CMR INSTITUTE OF TECHNOLOGY

DEPARTMENT OF ECE|TCE

DIGITAL SIGNAL PROCESSING.

ASSIGNMENT ON MODULE-1

DISCRETE FOURIER TRANSFORM.

1 Explain frequency domain sampling and reconstruction of discrete time signals.

To perform frequency domain analysis of a discrete time signal $x(n)$, we compute discrete time Fourier transform $X(\omega)$ of the signal $x(n)$.

However, $X(\omega)$ is a continuous function of frequency, ω and therefore it cannot be processed with digital signal processors.

Therefore we consider sampling of $X(\omega)$ which leads to discrete Fourier transf-

-orm (DFT).

Consider a discrete time aperiodic signal $x(n)$ with discrete time Fourier Transform $X(\omega)$. (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \ldots \quad (1)$$

Let us take N equidistant samples of $X(\omega)$ in the interval $0 \leq \omega < 2\pi$.

ie, we have to sample $X(\omega)$ at

$$\omega = 0,\ \frac{2\pi}{N},\ \frac{2\pi}{N}2,\ \frac{2\pi}{N}3,\ \ldots,\ \frac{2\pi}{N}(N-1)$$

ie, at $\omega = \frac{2\pi}{N}k$, $k = 0, 1, 2, \ldots N-1$

The resulting samples of $X(\omega)$ can be represented as $X\left(\frac{2\pi}{N}k\right)$, $k = 0, 1, 2, \ldots N-1$.

Using (1), we may write,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$k = 0, 1, \ldots N-1$

$$= \ldots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} +$$

$n=0$

(3)

$$\sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi}{N}kn} +$$

$$\sum_{n=2N}^{3N-1} x(n) e^{-j\frac{2\pi}{N}kn} +$$

$$\cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN) e^{-j\frac{2\pi}{N}k(n+lN)}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN) e^{-j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}klN}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN) e^{-j\frac{2\pi}{N}kn} \quad \cdots (2)$$

$$\left[\because e^{-j\frac{2\pi}{N}klN} = e^{-j2\pi kl} = 1\right]$$

By interchanging the summations, (2) can be written as follows.

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n+lN)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (3)$$

$k = 0, 1, \ldots N-1$

Consider the signal $\sum_{l=-\infty}^{\infty} x(n+lN)$.

This signal is obtained by periodic repetition of $x(n)$ every N samples.

Let us denote $\sum_{l=-\infty}^{\infty} x(n+lN)$ as $x_p(n)$.

Clearly, $x_p(n)$ is periodic with period N.

Hence, (3) may be written as,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (4)$$

$0 \leq k \leq N-1$

We know that $x_p(n)$ is periodic with period N and its Fourier series representation is given by

$$x_p(n) = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn} \quad \ldots (5)$$

$0 \leq n \leq N-1$

where Fourier series coefficients are

given by,

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} \quad \ldots (6)$$

$0 \le k \le N-1$

Using (4), (6) may be written as.

$$a_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right), \quad 0 \le k \le N-1 \quad \ldots (7)$$

Using (7), (5) may be written as

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} \quad \ldots (8)$$

$0 \le n \le N-1$

(8) suggests that we can reconstruct $x_p(n)$, $0 \le n \le N-1$ from $X\left(\frac{2\pi}{N}k\right)$, $0 \le k \le N-1$.

It does not imply that we can reconstruct $x(n)$ from $X\left(\frac{2\pi}{N}k\right)$, $0 \le k \le N-1$.

But if $x(n)$ is of finite length L, and if $N \geq L$, then,

$$x(n) = \sum_{l=-\infty}^{\infty} x(n+lN), \quad 0 \le n \le N-1$$

$$\Rightarrow x(n) = x_p(n), \quad 0 \le n \le N-1 \quad \ldots (9)$$

If $N < L$, then,

$$x(n) \neq x_p(n), \qquad 0 \leq n \leq N-1$$

due to time domain aliasing.

Assuming that $N \geq L$, we can write,

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1 \quad \ldots (10)$$

Using (10) and denoting $X(\frac{2\pi}{N}k)$ as $X(k)$, we may write (8) as,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi}{N}kn} \quad \ldots (11)$$

$0 \leq n \leq N-1$

and (4) may be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (12)$$

$0 \leq k \leq N-1$

(12) represents discrete Fourier transform (DFT) of the signal $x(n)$, $0 \leq n \leq L-1$ and (11) represents inverse discrete Fourier transform (IDFT) of $X(k)$, $0 \leq k \leq N-1$.

2 Obtain the relationship between DFT and DTFS coefficients. (7)

Consider a discrete time periodic signal $x_p(n)$ with period N.

Discrete time Fourier series (DTFS) representation of $x_p(n)$ is given by,

$$x_p(n) = a_0 + a_1 e^{j\frac{2\pi}{N}n} + a_2 e^{j\frac{2\pi}{N}2n} + \cdots + a_{N-1} e^{j\frac{2\pi}{N}(N-1)n}$$

$$= \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} \quad \cdots (1)$$

where DTFS coefficients a_k, $0 \le k \le N-1$ are given by,

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} \quad \cdots (2)$$

$0 \le k \le N-1$

Let $x(n) = x_p(n)$, $0 \le n \le N-1$. $\cdots (3)$

Discrete Fourier transform (DFT) of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \cdots (4)$$

$0 \le k \le N-1$

Comparing (2) and (4) and letting

$x(n) = x_p(n)$, $0 \le n \le N-1$, we get (8)

$$a_K = \frac{1}{N} X(K), \quad 0 \le k \le N-1. \quad \ldots (5)$$

(5) gives the relationship between DFT and DTFS coefficients.

3 Obtain the relationship between DFT and DTFT.

Consider a discrete time signal $x(n)$, $0 \le n \le N-1$. with DFT $X(K)$, $0 \le k \le N-1$ and DTFT $X(\omega)$.

We know that,

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \ldots (1)$$

$0 \le k \le N-1$

$$X(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \quad \ldots (2)$$

Let us sample $X(\omega)$ at N equidistant frequencies. $\omega_K = \frac{2\pi}{N} k$, $0 \le k \le N-1$.

We get,

$$X(\omega)\Big| = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X(\omega)\Big|_{\omega=\frac{2\pi}{N}k} \quad n=0 \qquad k=0,1,\ldots N-1$$

$$= X(k) \quad , \quad 0 \le k \le N-1$$

Now, let us reconstruct $X(\omega)$ using $X(k)$, $0 \le k \le N-1$.

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}\, e^{-j\omega n}$$

(Using IDFT equation)

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn}\, e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi}{N}k\right)n} \quad \ldots (1)$$

Let $P(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} \quad \ldots (2)$

We know that,

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} , & \alpha \ne 1 \\ N , & \alpha = 1 \end{cases} \quad \ldots (3)$$

Using (3) we may simplify (2) as

Using (3), we may simplify ... as follows.

$$P(\omega) = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\frac{\omega N}{2}}\left(e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= e^{-j\frac{\omega}{2}(N-1)} \frac{2j\sin\left(\frac{\omega N}{2}\right)}{2j\sin\left(\frac{\omega}{2}\right)}$$

$$= e^{-j\frac{\omega}{2}(N-1)} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \ldots (4)$$

Using the definition of $P(\omega)$ given by (2), we may write (1) as follows.

$$X(\omega) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, P\left(\omega - \frac{2\pi}{N}K\right) \quad \ldots (5)$$

$$0 \le K \le N-1$$

Hence, we can reconstruct $X(\omega)$ from $X(K)$, $0 \le K \le N-1$. (11)

4 Discuss the relationship between DFT and z-transform.

Cosider a sequence $x(n)$, $0 \le n \le N-1$. with DFT $X(K)$, $0 \le k \le N-1$ and z-transform $X(z)$ with ROC $|z| \neq 0$.

We know that,

$$\underset{0 \le k \le N-1}{X(K)} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (1)$$

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}, \quad ROC: |z| \neq 0 \quad \ldots (2)$$

Let us evaluate $X(z)$ @ $z = e^{j\frac{2\pi}{N}k}$, $0 \le k \le N-1$

$$X(z)\Big|_{\substack{z = e^{j\frac{2\pi}{N}k} \\ 0 \le k \le N-1}} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= X(K). \quad \ldots (3)$$

Hence, by evaluating $X(z)$ @ N equispaced points on unit circle in z-plane, we can obtain $X(k), 0 \le k \le N-1$.

Now, let us try to obtain $X(z)$ from $X(k), 0 \le k \le N-1$.

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} z^{-n}$$

(Using IDFT formula)

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k} z^{-1}\right)^{n} \quad \ldots (4)$$

We know that,

$$\sum_{n=0}^{N-1} \alpha^{n} = \begin{cases} \dfrac{1-\alpha^{N}}{1-\alpha}, & \text{for } \alpha \ne 1 \\ N, & \text{for } \alpha = 1 \end{cases} \quad \ldots (5)$$

Using (5), we may simplify (4) as follows.

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{1-\left(e^{j\frac{2\pi}{N}k} z^{-1}\right)^{N}}{1-e^{j\frac{2\pi}{N}k} z^{-1}}$$

$$\sum^{N-1} \quad 1 - z^{-N}$$

$$= \frac{1}{N} \sum_{k=0} X(k) \frac{}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

(13)

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

Hence, we can reconstruct $X(z)$ from $X(k)$, $0 \le k \le N-1$.

5 Find the z-transform of the sequence $x(n) = (0.5, 0, 0.5, 0)$. Using the z-transform, evaluate the DFT of $x(n)$.

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$= 0.5 + 0.5 z^{-2}, \quad \text{ROC: } |z| \neq 0$$

To obtain $X(k)$, $0 \le k \le 3$ from $X(z)$, we have to put $z = e^{j\frac{2\pi}{N}k}$

$$= e^{j\frac{2\pi}{4}k}$$

$$= e^{j\frac{\pi}{2}k}, \quad 0 \le k \le 3$$

$$\therefore X(k) = 0.5 + 0.5 \left(e^{j\frac{\pi}{2}k}\right)^{-2}, \quad 0 \le k \le 3$$

$0 \le k \le 3$

$$= 0.5 + 0.5 e^{-j\pi k}$$

$$= 0.5 + 0.5$$

$$\therefore X(0) = 0.5 + 0.5 e^{-j\pi(0)}$$

$$= 1$$

$$X(1) = 0.5 + 0.5 e^{-j\pi(1)}$$

$$= 0.5 + 0.5(-1)$$

$$= 0$$

$$X(2) = 0.5 + 0.5 e^{-j\pi(2)}$$

$$= 0.5 + 0.5$$

$$= 1$$

$$X(3) = 0.5 + 0.5 e^{-j\pi(3)}$$

$$= 0.5 + 0.5(-1)$$

$$= 0$$

$$\therefore X(k) = (1, 0, 1, 0)$$

$$0 \le k \le 3$$

6 State and prove Parseval's theorem for finite length energy signals.

Consider a finite length energy signal

Consider a ... of length N

$x(n)$, $0 \le n \le N-1$.

Let E be the energy of $x(n)$. (15)

Then, according to Parseval's theorem,

$$E = \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

Proof:

$$E = \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \sum_{n=0}^{N-1} x(n)\, x^*(n)$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn} x^*(n)$$

(using IDFT equation for $x(n)$)

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N}kn} \quad \ldots (1)$$

We know that,

$$\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = X(K), \quad 0 \le K \le N-1$$

$$\therefore \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N}kn} = X^*(K), \quad 0 \le K \le N-1$$

$\therefore$ (1) can be written as,

$$E = \frac{1}{N} \sum_{k=0}^{N-1} X(k) X^*(k)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Hence proved.

7 Prove that if $x(n)$ is real, then

a) $X(k) = X^*(N-k)$

b) $X(0)$ is real

c) $X(\frac{N}{2})$ is real for even N.

a) $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \ldots (1)$$

$0 \le k \le N-1$

$$\therefore X(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(N-k)n}$$

$0 \le k \le N-1$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Nn} e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n) e^{j\frac{2\pi}{N}kn} \quad \ldots (2)$$

$$\left(\because e^{-j2\pi n} = 1\right)$$

$$\therefore X^*(N-k) = \sum_{n=0}^{N-1} x^*(n)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (3)$$

If $x(n)$ is real, then $x^*(n) = x(n)$.

Then
$$X^*(N-k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= X(k) \quad \ldots (4)$$

b) $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$0 \le k \le N-1$

$$\therefore X(0) = \sum_{n=0}^{N-1} x(n)\, e^{0}$$

$$= \sum_{n=0}^{N-1} x(n)$$

$$= x(0) + x(1) + \cdots + x(N-1)$$

$\Rightarrow$ If $x(n)$ is real, then $X(0)$ will also be real.

c) $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$0 \le k \le N-1$

$$\sum^{N-1} \quad -j\frac{2\pi}{N}\frac{N}{2}n$$

$$\therefore X\left(\frac{N}{2}\right) = \sum_{n=0} x(n) e$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\pi n}$$

$$= x(0) - x(1) + x(2) - x(3) + \dots - x(N-1)$$

$\Rightarrow$ if $x(n)$ is real, $X\left(\frac{N}{2}\right)$ will also be real.

8 Find the 6-point DFT of the sequence $x(n) = (1, 1, 2, 2, 3, 3)$. Plot magnitude spectrum and phase spectrum.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$0 \leq k \leq N-1$

Here, $N = 6$.

$$X(0) = 1+1+2+2+3+3$$

$$= 12$$

$$X(1) = \sum_{n=0}^{6-1} x(n) e^{-j\frac{2\pi}{6}1n}$$

$$= \sum_{n=0}^{5} x(n) e^{-j\frac{\pi}{3}n}$$

$$= x(0) e^{0} + x(1) e^{-j\frac{\pi}{3}} + x(2) e^{-j\frac{2\pi}{3}} + x(3) e^{-j\frac{\pi}{3}3}$$

$$+ x(4)e^{-j\frac{\pi}{3}4} + x(5)e^{-j\frac{\pi}{3}5}$$

$$= 1 + 1(0.5-0.866j) + 2(-0.5-j0.866) + 2(-1) + 3(-0.5+j0.866j) + 3(0.5+j0.866)$$

$$= -1.5 + j2.5981$$

$$X(2) = \sum_{n=0}^{6-1} x(n)e^{-j\frac{2\pi}{6}2n}$$

$$= \sum_{n=0}^{5} x(n)e^{-j\frac{2\pi}{3}n}$$

$$= x(0)e^{0} + x(1)e^{-j\frac{2\pi}{3}} + x(2)e^{-j\frac{2\pi}{3}2} + x(3)e^{-j\frac{2\pi}{3}3} + x(4)e^{-j\frac{2\pi}{3}4} + x(5)e^{-j\frac{2\pi}{3}5}$$

$$= 1 + 1(-0.5-j0.866) + 2(-0.5+j0.866) + 2(1) + 3(-0.5-j0.866) + 3(-0.5+j0.866)$$

$$= -1.5 + 0.866j$$

$$X(3) = X\left(\frac{N}{2}\right)$$

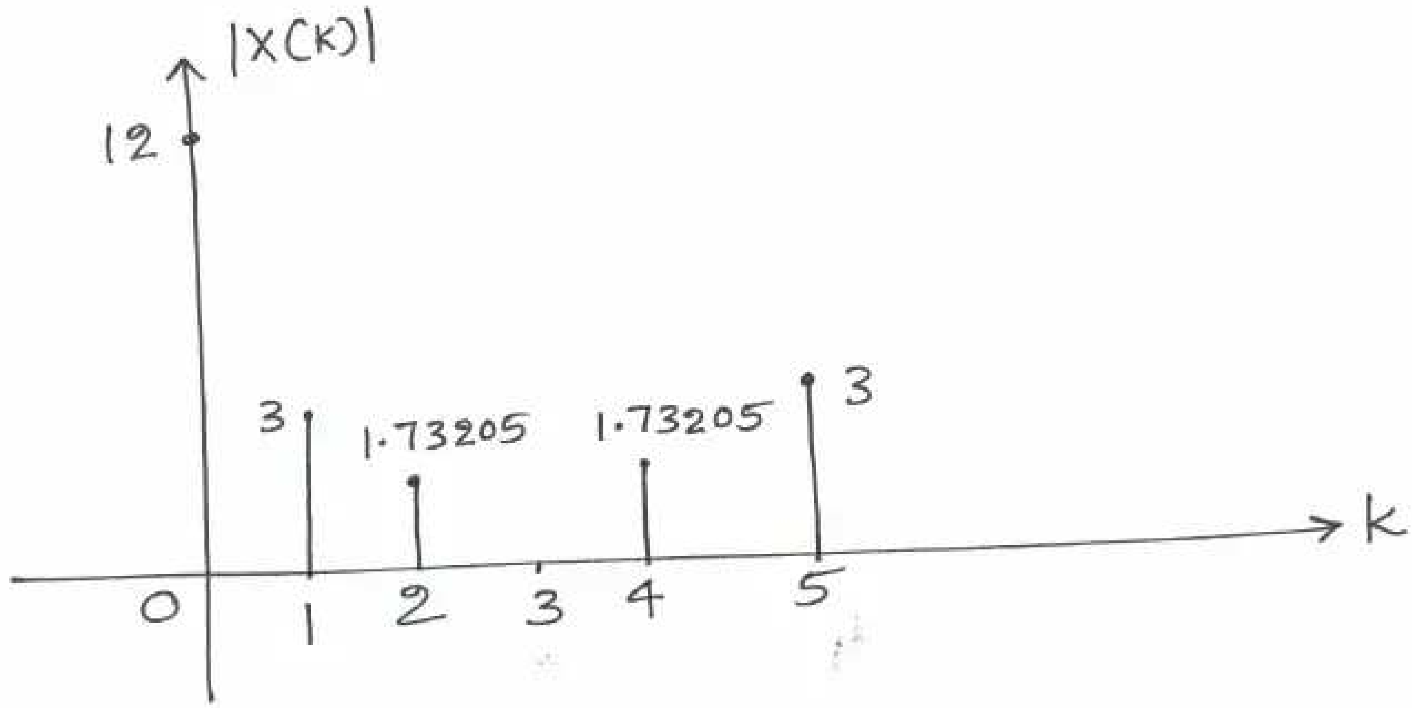
$$= 1-1+2-2+3-3$$
$$= 0$$

$$X(4) = X^*(6-4)$$
$$= X^*(2)$$
$$= -1.5 - 0.866j$$

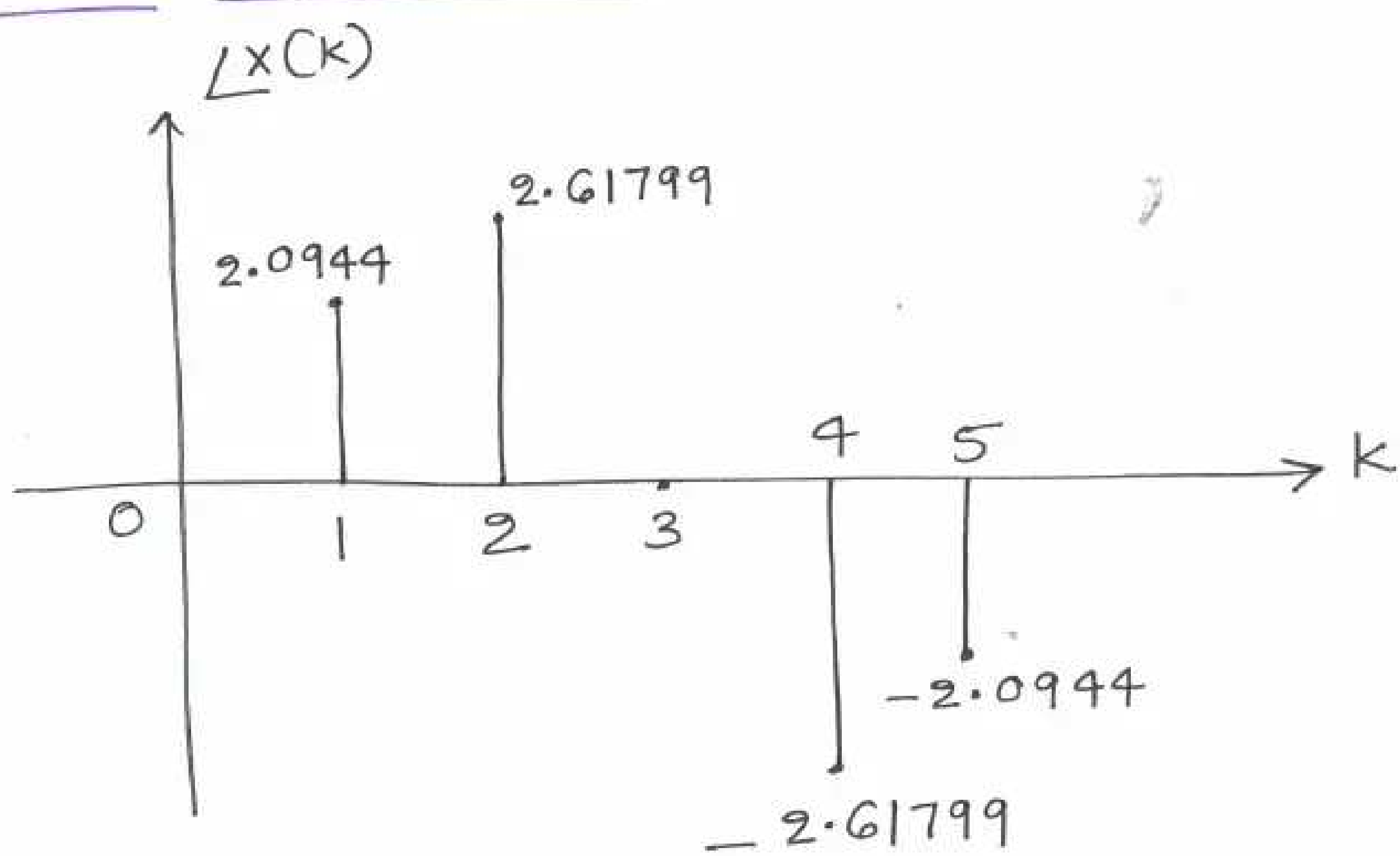
$$X(5) = X^*(6-5)$$
$$= X^*(1)$$
$$= -1.5 - 2.5981j$$

k	$X(k)$	$\lvert X(k) \rvert$	$\angle X(k)$
0	12	12	0
1	$-1.5+j2.5981$	3	2.0944
2	$-1.5+j0.866$	1.73205	2.61799
3	0	0	0
4	$-1.5-j0.866$	1.73205	-2.61799
5	$-1.5-j2.5981$	3	-2.0944

Magnitude Spectrum



Phase Spectrum



9 Compute the 8-point DFT of

$$x(n) = \begin{cases} \frac{1}{4} & \text{for } 0 \le n \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Plot magnitude spectrum and phase spectrum.

$x(n) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right)$

$x(n) = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$

$$X(0) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= 1.$$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$0 \leq k \leq N-1$

$$\therefore X(1) = \sum_{n=0}^{8-1} x(n) e^{-j\frac{2\pi}{8}1n}$$

$$= \sum_{n=0}^{7} x(n) e^{-j\frac{\pi}{4}n}$$

$$= x(0)e^{0} + x(1)e^{-j\frac{\pi}{4}} + x(2)e^{-j\frac{\pi}{2}} + x(3)e^{-j\frac{3\pi}{4}}$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) + \frac{1}{4}(-j) + \frac{1}{4}\left(\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$= 0.25 - j\,0.6035$$

$$X(2) = \sum_{n=0}^{8-1} x(n) e^{-j\frac{2\pi}{8}2n}$$

$$= \sum_{n=0}^{7} x(n) e^{-j\frac{\pi}{2}n}$$

$$x(0)e^{0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\frac{\pi}{2}2} + x(3)e^{-j\frac{\pi}{2}3}$$

$= x(0)e \ldots$

$= \frac{1}{4} + \frac{1}{4}(-j) + \frac{1}{4}(-1) + \frac{1}{4}(j)$

$= 0$

$$X(3) = \sum_{n=0}^{8-1} x(n) e^{-j\frac{2\pi}{8}3n}$$

$$= \sum_{n=0}^{7} x(n) e^{-j\frac{3\pi}{4}n}$$

$$= x(0)e^{0} + x(1)e^{-j\frac{3\pi}{4}} + x(2)e^{-j\frac{3\pi}{4}2} + x(3)e^{-j\frac{3\pi}{4}3}$$

$$= \frac{1}{4} + \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) + \frac{1}{4}(j) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$= 0.25 - j0.1035$$

$$X(4) = X\left(\frac{N}{2}\right)$$

$$= \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4}$$

$$= 0$$

$$X(5) = X^{*}(8-5)$$

$$= X^*(3)$$

$$= 0.25 + j\,0.1035$$

$$X(6) = X^*(8-6)$$

$$= X^*(2)$$

$$= 0$$

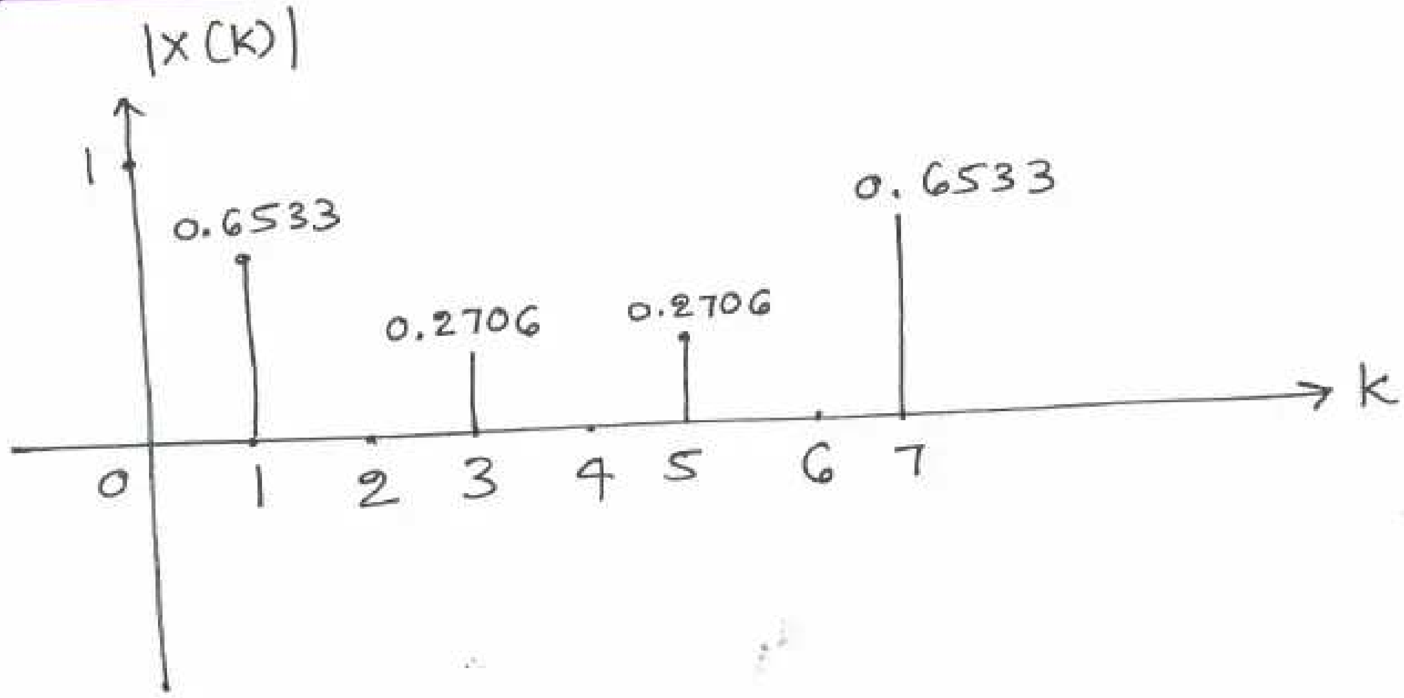
$$X(7) = X^*(8-7)$$

$$= X^*(1)$$

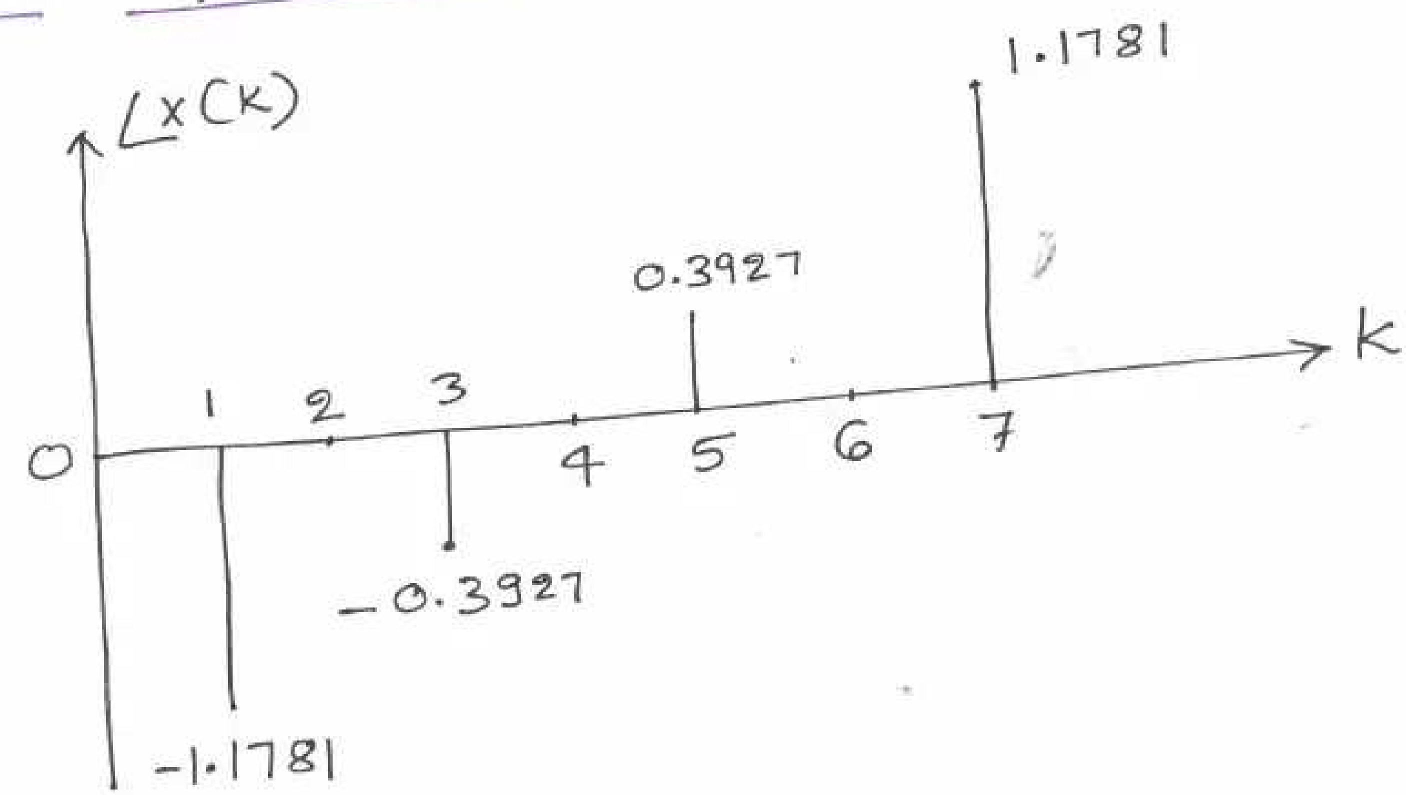
$$= 0.25 + j\,0.6035$$

k	$X(k)$	$\lvert X(k)\rvert$	$\angle X(k)$
0	1	1	0
1	$0.25 - j0.6035$	0.6533	-1.1781
2	0	0	0
3	$0.25 - j0.1035$	0.2706	-0.3927
4	0	0	0
5	$0.25 + j0.1035$	0.2706	0.3927
6	0	0	0
7	$0.25 + j0.6035$	0.6533	1.1781

Magnitude Spectrum



Phase Spectrum



10 Find the IDFT of

$$X(K) = (5, 0, 1-j, 0, 1, 0, 1+j, 0)$$

$N = 8$

$$x(0) = \frac{1}{8}\left(5+0+1-j+0+1+0+1+j+0\right)$$

$$= 1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}$$

$0 \le n \le N-1$

$$x(1) = \frac{1}{8} \sum_{k=0}^{8-1} X(k)\, e^{j\frac{2\pi}{8}1\,k}$$

$$= \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j\frac{\pi}{4}k}$$

$$= \frac{1}{8}\Big[X(0)e^{0} + X(1)e^{j\frac{\pi}{4}} + X(2)e^{j\frac{2\pi}{4}} + X(3)e^{j\frac{3\pi}{4}} + X(4)e^{j\frac{4\pi}{4}} + X(5)e^{j\frac{5\pi}{4}} + X(6)e^{j\frac{6\pi}{4}} + X(7)e^{j\frac{7\pi}{4}}\Big]$$

$$= \frac{1}{8}\Big[5 + (1-j)(j) + 1(-1) + (1+j)(-j)\Big]$$

$$= \frac{1}{8}\Big[5 + j + 1 - 1 - j + 1\Big]$$

$$= \frac{1}{8}\Big[6\Big]$$

$$= 0.75$$

$$\sum^{8-1} \quad e^{j\frac{2\pi}{8}2k}$$

$$x(2) = \frac{1}{8} \sum_{k=0}^{8-1} X(k)\, e$$

$$= \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j\frac{\pi}{2}k}$$

$$= \frac{1}{8}\left[5 + (1-j)\, e^{j\frac{\pi}{2}2} + (1)\, e^{j\frac{\pi}{2}4} + (1+j)\, e^{j\frac{\pi}{2}6}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-1) + 1 + (1+j)(-1)\right]$$

$$= \frac{1}{8}\left[5 - 1 + j + 1 - 1 - j\right]$$

$$= \frac{1}{8}\left[4\right]$$

$$= 0.5$$

$$x(3) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j\frac{2\pi}{8}3k}$$

$$= \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j\frac{3\pi}{4}k}$$

$$= \frac{1}{8}\left[5 + (1-j)\, e^{j\frac{3\pi}{4}2} + 1\left(e^{j\frac{3\pi}{4}4}\right) + (1+j)\, e^{j\frac{18\pi}{4}}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-j) + (-1) + (1+j)(j)\right]$$

$$= \frac{1}{8}\left[5-j-1-1+j-1\right]$$

$$= 0.25$$

$$x(4) = \frac{1}{8}\left[5-0+(1-j)-0+1-0+(1+j)-0\right]$$

$$= \frac{1}{8}\left[8\right]$$

$$= 1$$

$$x(5) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\frac{2\pi}{8}k5}$$

$$= \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\frac{5\pi}{4}k}$$

$$= \frac{1}{8}\left[5+(1-j)e^{j\frac{5\pi}{4}2}+1\,e^{j\frac{5\pi}{4}4}+(1+j)\,e^{j\frac{5\pi}{4}6}\right]$$

$$= \frac{1}{8}\left[5+(1-j)(j)+1(-1)+(1+j)(-j)\right]$$

$$= \frac{1}{8}\left[5+j+1-1-j+1\right]$$

$$= 0.75$$

$$= 0.15$$

$$x(6) = \frac{1}{8}\sum_{k=0}^{8-1} X(k)e^{j\frac{2\pi}{8}6k}$$

$$= \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j\frac{3}{2}\pi k}$$

$$= \frac{1}{8}\left[5 + (1-j)(e^{j3\pi}) + 1\,e^{j6\pi} + (1+j)e^{j9\pi}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-1) + 1 + (1+j)(-1)\right]$$

$$= \frac{1}{8}\left[5 - 1 + j + 1 - 1 - j\right]$$

$$= 0.5$$

$$x(7) = \frac{1}{8}\sum_{k=0}^{8-1} X(k)e^{j\frac{2\pi}{8}7k}$$

$$= \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j\frac{7\pi}{4}k}$$

$$= \frac{1}{8}\left[5 + (1-j)e^{j\frac{7\pi}{4}2} + 1\,e^{j\frac{7\pi}{4}4} + (1+j)e^{j\frac{7\pi}{4}6}\right]$$

$$\left[\ldots (1-j)e^{j\frac{7\pi}{2}} + 1\,e^{j7\pi} + (1+j)e^{j21\frac{\pi}{2}}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)e \quad + e \quad \right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-j) + (-1) + (1+j)(j)\right]$$

$$= \frac{1}{8}\left[5 - j - 1 - 1 + j - 1\right]$$

$$= 0.25.$$

Hence,

$$x(n) = (1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25)$$

11 Compute the 4-point DFT of $x(n) = (0, 1, 2, 3)$

$$\underset{0 \le k \le 3}{X(k)} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

$$W_4^0 = e^0 = 1$$

$$W_4^1 = e^{-j\frac{2\pi}{4}1} = -j$$

$$W_4^2 = e^{-j\frac{2\pi}{4}2} = -1$$

$$W_4^3 = e^{-j\frac{2\pi}{4}3} = j$$

$$W_4^4 = e^{-j\frac{2\pi}{4}4} = 1$$

$$W_4^6 = W_4^4 \, W_4^2 = W_4^2 = -1$$

$$W_4^9 = W_4^8 \, W_4^1 = W_4^1 = -j$$

$$\underset{0 \le k \le 3}{X(k)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

12 Compute the 3-point DFT of

$x(n) = u(n) - u(n-3)$

(32)

$x(n) = u(n) - u(n-3)$

$= (1, 1, 1)$

$$\underset{0 \le k \le 3}{X(k)} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$W_N^k = e^{-j\frac{2\pi}{N}k}$

$W_3^0 = 1$

$W_3^1 = e^{-j\frac{2\pi}{3}1} = -0.5 - j0.866$

$W_3^2 = e^{-j\frac{2\pi}{3}2} = -0.5 + j0.866$

$W_3^3 = 1$

$W_3^4 = W_3^3 W_3^1 = W_3^1 = -0.5 - j0.866$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$\underset{0\le k\le 2}{X(k)} = \begin{bmatrix} 1 & -0.5-j0.866 & -0.5+j0.866 \\ 1 & -0.5+j0.866 & -0.5-j0.866 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 \\ 1-0.5-j0.866-0.5+j0.866 \\ 1-0.5+j0.866-0.5-j0.866 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

14 Compute the IDFT of $X(k) = (6, -2+2j, -2, -2-2j)$

$$\underset{0\le n\le 3}{x(n)} = \frac{1}{N} \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-2+2j-2-2-2j \\ 6+j(-2+2j)+2+(2+2j)j \\ 6-(-2+2j)-2+2+2j \\ 6-j(-2+2j)+2+j(-2-2j) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 0 \\ 6-2+2-2 \\ 6+2-2+2 \\ 6+2+2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

15 Compute the IDFT of $X(K) = (3, -2+2j, -2-2j)$

$$\underset{0 \le n \le 2}{x(n)} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5+j0.866 & -0.5-j0.866 \\ 1 & -0.5-j0.866 & -0.5+j0.866 \end{bmatrix}\begin{bmatrix} 3 \\ -2+2j \\ -2-2j \end{bmatrix}$$

$$\begin{bmatrix} 3-2+2j-2-2j \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 - 0.732 - 0.732 \\ 3 + 2.732 + 2.732 \end{bmatrix}$$

$$= \begin{bmatrix} -0.333 \\ 0.512 \\ 2.8213 \end{bmatrix}$$

16 Consider the sequence $x(n) = (2,1,1,0,3,2,0,3,4,6)$ with 10-point DFT $X(k)$. Evaluate the following without explicitly computing DFT.

i) $X(0)$

ii) $X(5)$

iii) $\sum_{k=0}^{9} X(k)$

iv) $\sum_{k=0}^{9} |X(k)|^2$

v) $\sum_{k=0}^{9} e^{j\frac{4\pi}{5}k} X(k)$

i) $X(0) = 2+1+1+3+2+3+4+6$

$= 22$

ii) $X(5) = X\left(\frac{N}{2}\right)$

$= 2-1+1-0+3-2+0-3+4-6$

$= -2$

iii) We know that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad , \quad 0 \le n \le N-1$$

$$\therefore \quad x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)$$

$$\therefore \quad \sum_{k=0}^{N-1} X(k) = N\, x(0)$$

$$\therefore \quad \sum_{k=0}^{9} X(k) = 10\, x(0)$$

$$= 20$$

iv) According to Parseval's theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\therefore \quad \sum_{k=0}^{N-1} |X(k)|^2 = N \sum_{n=0}^{N-1} |x(n)|^2$$

$$\therefore \quad \sum_{k=0}^{9} |X(k)|^2 = 10 \sum_{n=0}^{9} |x(n)|^2$$

$$= 10\left(2^2+1^2+1^2+3^2+2^2+3^2+4^2+6^2\right)$$

$$= 10\,(80)$$

$$= 800$$

v) $\displaystyle\sum_{k=0}^{9} e^{j\frac{4\pi}{5}k} X(k)$

$$\sum^{N-1} \quad e^{j\frac{2\pi}{N}kn}$$

$$x(n) = \frac{1}{N} \sum_{k=0} X(k) e$$

$0 \le n \le N-1$

$$= \frac{1}{10} \sum_{K=0}^{10-1} X(K) e^{j\frac{2\pi}{10}kn.}$$

$$= \frac{1}{10} \sum_{K=0}^{9} X(K) e^{j\frac{2\pi}{10}kn} \quad \ldots (1)$$

$$\sum_{n=0}^{9} e^{j\frac{4\pi}{5}k} X(K) = 10 \times \frac{1}{10} \sum_{k=0}^{9} X(k) e^{j\frac{2\pi}{10}4k}$$

$$= 10 . \times x(4) \qquad \text{(comparing with (1))}$$

$$= 30$$

17 Let $X(K)$ be the 14-point DFT of a length 14 real sequence. The first 8 samples of $X(K)$ are given by

$X(0) = 12$ $\qquad$ $X(1) = -1+j3$

$X(2) = 3+j4$ $\qquad$ $X(3) = 1-j5$

$X(4) = -2+j2$ $\qquad$ $X(5) = 6+j3$

$X(6) = -2-j3$ $\qquad$ $X(7) = 10$

Determine the remaining samples of $X(K)$. Evaluate the following functions witho.

ut explicitly computing $x(n)$

i) $x(0)$

ii) $x(7)$

iii) $\sum_{n=0}^{13} x(n)$

iv) $\sum_{n=0}^{13} |x(n)|^2$

v) $\sum_{n=0}^{13} x(n) e^{-j\frac{4\pi}{7}n}$

$x(n)$ is real.

$\therefore X(k) = X^*(N-k)$

$\therefore X(8) = X^*(14-8) = -2+j3$

$X(9) = X^*(14-9) = 6-j3$

$X(10) = X^*(14-10) = -2-j2$

$X(11) = X^*(14-11) = 1+j5$

$X(12) = X^*(14-12) = 3-j4$

$X(13) = X^*(14-13) = -1-j3$

i) $x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)$

$N-1$

$$= \frac{1}{14} \sum_{K=0} X(K)$$

$$= \frac{1}{14}(32)$$

$$= \frac{16}{7}$$

ii) $$x(7) = \frac{1}{14}\left(X(0) - X(1) + X(2) - X(3) + \cdots - X(13)\right)$$

$$= \frac{1}{14}(-12)$$

$$= -\frac{6}{7}$$

iii) $$\sum_{n=0}^{13} x(n).$$

We know that $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$ $\quad 0 \le k \le N-1$

$$\therefore X(0) = \sum_{n=0}^{N-1} x(n)$$

$$\therefore \sum_{n=0}^{13} x(n) = X(0)$$

$$= 12$$

iv) $$\sum_{n=0}^{13} |x(n)|^2 = \frac{1}{14} \sum_{K=0}^{13} |X(K)|^2$$

$$= \frac{1}{14}(498)$$

$$= 35.5714$$

v) $$\sum_{n=0}^{13} x(n) e^{-j\frac{4\pi}{7}n} = \sum_{n=0}^{13} x(n) e^{-j\frac{2\pi}{14}4n}$$

$$= X(4)$$ 40

$$= -2+j2$$

18 Compute the DFT of $x(n) = 0.5^n, 0 \le n \le 3$ by evaluating $x(n) = a^n$ for $0 \le n \le N-1$

$$\underset{0 \le k \le N-1}{X(k)} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi}{N}k}\right)^n \quad \ldots (1)$$

We know that.

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \text{for } \alpha \ne 1 \\ N, & \text{for } \alpha = 1 \end{cases} \quad \ldots (2)$$

$\therefore$ (1) can be simplified as follows.

$$X(k) = \frac{1 - \left(a e^{-j\frac{2\pi}{N}k}\right)^N}{\ldots}$$

$0 \le k \le N-1$ $\quad 1 - a e^{-j\frac{2\pi}{N}k}$

$$= \frac{1 - a^N e^{-j\frac{2\pi}{N}kN}}{1 - a e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{1 - a^N}{1 - a e^{-j\frac{2\pi}{N}k}} \quad \left(\because e^{-j2\pi k} = 1\right) \quad \text{---} (3)$$

Given : $x(n) = 0.5^n, \; 0 \le n \le 3$

$\therefore N = 4, \; a = 0.5$

Using (3), we may write,

$$\underset{0 \le k \le 3}{X(k)} = \frac{1 - (0.5)^4}{1 - 0.5\, e^{-j\frac{2\pi}{4}k}}$$

$$= \frac{0.9375}{1 - 0.5\, e^{-j\frac{\pi}{2}k}}$$

$$\therefore X(k) = \left(1.875,\ 0.75-j0.375,\ 0.625,\ 0.75+0.375j\right)$$

(By substituting $k = 0, 1, 2, 3$)

19 Find the N-point DFT of $x(n) = 1,\ 0 \le n \le N-1$.

$$\underset{0 \le k \le N-1}{X(k)} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}k}\right)^n \quad \ldots (1)$$

We know that

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \text{for } \alpha \neq 1 \\ N, & \text{for } \alpha = 1 \end{cases} \quad \ldots (2)$$

Let $\alpha = e^{-j\frac{2\pi}{N}k}$

@ $k = 0$, $\alpha = 1$

$$\therefore X(k)\Big|_{k=0} = \sum_{n=0}^{N-1} \alpha^n \quad \text{where } \alpha = 1$$

$$= N. \quad \ldots (3)$$

for $k \neq 0$, $\alpha \neq 1$

$$\therefore \underset{k \neq 0}{X(k)} = \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}k}\right)^n$$

(43)

$$= \frac{1-\left(e^{-j\frac{2\pi}{N}k}\right)^N}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{1-1}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= 0. \quad \ldots (4).$$

Using (3) and (4), we may write

$$X(k) = \begin{cases} N & \text{for } k=0 \\ 0 & \text{for } 1 \leq k \leq N-1 \end{cases}$$

ie, $X(k) = N\,\delta(k)$, $0 \leq k \leq N-1$.

20 Compute the N-point DFT of

$x(n) = (-1)^n$, $0 \leq n \leq 7$

$$\sum^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad N=8$$

$$X(K) = \sum_{n=0} x(n) e \quad ,$$

$0 \leq k \leq N-1$

$$= \sum_{n=0}^{7} (-1)^n e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{7} (-1)^n e^{-j\frac{2\pi}{8}kn}$$

$$= \sum_{n=0}^{7} \left(-1\, e^{-j\frac{\pi}{4}k}\right)^n \quad \ldots (1)$$

Let $\alpha = -1\, e^{-j\frac{\pi}{4}k} \quad \ldots (2)$

We know that,

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \text{for } \alpha \neq 1 \\ N, & \text{for } \alpha = 1 \end{cases} \quad \ldots (3)$$

@ $k=4$, $\alpha = -1\, e^{-j\frac{\pi}{4}4}$

$= 1 \quad \ldots (4)$

Then,

$X(K) = N$

$= 8 \quad \ldots (5)$

If $k \neq 4$, then,

$$X(k) = \sum_{n=0}^{7} \left(-1\, e^{-j\frac{\pi}{4}k}\right)^n$$

(45)

$$= \frac{1-\left(-1\, e^{-j\frac{\pi}{4}k}\right)^N}{1-\left(-1\, e^{-j\frac{\pi}{4}k}\right)}$$

$$= \frac{1-\left(-1\, e^{-j\frac{\pi}{4}k}\right)^8}{1+e^{-j\frac{\pi}{4}k}}$$

$$= 0. \qquad \left(\because e^{-j2\pi k} = 1\right)$$

$$\ldots (6)$$

Using (5) and (6), we may write,

$$\underset{0 \le k \le 7}{X(k)} = \begin{cases} 8 & \text{for } k=4 \\ 0 & \text{for } k \neq 4 \end{cases}$$

$$= 8\,\delta(k-4).$$

20 Compute the DFT of

$$x(n) = a^n, \quad 0 \le n \le N-1$$

$$0 < a < 1.$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$
$0 \le k \le N-1$

$$= \sum_{n=0}^{N-1} an\, e^{-j\frac{2\pi}{N}kn}$$

$$= a\sum_{n=0}^{N-1} n\, e^{-j\frac{2\pi}{N}kn} \quad \cdots (1)$$

We know that,

$$\sum_{n=0}^{N-1} n\alpha^n = \frac{-N\alpha^N + N\alpha^{N+1} + \alpha - \alpha^{N+1}}{(1-\alpha)^2} \quad \cdots (2)$$
$\alpha \neq 0, 1$

Using (2), we may simplify (1) as follows.

$$\underset{1 \le k \le N-1}{X(k)} = a\left[\frac{-N + N e^{-j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k} - e^{-j\frac{2\pi}{N}k}}{\left(1 - e^{-j\frac{2\pi}{N}k}\right)^2}\right]$$

$$= aN \frac{e^{-j\frac{2\pi}{N}k} - 1}{\left(e^{-j\frac{2\pi}{N}k} - 1\right)^2}$$

$$= \frac{aN}{e^{-j\frac{2\pi}{N}k} - 1}, \quad k \neq 0$$

Putting $k=0$ in (1), we get

$$X(0) = a \sum_{n=0}^{N-1} n$$

$$= a\,\frac{(N-1)(N)}{2}$$ (47)

$$\therefore \underset{0 \le k \le N-1}{X(k)} = \begin{cases} \dfrac{aN(N-1)}{2}, & \text{for } k=0 \\ \dfrac{aN}{e^{-j\frac{2\pi}{N}k} - 1}, & \text{for } k \neq 0 \end{cases}$$

21 Compute the N-point DFT of

$$x(n) = \delta(n).$$

$$\underset{0 \le k \le N-1}{X(k)} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= x(0)e^{0} + x(1)\,e^{-j\frac{2\pi}{N}k(1)} + x(2)\,e^{-j\frac{2\pi}{N}k(2)} + \dots + x(N-1)\,e^{-j\frac{2\pi}{N}k(N-1)}$$

$$= 1.$$

ie, $X(k) = (1, 1, 1, \dots 1)$

N times.

22 Compute the N-point DFT of

$$x(n) = e^{j\frac{2\pi}{N}ln}, \quad 0 \le n \le N-1$$

$$0 < l \le N-1$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$0 \le k \le N-1$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)n} \quad \ldots (1)$$

Let $\alpha = e^{-j\frac{2\pi}{N}(k-l)} \quad \ldots (2)$

We know that,

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \text{for } \alpha \ne 1 \\ N, & \text{for } \alpha = 1 \end{cases} \quad \ldots (3)$$

@ $k = l$, $\alpha = 1$

for $k \ne l$, $\alpha \ne 1$

@ $k = l$, $X(k) = N$. $\ldots (4)$

If $k \ne l$, $X(k) = \dfrac{1 - e^{-j\frac{2\pi}{N}(k-l)N}}{1 - e^{-j\frac{2\pi}{N}(k-l)}}$

$$= 0 \quad \ldots (5)$$

Using (4) and (5), we may write,

$$\underset{0 \le k \le N-1}{X(k)} = \begin{cases} N, & \text{for } k = l \\ 0, & \text{for } k \ne l. \end{cases}$$

$$= N\,\delta(k-l).$$

23 Compute the DFT of

$$x(n) = \cos\left(\frac{2\pi}{N} l n\right), \quad \begin{array}{l} 0 \le n \le N-1 \\ 0 < l \le N-1 \end{array}$$

$$x(n) = \cos\left(\frac{2\pi}{N} l n\right)$$

$$= \frac{1}{2} e^{j\frac{2\pi}{N} l n} + \frac{1}{2} e^{-j\frac{2\pi}{N} l n}, \quad 0 \le n \le N-1$$

$$\therefore \underset{0 \le k \le N-1}{X(k)} = \frac{N}{2}\delta(k-l) + \frac{N}{2}\delta(k+l-N)$$

24 Compute the DFT of

$$x(n) = \sin\left(\frac{2\pi}{N} l n\right), \quad \begin{array}{l} 0 \le n \le N-1 \\ 0 < l \le N-1 \end{array}$$

$$x(n) = \sin\left(\frac{2\pi}{N} l n\right), \quad 0 \le n \le N-1$$

$$= \frac{1}{2j} e^{j\frac{2\pi}{N}ln} - \frac{1}{2j} e^{-j\frac{2\pi}{N}ln}$$

$$\therefore \underset{0 \le k \le N-1}{X(k)} = \frac{N}{2j}\delta(k-l) - \frac{N}{2j}\delta(k+l-N)$$

25 Compute the N-point DFT of

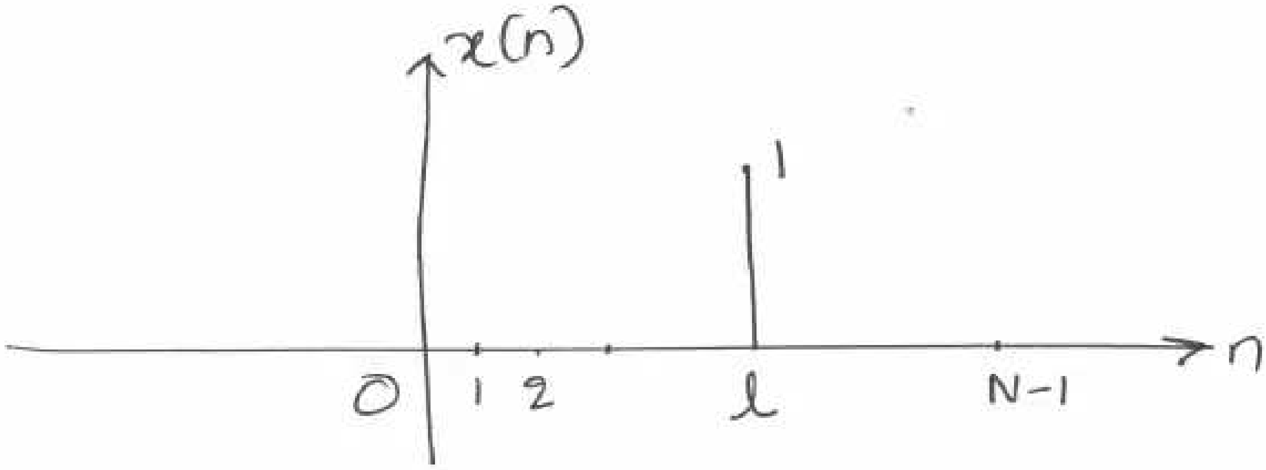
$$x(n) = \delta(n-l)$$

where $0 < l \le N-1$.

$$\underset{0 \le k \le N-1}{X(k)} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \delta(n-l) e^{-j\frac{2\pi}{N}kn}$$

$$= e^{-j\frac{2\pi}{N}kl}$$



26 Compute the IDFT of

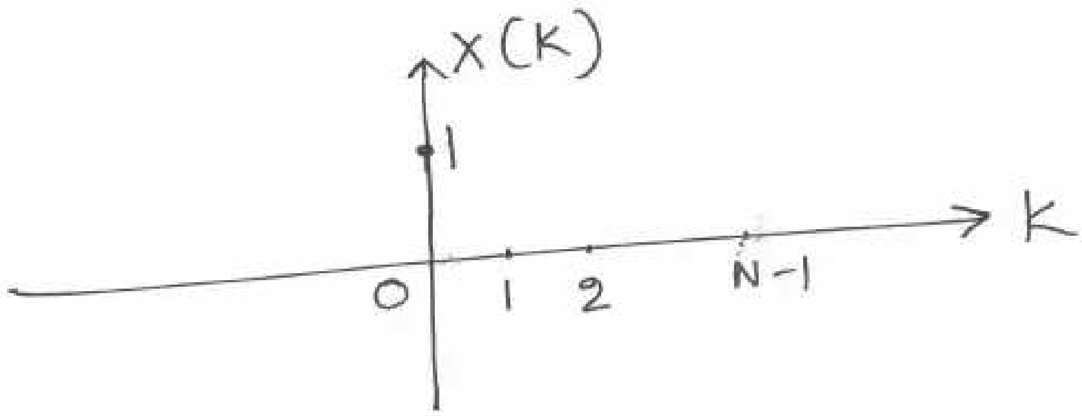
$$X(k) = \delta(k), \quad 0 \le k \le N-1$$

$$\underset{0 \le n \le N-1}{x(n)} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

(51)

$$= \frac{1}{N} \sum_{k=0}^{N-1} \delta(k) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}$$



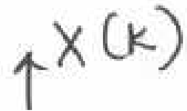
27 Compute the IDFT of

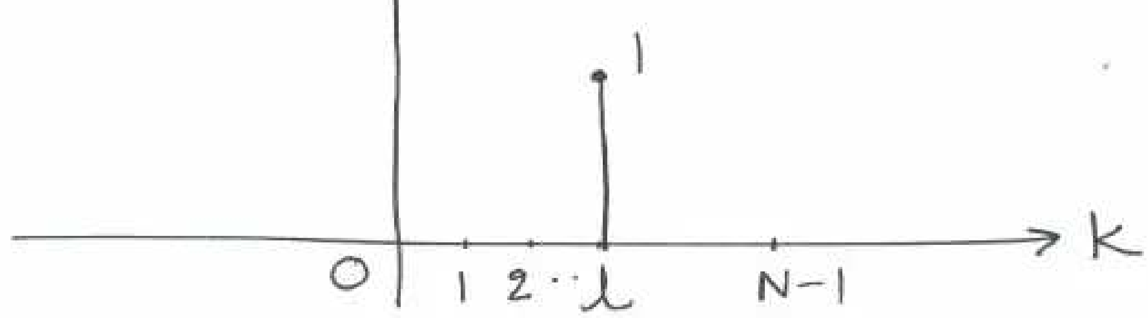
$$\underset{0 \le k \le N-1}{X(k)} = \delta(k-l), \quad 0 < l \le N-1$$

$$\underset{0 \le n \le N-1}{x(n)} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \delta(k-l) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} e^{j\frac{2\pi}{N}ln}$$





(52)

28 Compute the 8-point DFT of

$$x(n) = (9,1,1,1,1,1,1,1)$$

$x(n)$ can be written as,

$$x(n) = 1 + 8\delta(n) \quad , 0 \le n \le 7$$

$$\therefore \underset{0 \le k \le N-1}{X(k)} = \text{DFT of } 1 + \text{DFT of } 8\delta(n)$$

In problem #19, we have proved that

$$\text{DFT of } 1 = N\delta(k), \quad 0 \le k \le N-1$$

In problem #21, we have proved that

$$\text{DFT of } \delta(n) = 1$$

$$\therefore \underset{0 \le k \le N-1}{X(k)} = N\delta(k) + 8 \times 1$$

$$= 8\delta(k) + 8$$

$$= (16,8,8,8,8,8,8,8)$$

29 Compute the IDFT of

$$X(k) = (9,1,1,1,1,1,1,1)$$

$X(k)$ can be written as

$$X(k) = 1 + 8\,\delta(k), \quad 0 \le k \le 7$$

We know that

$$\text{DFT of } \delta(n) = 1 \quad , \quad 0 \le k \le N-1$$

$$\therefore \text{IDFT of } 1 = \delta(n) \quad , \quad 0 \le n \le N-1$$ (53)

$$\text{IDFT of } \delta(k) = \frac{1}{N} \quad , \quad 0 \le n \le N-1$$

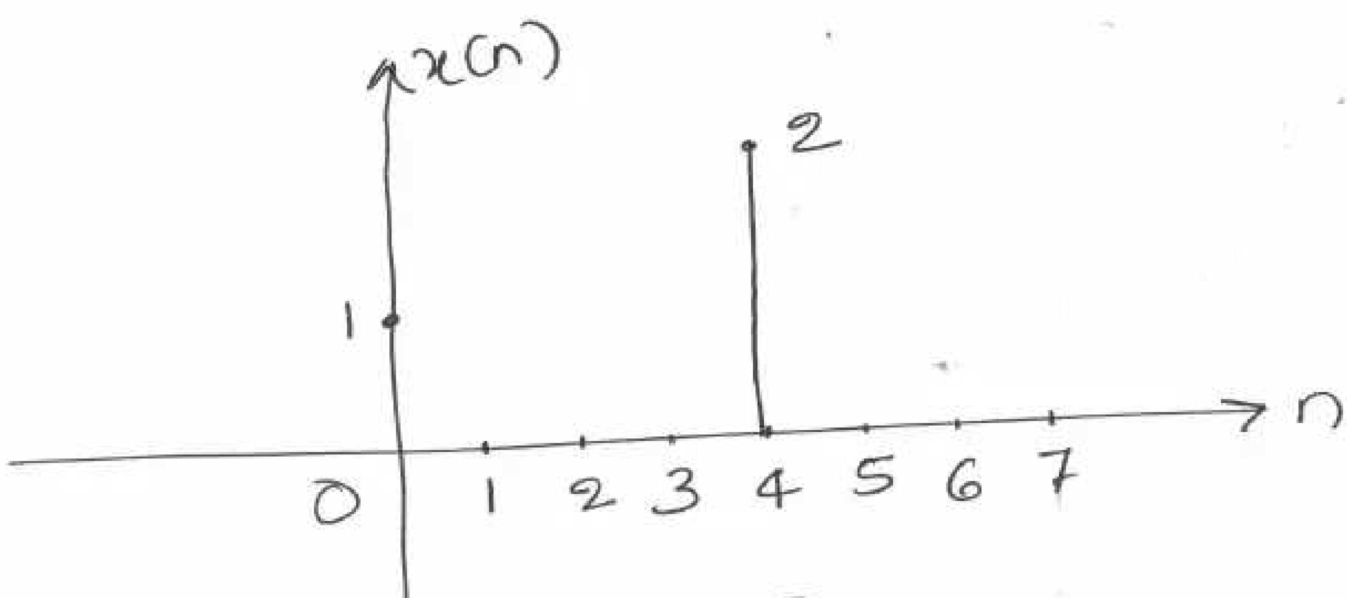
$$\therefore \quad \underset{0 \le n \le 7}{x(n)} = \delta(n) + 8 \cdot \frac{1}{8}$$

$$= \delta(n) + 1$$

$$= (2, 1, 1, 1, 1, 1, 1, 1)$$

30 Compute the 8-point DFT of

$$x(n) = \delta(n) + 2\,\delta(n-4) \quad , \quad 0 \le n \le 7$$



$$\underset{0 \le k \le 7}{X(k)} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= x(0)e^{0} + x(4)e^{-j\frac{2\pi}{8}k(4)}$$

$$-j\pi k$$

$$= 1 + 2e^{-j\pi k}$$

31 state and prove periodicity property of DFT.

statement:

If $x(n) \xleftrightarrow[DFT]{N} X(K)$, then

$$x(n+N) = x(n), \quad \forall n$$

$$X(K+N) = X(K), \quad \forall k.$$

Proof:

We know that,

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$\therefore X(K+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(K+N)n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}Nn}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad (e^{-j2\pi n} = 1)$$

$$= X(K), \quad \forall k.$$

that

We know that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

$$\therefore x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k(n+N)}$$ (55)

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \; e^{j\frac{2\pi}{N}kN}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad (\because e^{j2\pi k} = 1)$$

$$= x(n) \; , \; \forall n$$

32 State and prove linearity property of DFT.

Statement:

If $x_1(n) \xleftrightarrow[DFT]{N} X_1(k)$ and

$x_2(n) \xleftrightarrow[DFT]{N} X_2(k)$

then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[DFT]{N} a_1 X_1(k) + a_2 X_2(k)$$

where a_1, a_2 are constants.

Proof:

DFT of $a_1 x_1(n) + a_2 x_2(n)$

$$= \sum_{n=0}^{N-1} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\frac{2\pi}{N}kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$= a_1 X(K) + a_2 X_2(K).$$ 56

33 Discuss the symmetry properties of DFT.

We know that,

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \ldots (1)$$

$0 \le K \le N-1$

Let $x_r(n)$ be the real part of $x(n)$ and $x_i(n)$ be the imaginary part of $x(n)$.

Then, $$x(n) = x_r(n) + j x_i(n) \quad \ldots (2)$$

$$\therefore X(K) = \sum_{n=0}^{N-1} \left[x_r(n) + j x_i(n)\right] e^{-j\frac{2\pi}{N}kn}$$

$0 \le K \le N-1$

$$= \sum_{n=0}^{N-1} \left[x_r(n) + j x_i(n)\right]\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] \quad \ldots (3)$$

Separating real part and imaginary part, we get,

$$X_R(K) = \sum_{n=0}^{N-1} x_r(n)\cos\left(\tfrac{2\pi}{N}kn\right) + x_i(n)\sin\left(\tfrac{2\pi}{N}kn\right) \quad \ldots (4)$$

$0 \le k \le N-1$

$$X_I(k) = \sum_{n=0}^{N-1} x_i(n)\cos\left(\frac{2\pi}{N}kn\right) - x_r(n)\sin\left(\frac{2\pi}{N}kn\right)$$

$0 \le k \le N-1$ $\quad\ldots(5)$

i) If $x(n)$ is real, then

$$x_i(n) = 0 \quad \ldots (6)$$

$$\therefore X_R(k) = \sum_{n=0}^{N-1} x_r(n)\cos\left(\frac{2\pi}{N}kn\right) \quad \ldots (7)$$

$0 \le k \le N-1$

$$X_I(k) = -\sum_{n=0}^{N-1} x_r(n)\sin\left(\frac{2\pi}{N}kn\right) \quad \ldots (8)$$

$0 \le k \le N-1$

ii) If $x(n)$ is real and circularly even, then

$x_r(n)\sin\left(\frac{2\pi}{N}kn\right)$ will be circularly odd

and $\sum_{n=0}^{N-1} x_r(n)\sin\left(\frac{2\pi}{N}kn\right) = 0$

$$\therefore X_I(k) = 0$$

$$\therefore X(k) = X_R(k)$$

$0 \le k \le N-1$

$$= \sum_{n=0}^{N-1} x_r(n)\cos\left(\frac{2\pi}{N}kn\right)$$

$$= \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi}{N}kn\right) \quad \ldots (9)$$

Hence, if $x(n)$ is real and circularly even, then $X(K)$ is real and circularly even.

iii) If $x(n)$ is real and circularly odd, then

$x_r(n) \sin\left(\frac{2\pi}{N}kn\right)$ will be circularly even and $x_r(n) \cos\left(\frac{2\pi}{N}kn\right)$ will be circularly odd.

$$\therefore \sum_{n=0}^{N-1} x_r(n) \cos\left(\frac{2\pi}{N}kn\right) = 0$$

$$\therefore X_R(K) = 0$$

$$\therefore X(K) = X_R(K) + j X_I(k)$$

$$= -j \sum_{n=0}^{N-1} x_r(n) \sin\left(\frac{2\pi}{N}kn\right) \quad \ldots$$

$$= -j \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi}{N}kn\right) \quad \ldots (10)$$

Hence, if $x(n)$ is real and circularly odd, then $X(K)$ is imaginary and circularly odd.

imaginary, then

iv) If $x(n)$ is purely imaginary

$$x_r(n) = 0 \quad \cdots (11)$$

(4) and (5) may be modified as follows.

$$X_R(k) = \sum_{n=0}^{N-1} x_i(n) \sin\left(\frac{2\pi}{N} kn\right) \quad \cdots (12)$$

$0 \leq k \leq N-1$

$$X_I(k) = \sum_{n=0}^{N-1} x_i(n) \cos\left(\frac{2\pi}{N} kn\right) \quad \cdots (13)$$

$0 \leq k \leq N-1$

v) If $x(n)$ is purely imaginary and circularly even, then

$x_i(n) \sin\left(\frac{2\pi}{N} kn\right)$ .. will be circularly odd.

$$\therefore \sum_{n=0}^{N-1} x_i(n) \sin\left(\frac{2\pi}{N} kn\right) = 0 \quad \cdots (14)$$

$$\therefore X_R(k) = 0 \quad \cdots (15)$$

$$\therefore X(k) = j X_I(k)$$

$$= j \sum_{n=0}^{N-1} x_i(n) \cos\left(\frac{2\pi}{N} kn\right) \quad \cdots (15)$$

$$\cancel{j \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi}{N} kn\right)}$$

Hence, if $x(n)$ is purely imaginary

and circularly even, then $X(K)$ is purely imaginary and circularly even.

vi) If $x(n)$ is purely imaginary and circularly odd, then (60)

$x_i(n)\cos\left(\frac{2\pi}{N}kn\right)$ will be circularly odd.

$$\therefore \sum_{n=0}^{N-1} x_i(n)\cos\left(\frac{2\pi}{N}kn\right) = 0$$

$$\therefore \quad X_I(K) = 0$$

$$\therefore \quad X(K) = X_R(K)$$

$$= \sum_{n=0}^{N-1} x_i(n)\sin\left(\frac{2\pi}{N}kn\right) \quad \ldots (16)$$

Hence, if $x(n)$ is purely imaginary and circularly odd, then $X(K)$ will be real and circularly odd.

34 Consider a sequence $x(n) = (0,1,2,3,4)$.

a) Determine the sequence $y(n)$ with 6-point DFT $Y(K) = \text{Real}[X(K)]$

b) Determine the sequence $v(n)$ with 6-point DFT $V(K) = \text{Imaginary}[X(K)]$

6- point DFT

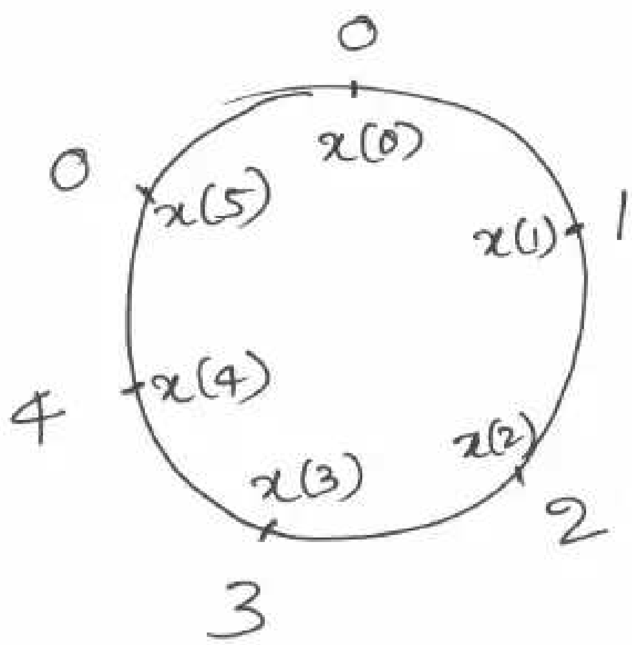
a) $Y(k) = \text{Real}[X(k)]$

$\therefore y(n) =$ even part of $x(n)$.

$$y(n) = \frac{x(n)_N + x(-n)_N}{2}$$

$x(n) = (0, 1, 2, 3, 4, 0)$

To find $x(-n)_N$, arrange the samples of $x(n)$ on the circumference of a circle in clockwise direction and read them in anticlockwise direction



$\therefore x(-n)_N = (0, 0, 4, 3, 2, 1)$

$$\therefore y(n) = \frac{x(n) + x(-n)_N}{2}$$

$= (0,1,2,3,4,0) + (0,0,4,3,2,1)$

$$= \frac{\quad}{2}$$

$$= (0, 0.5, 3, 3, 3, 0.5)$$

b) $V(K) = \text{Imaginary}[X(K)]$

$\therefore v(n) = \text{odd part of } x(n)$

$$= \frac{x(n) - x(-n)_N}{2}$$

$$= \frac{(0,1,2,3,4,0) - (0,0,4,3,2,1)}{2}$$

$$= (0, 0.5, -1, 0, 1, -0.5)$$

35 Show that multiplication of two DFTs leads to circular convolution of respective time sequences.

Let

$$x_1(n) \xleftrightarrow[\text{DFT}]{N} X_1(K) \quad \text{and}$$

$$x_2(n) \xleftrightarrow[\text{DFT}]{N} X_2(K).$$

Let $X_3(K) = X_1(K)\,X_2(K) \quad \ldots (1)$

Taking IDFT of $X_3(K)$, we get

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K)\, e^{j\frac{2\pi}{N}km}$$

$0 \le m \le N-1$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j\frac{2\pi}{N}km}$$

(63)

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} \times \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl} \; e^{j\frac{2\pi}{N}km}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-l-n)} \quad \ldots (2)$$

Now, consider

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(m-l-n)k}$$

We know that

$$\sum_{k=0}^{N-1} \alpha^k = \begin{cases} \frac{1-\alpha^N}{1-\alpha}, & \text{for } \alpha \neq 1 \\ N, & \text{for } \alpha = 1 \end{cases}$$

Let $\alpha = e^{j\frac{2\pi}{N}(m-l-n)}$

If $m-l-n = pN$ where p is an integer,

then

$\alpha = 1$

ie, when $l = m-n-pN$, $\alpha = 1$.

If $\alpha = 1$, then

$$\sum_{k=0}^{N-1} \alpha^k = N.$$

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$$\therefore \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(m-l-n)k} = N \quad \text{when } m-l-n = pN. \quad \ldots (3)$$

If $m-l-n \neq pN$, then

$$e^{j\frac{2\pi}{N}(m-l-n)} \neq 1$$

$$\therefore \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(m-l-n)k} = \frac{1 - e^{j\frac{2\pi}{N}(m-l-n)N}}{1 - e^{j\frac{2\pi}{N}(m-l-n)}}$$

$$= \frac{1 - e^{j2\pi(m-l-n)}}{1 - e^{j\frac{2\pi}{N}(m-l-n)}}$$

$$= \frac{1 - 1}{1 - e^{j\frac{2\pi}{N}(m-l-n)}}$$

$$= 0 \quad \ldots (4)$$

$$\therefore \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(m-l-n)k} = \begin{cases} N, & \text{when } m-l-n = pN \\ & \text{ie, when } l = m-n-pN \\ 0, & \text{when } l \neq m-n-pN \end{cases} \quad \ldots (5)$$

Using (5), we may simplify (2) as follows

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n-pN)\, N$$

$0 \le m \le N-1$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n)_N) \quad \ldots (6)$$

The sum on the RHS of (6) is called circular convolution sum.

Hence, multiplication of two DFTs has resulted in circular convolution of time domain sequences.

36. Compute the circular convolution of $x(n) = (1, 2, 3, 1)$ and $h(n) = (4, 3, 2, 2)$ using

a) Matrix method (Time domain approach)

b) Graphical method

c) DFT-IDFT method (Stockham's method)

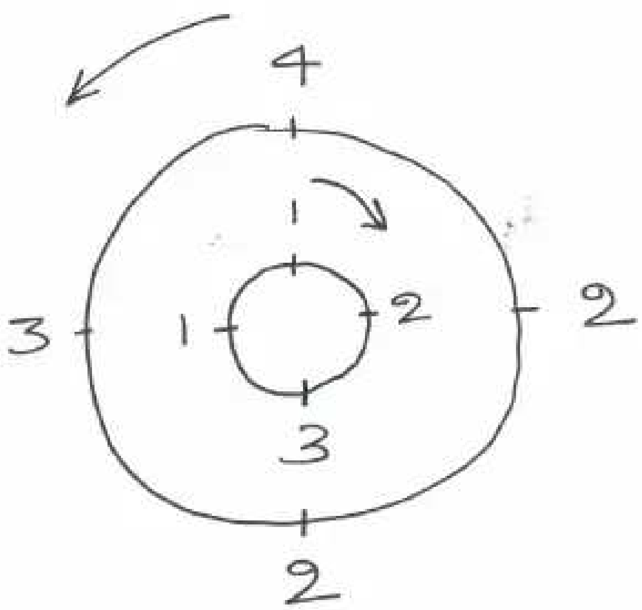
a) Matrix method (Time domain method)

$y(n) = x(n)$ ④ $h(n)$

$$= \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

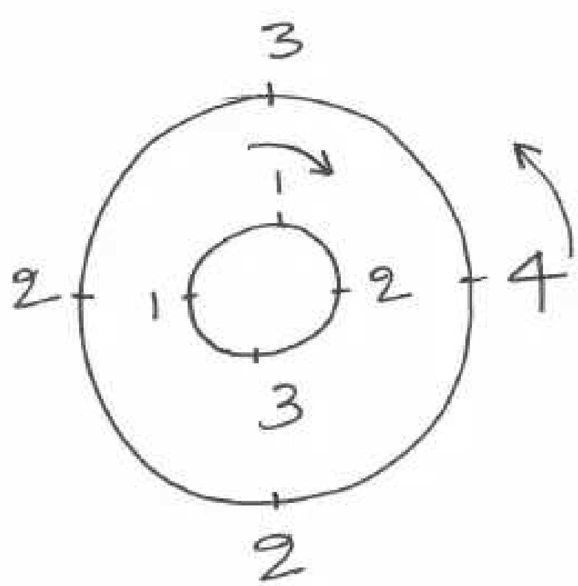
b) Graphical method



$y(0) = 1\times4 + 2\times2 + 3\times2 + 3\times1$

$= 4 + 4 + 6 + 3$

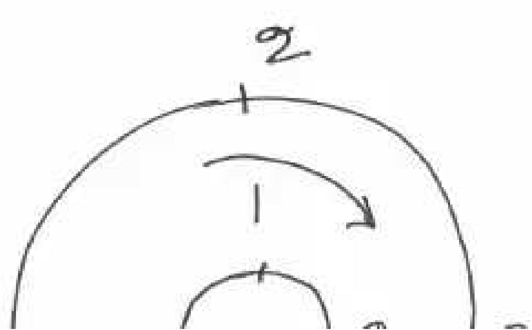
$= 17$



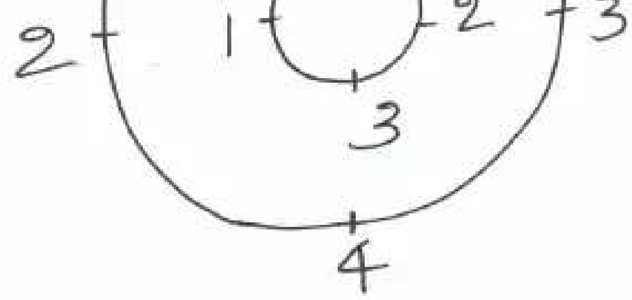
$y(1) = 1\times3 + 2\times4 + 3\times2 + 1\times2$

$= 3 + 8 + 6 + 2$

$= 19$

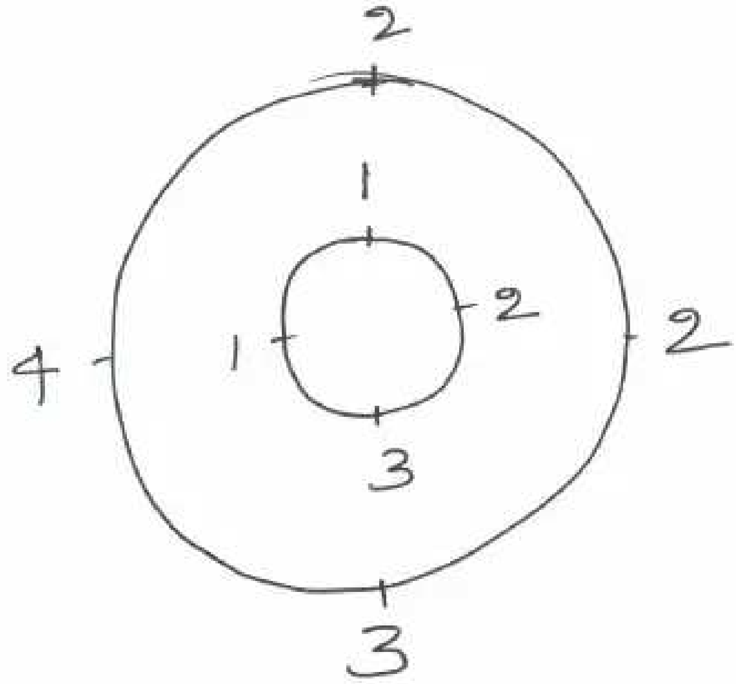


$y(2) = 1\times2 + 2\times3 + 3\times4 + 1\times2$



$= 2+6+12+2$

$= 22$



$y(4) = 1\times2 + 2\times2 + 3\times3 + 4\times1$

$= 2+4+9+4$

$= 19$

c) DFT- IDFT method

(Stockham's method)

(Frequency domain method)

$$\underset{0\le k\le 3}{X(k)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\begin{bmatrix} 4 \end{bmatrix}$$

$$H(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$0 \leq k \leq 3$

$$= \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

$$X(K)\,H(K) = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix} \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

$$= \begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$\text{IDFT of } X(K)H(K) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

37 Let $x(n) = (3,1,0,1)$. obtain the sequence $y(n)$ whose 5-point DFT is

$$Y(K) = [X(K)]^2$$

where $X(K)$ is 5-point DFT of $x(n)$. 69

$$Y(K) = [X(K)]^2$$

$$= X(K)\,X(K)$$

$$\therefore y(n) = x(n) \; 5 \; x(n)$$

$$= \begin{bmatrix} 3 & 0 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 7 \\ 1 \\ 6 \\ 2 \end{bmatrix}$$

38 Find the circular convolution of

$$x_1(n) = \cos\left(\frac{2\pi}{N} k_0 n\right), \quad 0 \le n \le N-1$$

and

$$x_2(n) = \sin\left(\frac{2\pi}{N} k_0 n\right), \quad 0 \le n \le N-1$$

method.

DFT- IDFT method

$$X_1(k) = \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta(k+k_0-N)$$

$0 \le k \le N-1$

(see problem #23)

$$X_2(k) = \frac{N}{2j}\delta(k-k_0) - \frac{N}{2j}\delta(k+k_0-N)$$

$0 \le k \le N-1$

(see problem #24)

$$X_1(k)\,X_2(k) = \frac{N}{2}\,\frac{N}{2j}\,\delta(k-k_0) - \frac{N}{2}\,\frac{N}{2j}\,\delta(k+k_0-N)$$

$$= \frac{N}{2}\left[\frac{N}{2j}\delta(k-k_0) - \frac{N}{2j}\delta(k+k_0-N)\right]$$

$$\text{IDFT of } X_1(k)X_2(k) = \frac{N}{2}\sin\left(\frac{2\pi}{N}k_0 n\right)$$

$0 \le n \le N-1$

$$\therefore\ x_1(n) \; Ⓝ \; x_2(n) = \frac{N}{2}\sin\left(\frac{2\pi}{N}k_0 n\right), \quad 0 \le n \le N-1$$

39 Compute the linear convolution of

$x(n) = (1,2,3,1)$ and

$h(n) = (4,3,2)$

using circular convolution.

Length of $h(n) = 3$.

$\therefore$ Let us append 2 zeroes to $x(n)$

$$x(n) = (1,2,3,1,0,0)$$

Length of $x(n) = 4$

$\therefore$ Let us append 3 zeroes to $h(n)$

$$\therefore h(n) = (4,3,2,0,0,0)$$

Now, let us perform circular convolution of $x(n)$ and $h(n)$.

The result will be same as linear convolution.

$$y(n) = \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 2 \\ 2 & 1 & 0 & 0 & 1 & 3 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 11 \\ 20 \\ 17 \\ 9 \\ 2 \end{bmatrix}$$