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Differentiate b/wn DT & CT signals
Differentiate b/wn Analog & Digital signals.

Discuss CTFS DTFS CTFT & DTFT

Qn why Fourier Transforms required.?

Ans: To obtain frequency domain representation from time domain representation of a signal.

TIME DOMAIN. FREQUENCY DOMAIN.

$$x(n) \longrightarrow \boxed{DTFT\{\ \}} \longrightarrow X(e^{j\omega})$$

DT NON PERIODIC

CT PERIODIC with period 2π.

The frequency Analysis of DT signals $x(n)$ is done by the DSP processor or a general purpose digital computer. The i/p to these devices should be digital/discrete. But the frequency domain representation of DT signal $x(n)$ is $X(e^{j\omega})$ which is continuous in nature which can't be handled by digital machines like DSP processor or computer. Therefore there is need for converting $X(e^{j\omega})$ a continuous function of ω to discrete.

This can be accoplished by sampling $X(e^{j\omega})$ over one complete cycle (2π) at regular sampling intervals.

$$x(n) \longrightarrow \boxed{DTFT\{\}} \xrightarrow{X(e^{j\omega})} \boxed{Sampler.} \longrightarrow X(k)$$

DISCRETE TIME — CONTINUOUS TIME — DISCRETE.

The sampled version of $X(e^{j\omega})$ is known as DISCRETE FOURIER TRANSFORM [DFT] denoted as $X(k)$.

DFT is a powerful tool for performing frequency analysis of DT signals.

FREQUENCY DOMAIN SAMPLING : The DISCRETE FOURIER TRANSFORM (DFT)

The DTFT of a DT signal $x(n)$ is given as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow (1)$$

$X(\omega)$ is continuous & periodic function of ω.

Suppose that $X(\omega)$ is sampled in frequency with sampling interval 'δ_ω' radian.

Since $X(\omega)$ is periodic with period 2π, we consider the samples only over one complete cycle.

Assume we take N number of samples over interval $0 \le \omega \le 2\pi$ then the sampling interval becomes $\delta_\omega = \frac{2\pi}{N}$.

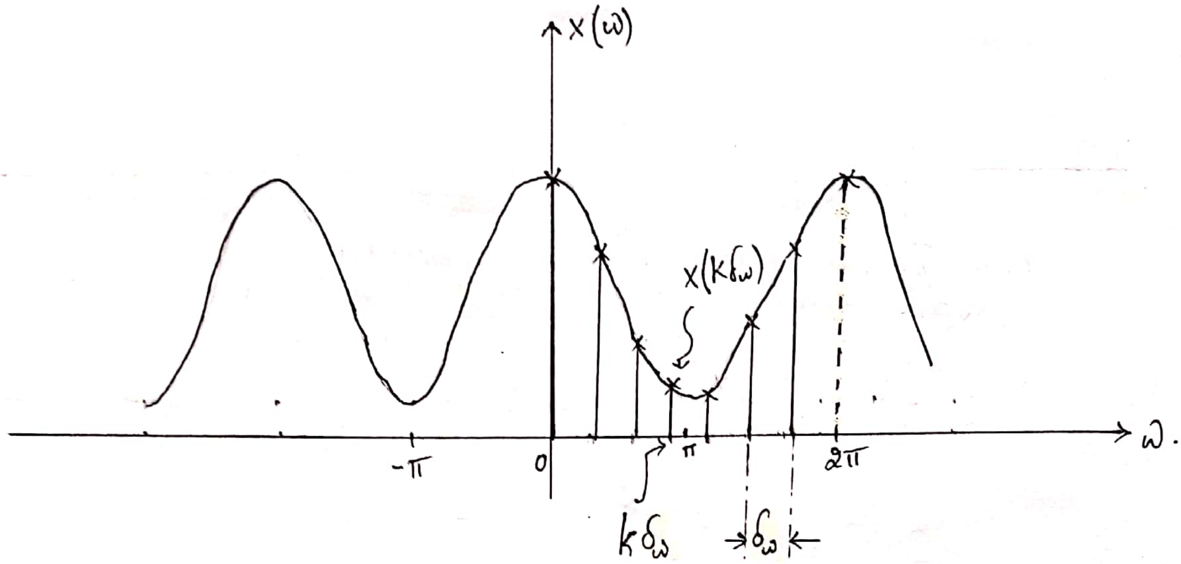

Figure: Frequency domain sampling of the Fourier Transform.

Evaluating Eqn ① at $\omega = \frac{2\pi}{N}k$ we get

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi k}{N}n} \qquad k = 0\ 1\ 2\ \ldots\ N-1$$

The summation on RHS can be split into infinite number of summations, each consisting of N terms as

$$X\left(\frac{2\pi k}{N}\right) = \cdots\cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi k}{N}n}$$

$$+ \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N}n}$$

$$+ \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi k}{N}n} + \cdots\cdots$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi k}{N}n} \longrightarrow ②$$

Put $n - lN = m$ in second summation

$m = n - lN \Rightarrow n = m + lN$

If $n = lN$ $\quad m = 0$

If $n = lN + N - 1$ $\quad m = N-1$

Therefore Eqn (2) becomes

$$x\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} x(m+lN)\, e^{-j\frac{2\pi k}{N}(m+lN)}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} x(m+lN)\, e^{-j\frac{2\pi k}{N} m}, \quad \cancel{e^{-j\frac{2\pi k}{N} lN}} \to 1$$

Now change the variable m to n & change the order of summation.

$$x\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n+lN)\right] e^{-j\frac{2\pi k}{N} n} \qquad k = 0,1,2 \ldots N-1$$

$$x\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi k}{N} n} \longrightarrow (3) \qquad k = 0,1,2 \ldots N-1$$

where $x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN) = \sum_{l=-\infty}^{\infty} x(n-lN) \longrightarrow (4)$.

$x_p(n)$ is a signal obtained by periodic representation of $x(n)$ every N samples $\therefore$ it is periodic with fundamental period N.

Now Expressing $x_p(n)$ as a Fourier series

$$x_p(n) = \sum_{k=0}^{N-1} c(k)\, e^{j\overset{w_0}{\left(\frac{2\pi k}{N}\right)} n} \qquad n = 0,1, \ldots N-1 \longrightarrow (5)$$

with Fourier coefficients

$$c(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi k}{N} n} \qquad k = 0\ 1 \ldots N-1 \longrightarrow (6)$$

comparing Eqn. (3) and (6)

$$x\left(\frac{2\pi k}{N}\right) = N\, c(k)$$

$$c(k) = \frac{1}{N}\, x\left(\frac{2\pi k}{N}\right) \longrightarrow (7). \qquad k = 0\ 1 \ldots N-1$$

Put (7) in (5)

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi k n/N} \qquad n = 0, 1, \ldots N-1 \longrightarrow (8).$$

Eqn (8) represents the relation b/wn $x_p(n)$ & $X(\omega)$.

To Compute $x(n)$ & hence $X(\omega)$ from $x_p(n)$:

Since $x_p(n)$ is periodic extension of $x(n)$ [Eqn (4)], we can recover $x(n)$ from $x_p(n)$ only if there is no 'TIME DOMAIN ALIASING' b/wn $x(n)$ and $x_p(n)$ i.e., $N \geq L$ refer figure (2).

where N - period of $x_p(n)$

L - Length of non periodic signal $x(n)$

$x(n)$ is recovered from $x_p(n)$ without any ambiguity as follows

$$x(n) = x_p(n) \qquad 0 \leq n \leq N-1 \quad \text{with} \quad N \geq L$$

On the otherhand if $N < L$, it is not possible to recover $x(n)$ from $x_p(n)$ due to time domain aliasing

Thus, the spectrum $X(\omega)$ of a non periodic DT sequence with finite duration L can be exactly recovered from its samples at frequencies $\omega_k = \frac{2\pi}{N}k$ if $N \geq L$

The pro is to compute $x_p(n)$ $n = 0, 1 \ldots N-1$ from Eqn (8). and then

$$x(n) = \begin{cases} x_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere.} \end{cases}$$

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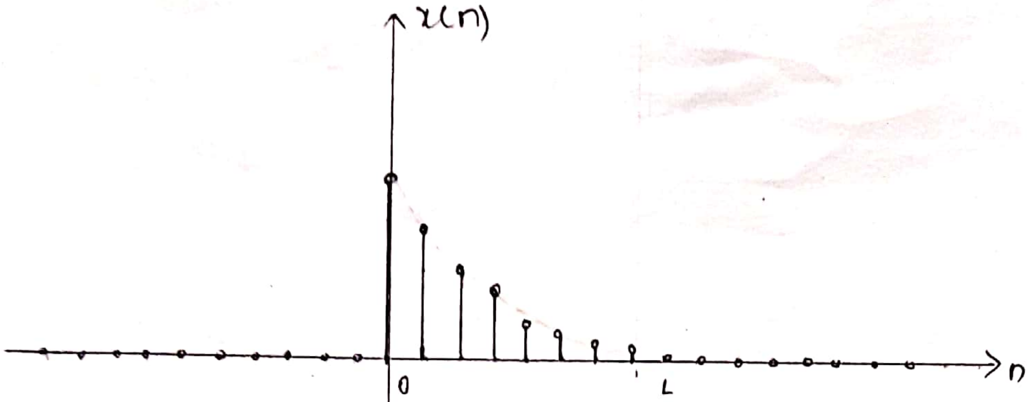
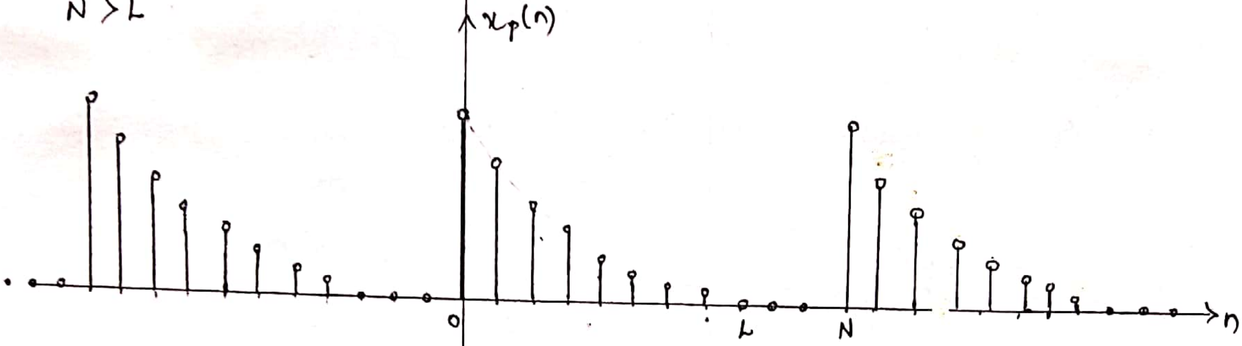
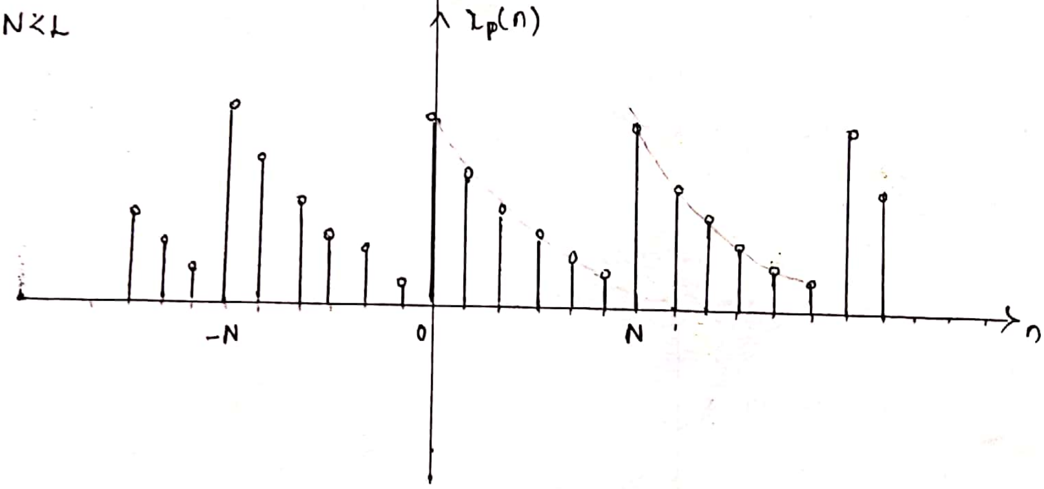
and finally, $X(\omega)$ can be computed as $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$.



N > L



N < L



W.K.T $x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{\frac{j2\pi k}{N} n}$ $\quad 0 \le n \le N-1$

Assuming $N \geq L$; $x_p(n) = x(n)$

above Eqn becomes.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{\frac{j2\pi k}{N} n} \quad 0 \le n \le N-1 \longrightarrow (1).$$

W.K.T DFT of $x(n)$

$$X(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \longrightarrow (2).$$

Put (1) in (2).

$$X(\omega) = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{\frac{j2\pi k}{N} n} e^{-j\omega n}.$$

Taking 2nd summation outside & rearranging the terms.

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi k}{N}\right) n}\right] \longrightarrow (3).$$

Recall $\sum_{n=0}^{p} a^n = \frac{1-a^{p+1}}{1-a}$ $\quad |a| \neq 1$

$$\left[\text{Let } P(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right.$$

$$\left. P(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\omega/2\right)} e^{-j\omega(N-1)/2} \right] \longrightarrow (4).$$

using the relations in Eqn (4) in Eqn (3) we get

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) P\left(\omega - \frac{2\pi}{N}\right) \qquad N \geq L \longrightarrow (5).$$

The interpolation function $P(\omega)$ is not the familiar $\frac{\sin\theta}{\theta}$ but instead, it is a periodic counterpart of it, & it is due to periodic nature of $X(\omega)$. The phase shift in Eqn (4) reflects the fact that the signal $x(n)$ is a causal, finite duration sequence of length N.

The function $\frac{\sin\left(\frac{\omega N}{2}\right)}{N\sin\left(\frac{\omega}{2}\right)}$ is plotted in figure.

for $N = 5$

It is observed that the function $P(\omega)$ has the property

$$P\left(\frac{2\pi}{N}k\right) = \begin{cases} 1 & k = 0 \\ 0 & k = 1\ 2\ \cdots\ N-1 \end{cases}$$

Consequently the interpolation formula in Eqn (5). gives exactly the same values $X\left(\frac{2\pi k}{N}\right)$ for $\omega = \frac{2\pi}{N}k$

At all other frequencies the formula provides properly weighted linear combination of the original spectral samples.

Qn. Consider $x(n) = a^n u(n)$ $\quad 0 < a < 1$

The spectrum of this signal is sampled at frequencies $\omega_k = \frac{2\pi}{N} k$ $\quad k = 0, 1, \ldots, N-1$

Determine the reconstructed spectra for $a = 0.8$ when $N = 5$ and $N = 50$.

Soln.

Given $x(n) = a^n u(n)$ $\quad 0 < a < 1$

Taking DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

Sampling the spectrum $X(\omega)$ at N equidistant frequencies $\omega_k = \frac{2\pi k}{N}$ $\quad k = 0, 1, 2, \ldots, N-1$

We get the spectral samples.

(1a)

RELATIONSHIP of DFT with other Transforms

I Relation b/wn DFT and FS of a periodic sequence

$x(n)$: Non-periodic sequence with finite duration

The N point DFT of $x(n)$ [assuming $N > L$] is given as.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N} n} \qquad k = 0\ 1 \cdots N-1 \longrightarrow (1)$$

IDFT of $X(k)$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k}{N} n} \qquad n = 0\ 1 \cdots N-1 \longrightarrow (2).$$

Let $x_p(n)$ be the periodic replication of $x(n)$ with period 'N',

the DTFS of $x_p(n)$ is

$$x_p(n) = \sum_{k=0}^{N-1} c(k)\, e^{j\frac{2\pi k}{N} n} \qquad -\infty < n < \infty \longrightarrow (3).$$

where the Fourier series coefficients $c(k)$ are given as.

$$c(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi k}{N} n} \qquad k = 0\ 1 \cdots N-1 \longrightarrow (4).$$

Noting that

$$x(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

and comparing Eqn (1) & (4) we get

$$\boxed{X(k) = N \cdot c(k)} \longrightarrow (5)$$

$k = 0, 1\ 2 \cdots N-1$

or

$$c(k) = \frac{1}{N} X(k) \longrightarrow (6) \quad k = 0\, 1 \, \text{--} \; N-1$$

put (6) in (3)

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi k}{N} n} \longrightarrow (7)$$

$$n = 0\; 1 \cdots N-1$$

OR

$$x_p(n) = IDFT\{X(n)\} \longrightarrow (8)$$

OR

$$\boxed{x(n) = DFT\{x_p(n)\}}$$

Thus the n point DFT provides the exact line spectrum of a periodic sequence with fundamental period. N.

II Relation b/wn DFT & DTFT of x(n).

x(n) : Finite Energy non periodic sequence with length 'L'

DTFT of x(n) is given as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

DFT of x(n) is obtained by sampling $X(\omega)$ in frequency domain with sampling frequency $f_\omega = \frac{2\pi}{N}$.

i.e.,
$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi}{N}k}.$$

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad k = 0\; 1 \;\text{--}\; N-1$$

The spectral components $X(k)$ are the DFT coefficients of the periodic sequence $x_p(n)$ with period N given as.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$x_p(n)$ is obtained by aliasing $\{x(n)\}$ over the interval $0 \leq n \leq N-1$

Now a finite duration sequence $\hat{x}(n)$ extracted from $x_p(n)$ as.

$$\hat{x}(n) = \begin{cases} x_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases}$$

is equivalent to $x(n)$ only if $x(n)$ was of finite length $L \leq N$. in which case.

$$x(n) = \hat{x}(n). \qquad 0 \leq n \leq N-1$$

Only in this case IDFT of $X(k)$ yield the original sequence. $x(n)$.

<u>Relation b/wn DFT & Z.T of a sequence x(n).</u>

By definition the Z.T of a sequence $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow (1).$$

Let the length of the sequence $x(n)$ be $\leq N$.

then Eqn (1) $\Longrightarrow$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \longrightarrow (2).$$

By definition of IDFT substitute for $x(n)$ in above Eqn.

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} x(k)\, e^{\frac{j2\pi k}{N} n} \right] z^{-n}$$

OR

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}} z^{-1} \right)^{n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x(k) \frac{1 - \left(e^{\frac{j2\pi k}{N}} z^{-1} \right)^{N-1+1}}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x(k) \frac{1 - e^{\frac{j2\pi k}{\cancel{N}} \cdot \cancel{N}} z^{-N}}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x(k) \frac{1 - z^{-N}}{1 - e^{\frac{j2\pi k}{N}} z^{-1}} \qquad \because e^{j2\pi k} = 1$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{x(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

put $z = e^{j\omega}$.

$$X(z) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{x(k)}{1 - e^{\frac{j2\pi k}{N}} e^{-j\omega}}$$

$$\boxed{X(\omega) = X(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{x(k)}{1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)}}}$$

This expression for the FT is a polynomial (Lagrange) interpolation formula for $X(\omega)$ expressed in terms of the values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies

$$\omega_k = \frac{2\pi k}{N} \quad k = 0\ 1\ \cdots\ N-1$$

<u>Note:</u> Here the Z.T of $x(n)$ is $X(z)$ & the ROC of $X(z)$ is assumed to include the unit circle.

If $X(z)$ is sampled at the N equally spaced points on the unit circle

$$z_k = e^{\frac{j2\pi k}{N}} \quad k = 0\ 1\ \cdots\ N-1$$

we get

$$X(k) = X(z)\Big|_{z = e^{\frac{j2\pi k}{N}}} \quad k = 0\ 1\ 2\ \cdots\ N-1$$

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-\frac{j2\pi k}{N} n} \quad k = 0\ 1\ 2\ \cdots\ N-1$$

<u>Relation b/w DFT & FS coefficients of a CT signal.</u>

Let $x_a(t)$ be a CT periodic signal with fundamental period

$$T_0 = \frac{1}{F_0}$$

F_0 : fundamental period.

Expressing $x_a(t)$ as Fourier Series.

$$x_a(t) = \sum_{k=-\infty}^{\infty} c(k)\, e^{jk\Omega t}$$

$$= \sum_{k=-\infty}^{\infty} c(k)\, e^{\frac{j2\pi k t}{T_0}}$$

$$x_a(t) = \sum_{k=-\infty}^{\infty} c(k)\, e^{j2\pi k t F_0} \qquad \because \frac{1}{T_0} = F_0.$$

where $c(k)$ are the Fourier coefficients

If $x_a(t)$ is sampled at uniform rate

$F_s = \frac{N}{T_0} = \frac{1}{T_s}$ we get the DT sequence $x(n)$ as

$$x(n) = x_a(nT_s) = \sum_{k=-\infty}^{\infty} c(k)\, e^{j2\pi k F_0 n T_s}$$

$$x(n) = \sum_{k=-\infty}^{\infty} c(k)\, e^{j2\pi k F_0 n \frac{T_0}{N}} \qquad \because \frac{1}{T_s} = \frac{N}{T_0}$$

$$= \sum_{k=-\infty}^{\infty} c(k)\, e^{j\frac{2\pi k}{N} \cancel{F_0} n \cdot \frac{1}{\cancel{F_0}}}$$

$$x(n) = \sum_{k=-\infty}^{\infty} c(k)\, e^{j\frac{2\pi k}{N} n}$$

THE DFT as a LINEAR TRANSFORMATION.

We know that the N- point DFT of a sequence ~~x(n)~~) $x(n)$ is

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N} n} \longrightarrow (1)$$

$k = 0\ 1 \cdots N-1$

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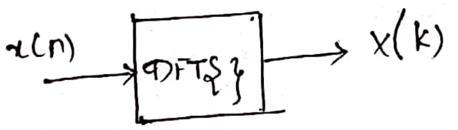


$k = 0\ 1\ 2 \cdots N-1$

$n = 0, 1, \cdots N-1$

The corresponding IDFT is

$$IDFT\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k}{N} n} \longrightarrow (2).$$

$n = 0\ 1 \cdots N-1$

Note that $x(n)$ & $X(k)$ are of same length equal to N. Hence it is known as transform length.

Let $W_N = e^{-j\frac{2\pi}{N}}$ which is an N th root of unity (or referred to as complex basis function)

$\therefore$ Eqn (1) & (2) $\Longrightarrow$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad k = 0, 1, 2 \ldots N-1 \rightarrow (3)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad n = 0\ 1\ 2 \ldots N-1 \rightarrow (4).$$

Expanding Eqn (3).

$$X(k) = x(0)\, W_N^{k(0)} + x(1)\, W_N^{k(1)} + \ldots\ldots + x(N-1)\, W_N^{k(N-1)}$$

$$k = 0\ 1\ 2 \ldots N-1$$

Now for $k = 0$.

$$X(0) = x(0)\, W_N^{0(0)} + x(1)\, W_N^{0(1)} + \ldots + x(N-1)\, W_N^{0(N-1)}$$

for $k = 1$

$$X(1) = x(0)\, W_N^{1(0)} + x(1)\, W_N^{1(1)} + \ldots\ldots + x(N-1)\, W_N^{1(N-1)}$$

$\vdots$

$k = N-1$

$$X(N-1) = x(0)\, W_N^{(N-1)(0)} + x(1)\, W_N^{(N-1)(1)} + \ldots + x(N-1)\, W_N^{(N-1)(N-1)}$$

It is observed that to compute each point of the DFT N complex multiplications and $(N-1)$ complex additions are required.

Therefore in the process of computation of N point DFT a sequence $x(n)$ a total of

N^2 COMPLEX MULTIPLICATION. and

$N(N-1)$ COMPLEX ADDITIONS

are required.

Writing above set of equations in matrix form.

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

OR

$$[X]_N = [W]_N [x]_N \longrightarrow (1)$$

$$[X(K)]_N = [W]_N [x(n)]_N$$

$[W]_N$ is the matrix of the linear combination. (or) linear transformation.

Also it is a SYMMETRIC MATRIX

<u>Note</u>: If inverse of $[W]_N$ exists then Eqn (1) $\Rightarrow$

$$[x]_N = [W]_N^{-1} [X]_N \longrightarrow (2).$$

Now writing IDFT in matrix form

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$$

(OR)

$$[x(n)]_N = \frac{1}{N}[W_N]^*[X(k)]_N \longrightarrow (3)$$

$[W_N]^*$ is the complex conjugate of $[W_N]$

on comparing Eqn (2) & (3).

$$[W_N]^{-1} = \frac{1}{N}[W]_N^*.$$

(OR)

$$W_N W_N^* = N[I]_N$$

where $[I]_N$ is an $N \times N$ identity matrix ∴ the matrix $[W]_N$ in the transformation is an orthogonal (unitary) matrix.

<u>Note:</u> writing W_N matrix for $N=4$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

∴ W_N matrix is symmetric.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Qn Compute the 4 point DFT of the sequence $x(n) = (1, 2, 3, 4)$ using linear transformation method / Matrix method.

soln. W.K.T

$$[X(k)]_N = [W]_N [x(n)]_N.$$

$$[X(k)]_4 = [W]_4 [x(n)]_4$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2 + 3 + 4 \\ 1 - 2j - 3 + 4j \\ 1 - 2 + 3 - 3 \\ 1 + 2j - 3 - 4j \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2j-2j \end{bmatrix}$$

$$X(k) = [10, -2+2j, -2, -2-2j]$$

Note:

i) $W_N^{kn} = e^{-j\frac{2\pi k}{N} \cdot n}$

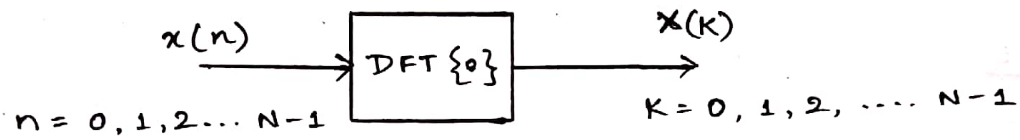
ii) $W_N = e^{-j\frac{2\pi}{N}}$

OR $W_N = \left(e^{-j2\pi}\right)^{\frac{1}{N}}$; $e^{-j2\pi} = 1$

W_N is a complex basis function which is also called 'twiddle factor' OR 'Nth root of unity'

DISCRETE FOURIER TRANSFORM [DFT].

If $x(n)$ is a sequence defined only over the interval, $0, 1, 2, \ldots N-1$, then DFT of the sequence $x(n)$ is defined as follows -



$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-jk\omega_0 n} \quad \text{——} \quad (1)$$

Note that $x(n)$ and $X(K)$ are of the same length equal to 'N'. Hence, 'N' is known as Transform Length.

Let us define $\omega_0 = \frac{2\pi}{N}$, substituting the above value of 'ω_0'.

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In equation (1), we get -

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-jk \cdot \frac{2\pi}{N} \cdot n} \quad \text{——} \quad (2)$$

Let us now introduce a complex basis function defined by $W_N = e^{-j \cdot \frac{2\pi}{N}}$.

Making use of the above complex function, in eq. (2), we get -

$$X(K) = \sum_{n=0}^{N-1} x(n).\, W_N^{Kn} \qquad \text{—— (3)}$$

$$K = 0, 1, 2 \ldots N-1$$

The inverse DFT of the sequence $X(K)$ is defined as follows -

$$\text{IDFT}\{X(K)\} = x(n) \triangleq \frac{1}{N}\sum_{K=0}^{N-1} X(K).\, W_N^{-Kn} \qquad \text{—— (4)}$$

$$n = 0, 1, 2, 3 \ldots N-1$$

NOTE :

1) The complex basis function is periodic with a period equal to 'N'.

That is, $W_N^{a} = W_N^{a+N}$, where a is some fixed integer.

Proof : $W_N \triangleq e^{-j.\frac{2\pi}{N}}$

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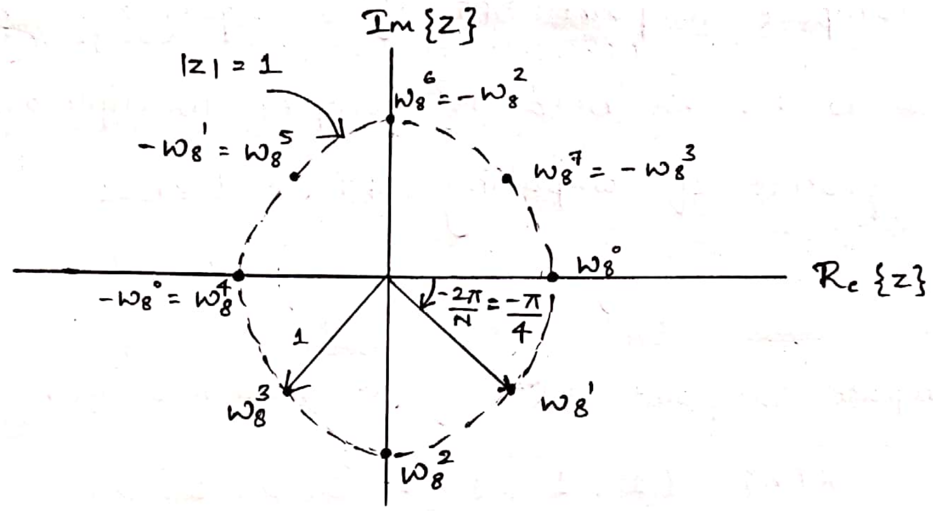
$$W_N^{a+N} = e^{-j\frac{2\pi}{N}(a+N)}$$

$$= e^{-j.\frac{2\pi}{N}.a} \cdot e^{-j2\pi}$$

$$= W_N^{a} \qquad (\because e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1)$$

Eg : i) $W_8^{10} = W_8^{2}$ ii) $W_8^{21} = W_8^{5}$

2) The complex basis function W_N for $N = 8$ is as shown below :



$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

$$\therefore W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 1 \angle{-\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = 1 \angle{-\pi/2} = -j$$

$$W_8^3 = e^{-j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^4 = e^{-j\pi} = -1$$

$$W_8^5 = e^{-j\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

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$$W_8^6 = +j$$

$$W_8^7 = e^{-j\frac{7\pi}{4}} = \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}$$

$$W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

To compute N-point DFT of a sequence, $x(n)$ of length equal to N, we need N^2 complex multiplications. Hence, the process of computing DFT is tedious.

1. Compute 8-point DFT of a sequence $x(n)$ given below.

$$x(n) = (\underset{\uparrow}{1}, 1, 1, 1, 0, 0, 0, 0)$$

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn}$$

Here, $N = 8$.

$$\therefore X(K) = \sum_{n=0}^{7} x(n) \cdot W_8^{Kn}$$

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$$= 1 + W_8^{K} + W_8^{2K} + W_8^{3K}$$

$$K = 0, 1, 2, \ldots 7.$$

<u>NOTE</u> : $W_8^0 = 1$, $W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$, $W_8^2 = j$,

$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$, $W_8^4 = -1$, $W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

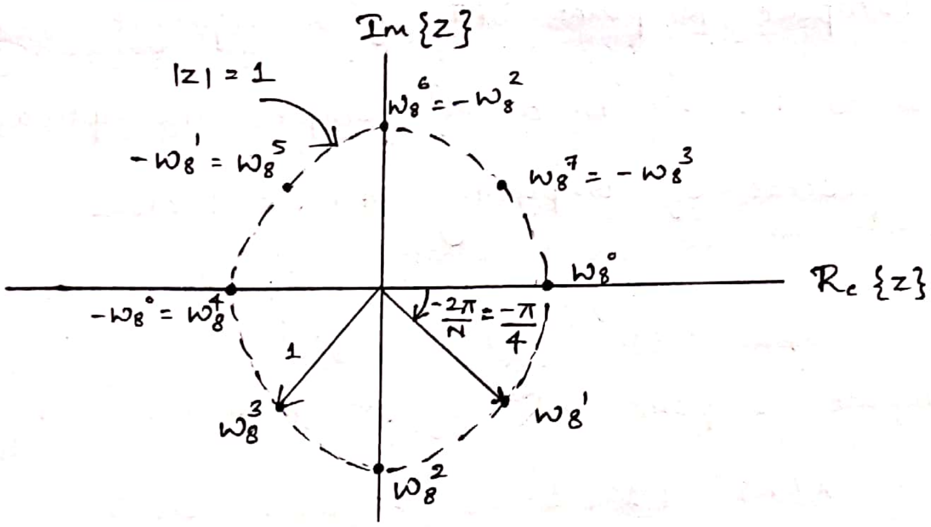
$W_8^6 = +j$, $W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$$X(0) = 1 + 1 + 1 + 1 = 4.$$

$$X(1) = 1 + W_8^1 + W_8^2 + W_8^3 = 1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + (-j) - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$X(1) = 1 - j\,2.414.$$

$$X(2) = 1 + W_8^2 + W_8^4 + W_8^6 = 1 - j - 1 + j = 0.$$



$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

$$\therefore W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 1\angle{-\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = 1\angle{-\pi/2} = -j$$

$$W_8^3 = e^{-j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^4 = e^{-j\pi} = -1$$

$$W_8^5 = e^{-j\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

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$$W_8^6 = +j$$

$$W_8^7 = e^{-j\frac{7\pi}{4}} = \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4}$$

$$W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

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$$X(3) = 1 + W_8^3 + W_8^6 + W_8^1 = 1 - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + j + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$X(3) = 1 - j\,0.414.$$

$$X(4) = 1 + W_8^4 + W_8^0 + W_8^4 = 1 - 1 + 1 - 1 = 0.$$

$$X(5) = 1 + W_8^5 + W_8^2 + W_8^7 = 1 - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - j + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$X(5) = 1 + j0.414.$$

$$X(6) = 1 + W_8^6 + W_8^4 + W_8^2 = 1 + j - 1 - j = 0.$$

$$X(7) = 1 + W_8^7 + W_8^6 + W_8^5 = 1 + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + j - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$X(7) = 1 + j2.414.$$

<u>NOTE</u> : DFT of a real sequence can be complex.

2: Find the 4-point DFT of the sequence $x(n)$ given below - Using $X(K)$ found above, find $x(n)$ using the definition of inverse DFT.

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$$x(n) = (\underset{\uparrow}{1}, 0, 1, 0)$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad K = 0, 1, 2 \ldots N-1.$$

Here, $N = 4$.

$$\therefore X(K) = \sum_{n=0}^{3} x(n)\, W_4^{Kn} \qquad K = 0, 1, 2, 3.$$

NOTE : $W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\pi/2}$

$$W_4^0 = 1 \quad , \quad W_4^1 = e^{-j\pi/2} = -j$$

$$W_4^2 = e^{-j\pi} = -1 = -W_4^0$$

$$W_4^3 = e^{-j3\pi/2} = +j = -W_4^1$$

$$X(K) = 1 + W_4^{2K} \qquad K = 0, 1, 2, 3.$$

$$\Rightarrow \quad X(0) = 1 + W_4^0 = 2$$

$$X(1) = 1 + W_4^2 = 1 - 1 = 0.$$

$$X(2) = 1 + W_4^0 = 2$$

$$X(3) = 1 + W_4^2 = 1 - 1 = 0.$$

$$x(n) \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n = 0, 1, \ldots N-1.$$

Here, $N = 4$.

$$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K) \cdot W_4^{-Kn} \qquad n = 0, 1, 2, 3.$$

$$\Rightarrow \quad x(n) = \frac{1}{4} \left\{ X(0) + X(1)\, W_4^{-n} + X(2)\, W_4^{-2n} + X(3)\, W_4^{-3n} \right\}$$

$$= \frac{1}{4} \left\{ 2 + 0 + 2\, W_4^{-2n} + 0 \right\}$$

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NOTE : $W_4^{-a} = (W_4^a)^*$

$$W_4^{-a} \triangleq e^{-j\frac{2\pi}{4}(-a)} = e^{+j2\pi/4\, a}$$

$$W_4^{-a} = \left(e^{-j\frac{2\pi}{4} a} \right)^*$$

$$= (W_4^a)^*$$

① $W_4^{-0} = (W_4^0)^* = 1$

② $W_4^{-1} = (W_4^1)^* = j$

③ $W_4^{-2} = (W_4^2)^* = -1$

④ $W_4^{-3} = (W_4^3) = -j$

WKT, $x(n) = \frac{1}{4}[2 + 2W_4^{-2n}]$ $0 \le n \le 3$.

$\therefore x(0) = \frac{1}{4}(2+2) = 1$

$x(1) = \frac{1}{4}(2 + 2W_4^{-2}) = 0.$

$x(2) = \frac{1}{2}(2 + 2W_4^{-0}) = 1$

$x(3) = \frac{1}{2}(2 + 2W_4^{-2}) = 0.$

3. Compute 3-point DFT of the sequence $x(n) = (1, 0, 1)$ using Inverse DFT relation.

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$W_N \triangleq e^{-j\frac{2\pi}{N}}$

$W_3 = e^{-j\frac{2\pi}{3}}$

$\therefore W_3^0 = 1 \Rightarrow W_3^{-0} = 1$

$W_3^1 = e^{-j\frac{2\pi}{3}} = 1\angle{-2\pi/3} = -0.5 - j0.866$

$\Rightarrow W_3^{-1} = -0.5 + j0.866$

$W_3^2 = e^{-j\frac{4\pi}{3}} = 1\angle{-4\pi/3} = -0.5 + j0.866$

$$\Rightarrow W_3^{-2} = -0.5 - j0.866$$

$$x(n) \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n = 0, 1, 2, \ldots N-1$$

$$\Rightarrow x(n) = \frac{1}{3} \sum_{K=0}^{2} X(K) \cdot W_3^{-Kn}$$

$$\Rightarrow 3x(n) = \sum_{K=0}^{2} X(K) \cdot W_3^{-Kn} \qquad 0 \le n \le 2.$$

$$= X(0) + X(1) W_3^{-n} + X(2) W_3^{-2n}$$

$$\therefore X(0) + X(1) + X(2) = 3x(0) = 3 \quad \text{——— (1)}$$

$$X(0) + X(1) W_3^{-1} + X(2) W_3^{-2} = 3x(1) = 0 \quad \text{——— (2)}$$

$$X(0) + X(1) W_3^{-2} + X(2) W_3^{-1} = 3x(0) = 3 \quad \text{——— (3)}$$

Solving equations (1), (2) and (3), find the values of $X(0)$, $X(1)$ and $X(2)$.

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$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

① $W_4^{-0} = (W_4^0)^* = 1$

② $W_4^{-1} = (W_4^1)^* = j$

③ $W_4^{-2} = (W_4^2)^* = -1$

④ $W_4^{-3} = (W_4^3) = -j$

WKT, $x(n) = \frac{1}{4}[2 + 2W_4^{-2n}]$ $\quad 0 \le n \le 3.$

$\therefore x(0) = \frac{1}{4}(2+2) = 1$

$x(1) = \frac{1}{4}(2 + 2W_4^{-2}) = 0.$

$x(2) = \frac{1}{2}(2 + 2W_4^{-0}) = 1$

$x(3) = \frac{1}{2}(2 + 2W_4^{-2}) = 0.$

3. Compute 3-point DFT of the sequence $x(n) = (1, 0, 1)$ using Inverse DFT relation.

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$W_N \triangleq e^{-j\frac{2\pi}{N}}$

$W_3 = e^{-j\frac{2\pi}{3}}$

$\therefore W_3^0 = 1 \Rightarrow W_3^{-0} = 1$

$W_3^1 = e^{-j\frac{2\pi}{3}} = 1 \angle -2\pi/3 = -0.5 - j0.866$

$\Rightarrow W_3^{-1} = -0.5 + j0.866$

$W_3^2 = e^{-j\frac{4\pi}{3}} = 1 \angle -4\pi/3 = -0.5 + j0.866$

$$\Rightarrow W_3^{-2} = -0.5 - j0.866$$

$$x(n) \triangleq \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n = 0, 1, 2, \ldots N-1$$

$$\Rightarrow x(n) = \frac{1}{3} \sum_{K=0}^{2} X(K) \cdot W_3^{-Kn}$$

$$\Rightarrow 3x(n) = \sum_{k=0}^{2} X(k) \cdot W_3^{-kn} \qquad 0 \leq n \leq 2.$$

$$= X(0) + X(1)\, W_3^{-n} + X(2)\, W_3^{-2n}.$$

$$\therefore X(0) + X(1) + X(2) = 3x(0) = 3 \quad \text{—— } (1)$$

$$X(0) + X(1)\, W_3^{-1} + X(2)\, W_3^{-2} = 3x(1) = 0 \quad \text{—— } (2)$$

$$X(0) + X(1)\, W_3^{-2} + X(2)\, W_3^{-1} = 3x(0) = 3 \quad \text{—— } (3)$$

Solving equations (1), (2) and (3), find the values of $X(0)$, $X(1)$ and $X(2)$.

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$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

4. Prove that $\sum_{n=0}^{N-1} W_N^{kn} = N\delta(k)$ (OR) Find N point DFT of unit step function $x(n) = u(n)$

$$LHS = \sum_{n=0}^{N-1} (W_N^{k})^n$$

Soln: $DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$DFT\{u(n)\} = \sum_{n=0}^{N-1} u(n) W_N^{kn}$; Since $u(n)=1$

Recall : $\sum_{n=0}^{N-1} a^n = \frac{a^N - 1}{a-1}$, $a \neq 1$

$DFT\{u(n)\} = \sum_{n=0}^{N-1} W_N^{kn}$

$DFT\{u(n)\} = N\delta(k)$

$$LHS = \sum_{n=0}^{N-1} (W_N^{k})^n \quad \text{——— (1)}$$

$$= \frac{W_N^{KN} - 1}{W_N^{K} - 1} \quad ; \quad K \neq 0$$

$$= 0.$$

NOTE : $W_N^{KN} = \left(e^{-j\frac{2\pi}{N}}\right)^{KN}$

$$= e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k = 1.$$

In eqn. (1), put $k=0$

Then, $LHS = \sum_{n=0}^{N-1} 1 = N$

$$\therefore \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} 0 & , k \neq 0 \\ N & , k = 0 \end{cases} \quad \text{(or)} \quad N\delta(k).$$

5. Sequences $x(n)$ and $X(k)$ are implicite periodic, with a period equal to N.

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$$X(k) \triangleq \sum_{n=0}^{N-1} x(n).W_N^{kn}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n).\ W_N^{(k+N)n}$$

4. Prove that $\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(K)$ (OR) Find N point DFT of unit step function $x(n) = u(n)$

$$LHS = \sum_{n=0}^{N-1} (W_N^{K})^{n}$$

Recall : $\sum_{n=0}^{N-1} a^n = \frac{a^N - 1}{a-1}$, $a \neq 1$

$$LHS = \sum_{n=0}^{N-1} (W_N^{k})^{n} \quad \text{——— (1)}$$

$$= \frac{W_N^{KN} - 1}{W_N^{K} - 1} \quad ; \quad K \neq 0$$

$$= \underline{\underline{0.}}$$

NOTE : $W_N^{KN} = \left(e^{-j\frac{2\pi}{N}}\right)^{KN}$

$$= e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k = 1.$$

In eqn. (1), put $k=0$

Then, $LHS = \sum_{n=0}^{N-1} 1 = N$

$$\therefore \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} 0 & , k \neq 0 \\ N & , k = 0 \end{cases} \quad \text{(or)} \quad N\,\delta(k).$$

soln. $DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$DFT\{u(n)\} = \sum_{n=0}^{N-1} u(n)\, W_N^{kn}$; since $u(n) = 1$

$DFT\{u(n)\} = \sum_{n=0}^{N-1} W_N^{kn}$

$DFT\{u(n)\} = N\,\delta(k)$

5. Sequences $x(n)$ and $X(K)$ are implicite periodic, with a period equal to N.

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n).W_N^{kn}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n).\,W_N^{(K+N)n}$$

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$$= \sum_{n=0}^{N-1} x(n)\ W_N^{kn} \cdot W_N^{Nn}$$

$$\because\ W_N^{Nn} = e^{-j\frac{2\pi}{N}(Nn)} = e^{-j2\pi n} = 1$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = X(K)$$

$$\Rightarrow \boxed{X(K+N) = X(K)}$$

Hence, $X(k)$ is implicite periodic with a period equal to N.

$$x(n) \triangleq \frac{1}{N} \sum_{k=0}^{N-1} X(K)\ W_N^{-kn}$$

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$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(K)\ W_N^{-kn} \cdot W_N^{-KN}$$

$$\because\ W_N^{-KN} = e^{-j\frac{2\pi}{N}(-KN)} = e^{+j2\pi K} = 1 \text{ , we get.}$$

$$x(n+N) = \frac{1}{N} \sum_{n=0}^{N-1} X(K)\ W_N^{-kn} = \underline{\underline{x(n)}}$$

$$\boxed{x(n+N) = x(n)}$$

6. Compute 4-point DFT of the sequence $x(n) = 2^n$, $0 \le n \le 3$. Also find $X(-1)$ and $X(5)$.

Given, $x(n) = 2^n$, $0 \le n \le 3$.

$$\Rightarrow x(n) = (\underset{\uparrow}{1}, 2, 4, 8)$$

$$X(K) \triangleq \sum_{n=0}^{3} x(n)\, W_4^{kn} \qquad 0 \le K \le 3.$$

Recall : $W_4^0 = 1,\ W_4^{+1} = -j,\ W_4^{+2} = -1,\ W_4^{+3} = +j$

$$X(K) = x(0) + x(1) W_4^{K} + x(2) W_4^{2K} + x(3) W_4^{3K}$$

$$= 1 + 2W_4^{k} + 4W_4^{2K} + 8W_4^{3K}$$

$\Rightarrow X(0) = 1 + 2 + 4 + 8 = 15.$

$$X(1) = 1 + 2W_4^1 + 4W_4^2 + 8W_4^3 = 1 + 2(-j) + 4(-1) + 8(j)$$

$$= 1 - 2j - 4 + 8j$$

$$= -3 + 6j$$

$$X(2) = 1 + 2W_4^2 + 4W_4^0 + 8W_4^2 = 1 + 2(-1) + 4(1) + 8(-1)$$

$$= 1 - 2 + 4 - 8 = -5.$$

$$X(3) = 1 + 2W_4^{+3} + 4W_4^2 + 8W_4^1 = 1 + 2(j) + 4(-1) + 8(-j)$$

$$= -3 - 6j$$

$X(K) = (15, -3+j6, -5, -3-j6).$

$\therefore X(-1) = X(-1+4) = X(3) = -3 - j6.$

$X(5) = X(1) = -3 + j6.$

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OR.

Periodic extension of $X(K)$.

15, −3+j6, −5, −3−j6	15, −3+j6, −5, −3−j6	15, −3+j6, −5, −3−j6
↑ K=−1 (at −3−j6)	↑ K=0 (at 15)	↑ K=5 (at −3+j6)

USEFUL PROPERTIES OF DFT.

1. **LINEARITY.**

If DFT $\{x_1(n)\} = X_1(K)$ and DFT $\{x_2(n)\} = X_2(K)$, then

DFT $\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(K) + a_2 X_2(K)$.

Proof :

$$\text{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1} \left(a_1 x_1(n) + a_2 x_2(n)\right) W_N^{KN}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{KN} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{KN}$$

$$= a_1 X_1(K) + a_2 X_2(K).$$

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→ Compute 4-point DFT of the sequence given below using linearity property.

$$x(n) = \cos\left(\frac{\pi}{4} n\right) + \sin\left(\frac{\pi}{4} n\right) \qquad 0 \le n \le 3.$$

Sol^n: Let $x(n) = x_1(n) + x_2(n)$

where $x_1(n) = \cos\left(\frac{\pi}{4} n\right)$ and $x_2(n) = \sin\left(\frac{\pi}{4} n\right)$.

n	$x_1(n)$	$x_2(n)$
0	1	0
1	$1/\sqrt{2}$	$1/\sqrt{2}$
2	0	1
3	$-1/\sqrt{2}$	$1/\sqrt{2}$

$$X(K) \triangleq \sum_{n=0}^{3} x(n)\, W_4^{kn} \qquad 0 \le k \le 3.$$

Recall : $W_4^0 = 1$, $W_4^{+1} = -j$, $W_4^{+2} = -1$, $W_4^{+3} = +j$

$$X(K) = x(0) + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k}$$

$$= 1 + 2W_4^{k} + 4W_4^{2k} + 8W_4^{3k}$$

$\Rightarrow \quad X(0) = 1+2+4+8 = 15.$

$$X(1) = 1 + 2W_4^1 + 4W_4^2 + 8W_4^3 = 1 + 2(-j) + 4(-1) + 8(j)$$

$$= 1 - 2j - 4 + 8j$$

$$= -3 + 6j$$

$$X(2) = 1 + 2W_4^2 + 4W_4^0 + 8W_4^2 = 1 + 2(-1) + 4(1) + 8(-1)$$

$$= 1 - 2 + 4 - 8 = -5.$$

$$X(3) = 1 + 2W_4^{+3} + 4W_4^2 + 8W_4^1 = 1 + 2(j) + 4(-1) + 8(-j)$$

$$= -3 - 6j$$

$X(K) = (15, -3+j6, -5, -3-j6).$

$\therefore \quad X(-1) = X(-1+4) = X(3) = -3-j6.$

$X(5) = X(1) = -3+j6.$

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OR.

Periodic extension of $X(K)$.

15, -3+j6, -5, -3-j6 ¦ 15, -3+j6, -5, -3-j6 ¦ 15, -3+j6, -5, -3-j6

↑ K=-1 ↑ K=0 ↑ K=5.

Recall : $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = +j$

$$X_1(K) = \sum_{n=0}^{3} x_1(n) W_4^{kn} = 1 + \frac{1}{\sqrt{2}} W_4^{k} - \frac{1}{\sqrt{2}} W_4^{3k}$$

$$X_2(K) = \sum_{n=0}^{3} x_2(n) W_4^{kn} = 0 + \frac{1}{\sqrt{2}} W_4^{K} + W_4^{2K} + \frac{1}{\sqrt{2}} W_4^{3K}$$

$$\therefore X_1(0) = 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1.$$

$$X_1(1) = 1 + \frac{1}{\sqrt{2}} W_4^1 - \frac{1}{\sqrt{2}} W_4^3 = 1 + \frac{1}{\sqrt{2}}(-j) - \frac{1}{\sqrt{2}}(j)$$

$$= 1 - \frac{1}{\sqrt{2}}j - \frac{1}{\sqrt{2}}j = 1 - \sqrt{2}j$$

$$X_1(2) = 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^2 = 1 + \frac{1}{\sqrt{2}}(-1) - \frac{1}{\sqrt{2}}(-1) = 1$$

$$X_1(3) = 1 + \frac{1}{\sqrt{2}} W_4^3 - \frac{1}{\sqrt{2}} W_4^1 = 1 + \frac{1}{\sqrt{2}}(j) - \frac{1}{\sqrt{2}}(-j)$$

$$= 1 + j\sqrt{2}.$$

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$$X_2(0) = 1 + 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2.414.$$

$$X_2(1) = 0 + \frac{1}{\sqrt{2}} W_4^1 + W_4^2 + W_4^3 \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-j) - 1 + \frac{1}{\sqrt{2}}(j)$$

$$= -1$$

$$X_2(2) = \frac{1}{\sqrt{2}} W_4^2 + W_4^0 + \frac{1}{\sqrt{2}} W_4^2 = \frac{1}{\sqrt{2}}(-1) + 1 + \frac{1}{\sqrt{2}}(-1) = -0.414$$

$$X_2(3) = \frac{1}{\sqrt{2}} W_4^3 + W_4^2 + \frac{1}{4} W_4^1 = -1.$$

Applying Linearity property, we get

$$X(K) = X_1(K) + X_2(K) \qquad 0 \le K \le 3.$$

$$\therefore\ X(0) = X_1(0) + X_2(0) = 1 + 2.414 = 3.414$$

$$X(1) = X_1(1) + X_2(1) = 1 - \sqrt{2}j - 1 = -\sqrt{2}j$$

$$X(2) = X_1(2) + X_2(2) = 1 - 0.414 = 0.586$$

$$X(3) = X_1(3) + X_2(3) = 1 + j\sqrt{2} - 1 = \sqrt{2}j$$

2. CIRCULAR SHIFT (OR) CIRCULAR TRANSLATION.

Let $x(n)$ be a sequence defined for all values of 'n', the shifted version of $x(n)$ is written as $x(n-n_0)$ and this means that $x(n)$ is shifted to right by a time index equal to 'n_0'. When dealing with DFT, a sequences are defined only for finite values of 'n'

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i.e, $n = 0, 1, \ldots N-1.$

Hence, if a regular shift is attempted, the last few samples would be lost and a part of initial sequence become undefined.

$\therefore$ While dealing with DFT, a special type of translation known as Circular translation must be used. The circular translation of a finite length sequence $x(n)$ shifted to right by an amount 'n_0' is denoted by

$$x((n-n_0))_N \quad \text{(or)} \quad x(n-n_0) \text{ modulo } N.$$

The sequence $x((n-n_o))_N$ means move last 'n_o' samples of $x(n)$ to the beginning.

Similarly, $x((n+n_o))_N$ means move first 'n_o' samples to the end.

Another way of visualizing circular translation is as follows-

Get the periodic extension of finite length sequence $x(n)$. Usually, 3 to 4 extensions are sufficient. Give a linear shift to this extended periodic sequence. Then extract 1 period from the extended sequence. The extracted sequence will be a circularly shifted sequence.

3. TIME SHIFT PROPERTY :

If DFT $\{x(n)\} = X(K)$, then –

$$\boxed{\text{DFT}\ \{x((n-m))_N\} = W_N^{Km}\ X(K)}$$

Proof : $x(n) \triangleq \frac{1}{N}\sum_{K=0}^{N-1} X(K)\cdot W_N^{-Kn}$

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$$x(n-m) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\cdot W_N^{-K(n-m)}$$

Since the shift is circular, we can write

$$x((n-m))_N = \frac{1}{N}\sum_{K=0}^{N-1} [X(K).W_N^{Km}]\ W_N^{-Kn}$$

$\therefore$ IDFT $\{X(K).\ W_N^{Km}\} = x((n-m))_N$

DFT $\{x((n-m))_N\} = W_N^{Km}\ X(K).$

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→ Compute a 4-point DFT of the sequence –

$x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$.

Also find $Y(K)$ if $y(n) = x((n-2))_4$.

$$X(K) \triangleq \sum_{n=0}^{3} x(n).\, W_4^{kn} \qquad K = 0, 1, 2 \ldots 3.$$

$$\Rightarrow X(K) = \sum_{n=0}^{3} \left[\delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3) \right] W_4^{kn}$$

$$= W_4^{kn}\Big|_{n=0} + 2W_4^{kn}\Big|_{n=1} + 3W_4^{kn}\Big|_{n=2} + 4W_4^{kn}\Big|_{n=3}$$

$$X(K) = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k} \qquad K = 0, 1, 2, 3.$$

$$x(n) = (\underset{\uparrow}{1}, 2, 3, 4)$$

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$$W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = +j$$

$$X(0) = 1 + 2 + 3 + 4 = 10$$

$$X(1) = 1 + 2W_4^1 + 3W_4^2 + 4W_4^3 = 1 + 2(-j) + 3(-1) + 4(j)$$

$$= -2 + j2.$$

$$X(2) = 1 + 2W_4^2 + 3W_4^0 + 4W_4^2$$

$$= 1 + 2(-1) + 3(1) + 4(-1) = 1 - 2 + 3 - 4$$

$$= -2.$$

$$X(3) = 1 + 2W_4^3 + 3W_4^2 + 4W_4^1$$

$$= -2 - j2$$

$$\therefore X(K) = (10, -2+j2, -2, -2-j2).$$

Given, $y(n) = x((n-2))_4$

Recall time-shift property :

$$DFT\{x((n-m))_N\} = W_N^{km} X(K)$$

$$\Rightarrow \quad Y(K) = W_4^{2k} X(K)$$

$$Y(0) = W_4^0 X(0) = 10$$

$$Y(1) = W_4^2 X(1) = -1 \times (-2+2j) = 2-2j$$

$$Y(2) = W_4^4 X(2) = 1 \times -2 = -2$$

$$Y(3) = W_4^6 X(3) = -1 \times (-2-j2) = 2+j2.$$

$$DFT\{y(n)\} = DFT\{x((n-m))_N\}$$
$$= Y(K)$$
$$= W^{2K} X(K)$$

4. FREQUENCY SHIFT PROPERTY.

If $DFT\{x(n)\} = X(K)$, then -

$$\boxed{DFT\{W_N^{-ln}\, x(n)\} = X((K-l))_N}$$

Proof : WKT, $X(K) \triangleq \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

$$X(K-l) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{(K-l)n}$$

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The quantity $(K-l)$ is modulo N type, we can write

$$X((K-l))_N = \sum_{n=0}^{N-1} [x(n) \cdot W_N^{-ln}]\, W_N^{kn}$$

$$\therefore DFT\{x(n) \cdot W_N^{-ln}\} = X((K-l))_N$$

→ Compute 4-point DFT of the sequence –

$x(n) = (\underset{\uparrow}{1}, -1, -1, 1)$. Also find $y(n)$ if $Y(K) = X((K-2))_4$

Sol^n :
$$X(K) \triangleq \sum_{n=0}^{3} x(n)\, W_4^{Kn} \qquad 0 \le K \le 3.$$

$$= 1 - W_4^{K} - W_4^{2K} - W_4^{3K}$$

NOTE : $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = j$

$$X(0) = 1 - 1 - 1 + 1 = 0$$
$$X(1) = 1 - W_4^1 - W_4^2 + W_4^3 = 1 + j + 1 + j = 2 + 2j$$
$$X(2) = 1 - W_4^2 - W_4^0 + W_4^2 = 1 + 1 - 1 - 1 = 0$$
$$X(3) = 1 - W_4^3 - W_4^2 + W_4^1 = 1 - j + 1 - j = 2 - 2j$$

Given, $Y(K) = X((K-2))_4$

RECALL : $DFT\{W_N^{-ln} x(n)\} = X((K-l))_N$

$IDFT\{X((K-l))_N\} = W_N^{-ln} x(n)$

$\therefore y(n) = W_N^{-ln} \cdot x(n)$.

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$$\Rightarrow y(0) = W_N^{-0} \cdot x(0) = (W_N^0)^* \cdot x(0)$$
$$y(0) = 1\,(1) = 1.$$

$$y(1) = W_N^{-2} \cdot x(1) = (W_N^2)^* \cdot x(1)$$
$$y(1) = -1\,(-1) = 1.$$

$$y(2) = W_N^{-0}\, x(2) = (W_4^0)^* \cdot x(2)$$
$$y(2) = -1$$

$$y(3) = W_4^{-2} \cdot x(3) = (W_4^2)^* \cdot x(3) = -1$$

$$\therefore y(n) = (\underset{\uparrow}{1}, 1, -1, -1)$$

5. SYMMETRY PROPERTY FOR REAL SEQUENCES :

Let $x(n)$ be a real sequence with DFT $\{x(n)\} = X(K)$,

then –

$$\boxed{X^*(K) = X(N-K)} \quad \text{OR} \quad X(k) = X^*(N-k)$$

proof : $X(k) \triangleq \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$\Rightarrow X^*(K) = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{-kn}$$

$\because$ $x(n)$ is a real sequence, we can write

$$X^*(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{-kn} \times 1$$

Since $W_N^{Nn} = e^{-j\frac{2\pi}{N} \times (Nn)} = e^{-j2\pi n} = 1$, we get

$$X^*(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{-kn} \cdot W_N^{Nn}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)n}$$

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$$\boxed{X^*(K) = X(N-K)}$$

→ Compute 4-point DFT of the sequence $x(n) = \cos\left(\frac{\pi}{4} n\right)$ and check for symmetry property.

Sol^n. Given, $x(n) = \cos\left(\frac{\pi}{4} n\right)$, $0 \le n \le 3$.

$$\Rightarrow x(n) = \left(1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$X(K) = \sum_{n=0}^{3} W_N^{kn}\, x(n). \qquad K = 0, 1, 2, 3.$$

$$\Rightarrow X(K) = 1 + \frac{1}{\sqrt{2}} W_4^{K} - \frac{1}{\sqrt{2}} W_4^{3K}$$

Recall : $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = +j$

$$\therefore X(0) = 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1$$

$$X(1) = 1 + \frac{1}{\sqrt{2}} W_4^1 - \frac{1}{\sqrt{2}} W_4^3 = 1 + \frac{1}{\sqrt{2}}(-j) - \frac{1}{\sqrt{2}}(j)$$

$$= 1 - \sqrt{2}j$$

$$X(2) = 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^2 = 1 + \frac{1}{\sqrt{2}}(-1) - \frac{1}{\sqrt{2}}(-1)$$

$$= 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1$$

$$X(3) = 1 + \frac{1}{\sqrt{2}} W_4^3 - \frac{1}{\sqrt{2}} W_4^1$$

$$= 1 + \frac{1}{\sqrt{2}}(j) - \frac{1}{\sqrt{2}}(-j) = 1 + \sqrt{2}j$$

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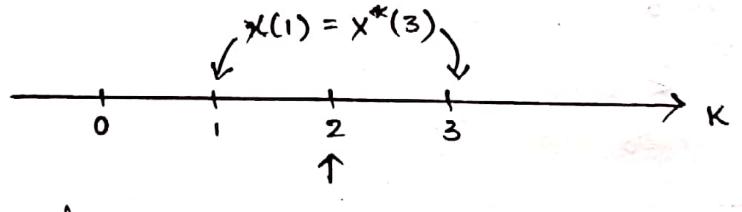

Symmetry about $FI = \frac{N}{2} = 2.$

→ Compute 5 point DFT of the sequence $x(n) = (1, 1, 1, 0, 0)$
Also check for symmetry property.

Sol^n : $x(n) = (\underset{\uparrow}{1}, 1, 1, 0, 0)$

WKT, $W_5 = e^{-j2\pi/5}$

$\Rightarrow W_5^0 = 1$

$W_5^1 = e^{-j2\pi/5} = 1\angle -2\pi/5 = 0.309 - j0.951.$

$W_5^2 = e^{-j4\pi/5} = 1\angle -4\pi/5 = -0.809 - j0.587$

$W_5^3 = e^{-j6\pi/5} = 1\angle -6\pi/5 = -0.809 + j0.587$

$W_5^4 = e^{-j8\pi/5} = 1\angle -8\pi/5 = 0.309 + j0.951.$

FI = 2

$$X(K) = \sum_{n=0}^{4} x(n)\, W_5^{kn}$$

$$= 1 + W_5^{K} + W_5^{2K} \qquad 0 \le K \le 4$$

$\therefore\ X(0) = 1+1+1 = 3$

$X(1) = 1 + W_5^1 + W_5^2 = 1 + 0.309 - j0.951 - 0.809 - j0.587.$

$X(1) = 0.5 - j1.538.$

$X(2) = 1 + W_5^2 + W_5^4 = 1 - 0.809 - j0.587 + 0.309 + j0.951$

$X(2) = 0.5 + j0.364.$

FI =2.5

$X(3) = 1 + W_5^3 + W_5^1 = 1 - 0.809 + j0.587 + 0.309 - j0.951$

$X(3) = 0.5 - j0.364$

$X(4) = 1 + W_5^4 + W_5^3 = 1 + 0.309 + j0.951 - 0.809 + j0.587$

$X(4) = 0.5 + j1.538.$

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Symmetry property :

$X(K) = X^*(N-K)$

$\Rightarrow X(K) = X^*(5-K)$

Here, we find that

$X(1) = X^*(4)$

and $X(2) = X^*(3).$

//

6. COMPLEX CONJUGATE PROPERTY :

Let $x(n)$ be a complex conjugate sequence having an n-point DFT given by $X(K)$, then -

$$DFT\{x^*(n)\} = X^*((-K))_N = X^*(N-K)$$

Proof : $$X^*(K) = \sum_{n=0}^{N-1} x^*(n) . W_N^{-Kn} \quad \text{——— (1)}$$

Changing K to −K in (1), we get.

$$X^*(-K) = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{Kn}$$

Since folding is circular, we can write -

$$X^*((-K))_N = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{Kn}$$

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$$\therefore DFT\{x^*(n)\} = X^*((-K))_N \quad \text{——— (2)}$$

changing K to N−K in eq. (1), we get

$$X^*(N-K) = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{-(N-K)n}$$

$$= \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{Kn} \cdot \underbrace{W_N^{-Nn}}_{\rightarrow 1}$$

$$\therefore DFT\{x^*(n)\} = X^*(N-K) \quad \text{——— (3)}$$

From (2) and (3), we can write -

$$DFT\{x^*(n)\} = X^*((-K))_N = \underline{\underline{X^*(N-K)}}$$

→ Find 4-point DFT of the sequence → $x(n) = \cos\left(\frac{\pi}{4}n\right) + j\sin\left(\frac{\pi}{2}n\right)$, $0 \le n \le 3$. Also, find DFT of $\{x^*(n)\}$.

Sol^n : Given, $x(n) = \cos\left(\frac{\pi}{4}n\right) + j\sin\left(\frac{\pi}{2}n\right)$

$$\Rightarrow x(n) = \left(1, \frac{1}{\sqrt{2}} + j1, 0, \frac{-1}{\sqrt{2}} - j1\right)$$

Recall : $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = +j$

WKT, $X(K) \triangleq \sum_{n=0}^{3} x(n) \cdot W_4^{Kn}$ $K = 0, 1, 2, 3.$

$$\Rightarrow X(K) = 1 + \left(\frac{1}{\sqrt{2}} + j1\right) W_4^K - \left(\frac{1}{\sqrt{2}} + j1\right) W_4^{3K}$$

$$\therefore X(0) = 1 + \frac{1}{\sqrt{2}} + j1 - \frac{1}{\sqrt{2}} - j1 = 1.$$

$$X(1) = 1 + \left(\frac{1}{\sqrt{2}} + j1\right) W_4^1 - \left(\frac{1}{\sqrt{2}} + j1\right) W_4^3$$

$$= 1 + \left(\frac{1}{\sqrt{2}} + j1\right)(-j) - \left(\frac{1}{\sqrt{2}} + j1\right)(+j)$$

$$= 1 - \frac{1}{\sqrt{2}}j + 1 - \frac{1}{\sqrt{2}}j + 1 = 3 - \sqrt{2}j$$

$$X(2) = 1 + \left(\frac{1}{\sqrt{2}} + j1\right) W_4^2 - \left(\frac{1}{\sqrt{2}} + j1\right) W_4^2$$

$$= 1 + \left(\frac{1}{\sqrt{2}} + j1\right)(-1) - \left(\frac{1}{\sqrt{2}} + j1\right)(-1) = 1$$

$$X(3) = 1 + \left(\frac{1}{\sqrt{2}} + j1\right) W_4^3 - \left(\frac{1}{\sqrt{2}} + j1\right) W_4^1$$

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$$= 1 + \left(\frac{1}{\sqrt{2}} + j1\right)(j) - \left(\frac{1}{\sqrt{2}} + j1\right)(-j)$$

$$= -1 + j\sqrt{2}$$

K	X(K)	X*(K)
0	1	1
1	$3-j\sqrt{2}$	$3+j\sqrt{2}$
2	1	1
3	$-1+j\sqrt{2}$	$-1-j\sqrt{2}$

Let $y(n) = x^*(n)$

then, $Y(K) = \text{DFT}\{x^*(n)\}$

$$= X^*((-K))_4$$

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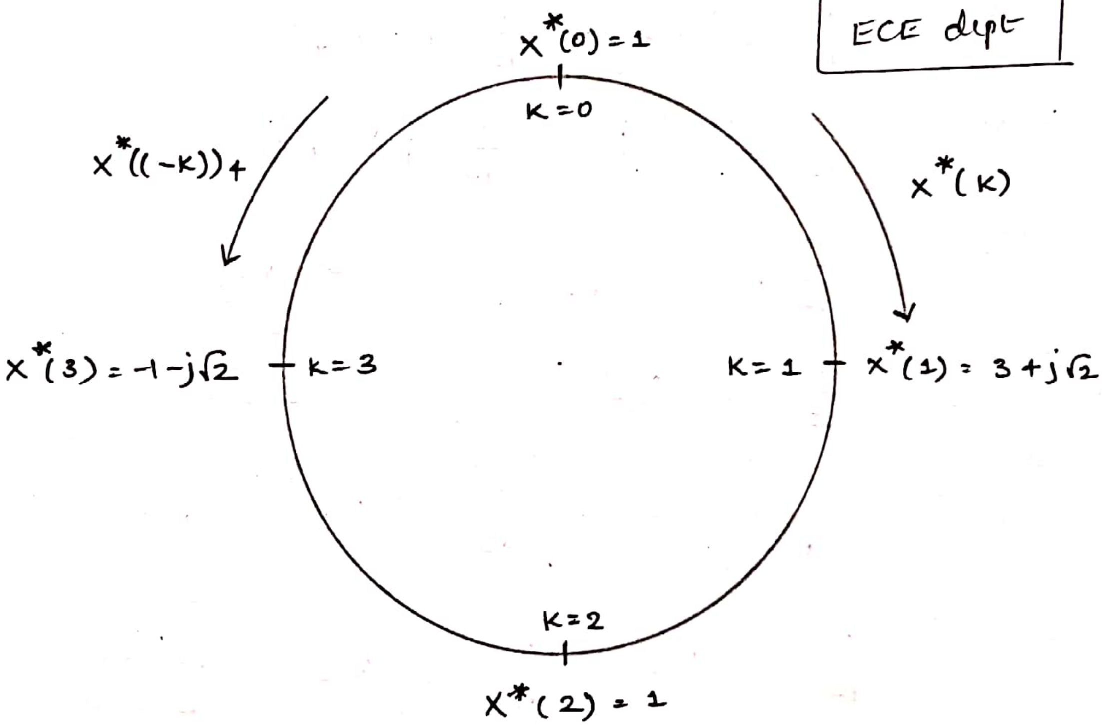


$$Y(K) = (1, -1-j\sqrt{2}, 1, 3+j\sqrt{2})$$

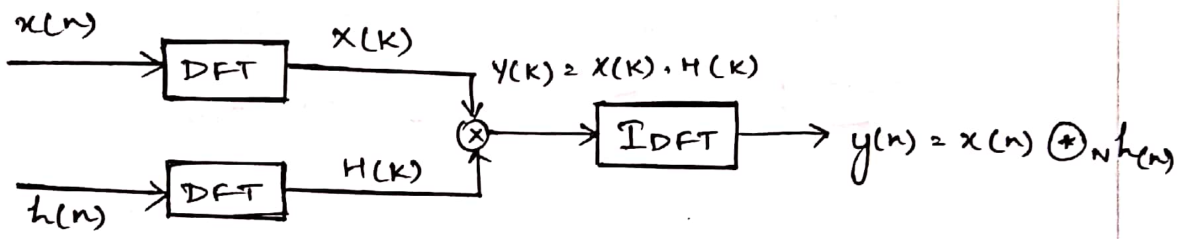
7. CIRCULAR CONVOLUTION

Let $x(n)$ and $h(n)$ be 2 sequences of length 'N' each. Then,

$$y(n) = x(n) \circledast_N h(n) \triangleq \sum_{m=0}^{N-1} x(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{N-1} x((n-m))_N\, h(m)$$

Proof :



The proof of circular convolution is obtained by referring to the block diagram shown above.

WKT, $y(n) = \text{IDFT}\{Y(K)\}$

$$= \frac{1}{N} \sum_{K=0}^{N-1} Y(K) \cdot W_N^{-Kn} \qquad n = 0, 1, \ldots N-1.$$

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$$\Rightarrow y(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, H(K)\, W_N^{-Kn} \quad \text{——} \quad (1)$$

Also,

$$X(K) = \sum_{i=0}^{N-1} x(i)\, W_N^{iK} \quad \text{——} \quad (2) \qquad K = 0, 1, \ldots N-1$$

and

$$H(K) = \sum_{m=0}^{N-1} h(m) \cdot W_N^{mK} \quad \text{——} \quad (3) \qquad K = 0, 1, \ldots N-1$$

Substituting equations ② and ③ in ①, we get -

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} \cdot \sum_{i=0}^{N-1} x(i) \cdot W_N^{ik} \sum_{m=0}^{N-1} h(m) W_N^{mk} W_N^{-kn}$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} x(i) \sum_{m=0}^{N-1} h(m) \sum_{k=0}^{N-1} W_N^{(i-(n-m))k}$$

Since, $\sum_{k=0}^{N-1} W_N^{(i-(n-m))k} = N\delta(i-(n-m))$, we get -

$$y(n) = \frac{1}{N} \sum_{i=0}^{N-1} x(i) \sum_{m=0}^{N-1} h(m) \, N. \, \delta(i-(n-m))$$

$$= \sum_{m=0}^{N-1} h(m) \sum_{i=0}^{N-1} x(i). \, \delta(i-(n-m))$$

$$y(n) = \sum_{m=0}^{N-1} x(i) \Big|_{i=n-m} \cdot h(m)$$

$$= \sum_{m=0}^{N-1} h(m) \cdot x(n-m).$$

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Since the quantity $(n-m)$ is modulo n type, the above equation can be written as

$$y(n) = \sum_{m=0}^{N-1} h(m) \, x((n-m))_N$$

Since circular convolution is commutative in nature, the above equation can be written as -

$$y(n) = \sum_{m=0}^{N-1} h((n-m))_N \, x(m).$$

→ 1) Compute 4- point circular convolution of the sequences $x(n)$ and $h(n)$. given below :

$x(n) = (1, 1, 1, 2)$ and $h(n) = (4, 6, 2, 1)$

METHOD (1) :

Let $y(n) = x(n) \circledast_4 h(n)$

$$y(n) = \sum_{m=0}^{3} x(m) \cdot h((n-m))_4$$

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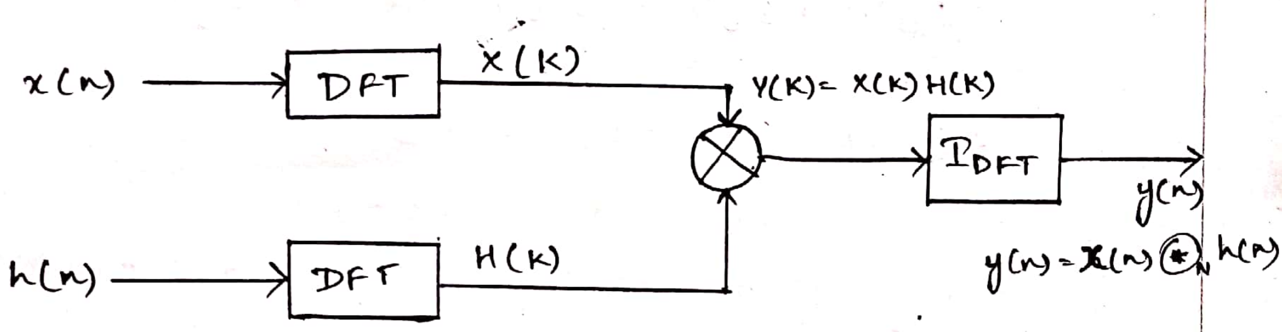
n	$x(m)$	$h((n-m))_4$	$y(n)$
0	(1, 1, 1, 2)	(4, 1, 2, 6)	4+1+12+2 = 19
1	(1, 1, 1, 2)	(6, 4, 1, 2)	6+4+1+4 = 15
2	(1, 1, 1, 2)	(2, 6, 4, 1)	2+6+4+2 = 14
3	(1, 1, 1, 2)	(1, 2, 6, 4)	1+2+6+8 = 17

$\therefore y(n) = (\underset{\uparrow}{19}, 15, 14, 17)$

METHOD (2) : STOCKHAM'S METHOD (OR) FREQUENCY DOMAIN APPROACH.



$$X(K) = \sum_{n=0}^{3} x(n)\, W_4^{Kn}$$

$$= 1 + W_4^{K} + W_4^{2K} + 2W_4^{3K}$$

$$H(K) = \sum_{n=0}^{3} h(n) \cdot W_4^{Kn}$$

$$= 4 + 6W_4^{K} + 2W_4^{2K} + W_4^{3K}$$

$$Y(K) = 4 + 6W_4^{K} + 2W_4^{2K} + 2W_4^{3K} + 6W_4^{2K} + 4W_4^{K} + 2W_4^{3K} + W_4^{0K} + 4W_4^{2K} + 6W_4^{3K} + 2W_4^{0K} + W_4^{K} + 8W_4^{3K} + 12W_4^{0K} + 4W_4^{K} + 2W_4^{2K}.$$

$$= 19 + 15W_4^{K} + 14W_4^{2K} + 17W_4^{3K}$$

$$\therefore\ y(n) = 19 + 15\,\delta(n-1) + 14\,\delta(n-2) + 17\,\delta(n-3)$$

$$y(n) = (\underset{\uparrow}{19}, 15, 14, 17)$$

NOTE: $DTF\{\delta(n-n_0)\}$

$$= \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{Kn}$$

$$= W_N^{Kn}\Big|_{n=n_0} = W_N^{Kn_0}$$

$$\therefore\ IDFT\{W_N^{Kn_0}\} = \delta(n-n_0)$$

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METHOD 3 :

Let $y_l(n) = x(n) * h(n)$

$$y_l(n) = [\delta(n) + \delta(n-1) + \delta(n-2) + 2\delta(n-3)] * [4\delta(n) + 6\,\delta(n-1) + 2\,\delta(n-2) + \delta(n-3)]$$

$$= 4\,\delta(n) + 6\delta(n-1) + 2\delta(n-2) + \delta(n-3) + 4\,\delta(n-1) + 6\delta(n-2) + 2\delta(n-3) + \delta(n-4) + 4\,\delta(n-2) + 6\delta(n-3) + 2\delta(n-4) + \delta(n-5) + 8\delta(n-3) + 12\delta(n-4) + 4\delta(n-5) + 2\delta(n-6).$$

$= 4\,\delta(n) + 10\,\delta(n-1) + 12\,\delta(n-2) + 17\,\delta(n-3) + 15\,\delta(n-4) + 5\,\delta(n-5)$

$+ 2\,\delta(n-6).$

$y_l(n) = (4, 10, 12, 17, 15, 5, 2).$

$y_c(n) = (15+4, 10+5, 12+2, 17).$

$y_c(n) = (19, 15, 14, 17)$
$\uparrow$

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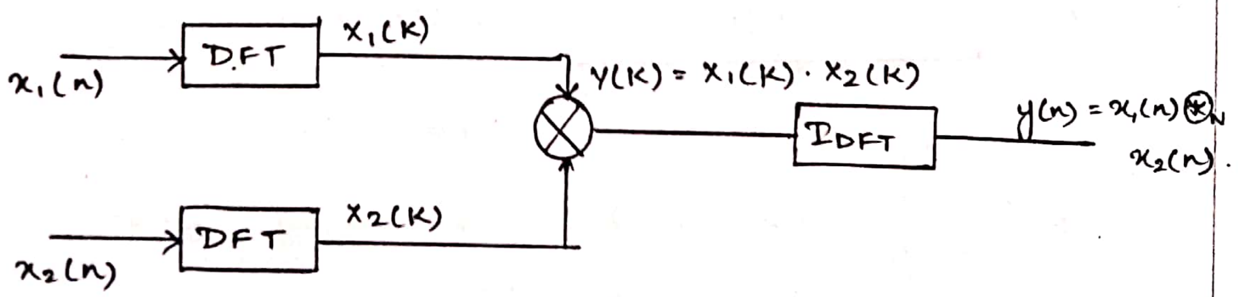
$\therefore$ Circular convolution = Linear convolution + Aliasing.

2) Compute 4-point circular convolution of the two sequences given below. –

$x_1(n) = (1, 2, 3, 4)$ and $x_2(n) = (1, 6, 7, 2).$
(arrows under 2 and 6)



$$X_1(K) = \sum_{n=-1}^{2} x_1(n) \cdot W_4^{kn}$$

$$= W_4^{-K} + 2 + 3W_4^{K} + 4W_4^{2K}$$

Similarly, $X_2(K) = W_4^{-K} + 6 + 7W_4^{K} + 2W_4^{2K}$

$Y(K) = X_1(K) \cdot X_2(K)$

$$= W_4^{2K} + 6W_4^{-K} + 7 + 2W_4^{K} + 2W_4^{-K} + 12 + 14W_4^{K} + 4W_4^{2K}$$
$$+ 3 + 18W_4^{K} + 21W_4^{2K} + 6W_4^{-K} + 4W_4^{K} + 24W_4^{2K} + 28W_4^{-K} + 8$$

$$= 42W_4^{-K} + 30 + 38W_4^{K} + 50W_4^{2K}$$

$$\Rightarrow y(n) = (42, \underset{\uparrow}{30}, 38, 50)$$

Method ②:

$x_1(-1) = 1 \Rightarrow x_1(-1+4) = x_1(3) = 1$

$x_2(-1) = 1 \Rightarrow x_2(3) = 1$

$\therefore x_1(n) = (\underset{\uparrow}{2}, 3, 4, 1) \quad x_2(n) = (\underset{\uparrow}{6}, 7, 2, 1)$.

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$$y(n) = x_1(n) \circledast_4 x_2(n) = \sum_{m=0}^{3} x_1(m)\, x_2((n-m))_4$$

n	$x_1(m)$	$x_2((n-m))_4$	$y(n)$
0	(2, 3, 4, 1)	(6, 1, 2, 7)	12+3+8+7 = 30
1	(2, 3, 4, 1)	(7, 6, 1, 2)	14+18+4+2 = 38
2	(2, 3, 4, 1)	(2, 7, 6, 1)	4+21+24+1 = 50
3.	(2, 3, 4, 1)	(1, 2, 7, 6)	2+6+28+6 = 42

$$y(n) = (\underset{\uparrow}{30}, 38, 50, 42)$$

$$y(3) = 42 \Rightarrow y(3-4) = y(-1) = 42$$

$$\therefore y(n) = (42, \underset{\uparrow}{30}, 38, 50)$$

3). Given $x_1(n) = (1, 6, \underset{\uparrow}{7}, 8, 5)$ and $x_2(n) = (1, 2, \underset{\uparrow}{3}, 4, 5)$

Find $y(n) = x_1(n) \circledast_5 x_2(n)$.

$x_1(n) = (\underset{\uparrow}{7}, 8, 5, 1, 6)$

$x_2(n) = (\underset{\uparrow}{3}, 4, 5, 1, 2)$

n	-3	-2	-1	0	1	2	3	4	5	6
$x_1(n)$	5	1	6	7	8	5	1	6	7	8
$x_2(n)$	5	1	2	3	4	5	1	2	3	4

$$y(n) = x_1(n) \circledast_5 x_2(n) = \sum_{m=0}^{4} x_1(m) \cdot x_2((n-m))_5$$

n	$x_1(m)$	$x_2((n-m))_4$	$y(n)$
0	(7, 8, 5, 1, 6)	(3, 2, 1, 5, 4)	21 + 16 + 5 + 5 + 24 = 71.
1	(7, 8, 5, 1, 6)	(4, 3, 2, 1, 5)	28 + 24 + 10 + 1 + 30 = 93
2.	(7, 8, 5, 1, 6)	(5, 4, 3, 2, 1)	35 + 32 + 15 + 2 + 6 = 90
3.	(7, 8, 7, 1, 6)	(1, 5, 4, 3, 2)	7 + 40 + 20 + 3 + 12 = 82
4.	(7, 8, 5, 1, 6)	(2, 1, 5, 4, 3)	14 + 8 + 25 + 4 + 18 = 69.

$y(n) = (\underset{\uparrow}{71}, 93, 90, 82, 69)$

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$\therefore y(n) = (82, 69, \underset{\uparrow}{71}, 93, 90)$

4) Perform linear convolution of the 2 sequences given below.

a) $x_1(n) = (\underset{\uparrow}{1}, 2, 3)$ $x_2(n) = (\underset{\uparrow}{1}, 6, 7)$.

b) Perform linear convolution of the sequences $x_1(n)$ and $x_2(n)$ through circular convolution.

Sol^n : Part (b) :

$N_1 \rightarrow$ Non - zero length of $x_1(n) = 3$.

$N_2 \rightarrow$ Non - zero length of $x_2(n) = 3$.

Let 'N' be the lengths of $x_1(n)$ and $x_2(n)$ for which circular convolution is identical to linear convolution and this length is given by

$$N \geq N_1 + N_2 - 1$$

Minimum value of $N = N_1 + N_2 - 1$
$= 3 + 3 - 1 = 5$.

Let us increase the lengths of $x_1(n)$ and $x_2(n)$ to 5 by padding them with zeroes.

$x_1(n) = (1, 2, 3, 0, 0)$, $x_2(n) = (1, 6, 7, 0, 0)$.

Let $y_c(n) = x_1(n) \circledast_5 x_2(n)$

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$$= \sum_{n=0}^{4} x_1(m) \cdot x_2((n-m))_5$$

$$\begin{bmatrix} y_c(0) \\ y_c(1) \\ y_c(2) \\ y_c(3) \\ y_c(4) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 7 & 6 \\ 6 & 1 & 0 & 0 & 7 \\ 7 & 6 & 1 & 0 & 0 \\ 0 & 7 & 6 & 1 & 0 \\ 0 & 0 & 7 & 6 & 1 \end{bmatrix}}_{\text{circulant Matrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 22 \\ 32 \\ 21 \end{bmatrix}$$

$y_c(n) = (\underset{\uparrow}{1}, 8, 22, 32, 21)$

Part (a) :

$$y_1(n) = x_1(n) * x_2(n).$$

$$y_1(n) = [\delta(n) + 2\delta(n-1) + 3\delta(n-2)] * [\delta(n) + 6\delta(n-1) + 7\delta(n-2)]$$

$$= \delta(n) + 6\delta(n-1) + 7\delta(n-2) + 2\delta(n-1) + 12\delta(n-2) +$$

$$14\delta(n-3) + 3\delta(n-2) + 18\delta(n-3) + 21\delta(n-4)$$

$$y_2(n) = \delta(n) + 8\delta(n-1) + 22\delta(n-2) + 32\delta(n-3) + 21\delta(n-4).$$

Hence, $y_c(n) = y_2(n)$.

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5) Given $x_1(n) = (1, 4, 6, 0, 7)$ and $x_2(1, 3)$.

a) Perform Linear convolution of $x_1(n)$ and $x_2(n)$.

b). Perform circular convolution of $x_1(n)$ and $x_2(n)$ and get the result of part (a).

Part (b) : $N_1 \rightarrow$ Non-zero length of $x_1(n) = 5$.

$N_2 \rightarrow$ Non-zero length of $x_2(n) = 2$

Minimum value of $N = N_1 + N_2 - 1$

$$= 5 + 2 - 1 = 6.$$

Let us increase the lengths of $x_1(n)$ and $x_2(n)$ to 6 by padding them with zeroes.

$x_1(n) = (1, 4, 6, 0, 7, 0)$ & $x_2(n) = (1, 3, 0, 0, 0, 0)$.

Let $y_c(n) = x_1(n) \circledast_6 x_2(n)$.

$$= \sum_{n=0}^{5} x_1(m) \cdot x_2((n-m))_6 .$$

$$\begin{bmatrix} y_c(0) \\ y_c(1) \\ y_c(2) \\ y_c(3) \\ y_c(4) \\ y_c(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 6 \\ 0 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 18 \\ 18 \\ 7 \\ 21 \end{bmatrix}$$

$$y_c(n) = (\underset{\uparrow}{1}, 7, 18, 18, 7, 21).$$

Part (a) :

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6) a) Perform 4-point circular convolution of the 2 sequences given below –

$x_1(n) = (\underset{\uparrow}{1}, 2, 4, 6)$ and $x_2(n) = (\underset{\uparrow}{6}, 1, 7, 2)$

b) Perform linear convolution of $x_1(n)$ and $x_2(n)$ and get the result of part (a).

Part (a) : $y_c(n) = x_1(n) \circledast_4 x_2(n)$.

$$= \sum_{n=0}^{3} x_1(m) \cdot x_2((n-m))_4 .$$

$$\begin{bmatrix} y_c(0) \\ y_c(1) \\ y_c(2) \\ y_c(3) \end{bmatrix} = \begin{bmatrix} 6 & 2 & 7 & 1 \\ 1 & 6 & 2 & 7 \\ 7 & 1 & 6 & 2 \\ 2 & 7 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6+4+28+6 = 44 \\ 1+12+8+48 = 63 \\ 7+2+24+12 = 45 \\ 2+14+4+36 = 56 \end{bmatrix}$$

$\therefore y_c(n) = (\underset{\uparrow}{44}, 63, 45, 56)$

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Part (b) : Let $y_l(n) = x_1(n) * x_2(n)$.

$$y_l(n) = [\delta(n) + 2\delta(n-1) + 4\delta(n-2) + 6\delta(n-3)] * [6\delta(n) + \delta(n-1) + 7\delta(n-2) + 2\delta(n-3)].$$

$$= 6\delta(n) + \delta(n-1) + 7\delta(n-2) + 2\delta(n-3) + 12\delta(n-1) + 2\delta(n-2) + 14\delta(n-3) + 4\delta(n-4) + 24\delta(n-2) + 4\delta(n-3) + 28\delta(n-4) + 8\delta(n-5) + 36\delta(n-3) + 6\delta(n-4) + 48\delta(n-5) + 12\delta(n-6).$$

$$= 6\delta(n) + 13\delta(n-1) + 33\delta(n-2) + 56\delta(n-3) + 38\delta(n-4) + 50\delta(n-5) + 12\delta(n-6).$$

$y_l(n) = (6, 13, 33, 56, 38, 50, 12).$

$y_l(n) = (38+6, 50+13, 33+12, 56).$

$y_l(n) = (\underset{\uparrow}{44}, 63, 45, 56).$

Note: Circular convolution = Linear convolution + Aliasing.

7) Find N-point DFT of the sequence -

$$x(n) = a^n \quad 0 \leq n \leq N-1$$

Sol^n :

$$X(K) \triangleq \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn}$$

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} a^n \cdot W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} [a W_N^K]^n$$

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Recall : $\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1} \quad ; \quad b \neq 0.$

$$\therefore X(K) = \frac{a^N W_N^{KN} - 1}{a W_N^K - 1}$$

$$X(K) = \frac{a^N - 1}{a \cdot W_N^K - 1}$$

8) Compute N-point DFT of the sequence -

$$x(n) = a n \quad , \quad 0 \le n \le N-1$$

Sol^n : $X(k) \triangleq \sum_{n=0}^{N-1} x(n) . W_N^{kn}$

$$\Rightarrow X(k) = \sum_{n=0}^{N-1} a n . W_N^{kn}$$

$$= a \sum_{n=0}^{N-1} n \left[W_N^{K}\right]^n \quad \text{—} \quad 1.$$

wkt, $s = \sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}$

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$$\Rightarrow \frac{ds}{db} = \sum_{n=0}^{N-1} n . b^{n-1} = \frac{(b-1) N b^{N-1} - (b^N - 1)}{(b-1)^2}$$

$$\Rightarrow \sum_{n=0}^{N-1} n b^n = b \left[\frac{N b^N - N b^{N-1} - b^N + 1}{(b-1)^2}\right]$$

$$\Rightarrow \sum_{n=0}^{N-1} n b^n = b \left[\frac{b^N (N-1) - N b^{N-1} + 1}{(b-1)^2}\right]$$

Put $b = W_N^K$, we get

$$\sum_{n=0}^{N-1} n (W_N^K)^n = W_N^K \left[\frac{W_N^{KN}(N-1) - N W_N^{KN} \cdot W_N^{-K} + 1}{(W_N^K - 1)^2}\right]$$

$$= W_N^K \left\{\frac{N - \cancel{1} - N W_N^{-K} + \cancel{1}}{(W_N^K - 1)^2}\right\}$$

$$= \frac{N(\cancel{W_N^k} - 1)}{(W_N^k - 1)^{\cancel{2}}} = \frac{N}{(W_N^k - 1)} \quad ; \; k \neq 0 \qquad \text{--- (2)}$$

Put equation (2) in (1), we get,

$$X(K) = \frac{a \cdot N}{W_N^k - 1} \quad ; \quad k \neq 0.$$

Letting $k = 0$ in eq. (1), we get -

$$X(0) = a \sum_{n=0}^{N-1} n.$$

$$= \frac{a N(N-1)}{2}$$

Summerising, we have -

$$X(K) = \begin{cases} \dfrac{aN}{W_N^k - 1} & ; \; k \neq 0. \\[2ex] \dfrac{a N(N-1)}{2} & ; \; k = 0. \end{cases}$$

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9) Compute N-point DFT of the sequence -

$$x(n) = \cos w_0 n \quad ; \quad w_0 = \frac{2\pi k_0}{N} \, , \quad 0 \leq n \leq N-1.$$

Soln : $X(K) \triangleq \sum_{n=0}^{N-1} x(n) \, W_N^{kn}$

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} \cos w_0 n \cdot W_N^{kn}$$

Given, $x(n) = \cos\left(\frac{2\pi K_0}{N} \cdot n\right)$

$$x(n) = \frac{1}{2} e^{j\frac{2\pi}{N}K_0 n} + \frac{1}{2} e^{-j\frac{2\pi}{N}\cdot K_0 n}$$

Since $W_N = e^{-j\frac{2\pi}{N}}$, we get -

$$x(n) = \frac{1}{2} W_N^{-K_0 n} + \frac{1}{2} W_N^{K_0 n}$$

WKT, $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$

$$= \frac{1}{2} \sum_{n=0}^{N-1} W_N^{-K_0 n} \cdot W_N^{Kn} + \frac{1}{2} \sum_{n=0}^{N-1} W_N^{K_0 n} \cdot W_N^{Kn}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} W_N^{(K-K_0)n} + \frac{1}{2} \sum_{n=0}^{N-1} W_N^{(K+K_0)n}$$

$$= \begin{cases} \frac{N}{2}, & K = K_0 \\ \frac{N}{2}, & K = -K_0 \text{ or } K = -K_0 + N \\ 0, & \text{otherwise} \end{cases}$$

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(or),

$$X(K) = \frac{N}{2} \delta(K - K_0) + \frac{N}{2} \delta(K - (N - K_0))$$

10) Compute N- point DFT of the sequence -

$$x(n) = \frac{1}{2} + \cos^2\left(\frac{2\pi n}{N}\right), \quad 0 \le n \le N-1$$

Soln : Given , $x(n) = \frac{1}{2} + \cos^2\left(\frac{2\pi n}{N}\right)$

$$\Rightarrow x(n) = \frac{1}{2} + \left\{\frac{1}{2} e^{j\frac{2\pi}{N}\cdot n} + \frac{1}{2} e^{-j\frac{2\pi}{N}\cdot n}\right\}^2$$

$$= \frac{1}{2} + \frac{1}{4} W_N^{-2n} + \frac{1}{4} W_N^{2n} + \frac{1}{2}$$

$$= 1 + \frac{1}{4} W_N^{-2n} + \frac{1}{4} W_N^{2n}$$

WKT, $X(K) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{Kn}$

$$\therefore X(K) = \sum_{n=0}^{N-1} W_N^{Kn} + \frac{1}{4}\sum_{n=0}^{N-1} W_N^{(K-2)n} + \frac{1}{4}\sum_{n=0}^{N-1} W_N^{(K-2)n}$$

$$= \begin{cases} N & ; \quad K=0 \\ \frac{N}{4} & ; \quad K=2 \\ \frac{N}{4} & ; \quad K=-2 \text{ or } -2+N \\ 0 & ; \quad \text{otherwise} \end{cases}$$

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