* CHANNEL:

[A Channel is defined as the medium through which the coded signals generated by an information source are transmitted.]

In general, the input to the channel is a symbol belonging to an alphabet 'A' with 'r' symbols, the outputs of channel is a symbol belonging to an alphabet 'B' with 's' symbols.

i/p → Channel → o/p

i/p channel (A) 'r'

o/p channel ('B') ('s')

Due to errors in channel, the o/p symbols may differ from i/p symbols.

> REPRESENTATION OF A CHANNEL:

A Communication channel may be represented by a set of i/p alphabets $A = \{a_1\ a_2\ a_3 \ldots\ a_r\}$ Consisting of 'r' symbols & set of o/p alphabets $B = \{b_1\ b_2\ b_3 \ldots\ b_s\}$ consisting of 's' symbols & a set of conditional probability $P(b_j/a_i)$ with $i = 1$ to r & $j = 1$ to s.

$$A \left.\begin{matrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \end{matrix}\right\} \longleftarrow P(b_j/a_i) \longrightarrow \left\{\begin{matrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_s \end{matrix}\right. B$$

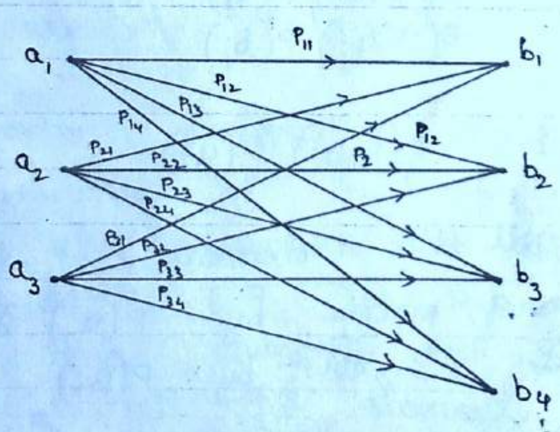
The conditional probabilities come into existence due to presence of noise in the channel. Because of noise, there will be some amount of uncertainity in the reception of any symbol. For this reason, there are 's' number of symbols at receiver from 'r' symbols at transmitter.

Totally, there are $r \times s$ conditional probabilities represented in a matrix form which is called as CHANNEL MATRIX or NOISE MATRIX.

$$
\begin{array}{c c c c c c c}
 & & b_1 & b_2 & b_3 & \cdots & b_s \\
 & & & & & & \\
\end{array}
$$

$$
P(b_j/a_i) = P(B/A) = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_r \end{array}
\begin{bmatrix}
P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\
P(b_1/a_2) & P(b_2/a_2) & P(b_3/a_2) & \cdots & P(b_s/a_2) \\
P(b_1/a_3) & P(b_2/a_3) & P(b_3/a_3) & \cdots & P(b_s/a_3) \\
P(b_1/a_4) & P(b_2/a_4) & P(b_3/a_4) & \cdots & P(b_s/a_4) \\
\vdots & \vdots & \vdots & & \vdots \\
P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & & P(b_s/a_r)
\end{bmatrix}
$$

This can be represented in the channel diagram also. For example, consider a source $A = \{a_1\ a_2\ a_3\}$ & output alphabet $B = \{b_1\ b_2\ b_3\ b_4\}$ which can be represented in a channel diagram as shown below:



The symbols received at b_4 are due to noise present in channel. The line joining 'a_1' with 'b_1' is shown as $P_{11} = P(b_1/a_1)$ i.e., probability of receiving b_1 when a_1 is transmitted.

$P_{12} = P(b_2/a_1) \rightarrow$ probability of receiving b_2 when a_1 is transmitted.

It is evident that when a_1 is transmitted, it can be received as any one of the output symbols $\{b_1\ b_2\ b_3\ b_4\}$

$$\therefore \quad P_{11} + P_{12} + P_{13} + P_{14} = 1$$

$$\Rightarrow \quad P(b_1/a_1) + P(b_2/a_1) + P(b_3/a_1) + P(b_4/a_1) = 1$$

In general,

$$\sum_{j=1}^{s} P(b_j/a_i) = 1 \quad \text{for } i = 1 \text{ to } r$$

※ Thus, sum of all the elements in any row of the channel matrix is equal to UNITY.

* JOINT PROBABILITY:

Joint Probability b/w any i/p symbol a_i & any o/p symbol b_j is given by $P(a_i \cap b_j) = P(a_i, b_j)$

$$P(a_i \cap b_j) = P(b_j/a_i)\, P(a_i)$$
$$= P(a_i/b_j)\, P(b_j)$$

Consider,

$$P(a_i, b_j) = P(b_j/a_i)\, P(a_i)$$

Multiply all the elements of the first row of channel matrix by $P(a_1)$ & 2^{nd} row by $P(a_2)$ & r^{th} row by $P(a_r)$

Then the matrix obtained is of the form:

$$
\begin{array}{c|ccccc}
 & b_1 & b_2 & b_3 & \cdots & b_s \\
\hline
a_1 & P(b_1/a_1)P(a_1) & P(b_2/a_1)P(a_1) & P(b_3/a_1)P(a_1) & \cdots & P(b_s/a_1)P(a_1) \\
a_2 & P(b_1/a_2)P(a_2) & P(b_2/a_2)P(a_2) & P(b_3/a_2)P(a_2) & \cdots & P(b_s/a_2)P(a_2) \\
a_3 & P(b_1/a_3)P(a_3) & P(b_2/a_3)(P(a_3) & P(b_3/a_3)P(a_3) & \cdots & P(b_s/a_3)P(a_3) \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
a_r & P(b_1/a_r)P(a_r) & P(b_2/a_r)P(a_r) & P(b_3/a_r)P(a_r) & & P(b_s/a_r)P(a_r)
\end{array}
$$

$P(b_j/a_i)\,P(a_i) = P(a_i, b_j)$ = the matrix above.

$$
P(a_i, b_j) = \begin{array}{c|ccccc}
 & b_1 & b_2 & b_3 & \cdots & b_s \\
\hline
a_1 & P(a_1, b_1) & P(a_1, b_2) & P(a_1, b_3) & \cdots & P(a_1, b_s) \\
a_2 & P(a_2, b_1) & P(a_2, b_2) & P(a_2, b_3) & \cdots & P(a_2, b_s) \\
a_3 & P(a_3, b_1) & P(a_3, b_2) & P(a_3, b_3) & \cdots & P(a_3, b_s) \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
a_r & P(a_r, b_1) & P(a_r, b_2) & P(a_r, b_3) & \cdots & P(a_r, b_s)
\end{array}
$$

The above matrix whose elements are various Joint Probabilities b/w i/p & o/p symbols is called JOINT PROBABILITY MATRIX (JPM) denoted by $P(a_i, b_j)$

The important properties of JPM are:

> The probabilities of i/p symbols can be obtained by adding the elements of JPM row wise.

> The probabilities of o/p symbols are obtained by adding the elements of JPM column wise.

> The sum of all the elements of JPM is always equal to unity.

* PROPERTIES:

Consider the source alphabet $A = \{a_1\ a_2\ a_3 \ldots a_r\}$ & o/p alphabet $B = \{b_1\ b_2\ b_3 \ldots b_s\}$

i) The source entropy is given by

$$H(A) = \sum_{i=1}^{r} P_{a_i} \log_2\left(\frac{1}{P_{a_i}}\right)$$

ii) The entropy of receiver or o/p is given by

$$H(B) = \sum_{j=1}^{s} P_{b_j} \log_2\left(\frac{1}{P_{b_j}}\right)$$

iii) If all the symbols are equi-probable, then maximum source entropy,

$$H(A)_{MAX} = \log_2 r$$

iv) CONDITIONAL ENTROPY:

The entropy of i/p symbols $\{a_1\ a_2\ a_3 \ldots a_r\}$ after the transmission & reception of particular o/p symbol b_j is defined as Conditional Entropy, denoted by $H(A/b_j)$

$$H(A/b_j) = \sum_{i=1}^{r} P(a_i/b_j) \log_2 \frac{1}{P(a_i/b_j)}$$

If the avg. value of all the conditional probability is taken as j varies from 1 to s is denoted by

$$H(A/B) = \sum_{j=1}^{s} P(b_j)\, H(A/b_j)$$

$$= \sum_{j=1}^{s} \sum_{i=1}^{r} P(b_j)\, P(a_i/b_j) \log_2 \frac{1}{P(a_i/b_j)}$$

$$\boxed{H(A/B) = \sum_{j=1}^{s} \sum_{i=1}^{r} P(a_i, b_j) \log_2 \frac{1}{P(a_i/b_j)}}$$

is Conditional entropy of transmitter

Similarly,

$$\boxed{H(B/A) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(b_j/a_i)}}$$

is conditional entropy of receiver

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

is joint conditional probability

P) A transmitter produces 3 symbols A, B and C which are related to each other with the probabilities as shown below. Find Entropy of transmitter & Conditional entropy of receiver.

$P(A) = 9/27$; $P(B) = 16/27$; $P(C) = 2/27$

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 4/5 & 1/5 \\ B & 1/2 & 1/2 & 0 \\ C & 1/2 & 2/5 & 1/10 \end{array}$$

$X = \{A, B, C\}$ be input (source)

$Y = \{A, B, C\}$ be output (receiver)

i) $H(X) = \sum_{i=1}^{3} P_i \log_2 \left(\frac{1}{P_i}\right)$

$= \left(\frac{9}{27}\right) \log_2 \left(\frac{27}{9}\right) + \left(\frac{16}{27}\right) \log_2 \left(\frac{27}{16}\right) + \left(\frac{2}{27}\right) \log_2 \left(\frac{27}{2}\right)$

$= \underline{\underline{1.2538}}$ bits/symbol

ii) $H(Y/X) = \sum_{i=1}^{3} \sum_{j=1}^{3} P(x_i, y_j) \cdot \log_2 \frac{1}{P(y_j/x_i)}$

To find JPM : $P(X, Y)$

$$\begin{array}{c} 9/27 \ A \\ 16/27 \ B \\ 2/27 \ C \end{array} \begin{bmatrix} 0(9/27) & (4/5)(9/27) & (1/5)(9/27) \\ (1/2)(16/27) & (1/2)(16/27) & 0(16/27) \\ (1/2)(2/27) & (2/5)(2/27) & (1/10)(2/27) \end{bmatrix} = \begin{bmatrix} 0 & 36/135 & 9/135 \\ 8/27 & 8/27 & 0 \\ 1/27 & 4/135 & 1/135 \end{bmatrix}$$

Lt
3/1

$$H(Y/X) = 0 + \left(\tfrac{36}{135}\right)\log_2\left(\tfrac{5}{4}\right) + \left(\tfrac{9}{135}\right)\log_2(5) + \left(\tfrac{8}{27}\right)\log_2(2)$$
$$+\left(\tfrac{8}{27}\right)\log_2(2) + \left(\tfrac{1}{27}\right)\log_2(2) + \left(\tfrac{4}{135}\right)\log_2\left(\tfrac{5}{2}\right)$$
$$+\left(\tfrac{1}{135}\right)\log_2(10)$$
$$H(Y/X) = \underline{\underline{0.934}} \text{ bits/symbol}$$

P) A transmitter transmits 5 symbols with probabilities $\{0.2, 0.3, 0.2, 0.1, 0.2\}$. Given the channel matrix $P(Y/X)$ as shown below, calculate $H(Y)$ & $H(X,Y)$

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$H(Y) = \sum_{j=1}^{4} P_{y_i} \log_2\left(\frac{1}{P_{y_i}}\right)$$

To find JPM:

$$\begin{array}{cc} & \\ 0.2 & x_1 \\ 0.3 & x_2 \\ 0.2 & x_3 \\ 0.1 & x_4 \\ 0.2 & x_5 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1(0.2) & 0 & 0 & 0 \\ 1/4(0.3) & 3/4(0.3) & 0 & 0 \\ 0 & 1/3(0.2) & 2/3(0.2) & 0 \\ 0 & 0 & 1/3(0.1) & 2/3(0.1) \\ 0 & 0 & 1(0.2) & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0.075 & 0.225 & 0 & 0 \\ 0 & 0.0667 & 0.1333 & 0 \\ 0 & 0 & 0.0333 & 0.0667 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$JPM = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0.075 & 0.225 & 0 & 0 \\ 0 & 0.0667 & 0.1333 & 0 \\ 0 & 0 & 0.0333 & 0.0667 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$P(y_i) = [0.275, 0.2917, 0.3666, 0.0667]$$

$$H(Y) = 0.275 \log_2\left(\frac{1}{0.275}\right) + 0.2917 \log_2\left(\frac{1}{0.2917}\right) + 0.3666 \log_2\left(\frac{1}{0.3666}\right) + 0.0667 \log_2\left(\frac{1}{0.0667}\right)$$

$$H(Y) = \underline{1.822} \text{ bits/symbol}$$

$$H(X;Y) = 0.2 \log_2 0.2^{-1} + 0.075 \log_2 (0.075)^{-1} + 0.225 \log_2 (0.225)^{-1} + 0.0667 \log_2 (0.0667)^{-1}$$
$$+ 0.1333 \log_2 (0.1333)^{-1} + 0.0333 \log_2 (0.0333)^{-1} + 0.0667 \log_2 (0.0667)^{-1} + 0.2 \log_2 (0.2)^{-1}$$

$$H(X,Y) = 2.7657 \text{ bits/symbol}$$

* THEOREM :-

Prove that i) $H(X,Y) = H(X) + H(Y/X)$

ii) $H(X,Y) = H(Y) + H(X/Y)$

$H(A,B) = H(A) + H(B/A)$

$H(A,B) = H(B) + H(A/B)$

Consider

$$P(a_i, b_j) = P(b_j/a_i)\, P(a_i) \quad \text{—— (i)}$$
$$= P(a_i/b_j)\, P(b_j) \quad \text{—— (ii)}$$

Consider (i) $\Rightarrow$ $P(a_i, b_j) = P(b_j/a_i)\, P(a_i)$

Taking summation on both sides as j varies from 1 to s, we get

$$\sum_{j=1}^{s} P(a_i, b_j) = \sum_{j=1}^{s} \overset{1}{P(b_j/a_i)}\, P(a_i)$$

$$\therefore \sum_{j=1}^{s} P(a_i, b_j) = P(a_i) \quad \text{—— (iii)}$$

Consider,

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left\{\frac{1}{P(a_i, b_j)}\right\}$$

From (i)

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left\{\frac{1}{P(b_j/a_i) P(a_i)}\right\}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \left\{ \log_2 \left(\frac{1}{P(b_j/a_i)}\right) + \log_2 \left(\frac{1}{P(a_i)}\right)\right\}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(b_j/a_i)} + \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left(\frac{1}{P(a_i)}\right)$$

From (iii),

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{1}{P(b_j/a_i)} + \sum_{i=1}^{r} P(a_i) \log_2 \left(\frac{1}{P(a_i)}\right)$$

$$= H(B/A) + H(A)$$

$$\therefore \boxed{H(A,B) = H(B/A) + H(A)}$$

Consider (ii),

$$P(a_i, b_j) = P(a_i/b_j) P(b_j)$$

Taking summation as i varies from 1 to r is

$$\sum_{i=1}^{r} P(a_i, b_j) = \sum_{i=1}^{r} P(\cancel{a_i/b_j})^{1} P(b_j)$$

$$\therefore \sum_{i=1}^{r} P(a_i, b_j) = P(b_j) \quad \text{— (iv)}$$

Consider,

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left\{\frac{1}{P(a_i, b_j)}\right\}$$

from (ii)

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left\{\frac{1}{P(a_i/b_j)\, P(b_j)}\right\}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left(\frac{1}{P(a_i/b_j)}\right) + \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left(\frac{1}{P(b_j)}\right)$$

From (iv)

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \left\{\frac{1}{P(a_i/b_j)}\right\} + \sum_{j=1}^{s} P(b_j) \log_2 \left(\frac{1}{P(b_j)}\right)$$

$$= H(A/B) + H(B)$$

$$\therefore \boxed{H(A,B) = H(A/B) + H(B)}$$

* MUTUAL INFORMATION :-

When an average amount of information $H(x)$ is transmitted over a noisy channel, then an amount of information $H(x/y)$ is lost in the channel. The balance of the information at the receiver is defined as MUTUAL INFORMATION. $(I(x,y))$

$$\boxed{\begin{aligned} I(x,y) &= H(x) - H(x/y) \\ &= H(y) - H(y/x) \end{aligned}}$$

Consider particular symbols, x_i and y_j. The information associated with the transmitted symbol x_i is given by $I(x_i) = \log \left\{\frac{1}{P(x_i)}\right\}$

Similarly, the information associated with x_i after receiving y_j is $I(x_i/y_j) = \log \frac{1}{P(x_i/y_j)}$

The difference b/w the 2 is the information gained through the channel.

$$I(x_i, y_j) = \log\left\{\frac{1}{P(x_i)}\right\} - \log\left\{\frac{1}{P(x_i/y_j)}\right\}$$

$$= \log\left\{\frac{P(x_i/y_j)}{P(x_i)}\right\}$$

$$\therefore \boxed{I(x_i, y_j) = \log \frac{P(x_i/y_j)}{P(x_i)}}$$

we know that,

$$P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$$
$$= P(x_i/y_j)\, P(y_j)$$

$$\Rightarrow P(x_i/y_j) = \frac{P(x_i, y_j)}{P(y_j)}$$

$$I(x_i, y_j) = \log \frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

$$\boxed{\cancel{I(x_i, y_j) = \log \frac{P(x_i, y_j)}{P(x_i)P(y_j)}}} \quad \text{——} \cancel{(i)}$$

$$\boxed{I(x_i, y_j) = \log \frac{P(x_i, y_j)}{P(x_i)P(y_j)}} \quad \text{——} (i)$$

PROPERTIES :-

i) Consider (i) $I(x_i, y_j) = \log \left\{ \frac{P(x_i, y_j)}{P(x_i) P(y_j)} \right\}$

we know that

$$P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$$

$$\Rightarrow I(x_i, y_j) = \log \frac{P(y_j/x_i)}{P(y_j)}$$

$$= \log \left\{ \frac{1}{P(y_j)} \right\} - \log \left\{ \frac{1}{P(y_j/x_i)} \right\}$$

$$= I(y_j) - I(y_j/x_i) = I(y_j, x_i)$$

$$\therefore \boxed{I(x_i, y_j) = I(y_j, x_i)}$$

Thus, mutual information is symmetric.

ii) Averaging all the symbols, (x_i, y_j) we can obtain the average information gain of the receiver.

That is, $I(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{r} \sum_{j=1}^{s} I(x_i, y_j)\, P(x_i, y_j)$

From (i),

$$I(X, Y) = \sum_{i=1}^{r} \sum_{j=1}^{s} \log \left\{ \frac{P(x_i, y_j)}{P(x_i) P(y_j)} \right\} P(x_i, y_j) \quad \text{— (ii)}$$

a) Consider

$$I(X, Y) = \sum_{i=1}^{r} \sum_{j=1}^{s} \left[P(x_i, y_j) \log \left\{ \frac{P(x_i, y_j)}{P(x_i) P(y_j)} \right\} \right]$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} P(x_i, y_j) \left\{ \log \left\{ \frac{1}{P(x_i)} \right\} + \log \left\{ \frac{1}{P(y_j)} \right\} - \log \left\{ \frac{1}{P(x_i, y_j)} \right\} \right\}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(x_i)} + \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(y_j)}$$
$$- \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(x_i, y_j)}$$

But,
$$P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$$
$$= P(x_i/y_j)\, P(y_j)$$

$$\therefore I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(y_j/x_i)\, P(x_i) \log \frac{1}{P(x_i)}$$
$$+ \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i/y_j)\, P(y_j) \log \frac{1}{P(y_j)}$$
$$- \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(x_i, y_j)}$$

But $\sum_{j=1}^{s} P(y_j/x_i) = 1$ & $\sum_{i=1}^{r} P(x_i/y_j) = 1$

$$\therefore I(X,Y) = \sum_{i=1}^{r} P(x_i) \log \frac{1}{P(x_i)} + \sum_{j=1}^{s} P(y_j) \log \frac{1}{P(y_j)}$$
$$- H(X,Y)$$

$$\boxed{I(X,Y) = H(X) + H(Y) - H(X,Y)}$$

b) Consider
$$I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \left\{\frac{P(x_i, y_j)}{P(x_i)\, P(y_j)}\right\}$$

We know that
$$P(x_i, y_j) = P(x_i/y_j)\, P(y_j)$$

$$\therefore\ I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \left\{ \frac{P(x_i/y_j)}{P(x_i)} \right\}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \left\{ \log \left(\frac{1}{P(x_i)}\right) - \log\left(\frac{1}{P(x_i/y_j)}\right) \right\}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(x_i)} - \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \left(\frac{1}{P(x_i/y_j)}\right)$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} P\left(y_j/x_i\right) P(x_i) \log \frac{1}{P(x_i)} - H(X/Y)$$

But $\sum_{j=1}^{s} P\left(y_j/x_i\right) = 1$

$$\therefore\ I(X,Y) = \sum_{i=1}^{r} P(x_i) \log \frac{1}{P(x_i)} - H(X/Y)$$

$$= H(X) - H(X/Y)$$

$$\therefore\ \boxed{I(X,Y) = H(X) - H(X/Y)}$$

c) Consider

$$I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \left\{ \frac{P(x_i, y_j)}{P(x_i)\,P(y_j)} \right\}$$

We know that

$$P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$$

$$\therefore\ I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \left\{ \frac{P(y_j/x_i)}{P(y_j)} \right\}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(y_j)} - \sum_{i=1}^{r}\sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(y_j/x_i)}$$

~~But~~

$$I(X,Y) = \sum_{i=1}^{r}\sum_{j=1}^{s} P\left(x_i/y_j\right) P(y_j) \log \frac{1}{P(y_j)} - H(Y/X)$$

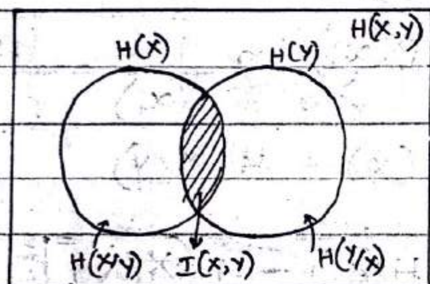
But $\sum_{i=1}^{r} P(x_i/y_j) = 1$

$$\therefore I(X,Y) = \sum_{j=1}^{s} P(y_j) \log \frac{1}{P(y_j)} - H(Y/X)$$

$$= H(Y) - H(Y/X)$$

$$\therefore \boxed{I(X,Y) = H(Y) - H(Y/X)}$$

* VENN DIAGRAM REPRESENTATION :-



Q) A transmitter has an alphabet consisting of 5 source symbols & receiver of 4 symbols. Find, $H(X)$, $H(Y)$, $H(Y/X)$, $H(X/Y)$ and $I(X,Y)$ for the following JPM given below.

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.25 & 0 & 0 & 0 \\ x_2 & 0.1 & 0.3 & 0 & 0 \\ x_3 & 0 & 0.05 & 0.1 & 0 \\ x_4 & 0 & 0 & 0.05 & 0.1 \\ x_5 & 0 & 0 & 0.05 & 0 \end{array}$$

Adding row-wise,

$$P(x_i) = \{0.25, 0.4, 0.15, 0.15, 0.05\}$$

Adding column-wise

$$P(y_j) = \{0.35, 0.35, 0.2, 0.1\}.$$

$$H(X) = \sum_{i=1}^{5} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= 0.25 \log_2\left(\frac{1}{0.25}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.15 \times 2 \times \log_2\left(\frac{1}{0.15}\right)$$

$$+ 0.05 \log_2\left(\frac{1}{0.05}\right)$$

$$H(X) = \underline{2.066} \text{ bits/symbol}$$

$$H(Y) = \sum_{j=1}^{4} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$$= 2 \times 0.35 \log_2\left(\frac{1}{0.35}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$$H(Y) = \underline{1.8568} \text{ bits/symbol}$$

$$H(X,Y) = \sum_{i=1}^{5} \sum_{j=1}^{4} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= 0.25 \log_2\left(\frac{1}{0.25}\right) + 3 \times 0.1 \log_2\left(\frac{1}{0.1}\right) + 3 \times 0.05 \log_2\left(\frac{1}{0.05}\right)$$

$$+ 0.3 \log_2\left(\frac{1}{0.3}\right)$$

$$H(X,Y) = \underline{2.666} \text{ bits/symbol}$$

$$H(X/Y) = H(X,Y) - H(Y)$$

$$= 2.666 - 1.8568$$

$$H(X/Y) = \underline{0.8092} \text{ bits/symbol}$$

$$H(Y/X) = H(X,Y) - H(X)$$
$$= 2.666 - 2.066$$
$$H(Y/X) = \underline{0.6} \text{ bits/symbol}$$

$$I(X,Y) = H(X) - H(X/Y)$$
$$= 2.066 - 0.8092$$
$$= \underline{1.2568} \text{ bits/symbol}$$

* CHANNEL CAPACITY :-

It is known that average information content of the source is $H(x) = \sum_{i=1}^{q} P(x_i) \log_2 \frac{1}{P(x_i)}$

Average amount of information per symbol going into the channel, $R_{in} = H(x) \cdot r_s$ bits/sec.

Due to the error, it is not possible to reconstruct the input symbol sequence with certainity on the recovered sequence. Therefore, source information is lost due to the errors.

Therefore, avg rate of information transmission is given by $\boxed{R_t = I(x,y) \cdot r_s}$ bits/sec

The Capacity of a discrete memoryless noisy channel is defined as maximum possible rate of information transmission through the channel. This maximum rate of transmission occurs when the source is matched to the channel.

$$\therefore \quad C = \text{Max } R_t$$
$$= \text{Max } \left[I(x,y)\, r_s \right]$$
$$= \text{Max} \left\{ \left[H(x) - H(x/y) \right] r_s \right\}$$

* SHANNON'S THEOREM ON CHANNEL CAPACITY :-

It is known that rate of information transmission is given by $R_t = [H(x) - H(x/y)]\, r_s$ and channel capacity $C = Max[R_t]$.

Shannon's second theorem is stated in 2 ways:

1> POSITIVE STATEMENT : It states that when the rate of information transmission $R_t \leq C$, there exists a coding technique which enables the transmission over channel with smaller probability of error.

2> NEGATIVE STATEMENT: It states that if $R_t > C$, then the reliable transmission of the information is not possible without errors i.e, errors cannot be controlled by any coding technique.

Channel efficiency.

$$\%\eta_{ch} = \frac{R_t}{C} \times 100$$

$$= \frac{I(x,y) \cdot r_s}{Max[I(x,y)] \cdot r_s} \times 100$$

$$\boxed{\%\eta_{ch} = \frac{H(x) - H(x/y)}{Max[H(x) - H(x/y)]} \times 100}$$

$$\text{Redundancy} = 1 - \eta_{ch}$$

* TYPES OF CHANNELS :-

i> UNIFORM CHANNEL OR SYMMETRIC CHANNEL :

Symmetric channel is defined as the channel in which the channel matrix has 2^{nd} & subsequent rows, the same elements as the first row, but in different order.

3/4

Ex: $P(y/x) = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$; $P(y/x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$

In general, symmetric channel can be represented as

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots\cdots & y_s \end{array} \\ \begin{bmatrix} P_1 & P_2 & P_3 & \cdots\cdots & P_s \\ P_3 & P_2 & P_1 & \cdots\cdots & P_5 \\ P_s & P_2 & P_5 & \cdots\cdots & P_9 \\ \vdots & \vdots & \vdots & & \vdots \\ P_5 & P_2 & P_s & \cdots\cdots & P_1 \end{bmatrix} \end{array}$$

$P_1, P_2, P_3 \ldots P_s$ are conditional probabilities whose permutation & combinations are present in other rows.
It is known that

$$\sum_{j=1}^{s} P(y_j/x_i) = 1$$

and $$\sum_{j=1}^{s} P_j = 1$$

All the 'r' rows have the same elements $P_1, P_2 \ldots P_s$.

We know that the conditional entropy of receiver is given by

$$H(y/x) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(x_i, y_j) \log_2 \left\{ \frac{1}{P(y_j/x_i)} \right\}$$

$$= \sum_{j=1}^{s} \sum_{i=1}^{r} \cancel{P(x_i)}^{\,1}\, P(y_j/x_i) \log_2 \left\{ \frac{1}{P(y_j/x_i)} \right\}$$

$$= \sum_{j=1}^{s} P(y_j/x_i) \log_2 \left\{ \frac{1}{P(y_j/x_i)} \right\}$$

But $\sum_{j=1}^{s} P(y_j/x_i) \log 1/P(y_j/x_i)$ is entropy of each row.

$$\therefore \boxed{H(Y/X) = h}$$

where $h \rightarrow$ entropy of any single row.

The channel capacity with $r_s = 1$ bit/sec is given by

$$C = \text{Max}\,[R_t]$$
$$= \text{Max}\,[I(x,y)]\, r_s$$
$$= \text{Max}\,[I(x,y)]$$
$$= \text{Max}\,[H(Y) - H(Y/X)]$$
$$= \text{Max}\,[H(Y)] - \text{Max}\,(h)$$
$$\therefore \boxed{C = \text{Max}\, H(Y) - h}$$

$H(y)$ is the entropy of o/p symbol which becomes maximum if & only if all the received symbols become equi-probable.

Since these are 's' output symbols,

$$\text{Max}\,[H(y)] = \log_2 s$$
$$\therefore \boxed{C = \log_2 s - h}$$

P) For a channel matrix shown below, find the channel capacity:

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{bmatrix} \end{array}$$

It is a symmetric or uniform channel. 3/4

$\therefore$ Channel Capacity, $C = \log_2 s - h$

Let $r_s = 1$ bit/sec

$$h = \sum_{j=1}^{3} P_j \log_2 \left(\frac{1}{P_j}\right) = \frac{1}{2}\log_2 2 + \frac{1}{3}\log_2 3 + \frac{1}{6}\log_2 6$$
$$= \underline{\underline{1.459}} \text{ bits/symbol}$$

$$C = \log_2 s - h$$

$$= \log_2 3 - 1.459$$

$$C = 0.126 \text{ bits/sec}$$

P) Repeat the above problem for channel matrix given below with $r_s = 1000$ symbols/sec.

$$P(Y/X) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0.3 & 0.2 & 0.1 \\ x_2 & 0.4 & 0.1 & 0.3 & 0.2 \\ x_3 & 0.1 & 0.2 & 0.4 & 0.3 \end{array}$$

$$h = 0.4 \log_2 \left(\frac{1}{0.4}\right) + 0.3 \log_2 \left(\frac{1}{0.3}\right) + 0.2 \log_2 \left(\frac{1}{0.2}\right) + 0.1 \log_2 \left(\frac{1}{0.1}\right)$$

$$= 1.846 \text{ bits/symbol}$$

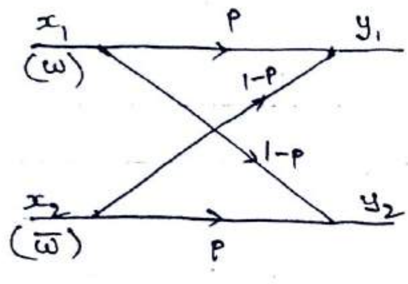
$$C = [\log_2 s - h]\, r_s$$

$$= [\log_2 4 - 1.846]\, 1000$$

$$\boxed{C = 154 \text{ bits/sec}}$$

* BINARY SYMMETRIC CHANNEL :-

The binary symmetric Channel is one of the most commonly & widely used channel whose channel diagram is shown below.



From the above diagram, channel matrix can be written as

$$P(Y/X) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & p & 1-p \\ x_2 & 1-p & p \end{array} = \begin{bmatrix} p & \bar{p} \\ \bar{p} & p \end{bmatrix}$$

The matrix is a symmetric matrix. Hence, the channel is BINARY SYMMETRIC CHANNEL:

> CHANNEL CAPACITY : It is known that

$$C = \cancel{\log}\ \mathrm{Max}\left[H(Y) - H(Y/X)\right] r_s$$

For symmetric channel, $H(Y/X) = h = p\log\left(\frac{1}{p}\right) + \bar{p}\log\left(\frac{1}{\bar{p}}\right)$

Since it is a binary symmetric channel,

$$H(Y)_{max} = \log_2 s = \log_2 2 = 1$$ 14

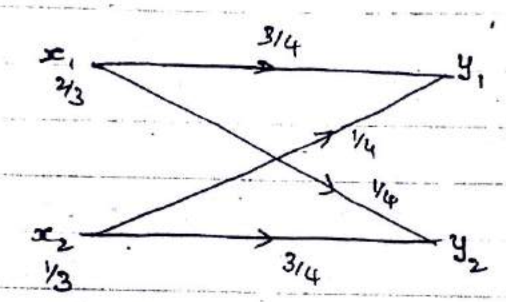
$$\therefore \boxed{C = 1 - h} \text{ bits/sec}$$

P) A binary symmetric channel has the following noise matrix with source probabilities

$$P(x_1) = 2/3 \quad ; \quad P(x_2) = 1/3$$

$$P(Y/X) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Find, $H(x)$, $H(y)$, $I(x,y)$, C, η_{ch}



$$H(X) = \sum_{i=1}^{2} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= \frac{2}{3} \log_2\left(\frac{3}{2}\right) + \left(\frac{1}{3}\right) \log_2(3)$$

$H(X) = 0.9183$ bits/symbol

JPM $\Rightarrow$ $\begin{bmatrix} 2/4 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$

$\therefore P(y_1) = 7/12 \quad ; \quad P(y_2) = 5/12$

$$H(Y) = \sum_{j=1}^{2} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$$= \frac{7}{12} \log_2\left(\frac{12}{7}\right) + \frac{5}{12} \log_2\left(\frac{12}{5}\right)$$

$H(Y) = 0.9799$ bits/symbol

$$I(x,y) = H(Y) - H(Y/X)$$

$$H(Y/X) = h = p \log \frac{1}{p} + \bar{p} \log \frac{1}{\bar{p}}$$

$$= \frac{3}{4} \log_2\left(\frac{4}{3}\right) + \frac{1}{4} \log_2 4$$

$$= \underline{0.8113} \text{ bits/symbol}$$

$$I(x,y) = H(Y) - H(Y/X)$$

$$= 0.9799 - 0.8113 = \underline{0.1686} \text{ bits/symbol}$$

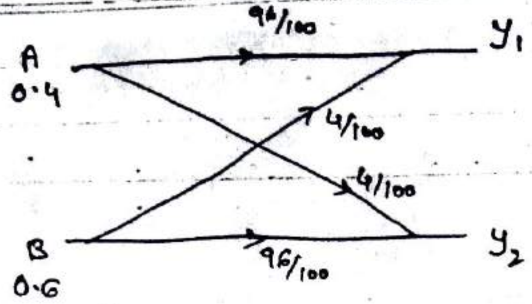
$$\eta_{ch} = \frac{R_t}{C} \times 100 = \frac{I(x,y)\, r_s}{1-h} \times 100$$

Let $r_s = 1$ bit/symbol

$$\therefore \eta_{ch} = \frac{0.1686}{1-0.8113} \times 100 = \frac{0.1686}{0.1887} \times 100$$

$$\boxed{\eta_{ch} = 89.35\ \%}$$

P) A message source produces 2 independent symbols A & B, with probabilities $P(A) = 0.4$ & $P(B) = 0.6$. Calculate efficiency of source & its redundancy if its symbols are received in an avg with 4 in every 100 symbols in error.
Calculate the transmission rate of the system.



$$\text{Channel matrix} = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} y_1 & y_2 \\ \left[\begin{matrix} 96/100 & 4/100 \\ 4/100 & 96/100 \end{matrix}\right. & \left.\right] \end{array} = \begin{bmatrix} 0.96 & 0.04 \\ 0.04 & 0.96 \end{bmatrix}$$

$$JPM = \begin{bmatrix} 0.384 & 0.016 \\ 0.024 & 0.576 \end{bmatrix}$$

$P(y_1) = 0.408$; $P(y_2) = 0.592$

$$H(Y) = 0.408 \log_2\left(\frac{1}{0.408}\right) + 0.592 \log_2\left(\frac{1}{0.592}\right)$$

$H(Y) = 0.9754$ bits/symbol

$$H(Y/X) = h = 0.96 \log_2\left(\frac{1}{0.96}\right) + 0.04 \log_2\left(\frac{1}{0.04}\right)$$

$= 0.2423$ bits/symbol

$I(x,y) = H(Y) - H(Y/X)$

$= 0.7331$ bits/symbol

$R_t = I(x,y)\, r_s$

Let $r_s = 100$ symbols/sec

$\therefore R_t = 73.31$ bits/sec