

$$h(n) = h(M-1-n) \quad i.e.,$$

$$h(0) = h(M-1); h(1) = h(M-2); \dots; h(M-1) = h(0);$$

then

$$H(z) = h(M-1) + h(M-2)z^{-1} + \dots + h(1)z^{-(M-2)} + h(0)z^{-(M-1)}$$

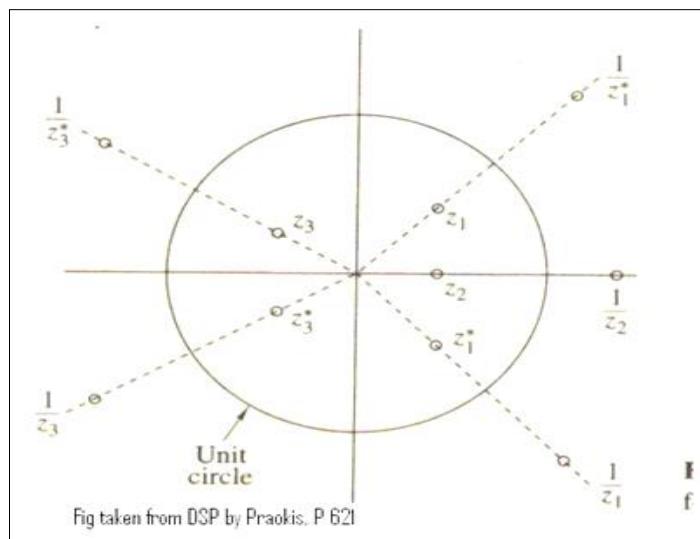
$$H(z) = z^{-(M-1)}[h(M-1)z^{(M-1)} + h(M-2)z^{(M-2)} + \dots + h(1)z + h(0)]$$

$$H(z) = z^{-(M-1)} \left[\sum_{n=0}^{M-1} h(n)(z^{-1})^{-n} \right] = z^{-(M-1)} H(z^{-1})$$

This shows that if $z = z_1$ is a zero then $z = z_1^{-1}$ is also a zero

The different possibilities:

1. If $z_1 = 1$ then $z_1 = z_1^{-1} = 1$ is also a zero implying it is one zero
2. If the zero is real and $|z| < 1$ then we have pair of zeros
3. If zero is complex and $|z|=1$ then we again have pair of complex zeros.
4. If zero is complex and $|z| \neq 1$ then we have two pairs of complex zeros



The plot above shows distribution of zeros for a Linear – phase FIR filter. As it can be seen there is pattern in distribution of these zeros.

7.7 Methods of designing FIR filters:

The standard methods of designing FIR filter can be listed as:

1. Fourier series based method
2. Window based method
3. Frequency sampling method

7.7.1 Design of Linear Phase FIR filter based on Fourier Series method:

Motivation: Since the desired freq response $H_d(e^{j\omega})$ is a periodic function in ω with period 2π , it can be expressed as Fourier series expansion

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

where $h_d(n)$ are fourier series coefficients

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

This expansion results in impulse response coefficients which are infinite in duration and non causal. It can be made finite duration by truncating the infinite length. The linear phase can be obtained by introducing symmetric property in the filter impulse response, i.e., $h(n) = h(-n)$. It can be made causal by introducing sufficient delay (depends on filter length)

7.7.2 Stepwise procedure:

1. From the desired freq response using inverse FT relation obtain $h_d(n)$
2. Truncate the infinite length of the impulse response to finite length with (assuming M odd)

$$\begin{aligned} h(n) &= h_d(n) \text{ for } -(M-1)/2 \leq n \leq (M-1)/2 \\ &= 0 \text{ otherwise} \end{aligned}$$

3. Introduce $h(n) = h(-n)$ for linear phase characteristics
4. Write the expression for $H(z)$; this is non-causal realization
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

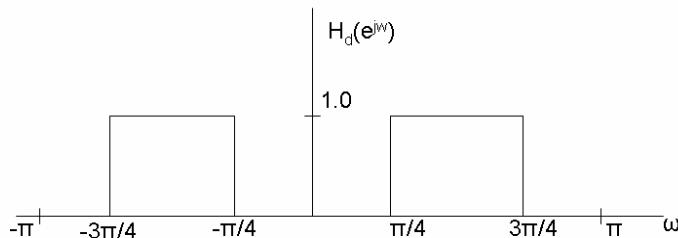
Exercise Problems

Problem 1 : Design an ideal bandpass filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4}$$

$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $M = 11$ and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{\pi n} \left[\sin \frac{3\pi}{4} n - \sin \frac{\pi}{4} n \right] \quad -\infty \leq n \leq \infty$$

truncating to 11 samples we have $h(n) = h_d(n)$ for $|n| \leq 5$

$$= 0 \quad \text{otherwise}$$

For $n = 0$ the value of $h(n)$ is separately evaluated from the basic integration

$$h(0) = 0.5$$

Other values of $h(n)$ are evaluated from $h(n)$ expression

$$h(1)=h(-1)=0$$

$$h(2)=h(-2)=-0.3183$$

$$h(3)=h(-3)=0$$

$$h(4)=h(-4)=0$$

$$h(5)=h(-5)=0$$

The transfer function of the filter is

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{(N-1)/2} [h(n)\{z^n + z^{-n}\}] \\ &= 0.5 - 0.3183(z^2 + z^{-2}) \end{aligned}$$

the transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-5}[0.5 - 0.3183(z^2 + z^{-2})] \\ &= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7} \end{aligned}$$

the filter coeff are

$$\begin{aligned} h'(0) &= h'(10) = h'(1) = h'(9) = h'(2) = h'(8) = h'(4) = h'(6) = 0 \\ h'(3) &= h'(7) = -0.3183 \\ h'(5) &= 0.5 \end{aligned}$$

The magnitude response can be expressed as

$$|H(e^{j\omega})| = \sum_{n=1}^{(N-1)/2} a(n) \cos \omega n$$

comparing this exp with

$$|H(e^{j\omega})| = z^{-5}[h(0) + 2 \sum_{n=1}^5 h(n) \cos \omega n]$$

We have

$$a(0)=h(0)$$

$$a(1)=2h(1)=0$$

$$a(2)=2h(2)=-0.6366$$

$$a(3)=2h(3)=0$$

$$a(4)=2h(4)=0$$

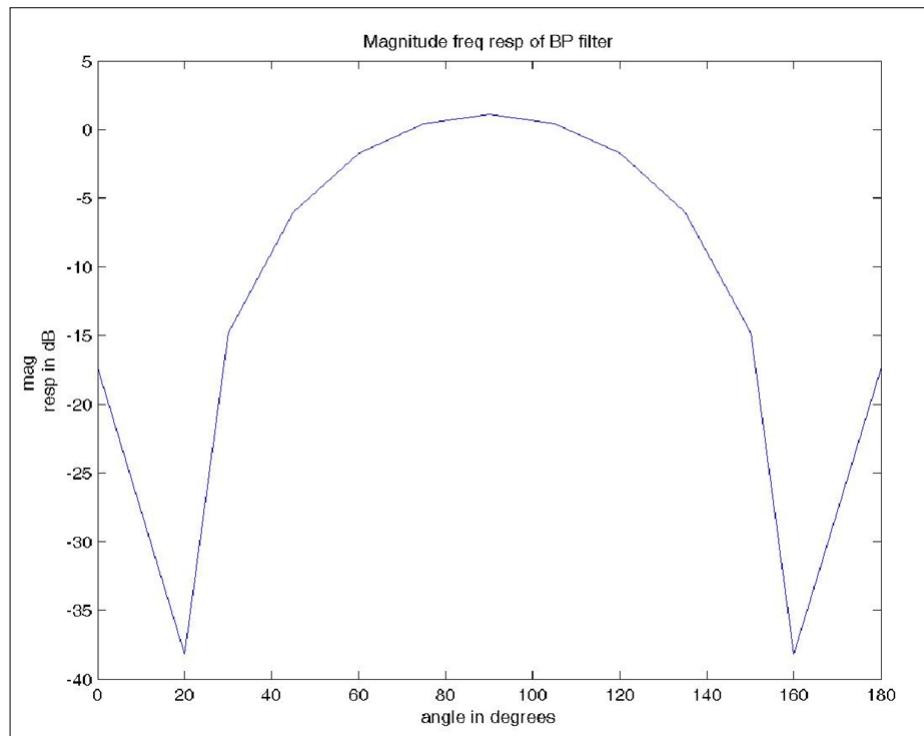
$$a(5)=2h(5)=0$$

The magnitude response function is

$$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega \text{ which can plotted for various values of } \omega$$

ω in degrees =[0 20 30 45 60 75 90 105 120 135 150 160 180];

$|H(e^{j\omega})|$ in dBs = [-17.3 -38.17 -14.8 -6.02 -1.74 0.4346 1.11 0.4346 -1.74 -6.02 -14.8 -38.17 -17.3];



Problem 2: Design an ideal lowpass filter with a freq response

$$H_d(e^{j\omega}) = 1 \quad \text{for } -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2}$$

$$= 0 \quad \text{for } \frac{\pi}{2} \leq |\omega| \leq \pi$$

Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response

From the freq response we can determine $h_d(n)$,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \frac{\sin \frac{\pi n}{2}}{\pi n} \quad -\infty \leq n \leq \infty \quad \text{and} \quad n \neq 0$$

Truncating $h_d(n)$ to 11 samples

$$h(0) = 1/2$$

$$h(1) = h(-1) = 0.3183$$

$$h(2) = h(-2) = 0$$

$$h(3) = h(-3) = -0.106$$

$$h(4)=h(-4)=0$$

$$h(5)=h(-5)=0.06366$$

The realizable filter can be obtained by shifting $h(n)$ by 5 samples to right $h'(n)=h(n-5)$

$$h'(n)=[0.06366, 0, -0.106, 0, 0.3183, 0.5, 0.3183, 0, -0.106, 0, 0.06366];$$

$$H'(z) = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

Using the result of magnitude response for M odd and symmetry

$$H_r(e^{j\omega}) = [h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} h(n) \cos \omega (\frac{M-1}{2} - n)]$$

$$|H_r(e^{j\omega})| = |[0.5 + 0.6366 \cos \omega - 0.212 \cos 3\omega + 0.127 \cos 5\omega]|$$

Problem 3 :

Design an ideal band reject filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \quad \text{for } |\omega| \leq \frac{\pi}{3} \text{ and } |\omega| \geq \frac{2\pi}{3}$$

$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $M = 11$ and plot the frequency response

$$\text{Ans: } h(n) = [0 \ -0.1378 \ 0 \ 0.2757 \ 0 \ 0.667 \ 0 \ 0.2757 \ 0 \ -0.1378 \ 0];$$

7.8 Window based Linear Phase FIR filter design

The other important method of designing FIR filter is by making use of windows. The arbitrary truncation of impulse response obtained through inverse Fourier relation can lead to distortions in the final frequency response. The arbitrary truncation is equivalent to multiplying infinite length function with finite length rectangular window, i.e.,

$$h(n) = h_d(n) w(n) \text{ where } w(n) = 1 \text{ for } n = \pm(M-1)/2$$

The above multiplication in time domain corresponds to convolution in freq domain, i.e.,

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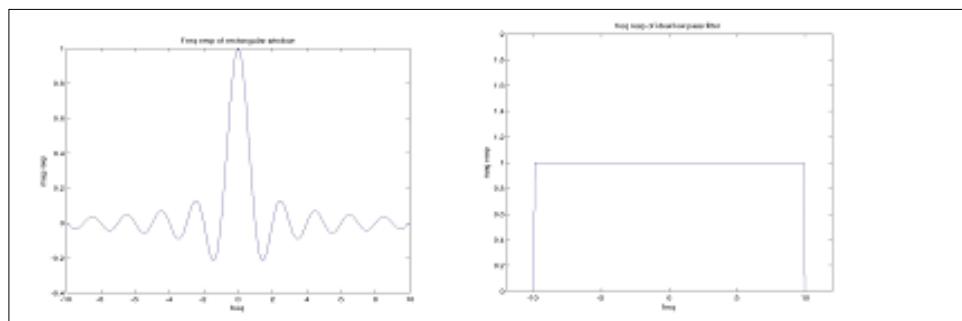
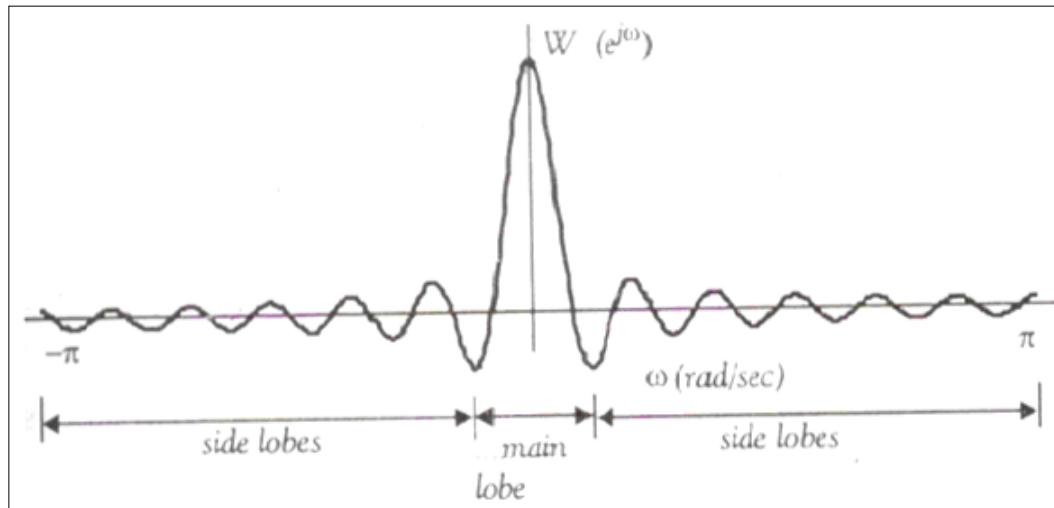
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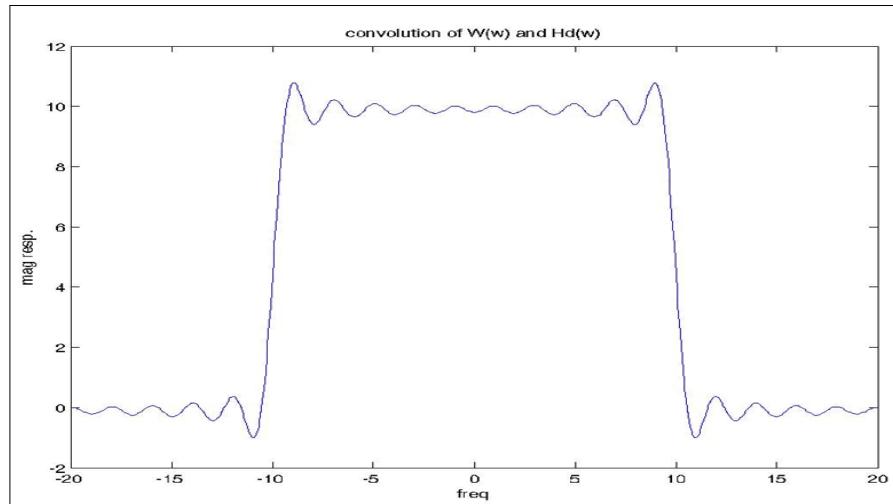
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega}) \text{ where } W(e^{j\omega}) \text{ is the FT of window function } w(n).$$

The FT of $w(n)$ is given by

$$W(e^{j\omega}) = \frac{\sin(\omega M / 2)}{\sin(\omega / 2)}$$

The whole process of multiplying $h(n)$ by a window function and its effect in freq domain are shown in below set of figures.





Suppose the filter to be designed is Low pass filter then the convolution of ideal filter freq response and window function freq response results in distortion in the resultant filter freq response. The ideal sharp cutoff chars are lost and presence of ringing effect is seen at the band edges which is referred to Gibbs Phenomena. This is due to main lobe width and side lobes of the window function freq response. The main lobe width introduces transition band and side lobes results in rippling characters in pass band and stop band. Smaller the main lobe width smaller will be the transition band. The ripples will be of low amplitude if the peak of the first side lobe is far below the main lobe peak.

7.8.1 How to reduce the distortions?

1. Increase length of the window

- as M increases the main lob width becomes narrower, hence the transition band width is decreased

-With increase in length the side lobe width is decreased but height of each side lobe increases in such a manner that the area under each sidelobe remains invariant to changes in M. Thus ripples and ringing effect in pass-band and stop-band are not changed.

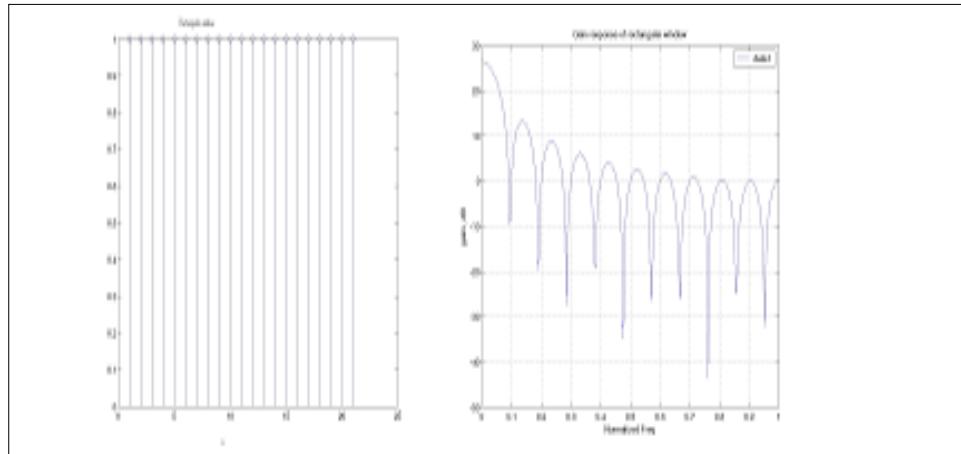
2. Choose windows which tapers off slowly rather than ending abruptly - Slow tapering reduces ringing and ripples but generally increases transition width since main lobe width of these kind of windows are larger.

7.8.2 What is ideal window characteristics?

Window having very small main lobe width with most of the energy contained with it (i.e.,ideal window freq response must be impulsive).Window design is a mathematical problem, more complex the window lesser are the distortions. Rectangular window is one of the simplest window in terms of computational complexity. Windows better than rectangular window are, Hamming, Hanning, Blackman, Bartlett, Traingular,Kaiser. The different window functions are discussed in the following section.

7.8.3 Rectangular window: The mathematical description is given by,

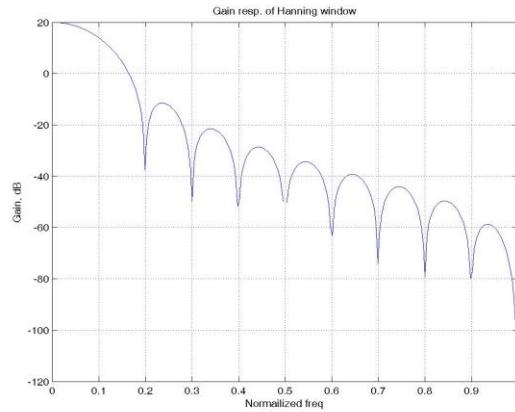
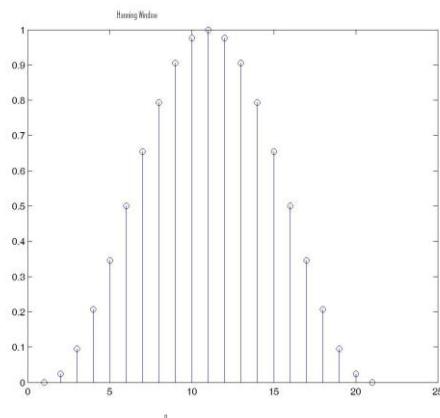
$$w_r(n) = 1 \text{ for } 0 \leq n \leq M - 1$$



7.8.4 Hanning windows:

It is defined mathematically by,

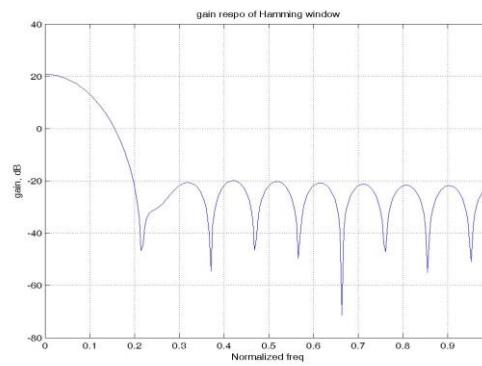
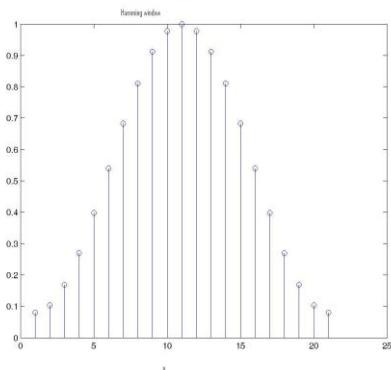
$$w_{han}(n) = 0.5(1 - \cos \frac{2\pi n}{M-1}) \text{ for } 0 \leq n \leq M - 1$$



7.8.5 Hamming windows:

This window function is given by,

$$w_{ham}(n) = 0.54 - 0.46 \cos \frac{2\pi n}{M-1} \text{ for } 0 \leq n \leq M-1$$



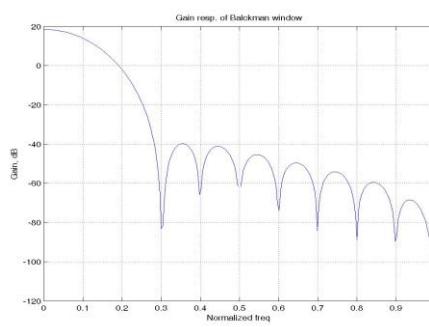
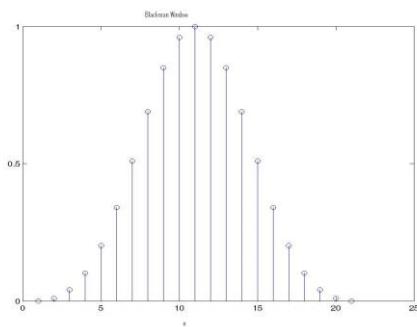
7.8.6 Blackman windows:

This window function is given by,

$$w_{blk}(n) = 0.42 - 0.5 \cos \frac{2\pi n}{M-1} + 0.08 \cos \frac{4\pi n}{M-1} \text{ for } 0 \leq n \leq M-1$$

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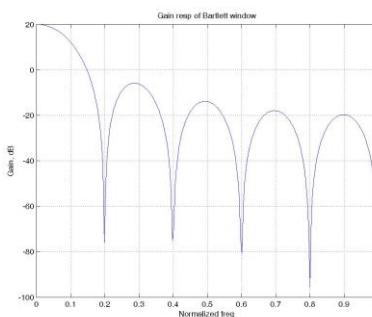
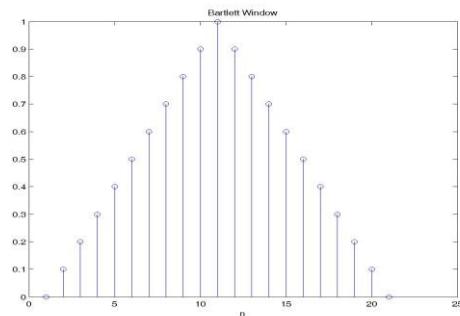
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7.8.7 Bartlett (Triangular) windows:

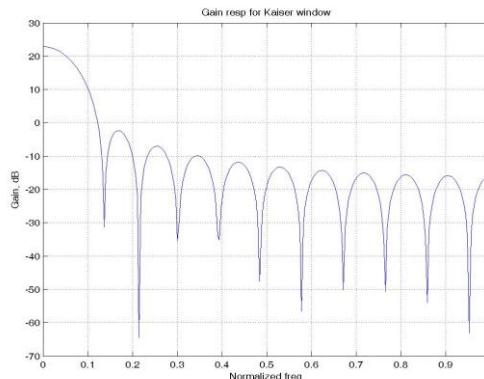
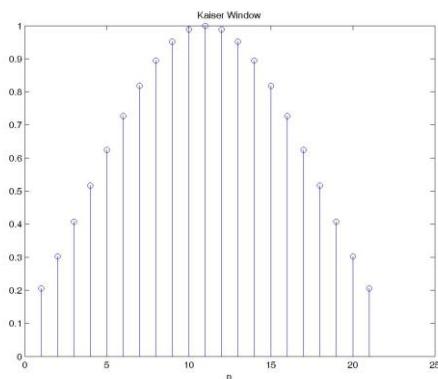
The mathematical description is given by,

$$w_{bart}(n) = 1 - \frac{2|n - \frac{M-1}{2}|}{M-1} \quad \text{for } 0 \leq n \leq M-1$$



7.8.8 Kaiser windows: The mathematical description is given by,

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]} \quad \text{for } 0 \leq n \leq M-1$$



Type of window	Appr. Transition width of the main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-27
Hanning	$8\pi/M$	-32
Hamming	$8\pi/M$	-43
Blackman	$12\pi/M$	-58

Looking at the above table we observe filters which are mathematically simple do not offer best characteristics. Among the window functions discussed Kaiser is the most complex one in terms of functional description whereas it is the one which offers maximum flexibility in the design.

7.8.9 Procedure for designing linear-phase FIR filters using windows:

1. Obtain $h_d(n)$ from the desired freq response using inverse FT relation
2. Truncate the infinite length of the impulse response to finite length with

(assuming M odd) choosing proper window

$$h(n) = h_d(n)w(n) \text{ where}$$

$w(n)$ is the window function defined for $-(M-1)/2 \leq n \leq (M-1)/2$

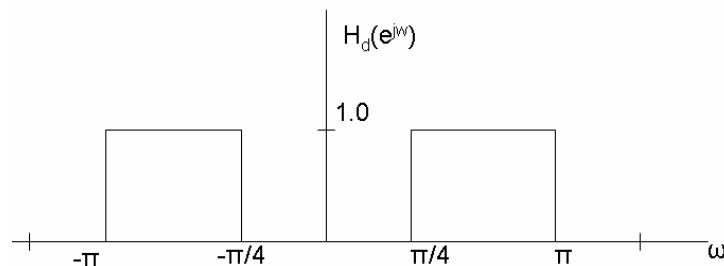
3. Introduce $h(n) = h(-n)$ for linear phase characteristics
4. Write the expression for $H(z)$; this is non-causal realization
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

Exercise Problems

Prob 1: Design an ideal highpass filter with a frequency response:

$$\begin{aligned} H_d(e^{j\omega}) &= 1 \quad \text{for } \frac{\pi}{4} \leq |\omega| \leq \pi \\ &= 0 \quad |\omega| < \frac{\pi}{4} \end{aligned}$$

using a hanning window with $M = 11$ and plot the frequency response.



$$h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right]$$

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$$h_d(n) = \frac{1}{\pi n} [\sin \pi n - \sin \frac{\pi n}{4}] \quad \text{for } -\infty \leq n \leq \infty \quad \text{and} \quad n \neq 0$$

$$h_d(0) = \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} d\omega + \int_{\pi/4}^{\pi} d\omega \right] = \frac{3}{4} = 0.75$$

$$h_d(1) = h_d(-1) = -0.225$$

$$h_d(2) = h_d(-2) = -0.159$$

$$h_d(3) = h_d(-3) = -0.075$$

$$h_d(4) = h_d(-4) = 0$$

$$h_d(5) = h_d(-5) = 0.045$$

The hamming window function is given by

$$w_{hn}(n) = 0.5 + 0.5 \cos \frac{2\pi n}{M-1} \quad -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right)$$

$$= 0 \quad \text{otherwise}$$

for $N = 11$

$$w_{hn}(n) = 0.5 + 0.5 \cos \frac{\pi n}{5} \quad -5 \leq n \leq 5$$

$$w_{hn}(0) = 1$$

$$w_{hn}(1) = w_{hn}(-1) = 0.9045$$

$$w_{hn}(2) = w_{hn}(-2) = 0.655$$

$$w_{hn}(3) = w_{hn}(-3) = 0.345$$

$$w_{hn}(4) = w_{hn}(-4) = 0.0945$$

$$w_{hn}(5) = w_{hn}(-5) = 0$$

$$h(n) = w_{hn}(n)h_d(n)$$

$$h(n) = [0 \ 0 \ -0.026 \ -0.104 \ -0.204 \ 0.75 \ -0.204 \ -0.104 \ -0.026 \ 0 \ 0]$$

$$h'(n) = h(n-5)$$

$$H'(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$

Using the equation

$$H_r(e^{jw}) = [h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \cos \omega \left(\frac{M-1}{2} - n\right)]$$

$$H_r(e^{jw}) = 0.75 + 2 \sum_{n=0}^4 h(n) \cos \omega (5-n)$$

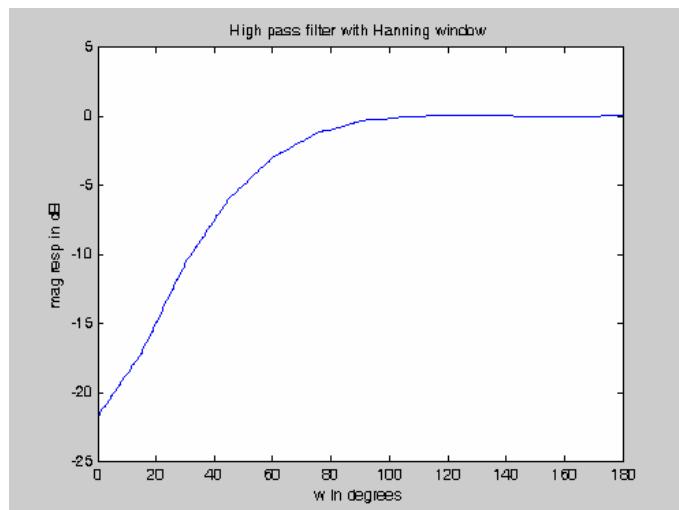
The magnitude response is given by,

$$|H_r(e^{j\omega})| = |0.75 - 0.408\cos\omega - 0.208\cos2\omega - 0.052\cos3\omega|$$

$$\omega \text{ in degrees} = [0 \ 15 \ 30 \ 45 \ 60 \ 75 \ 90 \ 105 \ 120 \ 135 \ 150 \ 165 \ 180]$$

$$|H(e^{j\omega})| \text{ in dBs} = [-21.72 \ -17.14 \ -10.67 \ -6.05 \ -3.07 \ -1.297 \ -0.3726]$$

$$-0.0087 \ 0.052 \ 0.015 \ 0 \ 0 \ 0.017]$$



Prob 2 : Design a filter with a frequency response:

$$H_d(e^{j\omega}) = e^{-j3\omega} \quad \text{for } -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4}$$

$$= 0 \quad \frac{\pi}{4} < |\omega| \leq \pi$$

using a Hanning window with $M = 7$

Soln:

The freq resp is having a term $e^{-j\omega(M-1)/2}$ which gives $h(n)$ symmetrical about $n = M-1/2 = 3$ i.e we get a causal sequence.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)}$$

$$\text{this gives } h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The Hanning window function values are given by

$$w_{hn}(0) = w_{hn}(6) = 0$$

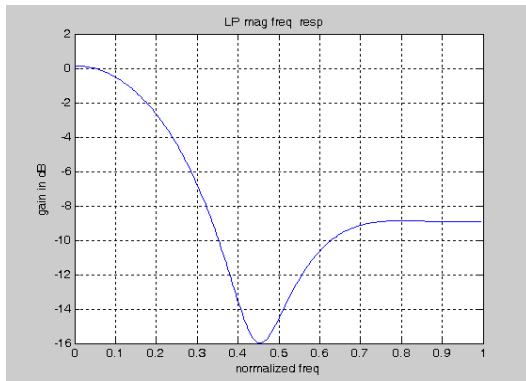
$$w_{hn}(1) = w_{hn}(5) = 0.25$$

$$w_{hn}(2) = w_{hn}(4) = 0.75$$

$$w_{hn}(3) = 1$$

$$h(n) = h_d(n) w_{hn}(n)$$

$$h(n) = [0 \ 0.03975 \ 0.165 \ 0.25 \ 0.165 \ 0.3975 \ 0]$$



7.9 Design of Linear Phase FIR filters using Frequency Sampling method

7.9.1 Motivation: We know that DFT of a finite duration DT sequence is obtained by sampling FT of the sequence then DFT samples can be used in reconstructing original time domain samples if frequency domain sampling was done correctly. The samples of FT of $h(n)$ i.e., $H(k)$ are sufficient to recover $h(n)$.

Since the designed filter has to be realizable then $h(n)$ has to be real, hence even symmetry properties for mag response $|H(k)|$ and odd symmetry properties for phase response can be applied. Also, symmetry for $h(n)$ is applied to obtain linear phase chas.

Fro DFT relationship we have

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j2\pi kn/N} \quad \text{for } n = 0, 1, \dots, N-1$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N} \quad \text{for } k = 0, 1, \dots, N-1$$

Also we know $H(k) = H(z)|_{z=e^{j2\pi kn/N}}$

The system function $H(z)$ is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Substituting for $h(n)$ from IDFT relationship

$$H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j2\pi kn/N} z^{-1}}$$

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Since $H(k)$ is obtained by sampling $H(e^{j\omega})$ hence the method is called Frequency Sampling Technique.

Since the impulse response samples or coefficients of the filter has to be real for filter to be realizable with simple arithmetic operations, properties of DFT of real sequence can be used. The following properties of DFT for real sequences are useful:

$$H^*(k) = H(N-k)$$

$|H(k)| = |H(N-k)|$ - magnitude response is even

$\theta(k) = -\theta(N-k)$ – Phase response is odd

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j2\pi kn/N} \text{ can be rewritten as (for N odd)}$$

$$\begin{aligned} h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{N-1} H(k) e^{j2\pi kn/N} \right] \\ h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{N-1/2} H(k) e^{j2\pi kn/N} + \sum_{k=N-1/2}^{N-1} H(k) e^{j2\pi kn/N} \right] \end{aligned}$$

Using substitution $k = N - r$ or $r = N - k$ in the second substitution with r going from now $(N-1)/2$ to 1 as k goes from 1 to $(N-1)/2$

$$\begin{aligned} h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H(N-k) e^{-j2\pi kn/N} \right] \\ h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H^*(k) e^{-j2\pi kn/N} \right] \\ h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} (H(k) e^{j2\pi kn/N})^* \right] \\ h(n) &= \frac{1}{N} \left[H(0) + \sum_{k=1}^{(N-1)/2} (H(k) e^{j2\pi kn/N} + (H(k) e^{j2\pi kn/N})^*) \right] \\ h(n) &= \frac{1}{N} \left[H(0) + 2 \sum_{k=1}^{(N-1)/2} \operatorname{Re}(H(k) e^{j2\pi kn/N}) \right] \end{aligned}$$

Similarly for N even we have

$$h(n) = \frac{1}{N} \left[H(0) + 2 \sum_{k=1}^{(N-1)/2} \operatorname{Re}(H(k) e^{j2\pi kn/N}) \right]$$

Using the symmetry property $h(n) = h(N-1-n)$ we can obtain Linear phase FIR filters using the frequency sampling technique.

Exercise problems

Prob 1 : Design a LP FIR filter using Freq sampling technique having cutoff freq of $\pi/2$ rad/sample. The filter should have linear phase and length of 17.

The desired response can be expressed as

$$\begin{aligned} H_d(e^{j\omega}) &= e^{-j\omega(\frac{M-1}{2})} \quad \text{for } |\omega| \leq \omega_c \\ &= 0 \quad \text{otherwise} \end{aligned}$$

with $M = 17$ and $\omega_c = \pi/2$

$$\begin{aligned} H_d(e^{j\omega}) &= e^{-j\omega 8} \quad \text{for } 0 \leq \omega \leq \pi/2 \\ &= 0 \quad \text{for } \pi/2 \leq \omega \leq \pi \end{aligned}$$

Selecting $\omega_k = \frac{2\pi k}{M} = \frac{2\pi k}{17}$ for $k = 0, 1, \dots, 16$

$$\begin{aligned} H(k) &= H_d(e^{j\omega}) \Big|_{\omega=\frac{2\pi k}{17}} \\ H(k) &= e^{-j\frac{2\pi k}{17} 8} \quad \text{for } 0 \leq \frac{2\pi k}{17} \leq \frac{\pi}{2} \\ &= 0 \quad \text{for } \frac{\pi}{2} \leq \frac{2\pi k}{17} \leq \pi \\ H(k) &= e^{-j\frac{16\pi k}{17}} \quad \text{for } 0 \leq k \leq \frac{17}{4} \\ &= 0 \quad \text{for } \frac{17}{4} \leq k \leq \frac{17}{2} \end{aligned}$$

The range for "k" can be adjusted to be an integer such as

$$0 \leq k \leq 4$$

$$\text{and } 5 \leq k \leq 8$$

The freq response is given by

$$H(k) = e^{-j\frac{2\pi k}{17}8} \quad \text{for } 0 \leq k \leq 4$$

$$= 0 \quad \text{for } 5 \leq k \leq 8$$

Using these value of $H(k)$ we obtain $h(n)$ from the equation

$$h(n) = \frac{1}{M} (H(0) + 2 \sum_{k=1}^{(M-1)/2} \operatorname{Re}(H(k) e^{j2\pi kn/M}))$$

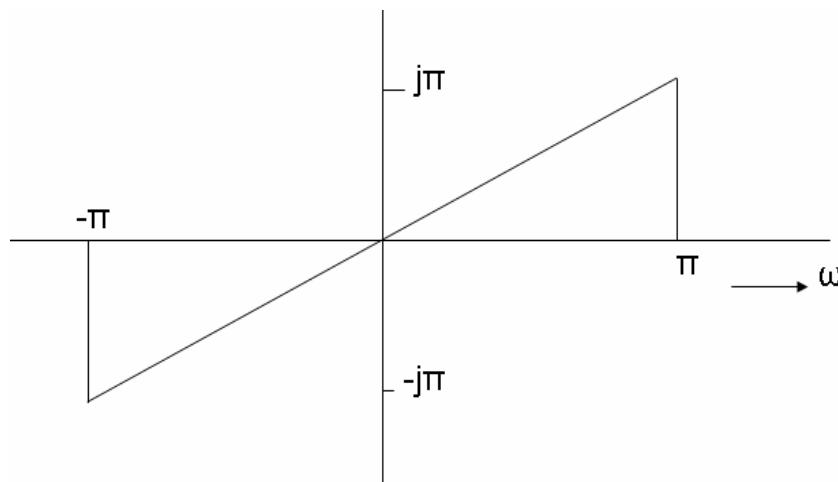
i.e., $h(n) = \frac{1}{17} (1 + 2 \sum_{k=1}^4 \operatorname{Re}(e^{-j16\pi k/17} e^{j2\pi kn/17}))$

$$h(n) = \frac{1}{17} (H(0) + 2 \sum_{k=1}^4 \cos(\frac{2\pi k(8-n)}{17})) \quad \text{for } n = 0, 1, \dots, 16$$

- Even though k varies from 0 to 16 since we considered ω varying between 0 and $\pi/2$ only k values from 0 to 8 are considered
- While finding $h(n)$ we observe symmetry in $h(n)$ such that n varying 0 to 7 and 9 to 16 have same set of $h(n)$

7.10 Design of FIR Differentiator

Differentiators are widely used in Digital and Analog systems whenever a derivative of the signal is needed. Ideal differentiator has pure linear magnitude response in the freq range $-\pi$ to $+\pi$. The typical frequency response characteristics is as shown in the below figure.



Problem 2: Design an Ideal Differentiator using a) rectangular window and b)Hamming window with length of the system = 7.

Solution:

As seen from differentiator frequency chars. It is defined as

$$H(e^{j\omega}) = j\omega \quad \text{between } -\pi \text{ to } +\pi$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = \frac{\cos \pi n}{n} \quad -\infty \leq n \leq \infty \quad \text{and} \quad n \neq 0$$

The $h_d(n)$ is an odd function with $h_d(n) = -h_d(-n)$ and $h_d(0) = 0$

a) rectangular window

$$h(n) = h_d(n)w_r(n)$$

$$h(1) = -h(-1) = h_d(1) = -1$$

$$h(2) = -h(-2) = h_d(2) = 0.5$$

$$h(3) = -h(-3) = h_d(3) = -0.33$$

$h'(n) = h(n-3)$ for causal system
thus,

$$H'(z) = 0.33 - 0.5z^{-1} + z^{-2} - z^{-4} + 0.5z^{-5} - 0.33z^{-6}$$

Also from the equation

$$H_r(e^{j\omega}) = 2 \sum_{n=0}^{(M-3)/2} h(n) \sin \omega \left(\frac{M-1}{2} - n \right)$$

For $M=7$ and $h'(n)$ as found above we obtain this as

$$H_r(e^{j\omega}) = 0.66 \sin 3\omega - \sin 2\omega + 2 \sin \omega$$

$$H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.66 \sin 3\omega - \sin 2\omega + 2 \sin \omega)$$

b) Hamming window

$$h(n) = h_d(n)w_h(n)$$

where $w_h(n)$ is given by

$$w_h(n) = 0.54 + 0.46 \cos \frac{2\pi n}{(M-1)} \quad -(M-1)/2 \leq n \leq (M-1)/2 \\ = 0 \quad \text{otherwise}$$

For the present problem

$$w_h(n) = 0.54 + 0.46 \cos \frac{\pi n}{3} \quad -3 \leq n \leq 3$$

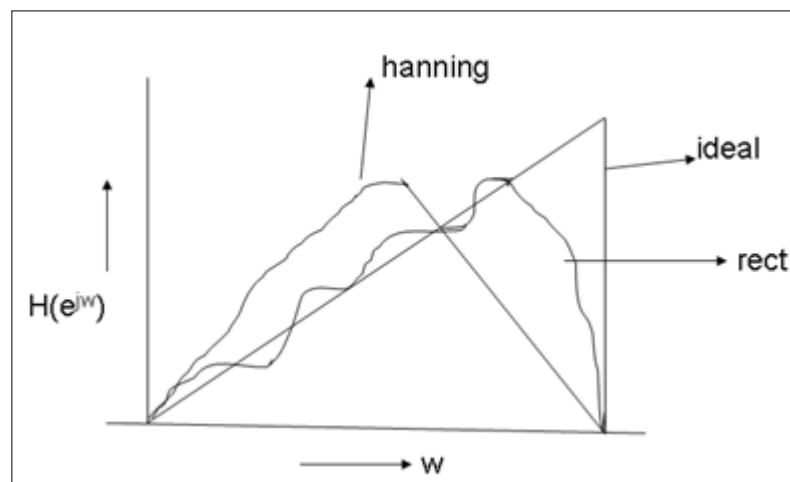
The window function coefficients are given by for n=-3 to +3

$$Wh(n) = [0.08 \ 0.31 \ 0.77 \ 1 \ 0.77 \ 0.31 \ 0.08]$$

$$\text{Thus } h'(n) = h(n-5) = [0.0267, -0.155, 0.77, 0, -0.77, 0.155, -0.0267]$$

Similar to the earlier case of rectangular window we can write the freq response of differentiator as

$$H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.0534 \sin 3\omega - 0.31 \sin 2\omega + 1.54 \sin \omega)$$



We observe

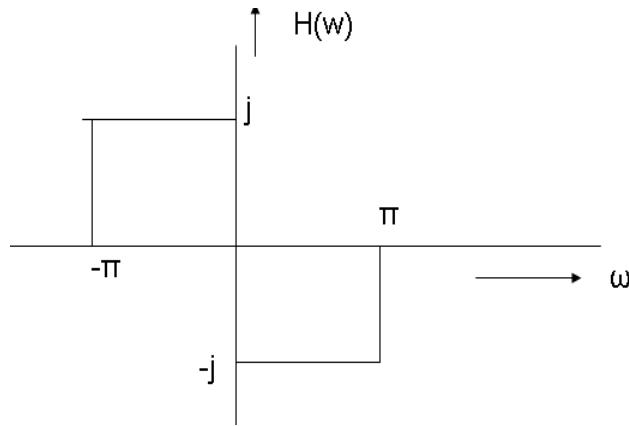
- With rectangular window, the effect of ripple is more and transition band width is small compared with hamming window
- With hamming window, effect of ripple is less whereas transition band is more

7.11 Design of FIR Hilbert transformer:

Hilbert transformers are used to obtain phase shift of 90 degree. They are also called j operators. They are typically required in quadrature signal processing. The Hilbert transformer

is very useful when out of phase component (or imaginary part) need to be generated from available real component of the signal.

Problem 3: Design an ideal Hilbert transformer using a) rectangular window and b) Blackman Window with M = 11



Solution:

As seen from freq chars it is defined as

$$\begin{aligned} H_d(e^{j\omega}) &= j \quad -\pi \leq \omega \leq 0 \\ &= -j \quad 0 \leq \omega \leq \pi \end{aligned}$$

The impulse response is given by

$$h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^0 j e^{j\omega n} d\omega + \int_0^\pi -j e^{j\omega n} d\omega \right] = \frac{(1 - \cos \pi n)}{\pi n} \quad -\infty \leq n \leq \infty \quad \text{except } n = 0$$

At $n = 0$ it is $h_d(0) = 0$ and $h_d(n)$ is an odd function

a) Rectangular window

$$h(n) = h_d(n) \quad w_r(n) = h_d(n) \text{ for } -5 \leq n \leq 5$$

$$h'(n) = h(n-5)$$

$$h(n) = [-0.127, 0, -0.212, 0, -0.636, 0, 0.636, 0, 0.212, 0, 0.127]$$

$$H_r(e^{j\omega}) = 2 \sum_{n=0}^4 h(n) \sin \omega(5-n)$$

$$H(e^{j\omega}) = j |H_r(e^{j\omega})| = j\{0.254 \sin 5\omega + 0.424 \sin 3\omega + 1.272 \sin \omega\}$$

b) Blackman Window

window function is defined as

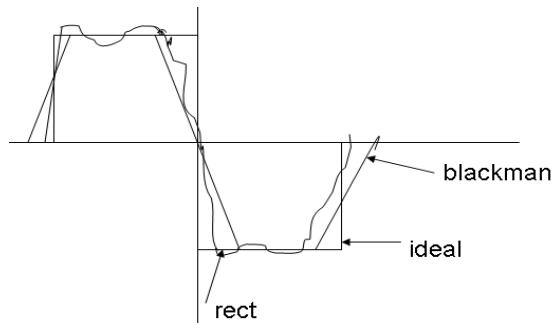
$$w_b(n) = 0.42 + 0.5 \cos \frac{\pi n}{5} + 0.08 \cos \frac{2\pi n}{5} \quad -5 \leq n \leq 5$$

$$= 0 \quad \text{otherwise}$$

$$W_b(n) = [0, 0.04, 0.2, 0.509, 0.849, 1, 0.849, 0.509, 0.2, 0.04, 0] \text{ for } -5 \geq n \geq 5$$

$$h'(n) = h(n-5) = [0, 0, -0.0424, 0, -0.5405, 0, 0.5405, 0, 0.0424, 0, 0]$$

$$H(e^{j\omega}) = -j[0.0848 \sin 3\omega + 1.0810 \sin \omega]$$



Recommended questions with solution

Question1

Design an FIR linear phase, digital filter approximating the ideal frequency response

$$H_d(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{\pi}{6} \\ 0, & \text{for } \frac{\pi}{6} < |\omega| \leq \pi \end{cases}$$

- (a) Determine the coefficients of a 25-tap filter based on the window method with a rectangular window.
- (b) Determine and plot the magnitude and phase response of the filter.
- (c) Repeat parts (a) and (b) using the Hamming window.
- (d) Repeat parts (a) and (b) using a Bartlett window.

Solution:-

(a) To obtain the desired length of 25, a delay of $\frac{25-1}{2} = 12$ is incorporated into $H_d(w)$. Hence,

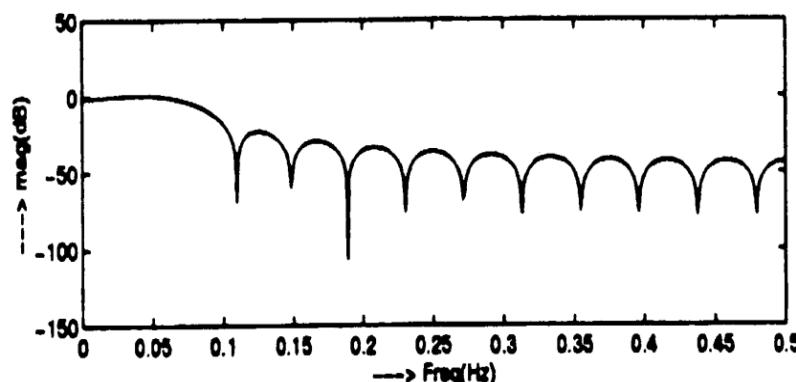
$$\begin{aligned} H_d(w) &= 1e^{-j12w}, \quad 0 \leq |w| \leq \frac{\pi}{6} \\ &= 0, \quad \text{otherwise} \end{aligned}$$

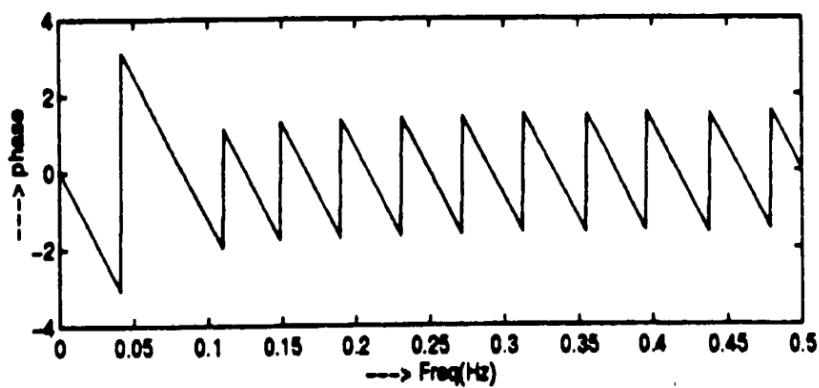
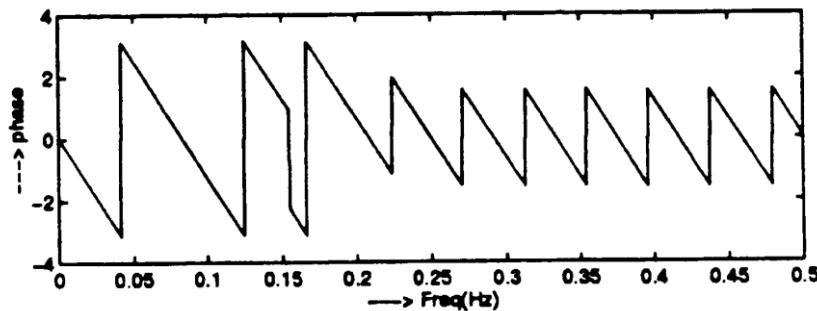
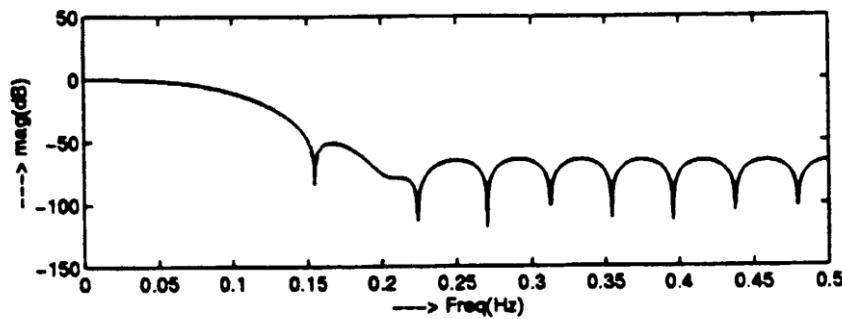
$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} H_d(w)e^{-jwn} dw \\ &= \frac{\sin \frac{\pi}{6}(n-12)}{\pi(n-12)} \end{aligned}$$

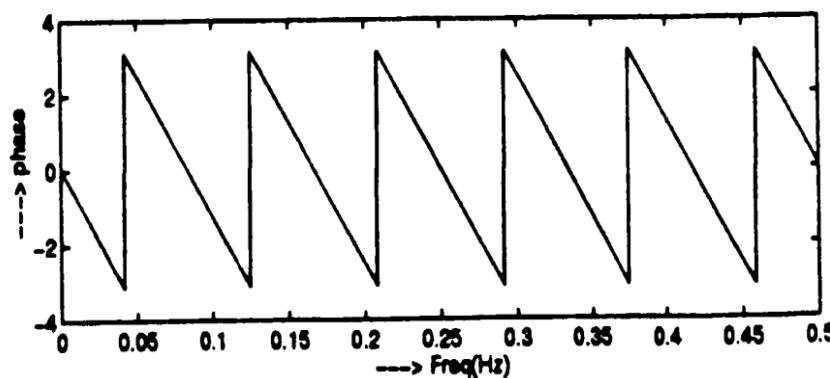
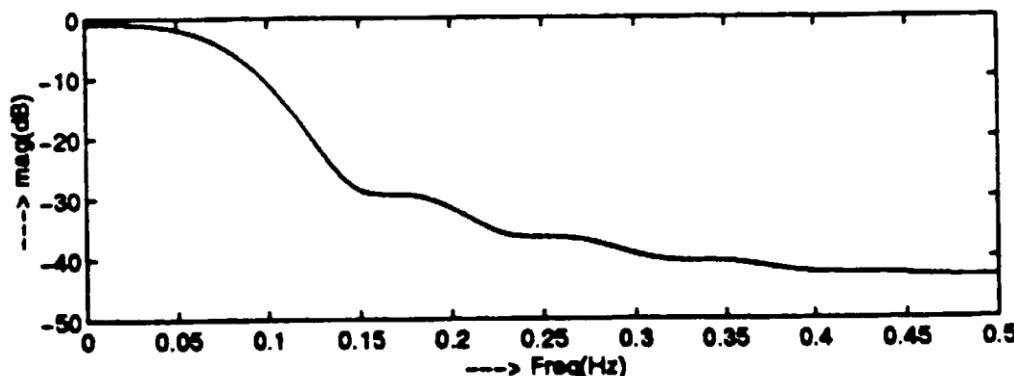
$$\text{Then, } h(n) = h_d(n)w(n)$$

where $w(n)$ is a rectangular window of length $N = 25$.

(b) Magnitude plot



Phase plot**(c) Hamming window****(d) Bartlett window**



Question 2

Determine the unit sample response $\{h(n)\}$ of a linear-phase FIR filter of length $M = 4$ for which the frequency response at $\omega = 0$ and $\omega = \pi/2$ is specified as

$$H_r(0) = 1 \quad H_r\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

Solution:-

$$\begin{aligned} M &= 4, \quad H_r(0) = 1, \quad H_r\left(\frac{\pi}{2}\right) = \frac{1}{2} \\ H_r(\omega) &= 2 \sum_{n=0}^{M-1} h(n) \cos\left[\omega\left(\frac{M-1}{2} - n\right)\right] \\ &= 2 \sum_{n=0}^1 h(n) \cos\left[\omega\left(\frac{3}{2} - n\right)\right] \end{aligned}$$

$$\begin{aligned} \text{At } \omega = 0, H_r(0) = 1 &= 2 \sum_{n=0}^1 h(n) \cos[0] \\ 2[h(0) + h(1)] &= 1 \quad (1) \end{aligned}$$

$$\text{At } w = \frac{\pi}{2}, H_r\left(\frac{\pi}{2}\right) = \frac{1}{2} = 2 \sum_{n=0}^1 h(n) \cos\left[\frac{\pi}{2}\left(\frac{3}{2} - n\right)\right]$$

$$-h(0) + h(1) = 0.354 \quad (2)$$

Solving (1) and (2), we get

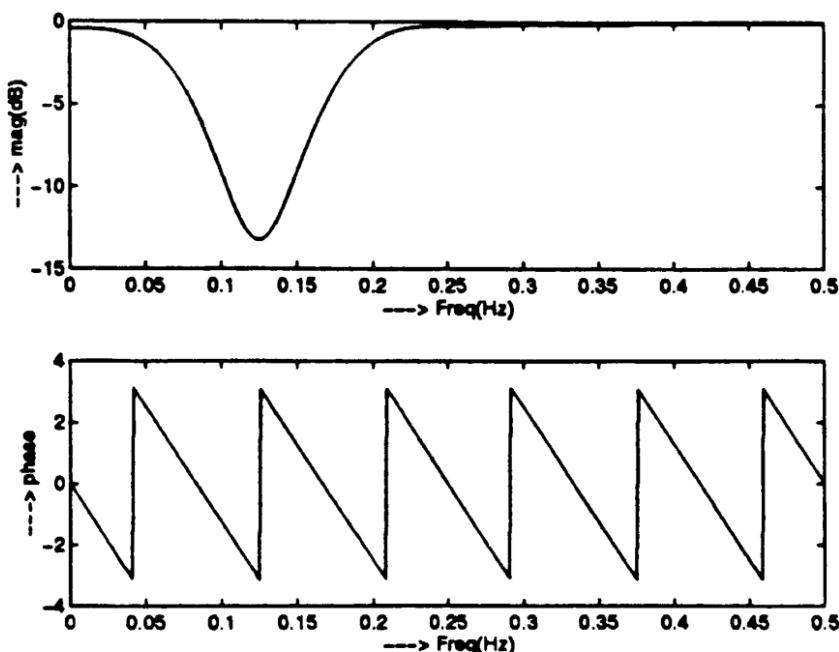
$$h(0) = 0.073 \text{ and}$$

$$h(1) = 0.427$$

$$h(2) = h(1)$$

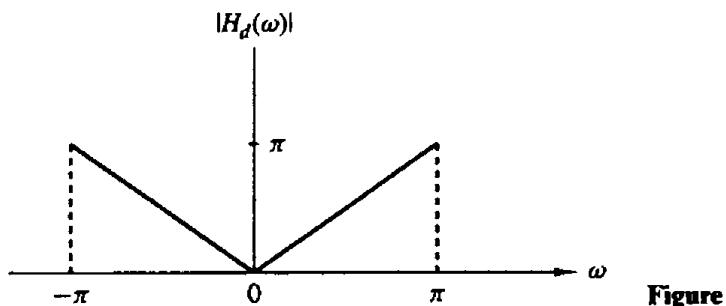
$$h(3) = h(0)$$

$$\text{Hence, } h(n) = \{0.073, 0.427, 0.427, 0.073\}$$

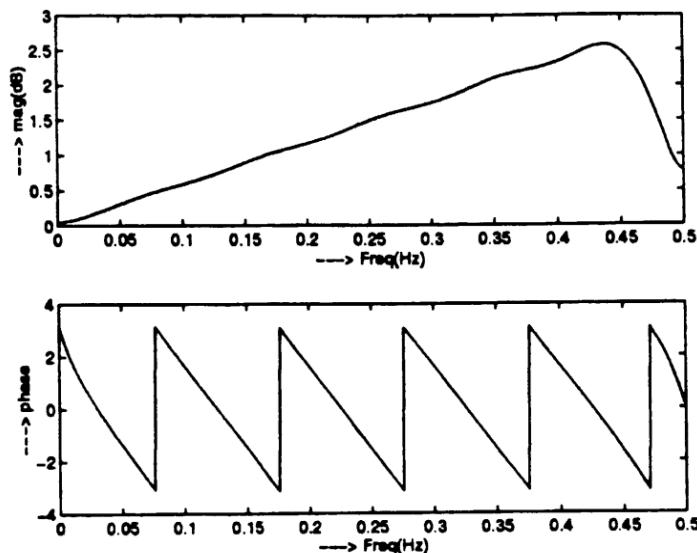


Question 3

Use the window method with a Hamming window to design a 21-tap differentiator as shown in Fig. P8.9. Compute and plot the magnitude and phase response of the resulting filter.

**Figure P8.9****Solution:-**

$$\begin{aligned}
 H_d(\omega) &= we^{-j10\omega}, \quad 0 \leq \omega \leq \pi \\
 &= -we^{-j10\omega}, \quad -\pi \leq \omega \leq 0 \\
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{-j\omega n} d\omega \\
 &= \frac{\cos\pi(n-10)}{(n-10)}, \quad n \neq 10 \\
 &= 0, \quad n = 10 \\
 h_d(n) &= \frac{\cos\pi(n-10)}{(n-10)}, \quad 0 \leq n \leq 20, n \neq 10 \\
 &= 0, \quad n = 10
 \end{aligned}$$

Magnitude and phase response

Question 4

A digital low-pass filter is required to meet the following specifications:

Passband ripple: ≤ 1 dB

Passband edge: 4 kHz

Stopband attenuation: ≥ 40 dB

Stopband edge: 6 kHz

Sample rate: 24 kHz

The filter is to be designed by performing a bilinear transformation on an analog system function. Determine what order Butterworth, Chebyshev, and elliptic analog designs must be used to meet the specifications in the digital implementation.

Solu

tion:-

From the design specifications we obtain

$$\epsilon = 0.509$$

$$\delta = 99.995$$

$$f_p = \frac{4}{24} = \frac{1}{6}$$

$$f_s = \frac{6}{24} = \frac{1}{4}$$

$$\text{Assume } t = 1. \text{ Then, } \Omega_p = 2\tan\frac{\omega_p}{2}$$

$$= 2\tan\pi f_p = 1.155$$

$$\text{and } \Omega_s = 2\tan\frac{\omega_s}{2}$$

$$= 2\tan\pi f_s = 2$$

$$\eta = \frac{\delta}{\epsilon} = 196.5$$

UNIT 8**Design of IIR Filters from Analog Filters****CONTENTS:-**

DESIGN OF IIR FILTERS FROM ANALOG FILTERS (BUTTERWORTH AND CHEBYSHEV) - IMPULSE INVARIANCE METHOD. MAPPING OF TRANSFER FUNCTIONS: APPROXIMATION OF DERIVATIVE (BACKWARD DIFFERENCE AND BILINEAR TRANSFORMATION) METHOD, MATCHED Z TRANSFORMS, VERIFICATION FOR STABILITY AND LINEARITY DURING MAPPING 7 HRS

RECOMMENDED READINGS:-

1. DIGITAL SIGNAL PROCESSING – PRINCIPLES ALGORITHMS & APPLICATIONS, PROAKIS & MONALAKIS, PEARSON EDUCATION, 4TH EDITION, NEW DELHI, 2007.
2. DISCRETE TIME SIGNAL PROCESSING, OPPENHEIM & SCHAFER, PHI, 2003.
3. DIGITAL SIGNAL PROCESSING, S. K. MITRA, TATA MC-GRAW HILL, 2ND EDITION, 2004.

UNIT - 8

DESIGN OF IIR FILTERS FROM ANALOG FILTERS
(BUTTERWORTH AND CHEBYSHEV)

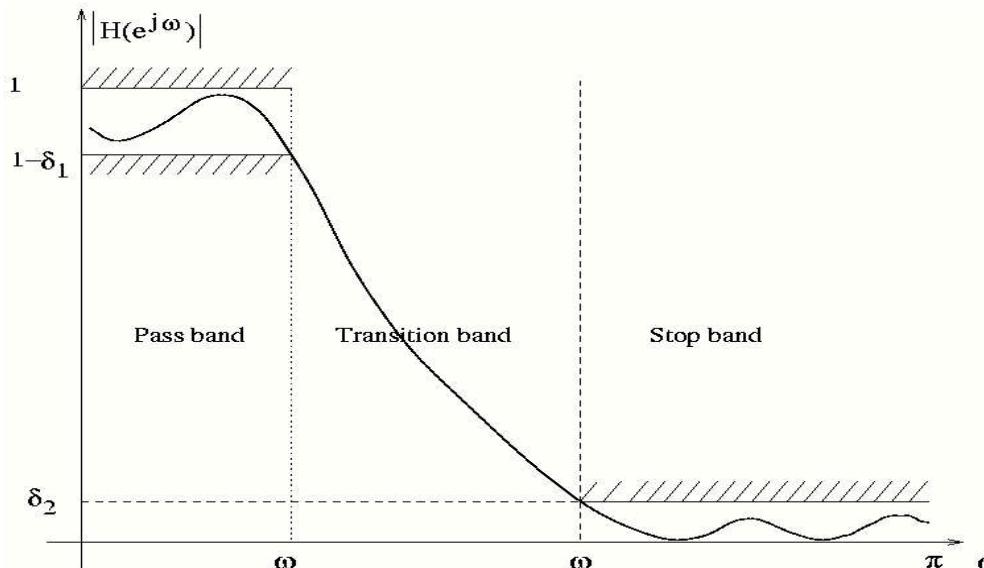
8.1 Introduction

A digital filter is a linear shift-invariant discrete-time system that is realized using finite precision arithmetic. The design of digital filters involves three basic steps:

- The specification of the desired properties of the system.
- The approximation of these specifications using a causal discrete-time system.
- The realization of these specifications using finite precision arithmetic.

These three steps are independent; here we focus our attention on the second step.

The desired digital filter is to be used to filter a digital signal that is derived from an analog signal by means of periodic sampling. The specifications for both analog and digital filters are often given in the frequency domain, as for example in the design of low pass, high pass, band pass and band elimination filters. Given the sampling rate, it is straight



forward to convert from frequency specifications on an analog filter to frequency specifications on the corresponding digital filter, the analog frequencies being in terms of Hertz and digital frequencies being in terms of radian frequency or angle around the unit circle with

the point $Z=-1$ corresponding to half the sampling frequency. The least confusing point of view toward digital filter design is to consider the filter as being specified in terms of angle around the unit circle rather than in terms of analog frequencies.

Figure 7.1: Tolerance limits for approximation of ideal low-pass filter

A separate problem is that of determining an appropriate set of specifications on the digital filter. In the case of a low pass filter, for example, the specifications often take the form of a tolerance scheme, as shown in Fig. 4.1

$$1 - \delta_1 \leq |H(e^{j\omega})| \leq 1, \quad |\omega| \leq \omega_p$$

$$|H(e^{j\omega})| \leq \delta_2, \quad \omega_s \leq |\omega| \leq \pi$$

Many of the filters used in practice are specified by such a tolerance scheme, with no constraints on the phase response other than those imposed by stability and causality requirements; i.e., the poles of the system function must lie inside the unit circle. Given a set of specifications in the form of Fig. 7.1, the next step is to find a discrete time linear system whose frequency response falls within the prescribed tolerances. At this point the filter design problem becomes a problem in approximation. In the case of infinite impulse response (IIR) filters, we must approximate the desired frequency response by a rational function, while in the finite impulse response (FIR) filters case we are concerned with polynomial approximation.

7.2 Design of IIR Filters from Analog Filters:

The traditional approach to the design of IIR digital filters involves the transformation of an analog filter into a digital filter meeting prescribed specifications. This is a reasonable approach because:

- The art of analog filter design is highly advanced and since useful results can be achieved, it is advantageous to utilize the design procedures already developed for analog filters.
- Many useful analog design methods have relatively simple closed-form design formulas.

Therefore, digital filter design methods based on analog design formulas are rather simple to implement.

An analog system can be described by the differential equation

$$\sum_{k=0}^N c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^M d_k \frac{d^k x_a(t)}{dt^k} \quad \text{-----7.1}$$

And the corresponding rational function is

$$H_a(s) = \frac{\sum_{k=0}^M d_k s^k}{\sum_{k=0}^N c_k s^k} = \frac{y_a(s)}{x_a(s)} \quad \text{-----7.2}$$

The corresponding description for digital filters has the form

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k) \quad \text{-----7.3}$$

and the rational function

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{Y(z)}{X(z)} \quad \text{-----7.4}$$

In transforming an analog filter to a digital filter we must therefore obtain either $H(z)$ or $h(n)$ (inverse Z-transform of $H(z)$ i.e., impulse response) from the analog filter design. In such transformations, we want the imaginary axis of the S-plane to map into the finite circle of the Z-plane, a stable analog filter should be transformed to a stable digital filter. That is, if the analog filter has poles only in the left-half of S-plane, then the digital filter must have poles only inside the unit circle. These constraints are basic to all the techniques discussed

7.3 IIR Filter Design by Impulse Invariance:

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This technique of transforming an analog filter design to a digital filter design corresponds to choosing the unit-sample response of the digital filter as equally spaced samples of the impulse response of the analog filter. That is,

$$h(n) = h_a(nT) \quad \text{-----7.5}$$

Where T is the sampling period. Because of uniform sampling, we have

$$H(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(j\Omega + j\frac{2\pi}{T}k) \quad \text{-----7.6}$$

Or

$$H(z) |_{z=e^{j\omega T}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(s + j\frac{2\pi}{T}k) \quad \text{-----7.7}$$

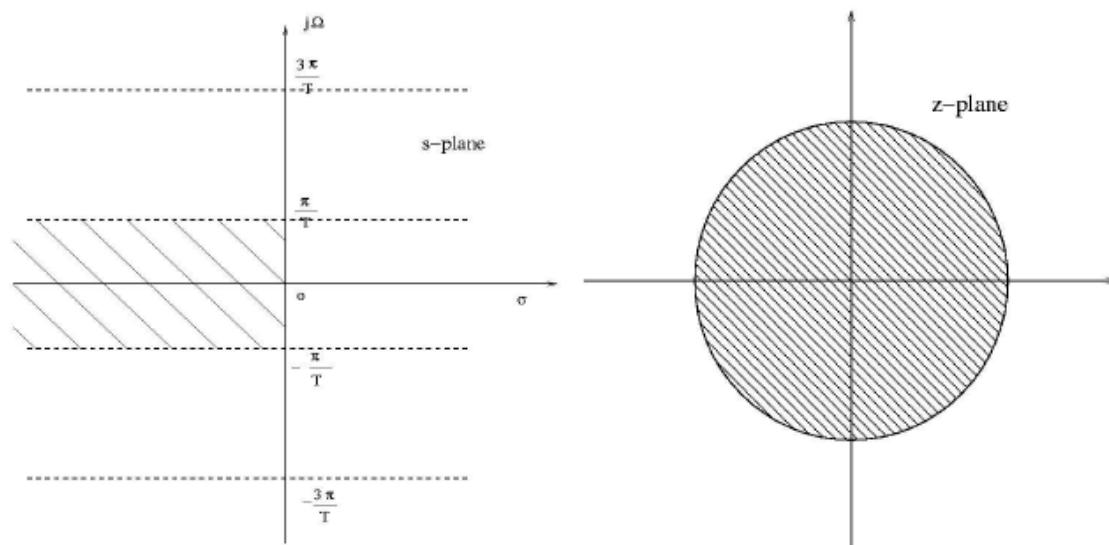


Figure 7.2: Mapping of s-plane into z-plane

Where $s = j\omega$ and $\Omega = \omega/T$, is the frequency in analog domain and ω is the frequency in digital domain.

From the relationship $Z = e^{sT}$ it is seen that strips of width $2\pi/T$ in the S-plane map into the entire Z-plane as shown in Fig. 7.2. The left half of each S-plane strip maps into interior of the unit circle, the right half of each S-plane strip maps into the exterior of the unit circle, and the imaginary axis of length $2\pi/T$ of S-plane maps on to once round the unit circle of Z-plane. Each horizontal strip of the S-plane is overlaid onto the Z-plane to form the digital filter function from analog filter function. The frequency response of the digital filter is related to the frequency response of the

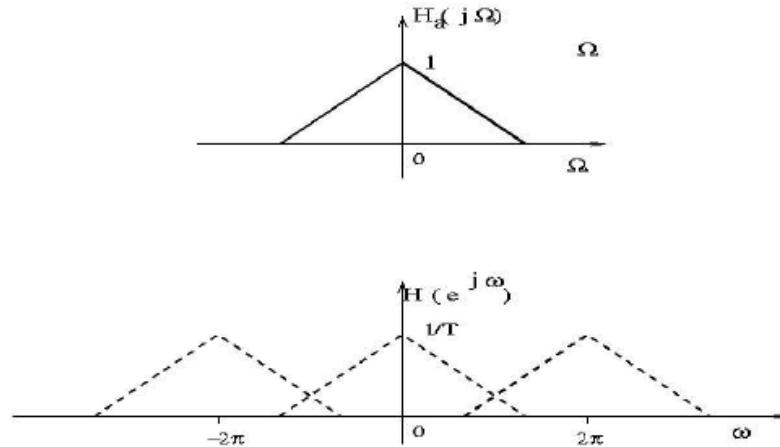


Figure 7.3: Illustration of the effects of aliasing in the impulse invariance technique

analog filter as

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(j\frac{\omega}{T} + j\frac{2\pi}{T}k) \quad \text{-----7.8}$$

From the discussion of the sampling theorem it is clear that if and only if

$$H_a(j\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T}$$

Then

$$H(e^{j\omega}) = \frac{1}{T} H_a(j\frac{\omega}{T}), \quad |\omega| \leq \pi$$

Unfortunately, any practical analog filter will not be band limited, and consequently there is interference between successive terms in Eq. (7.8) as illustrated in Fig. 7.3. Because of the aliasing that occurs in the sampling process, the frequency response of the resulting digital filter will not be identical to the original analog frequency response. To get the filter design procedure, let us consider the system function of the analog filter expressed in terms of a partial-fraction expansion

$$H_a(s) = \sum_{k=1}^N \frac{A_k}{s - s_k} \quad \text{-----7.9}$$

The corresponding impulse response is

$$h_a(t) = \sum_{k=1}^N A_k e^{s_k t} U(t) \quad \dots \quad 7.10$$

And the unit-sample response of the digital filter is then

$$h(n) = h_a(nT) = \sum_{k=1}^N A_k e^{s_k nT} u(n) = \sum_{k=1}^N A_k (e^{s_k T})^n U(n) \quad \dots \quad 7.11$$

The system function of the digital filter $H(z)$ is given by

$$H(z) = \sum_{k=1}^N \frac{A_k}{(1 - \exp^{s_k T} z^{-1})} \quad \dots \quad 7.12$$

In comparing Eqs. (7.9) and (7.12) we observe that a pole at $s=s_k$ in the S-plane transforms to a pole at $\exp^{s_k T}$ in the Z-plane. It is important to recognize that the impulse invariant design procedure does not correspond to a mapping of the S-plane to the Z-plane.

8.4 IIR Filter Design By Approximation Of Derivatives:

A second approach to design of a digital filter is to approximate the derivatives in Eq. (4.1) by finite differences. If the samples are closer together, the approximation to the derivative would be increasingly accurate. For example, suppose that the first derivative is approximated by the first backward difference

$$\frac{dy_a(t)}{dt} \Big|_{t=nT} \longrightarrow \nabla^{(1)}[y(n)] = \frac{y(n) - y(n-1)}{T} \quad \dots \quad 7.13$$

Where $y(n)=y(nT)$. Approximation to higher-order derivatives are obtained by repeated application of Eq. (7.13); i.e.,

$$\frac{d^k y_a(t)}{dt^k} \Big|_{t=nT} = \frac{d}{dt} \left(\frac{d^{k-1} y_a(t)}{dt^{k-1}} \right) \Big|_{t=nT} \longrightarrow \nabla^{(k)}[y(n)] = \nabla^{(1)}[\nabla^{(k-1)}[y(n)]] \quad \dots \quad 7.14$$

For convenience we define

$$\nabla^{(0)}[y(n)] = y(n) \quad \dots \quad 7.15$$

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Applying Eqs. (7.13), (7.14) and (7.15) to (7.1), we obtain

$$\sum_{k=0}^N c_k \nabla^{(k)}[y(n)] = \sum_{k=0}^M d_k \nabla^{(k)}[x(n)] \quad \text{-----7.16}$$

Where $y(n) = ya(nT)$ and $x(n) = xa(nT)$. We note that the operation $\Delta^{(1)}[\]$ is a linear shift-invariant operator and that $\Delta^{(k)}[\]$ can be viewed as a cascade of (k) operators $\Delta^{(1)}[\]$. In particular

$$Z[\nabla^{(1)}[x(n)]] = \left[\frac{1 - z^{-1}}{T} \right] X(z)$$

And

$$Z[\nabla^{(1)}[x(n)]] = \left[\frac{1 - z^{-1}}{T} \right]^k X(z)$$

Thus taking the Z-transform of each side in Eq. (7.16), we obtain

$$H(z) = \frac{\sum_{k=0}^M d_k \left[\frac{1-z^{-1}}{T} \right]^k}{\sum_{k=0}^N N c_k \left[\frac{1-z^{-1}}{T} \right]^k} \quad \text{-----7.17}$$

Comparing Eq. (7.17) to (7.2), we observe that the digital transfer function can be obtained directly from the analog transfer function by means of a substitution of variables

$$s = \frac{1 - z^{-1}}{T} \quad \text{-----7.18}$$

So that, this technique does indeed truly correspond to a mapping of the S-plane to the Z-plane, according to Eq. (7.18). To investigate the properties of this mapping, we must express z as a function of s , obtaining

$$z = \frac{1}{1 - sT}$$

Substituting $s = j\Omega$, i.e., imaginary axis in S-plane

$$\begin{aligned}
 z &= \frac{1}{1 - j\Omega T} \\
 &= \frac{1}{1 - j\Omega T} + \frac{1}{2} - \frac{1}{2} \\
 &= \frac{1}{2} + \frac{1}{2} \left[\frac{1 + j\Omega T}{1 - j\Omega T} \right] \\
 &= \frac{1}{2} \left[1 + \frac{1 + j\Omega T}{1 - j\Omega T} \right] \\
 &= \frac{1}{2} \left[1 + e^{j2\tan^{-1}(\Omega T)} \right]
 \end{aligned}$$

-----7.19

Which corresponds to a circle whose center is at $z = 1/2$ and radius is $1/2$, as shown in Fig. 7.4. It is easily verified that the left half of the S-plane maps into the inside of the small circle and the right half of the S-plane maps onto the outside of the small circle. Therefore, although the requirement of mapping the $j\Omega$ -axis to the unit circle is not satisfied, this mapping does satisfy the stability condition.

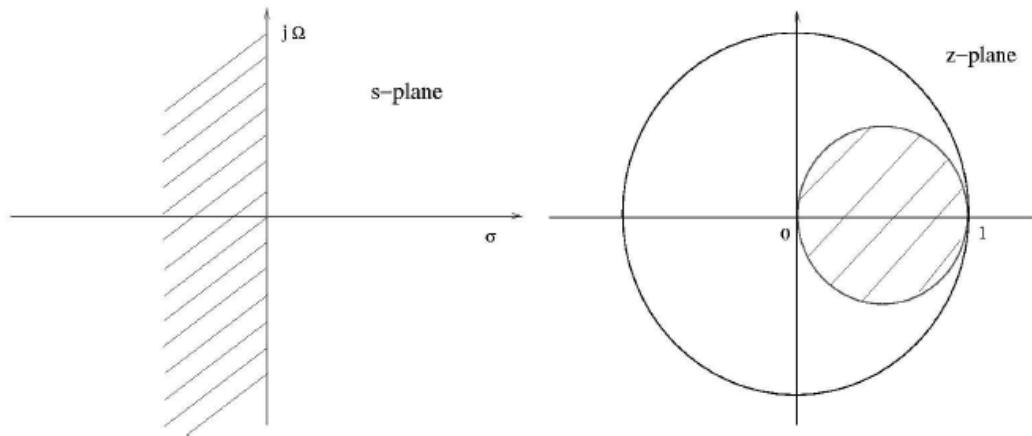


Figure 4.4: Mapping of s-plane to z-plane corresponding to first backward-difference approximation to the derivative

In contrast to the impulse invariance technique, decreasing the sampling period T , theoretically produces a better filter since the spectrum tends to be concentrated in a very small region of the unit circle. These two procedures are highly unsatisfactory for anything but low pass filters. An alternative approximation to the derivative is a forward difference and it provides a mapping into the unstable digital filters.

8.5 IIR Filter Design By The Bilinear Transformation:

Digital Signal Processing

10EC52

In the previous section a digital filter was derived by approximating derivatives by differences. An alternative procedure is based on integrating the differential equation and then using a numerical approximation to the integral. Consider the first - order equation

$$c_1 y'_a(t) + c_0 y_a(t) = d_0 x_a(t) \quad \dots \quad 7.20$$

Where $y'_a(t)$ is the first derivative of $y_a(t)$. The corresponding analog system function is

$$H_a(s) = \frac{d_0}{c_0 + c_1 s}$$

We can write $y_a(t)$ as an integral of $y'_a(t)$, as in

$$y_a(t) = \int_{t_0}^t y'_a(\tau) d\tau + y_a(t_0)$$

In particular, if $t = nT$ and $t_0 = (n - 1)T$,

$$y_a(nT) = \int_{(n-1)T}^{nT} y'_a(\tau) d\tau + y_a((n-1)T)$$

If this integral is approximated by a trapezoidal rule, we can write

$$y_a(nT) = y_a((n-1)T) + \frac{T}{2} [y'_a(nT) + y'_a((n-1)T)] \quad \dots \quad 7.21$$

However, from Eq. (7.20),

$$y'_a(nT) = -\frac{c_0}{c_1} y_a(nT) + \frac{d_0}{c_1} x_a(nT)$$

Substituting into Eq. (4.21) we obtain

$$[y(n) - y(n-1)] = \frac{T}{2} \left[-\frac{c_0}{c_1} (y(n) + y(n-1)) + \frac{d_0}{c_1} (x(n) + x(n-1)) \right]$$

Where $y(n) = y(nT)$ and $x(n) = x(nT)$. Taking the Z-transform and solving for $H(z)$ gives

$$H(z) = \frac{Y(z)}{X(z)} = \frac{d_0}{c_0 + c_1 \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} \quad \dots \quad 7.22$$

From Eq. (7.22) it is clear that $H(z)$ is obtained from $H_a(s)$ by the substitution

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{-----7.23}$$

That is,

$$H(z) = H_a(s) \Big|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} \quad \text{-----7.24}$$

This can be shown to hold in general since an N^{th} - order differential equation of the form of Eq. (7.1) can be written as a set of N first-order equations of the form of Eq. (7.20). Solving Eq. (7.23) for z gives

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} \quad \text{-----7.25}$$

The invertible transformation of Eq. (7.23) is recognized as a bilinear transformation. To see that this mapping has the property that the imaginary axis in the s -plane maps onto the unit circle in the z -plane, consider $z = e^{j\omega}$, then from Eq. (7.23), s is given by

$$\begin{aligned} s &= \frac{2}{T} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \\ &= \frac{2}{T} \frac{j \sin(\omega/2)}{\cos(\omega/2)} \\ &= \frac{2}{T} j \tan(\omega/2) \\ &= \sigma + j\Omega \end{aligned}$$

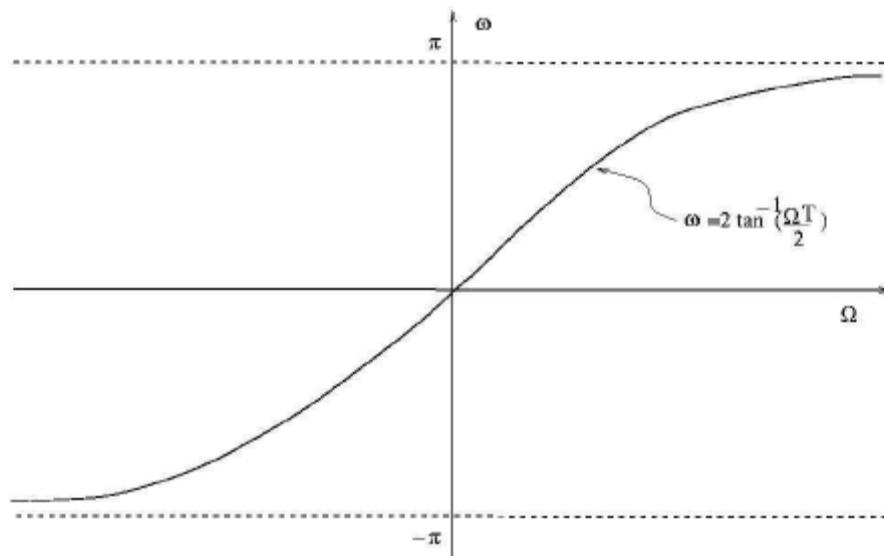


Figure 7.5: Mapping of analog frequency axis onto the unit circle using the bilinear Transformation

Thus for z on the unit circle, $\sigma = 0$ and Ω and ω are related by

$$\begin{aligned} T \Omega/2 &= \tan(\omega/2) \\ \text{or} \\ \omega &= 2 \tan^{-1}(T \Omega/2) \end{aligned}$$

This relationship is plotted in Fig. (7.5), and it is referred as frequency warping. From the figure it is clear that the positive and negative imaginary axis of the s-plane are mapped, respectively, into the upper and lower halves of the unit circle in the z-plane. In addition to the fact that the imaginary axis in the s-plane maps into the unit circle in the z-plane, the left half of the s-plane maps to the inside of the unit circle and the right half of the s-plane maps to the outside of the unit circle, as shown in Fig. (7.6). Thus we see that the use of the bilinear transformation yields stable digital filter from analog filter. Also this transformation avoids the problem of aliasing encountered with the use of impulse invariance, because it maps the entire imaginary axis in the s-plane onto the unit circle in the z-plane. The price paid for this, however, is the introduction of a distortion in the frequency axis.

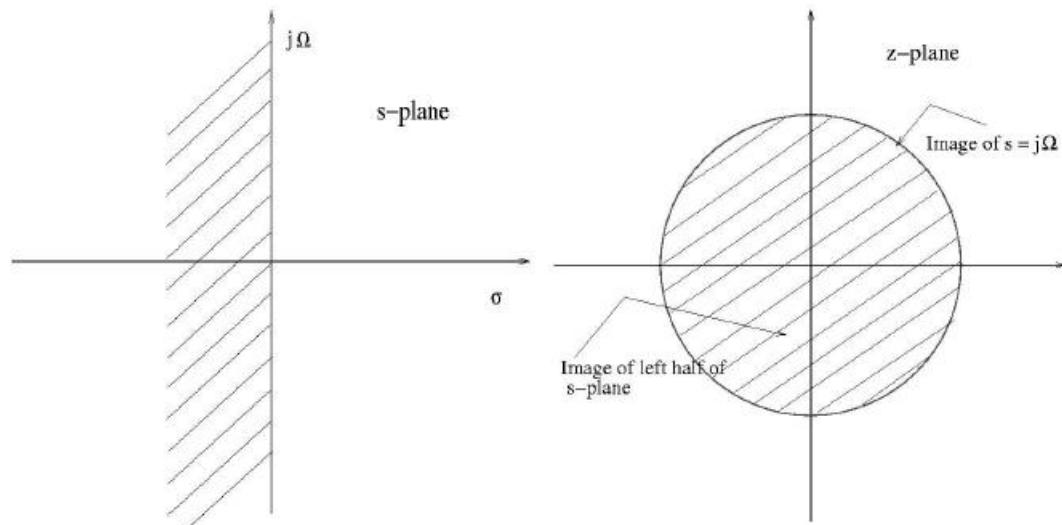


Figure 4.6: Mapping of the s-plane into the z-plane using the bilinear transformation

8.6 The Matched-Z Transform:

Another method for converting an analog filter into an equivalent digital filter is to map the poles and zeros of $H_a(s)$ directly into poles and zeros in the z-plane. For analog filter

$$H_a(s) = \frac{\prod_{k=1}^M (s - s_k)}{\prod_{k=1}^N (s - p_k)} \quad \text{---7.26}$$

the corresponding digital filter is

$$H(z) = \frac{\prod_{k=1}^M (1 - e^{z_k T} z^{-1})}{\prod_{k=1}^N (1 - e^{p_k T} z^{-1})} \quad \text{---7.27}$$

Where T is the sampling interval. Thus each factor of the form $(s-a)$ in $H_a(s)$ is mapped into the factor $(1 - e^{aT} z^{-1})$.

Recommended questions with solution

Question 1

Design a digital band pass filter from a 2nd order analog low pass Butterworth prototype filter using bilinear transformation. The lower and upper cut-off frequencies for band pass filter are $5\pi/12$ and $7\pi/12$. Assume $T = 2$ sec. [12]

Sol. :

$$\omega_l = \frac{5\pi}{12}$$

$$\Omega_l = \frac{2}{T} \tan \frac{\omega_l}{2}$$

For

$$T = 2$$

$$\Omega_l = \tan \frac{\omega_l}{2}$$

$$\omega_u = \frac{7\pi}{12}$$

$$\therefore \Omega_u = \tan \frac{\omega_u}{2}$$

Analog low pass to band pass

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \quad \dots (1)$$

Analog prototype is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \quad \dots (2)$$

Putting equation (1) in equation (2) and then to get bilinear analog to digital

$$s \rightarrow \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}}$$

Combining above two steps we get

$$s \rightarrow \frac{\left(\frac{1-z^{-1}}{1+z^{-1}} \right)^2 + \Omega_l \Omega_u}{\left(\frac{1-z^{-1}}{1+z^{-1}} \right)(\Omega_u - \Omega_l)} = \frac{(1-z^{-1})^2 + \Omega_l \Omega_u (1+z^{-1})^2}{(1-z^{-2})(\Omega_u - \Omega_l)}$$

$$\therefore H(z) = \frac{1}{\left[\frac{(1-z^{-1})^2 + \Omega_l \Omega_u (1+z^{-1})^2}{(1-z^{-2})(\Omega_u - \Omega_l)} \right]^2 + \sqrt{2} \left[\frac{(1-z^{-1})^2 + \Omega_l \Omega_u (1+z^{-1})^2}{(1-z^{-2})(\Omega_u - \Omega_l)} \right] + 1}$$

Question 2

Show that the bilinear transformation maps.

- i) The $j\Omega$ axis in s-plane onto the unit circle, $|z| = 1$.
- ii) The left half s-plane, $\operatorname{Re}(s) < 0$ inside the unit circle, $|z < 1|$.

Sol. :

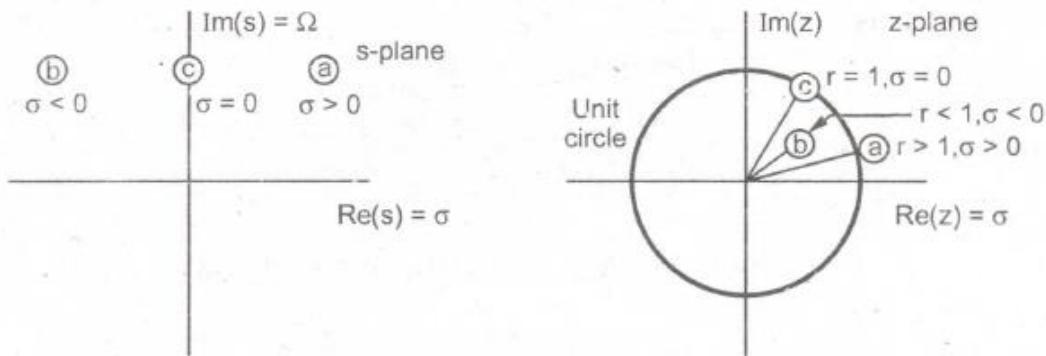


Fig. 3

Here

$$s = \sigma + j\Omega \quad \text{and} \quad z = re^{j\omega}$$

Question 3

Fig. 4 shows the frequency response of an infinite-length ideal multi-band real filter. Find. $h(n)$, impulse response of this filter. Present the sketch of implementation of $\omega(n) h(n)$ (Truncated impulse response of this filter) via block diagram. Where $\omega(n)$ is a finite length window sequence ? [12]

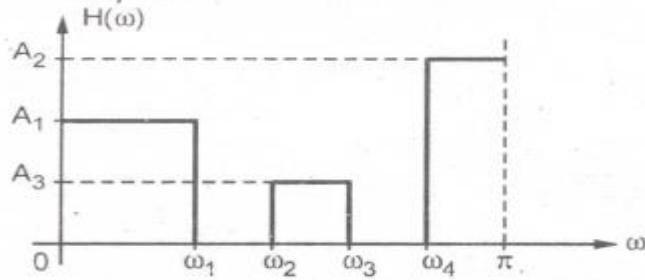


Fig. 4

Question 4

We are interested to design an FIR filter with a stopband attenuation of 64 dB and $\Delta\omega=0.05\pi$ using windows. Provide the means to achieve precisely this attenuation using suitable window function. [3]

Sol. : Hamming window will satisfy the stopband attenuation requirement i.e. 64 dB. Because it has lower transition width.

Hamming window function is given by :

$$\omega(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad 0 \leq n \leq N-1$$

Question 5

The transfer function of analog low pass filter is given by $H(s) = \frac{(s-1)}{(s^2-1)(s^2+s+1)}$.

Find $H(z)$ using impulse invariance method. Take $T = 1$ sec. [6]

Sol. :

$$\begin{aligned} H(s) &= \frac{(s-1)}{(s^2-1)(s^2+s+1)} \\ &= \frac{(s-1)}{(s+1)(s-1)(s^2+s+1)} \\ &= \frac{1}{(s+1)(s^2+s+1)} \\ &= \frac{1}{(s+1)(s+0.5-j0.866)(s+0.5+j0.866)} \\ &= \frac{C_1}{s+1} + \frac{C_2}{s+0.5-j0.866} + \frac{C_2^*}{s+0.5+j0.866} \end{aligned}$$

Using practical fraction expansion, we get

$$C_1 = 1, \quad C_2 = 0.577e^{-j2.62} \quad \text{and} \quad C_2^* = 0.577e^{j2.62}$$

$$\therefore H(s) = \frac{1}{s+1} + \frac{0.577e^{-j2.62}}{s+0.5-j0.866} + \frac{0.577e^{j2.62}}{s+0.5+j0.866}$$

The three poles are :

$$s_1 = -1, \quad s_2 = -0.5 + j0.866 \quad \text{and} \quad s_3 = -0.5 - j0.866$$

We know that,

$$\begin{aligned}
 H(z) &= \sum_{i=1}^3 \frac{C_i}{1 - e^{s_i T} z^{-1}} \\
 &= \frac{C_1}{1 - e^{s_1 T} z^{-1}} + \frac{C_2}{1 - e^{s_2 T} z^{-1}} + \frac{C_3}{1 - e^{s_3 T} z^{-1}}
 \end{aligned}$$

Here $C_3 = C_2^*$

$$\begin{aligned}
 \therefore H(z) &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{0.577 e^{-j2.62}}{1 - e^{(-0.5+j0.866)T} z^{-1}} + \frac{0.577 e^{j2.62}}{1 - e^{(0.5-j0.866)T} z^{-1}} \\
 &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{0.577 e^{-j2.62}}{1 - e^{-0.5T} e^{j0.866T} z^{-1}} + \frac{0.577 e^{j2.62}}{1 - e^{-0.5T} e^{-j0.866T} z^{-1}} \\
 &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{2(0.577) \cos(-2.62) - 2(0.577) e^{-0.5T} z^{-1} \cos(-2.62 - 0.866T)}{1 - 2e^{-0.5T} \cos(0.866T) z^{-1} + e^{-T} z^{-2}}
 \end{aligned}$$

Multiplying the numerator and denominator of first term on RHS by z and by τ^2 for second term on RHS, above equation becomes,

$$H(z) = \frac{z}{z - e^{-T}} + \frac{-z^2 - 1.154 e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) z}{z^2 - 2e^{-0.5T} \cos(0.866T) z + e^{-T}}$$

In terms of sampling interval $T = 1$, transfer function is,

$$H(z) = \frac{b_0 z^{-1} + b_1 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

$$\text{Where } b_0 = -2e^{-0.5T} \cos(0.866T) + e^{-T} + 1.154 e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) = 1.0773$$

$$b_1 = e^{-T} + 1.154 e^{-1.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) = 0.1254$$

$$a_1 = e^{-T} + 2e^{-0.5T} \cos(0.866T) = 1.1538$$

$$a_2 = -e^{-T} - 2e^{-1.5T} \cos(0.866T) = -0.657$$

$$a_3 = e^{-2T} = 0.1353$$

Question 6

Design a linear phase high pass filter using the Hamming window for the following desired frequency response.

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & \frac{\pi}{6} \leq |\omega| \leq \pi \\ 0 & |\omega| < \frac{\pi}{6} \end{cases}$$

$\omega(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right)$, where N is the length of the Hamming window.

Sol. :

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\pi/6} e^{-j3\omega} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\pi/6}^{\pi} e^{-j3\omega} e^{j\omega n} d\omega \\ &= \frac{1}{\pi(n-3)} \left[\sin[\pi(n-3)] - \sin\left[\frac{\pi}{6}(n-3)\right] \right] n \neq 3 \end{aligned}$$

Also,

$$h_d(3) = \frac{1}{2\pi} \left(\frac{5\pi}{6} + \frac{5\pi}{6} \right) \frac{5}{6}$$

Let

$$N = 7$$

Impulse response of FIR filter is :

$$h(n) = h_d(n)\omega(n)$$

$$= \begin{cases} \left\{ \frac{1}{\pi(n-3)} \left[\sin[\pi(n-3)] - \sin\left[\frac{\pi}{6}(n-3)\right] \right] \right\} \left\{ 0.54 - 0.46 \cos\left(\frac{2\pi n}{6}\right) \right\} & n \neq 3 \\ \frac{5}{6} \left[0.54 - 0.46 \cos\left(\frac{2\pi n}{6}\right) \right] & n = 3 \end{cases}$$

n	$h_d(n)$	$\omega(n)$	$h(n)$
0	-0.1061	0.08	0.0085
1	-0.1378	0.31	0.0427
2	-0.1592	0.77	0.1226
3	0.8333	1	0.8333
4	0.1592	0.77	0.1226
5	0.1378	0.31	0.0427
6	0.1061	0.08	0.0085

Question 7

Design a digital lowpass Butterworth filter using bilinear transformation method to meet the following specifications. Take $T = 2$ sec.

Passband ripple ≤ 1.25 dB

Passband edge = 200 Hz

Stopband attenuation ≥ 15 dB

Stopband edge = 400 Hz

Sampling frequency = 2 kHz

[12]

Sol. :

$$\Omega_p = 2\pi \times 200 = 400\pi \text{ rad/sec}$$

$$\Omega_s = 2\pi \times 400 = 800\pi \text{ rad/sec}$$

$$T_s = \frac{1}{f_s} = \frac{1}{2000} \text{ sec}$$

$$\omega_p = \Omega_p T_s = 400\pi \times \frac{1}{2000} = 0.2\pi \text{ rad}$$

$$\omega_s = \Omega_s T_s = 800\pi \times \frac{1}{2000} = 0.4\pi \text{ rad}$$

Given : T = 2 sec.

$$\begin{aligned}\Omega'_P &= \frac{2}{T} \tan\left(\frac{\omega_P}{2}\right) = \tan\left(\frac{0.2\pi}{2}\right) = 0.3249 \\ \Omega'_s &= \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{0.4\pi}{2}\right) = 0.7265 \\ N &= \frac{\log\left[\left(10^{-k_p/10} - 1\right)/\left(10^{-k_s/10} - 1\right)\right]}{2\log(\Omega'_P/\Omega'_s)} \\ &= \frac{\log(0.3335/30.6228)}{2\log(0.3249/0.7265)} = 2.8083 \equiv 3 \\ \Omega_c &= \frac{\Omega'_P}{\left(10^{-k_p/10} - 1\right)^{1/2N}} = 1.7688\end{aligned}$$

Referring to normalized lowpass butterworth filter tables

$$H_3(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

The required prewarped analog filter is obtained by applying lowpass to lowpass transformation.

$$\begin{aligned}H_a(s) &= H_3\left(s\right|_{s \rightarrow \frac{s}{2}} \\ &= \frac{1}{(s^2 + s + 1)(s + 1)}\Big|_{s \rightarrow \frac{s}{2}} \\ &= \frac{1}{\left(\frac{s^2}{4} + \frac{s}{2} + 1\right)\left(\frac{s}{2} + 1\right)} \\ &= \frac{1}{\left(\frac{s^2}{4} + \frac{s+2}{2}\right)\left(\frac{s+2}{2}\right)} \\ &= \frac{1}{\left(\frac{2s^2 + 4s + 8}{8}\right)\left(\frac{s+2}{2}\right)}\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\left(\frac{s^2 + 2s + 4}{4}\right)\left(\frac{s+2}{2}\right)} \\
 &= \frac{8}{(s^2 + 2s + 4)(s+2)} = \frac{8}{s^3 + 2s^2 + 2s^2 + 4s + 8} \\
 &= \frac{8}{s^3 + 4s^2 + 8s + 8}
 \end{aligned}$$

Applying bilinear transformation to $H_a(s)$

$$\begin{aligned}
 H(z) &= H_3(s)|_{s \rightarrow \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)} \\
 \therefore H(z) &= \frac{8}{\left(\frac{1-z^{-1}}{1+z^{-1}} \right)^3 + 4 \left(\frac{1-z^{-1}}{1+z^{-1}} \right)^2 + 8 \left(\frac{1-z^{-1}}{1+z^{-1}} \right) + 8}
 \end{aligned}$$