Chapter 3

The DFT and FFT

3.1 Introduction

The sampled discrete-time Fourier transform (DTFT) of a finite length, discrete-time signal is known as the *discrete Fourier transform* (DFT). The DFT contains a finite number of samples, equal to the number of samples N in the given signal. Computationally efficient algorithms for implementing the DFT go by the generic name of *fast Fourier transforms* (FFTs). This chapter describes the DFT and its properties, and its relationship to DTFT. The chapter concludes with a discussion of FFT algorithms for computing DFT and its inverse.

3.2 Definition of DFT and its Inverse

Let us consider a discrete time signal $x(n)$ having a finite duration, say in the range $0 \leqslant n \leqslant N-1$. The DTFT of this signal is

$$X(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \tag{3.1}$$

Let us sample $X(\omega)$ using a total of N equally spaced samples in the range: $\omega \in (0, 2\pi)$, so the sampling interval is $\frac{2\pi}{N}$. That is, we sample $X(\omega)$ using the frequencies

$$\omega = \omega_k = \frac{2\pi k}{N}, \quad 0 \leqslant k \leqslant N-1$$

The result is, by definition the DFT.

That is,

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n} \\ &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \end{aligned} \tag{3.2}$$

Equation (3.2) is known as N-point DFT analysis equation. Fig. 3.1 shows the Fourier transform of a discrete-time signal and its DFT samples.

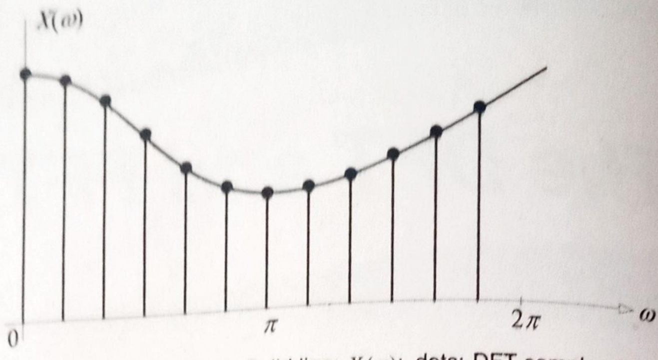


Fig. 3.1 Sampling of $X(\omega)$ to get $X(k)$. Solid line: $X(\omega)$; dots: DFT samples (shown for $N=12$).

While working with DFT, it is customary to introduce a complex quantity:

$$W_N = e^{-j\frac{2\pi}{N}}$$

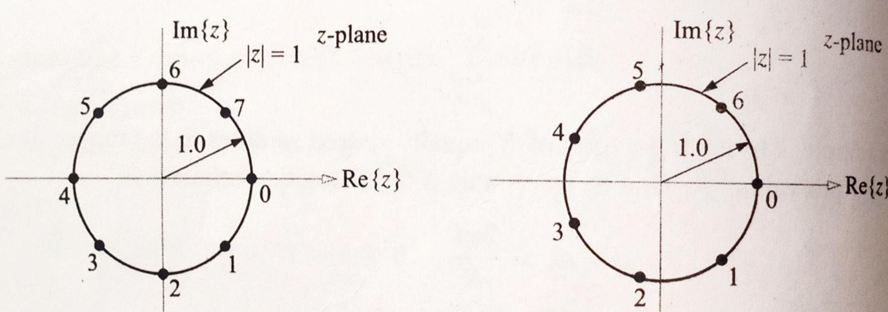
Also, it is very common to represent the DFT operation for a sequence $x(n)$ of length DFT $\{x(n)\}$. As a consequence of this notation, we can rewrite equation (3.2) as

$$X(k) = \text{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \qquad 0 \leqslant k \leqslant N-1$$

The complex quantity W_N is periodic with a period equal to N. That is,

$W_N^{a+N} = e^{-j\frac{2\pi}{N}(a+N)} = e^{-j\frac{2\pi}{N}a} = W_N^a$, where a is any integer.

Figs. 3.2(a) and (b) shows the sequence W_N^n for $0 \leqslant n \leqslant N-1$ in the z-plane for N be even and odd respectively.



(a) (b)

Fig. 3.2(a) The sequence W_N^n for even N. **(b)** The sequence W_N^n for odd N.

The sequence W_N^n for $0 \leqslant n \leqslant N-1$ lies on a circle of unit radius in the complex plane the phases are equally spaced, beginning at zero.

The formula given in the lemma to follow is a useful tool in deriving and analysing vari DFT oriented results.

3.2.1 Lemma

$$\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(k) = \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases} \tag{3.3}$$

Proof:

We know that

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}; \quad a \neq 1$$

Applying the above result to the left side of equation (3.3), we get

$$\begin{aligned}\sum_{n=0}^{N-1} (W_N^k)^n = \frac{1-W_N^{kN}}{1-W_N^k} &= \frac{1-e^{-j\frac{2\pi}{N}kN}}{1-e^{-j\frac{2\pi}{N}k}}; \quad k \neq 0 \\ &= \frac{1-1}{1-e^{-j\frac{2\pi k}{N}}} \\ &= 0, \quad k \neq 0\end{aligned}$$

when $k = 0$, the left side of equation (3.3) becomes

$$\sum_{n=0}^{N-1} W_N^{0\times n} = \sum_{n=0}^{N-1} 1 = N$$

Hence, we may write

$$\begin{aligned}\sum_{n=0}^{N-1} W_N^{kn} &= \begin{cases} N, & k=0 \\ 0, & k \neq 0 \end{cases} \\ &= N\delta(k), \quad 0 \leqslant k \leqslant N-1\end{aligned}$$

3.2.2 Inverse DFT

The DFT values $(X(k),\ 0 \leqslant k \leqslant N-1)$, uniquely define the sequence $x(n)$ through the inverse DFT formula (IDFT):

$$x(n) = \text{IDFT}\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \tag{3.4}$$

The above equation is known as the synthesis equation.

Proof:

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m) W_N^{km}\right] W_N^{-kn} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} W_N^{-(n-m)k}\right]\end{aligned}$$

It can be shown that

$$\sum_{k=0}^{N-1} W_N^{-(n-m)k} = \begin{cases} N, & n = m \\ 0, & n \neq m \end{cases}$$
$$= N\delta(n-m)$$

Hence,

$$\frac{1}{N}\sum_{k=0}^{N-1} x(m)N\delta(n-m)$$
$$= \frac{1}{N} \times Nx(m)\Big|_{m=n} \quad \text{(sifting property)}$$
$$= x(n)$$

3.2.3 Periodicity of $X(k)$ and $x(n)$

The N-point DFT and N-point IDFT are implicit periodic with period N. Eventhough $x(n)$ and $X(k)$ are sequences of length-N each, they can be shown to be periodic with a period N because the exponentials $W_N^{\pm kn}$ in the defining equations of DFT and IDFT are periodic with a period N. For this reason, $x(n)$ and $X(k)$ are called *implicit periodic sequences*. We reiterate the fact that for finite length sequences in DFT and IDFT analysis periodicity means implicit periodicity. This can be proved as follows:

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}$$

$$\Rightarrow \quad X(k+N) = \sum_{n=0}^{N-1} x(n)W_N^{(k+N)n}$$
$$= \sum_{n=0}^{N-1} x(n)W_N^{kn}\, W_N^{Nn}$$

Since, $W_N^{Nn} = e^{-j\frac{2\pi}{N}Nn} = e^{-j2\pi n} = 1$, we get

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$
$$= X(k)$$

Similarly,

$$x(n) \triangleq \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}$$

$$\Rightarrow \quad x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-k(n+N)}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\, W_N^{-kN}$$

Since, $W_N^{-kN} = e^{j\frac{2\pi}{N}kN} = e^{j2\pi k} = 1$, we get

$$\begin{aligned} x(n+N) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \\ &= x(n) \end{aligned}$$

Since, DFT and its inverse are both periodic with period N, it is sufficient to compute the results for one period (0 to $N-1$). We want to emphasize that both $x(n)$ and $X(k)$ have a starting index of zero.

A very important implication of $x(n)$ being periodic is, if we wish to find DFT of a periodic signal, we extract one period of the periodic signal and then compute its DFT.

Example 3.1 Compute the 8-point DFT of the sequence $x(n)$ given below:

$$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$$

Solution

The complex basis functions, W_8^n for $0 \leqslant n \leqslant 7$ lie on a circle of unit radius as shown in Fig. Ex.3.1.

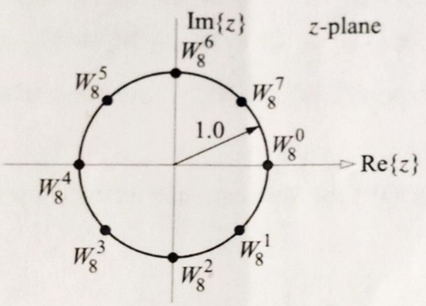


Fig. Ex.3.1 Sequence W_8^n for $0 \leqslant n \leqslant 8$.

Since $N = 8$, we get $W_8 = e^{-j\frac{2\pi}{8}}$.

Thus,

$$\begin{aligned} W_8^0 &= 1 \\ W_8^1 &= e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ W_8^2 &= e^{-j\frac{\pi}{2}} = -j \\ W_8^3 &= e^{-j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ W_8^4 &= -W_8^0 = -1 \end{aligned}$$

$$W_8^5 = -W_8^1 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$
$$W_8^6 = -W_8^2 = j$$
$$W_8^7 = -W_8^3 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

By definition, the DFT of $x(n)$ is

$$\begin{aligned}
X(k) &= \text{DFT}\{x(n)\} \\
&= \sum_{n=0}^{7} x(n) W_8^{kn} \\
&= 1 + 1 \times W_8^k + 1 \times W_8^{2k} + 1 \times W_8^{3k} \\
&= 1 + W_8^k + W_8^{2k} + W_8^{3k}, \; k = 0, 1, \cdots 7 \\
X(0) &= 1 + 1 + 1 + 1 = 4 \\
X(1) &= 1 + W_8^1 + W_8^2 + W_8^3 = 1 - j2.414 \\
X(2) &= 1 + W_8^2 + W_8^4 + W_8^6 = 0 \\
X(3) &= 1 + W_8^3 + W_8^6 + W_8^1 = 1 - j0.414 \\
X(4) &= 1 + W_8^4 + W_8^0 + W_8^4 = 0 \\
X(5) &= 1 + W_8^5 + W_8^2 + W_8^7 = 1 + j0.414 \\
X(6) &= 1 + W_8^6 + W_8^4 + W_8^2 = 0 \\
X(7) &= 1 + W_8^7 + W_8^6 + W_8^5 = 1 + j2.414
\end{aligned}$$

Please note the periodic property: $W_N^a = W_N^{a+N}$, where a is any integer.

Example 3.2 Compute the DFT of the sequence defined by $x(n) = (-1)^n$ for

a. $N = 3$,

b. $N = 4$,

c. N odd,

d. N even.

☐ **Solution**

$$\begin{aligned}
X(k) &= \text{DFT}\{x(n)\} \\
&= \sum_{n=0}^{N-1} (-1)^n W_N^{nk} \\
&= \sum_{n=0}^{N-1} [-W_N^k]^n \\
&= \frac{1 - (-1)^N}{1 + W_N^k} \quad \text{for } W_N^k \neq -1
\end{aligned}$$

a. $N = 3$

$$X(k) = \frac{2}{1 + W_3^k} = \frac{2}{1 + \cos\left(\frac{2\pi k}{3}\right) - j\sin\left(\frac{2\pi k}{3}\right)}, \quad 0 \leqslant k \leqslant 2$$

b. $N = 4$

$$X(k) = 0 \quad \text{for} \quad W_4^k \neq -1 \quad \text{or} \quad k \neq 2$$

With $k = 2$, we get

$$\begin{aligned} X(2) &= \sum_{n=0}^{3} (-1)^n W_4^{2n} \\ &= 1 - W_4^2 + W_4^4 - W_4^6 \\ &= 1 - (-1) + (-1)^2 - (-1)^3 = 4 \end{aligned}$$

Hence,

$$X(k) = 4\delta(k-2).$$

c. We know that

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

If

$$N = 2k, \quad \text{we get} \quad W_N^k = -1.$$

Since, N is odd no k exists. This means to say that $W_N^k \neq -1$ for all k from 0 to $N-1$.
Therefore,

$$\begin{aligned} X(k) &= \frac{2}{1 + W_N^k}, \quad 0 \leqslant k \leqslant N-1 \\ &= \frac{2}{1 + \cos\frac{2\pi k}{N} - j\sin\frac{2\pi k}{N}} \end{aligned}$$

d. N even: $W_N^k = -1$, if $k = \frac{N}{2}$.

$$X(k) = 0 \quad \text{for} \quad k \neq \frac{N}{2}$$

and

$$\begin{aligned} X\left(\frac{N}{2}\right) &= \sum_{n=0}^{N-1} \left[-W_N^{\frac{N}{2}}\right]^n \\ &= \sum_{n=0}^{N-1} (1) = N \end{aligned}$$

Hence,

$$X(k) = N\delta\left(k - \frac{N}{2}\right)$$

Example 3.3 Find the N-point DFT of the following sequences:

a. $x_1(n) = \delta(n)$

b. $x_2(n) = \delta(n - n_0)$

☐ **Solution**

a.

$$\begin{aligned} X_1(k) &= \text{DFT}\{x_1(n)\} \\ &= \sum_{n=0}^{N-1} x_1(n) W_N^{kn} \\ &= \sum_{n=0}^{N-1} \delta(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Using sifting property, we get

$$X_1(k) = W_N^{kn}\Big|_{n=0} = 1$$

b.

$$\begin{aligned} X_2(k) &= \text{DFT}\{x_2(n)\} \\ &= \sum_{n=0}^{N-1} x_2(n) W_N^{kn} \\ &= \sum_{n=0}^{N-1} \delta(n-n_0) W_N^{kn} \end{aligned}$$

Using sifting property, we get

$$\begin{aligned} X_2(k) &= W_N^{kn}\Big|_{n=n_0} \\ &= W_N^{kn_0}, \qquad 0 \leqslant k \leqslant N-1 \\ &= e^{-j\frac{2\pi}{N}kn_0} \end{aligned}$$

Example 3.4 Find the N-point DFT of the sequence

$$x(n) = e^{j\omega m n}, \qquad 0 \leqslant n \leqslant N-1$$

☐ **Solution**

$$\begin{aligned} X(k) &= \text{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mn} W_N^{kn} \qquad \left(\because \omega = \frac{2\pi}{N}\right) \end{aligned}$$

$$= \sum_{n=0}^{N-1} W_N^{-mn} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} W_N^{(k-m)n}$$

We know that
$$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b - 1}, \quad b \neq 1$$

Hence,
$$X(k) = \frac{W_N^{(k-m)N} - 1}{W_N^{k-m} - 1}, \quad k \neq m$$

$$= \frac{W_N^{kN} W_N^{-mN} - 1}{W_N^{k-m} - 1}$$

$$= \frac{1 \times 1 - 1}{W_N^{k-m} - 1} = 0, \quad k \neq m$$

When, $k = m$,

$$X(m) = \sum_{n=0}^{N-1} 1 = N$$

Hence,
$$X(k) = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases}$$

or
$$X(k) = N\delta(k - m), \qquad 0 \leqslant m \leqslant N - 1$$

Example 3.5 Compute the N-point DFT of the sequence,

$$x(n) = a^n, \qquad 0 \leqslant n \leqslant N - 1$$

☐ Solution

$$X(k) = \text{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a^n W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a W_N^k)^n$$

We know that
$$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b - 1}, \quad b \neq 1$$

Hence,
$$X(k) = \frac{a^N W_N^{kN} - 1}{aW_N^k - 1}$$
$$= \frac{a^N - 1}{aW_N^k - 1}, \quad 0 \leqslant k \leqslant N - 1$$

Example 3.6 Compute the N-point DFT of the sequence,

$$x(n) = an, \quad 0 \leqslant n \leqslant N - 1$$

☐ **Solution**

We know that
$$X(k) = \text{DFT}\{x(n)\}$$
$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}$$
$$= a \sum_{n=0}^{N-1} n W_N^{kn}, \quad 0 \leqslant k \leqslant N - 1 \tag{3.5}$$

Let,
$$S = \sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b - 1}, \quad b \neq 1$$

Differentiating both the sides of the above equation with respect to b, we get

$$\frac{a}{b} = \frac{b\frac{d}{dt}(a) - a\frac{d}{dt}(b)}{b^2}$$

$$\sum_{n=0}^{N-1} n b^{n-1} = \frac{(b-1)N\, b^{N-1} - (b^N - 1) \times 1}{(b-1)^2}$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} n b^n = \frac{b\left(Nb^N - Nb^{N-1} - b^N + 1\right)}{(b-1)^2}$$
$$= \frac{b\left(b^N(N-1) - Nb^{N-1} + 1\right)}{(b-1)^2}$$

Letting $b = W_N^k$ in the above expression, we get

$$\sum_{n=0}^{N-1} n W_N^{kn} = \frac{W_N^k \left[W_N^{kN}(N-1) - N W_N^{k(N-1)} + 1\right]}{\left[W_N^k - 1\right]^2}$$
$$= \frac{W_N^k\left[N - 1 - N W_N^{-k} + 1\right]}{\left[W_N^k - 1\right]^2} \quad \left(\because\ W_N^{kN} = \left(e^{-j\frac{2\pi}{N}kN}\right) = e^{-j2\pi k} = 1\right)$$
$$= \frac{N\left[W_N^k - 1\right]}{\left[W_N^k - 1\right]^2} = \frac{N}{W_N^k - 1}, \quad k \neq 0$$

When $k = 0$, equation (3.5) becomes

$$X(0) = a\sum_{n=0}^{N-1} n = \frac{aN(N-1)}{2}$$

Hence,
$$X(k) = \begin{cases} \frac{aN(N-1)}{2}, & k = 0 \\ \frac{aN}{W_N^k - 1}, & k \neq 0 \end{cases}$$

Example 3.7 a. Compute the N-point DFT of the sequence, $x(n) = 1,\ 0 \leqslant n \leqslant N-1$.
b. For $N = 5$, compute the DFT of $x_1(n) = (1, 1, 1, 0, 0)$ and compare the result with the DFT of $x_2(n) = (1, 1, 1)$ for $N = 3$.

Solution

a.
$$\begin{aligned} X(k) &= \text{DFT}\{x(n)\} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} 1 \times (W_N^k)^n \end{aligned}$$

We know that
$$\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b - 1}, \quad b \neq 1.$$

Hence,
$$X(k) = \frac{W_N^{kN} - 1}{W_N^k - 1} = \frac{1 - 1}{W_N^k - 1} = 0, \quad k \neq 0.$$

When $k = 0$, we get

$$X(0) = \sum_{n=0}^{N-1} 1 \times 1 = N$$

Hence,
$$X(k) = \begin{cases} N, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

or
$$X(k) = N\delta(k)$$

b. Given,
$$x_1(n) = (1, 1, 1, 0, 0)$$

$$N = 5$$

In the present context, $W_5 = e^{-j\frac{2\pi}{5}}$

$$
\begin{aligned}
\Rightarrow \qquad W_5^0 &= 1\\
W_5^1 &= 0.309 - j0.951\\
W_5^2 &= -0.809 - j0.587\\
W_5^3 &= -0.809 + j0.587\\
W_5^4 &= 0.309 + j0.951
\end{aligned}
$$

Hence,
$$
\begin{aligned}
X_1(k) = \mathrm{DFT}\{x_1(n)\} &= \sum_{n=0}^{4} x_1(n) W_5^{kn}, \quad 0 \leqslant k \leqslant 4\\
&= 1 + W_5^k + W_5^{2k}\\
\Rightarrow \qquad X_1(0) &= 1 + 1 + 1 = 3\\
X_1(1) &= 1 + W_5^1 + W_5^2 = 0.5 - j1.538\\
X_1(2) &= 1 + W_5^2 + W_5^4 = 0.5 + j0.364\\
X_1(3) &= 1 + W_5^3 + W_5^1 = 0.5 - j0.364\\
X_1(4) &= 1 + W_5^4 + W_5^3 = 0.5 + j1.538
\end{aligned}
$$

Also, given $\qquad x_2(n) = (1, 1, 1), \quad N = 3$

Now,
$$
\begin{aligned}
W_N &= e^{-j\frac{2\pi}{N}}\\
W_3 &= e^{-j\frac{2\pi}{3}}\\
\Rightarrow \qquad W_3^0 &= 1\\
W_3^1 &= -0.5 - j0.866\\
W_3^2 &= -0.5 + j0.866
\end{aligned}
$$

Hence,
$$
\begin{aligned}
X_2(k) &= \mathrm{DFT}\{x_2(n)\}\\
&= \sum_{n=0}^{2} x_2(n) W_N^{kn}, \qquad 0 \leqslant k \leqslant 2\\
&= W_3^{0k} + W_3^k + W_3^{2k}\\
&= 1 + W_3^k + W_3^{2k}\\
\Rightarrow \qquad X_2(0) &= 1 + 1 + 1 = 3\\
X_2(1) &= 1 + W_3^1 + W_3^2 = 0\\
X_2(3) &= 1 + W_3^2 + W_3^1 = 0
\end{aligned}
$$

Thus, we find that $X_1(k) \neq X_2(k)$.

Example 3.8 Compute the N-point DFT of the sequence:

$$x(n) = \cos(n\omega_0), \quad \omega_0 = \frac{2\pi}{N} k_0, \qquad 0 \leqslant n \leqslant N - 1.$$

Solution

Given,
$$x(n) = \cos(n\omega_0)$$
$$\Rightarrow \quad x(n) = \frac{1}{2}e^{jn\omega_0} + \frac{1}{2}e^{-jn\omega_0}$$
$$= \frac{1}{2}e^{jn\frac{2\pi}{N}k_0} + \frac{1}{2}e^{-jn\frac{2\pi}{N}k_0}$$
$$= \frac{1}{2}e^{-j\frac{2\pi}{N}(-k_0n)} + \frac{1}{2}e^{-j\frac{2\pi}{N}(k_0n)}$$
$$= \frac{1}{2}W_N^{-k_0n} + \frac{1}{2}W_N^{k_0n}$$

Hence,
$$X(k) = \text{DFT}\{x(n)\}, \quad 0 \leqslant k \leqslant N-1$$
$$= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k+k_0)n}$$
$$= \frac{1}{2}S_1 + \frac{1}{2}S_2 \tag{3.6}$$

To find S_1 and S_2:

$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n}$$
$$= \frac{W_N^{(k-k_0)N} - 1}{W_N^{(k-k_0)} - 1} = \frac{W_N^{kN}\,W_N^{-k_0N} - 1}{W_N^{(k-k_0)} - 1}$$
$$= \frac{1 \times 1 - 1}{W_N^{(k-k_0)} - 1} = 0, \qquad k \neq k_0$$

When $k = k_0$, we get

$$S_1 = \sum_{n=0}^{N-1}(1)^n = N$$

Hence,
$$S_1 = \sum_{n=0}^{N-1} W_N^{(k-k_0)n} = \begin{cases} 0, & k \neq k_0 \\ N, & k = k_0 \end{cases}$$

or
$$S_1 = N\delta(k-k_0)$$

Similarly,
$$S_2 = N\delta(k+k_0)$$
$$= N\delta\,[k-(N-k_0)]$$

Thus,
$$X(k) = \frac{1}{2}S_1 + \frac{1}{2}S_2$$
$$= \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta\,[k-(N-k_0)]$$

Example 3.9 Compute the inverse DFT of the sequence,

$$X(k) = (2, 1 + j, 0, 1 - j)$$

☐ **Solution**

$$\begin{aligned} x(n) &= \text{IDFT}\{X(k)\} \\ &\triangleq \frac{1}{N}\sum_{n=0}^{N-1} x(n) W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \end{aligned}$$

Please note that:

$$W_N^{-kn} = \left[W_N^{kn}\right]^*$$

Since, $N = 4$, we get

$$\begin{aligned} x(n) &= \frac{1}{4}\sum_{k=0}^{3} X(k) W_4^{-kn}, \quad 0 \leqslant n \leqslant 3 \\ &= \frac{1}{4}\left[X(0)W_4^{-0\times n} + X(1)W_4^{-n} + X(2)W_4^{-2n} + X(3)W_4^{-3n}\right] \\ &= \frac{1}{4}\left[2 + (1+j)W_4^{-n} + 0 + (1-j)W_4^{-3n}\right] \end{aligned}$$

Hence,

$$\begin{aligned} x(0) &= \frac{1}{4}[2 + (1+j) + (1-j)] = 1 \\ x(1) &= \frac{1}{4}\left[2 + (1+j)W_4^{-1} + (1-j)W_4^{-3}\right] \\ &= \frac{1}{4}[2 + (1+j)j + (1-j)(-j)] = 0 \\ x(2) &= \frac{1}{4}\left[2 + (1+j)W_4^{-2} + (1-j)W_4^{-6}\right] \end{aligned}$$

Because of periodicity, $W_4^{-6} = W_4^{-2}$.

Hence,

$$\begin{aligned} x(2) &= \frac{1}{4}[2 + (1+j)(-1) + (1-j)(-1)] = 0 \\ x(3) &= \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-9}\right] \end{aligned}$$

Because of periodicity, $W_4^{-9} = W_4^{-5} = W_4^{-1}$.

Hence,

$$\begin{aligned} x(3) &= \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-1}\right] \\ &= \frac{1}{4}[2 + (1+j)(-j) + (1-j)(j)] = 1 \end{aligned}$$

Hence,

$$x(n) = (1, 0, 0, 1)$$

3.3 Matrix Relation for Computing DFT

The defining relation for DFT of a finite length sequence $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Let us evaluate $X(k)$ for different values of k in the range $(0, N-1)$ as given below:

$$\begin{aligned}
X(0) &= W_N^0\, x(0) + W_N^0\, x(1) + \cdots + W_N^0\, x(N-1) \\
X(1) &= W_N^0\, x(0) + W_N^1\, x(1) + \cdots + W_N^{(N-1)}\, x(N-1) \\
X(2) &= W_N^0\, x(0) + W_N^2\, x(1) + \cdots + W_N^{2(N-1)}\, x(N-1) \\
&\vdots \\
X(N-1) &= W_N^0\, x(0) + W_N^{(N-1)}\, x(1) + \cdots + W_N^{(N-1)(N-1)}\, x(N-1)
\end{aligned}$$

Putting the N DFT equations in N unknowns in the matrix form, we get

$$\mathbf{X} = \mathbf{W}_N \mathbf{x} \tag{3.7}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $(N \times 1)$ matrices, and W_N is an $(N \times N)$ square matrix called the *DFT matrix*. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

The elements W_N^{kn} of $\mathbf{W}_N$ are called *complex basis functions or twiddle factors.*

Example 3.10 Compute the 4-point DFT of the sequence, $x(n) = (1, 2, 1, 0)$.

Solution

With $N = 4$, $W_4 = e^{-j\frac{2\pi}{4}} = -j$.

We know that $\mathbf{X} = \mathbf{W}_N\, \mathbf{x}$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Exploting the periodic property: $W_N^a = W_N^{a+N}$, where a is any integer the above matrix relation becomes

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Hence, $$X(k) = (4, -j2, 0, j2)$$

3.4 Matrix Relation for Computing IDFT

We know that
$$\mathbf{X} = \mathbf{W}_N\, \mathbf{x}$$

Premultiplying both the sides of the above equation by $\mathbf{W}_N^{-1}$, we get

$$\mathbf{W}_N^{-1}\mathbf{X} = \mathbf{W}_N^{-1}\mathbf{W}_N\, \mathbf{x}$$

$$\Rightarrow \quad \mathbf{W}_N^{-1}\mathbf{X} = \mathbf{x}$$

or

$$\mathbf{x} = \mathbf{W}_N^{-1}\,\mathbf{X} \tag{3.8}$$

In the above equation (3.8), $\mathbf{W}_N^{-1}$ is called *IDFT matrix*.

The defining equation for finding IDFT of a sequence $X(k)$ is

$$\begin{aligned} x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1 \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[W_N^{kn}\right]^* \end{aligned}$$

The first set of N IDFT equations in N unknowns may be expressed in the matrix form as

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^*\mathbf{X} \tag{3.9}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of $\mathbf{W}_N$. Comparison of equations (3.8) and (3.9) leads us to conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

This very important result shows that $\mathbf{W}_N^{-1}$ requires only conjugation of $\mathbf{W}_N$ multiplied by $\frac{1}{N}$, an obvious computational advantage. **The matrix relations (3.7) and (3.9) together define DFT as a linear transformation.**

3.5 Using the DFT to Find the IDFT

We know that

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad n = 0, 1, \cdots N-1$$

Taking complex conjugates on both the sides of the above equation, we get

$$\begin{aligned} x^*(n) &= \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\right]^* \\ \Rightarrow \quad x^*(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn} \end{aligned} \tag{3.10}$$

The right-hand side of equation (3.10) is recognized as the DFT of $X^*(k)$, so we can rewrite equation (3.10) as follows:

$$x^*(n) = \frac{1}{N}\,\mathrm{DFT}\{X^*(k)\} \tag{3.11}$$

Taking complex conjugates on both the sides of equation (3.11), we get

$$x(n) = \frac{1}{N}\left[\mathrm{DFT}\{X^*(k)\}\right]^*$$

The above result suggests that DFT algorithm itself can be used to find IDFT. In practice, this is indeed what is done.

Example 3.11 Find the IDFT of 4-point sequence,

$$X(k) = (4, -j2, 0, j2)$$

using the DFT.

☐ **Solution**

Let us first conjugate the sequence $X(k)$ to get $X^*(k) = (4, j2, 0, -j2)$. As a second step, we find the DFT of $X^*(k)$.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} X^*(0) \\ X^*(1) \\ X^*(2) \\ X^*(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

Finally, we get the conjugate of the above result and divide it by $N = 4$ to get IDFT of $X(k)$ as

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}(4, 8, 4, 0) = (1, 2, 1, 0)$$

Example 3.12 Consider a signal of length equal to 4 defined by

$$x(n) = (1, 2, 3, 1)$$

a. Compute the 4-point DFT by solving explicitly the 4-by-4 system of linear equations defined by the inverse DFT formula.
b. Verify the result of part (a) by finding $X(k)$ using the defining equation for DFT.

☐ **Solution**

We have,

$$\begin{aligned} \text{IDFT}\{X(k)\} &= x(n) \\ &\triangleq \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \end{aligned}$$

$$\Rightarrow \quad \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk}$$
$$= N\,x(n), \quad 0 \leqslant n \leqslant N-1$$

Since $N = 4$, we get

$$\sum_{k=0}^{3} X(k)e^{j\frac{2\pi}{4}nk} = 4x(n), \quad n = 0, 1, 2, 3$$

Hence, we get the following linear equations:

$$X(0) + X(1) + X(2) + X(3) = 4x(0) = 4$$
$$X(0) + X(1)\,e^{j\frac{\pi}{2}} + X(2)\,e^{j\pi} + X(3)\,e^{j\frac{3\pi}{2}} = 4x(1) = 8$$
$$X(0) + X(1)\,e^{j\pi} + X(2)\,e^{j2\pi} + X(3)\,e^{j3\pi} = 4x(2) = 12$$
$$X(0) + X(1)\,e^{j\frac{3\pi}{2}} + X(2)\,e^{j3\pi} + X(3)\,e^{j\frac{9\pi}{2}} = 4x(3) = 4$$

Putting the above set of linear equations in the matrix form, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\text{DFT}\,\{x(n)\} = X(k)$$
$$\triangleq \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

In the present context,

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

$$= x(0) + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k}$$

$$\Rightarrow \quad X(k) = 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

Evaluation of $X(k)$ needs the following complex basis functions:

$$W_4^0 = \left[e^{-j\frac{2\pi}{4}}\right]^0 = 1, \; W_4^1 = \left[e^{-j\frac{2\pi}{4}}\right]^1 = -j, \; W_4^2 = \left[e^{-j\frac{2\pi}{4}}\right]^2 = -1, \; W_4^3 = \left[e^{-j\frac{2\pi}{4}}\right]^3 = j.$$

Also,

$$W_N^a = W_N^{a+N}, \quad \text{where } a \text{ is any integer}$$

Hence,

$$X(0) = 1 + 2 + 3 + 1 = 7$$

$$X(1) = 1 + 2W_4^1 + 3W_4^2 + 3W_4^3 = -2 - j$$

$$X(2) = 1 + 2W_4^2 + 3W_4^4 + W_4^6$$

$$= 1 + 2W_4^2 + 3W_4^0 + W_4^2 = 1$$

$$X(3) = 1 + 2W_4^3 + 3W_4^6 + W_4^9$$

$$= 1 + 2W_4^3 + 3W_4^2 + W_4^1 = -2 + j$$

3.6 Concept of Circular Shift and Circular Symmetry

Let us consider a sequence $x(n)$ defined for all n, the translated version of $x(n)$ is written as $x(n - n_0)$, where n_0 represents the number of indices that the sequence $x(n)$ is translated to right. For a finite length sequence defined for $0 \leqslant n \leqslant N - 1$, if a regular shift is employed, parts of the sequence would fall outside the defined range for n and the first part of the sequence would be undefined. It was proved in the earlier discussions that although $x(n)$ is defined for $0 \leqslant n \leqslant N - 1$, $x(n)$ is implicit periodic with a period equal to N. Because of this implied periodic nature of $x(n)$, the fundamental form of sequence translation, useful in a mathematical sense, is a circular translation.

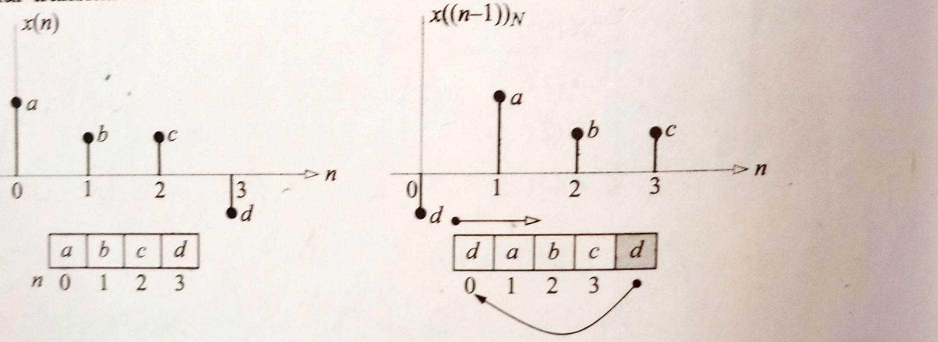


Fig. 3.3 Circular shift.

The circular shift or translation of $x(n)$ with $0 \leqslant n \leqslant N-1$ by an amount n_0 to right is denoted by $x((n-n_0))_N$. This operation tantamounts, wrapping the first part of the sequence that falls outside the range for n around to the first part of the sequence, or just a straight translation of the periodic extension outside of 0 to $N-1$ of the given sequence. In otherwords, any value that falls off the interval $(0, \ldots N-1)$ after the shift, it comes back from the other side as shown in Fig. 3.3.

Thus, we follow the guidelines given below for generating circularly shifted signal.

- To generate $x((n-n_0))_N$: Move the last n_0 samples of $x(n)$ to the beginning.
- To generate $x((n+n_0))_N$: Move the first n_0 samples of $x(n)$ to the end.

Circular folding generates the signal $x((-n))_N$ from $x(n)$. We fold $x(n)$, create the periodic extension of the folded signal and pick N samples of the periodic extension over $(0, N-1)$. In order to understand circular folding, let us consider a finite duration sequence,

$$x(n) = (\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4), \quad 0 \leqslant n \leqslant 3$$

Then,

$$x(-n) = (4\ \ 3\ \ 2\ \ \underset{\substack{\uparrow \\ n=0}}{1})$$

The periodic extension of $x(-n)$ is shown below:

$$\cdots \ \vdots\ 4\,3\,2\,1\ \vdots\ 4\,3\,2\,\underset{\substack{\uparrow \\ n=0}}{1}\ \vdots\ 4\,3\,2\,1\ \vdots\ \cdots$$

Let us now, pick 4 samples starting from $n = 0$. This results in a circularly folded sequence is given by

$$x((-n))_N = (1, 4, 3, 2)$$

Because of the implied periodicity of $x(n)$, it may be noted that $x((-n))_N = x(N-n)$. Thus, for the example considered,

$$\begin{aligned} x((-n))_N &= x(4-n), \qquad 0 \leqslant n \leqslant 3 \\ &= (x(4), x(3), x(2), x(1)) \end{aligned}$$

Since, $\quad x(n+N) = x(n)$, we have $x(4) = x(0)$.

Hence,

$$\begin{aligned} x((-n))_N &= (x(0), x(3), x(2), x(1)) \\ &= (\underset{\substack{\uparrow \\ n=0}}{1}, 4, 3, 2) \end{aligned}$$

Even symmetry of $x(n)$ with $0 \leqslant n \leqslant N-1$ requires that $x(n) = x((-n))_N$. Similarly, the odd symmetry of $x(n)$ with $0 \leqslant n \leqslant N-1$ requires that $x(n) = -x((-n))_N$.

3.7 Properties of DFT

In the following section, we shall discuss some of the important properties of the DFT. They strikingly similar to other frequency-domain transforms, but must always be used in keeping w implied periodicity (of both DFT and IDFT) in time and frequency-domains.

3.7.1 Linearity

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k), \qquad k = 0, 1, \cdots N-1$$

with $X_1(k)$ and $X_2(k)$ are the DFTs of the sequences $x_1(n)$ and $x_2(n)$, respectively, both lengths N.

Proof:

$$\underset{n=0,1,\ldots N-1}{x(n)} \longrightarrow \boxed{\text{DFT}\{\cdot\}} \longrightarrow \underset{k=0,1,\ldots N-1}{X(k)}$$

Fig. 3.4 DFT operation viewed as a system represented by an operator DFT{·}.

In Fig. 3.4, we have represented the DFT operation by an operator DFT $\{\cdot\}$. This figure always reminds us that the input, $x(n)$ and output, $X(k)$ are of same length, N. Hence, N is known as the transform length for the DFT operation.

We know that,
$$\text{DFT}\{x(n)\} \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Letting $x(n) = ax_1(n) + bx_2(n)$, we get

$$\begin{aligned}
\text{DFT}\{ax_1(n) + bx_2(n)\} &= \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)]\, W_N^{kn} \\
&= a\sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b\sum_{n=0}^{N-1} x_2(n) W_N^{kn} \\
&= aX_1(k) + bX_2(k), \quad 0 \leqslant k \leqslant N-1
\end{aligned}$$

Sometimes, we represent the linearity property as given below:

$$a\, x_1(n) + b\, x_2(n) \xleftrightarrow{\text{DFT}} a\, X_1(k) + b\, X_2(k)$$

Example 3.13 Find the 4-point DFT of the sequence,

$$x(n) = \cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{4}n\right)$$

Use linearity property.

Solution

Given $N = 4$.

We know that,
$$W_N = e^{-j\frac{2\pi}{N}} \quad \Rightarrow \quad W_4 = e^{-j\frac{\pi}{2}}$$

Hence,
$$\begin{aligned} W_4^0 &= 1 \\ W_4^1 &= e^{-j\frac{\pi}{2}} = -j \\ W_4^2 &= e^{-j\pi} = -1 \\ W_4^3 &= e^{-j\frac{3\pi}{2}} = j \end{aligned}$$

Let
$$x_1(n) = \cos\left(\frac{\pi}{4}n\right)$$

and
$$x_2(n) = \sin\left(\frac{\pi}{4}n\right)$$

Then, the values of $x_1(n)$ and $x_2(n)$ for $0 \leqslant n \leqslant 3$ are tabulated below:

n	$x_1(n) = \cos\left(\frac{\pi}{4}n\right)$	$x_2(n) = \sin\left(\frac{\pi}{4}n\right)$
0	1	0
1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
2	0	1
3	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

As the next step in the problem solving, we compute the 4-point DFTs, $X_1(k)$ and $X_2(k)$.

$$\begin{aligned} X_1(k) &= \text{DFT}\{x_1(n)\} \\ &\triangleq \sum_{n=0}^{3} x_1(n) W_4^{kn}, \quad k = 0, 1, 2, 3 \end{aligned}$$

$$\Rightarrow \quad X_1(k) = 1 + \frac{1}{\sqrt{2}} W_4^k + 0 - \frac{1}{\sqrt{2}} W_4^{3k}$$

Hence,
$$\begin{aligned} X_1(0) &= 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 \\ X_1(1) &= 1 + \frac{1}{\sqrt{2}} W_4^1 - \frac{1}{\sqrt{2}} W_4^3 = 1 - j1.414 \\ X_1(2) &= 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^6 \\ &= 1 + \frac{1}{\sqrt{2}} W_4^2 - \frac{1}{\sqrt{2}} W_4^2 = 1 \end{aligned}$$

$$
\begin{aligned}
X_1(3) &= 1 + \frac{1}{\sqrt{2}} W_4^3 - \frac{1}{\sqrt{2}} W_4^9 \\
&= 1 + \frac{1}{\sqrt{2}} W_4^3 - \frac{1}{\sqrt{2}} W_4^1 \\
&= 1 + j1.414
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
X_2(k) &= \text{DFT}\{x_2(n)\} \\
&\triangleq \sum_{n=0}^{3} x_2(n) W_4^{kn}
\end{aligned}
$$

$$
\Rightarrow \quad X_2(k) = \frac{1}{\sqrt{2}} W_4^k + W_4^{2k} + \frac{1}{\sqrt{2}} W_4^{3k}
$$

Hence,

$$
\begin{aligned}
X_2(0) &= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 2.414 \\
X_2(1) &= \frac{1}{\sqrt{2}} W_4^1 + W_4^2 + \frac{1}{\sqrt{2}} W_4^3 = -1 \\
X_2(2) &= \frac{1}{\sqrt{2}} W_4^2 + W_4^0 + \frac{1}{\sqrt{2}} W_4^6 \\
&= \frac{1}{\sqrt{2}} W_4^2 + W_4^0 + \frac{1}{\sqrt{2}} W_4^2 = -0.414 \\
X_2(3) &= \frac{1}{\sqrt{2}} W_4^3 + W_4^6 + \frac{1}{\sqrt{2}} W_4^9 \\
&= \frac{1}{\sqrt{2}} W_4^3 + W_4^2 + \frac{1}{\sqrt{2}} W_4^1 = -1
\end{aligned}
$$

Finally, applying the linearity property, we get

$$
\begin{aligned}
X(k) &= \text{DFT}\{x_1(n) + x_2(n)\} \\
&= X_1(k) + X_2(k) \\
&= (X_1(0) + X_2(0), X_1(1) + X_2(1), X_1(2) + X_2(2), X_1(3) + X_2(3)) \\
&= (\underset{\substack{\uparrow \\ k=0}}{3.414}, -j1.414, 0.586, j1.414)
\end{aligned}
$$

It may be noted that the arrow, $\uparrow$ explicitly represents the position index of $k = 0$ or $n = 0$ of a given sequence. The absence of this arrow also implicitly means that the first element in a sequence always has the index $k = 0$ or $n = 0$.

Example 3.14 Compute DFT $\{x(n)\}$ of the sequence given below using the linearity property.

$$
x(n) = \cosh an, \qquad 0 \leqslant n \leqslant N - 1
$$

Solution

Given
$$x(n) = \cosh an, \quad 0 \leqslant n \leqslant N-1$$

Then the N-point DFT of the sequence $x(n)$ is

$$\begin{aligned} X(k) = \text{DFT}\{x(n)\} &= \text{DFT}\{\cosh an\} \\ &= \text{DFT}\left\{\frac{1}{2}e^{an} + \frac{1}{2}e^{-an}\right\} \end{aligned}$$

$\cosh an = \frac{1}{2}e^{an} + \frac{1}{2}e^{-an}.$

Applying linearity property, we get

$$X(k) = \frac{1}{2}\text{DFT}\left\{e^{an}\right\} + \frac{1}{2}\text{DFT}\left\{e^{-an}\right\}, \quad 0 \leqslant n \leqslant N-1$$

We know from Example 3.5, that

$$\text{DFT}\{b^n\} = \frac{b^N - 1}{bW_N^k - 1}, \quad 0 \leqslant k \leqslant N-1$$

Hence,

$$\begin{aligned} X(k) &= \frac{1}{2}\left[\frac{e^{aN} - 1}{e^a W_N^k - 1} + \frac{e^{-aN} - 1}{e^{-a} W_N^k - 1}\right] \\ &= \frac{W_N^k\left[e^{a(N-1)} + e^{-a(N-1)} - e^{-a} - e^{-a}\right] - e^{-aN} - e^{-aN} + 2}{2\left[1 - W_N^k\left(e^a - e^{-a}\right) + W_N^k\right]} \\ &= \frac{1 - \cosh Na + W_N^k\left[\cosh(N-1)a - \cosh a\right]}{1 - 2W_N^k \cosh a + W_N^k}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

3.7.2 Circular time shift

$$\begin{aligned} \text{If} \qquad \text{DFT}\{x(n)\} &= X(k), \\ \text{then} \qquad \text{DFT}\{x((n-m))_N\} &= W_N^{mk}\, X(k), \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

Proof:

$$\underset{n = 0, 1, \cdots N-1}{x(n)} \longrightarrow \boxed{\text{DFT}\{\cdot\}} \longrightarrow \underset{k = 0, 1, \cdots N-1}{X(k)}$$

Fig. 3.5 DFT viewed as an operator.

Fig. 3.5 time and again reminds that $x(n)$ and $X(k)$ are of the same length, N. From the definition of inverse DFT, we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$\Rightarrow \qquad x(n-m) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-k(n-m)}$$

Since, the time shift is circular, we can write the above equation as

$$
\begin{aligned}
x((n-m))_N &= \frac{1}{N}\sum_{k=0}^{N-1}\left[X(k)W_N^{km}\right]W_N^{-kn} \\
\Rightarrow \quad \boxed{x((n-m))_N = \mathrm{IDFT}\left[X(k)W_N^{km}\right]} \\
\text{or} \quad \mathrm{DFT}\{x((n-m))_N\} &= W_N^{km}X(k)
\end{aligned}
$$

In terms of the transform pair, we can write the above equation as

$$x((n-m))_N \overset{\mathrm{DFT}}{\longleftrightarrow} W_N^{km}X(k)$$

Example 3.15 Find the 4-point DFT of the sequence, $x(n) = (1, -1, 1, -1)$. Also, using time shift property, find the DFT of the sequence, $y(n) = x((n-2))_4$.

Solution

Given, $N = 4$.

We know that

$$
\begin{aligned}
W_4^0 &= 1, \quad W_4^1 = -j \\
W_4^2 &= -1, \quad W_4^3 = j
\end{aligned}
$$

Hence,

$$
\begin{aligned}
X(k) &= \mathrm{DFT}\{x(n)\} \\
&= \sum_{n=0}^{3} x(n)W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\
&= 1 \times W_4^{0k} - 1 \times W_4^{k} + 1 \times W_4^{2k} - 1 \times W_4^{3k} \\
&= 1 - W_4^{k} + W_4^{2k} - W_4^{3k} \\
\Rightarrow \quad X(0) &= 1 - 1 + 1 - 1 = 0 \\
X(1) &= 1 - W_4^1 + W_4^2 - W_4^3 = 0 \\
X(2) &= 1 - W_4^2 + W_4^4 - W_4^6 \\
&= 1 - W_4^2 + W_4^0 - W_4^2 = 4 \\
X(3) &= 1 - W_4^3 + W_4^6 - W_4^9 \\
&= 1 - W_4^3 + W_4^2 - W_4^1 = 0
\end{aligned}
$$

Given,

$$y(n) = x((n-2))_4$$

Applying circular time shift property, we get

$$
\begin{aligned}
Y(k) &= W_4^{2k}\,X(k), \quad k = 0, 1, 2, 3 \\
\Rightarrow \quad Y(0) &= W_4^0\,X(0) = 0 \\
Y(1) &= W_4^2\,X(1) = 0
\end{aligned}
$$

$$Y(2) = W_4^4 X(2) = W_4^0 X(2) = 4$$
$$Y(3) = W_4^6 X(3) = W_4^2 X(3) = 0$$

Hence,
$$Y(k) = (\underset{\substack{\uparrow \\ k=0}}{0}, 0, 4, 0)$$

Example 3.16 Suppose $x(n)$ is a sequence defined on $0-7$ only as $(0, 1, 2, 3, 4, 5, 6, 7)$.

a. Illustrate $x((n-2))_8$.

b. If DFT $\{x(n)\} = X(k)$, what is the DFT$\{x((n-2))_8\}$?

Solution

a. Given
$$x(n) = (0, 1, 2, 3, 4, 5, 6, 7)$$

To generate $x((n-2))_8$, move the last 2 samples of $x(n)$ to the beginning.

That is,
$$x((n-2))_8 = (6, 7, 0, 1, 2, 3, 4, 5)$$

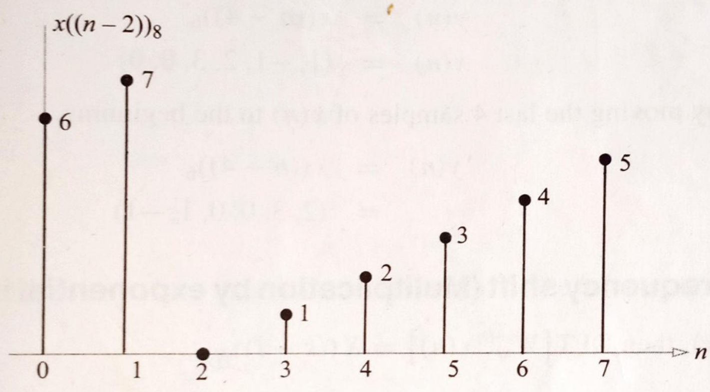


Fig. Ex.3.16 Sequence $x((n-2))_8$.

It should be noted that $x((n-2))_8$ is implicity periodic with a period $= N = 8$.

b. Let
$$y(n) = x((n-2))_8$$

Applying circular time shift property, we get

$$Y(k) = W_8^{2k} X(k)$$

Example 3.17 Let $X(k)$ donote a 6-point DFT of a length-6 real sequence, $x(n)$. The sequence is shown in Fig. Ex.3.17. Without computing the IDFT, determine the length-6 sequence, $y(n)$ whose 6-point DFT is given by, $Y(k) = W_3^{2k} X(k)$.

Fig. Ex.3.17 Sequence $x(n)$ for Example.3.17.

Solution

We may write

$$W_3^{2k} = e^{-j\frac{2\pi}{3}\times 2k}$$
$$= e^{-j\frac{2\pi}{6}\times 4k}$$

Hence,
$$W_3^{2k} = W_6^{4k}$$

It is given in the problem that

$$Y(k) = W_3^{2k} X(k)$$
$$\Rightarrow \quad Y(k) = W_6^{4k} X(k)$$

We know that
$$\text{DFT}\{x((n-m))_N\} = W_N^{mk} X(k)$$
$$\Rightarrow \quad \text{IDFT}\{W_N^{mk} X(k)\} = x((n-m))_N$$

Hence,
$$y(n) = x((n-4))_6$$

Since,
$$x(n) = (1, -1, 2, 3, 0, 0)$$

we get $x((n-4))_6$ by moving the last 4 samples of $x(n)$ to the beginning.

Hence,
$$y(n) = x((n-4))_6$$
$$= (2, 3, 0, 0, 1, -1)$$

3.7.3 Circular frequency shift (Mulitplication by exponential in time-do

If $\text{DFT}\{x(n)\} = X(k)$, then $\text{DFT}\left\{W_N^{-ln} x(n)\right\} = X((k-l))_N$.

Proof:

$$\underset{n=0,1,\cdots N-1}{x(n)} \longrightarrow \text{DFT}\{\cdot\} \longrightarrow \underset{k=0,1,\cdots N-1}{X(k)}$$

Fig. 3.6 DFT viewed as an operator.

In Fig. 3.6, DFT is viewed as an operator, that is, $\text{DFT}\{x(n)\} = X(k)$.
Using the defining equation, we have

$$X(k) = \text{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k-l) = \sum_{n=0}^{N-1} x(n) W_N^{(k-l)n}$$

Since, the shift in frequency is circular, we may write the above equation as

$$X((k-l))_N = \sum_{n=0}^{N-1} \left[x(n)W_N^{-ln}\right] W_N^{kn}$$

Hence,

$$\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$$

Example 3.18 Compute the 4-point DFT of the sequence $x(n) = (1, 0, 1, 0)$. Also, find $y(n)$ if $Y(k) = X((k-2))_4$.

☐ Solution

Given $N = 4$.

Also, $W_4^0 = 1$, $W_4^1 = -j$, $W_4^2 = -1$, $W_4^3 = j$.

The DFT of the sequence, $x(n)$ is

$$\begin{aligned}
X(k) &= \sum_{n=0}^{3} x(n)W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\
&= 1 \times W_4^{0k} + 0 + 1 \times W_4^{2k} + 0 \\
&= 1 + W_4^{2k} \\
\Rightarrow \quad X(0) &= 1 + 1 = 2 \\
X(1) &= 1 + W_4^2 = 0 \\
X(2) &= 1 + W_4^0 = 2 \\
X(3) &= 1 + W_4^2 = 0
\end{aligned}$$

Given

$$Y(k) = X((k-2))_4$$

We know that,

$$\boxed{\text{DFT}\{W_N^{-ln}x(n)\} = X((k-l))_N}$$

That is,

$$y(n) = W_N^{-ln}x(n) \overset{\text{DFT}}{\longleftrightarrow} Y(k) = X((k-l))_N$$

Hence,

$$\begin{aligned}
y(n) &= W_4^{-2n}\, x(n) \\
\Rightarrow \quad y(0) &= W_4^{-0}\, x(0) = 1 \\
y(1) &= W_4^{-2}\, x(1) = 0 \\
y(2) &= W_4^{-4}\, x(2) \\
&= W_4^{-0}\, x(2) = 1 \times 1 = 1 \\
y(3) &= W_4^{-6}\, x(3) = W_4^{-2}\, x(3) = 0
\end{aligned}$$

That is,

$$y(n) = (\underset{\underset{n=0}{\uparrow}}{1}, 0, 1, 0)$$

Example 3.19 In many signal processing applications, we often multiply an infinite length sequence by a window of length N. The time-domain expression for this window is

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right]$$

What is the DFT of the windowed sequence, $y(n) = x(n)w(n)$? Keep the answer interms of $X(k)$.

☐ Solution

Given,

$$\begin{aligned}
w(n) &= \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n - \frac{N}{2}\right)\right], \quad 0 \leqslant n \leqslant N-1 \\
\Rightarrow \quad w(n) &= \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}e^{j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)} + \frac{1}{2}e^{-j\frac{2\pi}{N}\left(n-\frac{N}{2}\right)}\right] \\
&= \frac{1}{2} + \frac{1}{4}e^{j\frac{2\pi n}{N}}e^{-j\pi} + \frac{1}{4}e^{-j\frac{2\pi n}{N}}e^{j\pi} \\
&= \frac{1}{2} + \frac{1}{4}W_N^{-n} \times (-1) + \frac{1}{4}W_N^{n} \times (-1) \\
&= \frac{1}{2} - \frac{1}{4}W_N^{-n} - \frac{1}{4}W_N^{n}
\end{aligned}$$

Given

$$\begin{aligned}
y(n) &= x(n)w(n) \\
\Rightarrow \quad y(n) &= \frac{1}{2}x(n) - \frac{1}{4}x(n)W_N^{-n} - \frac{1}{4}x(n)W_N^{n}
\end{aligned}$$

We know that, $\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$

Hence,

$$Y(k) = \frac{1}{2}X(k) - \frac{1}{4}X((k-1))_N - \frac{1}{4}X((k+1))_N$$

Example 3.20 Let $x(n)$ be a length-N sequence with N-point DFT $X(k)$. Determine the N-point DFTs of the following length-N sequences interms of $X(k)$.

a. $y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$

b. $y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$

☐ Solution

a. We know that

$$\text{DFT}\{x((n-m))_N\} = W_N^{mk}X(k)$$

Given

$$y_1(n) = \alpha x((n-m_1))_N + \beta x((n-m_2))_N$$

Hence,

$$\begin{aligned}
Y_1(k) &= \alpha\,\text{DFT}\{x((n-m_1))_N\} + \beta\text{DFT}\{x((n-m_2))_N\} \\
&= \alpha W_N^{m_1k}X(k) + \beta W_N^{m_2k}X(k)
\end{aligned}$$

b. Given

$$y_2(n) = \begin{cases} x(n), & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}\left[x(n) + (-1)^n x(n)\right]$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}\left[x(n) + e^{-j\pi n} x(n)\right]$$

$$= \frac{1}{2}\left[x(n) + e^{-j\frac{2\pi}{N}\frac{N}{2}n} x(n)\right]$$

$$\Rightarrow \quad y_2(n) = \frac{1}{2}\left[x(n) + W_N^{\frac{N}{2}n} x(n)\right]$$

$(-1)^n = e^{-j\pi n}$.

We know that $\quad \text{DFT}\{W_N^{-ln} x(n)\} = X((k-l))_N$

Hence, $$Y_2(k) = \frac{1}{2}\left[X(k) + X\left(\left(k + \frac{N}{2}\right)\right)_N\right]$$

3.7.4 Symmetry: real-valued sequences

If the sequence $x(n), n = 0, 1, \cdots N-1$ is *real*, then its DFT is such that

$$X(k) = X^*(N-k), \quad k = 0, 1, \cdots N-1$$

Proof:

We know that $$X(k) = \text{DFT}\{x(n)\}$$

$$\triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Taking conjugates on both the sides, we get

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Since $x(n)$ is real, we have $x^*(n) = x(n)$. As a consequence of this, the above equation reduces to

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}\, W_N^{Nn} \qquad (\text{since } W_N^{Nn} = 1)$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

Hence, $$X^*(k) = X(N-k)$$

The above equation conveys the message that the DFT of a real sequence possesses conjugate symmetry about the midpoint.

If N is odd, the conjugate symmetry is about $\frac{N}{2}$. The index, $k = \frac{N}{2}$ is called the *folding index*. This aspect is illustrated in Fig. 3.7.

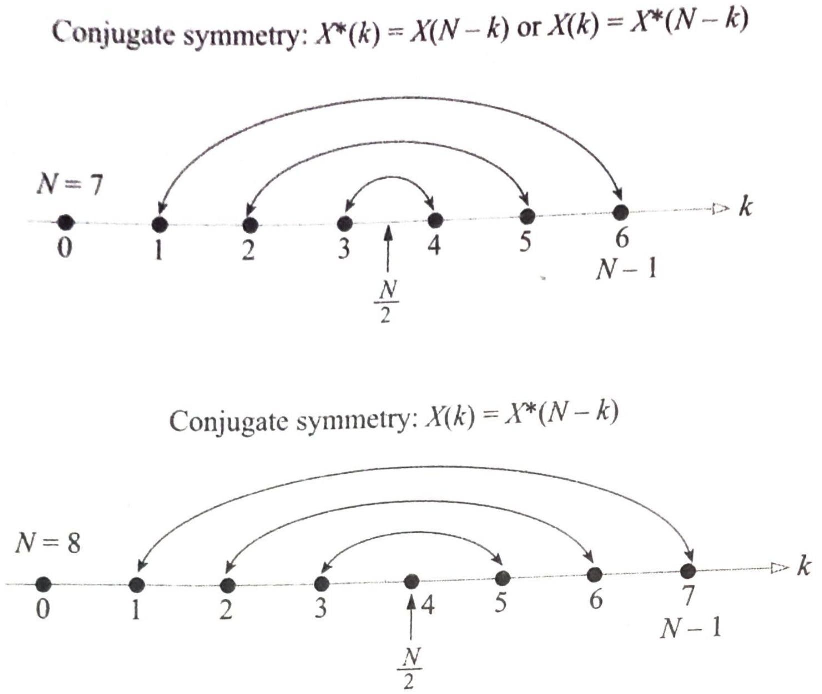


Fig. 3.7 Symmetry of $X(k)$ for $x(n)$ being real.

Conjugate symmetry implies that we need to compute only half of the DFT values to find the entire DFT sequences – a great labor saving help! A similar result holds good for IDFT also.

Example 3.21 Compute the 5-point DFT of the sequence, $x(n) = (1, 0, 1, 0, 1)$ and hence verify the symmetry property.

☐ Solution

We know that $\quad W_N = e^{-j\frac{2\pi}{N}}$

Since, $\quad N = 5, \quad W_5 = e^{-j\frac{2\pi}{5}}$

Hence,

$$W_5^0 = 1$$

$$W_5^1 = e^{-j\frac{2\pi}{5}} = 1\angle{-\tfrac{2\pi}{5}} = 0.309 - j0.951$$

$$W_5^2 = e^{-j\frac{4\pi}{5}} = 1\angle{-\tfrac{4\pi}{5}} = -0.809 - j0.587$$

$$W_5^3 = e^{-j\frac{6\pi}{5}} = 1\angle{-\tfrac{6\pi}{5}} = -0.809 + j0.587$$

$$W_5^4 = e^{-j\frac{8\pi}{5}} = 1\angle{-\tfrac{8\pi}{5}} = 0.309 + j0.951$$

By definition,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Since $N = 5$,

$$X(k) = \sum_{n=0}^{4} x(n) W_5^{kn}, \quad k = 0, 1, 2, 3, 4$$

$$\Rightarrow \quad X(k) = 1 + 0 + W_5^{2k} + 0 + W_5^{4k} = 1 + W_5^{2k} + W_5^{4k}$$

Hence,

$$X(0) = 1 + 1 + 1 = 3$$

$$X(1) = 1 + W_5^2 + W_5^4 = 0.5 + j0.364$$

$$X(2) = 1 + W_5^4 + W_5^8 = 1 + W_5^4 + W_5^3 = 0.5 + j1.538$$

$$X(3) = 1 + W_5^6 + W_5^{12} = 1 + W_5^1 + W_5^2 = 0.5 - j1.538$$

$$X(4) = 1 + W_5^8 + W_5^{16} = 1 + W_5^3 + W_5^1 = 0.5 - j0.364$$

Verification

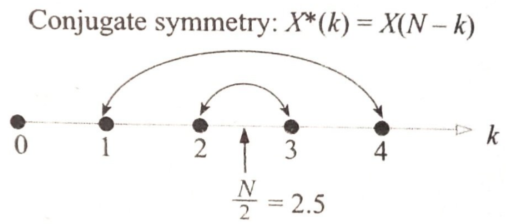


Fig. Ex.3.21 Symmetry of $X(k)$ for $x(n)$ being real and $N = 5$.

Since $x(n)$ is real,

$$X^*(k) = X(N-k) = X(5-k)$$

We find that $X^*(1) = X(4)$

and $X^*(2) = X(3)$

Hence, the symmetry property for $x(n)$ being real is verified.

Example 3.22 The first five points of the 8-point DFT of a real-valued sequence are $(0.25, 0.5 - j0.5, 0, 0.5 - j0.86, 0)$. Find the remaining three points.

☐ Solution

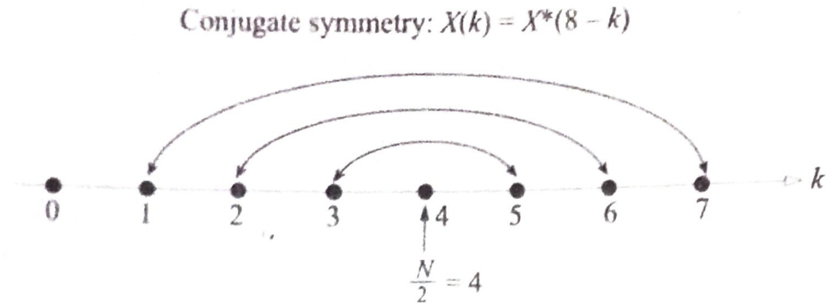


Fig. Ex.3.22 Symmetry of DFT for real signals ($N = 8$).

Since $x(n)$ is real-valued, we have

$$\begin{aligned} X(k) &= X^*(8-k), \quad k = 0, 1, \cdots 7 \end{aligned}$$

Hence,

$$\begin{aligned} X(5) &= X^*(3) = 0.5 + j0.86 \\ X(6) &= X^*(2) = 0 \\ X(7) &= X^*(1) = 0.5 + j0.5 \end{aligned}$$

Thus, the complete 8-point sequence, $X(k)$ is as tabulated below:

k	$X(k)$	k	$X(k)$
0	0.25	4	0
1	$0.5 - j0.5$	5	$0.5 + j0.86$
2	0	6	0
3	$0.5 - j0.86$	7	$0.5 + j0.5$

Example 3.23 Let $x(n)$ be a real sequence of length-N and its N-point DFT is given by $X(k)$. Show that:

a. $X(N-k) = X^*(k)$,

b. $X(0)$ is real, and

c. if N is even, $X\left(\frac{N}{2}\right)$ is real.

☐ Solution

a. The proof of this part is given in section 3.7.4. However, this being a very important property, we would like to prove this in a slightly different manner.

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{-kn} \quad (\because W_N^{Nn} = 1)$$

Since $x(n)$ is real, we can replace $x(n)$ by $x^*(n)$ in the above expression.

Thus, $$X(N-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Hence, $$X(N-k) = X^*(k)$$

b. $$X(k) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(0) = \sum_{n=0}^{N-1} x(n)$$

Since $x(n)$ is real, its summation over n is always real. Hence shown.

c. $$X(k) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

Letting $k = \frac{N}{2}$ in the above expression, we get

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n) W_N^{\frac{N}{2}n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}\frac{N}{2}n}$$

$$= \sum_{n=0}^{N-1} x(n)(-1)^n$$

Since $x(n)$ is real, the above summation gives always a real number. Hence shown.

3.7.5 Circular folding

If DFT$\{x(n)\} = X(k)$, then DFT$\{x((-n))_N\} = X((-k))_N$.

Proof:

By definition,

$$X(k) = \text{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

Substitute $m = N - n$

Then, $$X(k) = \sum_{m=N}^{1} x(N-m)W_N^{k(N-m)}$$

Because of the implicit periodicity of $x(n)$, the limits of summation can be changed as shown below.

$$X(k) = \sum_{m=0}^{N-1} x(N-m)W_N^{k(N-m)}$$

Since m is a dummy variable, it can be replaced by n.

Thus, $$X(k) = \sum_{n=0}^{N-1} x(N-n)W_N^{-kn} \qquad (\because W_N^{kN} = 1)$$

$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(N-n)W_N^{-(N-k)n}$$

$$\Rightarrow \quad X(N-k) = \sum_{n=0}^{N-1} x(N-n)W_N^{kn} \quad (\because W_N^{-Nn} = 1)$$

Hence, $$\text{DFT}\{x(N-n)\} = X(N-k)$$

or $$\text{DFT}\{x((-n))_N\} = X((-k))_N$$

Example 3.24 Compute the 4-point DFT of the sequence $x(n) = (1, 2, 1, 0)$. Also, find $Y(k)$ if

$$y(n) = x((-n))_N, 0 \leqslant k \leqslant 3$$

☐ Solution

We know that $$W_N = e^{-j\frac{2\pi}{N}}$$

Since $N = 4$, we get $$W_4 = e^{-j\frac{\pi}{2}}$$

Hence,
$$\begin{aligned} W_4^0 &= 1 \\ W_4^1 &= -j \\ W_4^2 &= -1 \\ W_4^3 &= j \end{aligned}$$

By definition,
$$\begin{aligned} \text{DFT}\{x(n)\} &= X(k) \\ &= \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3 \\ \Rightarrow \quad X(k) &= 1 + 2W_4^k + W_4^{2k} \end{aligned}$$

Hence,
$$\begin{aligned} X(0) &= 1 + 2 + 1 = 4 \\ X(1) &= 1 + 2W_4^1 + W_4^2 = -j2 \\ X(2) &= 1 + 2W_4^2 + W_4^4 = 1 + 2W_4^2 + W_4^0 = 0 \\ X(3) &= 1 + 2W_4^3 + W_4^6 = 1 + 2W_4^3 + W_4^2 = j2 \end{aligned}$$

Thus, $$X(k) = (4, -j2, 0, j2)$$

Since $x(n)$ is real, it may be noted that the symmetry property: $X(k) = X^*(N-k)$ is observed.

Given $$y(n) = x((-n))_N$$

Hence,
$$\begin{aligned} Y(k) &= X((-k))_N \\ &= X^*(k), \quad 0 \leqslant k \leqslant 3 \\ \Rightarrow \quad Y(k) &= (4, j2, 0, -j2) \end{aligned}$$

3.7.6 Symmetry: DFT of real even and real odd sequences

Let $x(n)$ be a length-N real sequence with an N-point DFT given by $X(k)$. If $x(n) = x_e(n) + x_o(n)$, where $x_e(n)$ is the even part and $x_o(n)$ is the odd part of the sequence $x(n)$, then DFT$\{x_e(n)\}$ is purely real and DFT$\{x_o(n)\}$ is purely imaginary.

***Proof*:**

We know that, $$x_e(n) \triangleq \frac{1}{2}[x(n) + x((-n))_N]$$

Hence,
$$\begin{aligned} \text{DFT}\{x_e(n)\} &= \frac{1}{2}\text{DFT}\{x(n)\} + \frac{1}{2}\text{DFT}\ \{x((-n))_N\} \\ \Rightarrow \quad \text{DFT}\{x_e(n)\} &= \frac{1}{2}X(k) + \frac{1}{2}X((-k))_N \end{aligned}$$

$$= \frac{1}{2}[X(k) + X((-k))_N]$$

$$= \frac{1}{2}\left[X(k) + X^*(k)\right]$$

Let,
$$X(k) = A + jB$$

Then,
$$X^*(k) = A - jB$$

Hence,
$$\text{DFT}\{x_e(n)\} = \frac{1}{2}[A + jB + A - jB]$$

$$\Rightarrow \quad \text{DFT}\{x_e(n)\} = A$$

Thus, we have proved that the DFT of a real even sequence is purely real.
By definition,

$$x_o(n) = \frac{1}{2}[x(n) - x((-n))_N]$$

$$\Rightarrow \quad X_o(k) = \frac{1}{2}\left[X(k) - X^*(k)\right]$$

$$\Rightarrow \quad X_o(k) = \frac{1}{2}[A + jB - A + jB]$$

Hence,
$$X_o(k) = jB$$

Thus, we find that the DFT of a real odd sequence is purely imaginary.

Example 3.25 Consider the following sequences of length-8 defined for $0 \leqslant n \leqslant 7$.

a. $x_1(n) = (2, 2, 2, 0, 0, 0, 2, 2)$

b. $x_2(n) = (2, 2, 0, 0, 0, 0, -2, -2)$

c. $x_3(n) = (0, 2, 2, 0, 0, 0, -2, -2)$

d. $x_4(n) = (0, 2, 2, 0, 0, 0, 2, 2)$

Which sequences have a real-valued 8-point DFT? Which sequences have an imaginary-valued 8-point DFT?

Solution

a. To circularly fold $x_1(n)$, enter the sequence $x_1(n)$ in the clockwise direction along the circumference of a circle with an equal spacing between successive points and read the sequence anticlockwise as shown in Fig. Ex.3.25.

Thus,
$$x_1((-n))_8 = (2, 2, 2, 0, 0, 0, 2, 2)$$

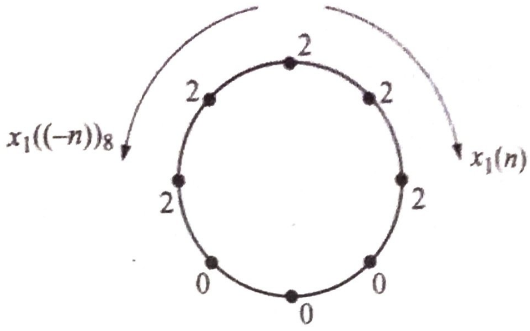


Fig. Ex.3.25 Concept of circular folding.

In the present context, we find that $x_1(n) = x_1((-n))_8$ and hence $x(n)$ is an even sequence. Also, $x(n)$ is a real sequence and hence $X(k)$ will be purely real.

b. Given
$$x_2(n) = (2, 2, 0, 0, 0, 0, -2, -2)$$
The circularly folded sequence is found to be

$$x_2((-n))_8 = (2, -2, -2, 0, 0, 0, 0, 2)$$

Since $x_2(n)$ is neither odd nor even, $\text{DFT}\{x_2(n)\} = X_2(k)$ is neither purely real nor purely imaginary.

c. Given
$$x_3(n) = (0, 2, 2, 0, 0, 0, -2, -2)$$
The circularly folded sequence is

$$x_3((-n))_8 = (0, -2, -2, 0, 0, 0, 2, 2)$$

Since $x_3(n) = -x_3((-n))_8$, the sequence $x_3(n)$ is an odd sequence. Also, $x_3(n)$ is a real sequence. Hence, $\text{DFT}\{x_3(n)\}$ is purely imaginary.

d. Given
$$x_4(n) = (0, 2, 2, 0, 0, 0, 2, 2)$$
The circularly folded sequence is

$$x_4((-n))_8 = (0, 2, 2, 0, 0, 0, 2, 2)$$

Since $x_4(n) = x_4((-n))_8$, it is an even sequence and being real, its DFT is purely real.

Example 3.26 If $x(n)$ is real and even, then show that its DFT reduces to the following form:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

☐ Solution

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) - j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Since, $x(n)$ is an even sequence and $\sin\left(\frac{2\pi kn}{N}\right)$ is an odd sequence, their product is an odd sequence. If this odd sequence is summed over one period of $x(n)$, the result is zero.

Hence,
$$X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

Example 3.27 If $x(n)$ is real and odd, then show that:

$$\begin{aligned} \text{DFT}\{x(n)\} &= X(k) \\ &= -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

☐ Solution

$$\begin{aligned} \text{DFT}\{x(n)\} &= X(k) \\ &\triangleq \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad 0 \leqslant k \leqslant N-1 \end{aligned}$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) - j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Since, $x(n)$ is an odd sequence and $\cos\left(\frac{2\pi kn}{N}\right)$ is an even sequence, their product is an odd sequence. If this odd sequence is summed over one period of $x(n)$, the result is zero.

Hence,
$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \leqslant k \leqslant N-1$$

3.7.7 DFT of a complex conjugate sequence

Let $x(n)$ be a complex sequence with

$$\text{DFT}\{x(n)\} = X(k), \quad 0 \leqslant k \leqslant N-1$$

Then,
$$\text{DFT}\{x^*(n)\} = X^*(N-k) = X^*((-k))_N$$

Proof:

By definition, the DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \tag{3.12}$$

Changing k to $-k$ gives

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn}$$

Since the folding is circular in nature, the above equation may be written as

$$X^*((-k))_N = \text{DFT}\{x^*(n)\} \tag{3.13}$$

Changing k to $N-k$ in equation (3.12) gives

$$\begin{aligned} X^*(N-k) &= \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{-Nn} W_N^{kn} \\ &= \sum_{n=0}^{N-1} x^*(n) W_N^{kn} \quad (\because W_N^{-Nn} = 1) \end{aligned}$$

$$\Rightarrow \quad \text{DFT}\{x^*(n)\} = X^*(N-k) \tag{3.14}$$

From equations (3.13) and (3.14), we can write

$$\text{DFT}\{x^*(n)\} = X^*((-k))_N = X^*(N-k)$$

Example 3.28 The 5-point DFT of a complex sequence $x(n)$ is given as

$$X(k) = (j, 1+j, 1+j2, 2+j2, 4+j)$$

Compute $Y(k)$, if $y(n) = x^*(n)$.

☐ **Solution**

$$\begin{aligned} Y(k) &= \text{DFT}\{y(n)\} \\ &= \text{DFT}\{x^*(n)\} \\ &= X^*((-k))_5 \end{aligned}$$

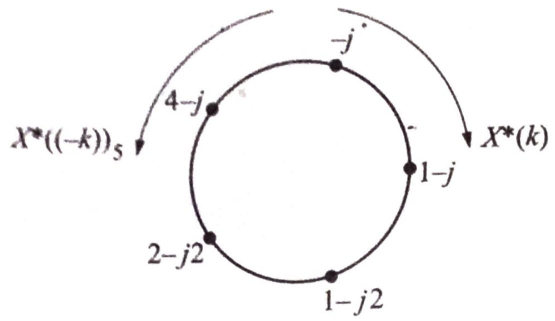


Fig. Ex.3.28 Concept of circular folding.

To find $X^*((-k))_5$, enter the sequence $X^*(k)$ on a circle clockwise and then read the sequence anticlockwise.

Thus,
$$Y(k) = (-j, 4-j, 2-j2, 1-j2, 1-j)$$

Example 3.29 Consider the sequence

$$x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

a. Find the 6-point DFT of the sequence $x(n)$.

b. Find the finite length sequence $y(n)$, which has a DFT equal to the real part of $X(k)$.

☐ Solution

a. We know that,
$$W_N = e^{-j\frac{2\pi}{N}}$$

Since $N = 6$, we get
$$W_6 = e^{-j\frac{2\pi}{6}}$$

Therefore,
$$\begin{aligned}
W_6^0 &= 1\\
W_6^1 &= \angle -\tfrac{2\pi}{6} = 0.5 - j0.866\\
W_6^2 &= 1\angle -\tfrac{4\pi}{6} = -0.5 - j0.866\\
W_6^3 &= 1\angle -\pi = -1\\
W_6^4 &= 1\angle -\tfrac{8\pi}{6} = -0.5 + j0.866\\
W_6^5 &= 1\angle -\tfrac{10\pi}{6} = 0.5 + j0.866
\end{aligned}$$

We know by definition that,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1$$

$$\Rightarrow \quad X(k) = \sum_{n=0}^{5} x(n) W_6^{kn}, \quad 0 \leqslant k \leqslant 5$$

$$= \sum_{n=0}^{5} [4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)] W_6^{kn}$$

Applying sifting property, we get

$$X(k) = 4W_6^{kn}\Big|_{n=0} + 3W_6^{kn}\Big|_{n=1} + 2W_6^{kn}\Big|_{n=2} + W_6^{kn}\Big|_{n=3}$$

$$= 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}, \quad k = 0, 1, 2, 3, 4, 5$$

Hence,

$$X(0) = 4 + 3 + 2 + 1 = 10$$

$$X(1) = 4 + 3W_6^1 + 2W_6^2 + W_6^3 = 3.5 - j4.33$$

$$X(2) = 4 + 3W_6^2 + 2W_6^4 + W_6^0 = 2.5 - j0.866$$

$$X(3) = 4 + 3W_6^3 + 2W_6^0 + W_6^3 = 2$$

$$X(4) = 4 + 3W_6^4 + 2W_6^2 + W_6^0 = 2.5 + j0.866$$

$$X(5) = 4 + 3W_6^5 + 2W_6^4 + W_6^3 = 3.5 + j4.33$$

Since $x(n)$ is a real sequence, we find that $X(k) = X^*(N-k)$ is satisfied.

b. Given

$$Y(k) = \text{Real}\{X(k)\}$$

$$\Rightarrow \quad Y(k) = \frac{1}{2}\left[X(k) + X^*(k)\right]$$

Hence,

$$y(n) = \frac{1}{2}[x(n)] + \frac{1}{2}\,\text{IDFT}\{X^*(k)\}$$

We know that

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \cdots N-1$$

$$\Rightarrow \quad X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \times 1$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \times W_N^{Nk} \quad (\because W_N^{Nk} = 1)$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{(N-n)k}$$

Put

$$N - n = m$$

Then,

$$X^*(k) = \sum_{m=N}^{1} x^*(N-m) W_N^{km}$$

Since m is a dummy variable, it can be replaced by n. Also because of implicit periodicity $x(n)$, the limit of summation can be changed as follows.

$$\begin{aligned}
X^*(k) &= \sum_{n=0}^{N-1} x^*(N-n) W_N^{kn} \\
\Rightarrow \quad X^*(k) &= \text{DFT}\{x^*(N-n)\} \\
&= \text{DFT}\{x^*((-n))_N\}
\end{aligned}$$

Hence,

$$x^*((-n))_N = \text{IDFT}\{X^*(k)\}$$

Thus, we get

$$\begin{aligned}
y(n) &= \frac{1}{2}[x(n)] + \frac{1}{2}\left[x^*((-n))_N\right] \\
\Rightarrow \quad y(n) &= \frac{1}{2}[(4, 3, 2, 1, 0, 0) + (4, 0, 0, 1, 2, 3)] \\
&= (4, 1.5, 1, 1, 1, 1.5)
\end{aligned}$$

3.7.8 Circular convolution in time

Let $x(n)$ and $h(n)$ be two sequences of length N.

Then,

$$\begin{aligned}
y(n) &= x(n) \circledast_N h(n) \\
&= \sum_{m=0}^{N-1} x((n-m))_N\, h(m), \quad 0 \leqslant n \leqslant N-1 \\
&= \sum_{m=0}^{N-1} x(m)\, h((n-m))_N
\end{aligned}$$

***Proof*:**

The above result can be proved by making use of the block diagram shown in Fig. 3.8.

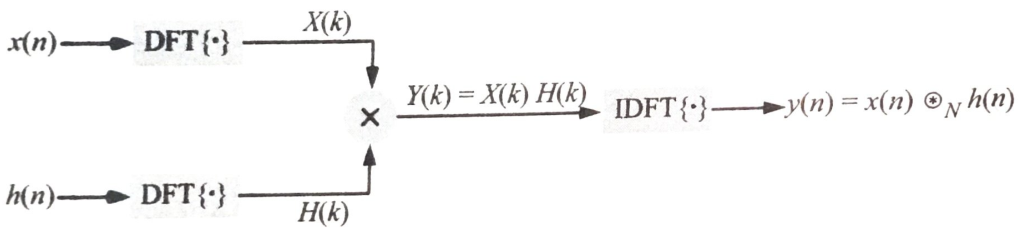


Fig. 3.8 Block diagram used for proving circular convolution.

From Fig. 3.8, we can write

$$Y(k) = X(k)\, H(k)$$

Hence,

$$\begin{aligned}
y(n) &= \text{IDFT}\{X(k)H(k)\} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k)H(k)W_N^{-kn}, \quad n = 0, 1, \cdots N-1 \qquad (3.15)
\end{aligned}$$

where

$$X(k) = \sum_{i=0}^{N-1} x(i) W_N^{ik}, \quad k = 0, 1, \cdots N-1$$

and
$$H(k) = \sum_{m=0}^{N-1} h(m) W_N^{mk}, \quad k = 0, 1, \cdots N-1$$

Substituting, the expressions for $X(k)$ and $H(k)$ in equation (3.15), we get

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{N-1} x(i) W_N^{ik} \sum_{m=0}^{N-1} h(m)\ W_N^{mk}\ W_N^{-nk}$$

Interchanging the order of summations, we get

$$y(n) = \frac{1}{N} \sum_{i=0}^{N-1} x(i) \sum_{m=0}^{N-1} h(m) \sum_{k=0}^{N-1} W_N^{(i+m-n)k}$$

The summation over k equals N when $i = n - m$ and zero for all other i.

Hence,
$$y(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(n-m)\ h(m) \times N$$

Since, the shift $(n - m)$ is circular, we may write the above equation as

$$\begin{aligned} y(n) &= \sum_{m=0}^{N-1} x((n-m))_N h(m) \\ &= h(n) \circledast_N x(n) = x(n) \circledast_N h(n) \quad (\because \text{ of commutative property, refer Example 3.35}) \end{aligned}$$

3.7.8.1 Circular convolution in time-domain is equivalent to multiplication in frequency-domain

$$\text{DFT}\{h(n) \circledast_N x(n)\} = H(k)X(k), \quad k = 0, 1, \cdots N-1$$

***Proof*:**

$$\begin{aligned} \text{DFT}\{h(n) \circledast_N x(n)\} &= \text{DFT}\left\{\sum_{l=0}^{N-1} h(l)\ x((n-l))_N\right\} \\ &= \sum_{l=0}^{N-1} h(l)\ \underbrace{W_N^{kl} X(k)}_{\text{DFT}\{x((n-l))_N\}} \\ &= H(k)X(k) \end{aligned}$$

Example 3.30 For $x_1(n)$ and $x_2(n)$ given below, compute $x_1(n) \circledast_N x_2(n)$. Take $N = 3$.

$$\begin{aligned} x_1(n) &= (1, 1, 1) \\ x_2(n) &= (1, -2, 2) \end{aligned}$$

☐ Solution

Let

$$y(n) = x_1(n) \circledast_N x_2(n), \quad N = 3$$
$$\triangleq \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$m = 0, 1 \cdots N-1$

Here, $N = 3$.

The table below demonstrates the computation of $y(n)$ using the above defining equation.

n	$x_1(m)$	$x_2((n-m))_N$	$y(n)$
0	(1, 1, 1)	(1, 2, −2)	$1 \times 1 + 1 \times 2 + 1 \times -2 = 1$
1	(1, 1, 1)	(−2, 1, 2)	$1 \times -2 + 1 \times 1 + 1 \times 2 = 1$
2	(1, 1, 1)	(2, −2, 1)	$1 \times 2 + 1 \times -2 + 1 \times 1 = 1$

Hence, $y(n) = (1, 1, 1)$.

Alternate method

In the Stockham's method of computing circular convolution, we make use of the following block diagram:

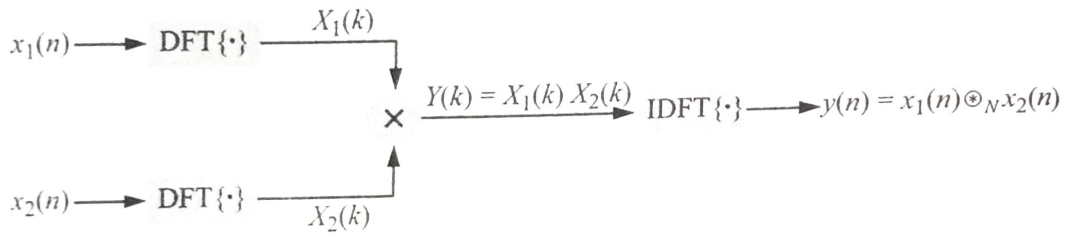


Fig. Ex.3.30 Stockham's method of performing circular convolution of two sequences of length N.

Let us first find $X_1(k)$ and $X_2(k)$.

$$X_1(k) = \sum_{n=0}^{2} x_1(n) W_3^{kn}$$
$$= 1 + 1 \times W_3^k + 1 \times W_3^{2k}, \quad 0 \leqslant k \leqslant 2$$

and

$$X_2(k) = \sum_{n=0}^{2} x_2(n) W_3^{kn}$$
$$= 1 - 2W_3^k + 2W_3^{2k}, \quad 0 \leqslant k \leqslant 2$$

Then

$$Y(k) = X_1(k) X_2(k)$$
$$= (1 + W_3^k + W_3^{2k}) \times (1 - 2W_3^k + 2W_3^{2k})$$
$$= 1 - 2W_3^k + 2W_3^{2k} + W_3^k - 2W_3^{2k} + 2W_3^{3k}$$
$$+ W_3^{2k} - 2W_3^{3k} + 2W_3^{4k}$$

$$\begin{aligned}
\text{Since,}\quad & W_3^{3k} &=& \; W_3^{0k} = 1 \\
\text{and}\quad & W_3^{4k} &=& \; W_4^{k} \\
\text{we get,}\quad & Y(k) &=& \; 1 + W_3^{k} + W_3^{2k} \\
\text{Hence,}\quad & y(n) &=& \; (1, 1, 1)
\end{aligned}$$

Example 3.31 For the sequences

$$x_1(n) = \cos\left(\frac{2\pi n}{N}\right), \quad x_2(n) = \sin\left(\frac{2\pi n}{N}\right), \quad 0 \leqslant n \leqslant N-1$$

find the N-point circular convolution $x_1(n) \circledast_N x_2(n)$.

□ **Solution**

Given

$$\begin{aligned}
x_1(n) &= \cos\left(\frac{2\pi n}{N}\right) \\
&= \frac{1}{2}e^{j\frac{2\pi n}{N}} + \frac{1}{2}e^{-j\frac{2\pi n}{N}} \\
&= \frac{1}{2}W_N^{-n} + \frac{1}{2}W_N^{n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{DFT}\{x_1(n)\} &= X_1(k) \\
&= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{-n}W_N^{kn} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{n}W_N^{kn} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k-1)n} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(k+1)n} \\
&= \frac{1}{2}N\delta(k-1) + \frac{N}{2}\delta(k+1)
\end{aligned}$$

$$\sum_{n=0}^{N-1} W_N^{(k-k_0)n} = N\delta(k-k_0)$$

Similarly,

$$\begin{aligned}
x_2(n) &= \sin\left(\frac{2\pi n}{N}\right) \\
&= \frac{1}{2j}e^{j\frac{2\pi n}{N}} - \frac{1}{2j}e^{-j\frac{2\pi n}{N}} \\
&= \frac{1}{2j}W_N^{-n} - \frac{1}{2j}W_N^{n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{DFT}\{x_2(n)\} &= X_2(k) \\
&= \frac{N}{2j}\delta(k-1) - \frac{N}{2j}\delta(k+1)
\end{aligned}$$

Let

$$y(n) = x_1(n) \circledast_N x_2(n)$$

Then,

$$\begin{aligned}
Y(k) &= X_1(k)X_2(k) \\
\Rightarrow \quad Y(k) &= \frac{N^2}{4j}[\delta(k-1) - \delta(k+1)]
\end{aligned}$$

Please note that, $\delta(k-1)\delta(k+1)=0$.

Hence,
$$y(n)=\frac{N}{2}\sin\left(\frac{2\pi n}{N}\right),\quad 0\leqslant n\leqslant N-1$$

Example 3.32 Find the 4-point circular convolution of the sequences,

$$x_1(n) = \underset{\uparrow}{(1,2,3,1)}$$

and

$$x_2(n) = \underset{\uparrow}{(4,3,2,2)}$$

using the time-domain approach and verify the result using frequency-domain approach.

☐ Solution

Time-domain approach

Let
$$y(n) = x_1(n) \circledast_N x_2(n),\quad N=4$$
$$\triangleq \sum_{m=0}^{N-1} x_1(m)x_2((n-m))_4,\quad 0\leqslant n\leqslant N-1$$

Time domain

n	$x_1(m)$	$x_2((n-m))_N$	$y(n)$
0	(1, 2, 3, 1)	(4, 2, 2, 3)	$1\times4+2\times2+3\times2+1\times3=17$
1	(1, 2, 3, 1)	(3, 4, 2, 2)	$1\times3+2\times4+3\times2+1\times2=19$
2	(1, 2, 3, 1)	(2, 3, 4, 2)	$1\times2+2\times3+3\times4+1\times2=22$
3	(1, 2, 3, 1)	(2, 2, 3, 4)	$1\times2+2\times2+3\times3+1\times4=19$

Frequency-domain approach

The circular convolution is done using frequency-domain approach by referring the block diagram shown below:

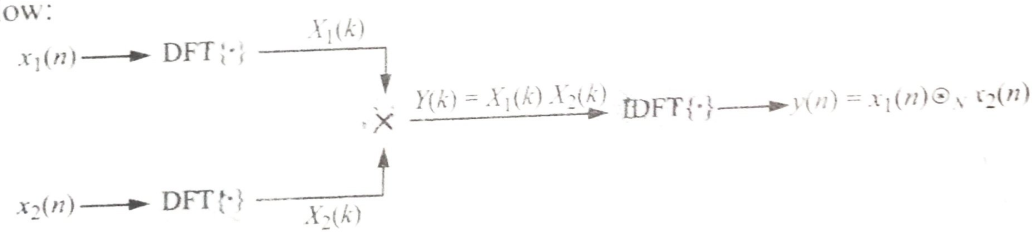


Fig. Ex.3.32 Block diagram for computing IDFT.

$$\text{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{3} x_1(n) W_4^{kn}$$
$$= 1 + 2W_4^k + 3W_4^{2k} + W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

$$\text{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{3} x_2(n) W_4^{kn}$$
$$= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

Hence,
$$\begin{aligned} Y(k) &= X_1(k)X_2(k) \\ &= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k} + 8W_4^k + 6W_4^{2k} + 4W_4^{3k} \\ &\quad +4W_4^{4k} + 12W_4^{2k} + 9W_4^{3k} + 6W_4^{4k} + 6W_4^{5k} \\ &\quad +4W_4^{3k} + 3W_4^{4k} + 2W_4^{5k} + 2W_4^{6k} \\ &= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k} + 4 + 8W_4^k + 6W_4^{2k} + 4W_4^{3k} \\ &\quad +6 + 6W_4^k + 12W_4^{2k} + 9W_4^{3k} + 3 + 2W_4^k + 2W_4^{2k} + 4W_4^{3k} \\ &\quad (\because W_4^{4k} = W_4^{0k} = 1, W_4^{5k} = W_4^k, W_4^{6k} = W_4^{2k}) \end{aligned}$$

frequency domain

Hence,
$$Y(k) = 17 + 19W_4^k + 22W_4^{2k} + 19W_4^{3k}, \quad 0 \leqslant k \leqslant 3$$

Taking IDFT, we get

$$y(n) = 17 + 19\delta(n-1) + 22\delta(n-2) + 19\delta(n-3)$$
$$\Rightarrow \quad y(n) = \{\underset{\uparrow}{17}, 19, 22, 19\}$$

Example 3.33 Let $g(n)$ and $h(n)$ be the two finite-length sequences of length-5 each. If $y_l(n)$ and $y_c(n)$ denote the linear and 5-point circular convolution of $g(n)$ and $h(n)$ respectively, express $y_c(n)$ interms of $y_l(n)$.

☐ Solution

Let $g(n) = \{g_0, g_1, g_2, g_3, g_4\}$

and $h(n) = \{h_0, h_1, h_2, h_3, h_4\}$

To find $y_c(n)$:

$$y_c(n) = g(n) \circledast_N h(n), \quad N = 5$$
$$\triangleq \sum_{m=0}^{4} g(m) h((n-m))_5, \quad 0 \leqslant n \leqslant 4$$

n	$g(m)$	$h((n-m))_5$	$y_c(n)$
0	$(g_0, g_1, g_2, g_3, g_4)$	$(h_0, h_4, h_3, h_2, h_1)$	$y_c(0) = g_0h_0 + g_1h_4 + g_2h_3 + g_3h_2 + g_4h_1$
1	$(g_0, g_1, g_2, g_3, g_4)$	$(h_1, h_0, h_4, h_3, h_2)$	$y_c(1) = g_0h_1 + g_1h_0 + g_2h_4 + g_3h_3 + g_4h_2$
2	$(g_0, g_1, g_2, g_3, g_4)$	$(h_2, h_1, h_0, h_4, h_3)$	$y_c(2) = g_0h_2 + g_1h_1 + g_2h_0 + g_3h_4 + g_4h_3$
3	$(g_0, g_1, g_2, g_3, g_4)$	$(h_3, h_2, h_1, h_0, h_4)$	$y_c(3) = g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0 + g_4h_4$
4	$(g_0, g_1, g_2, g_3, g_4)$	$(h_4, h_3, h_2, h_1, h_0)$	$y_c(4) = g_0h_4 + g_1h_3 + g_2h_2 + g_3h_1 + g_4h_0$

To find $y_l(n)$

linear convolution

$$y_l(n) = g(n)_\infty * h(n) \triangleq \sum_{m=-\infty}^{\infty} g(m)h(n-m)$$

i. $n = 0$

$$\begin{array}{lccccccccc} & & & & & \downarrow & & & & \\ g(m): & & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\ h(-m): & h_4 & h_3 & h_2 & h_1 & h_0 & & & & \\ \hline \end{array}$$

$$y_l(0) = g_0h_0$$

ii. $n = 1$

$$\begin{array}{lcccccccc} & & & & \downarrow & & & & \\ g(m): & & & & g_0 & g_1 & g_2 & g_3 & g_4 \\ h(1-m): & h_4 & h_3 & h_2 & h_1 & h_0 & & & \\ \hline \end{array}$$

$$y_l(1) = g_0h_1 + g_1h_0$$

iii. $n = 2$

$$\begin{array}{lccccccc} & & & \downarrow & & & & \\ g(m): & & & g_0 & g_1 & g_2 & g_3 & g_4 \\ h(2-m): & h_4 & h_3 & h_2 & h_1 & h_0 & & \\ \hline \end{array}$$

$$y_l(2) = g_0h_2 + g_1h_1 + g_2h_0$$

iv. $n = 3$

$$\begin{array}{lcccccc} & & \downarrow & & & & \\ g(m): & & g_0 & g_1 & g_2 & g_3 & g_4 \\ h(3-m)): & h_4 & h_3 & h_2 & h_1 & h_0 & \\ \hline \end{array}$$

$$y_l(3) = g_0h_3 + g_1h_2 + g_2h_1 + g_3h_0$$

v. $n = 4$

$$\begin{array}{lccccc} g(m): & g_0 & g_1 & g_2 & g_3 & g_4 \\ h(4-m): & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline \end{array}$$

$$y_l(4) = g_0h_4 + g_1h_3 + g_2h_2 + g_3h_1 + g_4h_0$$

vi. $n = 5$

$$\begin{array}{ll} g(m): & \overset{\downarrow}{g_0}\; g_1\; g_2\; g_3\; g_4 \\ h(5-m): & \quad\; h_4\; h_3\; h_2\; h_1\; h_0 \\ \hline & y_l(5) = g_1h_4 + g_2h_3 + g_3h_2 + g_4h_1 \end{array}$$

vii. $n = 6$

$$\begin{array}{ll} g(m): & \overset{\downarrow}{g_0}\; g_1\; g_2\; g_3\; g_4 \\ h(6-m): & \qquad\quad h_4\; h_3\; h_2\; h_1\; h_0 \\ \hline & y_l(6) = g_2h_4 + g_3h_3 + g_4h_2 \end{array}$$

viii. $n = 7$

$$\begin{array}{ll} g(m): & \overset{\downarrow}{g_0}\; g_1\; g_2\; g_3\; g_4 \\ h(7-m): & \qquad\qquad h_4\; h_3\; h_2\; h_1\; h_0 \\ \hline & y_l(7) = g_3h_4 + g_4h_3 \end{array}$$

ix. $n = 8$

$$\begin{array}{ll} g(m): & g_0\; g_1\; g_2\; g_3\; g_4 \\ h(8-m): & \qquad\qquad\quad h_4\; h_3\; h_2\; h_1\; h_0 \\ \hline & y_l(8) = g_4h_4 \end{array}$$

For $n > 8$, $y_l(n) = 0$

Also, for $n < 0$, $y_l(n) = 0$

Thus, we find that

$$\begin{aligned} y_c(0) &= y_l(0) + y_l(5) \\ y_c(1) &= y_l(1) + y_l(6) \\ y_c(2) &= y_l(2) + y_l(7) \\ y_c(3) &= y_l(3) + y_l(8) \\ y_c(4) &= y_l(4) \end{aligned}$$

Hence, circular convolution equals linear convolution plus aliasing.

Example 3.34 Fig. Ex.3.34 shows two finite-length sequences. Sketch their 6-point circular convolution.

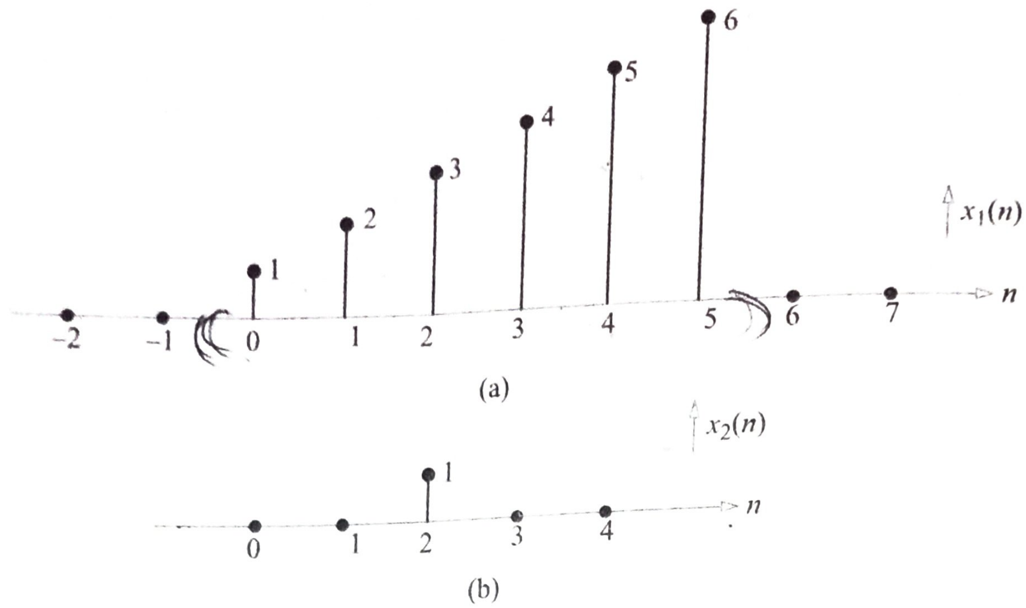


Fig. Ex.3.34 Sequences $x_1(n)$ and $x_2(n)$ for Example.3.34.

☐ Solution

Since $x_2(n)$ is a just shifted impulse, the circular convolution coincides with a circular shift of $x_1(n)$ by two points.

$$\begin{aligned} y(n) &= x_1(n) \circledast_6 x_2(n) \\ &= x_1(n) \circledast_6 \delta(n-2) \\ &= x_1((n-2))_6 = (\underset{\uparrow}{5}, 6, 1, 2, 3, 4) \end{aligned}$$

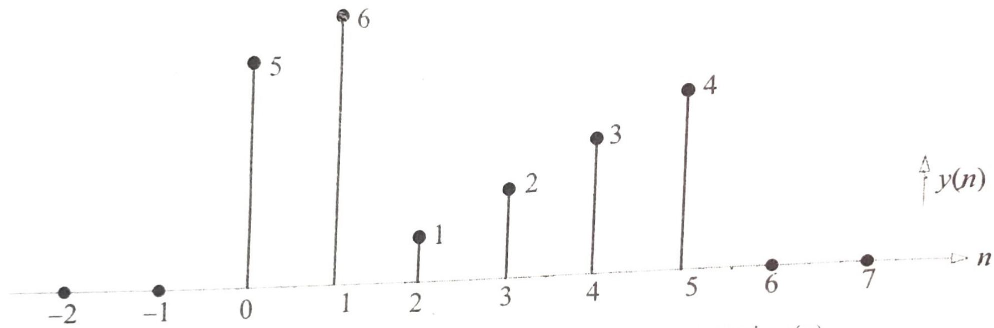


Fig. Ex.3.34(a) 6-point circular convolution of $x_1(n)$ and $x_2(n)$.

Alternate solution

$$\begin{aligned} y(n) &= x_1(n) \circledast_6 x_2(n) \\ &= \sum_{k=0}^{5} x_1(m) \quad x_2((n-m))_6 \end{aligned}$$

n	$x_1(m)$	$x_2((n-m))_6$	$y(n)$
0	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 0, 1, 0)	5
1	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 0, 0, 1)	6
2	(1, 2, 3, 4, 5, 6)	(1, 0, 0, 0, 0, 0)	1
3	(1, 2, 3, 4, 5, 6)	(0, 1, 0, 0, 0, 0)	2
4	(1, 2, 3, 4, 5, 6)	(0, 0, 1, 0, 0, 0)	3
5	(1, 2, 3, 4, 5, 6)	(0, 0, 0, 1, 0, 0)	4.

Example 3.35 Prove the commutative property of circular convolution.

That is, $$x(n) \circledast_N h(n) = h(n) \circledast_N x(n)$$

☐ **Solution**

a. $$x(n) \circledast_N h(n) \triangleq \sum_{m=0}^{N-1} x(m)h((n-m))_N$$

Let $$n - m = p$$

Then, $$x(n) \circledast_N h(n) = \sum_{p=n}^{n-N+1} x((n-p))_N\, h(p) \quad (\because n-p \text{ is a circular shift})$$

Since both the sequences are implicit periodic, the limits of summations can be changed as follows:

$$\begin{aligned} x(n) \circledast_N h(n) &= \sum_{p=0}^{N-1} h(p)x((n-p))_N \\ &= h(n) \circledast_N x(n) \end{aligned}$$

3.7.9 Multiplication in time

$$\text{DFT}\{x_1(n)x_2(n)\} = \frac{1}{N} X_1(k) \circledast_N X_2(k)$$

***Proof*:**

$$\text{DFT}\{x_1(n)x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n)x_2(n)W_N^{kn} \tag{3.16}$$

From the definition of inverse DFT, we have

$$x_2(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)W_N^{-ln} \tag{3.17}$$

Substituting equation (3.17) in equation (3.16), we get

$$\begin{aligned}
\text{DFT}\{x_1(n)x_2(n)\} &= \sum_{n=0}^{N-1} x_1(n)\frac{1}{N}\sum_{l=0}^{N-1} X_2(l)W_N^{-ln}W_N^{kn} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)\sum_{n=0}^{N-1} x_1(n)W_N^{(k-l)n} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)X_1((k-l))_N \\
&= \frac{1}{N}X_1(k) \circledast_N X_2(k)
\end{aligned}$$

Example 3.36 Find $Y(k)$, if $y(n) = x_1(n)x_2(n)$. Take $x_1(n) = (1, 1, 1, 1, 1, 1, 1, 1)$ and $x_2(n) = \cos(0.25\pi n), 0 \leqslant n \leqslant 7$.

☐ **Solution**

$$\begin{aligned}
X_1(k) \triangleq \text{DFT}\{x_1(n)\} &= \sum_{n=0}^{7} x_1(n)W_8^{kn} \\
&= \sum_{n=0}^{7} 1 \times W_8^{kn}
\end{aligned}$$

We know that $$\sum_{n=0}^{N-1} a^n = \frac{a^N - 1}{a-1};\ a \neq 1$$

Hence, $$X_1(k) = \frac{W_8^{8k}-1}{W_8^k - 1}$$ when k is not given.

$$= \begin{cases} 0, & k \neq 0 \\ 8, & k = 0 \end{cases}$$

Hence, $$X_1(k) = (8, 0, 0, 0, 0, 0, 0, 0)$$

Similarly, $$\begin{aligned}
X_2(k) = \text{DFT}\{x_2(n)\} &= \sum_{n=0}^{7} x_2(n)W_8^{kn} \\
&= \sum_{n=0}^{7} \cos\left(\frac{\pi}{4}n\right) W_8^{kn} \\
&= \sum_{n=0}^{7} \frac{1}{2}\left[e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right] W_8^{kn}
\end{aligned}$$

$$= \sum_{n=0}^{7} \frac{1}{2}\left[W_8^n + W_8^{-n}\right] W_8^{kn}$$

$$= \frac{1}{2}\sum_{n=0}^{7}\left[W_8^{(k+1)n} + W_8^{(k-1)n}\right]$$

We know that, $$\sum_{n=0}^{N-1} W_N^{(k-k_0)n} = N\delta(k-k_0)$$

Hence,
$$X_2(k) = \frac{1}{2}\left[8\delta(k+1) + 8\delta(k-1)\right]$$
$$= 4\delta(k+1) + 4\delta(k-1)$$
$$= \begin{cases} 4, & k = -1 \text{ or } -1+8 = 7 \\ 4, & k = 1 \\ 0, & \text{for all other } k \text{ in } 0 \leqslant k \leqslant 7 \end{cases}$$

Hence, $$X_2(k) = (0; 4, 0, 0, 0, 0, 0, 4)$$

Recall the property: $$\text{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{N} X_1(k) \circledast_N X_2(k)$$

Hence, $$\text{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{8}\left[\sum_{k=0}^{7} X_1(m) X_2((k-m))_8\right]$$

k	$X_1(m)$	$X_2((k-m))_8$	$\frac{1}{8}\left[\sum_{k=0}^{7} X_1(m)X_2((k-m))_8\right]$
0	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 4, 0, 0, 0, 0, 0, 4)	$\frac{1}{8}(0) = 0$
1	(8, 0, 0, 0, 0, 0, 0, 0)	(4, 0, 4, 0, 0, 0, 0, 0)	$\frac{1}{8}(8 \times 4) = 4$
2	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 4, 0, 4, 0, 0, 0, 0)	$\frac{1}{8}(0) = 0$
3	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 4, 0, 4, 0, 0, 0)	$\frac{1}{8}(0) = 0$
4	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 4, 0, 4, 0, 0)	$\frac{1}{8}(0) = 0$
5	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 4, 0, 4, 0)	$\frac{1}{8}(0) = 0$
6	(8, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 4, 0, 4)	$\frac{1}{8}(0) = 0$
7	(8, 0, 0, 0, 0, 0, 0, 0)	(4, 0, 0, 0, 0, 0, 4, 0)	$\frac{1}{8}(8 \times 4) = 4$

Hence, $$\text{DFT}\{x_1(n)x_2(n)\} = (0, 4, 0, 0, 0, 0, 0, 4)$$

Alternate method

We have,

$$\begin{aligned} X_2(k) &= (0, 4, 0, 0, 0, 0, 0, 4) \\ &= 4\delta(k-1) + 4\delta(k-7), \quad 0 \leqslant k \leqslant 7 \end{aligned}$$

Hence,

$$\begin{aligned} \text{DFT}\{x_1(n)x_2(n)\} &= \frac{1}{8}(X_1(k) \circledast_8 [4\delta(k-1) + 4\delta(k-7)]) \\ &= \frac{1}{2}[X_1((k-1))_8 + X_1((k-7))_8] \\ &= \frac{1}{2}[(0, 8, 0, 0, 0, 0, 0, 0) + (0, 0, 0, 0, 0, 0, 0, 8)] \\ &= (0, 4, 0, 0, 0, 0, 0, 4) \end{aligned}$$

3.7.10 Inner product (Parseval)

$$\sum_{n=0}^{N-1} x^*(n)y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)Y(k)$$

***Proof*:**

From the definition of IDFT, we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}$$

Taking conjugates on both the sides, we get

$$x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn}$$

Hence,

$$\begin{aligned} \sum_{n=0}^{N-1} x^*(n)y(n) &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{kn}\right)y(n) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\left(\sum_{n=0}^{N-1} y(n)W_N^{kn}\right) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)Y(k) \end{aligned}$$

Corollary:
If $y(n) = x(n)$, we get

$$\sum_{n=0}^{N-1} x^*(n)x(n) = \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)X(k)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Thus, we have proved that

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Example 3.37 Find the energy of the 4-point sequence,

$$x(n) = \sin\left(\frac{2\pi}{N}n\right), \quad 0 \leqslant n \leqslant 3$$

☐ Solution

***Method* 1:** Time-domain approach

Given,

$$x(n) = \sin\left(\frac{2\pi}{4}n\right)$$

$$= \sin\left(\frac{\pi}{2}n\right), \quad 0 \leqslant n \leqslant 3$$

$$\Rightarrow \quad x(n) = (0, 1, 0, -1)$$

Hence,

$$\boxed{E = \sum_{n=0}^{N-1} |x(n)|^2}$$

$$= \sum_{n=0}^{3} |x(n)|^2$$

$$= 1^2 + 1^2 = 2\ \text{J}$$

***Method* 2:** Frequency-domain approach

From Parseval's theorem, we have

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Hence, let us find $X(k)$.

$$X(k) = \text{DFT}\{x(n)\}$$

$$
\begin{aligned}
&= \sum_{n=0}^{3} \sin\left(\frac{\pi}{2}n\right) W_4^{kn} \\
&= \sum_{n=0}^{3} \frac{1}{2j}\left[e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right] W_4^{kn} \\
&= \frac{1}{2j}\sum_{n=0}^{3}\left[e^{j\frac{2\pi}{4}n} - e^{-j\frac{2\pi}{4}n}\right] W_4^{kn} \\
&= \frac{1}{2j}\sum_{n=0}^{3}\left[W_4^{-n} - W_4^{n}\right] W_4^{kn} \\
&= \frac{1}{2j}\left[\sum_{n=0}^{3} W_4^{(k-1)n} - \sum_{n=0}^{3} W_4^{(k+1)n}\right] \\
&= \frac{1}{2j}\left[4\delta(k-1) - 4\delta(k+1)\right] \\
&= \begin{cases} \frac{4}{2j}, & k = 1 \\ -\frac{4}{2j}, & k = -1 \text{ or } -1+4 = 3 \\ 0, & \text{for all other } k \text{ in the interval}(0, 3) \end{cases}
\end{aligned}
$$

Hence,
$$X(k) = \left(0, \frac{4}{2j}, 0, -\frac{4}{2j}\right)$$

Then,
$$
\begin{aligned}
E &= \frac{1}{4}\sum_{k=0}^{3}|X(k)|^2 = \frac{1}{4}\left[\frac{16}{4} + \frac{16}{4}\right] \\
&= \frac{1}{4} \times \frac{32}{4} = 2\text{ J}
\end{aligned}
$$

Example 3.38 Let $x(n) = (1, 2, 0, 3, -2, 4, 7, 5)$. Evaluate the following:

(a) $X(0)$, (b) $X(4)$, (c) $\sum_{k=0}^{7} X(k)$, (d) $\sum_{k=0}^{7} |X(k)|^2$

☐ **Solution**

a. By definition,

$$
\begin{aligned}
\text{DFT}\{x(n)\} &= X(k) \\
&= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leq k \leq N-1
\end{aligned}
$$

Letting $k = 0$, we get

$$
\begin{aligned}
X(0) &= \sum_{n=0}^{7} x(n) \\
&= 1 + 2 + 0 + 3 - 2 + 4 + 7 + 5 \\
&= 20
\end{aligned}
$$

b. Letting, $k = \frac{N}{2}$ in the expression for $X(k)$, we get

$$
\begin{aligned}
X\left(\frac{N}{2}\right) &= \sum_{n=0}^{N-1} x(n) W_N^{\frac{N}{2}n} \\
&= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}\frac{N}{2}n} \\
&= \sum_{n=0}^{N-1} x(n) e^{-j\pi n} \\
&= \sum_{n=0}^{N-1} x(n)(-1)^n
\end{aligned}
$$

Here, $N = 8$.

Hence $$X(4) = \sum_{n=0}^{7} x(n)(-1)^n = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$$

$$= 1 - 2 + 0 - 3 - 2 - 4 + 7 - 5 = -8$$

c. From the definition of inverse DFT, we have

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant n \leqslant N - 1$$

Letting $n = 0$ on both the sides, we get

$$x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)$$

Since $N = 8$, we get

$$\sum_{k=0}^{7} X(k) = 8\, x(0) = 8 \times 1 = 8$$

d. According to Parseval's theorem:

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\Rightarrow \quad \sum_{k=0}^{7} |X(k)|^2 = 8\sum_{n=0}^{7} |x(n)|^2$$
$$= 8(1+4+0+9+4+16+49+25)$$
$$= 864$$

Example 3.39 Let $X(k)$ be a 14-point DFT of length-14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by

$$X(0) = 12, \quad X(1) = -1 + j3, \quad X(2) = 3 + j4$$
$$X(3) = 1 - j5, \quad X(4) = -2 + j2, \quad X(5) = 6 + j3$$
$$X(6) = -2 - j3, \quad X(7) = 10$$

Find the remaining samples of $X(k)$. Also, evaluate the following:

(a) $x(0)$, (b) $x(7)$, (c) $\sum_{n=0}^{13} x(n)$, (d) $\sum_{n=0}^{13} |x(n)|^2$

☐ Solution

Since $x(n)$ is a real sequence, the following symmetry condition must be satisfied:

$$X(k) = X^*(N-k), \quad 0 \leqslant k \leqslant N-1$$

Since, $N = 14$,

$$X(k) = X^*(14-k), \quad 0 \leqslant k \leqslant 13$$

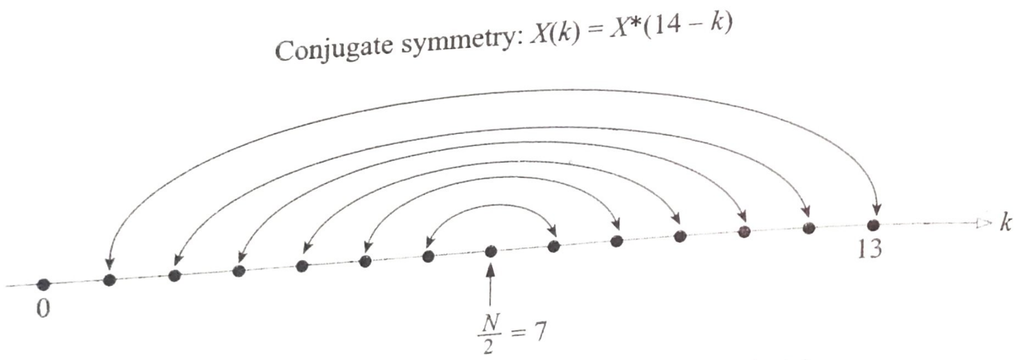


Fig. Ex.3.39 Conjugate symmetry of $X(k)$ for real $x(n)$.

Hence,

$$\begin{aligned} X(8) &= X^*(6) = -2 + j3 \\ X(9) &= X^*(5) = 6 - j3 \\ X(10) &= X^*(4) = -2 - j2 \\ X(11) &= X^*(3) = 1 + j5 \\ X(12) &= X^*(2) = 3 - j4 \\ X(13) &= X^*(1) = -1 - j3 \end{aligned}$$

The result is tabulated below:

k	$X(k)$	k	$X(k)$
0	12	7	10
1	$-1 + j3$	8	$-2 + j3$
2	$3 + j4$	9	$6 - j3$
3	$1 - j5$	10	$-2 - j2$
4	$-2 + j2$	11	$1 + j5$
5	$6 + j3$	12	$3 - j4$
6	$-2 - j3$	13	$-1 - j3$

a. From the definition of N-point inverse DFT, we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \leqslant k \leqslant N-1$$

Since $N = 14$, we get

$$x(n) = \frac{1}{14}\sum_{k=0}^{13} X(k)W_{14}^{-kn}, \quad 0 \leqslant k \leqslant 13$$

Letting $n = 0$ on both the sides of the above equation, we get

$$\begin{aligned} x(0) &= \frac{1}{14}\sum_{k=0}^{13} X(k) \\ &= \frac{1}{14}[X(0) + X(1) + \cdots + X(13)] \\ &= 2.2857 \end{aligned}$$

b. Letting $n = \frac{N}{2}$ in the expression for $x(n)$, we get

$$
\begin{aligned}
x\left(\frac{N}{2}\right) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-k\frac{N}{2}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k\frac{N}{2}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\pi k} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k)(-1)^k
\end{aligned}
$$

Since $N = 14$, we get

$$
\begin{aligned}
x(7) &= \frac{1}{14}\sum_{k=0}^{13} X(k)(-1)^k \\
&= -0.8571
\end{aligned}
$$

c. By definition:

$$
\begin{aligned}
\mathrm{DFT}\{x(n)\} = X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\
\Rightarrow \qquad X(k) &= \sum_{n=0}^{13} x(n) W_{14}^{kn}
\end{aligned}
$$

Letting $k = 0$ on both the sides of the above equation, we get

$$
X(0) = \sum_{n=0}^{13} x(n) = 12
$$

d. From Parseval's theorem:

$$
\begin{aligned}
\sum_{n=0}^{N-1} |x(n)|^2 &= \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \\
\Rightarrow \quad \sum_{n=0}^{13} |x(n)|^2 &= \frac{1}{14}\sum_{k=0}^{13} |X(k)|^2 \\
&= \frac{1}{14}\left(\begin{array}{c} 144 + 10 + 25 + 26 + 8 + 45 + 13 + 100 \\ +13 + 45 + 8 + 26 + 25 + 10 \end{array}\right) \\
&= 35.5714
\end{aligned}
$$

3.7.11 Circular correlation

Circular correlation of two length-N sequences $x(n)$ and $y(n)$ are defined as

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Then,

$$\begin{aligned} \mathrm{DFT}\{r_{xy}(l)\} &= R_{xy}(k) \\ &= X(k)Y^*(k) \end{aligned}$$

***Proof*:**

Using the definition of circular convolution, we have

$$r_{xy}(l) = x(l) \circledast_N y^*((-l))_N$$

Hence,

$$\begin{aligned} \mathrm{DFT}\{r_{xy}(l)\} &= R_{xy}(k) \\ &= X(k)Y^*(k) \end{aligned}$$

Example 3.40 For the sequences,

$$x_1(n) = \cos\left(\frac{2\pi}{N}n\right), \quad \text{and} \quad x_2(n) = \sin\left(\frac{2\pi}{N}n\right), \quad 0 \leqslant n \leqslant N-1$$

Compute the following:

a. circular correlation of $x_1(n)$ and $x_2(n)$,

b. circular autocorrelation of $x_1(n)$ and

c. circular autocorrelation of $x_2(n)$.

☐ **Solution**

Let us first find $X_1(k)$ and $X_2(k)$.

$$\begin{aligned} X_1(k) &= \sum_{n=0}^{N-1} x_1(n) W_N^{kn}, \quad 0 \leqslant k \leqslant N-1 \\ &= \sum_{n=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right] W_N^{kn} \\ &= \sum_{n=0}^{N-1} \frac{1}{2}\left[W_N^{-n} + W_N^{n}\right] W_N^{kn} \\ &= \frac{1}{2}\left[\sum_{n=0}^{N-1} W_N^{(k-1)n} + W_N^{(k+1)n}\right] \end{aligned}$$

$$= \frac{N}{2}[\delta(k-1)+\delta(k+1)]$$

Similarly,
$$X_2(k) = \frac{N}{2j}[\delta(k-1)-\delta(k+1)]$$

a. We know that,

$$\begin{aligned}
\text{DFT}\left\{r_{xy}(l)\right\} &= R_{xy}(k) = X(k)Y^*(k) \\
\Rightarrow \quad R_{x_1x_2}(k) &= X_1(k)X_2^*(k) \\
&= \frac{N}{2}[\delta(k-1)+\delta(k+1)] \times \frac{-N}{2j}[\delta(k-1)-\delta(k+1)] \\
&= \frac{-N^2}{4j}[\delta(k-1)-\delta(k+1)] \\
&= \frac{-N}{2} \times \frac{N}{2j}[\delta(k-1)-\delta(k+1)]
\end{aligned}$$

Taking IDFT, we get

$$r_{x_1x_2}(l) = \frac{-N}{2}\sin\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

b.
$$\begin{aligned}
R_{x_1x_1}(k) &= X_1(k)X_1^*(k) \\
\Rightarrow \quad R_{x_1x_1}(k) &= \frac{N}{2}[\delta(k-1)+\delta(k+1)] \times \frac{N}{2}[\delta(k-1)+\delta(k+1)] \\
&= \frac{N}{2} \times \frac{N}{2}[\delta(k-1)+\delta(k+1)]
\end{aligned}$$

Hence,
$$r_{x_1x_1}(l) = \frac{N}{2}\cos\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

c.
$$\begin{aligned}
R_{x_2x_2}(k) &= X_2(k)X_2^*(k) \\
&= \frac{N}{2j}[\delta(k-1)-\delta(k+1)] \times -\frac{N}{2j}[\delta(k-1)-\delta(k+1)] \\
&= -\frac{N}{2j} \times \frac{N}{2j}[\delta(k-1)+\delta(k+1)] \\
&= \frac{N}{2} \times \frac{N}{2}[\delta(k-1)+\delta(k+1)]
\end{aligned}$$

Hence,
$$r_{x_2x_2}(l) = \frac{N}{2}\cos\left(\frac{2\pi}{N}l\right), \quad 0 \leqslant l \leqslant N-1$$

3.8 Useful DFT Pairs

The DFT of finite sequences defined mathematically quite often results in very unwieldy expressions and explains the absence of many standard DFT pairs. However, the following DFT pairs are useful for many DFT and IDFT manipulations.

$$\delta(n) \xleftrightarrow{\text{DFT}} (1, 1, \cdots 1) \text{ (constant)}$$

$$\begin{aligned}
(1, 1, \cdots 1) \text{ (constant)} &\xleftrightarrow{\text{DFT}} (N, 0, \cdots 0) = N\delta(k) \\
a^n \text{(exponential)} &\xleftrightarrow{\text{DFT}} \frac{a^N - 1}{aW_N^k - 1} \\
\cos\left(\frac{2\pi n k_0}{N}\right) \text{ (sinusoid)} &\xleftrightarrow{\text{DFT}} \frac{N}{2}[\delta(k - k_0) + \delta(k + k_0)] \\
&= \frac{N}{2}[\delta(k - k_0) + \delta(k - (N - k_0))]
\end{aligned}$$

Example 3.41 Find the inverse DFT of the sequence given below:

$$X(k) = \begin{cases} 3, & k = 0 \\ 1, & k = 1, 2, \cdots 9 \end{cases}$$

☐ **Solution**

The given sequence, $X(k)$ may be written as

$$\begin{aligned}
X(k) &= 1 + 2\delta(k), \quad k = 0, 1, \cdots 9 \\
\Rightarrow \quad X(k) &= 1 + \frac{2}{10}[10\delta(k)]
\end{aligned}$$

We know that,

$$x_1(n) = \delta(n) \xleftrightarrow{\text{DFT}} X_1(k) = 1$$

and

$$x_2(n) = 1 \xleftrightarrow{\text{DFT}} X_2(k) = N\delta(k)$$

Hence,

$$x(n) = \delta(n) + \frac{1}{5}, \quad 0 \leqslant n \leqslant 9$$

3.9 *N*-point DFTs of Two Real Sequences Using a Single *N*-point DFT

Let $g(n)$ and $h(n)$ be two real sequences of length-N each with $G(k)$ and $H(k)$ denoting their respective N-point DFTs. These two N-point DFTs can be computed using a single N-point DFT, $X(k)$ of a length-N complex sequence, $x(n)$ defined as $x(n) = g(n) + jh(n)$.

The DFT operation is linear and hence,

$$X(k) = G(k) + jH(k)$$

Also,

$$g(n) = \frac{x(n) + x^*(n)}{2} \quad \text{and} \quad h(n) = \frac{x(n) - x^*(n)}{2j}$$

$$\Rightarrow \quad G(k) = \frac{1}{2}\left[\mathrm{DFT}\{x(n)\} + \mathrm{DFT}\{x^*(n)\}\right] = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

and

$$H(k) = \frac{1}{2j}\left[\mathrm{DFT}\{x(n)\} - \mathrm{DFT}\{x^*(n)\}\right] = \frac{1}{2j}\left[X(k) - X^*(N-k)\right]$$

Example 3.42 Find the 4-point DFTs of two sequences $g(n)$ and $h(n)$ defined below, using a single 4-point DFT.

$$g(n) = (1, 2, 0, 1)$$

and

$$h(n) = (2, 2, 1, 1)$$

☐ **Solution**

Let

$$x(n) = g(n) + jh(n), \quad 0 \leqslant n \leqslant 3$$

Hence,

$$x(n) = (1 + j2, 2 + j2, 0 + j1, 1 + j1)$$

$$\mathrm{DFT}\{x(n)\} \triangleq X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ 0+j1 \\ 1+j1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

$$\Rightarrow \quad X(k) = (4+j6, 2, -2, j2)$$

$$\Rightarrow \quad X^*(k) = (4-j6, 2, -2, -j2)$$

Hence,

$$G(k) = \frac{1}{2}\left[X(k) + X^*(4-k)\right]$$

$$= \frac{1}{2}[(4+j6, 2, -2, j2) + (4-j6, -j2, -2, 2)]$$

$$= (4, 1-j1, -2, 1+j)$$

and

$$H(k) = \frac{1}{2j}\left[X(k) - X^*(4-k)\right]$$

$$= \frac{1}{2j}[(4+j6, 2, -2, j2) - (4, -j6, -j2, -2, 2)]$$

$$= (6, 1-j, 0, 1+j)$$

Reinforcement Problems

RP-3.1 Two finite sequences

$$x(n) = [x(0), x(1), x(2), x(3)]$$

and

$$h(n) = [h(0), h(1), h(2), h(3)]$$

have DFTs given by

$$X(k) = \mathrm{DFT}\{x(n)\} = (1, j, -1, -j)$$

$$H(k) = \mathrm{DFT}\{h(n)\} = (0, 1+j, 1, 1-j)$$

Use the properties of the DFT and find the following:

a. $X_1(k) = \mathrm{DFT}\{x(3), x(0), x(1), x(2)\}$

b. $X_2(k) = \mathrm{DFT}\{h(0), -h(1), h(2), -h(3)\}$

c. $X_3(k) = \mathrm{DFT}\{y(n)\}$, where $y(n) = x(n) \circledast_4 h(n)$

d. $X_4(k) = \mathrm{DFT}\{x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)\}$

☐ Solution

a.

$$
\begin{aligned}
x_1(n) &= x((n-1))_4\\
\Rightarrow \quad X_1(k) &= W_4^k X(k), \quad 0 \leqslant k \leqslant 3\\
\text{Hence,} \quad X_1(0) &= W_4^0 X(0) = (1)(1) = 1\\
X_1(1) &= W_4^1 X(1) = (-j)(j) = 1\\
X_1(2) &= W_4^2 X(2) = (-1)(-1) = 1\\
X_1(3) &= W_4^3 X(3) = (j)(-j) = 1\\
\Rightarrow \quad X_1(k) &= (1, 1, 1, 1)
\end{aligned}
$$

b.

$$
\begin{aligned}
x_2(n) &= (h(0), -h(1), h(2), -h(3))\\
&= (-1)^n h(n) = e^{j\frac{2\pi}{4}2n}\\
&= W_4^{-2n} h(n)\\
\text{Hence,} \quad X_2(k) &= H((k-2))_4\\
&= (1, 1-j, 0, 1+j)
\end{aligned}
$$

c.

$$
\begin{aligned}
y(n) &= x(n) \circledast_4 h(n)\\
\Rightarrow \quad \text{DFT}\{y(n)\} &= X(k)H(k)\\
\Rightarrow \quad X_3(k) &= (1, j, -1, -j) \times (0, 1+j, 1, 1-j)\\
&= (0, -1+j, -1, -1-j)
\end{aligned}
$$

d. Let

$$
\begin{aligned}
x_4(n) &= (x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)), N = 8\\
X_4(k) &= \sum_{n=0}^{7} x_4(n) W_8^{kn}\\
&= \sum_{\substack{n=1\\ n,\ \text{even}}}^{6} x_4(n) W_8^{kn} + \sum_{\substack{n=1\\ n,\ \text{odd}}}^{7} x_4(n) W_8^{kn}
\end{aligned}
$$

Letting $n = 2r$ in the first sum and $n = 2r + 1$ in the second, we get

$$
\begin{aligned}
X_4(k) &= \sum_{r=0}^{3} x_4(2r) W_8^{2rk} + \sum_{r=0}^{3} x_4(2r+1) W_8^{(2r+1)k}\\
&= \sum_{r=0}^{3} x_4(2r) W_4^{rk} + W_8^k \sum_{r=0}^{3} x_4(2r+1) W_4^{rk}\\
&= X(k) + W_8^k H(k), \quad k = 0, 1, 2, 3
\end{aligned}
$$

For finding $X_4(k)$ for $k = 4, 5, 6, 7$ we use the above equation with the fact that $X(k)$ and $H(k)$ are periodic with a period equal to 4.

Hence

$$
\begin{aligned}
X_4(0) &= X(0) + W_8^0 H(0) = 1 \\
X_4(1) &= X(1) + W_8^1 H(1) = 1.414 + j \\
X_4(2) &= X(2) + W_8^2 H(2) = -1 - j \\
X_4(3) &= X(3) + W_8^3 H(3) = -1.414 - j \\
X_4(4) &= X(4) + W_8^4 H(4) \\
&= X(0) + W_8^4 H(0) = 1 \\
X_4(5) &= X(5) + W_8^5 H(5) \\
&= X(1) + W_8^5 H(1) = -1.414 + j \\
X_4(6) &= X(6) + W_8^6 H(6) \\
&= X(2) + W_8^6 H(2) = -1 + j \\
X_4(7) &= X(7) + W_8^7 H(7) \\
&= X(3) + W_8^7 H(3) = 1.414 - j
\end{aligned}
$$

Since $x_4(n)$ is a real sequence, the symmetry condition: $X(k) = X^*(8-k)$ is observed.

RP-3.2 Let $x(n)$ be a finite length sequence with $X(k) = (0, 1+j, 1, 1-j)$. Using the properties of DFT, find DFTs of the following sequences:

a. $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

b. $x_2(n) = \cos\left(\frac{\pi}{2}n\right) x(n)$

c. $x_3 = x((n-1))_4$

d. $x_4(n) = (0, 0, 1, 0) \circledast_4 x(n)$

Solution

a.

$$
\begin{aligned}
x_1(n) &= e^{j\frac{\pi}{2}n} x(n) \\
&= e^{j\frac{2\pi}{4}n} x(n) \\
&= W_4^{-n} x(n)
\end{aligned}
$$

Recall the property:

$$
\underline{\text{DFT}\left\{x(n) W_N^{-ln}\right\} = X((k-l))_N}
$$

Hence,

$$
\begin{aligned}
X_1(k) &= X((k-1))_4 \\
&= (1-j, 0, 1+j, 1)
\end{aligned}
$$

b.

$$
\begin{aligned}
x_2(n) &= \cos\left(\frac{\pi}{2}n\right) x(n) \\
&= \left[\frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n}\right] x(n)
\end{aligned}
$$

$$
\Rightarrow \quad x_2(n) = \frac{1}{2} W_4^{-n} x(n) + \frac{1}{2} W_4^{n} x(n)
$$

and

Recall the property:

$$\text{DFT}\left\{W_N^{-ln}y(n)\right\} = Y((k-l))_N$$

Hence,

$$\begin{aligned} X_2(k) &= \frac{1}{2}X((k-1))_4 + \frac{1}{2}X((k+1))_4 \\ &= \frac{1}{2}(1-j, 0, 1+j, 1) + (1+j, 1, 1-j, 0) \\ &= (1, 0.5, 1, 0.5) \end{aligned}$$

c. Given $x_3(n) = x((n-1))_4$

Recall the property:

$$\text{DFT}\{x((n-m))_N\} = W_N^{km}X(k)$$

Hence,

$$\begin{aligned} X_3(k) &= W_4^k X(k) \\ &= \left(e^{-j\frac{2\pi}{4}}\right)^k X(k) \\ &= \left(e^{-j\frac{\pi}{2}}\right)^k X(k) \\ &= (-j)^k X(k), \quad 0 \leqslant k \leqslant 3 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad X_3(0) &= X(0) = 0 \\ X_3(1) &= -jX(1) = -j(1+j) = 1-j \\ X_3(2) &= (-j)^2X(2) = -1(1) = -1 \\ X_3(3) &= (-j)^3X(3) = j(1-j) = 1+j \end{aligned}$$

Hence,

$$X_3(k) = (0, 1-j, -1, 1+j)$$

d.

$$\begin{aligned} x_4(n) &= (0, 0, 1, 0) \circledast_4 x(n) \\ &= \delta(n-2) \circledast_4 x(n) \\ &= x((n-2))_4 \end{aligned}$$

Hence,

$$X_4(k) = W_4^{2k}X(k) = (-1)^kX(k) = (0, -1-j, 1, -1+j)$$

RP- 3.3 The 4-point DFT of a real sequence $x(n)$ is $X(k) = (1, j, 1, -j)$. Using the properties of DFT, find the DFT of the following sequences:

a. $x_1(n) = (-1)^n x(n)$

b. $x_2(n) = x((n+1))_4$

c. $x_3(n) = x(4-n)$

☐ Solution

a. Given

$$\begin{aligned} x_1(n) &= (-1)^n x(n) \\ \Rightarrow \quad x_1(n) &= e^{+j\pi n}x(n) = e^{j\frac{2\pi}{4}2n}x(n) \\ &= W_4^{-2n}x(n) \end{aligned}$$

Recall the property:

$$\text{DFT}\left\{x(n)W_N^{-ln}\right\} = X((k-l))_N$$

Hence,

$$X_1(k) = X((k-2))_4$$
$$= (1, -j, 1, j)$$

b.

$$x_2(n) = x((n+1))_4$$
$$\Rightarrow \quad X_2(k) = W_4^{-k}X(k) = e^{-j\frac{2\pi}{4}(-k)}X(k)$$
$$= e^{j\frac{\pi}{2}k}X(k) = (j)^k X(k), \quad 0 \leqslant k \leqslant 3$$

Hence,

$$X_2(k) = (X(0), jX(1), -X(2), -jX(3))$$
$$= (1, -1, -1, -1)$$

c.

$$x_3(n) = x(4-n)$$
$$= x((-n))_4$$

Hence,

$$X_3(k) = X((-k))_4$$
$$= (1, -j, 1, j)$$

RP- 3.4 Let $x(n) = (1, 2, 3, 4)$ with $X(k) = (10, -2+2j, -2, -2-2j)$. Find the DFT of $x_1(n) = (1, 0, 2, 0, 3, 0, 4, 0)$ using minimum number of operations.

☐ Solution

Given

$$x_1(n) = (1, 0, 2, 0, 3, 0, 4, 0)$$

Then,

$$\text{DFT}\{x_1(n)\} = X_1(k)$$
$$= \sum_{n=0}^{7} x_1(n)\, W_8^{kn}$$
$$= \sum_{m=0}^{3} x(m)\, W_8^{2mk},$$

since $x_1(2n) = x(m)$ and $x_1(2n+1) = 0$.

Using the fact:

$$W_8^2 = e^{-j\frac{2\pi}{8}2} = W_4^1,$$

we can write,

$$X_1(k) = \sum_{m=0}^{3} x(m)\, W_4^{mk}$$
$$= X(k), \qquad k = 0, 1, 2, 3$$

Also,

$$X_1(k) = X(k+4) \quad \text{for} \quad k = 4, 5, 6, 7$$

because $X(k)$ is periodic with a period equal to $\frac{N}{2} = 4$.

Hence,

$$X_1(4) = X(8) = X(0) = 10$$
$$X_1(5) = X(9) = X(1) = -2+2j$$
$$X_1(6) = X(10) = X(2) = -2$$
$$X_1(7) = X(11) = X(3) = -2-2j$$

Thus, $\quad X_1(k) = (10, -2+2j, -2, -2-2j, 10, -2+2j, -2, -2-2j)$

Since $x_1(n)$ is a real sequence, we may notice that, $X_1(k) = X_1^*(8-k)$.

RP-3.5 The even part of a real sequence $x(n)$ is defined by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

Let $x(n)$ be a real sequence of finite length such that $x(n) = 0$ for $n < 0$ and $n \geq N$. Also, let $X(k)$ denote the N-point DFT of $x(n)$.

a. Is $\text{Re}\{X(k)\}$, the DFT of $x_e(n)$?

b. What is the inverse DFT of $\text{Re}\{X(k)\}$ in terms of $x(n)$?

☐ Solution

a. If the sequence $x(n)$ is of length N, then its even part $x_e(n)$ is of length $2N-1$. That is,

$$x_e(n) = \begin{cases} \frac{x(n)+x(-n)}{2}, & (-N+1) \leq n \leq (N-1) \\ 0, & \text{otherwise} \end{cases}$$

where $x(-n)$ implicitly represents $x((-n))_N$.

Hence, $\quad \text{DFT}\{x_e(n)\} = X_e(k)$

$$= \sum_{n=-N+1}^{N-1} \left[\frac{x(n)+x(-n)}{2}\right] W_{2N-1}^{kn}, \quad (-N+1) \leq k \leq (N-1)$$

$$= \sum_{n=-N+1}^{0} \frac{x(-n)}{2} W_{2N-1}^{kn} + \sum_{n=0}^{N-1} \frac{x(n)}{2} W_{2N-1}^{kn}$$

Let $m = -n$ in the first summation. Then, the above equation becomes

$$X_e(k) = \sum_{m=0}^{N-1} \frac{x(m)}{2} W_{2N-1}^{-km} + \sum_{n=0}^{N-1} \frac{x(n)}{2} W_{2N-1}^{kn}$$

Since m is a dummy variable, it can be replaced by n.

$$X_e(k) = \sum_{n=0}^{N-1} \frac{x(n)}{2} W_{2N-1}^{-kn} + \sum_{n=0}^{N-1} \frac{x(n)}{2} W_{2N-1}^{kn}$$

$$\Rightarrow \quad X_e(k) = \sum_{n=0}^{N-1} \frac{x(n)}{2} \left(e^{-j\frac{2\pi \times -kn}{N-1}} + e^{-j\frac{2\pi \times kn}{N-1}}\right)$$

$$= \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{2N-1}\right)$$

Also, $$\mathrm{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \leq k \leq N-1$$

$$\Rightarrow \quad \mathrm{Re}\{X(k)\} + j\,\mathrm{Im}\{X(k)\} = \sum_{n=0}^{N-1} x(n)\left(e^{-j\frac{2\pi kn}{N}}\right)$$

$$\Rightarrow \quad \mathrm{Re}\{X(k)\} + j\,\mathrm{Im}\{X(k)\} = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) + j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Comparing real and imaginary parts on both the sides, we get

$$\mathrm{Re}\{X(k)\} = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right)$$

and $$\mathrm{Im}\{X(k)\} = \sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Thus, we find that $$\mathrm{Re}\{X(k)\} \neq \mathrm{DFT}\{x_e(n)\}$$

b. $$\mathrm{Re}\{X(k)\} = \frac{X(k) + X^*(k)}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, W_N^{kn} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \qquad (\because x^*(n) = x(n))$$

Letting $n = N - m$ in the second summation, we get

$$\mathrm{Re}\{X(k)\} = \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, W_N^{kn} + \frac{1}{2}\sum_{m=N}^{1} x(N-m)\, W_N^{-k(N-m)}$$

Since $x(n)$ is implicit periodic, the limits of summation in the second summation is changed to $m = 0$ to $N-1$. Accordingly, the above equation becomes

$$\mathrm{Re}\{X(k)\} = \frac{1}{2}\sum_{n=0}^{N-1} x(n) W_N^{kn} + \frac{1}{2}\sum_{m=0}^{N-1} x(N-m) W_N^{-km} \qquad (\because W_N^{-kN} = 1)$$

Since m is a dummy variable, it can be replaced by n.

Then, $$\mathrm{Re}\{X(k)\} = \frac{1}{2}\sum_{n=0}^{N-1} [x(n) + x(N-n)]\, W_N^{kn}$$

$$\Rightarrow \quad \mathrm{Re}\{X(k)\} = \mathrm{DFT}\left\{\frac{1}{2}x(n) + \frac{1}{2}x(N-n)\right\}$$

Hence, $$\mathrm{IDFT}\{\mathrm{Re}\{X(k)\}\} = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[x(n) + x((-n))_N]$$

RP- 3.6 The two 8-point sequences $x_1(n)$ and $x_2(n)$ shown in Fig. RP.3.6 have DFTs $X_1(k)$ and $X_2(k)$ respectively. Determine the relationship between $X_1(k)$ and $X_2(k)$.

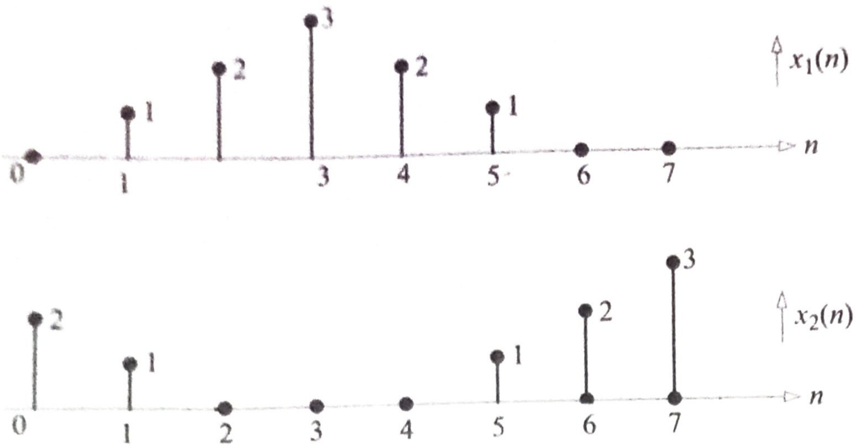


Fig. RP.3.6 Sequences $x_1(n)$ and $x_2(n)$ for RP.3.6.

Solution

The 8-point sequences $x_1(n)$ and $x_2(n)$ are related through a circular shift. Precisely,

$$x_2(n) = x_1((n-4))_8$$

Recall the property:

$$\text{DFT}\{x((n-m))_N\} = W_N^{km} X(k)$$

Hence,

$$\begin{aligned} X_2(k) &= \text{DFT}\{x_1((n-4))_8\} \\ &= W_8^{4k}\, X_1(k) \\ &= e^{-j\pi k}\, X_1(k) \\ \Rightarrow \quad X_2(k) &= (-1)^k\, X_1(k) \end{aligned}$$

RP- 3.7 If we perform DFT of an N-length sequence four times, what will be the resulting sequence?

Solution

Let

$$X(k) = \text{F}\{x(n)\}$$

Then,

$$\text{F}^{-1} = \frac{1}{N}\text{F} \Rightarrow \text{F} = N\text{F}^{-1}$$

Also, let $y(n)$ be a sequence obtained by applying the DFT operation four times to $x(n)$.

That is,

$$\begin{aligned} y(n) &= \text{F}\{\text{F}\{\text{F}\{\text{F}\{x(n)\}\}\}\} \\ \Rightarrow \quad y(n) &= N\text{F}^{-1}\{\text{F}\{N\text{F}^{-1}\{\text{F}\{x(n)\}\}\}\} \\ &= N^2\, x(n) \end{aligned}$$

RP- 3.8 The even samples of the 11-point DFT of a length-11 real sequence are given by

$$X(0) = 2, \quad X(2) = -1 - j3, \quad X(4) = 1 + j4,$$

$$X(6) = 9 + j3, \quad X(8) = 5, \quad X(10) = 2 + j2$$

Determine the missing odd samples of the DFT.

☐ Solution

Since $x(n)$ is a real sequence, the following symmetry condition must be satisfied:

$$\begin{aligned}
X(k) &= X^*(N-k) \\
\Rightarrow \quad X(k) &= X^*(11-k)
\end{aligned}$$

Hence,

$$\begin{aligned}
X(1) &= X^*(10) = 2 - j2 \\
X(3) &= X^*(8) = 5 \\
X(5) &= X^*(6) = 9 - j3 \\
X(7) &= X^*(4) = 1 - j4 \\
X(9) &= X^*(2) = -1 + j3 \\
X(11) &= X^*(0) = 2
\end{aligned}$$

RP- 3.9 Given the finite length sequence:

$$x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3).$$

Find the following:

a. 5-point DFT $X(k)$.

b. 5-point inverse DFT of $Y(k) = X^2(k)$ for $n = 0, 1, 2, 3, 4$.

☐ Solution

a. We have the finite-length sequence:

$$x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3)$$

Hence,

$$\begin{aligned}
\text{DFT}\{x(n)\} &= X(k) \\
&= \sum_{n=0}^{4} x(n)\, W_5^{kn} \\
&= \sum_{n=0}^{4} [2\delta(n) + \delta(n-1) + \delta(n-3)] W_5^{kn} \\
\Rightarrow \quad X(k) &= 2W_5^{kn}\Big|_{n=0} + W_5^{kn}\Big|_{n=1} + W_5^{kn}\Big|_{n=3} \quad \text{(sifting property)} \\
&= 2 + W_5^{k} + W_5^{3k}, \qquad 0 \le k \le 4
\end{aligned}$$

where

$$W_5^k = e^{-j\frac{2\pi}{5}k}$$

b.

$$\begin{aligned} Y(k) &= X^2(k) \\ &= 4 + 2W_5^k + 2W_5^{3k} + 2W_5^k + W_5^{2k} + W_5^{4k} \\ &\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \qquad 0 \leqslant k \leqslant 4 \end{aligned}$$

Using the periodicity of W_5, we have

$$W_5^{6k} = W_5^k$$

Hence,

$$Y(k) = 4 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + 2W_5^{4k}, \qquad 0 \leqslant k \leqslant 4$$

Taking IDFT on both the sides, we get

$$y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4), \qquad 0 \leqslant n \leqslant 4$$

RP- 3.10 Let

$$x(n) = \text{IDFT}\{X(k)\}, \text{ for } 0 \leqslant n,\ k \leqslant N-1$$

Find the relationship between $x(n)$ and the following IDFTs:

(a) $\text{IDFT}\{X^*(k)\}$, (b) $\text{IDFT}\{X((-k))_N\}$,

(c) $\text{IDFT}\{\text{Re}\{X(k)\}\}$, (d) $\text{IDFT}\{\text{Im}\{X(k)\}\}$ and

(e) apply all the above properties to the sequence $x(n) = \text{IDFT}\{1, -j4, j, -j2\}$.

Solution

a.

$$\begin{aligned} \text{IDFT}\{X^*(k)\} &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{-kn} \\ &= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)\, W_N^{kn}\right]^* \\ &= x^*((-n))_N \end{aligned}$$

b.

$$\begin{aligned} \text{IDFT}\{X((-k))_N\} &= \frac{1}{N}\sum_{k=0}^{N-1} X((-k))_N\, W_N^{-kn} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(N-k) W_N^{-kn} \end{aligned}$$

Letting $N - k = m$, we get

$$\text{IDFT}\{X((-k))_N\} = \frac{1}{N}\sum_{m=N}^{1} X(m)\, W_N^{-(N-m)n}$$

Since $X(m)$ is implicit periodic with a period N, the limits of summation can be changed as shown below.

$$\text{IDFT}\{X((-k))_N\} = \frac{1}{N}\sum_{m=0}^{N-1} X(m)\, W_N^{mn} \quad (\because\ W_N^{-Nn} = 1)$$

Since m is a dummy variable, it can be replaced by k. Thus, we get

$$\begin{aligned} \text{IDFT}\{X((-k))_N\} &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{kn} \\ &= x((-n))_N \end{aligned}$$

c.
$$\begin{aligned} \text{IDFT}\{\text{Re}\{X(k)\}\} &= \frac{1}{2}\,\text{IDFT}\{X(k)\} + \frac{1}{2}\,\text{IDFT}\{X^*(k)\} \\ &= \frac{1}{2}x(n) + \frac{1}{2}x^*((-n))_N \end{aligned}$$

d.
$$\begin{aligned} \text{IDFT}\{\text{Im}\{X(k)\}\} &= \frac{1}{2j}\,\text{IDFT}\{X(k)\} - \frac{1}{2j}\,\text{IDFT}\{X^*(k)\} \\ &= \frac{1}{2j}x(n) - \frac{1}{2j}x^*((-n))_N \end{aligned}$$

e. (i)
$$\text{IDFT}\{X^*(k)\} = x^*((-n))_N$$

Since, $x(n) = (1, -j4, j, -j2)$

we have $x^*(n) = (1, j4, -j, j2)$

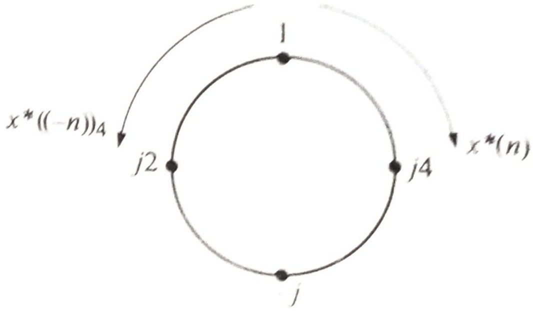


Fig. RP.3.10 Concept of circular folding.

To find $x^*((-n))_4$, the sequence $x^*(n)$ is entered clockwise, equally spaced locations on the circumference of a circle as shown in Fig. RP.3.10 and then read anticlockwise.

Hence,
$$\text{IDFT}\{X^*(k)\} = (1, j2, -j, j4)$$

(ii)
$$\begin{aligned} \text{IDFT}\{X((-k))_N\} &= x((-n))_N \\ &= (1, -j2, j, -j4) \end{aligned}$$

(iii)
$$\begin{aligned} \text{IDFT}\{\text{Re}\{X(k)\}\} &= \frac{1}{2}\left[x(n) + x^*((-n))_N\right] \\ &= \frac{1}{2}[(1, -j4, j, -j2) + (1, j2, -j, j4)] \\ &= \frac{1}{2}(2, -j2, 0, j2) \\ &= (1, -j, 0, j) \end{aligned}$$

(iv)

$$\begin{aligned}
\text{IDFT}\{\text{Im}\{x(n)\}\} &= \frac{1}{2j}\left[x(n) - x^*((-n))_N\right] \\
&= \frac{1}{2j}[(1, -j4, j, -j2) - (1, j2, -j, j4)] \\
&= \frac{1}{2j}[(0, -j6, j2, -j6)] \\
&= (0, -3, 1, -3).
\end{aligned}$$

RP- 3.11 Let $x(n) = (2, 3, 2, 1)$ and its DFT, $X(k) = (8, -j2, 0, j2)$. Find the DFT of the 12-point signal described by

$$x_1(n) = (x(n), x(n), x(n))$$

and the 12-point zero-interpolated signal $h(n) = x\left(\frac{n}{3}\right)$.

Solution

Fact: Replication in one domain corresponds to zero-interpolation in the other.

a. If a signal is replicated by P, its DFT is interpolated and scaled by P. That is, if $\text{DFT}\{x(n)\} = X(k)$, then

$$\text{DFT}\,\underbrace{\{x(n),\ x(n),\ \cdots x(n)\}}_{P\text{-fold replication}} = P\,X\left(\frac{k}{P}\right)$$

Hence,

$$\begin{aligned}
X_1(k) &= \text{DFT}\{x(n), x(n), x(n)\} \\
&= 3X\left(\frac{k}{3}\right) \\
&= (24, 0, 0, -j6, 0, 0, 0, 0, 0, j6, 0, 0)
\end{aligned}$$

b. If a signal is zero-interpolated by P, its DFT shows P-fold replication. That is,

$$\begin{aligned}
\text{DFT}\{x(n)\} &= X(k), \quad \text{then} \quad \text{DFT}\left\{x\left(\frac{n}{P}\right)\right\} \\
&= \underbrace{[X(k), X(k), \ \cdots X(k)]}_{P\text{-fold replication}}
\end{aligned}$$

Thus,

$$\begin{aligned}
H(k) &= (X(k), X(k), X(k)) \\
&= (8, -j2, 0, j2, 8, -j2, 0, j2, 8 - j2, 0, j2)
\end{aligned}$$

RP- 3.12 For DFT pair shown, compute the values of the boxed quantities using appropriate properties.

$$\left(\boxed{x_0}, 3, -4, 0, 2\right) \stackrel{\text{DFT}}{\longleftrightarrow} \left(5, \boxed{X_1}, -1.28 - j3.49, \boxed{X_3}, 8.78 - j1.4\right)$$

Solution

From the given time-domain and frequency-domain sequences, we find that the transform length, $N = 5$. Since $x(n)$ is a real sequence, the following symmetry property must be satisfied.

$$X(k) = X^*(N-k)$$

Hence,
$$X(1) = X_1 = X^*(4) = 8.78 + j1.4$$
$$X(3) = X_3 = X^*(2) = -1.28 + j3.49$$

Also,
$$x(n) = \frac{1}{5}\sum_{k=0}^{4} X(k)\, W_5^{-kn}, \qquad 0 \leqslant n \leqslant 4$$

Letting $n = 0$, we get

$$\begin{aligned} x(0) = x_0 &= \frac{1}{5}\sum_{k=0}^{4} X(k) \\ &= \frac{1}{5}[X(0) + X(1) + X(2) + X(3) + X(4)] \\ &= 4 \end{aligned}$$

Alternate method:

We know that
$$X(k) = \sum_{n=0}^{4} x(n) W_5^{kn}$$

Letting $k = 0$, we get

$$\begin{aligned} X(0) &= x(0) + x(1) + x(2) + x(3) + x(4) \\ \Rightarrow \quad 5 &= x_0 + 3 - 4 + 0 + 2 \\ \Rightarrow \quad x_0 &= 4 \end{aligned}$$

3.10 Digital Linear Filtering Using DFT

In an LTI system, the system response $y(n)$ is obtained by convoluting the input, $x(n)$ with the impulse response, $h(n)$ of the system.

That is,
$$y(n) = x(n) * h(n)$$

Taking DTFT on both the sides of the above equation, we get
$$Y(\omega) = X(\omega) H(\omega)$$

Since $X(\omega)$, $H(\omega)$ and $Y(\omega)$ are all functions of the continuous variable ω, the above multiplicative operation cannot be done on a digital computer. On the otherhand, if the linear convolution in the above equation is replaced by the circular convolution, then the DTFTs would be replaced by DFTs and the advantages of the digital world can be exploited. Hence, it is worth examining the condition for which circular convolution would give the same result as that of linear convolution.

In general, circular convolution yields a different result compared to linear convolution. This is due to the fact that a circularly shifted sequence is different from a linearly shifted sequence. Let us consider two finite length sequences of lengths N and M with $N > M$. The result of linear convolution is of length $M + N - 1$, whereas that of circular convolution is of length N. Our attempt is somehow to make the result of circular shift same as that of linear shift. If that happens, the two convolutions will give identical results. A circular shift of an N - point sequence is obtained by first linearly shifting the sequence and then rotating those elements that leave the original N-point domain ($0 \leqslant n \leqslant N-1$) into the original domain. These rotated elements are the ones that are in different positions in a linearly translated sequence. This gives us the clue that, if we insert an appropriate number of zeros at the end of the original sequence and then compute circular convolution, then we would rotate only zeros around and hence circular and linear translations will give identical results. The number of zeros to be padded must be such that the length of the sequences is increased to $N + M - 1$.

For example, consider the sequence, $x(n) = (4, 5, 6, 7), 0 \leqslant n \leqslant 3$. The linearly translated sequence $x(n-3) = (0, 0, 0, 4, 5, 6, 7)$. The circularly shifted sequence $x((n-3))_4 = (5, 6, 7, 4)$. Since, we have a shift of 3, zero-pad the original sequence as $x(n) = (4, 5, 6, 7, 0, 0, 0)$. Now, if we apply circular translation to right by a time index equal to 3, we get $x((n-3))_4 = (0, 0, 0, 4, 5, 6, 7)$. Thus, we find that $x(n-3) = x((n-3))_4$. We can now, formally arrive at the following conclusion.

Let us consider two sequences $x_1(n)$ and $x_2(n)$ of lengths N and M respectivley. Zero-pad each sequence so that they have a common length, $L = N + M - 1$. Let $X(k)$ and $H(k)$ represent the L-point DFTs of these sequences. Then, the L-point inverse DFT of $X(k)H(k)$ is equal to the linear convolution of $x(n)$ and $h(n)$.

Example 3.43 Find the circular (7-point) and linear convolution of the sequences,

$$x(n) = (1, 2, 7, -2, 3, -1, 5)$$

and

$$h(n) = (-1, 3, 5, -3, 1)$$

Solution

Length of $x(n)$ is $N = 7$.
Length of $h(n)$ is $M = 5$.

Linear convolution

$$x(n) = \delta(n) + 2\delta(n-1) + 7\delta(n-2) - 2\delta(n-3) + 3\delta(n-4) - \delta(n-5) + 5\delta(n-6)$$

and

$$h(n) = -\delta(n) + 3\delta(n-1) + 5\delta(n-2) - 3\delta(n-3) + \delta(n-4)$$

Let

$$y_l(n) = x(n) * h(n)$$

Then,

$$y_l(n) = [\delta(n) + 2\delta(n-1) + 7\delta(n-2) - 2\delta(n-3) + 3\delta(n-4) - \delta(n-5) + 5\delta(n-6)] * [-\delta(n) + 3\delta(n-1) + 5\delta(n-2) - 3\delta(n-3) + \delta(n-4)]$$

Recall the property:

$$\delta(n-\alpha) * \delta(n-\beta) = \delta[n-(\alpha+\beta)]$$

Hence,

$$\begin{aligned} y_l(n) &= -\delta(n) + \delta(n-1) + 4\delta(n-2) + 30\delta(n-3) + 21\delta(n-4) - 19\delta(n-5) \\ &\quad +20\delta(n-6) - \delta(n-7) + 31\delta(n-8) - 16\delta(n-9) + 5\delta(n-10) \end{aligned}$$

$$\Rightarrow \quad y_l(n) = (-1, 1, 4, 30, 21, -19, 20, -1, 31, -16, 5)$$

Circular convolution

Here $\quad x(n) = (1, 2, 7, -2, 3, -1, 5)$

and $\quad h(n) = (-1, 3, 5, -3, 1, 0, 0)$

The last two values of $h(n)$ are zero-padded and we then compute 7-point circular convolution of $x(n)$ and $h(n)$.

Let $\quad y_c(n) = x(n) \circledast_7 h(n)$

$$\Rightarrow \quad y_c(n) = \sum_{m=0}^{6} x(m)h((n-m))_7$$

n	$x(m)$	$h((n-m))_7$	$y_c(n)$
0	$(1, 2, 7, -2, 3, -1, 5)$	$(-1, 0, 0, 1, -3, 5, 3)$	-2
1	$(1, 2, 7, -2, 3, -1, 5)$	$(3, -1, 0, 0, 1, -3, 5)$	32
2	$(1, 2, 7, -2, 3, -1, 5)$	$(5, 3, -1, 0, 0, 1, -3)$	-12
3	$(1, 2, 7, -2, 3, -1, 5)$	$(-3, 5, 3, -1, 0, 0, 1)$	35
4	$(1, 2, 7, -2, 3, -1, 5)$	$(1, -3, 5, 3, -1, 0, 0)$	21
5	$(1, 2, 7, -2, 3, -1, 5)$	$(0, 1, -3, 5, 3, -1, 0)$	-19
6	$(1, 2, 7, -2, 3, -1, 5)$	$(0, 0, 1, -3, 5, 3, -1)$	20

Example 3.44 Given the following finite length sequence:

$$x(n) = (2, 1, 0, 1)$$
$$\underset{n=0}{\uparrow}$$

a. Find 5-point DFT $X(k)$.

b. We compute a 5-point IDFT of $Y(k) = X^2(k)$ to obtain a sequence $y(n)$.

(i) Determine the sequence $y(n)$ for $n = 0, 1, 2, 3, 4$.

(ii) If N-point DFTs are used in the two-step procedure, how should we choose N, so that $y(n) = x(n) * x(n)$ for $0 \leqslant n \leqslant N-1$?

Solution

a. Let

$$x(n) = (2, 1, 0, 1, 0)$$
$$\underset{n=0}{\uparrow}$$

The last sample in $x(n)$ is taken as zero to increase the length of the sequence to 5.

Hence, $$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{4} x(n)W_5^{kn}, \quad 0 \leqslant k \leqslant 4$$

$$\Rightarrow \quad X(k) = 2 + W_5^k + W_5^{3k}, \qquad 0 \leqslant k \leqslant 4$$

We know that $$W_N = e^{-j\frac{2\pi}{N}}$$

$$\Rightarrow \quad W_5 = e^{-j\frac{2\pi}{5}}$$

Hence, $$W_5^0 = 1, \quad W_5^1 = e^{-j\frac{2\pi}{5}} = \cos\left(\frac{2\pi}{5}\right) - j\sin\left(\frac{2\pi}{5}\right)$$
$$= 0.309 - j0.951$$

Similarly, $$W_5^2 = e^{-j\frac{4\pi}{5}} = -0.809 - j0.588$$
$$W_5^3 = e^{-j\frac{6\pi}{5}} = -0.809 + j0.588$$
$$W_5^4 = e^{-j\frac{8\pi}{5}} = 0.309 + j0.951$$

Let us now evaluate $X(k)$ for $0 \leqslant k \leqslant 4$.

$$\begin{aligned}
X(0) &= 2 + 1 + 1 = 4 \\
X(1) &= 2 + W_5^1 + W_5^3 = 1.5 - j0.363 \\
X(2) &= 2 + W_5^2 + W_5^6 \\
&= 2 + W_5^2 + W_5^1 = 1.5 - j1.539 \\
X(3) &= 2 + W_5^3 + W_5^9 \\
&= 2 + W_5^3 + W_5^4 = 1.5 + j1.539 \\
X(4) &= 2 + W_5^4 + W_5^{12} \\
&= 2 + W_5^4 + W_5^2 = 1.5 + j0.363
\end{aligned}$$

Since $x(n)$ is real, it may be noted that the symmetry property: $X(k) = X^*(5-k)$ is observed

b.
$$\begin{aligned}
Y(k) &= X^2(k) \\
&= 4 + 2W_5^k + 2W_5^{3k} + 2W_5^k + W_5^{2k} + W_5^{4k} \\
&\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \qquad 0 \leqslant k \leqslant 4
\end{aligned}$$

Recall the fact, $$W_5^{6k} = W_5^k$$

Hence, $$Y(k) = 4 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + 2W_5^{4k}$$

(i) We know that

$$\begin{aligned} \text{DFT}\{\delta(n-n_0)\} &= W_N^{kn_0}, \quad 0 \leqslant n_0 \leqslant N-1 \\ \Rightarrow \qquad \text{IDFT}\left\{W_N^{kn_0}\right\} &= \delta(n-n_0) \end{aligned}$$

Hence, IDFT of $Y(k)$ yields

$$y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4), \qquad 0 \leqslant n \leqslant 4$$

or equivalently, $$y(n) = (4, 5, 1, 4, 2)$$

(ii) We know that

$$y(n) = \text{IDFT}\{Y(k)\} = x(n) \circledast_N x(n)$$

The DFT and IDFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \geqslant 2M - 1$, where M is length of $x(n)$. In the present context, $M = 4$. Hence, $N \geqslant 7$.

Example 3.45 For the sequences, $x_1(n)$ and $x_2(n)$ shown in Fig. Ex.3.45:

a. Compute the circular convolution, $x_1(n) \circledast_N x_2(n)$ for $N = 4, 5, 6, 7, 8$.

b. Compute the linear convolution, $x_1(n) * x_2(n)$.

c. What value of N is necessary, so that linear and circular convolutions are the same on the N-point interval?

d. Determine the non-zero lengths of $x_1(n)$ and $x_2(n)$ and tell how could have found the results of part (c) without performing the calculations.

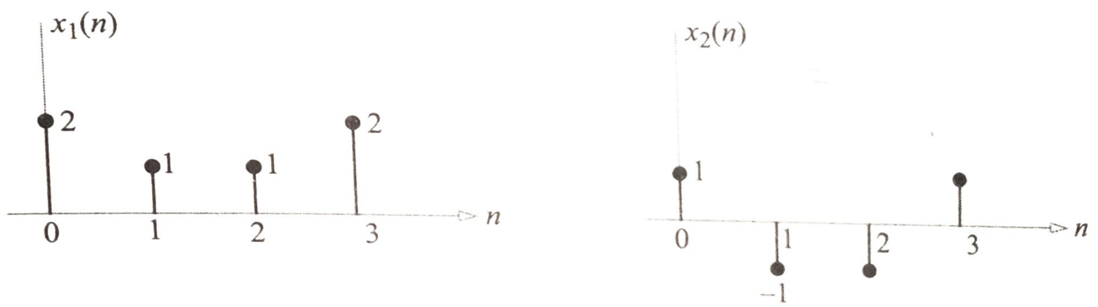


Fig. Ex.3.45 Sequences $x_1(n)$ and $x_2(n)$ for Example.3.45.

□ **Solution**

$$x_1(n) = (2, 1, 1, 2), \quad x_2(n) = (1, -1, -1, 1)$$

a. Let

$$y_c(n) = x_1(n) \circledast_N x_2(n)$$

$$\Rightarrow \quad y_c(n) = \sum_{m=0}^{N-1} x_1(m)x_2((n-m))_N$$

$N = 4$:

$$y_c(n) = \sum_{m=0}^{3} x_1(m)x_2((n-m))_4$$

n	$x_1(m)$	$x_2((n-m))_4$	$y_c(n)$
0	(2, 1, 1, 2)	(1, 1, −1, −1)	$2+1-1-2=0$
1	(2, 1, 1, 2)	(−1, 1, 1, −1)	$-2+1+1-2=-2$
2	(2, 1, 1, 2)	(−1, −1, 1, 1)	$-2-1+1+2=0$
3	(2, 1, 1, 2)	(1, −1, −1, 1)	$2-1-1+2=2$

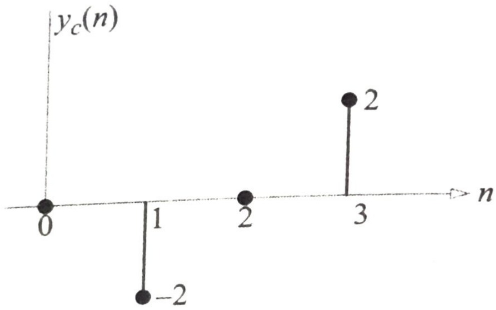


Fig. Ex.3.45(a) 4-point circular convolution of $x_1(n)$ and $x_2(n)$.

or equivalently $y_c(n) = (0, -2, 0, 2)$

$N = 5$:

$$x_1(n) = (2, 1, 1, 2, 0)$$
$$x_2(n) = (1, -1, -1, 1, 0)$$

$$y_c(n) = x_1(n) \circledast_5 x_2(n) = \sum_{n=0}^{4} x_1(m)\, x_2((n-m))_5$$

n	$x_1(m)$	$x_2((n-m))_5$	$y_c(n)$
0	(2, 1, 1, 2, 0)	(1, 0, 1, −1, −1)	$2+0+1-2+0=1$
1	(2, 1, 1, 2, 0)	(−1, 1, 0, 1, −1)	$-2+1+0+2+0=1$
2	(2, 1, 1, 2, 0)	(−1, −1, 1, 0, 1)	$-2-1+1+0+0=-2$
3	(2, 1, 1, 2, 0)	(1, −1, −1, 1, 0)	$2-1-1+2+0=2$
4	(2, 1, 1, 2, 0)	(0, 1, −1, −1, 1)	$0+1-1-2+0=-2$

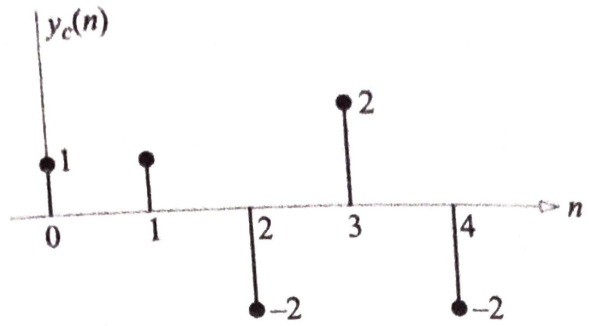


Fig. Ex.3.45(b) 5-point circular convolution of $x_1(n)$ and $x_2(n)$.

or equivalently $y_c(n) = (1, 1, -2, 2, -2)$

$N = 6$:

$$y_c(n) = \sum_{m=0}^{5} x_1(m)\, x_2((n-m))_6$$

n	$x_1(m)$	$x_2((n-m))_6$	$y_c(n)$
0	(2, 1, 1, 2, 0, 0)	(1, 0, 0, 1, −1, −1)	$2+0+0+2+0+0=4$
1	(2, 1, 1, 2, 0, 0)	(−1, 1, 0, 0, 1 − 1)	$-2+1+0+0+0+0=-1$
2	(2, 1, 1, 2, 0, 0)	(−1, −1, 1, 0, 0, 1)	$-2-1+1+0+0+0=-2$
3	(2, 1, 1, 2, 0, 0)	(1, −1, −1, 1, 0, 0)	$2-1-1+2+0+0=2$
4	(2, 1, 1, 2, 0, 0)	(0, 1, −1, −1, 1, 0)	$0+1-1-2+0+0=-2$
5	(2, 1, 1, 2, 0, 0)	(0, 0, 1, −1, −1, 1)	$0+0+1-2+0+0=-1$

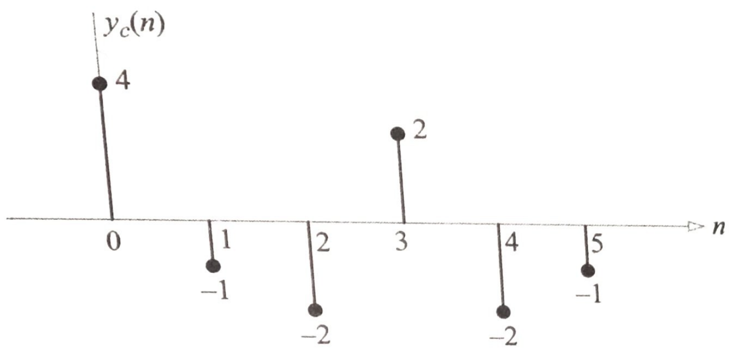


Fig. Ex.3.45(c) 6-point circular convolution of $x_1(n)$ and $x_2(n)$.

or equivalently, $y_c(n) = (4, -1, -2, 2, -2, -1)$

$N = 7$:

n	$x_1(m)$	$x_2((n-m))_7$	$y_c(n)$
0	(2, 1, 1, 2, 0, 0, 0)	(1, 0, 0, 0, 1, −1, −1)	2
1	(2, 1, 1, 2, 0, 0, 0)	(−1, 1, 0, 0, 0, 1, −1)	−1
2	(2, 1, 1, 2, 0, 0, 0)	(−1, −1, 1, 0, 0, 0, 1)	−2
3	(2, 1, 1, 2, 0, 0, 0)	(1, −1, −1, 1, 0, 0, 0)	2
4	(2, 1, 1, 2, 0, 0, 0)	(0, 1, −1, −1, 1, 0, 0)	−2
5	(2, 1, 1, 2, 0, 0, 0)	(0, 0, 1, −1, −1, 1, 0)	−1
6	(2, 1, 1, 2, 0, 0, 0)	(0, 0, 0, 1, −1, −1, 1)	2

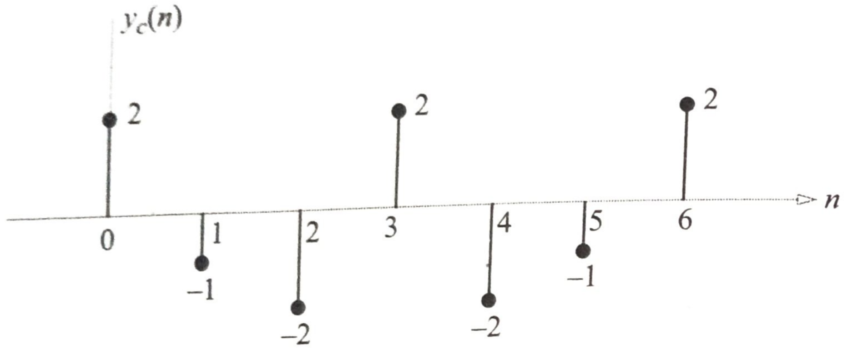


Fig. Ex.3.45(d) 7-point circular convolution of $x_1(n)$ and $x_2(n)$.

or equivalently, $y_c(n) = (2, -1, -2, 2, -2, -1, 2)$

$N = 8$:

n	$x_1(m)$	$x_2((n-m))_8$	$y_c(n)$
0	(2, 1, 1, 2, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 1, −1, −1)	2
1	(2, 1, 1, 2, 0, 0, 0, 0)	(−1, 1, 0, 0, 0, 0, 1, −1)	−1
2	(2, 1, 1, 2, 0, 0, 0, 0)	(−1, −1, 1, 0, 0, 0, 0, 1)	−2
3	(2, 1, 1, 2, 0, 0, 0, 0)	(1, −1, −1, 1, 0, 0, 0, 0)	2
4	(2, 1, 1, 2, 0, 0, 0, 0)	(0, 1, −1, −1, 1, 0, 0, 0)	−2
5	(2, 1, 1, 2, 0, 0, 0, 0)	(0, 0, 1, −1, −1, 1, 0, 0)	−1
6	(2, 1, 1, 2, 0, 0, 0, 0)	(0, 0, 0, 1, −1, −1, 1, 0)	2
7	(2, 1, 1, 2, 0, 0, 0, 0)	(0, 0, 0, 0, 1, −1, −1, 1)	0

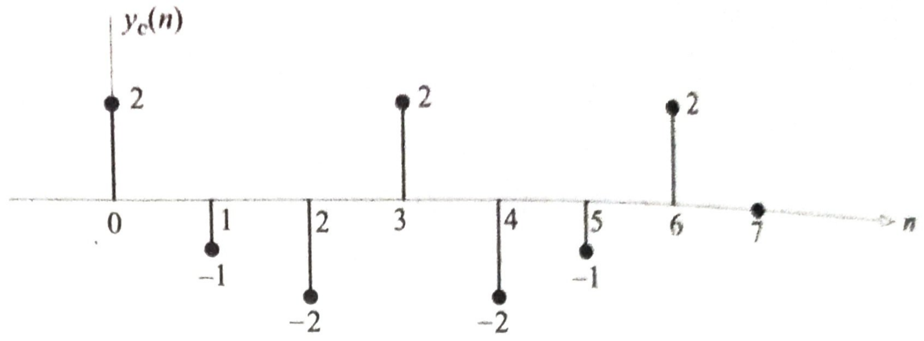


Fig. Ex.3.45(e) 8-point circular convolution of $x_1(n)$ and $x_2(n)$.

or equivalently, $y_c(n) = (2, -1, -2, 2, -2, -1, 2, 0)$

b. Let

$$\begin{aligned} y_l(n) &= x_1(n) * x_2(n) \\ \Rightarrow \quad y_l(n) &= [2\delta(n) + \delta(n-1) + \delta(n-2) + 2\delta(n-3)] * \\ &\quad [\delta(n) - \delta(n-1) - \delta(n-2) + \delta(n-3)] \\ &= 2\delta(n) - \delta(n-1) - 2\delta(n-2) + 2\delta(n-3) - 2\delta(n-4) \\ &\quad -\delta(n-5) + 2\delta(n-6) \end{aligned}$$

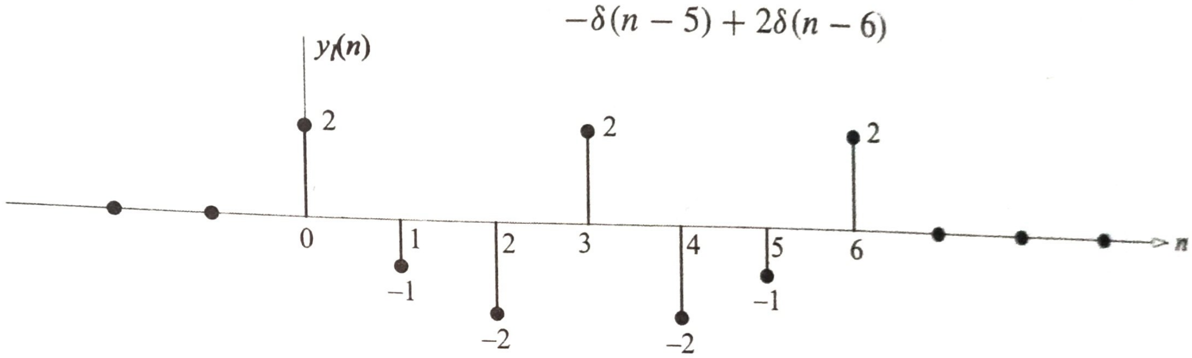


Fig. Ex.3.45(f) Linear convolution of $x_1(n)$ and $x_2(n)$.

c. If $N \geqslant 7$, the results of (a) indicate that linear and circular convolutions are same on the interval 0 to $N-1$.

d. For linear and circular convolutions to give identical results, $N \geqslant N_1 + N_2 - 1$.

Hence,

$$\begin{aligned} N &\geqslant 4 + 4 - 1 \\ \Rightarrow \quad N &\geqslant 7 \end{aligned}$$

Therefore, atleast 7 samples are required for equality of linear and circular convolution.

Example 3.46 Two length-4 sequences are defined below:

$$x(n) = \cos\left(\frac{\pi n}{2}\right), \quad n = 0, 1, 2, 3$$

$$h(n) = 2^n, \quad n = 0, 1, 2, 3$$

a. Calculate $x(n) \circledast_4 h(n)$ by doing circular convolution directly.

b. Calculate $x(n) \circledast_4 h(n)$ by doing linear convolution.

Solution

Let us compute $x(n)$ and $h(n)$ for $0 \leqslant n \leqslant 3$

n	$x(n)$	$h(n)$
0	1	1
1	0	2
2	−1	4
3	0	8

a. Let

$$
\begin{aligned}
y_c(n) &= x(n) \circledast_4 h(n) \\
&= \sum_{m=0}^{3} x(m)\, h((n-m))_4
\end{aligned}
$$

m	$x(m)$	$h((n-m))_4$	$y_c(n)$
0	$(1, 0, -1, 0)$	$(1, 8, 4, 2)$	$1+0-4+0=-3$
1	$(1, 0, -1, 0)$	$(2, 1, 8, 4)$	$2+0-8+0=-6$
2	$(1, 0, -1, 0)$	$(4, 2, 1, 8)$	$4+0-1+0=3$
3	$(1, 0, -1, 0)$	$(8, 4, 2, 1)$	$8+0-2+0=6$

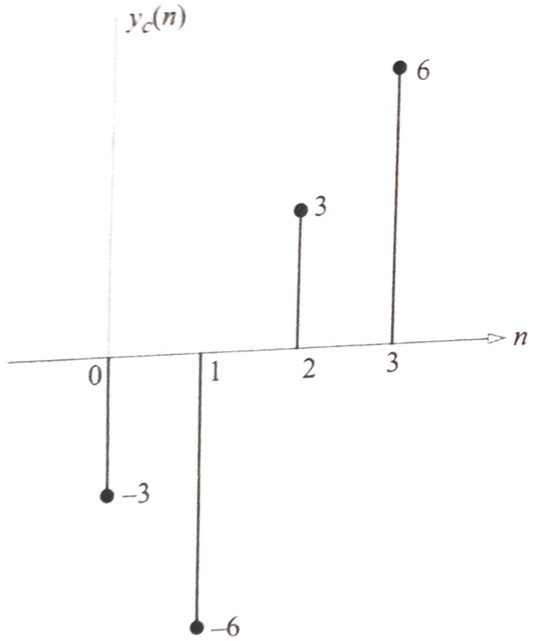


Fig. Ex.3.46(a) 4-point circular convolution of $x(n)$ and $h(n)$.

or equivalently, $y_c(n) = (-3, -6, 3, 6)$

b. Let

$$
\begin{aligned}
y_l(n) &= x(n) * h(n) \\
\Rightarrow \quad y_l(n) &= [\delta(n) - \delta(n-2)] * [\delta(n) + 2\delta(n-1) \\
&\quad +4\delta(n-2) + 8\delta(n-3)] \\
\Rightarrow \quad y_l(n) &= \delta(n) + 2\delta(n-1) + 4\delta(n-2) + 8\delta(n-3) \\
&\quad -\delta(n-2) - 2\delta(n-3) - 4\delta(n-4) - 8\delta(n-5) \\
\Rightarrow \quad y_l(n) &= \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 6\delta(n-3) \\
&\quad -4\delta(n-4) - 8\delta(n-5)
\end{aligned}
$$

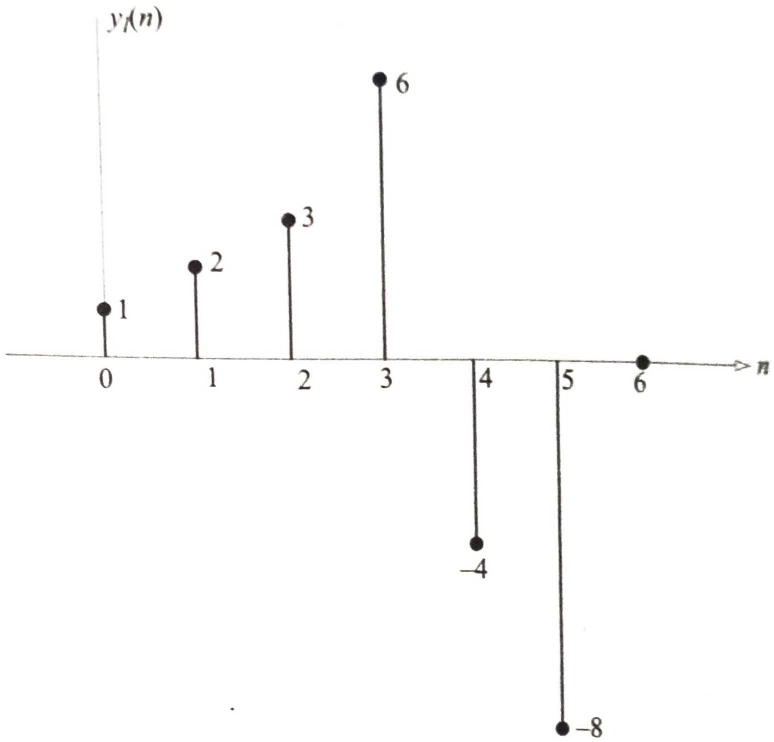


Fig. Ex.3.46(b) Linear convolution of $x(n)$ and $h(n)$.

or equivalently $\quad y_l(n) = (1, 2, 3, 6, -4, -8)$

Circular convolution is nothing but linear convolution plus aliasing. We need $N \geqslant 3+4-1=6$ to avoid aliasing. Since $N = 4$, there will be aliasing. Here, aliasing means the last two points ($n = 4, 5$) will wrap-around on top of the first two points giving a 4-point circular convolution.

That is,

$$
\begin{aligned}
y_c(0) &= y_l(0) + y_l(4) = 1 - 4 = -3 \\
y_c(1) &= y_l(1) + y_l(5) = 2 - 8 = -6 \\
y_c(2) &= y_l(2) = 3 \\
y_c(3) &= y_l(3) = 6
\end{aligned}
$$

3.11 Relationship of the DFT to Other Transforms

We want to reemphasize the fact that DFT is a very important computational tool for performing frequency-domain analysis of signals on digital signal processors. Therefore, it is very imperative to establish relationships between the DFT to other frequency-domain transforms.

3.11.1 Relationship to Fourier series coefficients of a periodic sequence

A periodic signal $x_p(n)$ with a fundamental period N can be represented in a Fourier series of the form:

$$x_p(n) = \sum_{k=0}^{N-1} C_k \, e^{j\frac{2\pi}{N}kn}, \ -\infty < n < \infty \tag{3.18}$$

In the above equation C_k is known as the Fourier coefficient and is computed using the equation given below:

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \cdots N-1$$

The defining equations for IDFT and DFT are as given below:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}, \quad 0 \leqslant n \leqslant N-1 \tag{3.19}$$

and

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad 0 \leqslant k \leqslant N-1$$

If we compare equations (3.18) and (3.19) with the defining equations for IDFT and DFT, we find that

$$C_k = \frac{X(k)}{N} \quad \text{or} \quad X(k) = NC_k$$

provided

$$x(n) = x_p(n) \quad \text{for} \quad 0 \leqslant n \leqslant N-1$$

Hence, except for a scaling factor, the DFS and DFT relations are identical.

3.11.2 Relationship to DTFT

The DTFT relation and its inverse are

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{3.20}$$

and

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \tag{3.21}$$

where $X(\omega)$ is periodic with a period equal to 2π. If $x(n)$ is a finite N-point sequence with $n = 0, 1, \cdots N-1$, then we obtain N samples of the DFT over one period at intervals of $\frac{2\pi}{N}$ as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad 0 \leqslant k \leqslant N-1$$

The above equation describes the DFT of $x(n)$ as a sampled version of its DTFT evaluated at the frequencies $\omega = \frac{2\pi k}{N}$, $k = 0, 1, \cdots N-1$. The DFT spectrum thus corresponds to the frequency range $0 \leqslant \omega < 2\pi$ and is plotted at the frequencies $\omega = \frac{2\pi k}{N}$, $k = 0, 1, \cdots N-1$.

To recover the finite-length sequence, $x(n)$ from N samples of $X(k)$, we use $d\omega \approx \frac{2\pi}{N}$ and $\omega \to \frac{2\pi k}{N}$ to approximate the integral relation defined by equation (3.21) as

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}} \times \frac{2\pi}{N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi nk}{N}} \end{aligned}$$

This is the inverse discrete Fourier transform. The periodicity of the IDFT implies that $x(n)$ actually corresponds to one period of a periodic signal.

If $x(n)$ is a finite length-N sequence with $0 \leqslant n \leqslant N-1$, the DFT $X(k)$ is an exact match to its DTFT $X(\omega)$ at

$$\omega = \frac{2\pi k}{N}, \quad k = 0, \cdots N-1$$

and the IDFT results in a perfect recovery of $x(n)$.

If $x(n)$ is a discrete periodic signal with period N, its scaled DFT, $\frac{1}{N}X(k)$ is an exact match to the impulse strengths in its DTFT, $X(\omega)$ at

$$\omega = \frac{2\pi k}{N}, \quad k = 0, 1, \cdots N-1$$

In this case too, the IDFT results in perfect recovery of one period of $x(n)$ over $0 \leqslant n \leqslant N-1$.

Summary: *The DFT, X(k) of an N-sample sequence x(n) is an exact match to its sampled DTFT. If x(n) is also periodic, the scaled DFT,* $\frac{1}{N}X(k)$ *is an exact match to its sampled DTFT.*

3.11.3 Relationship to $\mathcal{Z}$-transform

The double-sided $\mathcal{Z}$-transform of a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Let us assume that ROC includes the unit circle. If we let $z = e^{j\omega}$ and sample $X(e^{j\omega}) = X(\omega)$ at N equally spaced points on the unit circle $z = e^{j\frac{2\pi k}{N}}, 0 \leqslant k \leqslant N-1$, we get

$$\begin{aligned} X(k) &= X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}}, \qquad k = 0, 1, \cdots N-1 \\ &= \sum_{n=-\infty}^{n=\infty} x(n)z^{-n}\Bigg|_{z=e^{j\frac{2\pi k}{N}}} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi kn}{N}} \end{aligned}$$

The above equation is identical to the DTFT, $X(\omega)$ evaluated at N equally spaced frequencies,

$$\omega = \frac{2\pi k}{N}, \quad k = 0, 1, \cdots N-1$$

If the sequence $x(n)$ has a length-N or less, the sequence, $x(n)$ can be uniquely recovered from its N-point DFT, $X(k)$. Thus, its $\mathcal{Z}$-transform is uniquely determined from its N-point DFT. As a consequence of this, we can relate $X(z)$ and $X(k)$ as follows:

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n}$$

The IDFT of a sequence $X(k)$ is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}}$$

Hence,

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}}\right] z^{-n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^n \end{aligned}$$

Recall the formula: $\displaystyle\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \quad a \neq 1.$

Applying, the above formula to the inner summation in $X(z)$, we get

$$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-e^{j2\pi k}z^{-N}}{1-e^{j\frac{2\pi k}{N}}z^{-1}}\right]$$

$$\Rightarrow \quad X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi k}{N}}z^{-1}}\right]$$

$$= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\left[\frac{X(k)}{1-e^{j\frac{2\pi k}{N}}z^{-1}}\right]$$

The above equation when evaluated on the unit circle, gives the DTFT of $x(n)$ in terms of its DFT, $X(k)$ in the form

$$X(e^{j\omega}) = X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)}}$$

The above expression is known as *Polynomial (Lagrange) interpolation formula* for $X(\omega)$ expressed in terms of the values, $X(k)$ of the polynomial at a set of equally spaced discrete frequencies:

$$\omega = \frac{2\pi k}{N}, \quad k = 0, 1, \cdots N - 1.$$

Example 3.47 Given the following sequence:

$$x(n) = \begin{cases} e^{j\omega_0 n}, & 0 \leqslant n \leqslant N - 1 \\ 0, & \text{elsewhere} \end{cases}$$

a. Find the DTFT of $x(n)$.

b. Find the N-point DFT of $x(n)$.

c. Find the DFT of $x(n)$ for the case $\omega_0 = \frac{2\pi k_0}{N}$, where k_0 is an integer.

☐ Solution

a.

$$\begin{aligned} \text{DTFT}\{x(n)\} &= X(\omega) \\ &\triangleq \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ \Rightarrow \quad X(\omega) &= \sum_{n=0}^{N-1} e^{j\omega_0 n} e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{-j(\omega - \omega_0)n} \end{aligned}$$

Recall the formula:

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}, \quad a \neq 1$$

Hence,

$$\begin{aligned} X(\omega) &= \frac{1 - e^{-j(\omega - \omega_0)N}}{1 - e^{-j(\omega - \omega_0)}} \\ &= \frac{e^{-j(\omega - \omega_0)\frac{N}{2}}}{e^{-j\frac{(\omega - \omega_0)}{2}}} \left[\frac{\sin\left[(\omega - \omega_0)\frac{N}{2}\right]}{\sin\left[(\omega - \omega_0)\frac{1}{2}\right]} \right] \end{aligned}$$

$$= e^{-j(\omega-\omega_0)\left(\frac{N-1}{2}\right)} \frac{\sin\left[(\omega-\omega_0)\frac{N}{2}\right]}{\sin\left[(\omega-\omega_0)\frac{1}{2}\right]}$$

b. Recall the relation:

$$X(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}$$

Hence,

$$X(k) = e^{-j\left(\frac{2\pi k}{N}-\omega_0\right)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{1}{2}\right]}$$

c. Suppose $\omega_0 = \frac{2\pi k_0}{N}$, where, k_0 is an integer.

Then,

$$X(k) = e^{-j\frac{2\pi}{N}(k-k_0)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\pi\,(k-k_0)\right]}{\sin\left[\frac{\pi}{N}(k-k_0)\right]}$$

Example 3.48 Refer the finite length sequence, $x(n)$ shown in Fig. Ex.3.48 below. Let $X(z)$ be the $\mathcal{Z}$-transform of $x(n)$. If we sample $X(z)$ at

$$z = e^{j\left(\frac{2\pi}{4}\right)k}, \quad 0 \leqslant k \leqslant 3$$

we get

$$X(k) = X(z)\big|_{z=e^{j\left(\frac{2\pi}{4}\right)k}}, \quad 0 \leqslant k \leqslant 3$$

Sketch the sequence, $y(n)$ obtained as IDFT of $X(k)$.

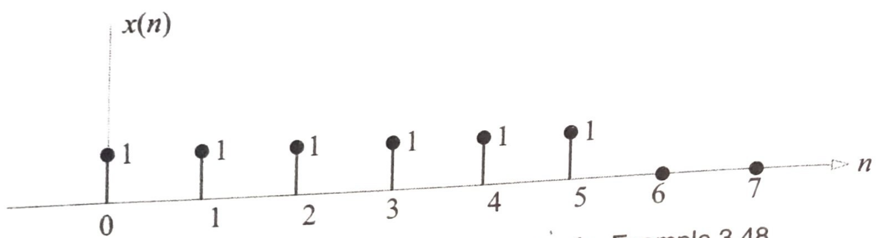


Fig. Ex.3.48 Finite length sequence $x(n)$ for Example.3.48.

Solution

$$x(n) = \begin{cases} 1, & 0 \leqslant n \leqslant 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{Z}\{x(n)\} \triangleq X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Since $x(n)$ is a finite length sequence,

$$X(z) = \sum_{n=0}^{5} 1 \times z^{-n}$$

If we substitute, $z = e^{j\left(\frac{2\pi}{4}\right)k}$, we get

$$X(k) = \sum_{n=0}^{5} 1 \times e^{-j\frac{2\pi kn}{4}}$$

$$\Rightarrow \qquad X(k) = \sum_{n=0}^{5} W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

As specified by the sampling of the $\mathcal{Z}$-transform, we have taken a 4-point DFT, however the original sequence is of length 6. Hence, there will be some aliasing when we find $y(n)$ as the IDFT of $X(k)$.

Perform the DFT,

$$X(k) = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}$$

Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^{k}$, two points are aliased.

Hence,
$$X(k) = 2 + 2W_4^{k} + W_4^{2k} + W_4^{3k}$$

Taking IDFT, we get

$$y(n) = 2\delta(n) + 2\delta(n-1) + \delta(n-2) + \delta(n-3)$$

Using the above equation, the sequence $y(n)$ is sketched as shown in Fig. Ex.3.48(a).

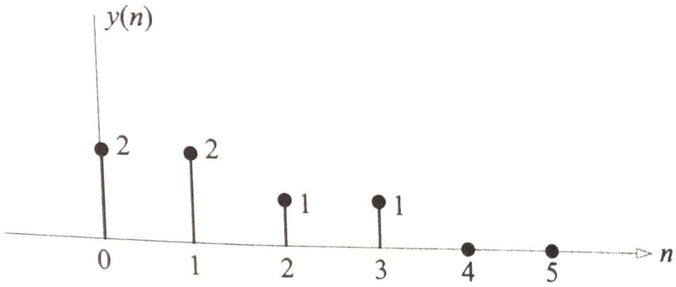


Fig. Ex.3.48(a) Sequence $y(n) = \text{IDFT}\{X(k)\}$.

Example 3.49 Let $X(\omega)$ denote the Fourier transform of the sequence $x(n) = \left(\frac{1}{2}\right)^n u(n)$. Let $x_1(n)$ denote a sequence of finite duration of length 10; that is, $x_1(n) = 0$, for $n < 0$ and $x_1(n) = 0$ for $n \geqslant 10$. The 10-point DFT of $x_1(n)$ denoted by $X_1(k)$, corresponds to 10 equally spaced samples of $X(\omega)$; that is, $X_1(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{10}}$. Determine $x_1(n)$.

Solution

To find DTFT of $x(n) = (\frac{1}{2})^n u(n)$:

$$X(\omega) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\Rightarrow \quad X(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

Recall the formula:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

Hence,

$$X(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

To find $X_1(k)$:

$$X_1(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{10}}, \quad 0 \leqslant k \leqslant 9$$

$$\Rightarrow \quad X_1(k) = \frac{1}{1 - \frac{1}{2}e^{-j\frac{2\pi k}{10}}}, \quad 0 \leqslant k \leqslant 9$$

$$= \frac{1}{1 - \frac{1}{2}W_{10}^k}$$

Recall the DFT pair:

$$a^n \underset{N\text{–point}}{\overset{\text{DFT}}{\longleftrightarrow}} \frac{1-a^N}{1-aW_N^k}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n \underset{10\text{–point}}{\overset{\text{DFT}}{\longleftrightarrow}} \frac{1-(\frac{1}{2})^{10}}{1-\frac{1}{2}W_{10}^k}$$

$$\Rightarrow \quad \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})^{10}} \underset{10\text{–point}}{\overset{\text{DFT}}{\longleftrightarrow}} \frac{1}{1-\frac{1}{2}W_{10}^k}$$

Hence the IDFT of

$$X_1(k) = \frac{1}{1-\frac{1}{2}W_{10}^k}$$

is

$$x_1(n) = \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})^{10}}, \quad 0 \leqslant n \leqslant 9$$

3.12 Fast Fourier Transform

The *fast Fourier transform* (FFT) refers to algorithms that compute the discrete Fourier transform (DFT) in a numerically efficient manner. Many such algorithms are available, however we will present only two such algorithms: decimation-in-time and decimation-in-frequency.

3.12.1 Decimation-in-time FFT

The decimation-in-time algorithm uses the *divide and conquer* approach. In the following presentation, the number of points is assumed as a power of 2, that is, $N = 2^p$. The decimation-in-time approach is one of breaking the N-point transform into two $\frac{N}{2}$-point transforms, then breaking each $\frac{N}{2}$-point transform into two $\frac{N}{4}$-point transforms, and continuing this process until two-point DFTs are obtained. In other words, the N-point DFT is performed as several 2-point DFTs.

Let $x(n)$ represent a sequence of length N, where N is a power of 2. Decimate this sequence into two sequences of length $\frac{N}{2}$, one composed of even-indexed values of $x(n)$ and the other of odd-indexed values of $x(n)$:

Given sequence:	$x(0),\ x(1), \cdots\cdots x(N-1)$
Even indexed:	$x(0),\ x(2), \cdots\cdots x(N-2)$
Odd indexed:	$x(1),\ x(3), \cdots\cdots x(N-1)$

We know that the DFT of an N-point sequence is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad 0 \leqslant k \leqslant N-1$$

Decomposing the sum into two parts, one for the even and one for the odd-indexed values, we get

$$X(k) = \sum_{\substack{n=0 \\ n,\ \text{even}}}^{N-2} x(n) W_N^{kn} + \sum_{\substack{n=1 \\ n,\ \text{odd}}}^{N-1} x(n) W_N^{kn}$$

Letting $n = 2r$ in the first summation and $n = 2r + 1$ in the second summation, the above equation can be written as

$n = 0$ $r = 0$ $n = N$ $r = \frac{N}{2} - 1$

$n = 2r + 1$ $n = 0$ $r =$ $\Rightarrow r = \frac{n-1}{2}$ $r = -1$

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k}$$

$n = N - 1$ $r = \frac{N-1-1}{2} = \frac{N-2}{2}$

Since,

$$W_N^{2rk} = e^{-j\frac{2\pi}{N} \times 2rk} = e^{-j\frac{2\pi rk}{N/2}} = W_{N/2}^{rk}$$

$\frac{N-2}{2} + 1 + 1$ $N - 2 + 2 + 2$

the above equation may be written as

$$\begin{aligned} X(k) &= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{N/2}^{rk} \\ &= G(k) + W_N^k\, H(k), \quad 0 \leqslant k \leqslant \frac{N}{2} - 1 \end{aligned} \tag{3.22}$$

where $G(k)$ and $H(k)$ are $\frac{N}{2}$-point DFTs of even-indexed samples and odd-indexed samples respectively. These DFTs yield

$$G(0),\ G(1), \cdots G\left(\frac{N}{2} - 1\right) \quad \text{and} \quad H(0),\ H(1), \cdots H\left(\frac{N}{2} - 1\right)$$

For the overall N-point DFT, we need $G(k)$ and $H(k)$ for $k = 0, 1, \cdots \frac{N}{2} - 1, \cdots N - 1$. However, $G(k)$ and $H(k)$ are periodic with period $\frac{N}{2}$, and so for $k = \frac{N}{2}, \cdots N - 1$, the values are same as those for $k = 0, 1, \cdots \frac{N}{2} - 1$.

Thus, the equations needed to compute N-point DFT of $x(n)$ are as follows:

$$X(k) = G(k) + W_N^k H(k), \quad 0 \leqslant k \leqslant \frac{N}{2} - 1 \tag{3.23}$$

$$X(k) = G\left(k + \frac{N}{2}\right) + W_N^k H\left(k + \frac{N}{2}\right), \quad \frac{N}{2} \leqslant k \leqslant N - 1 \tag{3.24}$$

The flow diagram for the first stage of the decomposition is shown in Fig. 3.9.

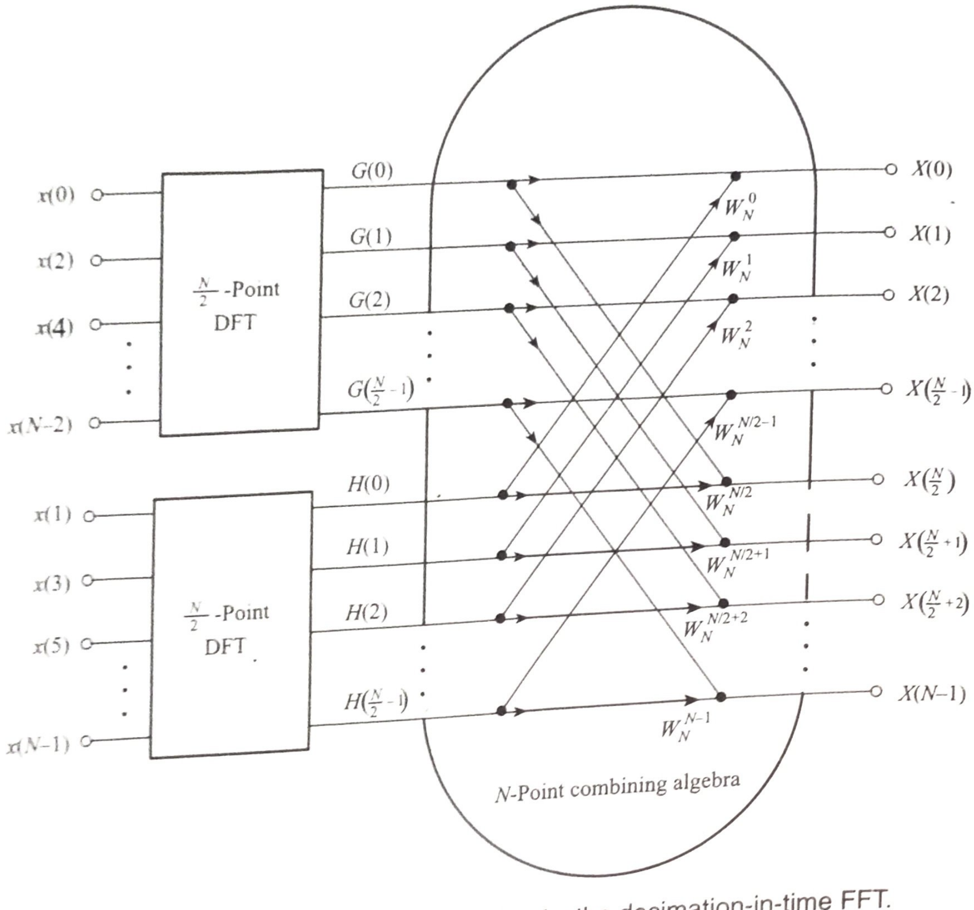


Fig. 3.9 First stage of the decomposition for the decimation-in-time FFT.

The total number of complex multiplications, η_1, required to evaluate the N-point transform with this first decimation is

$$\eta_1 = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = N + \frac{N^2}{2}$$

The first term on the right-side of the above equation is the number of complex multiplications for the direct calculation of $\frac{N}{2}$-point DFT of even-numbered samples, the second term is the number of complex multiplications for the direct calculation of $\frac{N}{2}$-point DFT of odd-numbered samples, and the third term is the number of complex multiplications required for the N-point combining algebra.

Each of the $\frac{N}{2}$-point sequences are further decimated into two sequences of length $\frac{N}{4}$ as shown in Fig. 3.10. Then, we get

$r = 2l$ $r = \frac{N}{2} - 1$
$r = 0$ $l = \frac{N}{4} - 1$
$l = 0$

$$\begin{aligned}
G(k) &= \sum_{r=0}^{\frac{N}{2}-1} g(r) W_{N/2}^{kr} \\
&= \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/2}^{2kl} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/2}^{k(2l+1)} \\
&= \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/4}^{kl} + W_{N/2}^{k} \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/4}^{kl} \\
&= A(k) + W_{N/2}^{k}\, B(k), \quad 0 \leqslant k \leqslant \frac{N}{4} - 1
\end{aligned}$$

odd Component of g(r)

even

since $A(k)$ and $B(k)$ are periodic with a period equal to $\frac{N}{4}$, we may find $\frac{N}{2}$-point transform $G(k)$ using the equation given below.

$$G(k) = \begin{cases} A(k) + W_{N/2}^{k}\, B(k), & 0 \leqslant k \leqslant \frac{N}{4} - 1 \\ A\left(k + \frac{N}{4}\right) + W_{N/2}^{k}\, B\left(k + \frac{N}{4}\right), & \frac{N}{4} \leqslant k \leqslant \frac{N}{2} - 1 \end{cases}$$

Similarly,

$$H(k) = \begin{cases} C(k) + W_{N/2}^{k}\, D(k), & 0 \leqslant k \leqslant \frac{N}{4} - 1 \\ C\left(k + \frac{N}{4}\right) + W_{N/2}^{k}\, D\left(k + \frac{N}{4}\right), & \frac{N}{4} \leqslant k \leqslant N - 1 \end{cases}$$

Often, in a flow diagram we write W_N^{2k} (as a matter of convenience) instead of $W_{\frac{N}{2}}^{k}$.

The flow diagram after second decimation-in-time is shown in Fig. 3.10. The total number of complex multiplications after second decimation is

$$\eta_2 = 4\left(\frac{N}{4}\right)^2 + 2\left(\frac{N}{2}\right) + N = \frac{N^2}{4} + 2N$$

The first term on the right-side of the above equation accounts for the number of complex multiplications necessary for the direct computation of four $\frac{N}{4}$-point DFTs, while the second and third terms represent the number of complex multiplications needed for combining algebras for the first and second decompositions.

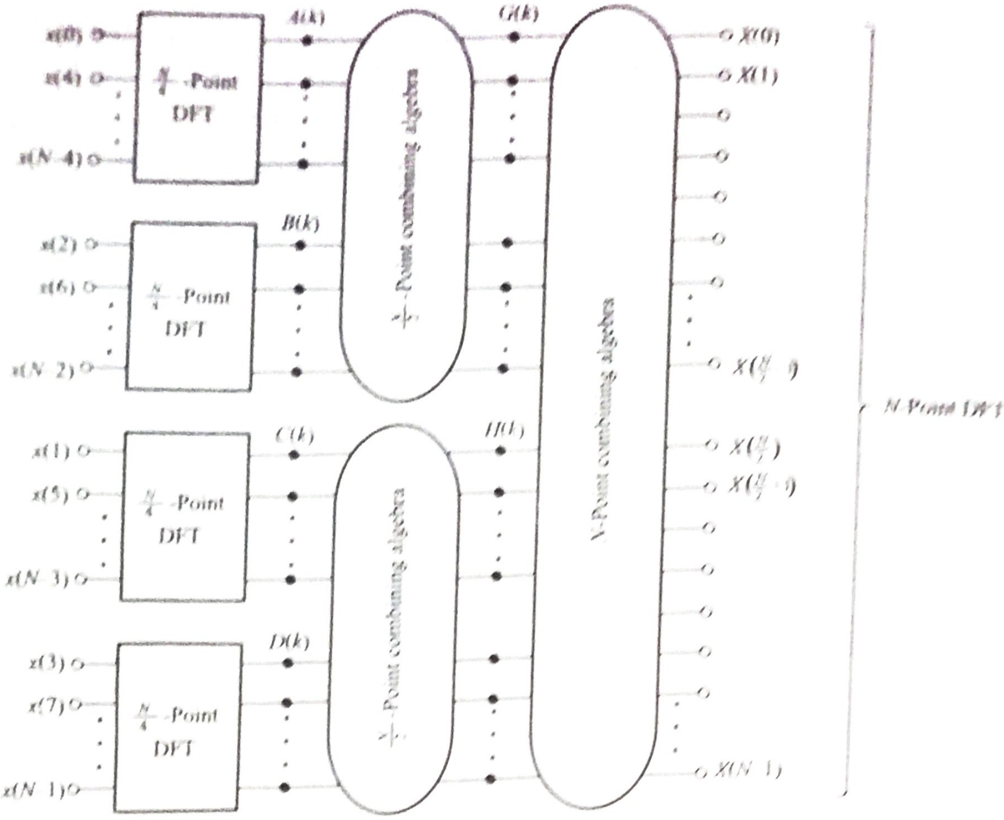


Fig. 3.10 Second stage of the decimation-in-time FFT.

We continue this process of decimation until we end up with 2-point DFTs. This would require $(p-1)$ stages when $N = 2^p$ or $p = \log_2 N$. The total number of complex multiplications becomes $N \log_2 N$ (there are $\log_2 N$ stages and each stage has N complex multiplications). The total number of complex multiplications have reduced from N^2 (direct evaluation) to $N \log_2 N$ (after decimation). The final conceptual decomposition for the decimation-in-time FFT is shown in Fig. 3.11.

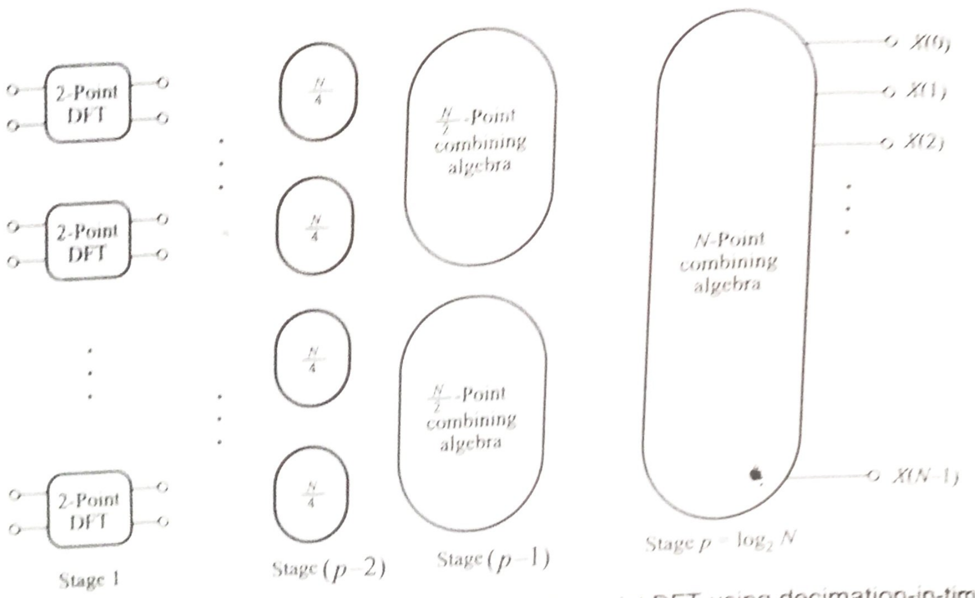


Fig. 3.11 Final conceptual decomposition for computation of N-point DFT using decimation-in-time FFT

The flow diagram for an 8-point decimation-in-time FFT algorithm is shown in Fig. 3.12.

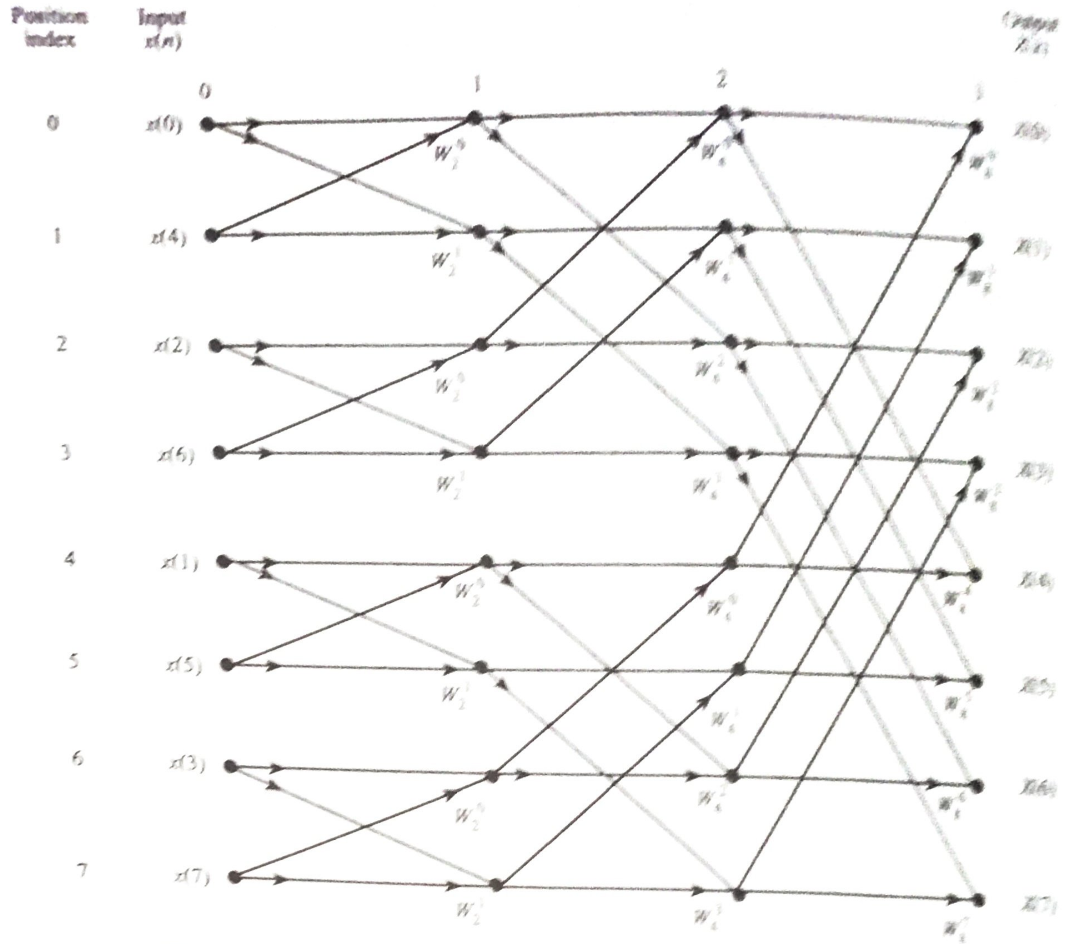


Fig. 3.12 The flow graph for an 8-point decimation-in-time FFT algorithm.

Fig. 3.13 shows the specimen butterfly or the basic building block for decimation-in-time FFT.

Fig. 3.13 Sample butterfly for decimation-in-time FFT.

The following observations can be made from Fig. 3.12.

1. The input data appears in the bit reversed order:

Position		*Binary equivalent*	*Bit reversed*	*Sequence*
5	→	101	101	$x(5)$
6	→	110	011	$x(3)$

2. The basic computational block in the flow diagram is called a *butterfly*. The power r of W_N is a variable and depends upon the position of the butterfly in the flow diagram.
3. The frequency-domain values are in the normal order.

3.12.2 Further reduction: Cooley-Tukey algorithm

The sample butterfly configuration shown in Fig. 3.13 can be further simplified to reduce the number of complex multiplications per butterfly by one. This is possible due to the fact,

$$W_N^{r+N/2} = e^{-j\frac{2\pi}{N}(r+\frac{N}{2})} = e^{-j\frac{2\pi r}{N}} e^{-j\pi} = -W_N^r$$

Thus, the sample butterfly shown in Fig. 3.13 can be further simplified into a butterfly shown in Fig. 3.14.

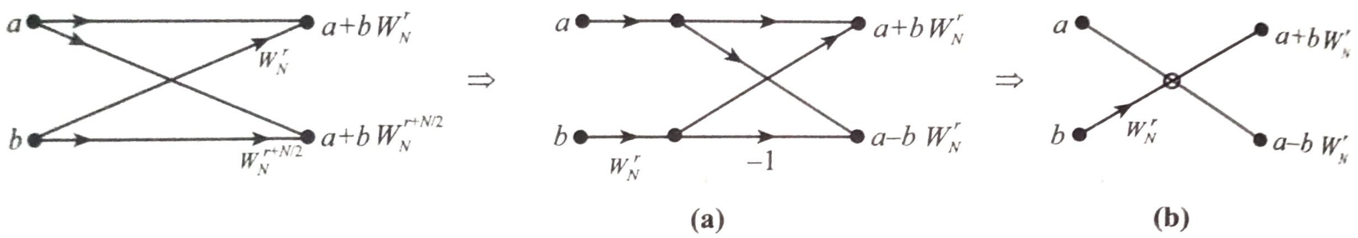


Fig. 3.14 Modified sample butterfies (a) and (b).

The number of complex multiplications required for calculating DIT-FFT algorithm based on Cooley-Tukey is $\frac{N}{2}\log_2 N$. Also, since there are two additions per butterfly, $\frac{N}{2}$ butterflies per stage and $\log_2 N$ stages, the total number of complex additions needed for computing a DFT using DIT-FFT algorithm is $N\log_2 N$.

The reduced complete 8-point DIT-FFT algorithm is shown in Fig. 3.15.

3.12.3 In-place computations

Other than speedy computations, another advantage that naturally comes from this algorithm is the reduction in storage requirement. Referring Fig. 3.15, we find that the algorithm is performed in stages. Each stage involves only $\frac{N}{2}$ butterflies and these butterflies operate on a pair of complex numbers and produce another pair of complex numbers. Once the output pair is calculated, we don't need the input pair anymore. Hence, the output pair can be stored in the same locations as the input pair. Therefore, for an N-point DFT, we need N complex registers or $2N$ real registers for the first stage. The same registers will be used for other stages to follow. This is called *in-place* computations in the DSP literature. If we refer to 8-point FFT, the butterfly computations are performed on the following pairs: $x(0)$ and $x(4)$, $x(2)$ and $x(6)$, $x(1)$ and $x(5)$, and $x(3)$ and $x(7)$. To begin with let us label the input samples as follows:

$$\begin{aligned} X_0(0) &= x(0) & X_0(4) &= x(1) \\ X_0(1) &= x(4) & X_0(5) &= x(5) \\ X_0(2) &= x(2) & X_0(6) &= x(3) \\ X_0(3) &= x(6) & X_0(7) &= x(7) \end{aligned}$$

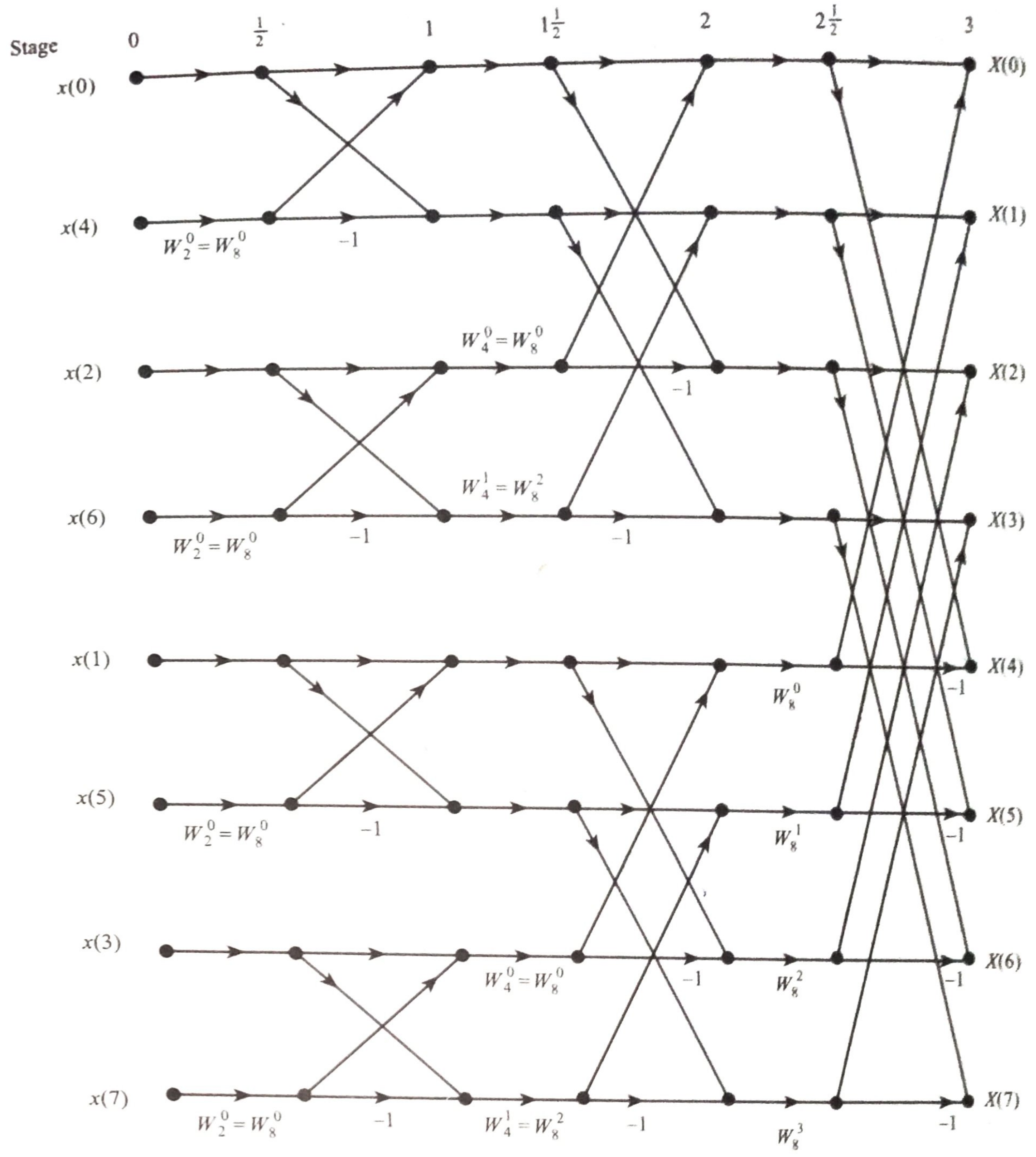


Fig. 3.15 The reduced flow graph for an 8-point DIT FFT algorithm.

Here, $X_m(n)$ refers to the data of the m^{th} stage. The butterflies in the first stage will yield the data $X_1(0)$, $X_1(1) \cdots X_1(7)$. Next, butterfly computations will be performed on the output of the first stage given by the pairs $X_1(0)$ and $X_1(2)$, $X_1(1)$ and $X_1(3)$, $X_1(4)$ and $X_1(6)$, and $X_1(5)$ and $X_1(7)$. The output of the second stage is the data set $X_2(0)$, $X_2(1) \cdots X_2(7)$. The butterflies in the third stage will be computed on the data pairs $X_2(0)$ and $X_2(4)$, $X_2(2)$ and $X_2(6)$, $X_2(1)$ and $X_2(5)$, and $X_2(3)$ and $X_2(7)$ to give the final DFT values.

The scheme of evaluation is pictorially shown in Fig. 3.16. As shown in Fig. 3.16, if we reverse the order of the bits, then we get the location where the data is to be stored. For example, $x(4)$ is represented as $x(100)$. If we reverse the bits, we get $x(001)$, which means that $x(4)$ will be stored in $X_0(1)$. Hence, if the input data samples $x(n)$ are in bit-reversed order, the final DFT is obtained in the natural order. It is easy to show that if the input data samples are stored in the normal order, then the final DFT is obtained in bit-reversed order.

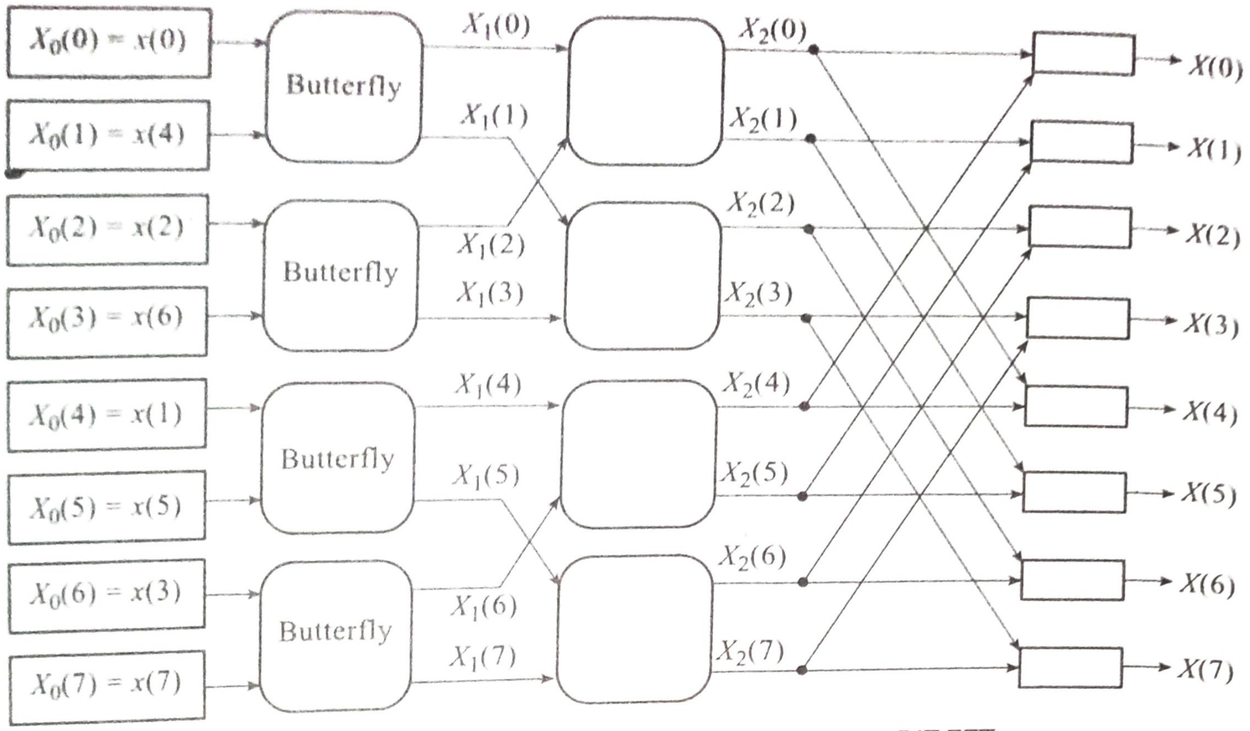


Fig. 3.16 Three-level computation diagram for an 8-point DIT-FFT.

Example 3.50 Find the 8-point DFT of a sequence $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$ using DIT-FFT radix-2 algorithm. Use the butterfly diagram given in Fig. Ex.3.50.

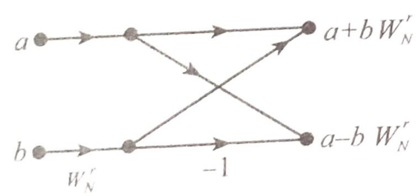


Fig. Ex.3.50 Specified butterfly for Example.3.50.

☐ Solution

The scale-factors W_8^r are as follows:

$$W_8^0 = 1 \qquad W_8^4 = -1$$

$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$W_8^2 = -j \qquad W_8^6 = +j$$

$$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

The decimation-in-time FFT algorithms are developed on 2 as the base or radix, and where the number of samples N and base 2 are related by the expression $N = 2^p$. This is the reason for calling the algorithms developed as radix-2 FFT algorithms.

The flow diagram is as shown in Fig. Ex. 3.50(b).

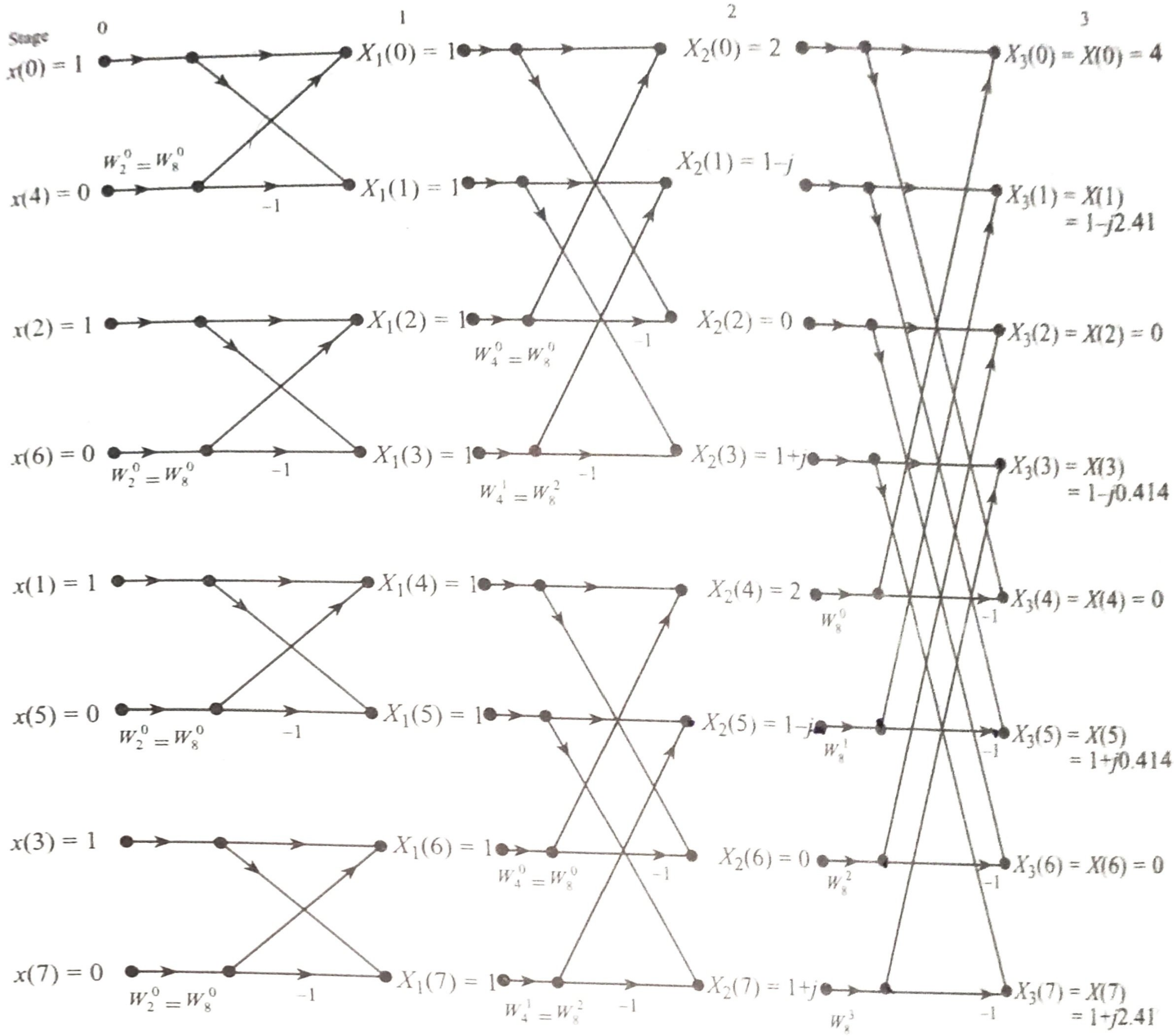


Fig. Ex.3.50(a) The flow graph for 8-point DFT using DIT-FFT algorithm.

To compute the output of first stage:

$$\begin{aligned}
X_1(0) &= 1 + 0 \times W_8^0 = 1 \\
X_1(1) &= 1 - 0 \times W_8^0 = 1 \\
X_1(2) &= 1 + 0 \times W_8^0 = 1 \\
X_1(3) &= 1 - 0 \times W_8^0 = 1 \\
X_1(4) &= 1 + 0 \times W_8^0 = 1 \\
X_1(5) &= 1 - 0 \times W_8^0 = 1 \\
X_1(6) &= 1 + 0 \times W_8^0 = 1 \\
X_1(7) &= 1 - 0 \times W_8^0 = 1
\end{aligned}$$

To compute the output of second stage:

$$\begin{aligned}
X_2(0) &= 1 + W_8^0 \times 1 = 2 \\
X_2(1) &= 1 + 1 \times W_8^2 = 1 - j \\
X_2(2) &= 1 - W_8^0 \times 1 = 0 \\
X_2(3) &= 1 - 1 \times W_8^2 = 1 + j \\
X_2(4) &= 1 + W_8^0 \times 1 = 2 \\
X_2(5) &= 1 + W_8^2 \times 1 = 1 - j \\
X_2(6) &= 1 - W_8^0 \times 1 = 0 \\
X_2(7) &= 1 - W_8^2 \times 1 = 1 + j
\end{aligned}$$

To compute the output of third stage

$$\begin{aligned}
X_3(0) &= X(0) = 2 + 2W_8^0 = 4 \\
X_3(1) &= X(1) = 1 - j + (1 - j)W_8^1 = 1 - j2.41 \\
X_3(2) &= X(2) = 0 + 0 \times W_8^2 = 0 \\
X_3(3) &= X(3) = (1 + j) + (1 + j)W_8^3 = 1 - j0.414 \\
X_3(4) &= X(4) = 2 - 2W_8^0 = 0 \\
X_3(5) &= X(5) = (1 - j) - (1 - j)W_8^1 = 1 + j0.414 \\
X_3(6) &= X(6) = 0 - 0 \times W_8^2 = 0 \\
X_3(7) &= X(7) = (1 + j) - (1 + j)W_8^3 = 1 + j2.414
\end{aligned}$$

Since $x(n)$ is a real sequence, we find that the symmetry property: $X(k) = X^*(8 - k)$ is satisfied.

Example 3.51 Find the 8-point DFT of the sequence,

$$x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$$

using DIT-FFT radix-2 algorithm. The basic computational block known as the butterfly should be as shown in Fig. Ex.3.51.

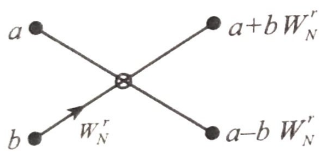


Fig. Ex.3.51 Specified butterfly for Example.3.51.

☐ Solution

The scale-factors W_8^r are as follows:

$$W_8^0 = 1 \qquad W_8^4 = -1$$

$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$W_8^2 = -j \qquad W_8^6 = +j$$

$$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

Also, recall that:

$$\begin{aligned}
W_4^0 &= W_8^0 \\
W_4^1 &= W_8^2 \\
W_2^0 &= W_8^0
\end{aligned}$$

The flow diagram is as shown in Fig. Ex. 3.51(a).

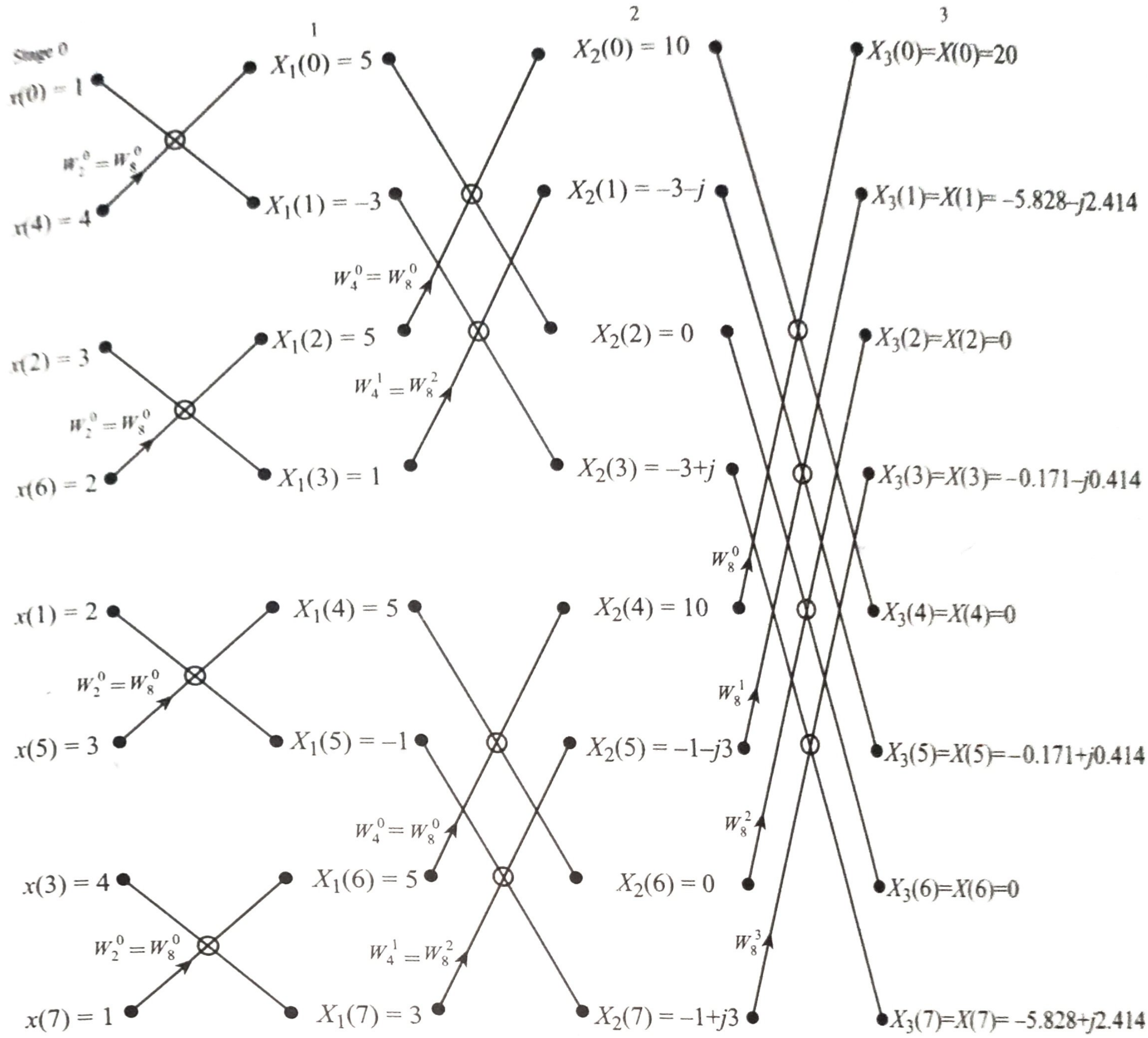


Fig. Ex.3.51(a) Reduced flow graph for an 8-point decimation-in-time FFT.

To compute the output of first stage

$$
\begin{aligned}
X_1(0) &= 1 + 4 \times W_8^0 = 5 \\
X_1(1) &= 1 - 4 \times W_8^0 = -3 \\
X_1(2) &= 3 + 2 \times W_8^0 = 5 \\
X_1(3) &= 3 - 2 \times W_8^0 = 1 \\
X_1(4) &= 2 + 3 \times W_8^0 = 5 \\
X_1(5) &= 2 - 3 \times W_8^0 = -1 \\
X_1(6) &= 4 + 1 \times W_8^0 = 5 \\
X_1(7) &= 4 - 1 \times W_8^0 = 3
\end{aligned}
$$

To compute the output of the second stage

$$
\begin{aligned}
X_2(0) &= 5 + 5 \times W_8^0 = 10 \\
X_2(1) &= -3 + 1 \times W_8^2 = -3 - j \\
X_2(2) &= 5 - 5 \times W_8^0 = 0 \\
X_2(3) &= -3 - 1 \times W_8^2 = -3 + j \\
X_2(4) &= 5 + 5 \times W_8^0 = 10 \\
X_2(5) &= -1 + 3 \times W_8^2 = -1 - j3 \\
X_2(6) &= 5 - 5 \times W_8^0 = 0 \\
X_2(7) &= -1 - 3 \times W_8^2 = -1 + j3
\end{aligned}
$$

To compute the output of the third stage

$$
\begin{aligned}
X_3(0) &= X(0) = 10 + 10 \times W_8^0 = 20 \\
X_3(1) &= X(1) = -3 - j + (-1 - 3j)W_8^1 = -5.828 - j2.414 \\
X_3(2) &= X(2) = 0 + 0 \times W_8^2 = 0 \\
X_3(3) &= X(3) = -3 + j + (-1 + 3j)W_8^3 = -0.171 - j0.414 \\
X_3(4) &= X(4) = 10 - 10 \times W_8^0 = 0 \\
X_3(5) &= X(5) = -3 - j - (-1 - 3j)W_8^1 = -0.171 + j0.414 \\
X_3(6) &= X(6) = 0 - 0 \times W_8^2 = 0 \\
X_3(7) &= X(7) = -3 + j - (-1 + 3j)W_8^3 = -5.828 + j2.414
\end{aligned}
$$

Since $x(n)$ is a real sequence, we find that the symmetry condition: $X(k) = X^*(8-k)$ is satisfied.

Example 3.52 a. Compute the 4-point DFT of the sequence $x(n) = (1, 0, 1, 0)$ using DIT-FFT radix-2 algorithm.

b. Find $x(n)$ for $X(k)$ found in part (a) by two different methods.

☐ Solution

The scale-factors are as follows:

$$W_4^0 = 1, \qquad W_4{}^1 = -j$$

Also,
$$W_4^{-0} = (W_4^0)^* = 1$$

and
$$W_4^{-1} = (W_4^1)^* = j$$

a. To find 4-point DFT $X(k)$

The signal flow diagram is shown in Fig. Ex.3.52(b). The computational block or the butterfly used is as shown in Fig. Ex.3.52(a), because it is very simple to draw and so is our preferred choice.

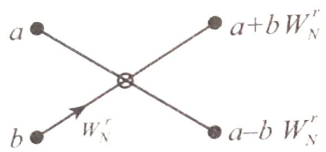


Fig. Ex.3.52(a) Preferred butterfly.

FACTS:

- Since $x(n)$ is a real sequence, the symmetry condition: $X(k) = X^*(N-k)$ must be observed
- Since $x(n)$ is real and $x(n) = x((-n))_4$, the DFT$\{x(n)\} = X(k)$ must be purely real.

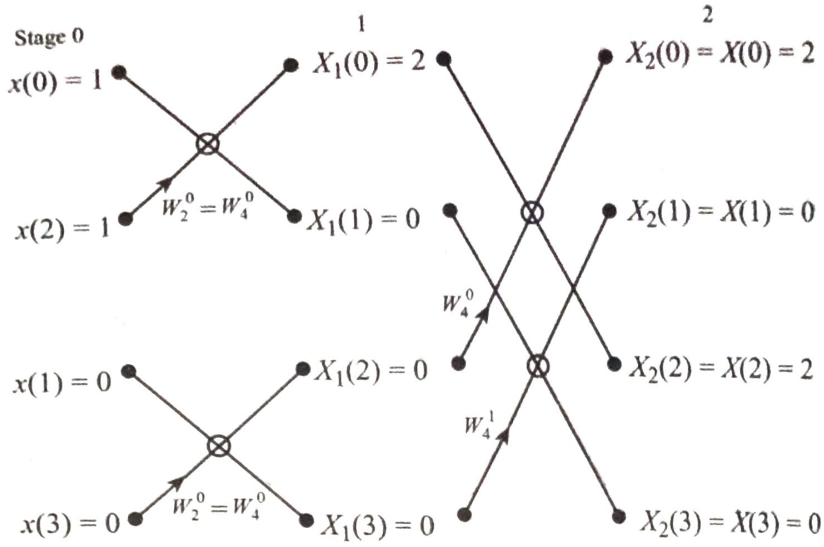


Fig. Ex.3.52(b) Flow graph for computing 4-point DFT.

Hence, $$X(k) = (2, 0, 2, 0)$$

b. **I method:** We know that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \leqslant n \leqslant N-1$$

The above equation says that DIT-FFT algorithm can be used to find IDFT of $X(k)$ by changing the power of W_N as negative and then dividing the final output by $\frac{1}{N}$.

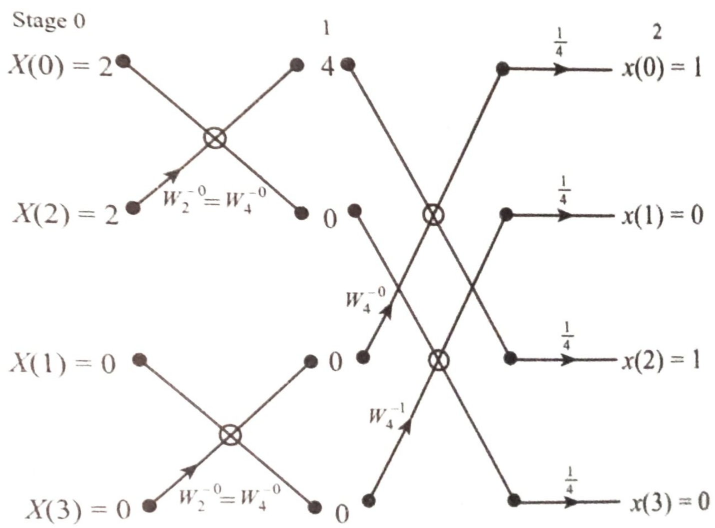


Fig. Ex.3.52(c) Flow graph for computing 4-point IDFT.

Hence, $$x(n) = (1, 0, 1, 0)$$

II method: We know that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

Taking conjugates on both the sides, we get

$$
\begin{aligned}
x^*(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{+kn} \\
&= \frac{1}{N}\,\text{IDFT}\{X^*(k)\} \\
&= \frac{1}{N}\,y(n), \quad \text{where } y(n) = \text{IDFT}\{X^*(k)\}
\end{aligned}
$$

Taking conjugates again, we get

$$x(n) = \frac{1}{N}\,y^*(n)$$

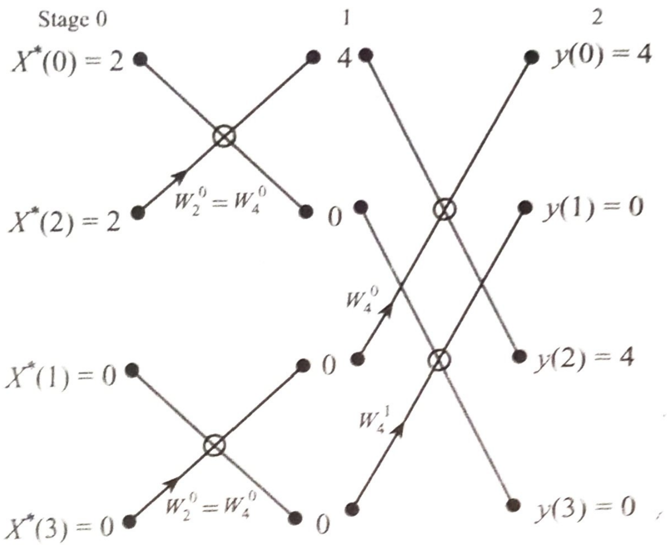


Fig. Ex.3.52(d) Flow graph for computing 4-point DFT of $X(k)$.

n	$x(n) = \frac{1}{4}y^*(n)$
0	$x(0) = \frac{1}{4} \times 4 = 1$
1	$x(1) = \frac{1}{4} \times 0 = 0$
2	$x(2) = \frac{1}{4} \times 4 = 1$
3	$x(3) = \frac{1}{4} \times 0 = 0$

Hence, $x(n) = (1, 0, 1, 0)$

Example 3.53 Given the sequences $x_1(n)$ and $x_2(n)$ below, compute the circular convolution $x_1(n) \circledast_N x_2(n)$ for $N = 4$. Use DIT-FFT algorithm.

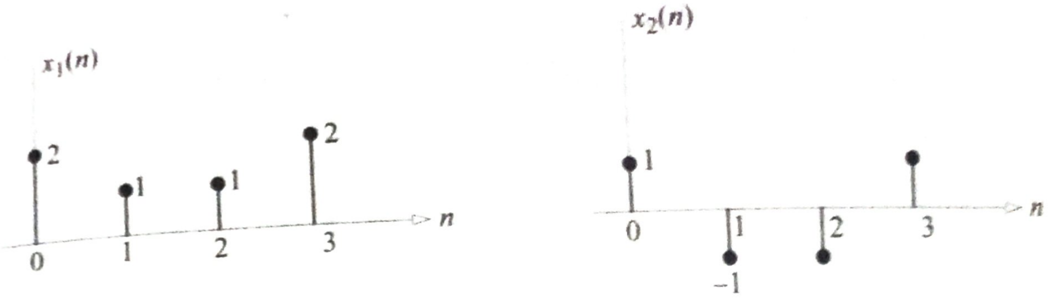


Fig. Ex.3.53 Sequences $x_1(n)$ and $x_2(n)$ for Example.3.53.

Solution

The block diagram used for finding the circular convolution of $x_1(n)$ and $x_2(n)$ using FFTs is shown in Fig. Ex.3.53(a).

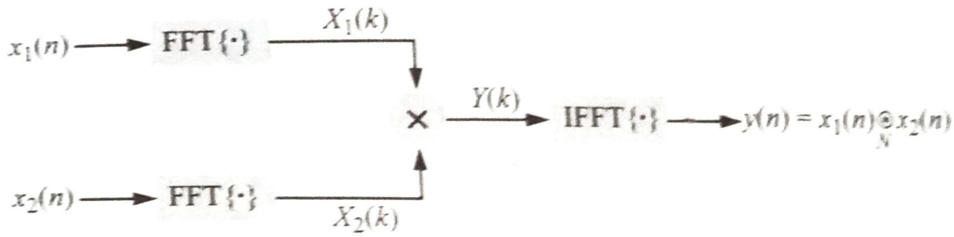


Fig. Ex.3.53(a) Block diagram for computing IDFT.

To find $X_1(k)$:

$$x_1(n) = (2, 1, 1, 2)$$

Let us find $X_1(k)$ using DIT-FFT algorithm. The corresponding signal flow graph is shown in Fig. Ex.3.53(b).

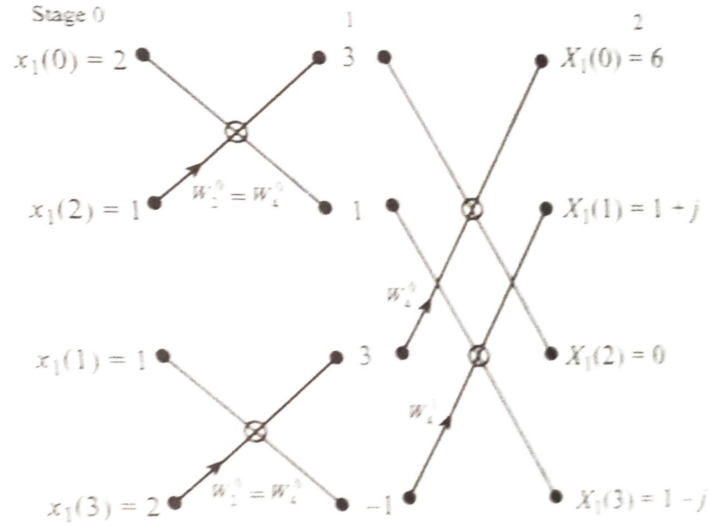


Fig. Ex.3.53(b) Flow graph for computing 4-point DFT.

Hence, $$X_1(k) = (6, 1+j, 0, 1-j)$$

To find $X_2(k)$:

$$x_2(n) = (1, -1, -1, 1)$$

Let us find $X_2(k)$ using DIT-FFT algorithm. The corresponding flow graph is shown in Fig. Ex.3.53(c).

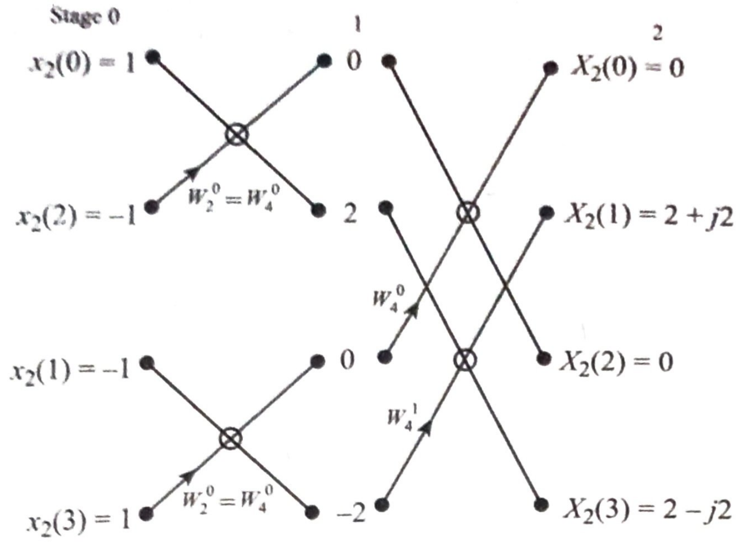


Fig. Ex.3.53(c) Flow graph for computing $X_2(k)$.

Hence,

$$X_2(k) = (0, 2 + 2j, 0, 2 - 2j)$$

To find $y(n) = x_1(n) \circledast_4 x_2(n)$

$$\begin{aligned} Y(k) &= X_1(k)X_2(k) \\ &= (6, 1 + j, 0, 1 - j) \times (0, 2 + 2j, 0, 2 - 2j) \\ &= (0, 4j, 0, -4j) \end{aligned}$$

The flow graph for finding IDFT of $Y(k)$ using DIT-FFT algorithm is shown in Fig. Ex.3.53(d).

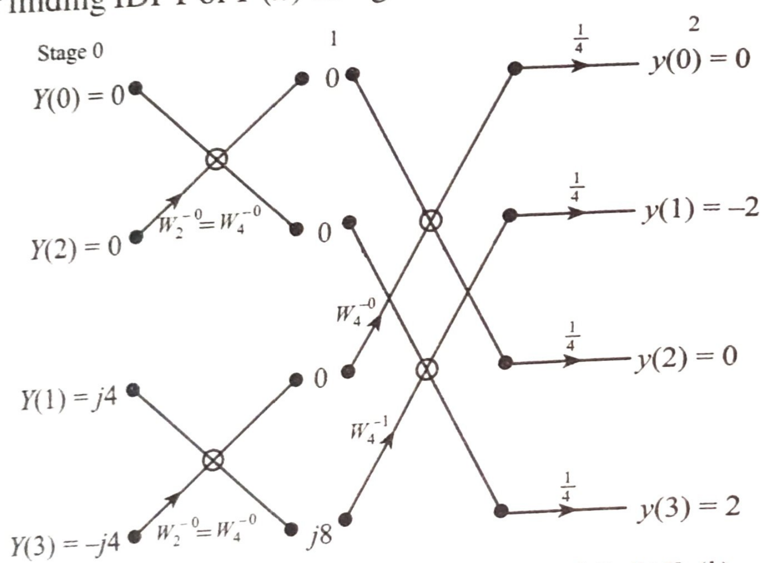


Fig. Ex.3.53(d) Flow graph for computing IDFT of $X_1(k)X_2(k)$.

Thus, we find that

$$y(n) = (0, -2, 0, 2)$$

Example 3.54 Draw the flow diagram for a 16-point radix-2 decimation-in-time FFT algorithm. Label all the multipliers appropriately.

Solution

The flow diagram for a 16-point radix-2 DIT-FFT algorithm is shown below.

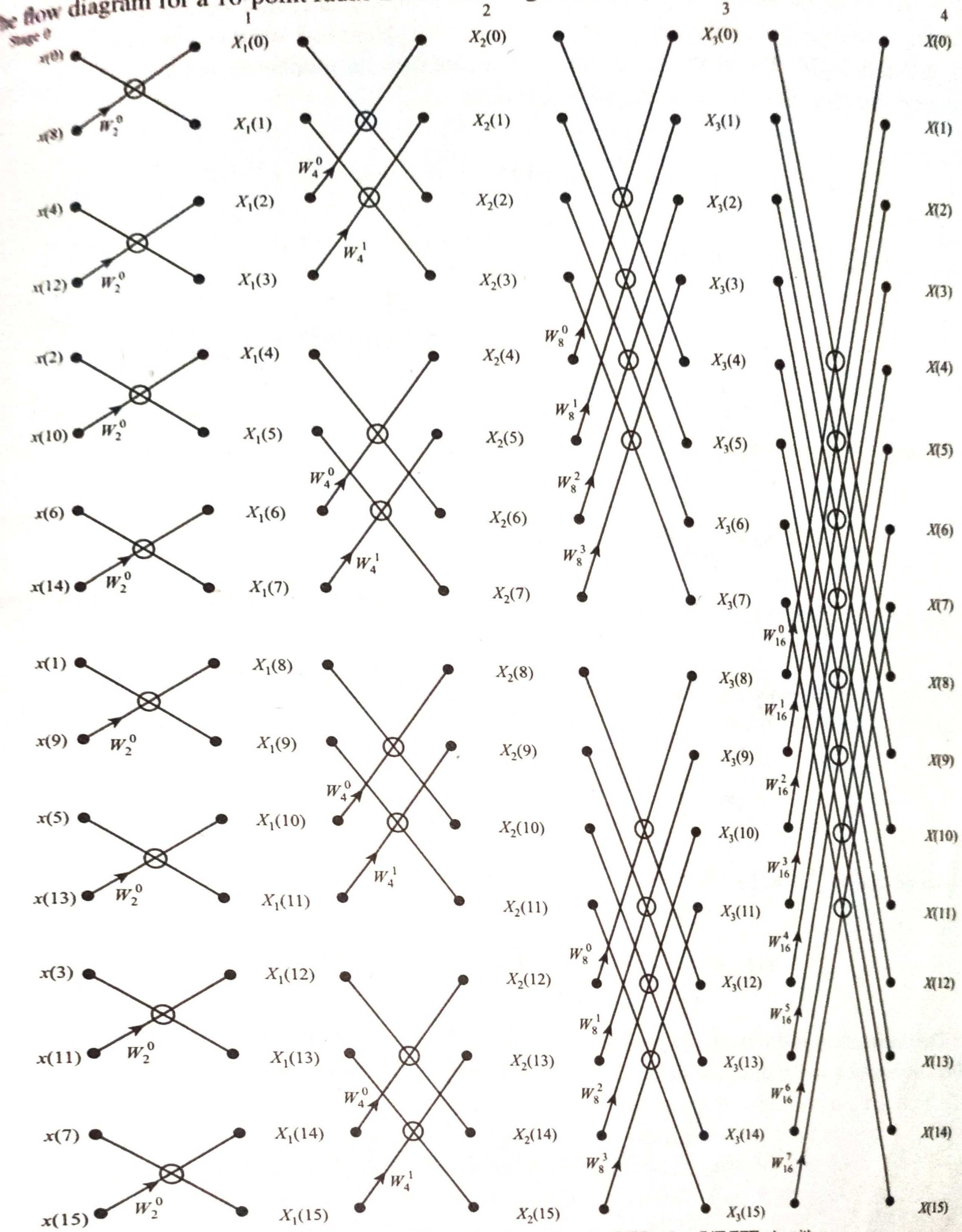


Fig. Ex.3.54 Flow graph for computing 16-point DFT using DIT-FFT algorithm.

3.12.4 Decimation-in-frequency FFT

In this algorithm, the output sequence $X(k)$ is divided into smaller and smaller subsequences in the same manner as in the decimation-in-time algorithm. Here too, we consider N to be a power of 2, so that $N = 2^P$. The goal is to separately evaluate the odd-numbered and the even-numbered frequency samples. We start with the DFT formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad 0 \leq k \leq N-1$$

Let us breakup $X(k)$ as follows:

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

Letting $r = n - \frac{N}{2}$ in the second summation, we get

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x\left(r + \frac{N}{2}\right) W_N^{k(r+N/2)} \tag{3.25}$$

Since r is a dummy variable, it can be replaced by n. Accordingly, the above equation becomes

$$\begin{aligned} X(k) &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{k(n+N/2)} \qquad (3.26) \\ &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{kn} \end{aligned}$$

Notice that $W_N^{kN/2} = (-1)^k$. Hence,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right] W_N^{kn} \tag{3.27}$$

The decimation-in-frequency is now done by taking the odd and even terms of $X(k)$. For even values of k, say $k = 2r$, and $r = 0, 1, \cdots \frac{N}{2} - 1$, equation (3.27) becomes

$$\begin{aligned} X(2r) &= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x\left(n + \frac{N}{2}\right)\right] W_N^{2rn} \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{rn} \qquad (3.28) \end{aligned}$$

For odd values of k, say $k = 2r + 1$ and $r = 0, 1, \cdots \frac{N}{2} - 1$, equation (3.27) becomes

$$
\begin{aligned}
X(2r+1) &= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r+1} x\left(n + \frac{N}{2}\right) \right] W_N^{n(2r+1)} \\
&= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \, W_N^{2rn} \qquad (3.29)
\end{aligned}
$$

Define
$$e(n) = x(n) + x\left(n + \frac{N}{2}\right) \qquad (3.30)$$

and
$$o(n) = \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \qquad (3.31)$$

Then, equations (3.28) and (3.29) become

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} e(n)\, W_{N/2}^{rn} \qquad (3.32)$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} o(n)\, W_{N/2}^{rn}, \quad r = 0, 1, \cdots \frac{N}{2} - 1 \qquad (3.33)$$

From equations (3.30) and (3.31), we see that once $e(n)$ and $o(n)$ are calculated, the even and odd indexed $X(k)$s are found from the $\frac{N}{2}$-point transforms of $e(n)$ and $o(n)$ as shown in Fig. 3.18 for $N = 8$. The basic computational block is a butterfly as shown in Fig. 3.17.

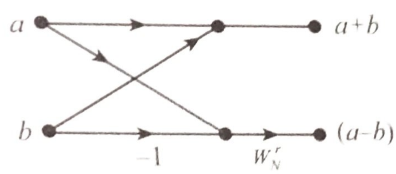


Fig. 3.17 Basic butterfly for decimation-in-frequency FFT.

In a similar manner, each $\frac{N}{2}$-point transform can be decomposed into two $\frac{N}{4}$-point transforms by again decimating the frequency outputs. If this process is continued down to 2-point DFTs, it is easily seen that $\frac{N}{2} \log_2 N$ complex multiplications and $N \log_2 N$ additions are required. Thus, the computational complexity is same as that for the decimation- in-time algorithm. In-place computations can also be done in this method for minimal storage requirements. The complete 8-point decimation-in-frequency FFT flow diagram is shown in Fig. 3.19.

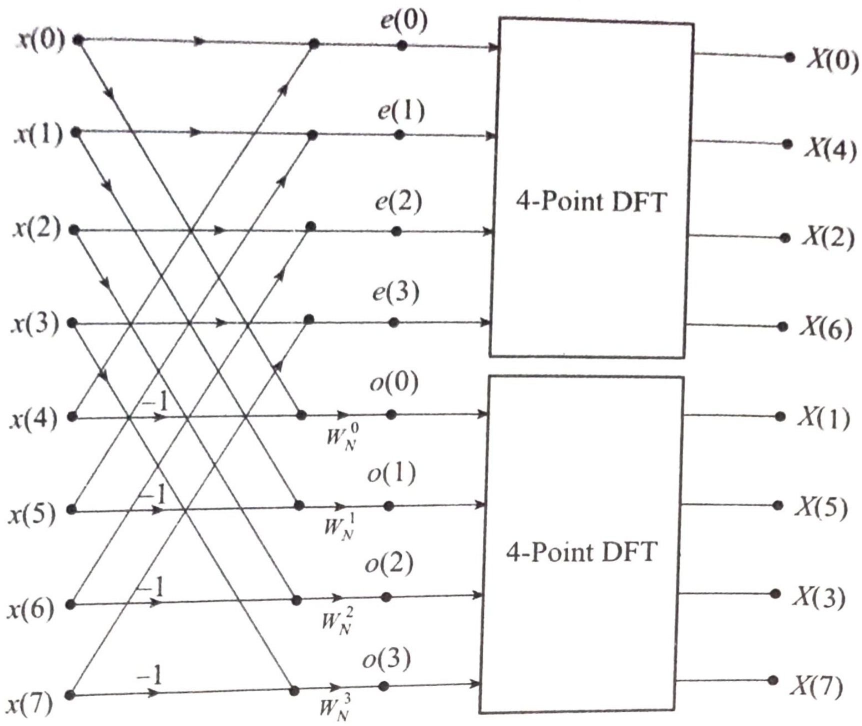


Fig. 3.18 Reduced flow graph for an 8-point decimation-in-frequency FFT.

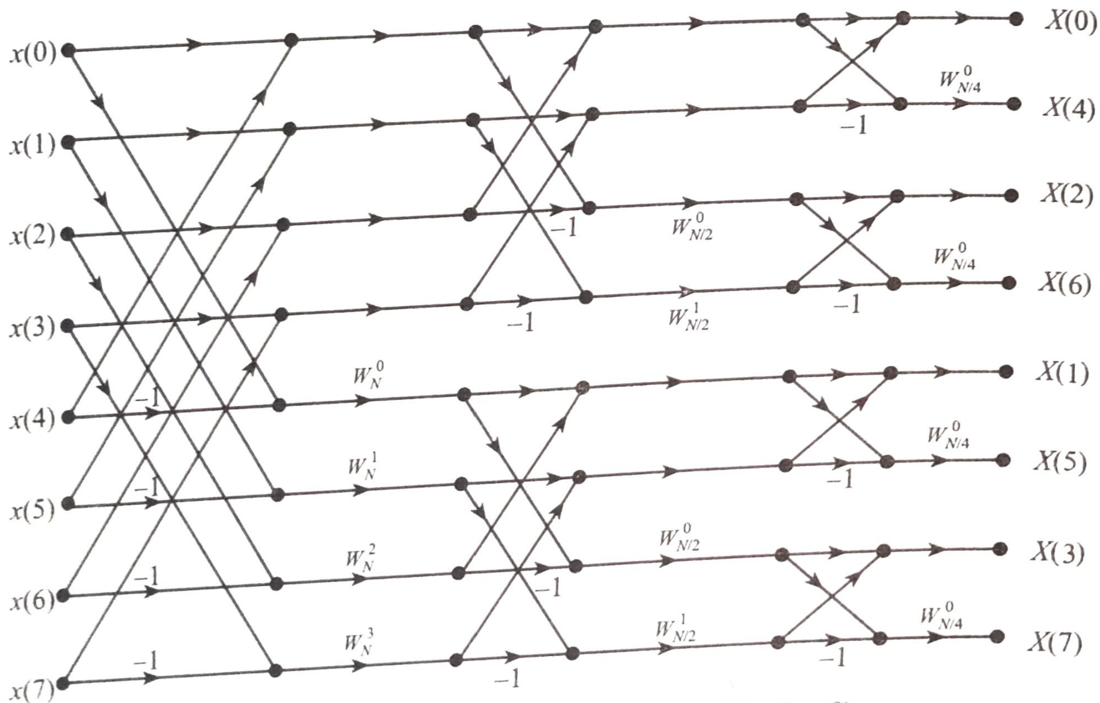


Fig. 3.19 Complete decimation-in-frequency FFT ($N = 8$).

Following observations are made:

i. the input is in the normal order, and

ii. the output is in the bit-reversed order.

We often use, as a matter of convenience, the butterfly shown in Fig. 3.20 as a computational block instead of that shown in Fig. 3.17.

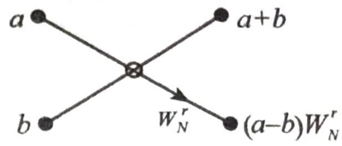


Fig. 3.20 A modified butterfly for DIF-FFT.

Example 3.55 Find the 8-point DFT of a real sequence

$$x(n) = (1, 2, 2, 2, 1, 0, 0, 0)$$

using decimation-in-frequency FFT algorithm.

☐ Solution

The scaling factors are as follows:

$$W_8^0 = 1,\quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}},\quad W_8^2 = -j,\quad W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

The basic computational block or the butterfly diagram used is as shown in Fig. Ex.3.55(a).

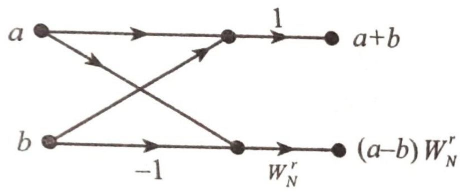


Fig. Ex.3.55(a) Preferred butterfly.

The flow diagram for $N = 8$ is shown in Fig. Ex.3.55(b).

To compute the output of the first stage:

$$\begin{aligned}
X_1(0) &= 1 + 1 = 2\\
X_1(4) &= 2 + 0 = 2\\
X_1(2) &= 2 + 0 = 2\\
X_1(6) &= 2 + 0 = 2\\
X_1(1) &= (1 - 1)\ W_8^0 = 0\\
X_1(5) &= (2 - 0)\ W_8^1 = 2\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] = \sqrt{2} - j\sqrt{2}\\
X_1(3) &= (2 - 0)\ W_8^2 = 2(-j) = -j2\\
X_1(7) &= (2 - 0)\ W_8^3 = 2\left[-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] = -\sqrt{2} - j\sqrt{2}
\end{aligned}$$

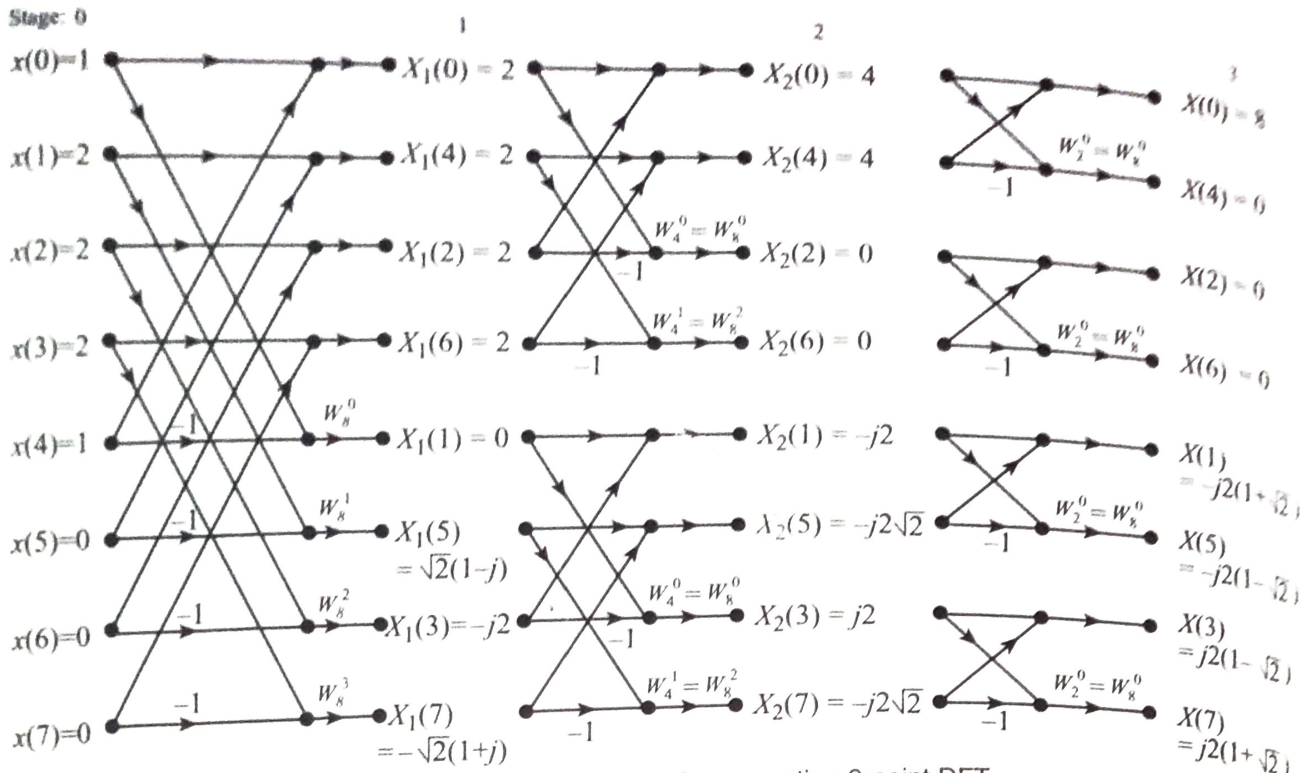


Fig. Ex.3.55(b) Flow graph for computing 8-point DFT.

To compute the output of the second stage:

$$\begin{aligned}
X_2(0) &= 2+2=4 \\
X_2(4) &= 2+2=4 \\
X_2(2) &= (2-2)W_8^0=0 \\
X_2(6) &= (2-2)W_8^0=0 \\
X_2(1) &= (0-j2)=-j2 \\
X_2(5) &= \sqrt{2}(1-j)-\sqrt{2}(1+j)=-j2\sqrt{2} \\
X_2(3) &= (0+j2)W_8^0=j2 \\
X_2(7) &= \left[\sqrt{2}(1-j)+\sqrt{2}(1+j)\right]W_8^2=-j2\sqrt{2}
\end{aligned}$$

To compute the output of third stage:

$$\begin{aligned}
X_3(0) = X(0) &= 4+4=8 \\
X_3(4) = X(4) &= (4-4)W_8^0=0 \\
X_3(2) = X(2) &= 0+0=0 \\
X_3(6) = X(6) &= (0-0)W_8^0=0 \\
X_3(1) = X(1) &= -j2-j2\sqrt{2}=-j2\left(1+\sqrt{2}\right) \\
X_3(5) = X(5) &= \left(-j2+j2\sqrt{2}\right)W_8^0=-j2\left(1-\sqrt{2}\right) \\
X_3(3) = X(3) &= j2-j2\sqrt{2}=j2\left(1-\sqrt{2}\right) \\
X_3(7) = X(7) &= \left(j2+j\sqrt{2}\right)W_8^0=j2\left(1+\sqrt{2}\right)
\end{aligned}$$

Tabulating the result in the normal order, we get

k	$X(k)$	k	$X(k)$
0	8	4	0
1	$-j2(1+\sqrt{2})$	5	$-j2(1-\sqrt{2})$
2	0	6	0
3	$j2(1-\sqrt{2})$	7	$j2(1+\sqrt{2})$

Since $x(n)$ is a real sequence, the symmetry condition, $X(k) = X^*(8-k)$ is observed.

Example 3.56 Find the convolution of $x(n)$ and $h(n)$ shown in Fig. Ex.3.56 below, using

a. the time-domain convolution operation,

b. the DFT and zero-padding, and

c. the radix-2 FFT and zero-padding.

Fig. Ex.3.56 Sequences $x(n)$ and $h(n)$ for Example.3.56.

☐ Solution

From Fig. Ex.3.56, we can write

$$x(n) = (\underset{\substack{\uparrow \\ n=0}}{1}, 2, 1) \quad \text{and} \quad h(n) = (\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3)$$

a.

$$\begin{aligned}
x(n) &= \delta(n) + 2\delta(n-1) + \delta(n-2) \\
\text{and} \quad h(n) &= \delta(n) + 2\delta(n-1) + 3\delta(n-2) \\
\text{Let} \quad y(n) &= x(n) * h(n) \\
\Rightarrow \quad y(n) &= [\delta(n) + 2\delta(n-1) + \delta(n-2)] \\
&\quad *[\delta(n) + 2\delta(n-1) + 3\delta(n-2)] \\
\Rightarrow \quad y(n) &= \delta(n) + 4\delta(n-1) + 8\delta(n-2) + 8\delta(n-3) + 3\delta(n-4) \\
\Rightarrow \quad y(n) &= (1, 4, 8, 8, 3)
\end{aligned}$$

if $\odot$ lar convolution aliasing takes place

b. The DFT requires both the sequences to be zero-padded to the convolution length:

$$N = N_1 + N_2 - 1 = 3 + 3 - 1 = 5$$

So, $$x_1(n) = (1, 2, 1, 0, 0)$$

and $$h_1(n) = (1, 2, 3, 0, 0)$$

$$\begin{aligned}
\text{DFT}\{x_1(n)\} = X_1(k) &= \sum_{n=0}^{4}[1 + 2\delta(n-1) + \delta(n-2)]W_5^{kn} \\
&= W_5^{kn}\big|_{n=0} + 2W_5^{kn}\big|_{n=1} + W_5^{kn}\big|_{n=2} \\
&= 1 + 2W_5^k + W_5^{2k}
\end{aligned}$$

$$\begin{aligned}
\text{DFT}\{h_1(n)\} = H_1(k) &= \sum_{n=0}^{4}[\delta(n) + 2\delta(n-1) + 3\delta(n-2)]W_5^{kn} \\
&= W_5^{kn}\big|_{n=0} + 2W_5^{kn}\big|_{n=1} + 3W_5^{kn}\big|_{n=2} \\
&= 1 + 2W_5^k + 3W_5^{2k}
\end{aligned}$$

Let, $$y(n) = x_1(n) \circledast_5 h_1(n)$$

$$\begin{aligned}
\Rightarrow \quad Y(k) &= X_1(k)\, H_1(k) \\
\Rightarrow \quad Y(k) &= [1 + 2W_5^k + W_5^{2k}][1 + 2W_5^k + 3W_5^{2k}] \\
&= 1 + 2W_5^k + 3W_5^{2k} + 2W_5^k + 4W_5^{2k} + 6W_5^{3k} \\
&\quad + W_5^{2k} + 2W_5^{3k} + 3W_5^{4k} \\
\Rightarrow \quad Y(k) &= 1 + 4W_5^k + 8W_5^{2k} + 8W_5^{3k} + 3W_5^{4k}
\end{aligned}$$

Taking IDFT yields $y(n) = (1, 4, 8, 8, 3)$, as before.

c. The radix-2 FFT requires both the sequences to be zero-padded to $N = 8$.

So, $$x_2(n) = (1, 2, 1, 0, 0, 0, 0, 0)$$
$$h_2(n) = (1, 2, 3, 0, 0, 0, 0, 0)$$

Let us employ DIF-FFT algorithm to compute $X_2(k)$ and $H_2(k)$. The basic computational block or the butterfly is as shown in Fig. Ex.3.56(a).

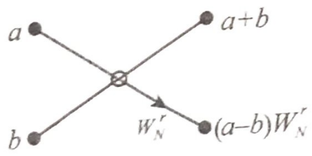


Fig. Ex.3.56(a) Preferred butterfly.

It may be recalled:

(i) $W_8^0 = 1,\ W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}},\ W_8^2 = -j,\ W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

(ii) $W_4^0 = W_2^0 = W_8^0 = 1, \quad W_4^1 = W_8^2 = -j$

To find $X_2(k)$:

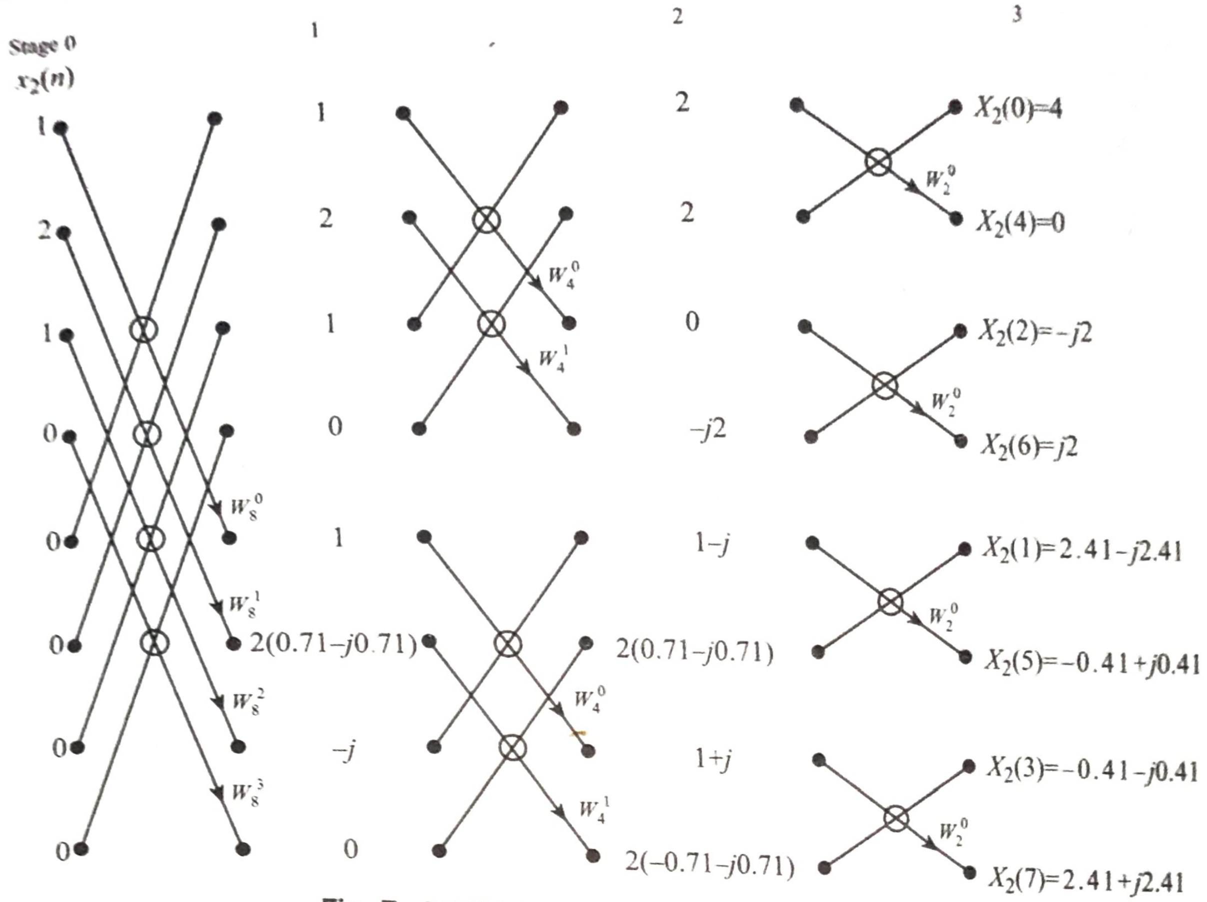


Fig. Ex.3.56(b) Flow diagram for computing 8-point DFT.

Tabulating the results in the normal order, we get

k	$X_2(k)$	k	$X_2(k)$
0	4	4	0
1	$2.41 - j2.41$	5	$-0.41 + j0.41$
2	$-j2$	6	$j2$
3	$-0.41 - j0.41$	7	$2.41 + j2.41$

Since $x_2(n)$ is a real sequence, we find that the symmetry property: $X(k) = X^*(N-k)$ is observed. Furthermore, since $x_2(n)$ is neither odd nor even, we find that $X_2(k)$ is neither purely imaginary nor purely real. It may be noted that the suffix of $X_2(k)$ in the output of stage 3 has nothing to do with the stage.

To find $H_2(k)$

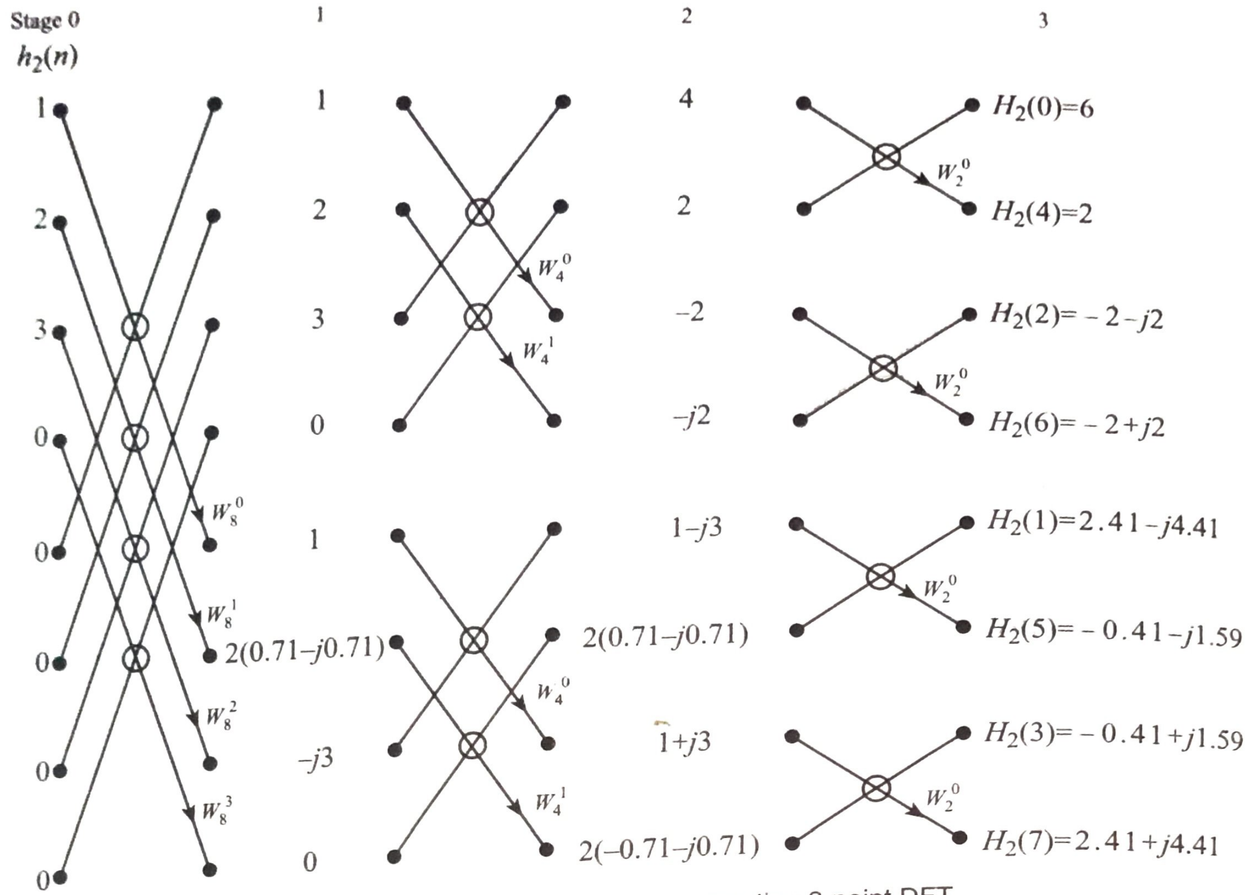


Fig. Ex.3.56(c) Flow diagram for computing 8-point DFT.

Tabulating the results in the normal order, we get

k	$H_2(k)$	k	$H_2(k)$
0	6	4	2
1	$2.41 - j4.41$	5	$-0.41 - j1.59$
2	$-2 - j2$	6	$-2 + j2$
3	$-0.41 + j1.59$	7	$2.41 + j4.41$

Since $h_2(n)$ is a real sequence, the symmetry property: $H_2(k) = H_2^*(8 - k)$ is observed. In addition, since $h_2(n)$ is neither odd nor even, we find that $H_2(k)$ is neither purely imaginary nor purely real. It may be noted that the suffix of $H_2(k)$ in the output of stage 3 has nothing to do with the stage.

To find $Y(k) = X_2(k)\ H_2(k)$:

k	$X_2(k)$	$H_2(k)$	$Y(k) = X_2(k)\ H_2(k)$
0	4	6	24
1	$2.41 - j2.41$	$2.41 - j4.41$	$-4.82 - j16.49$
2	$-j2$	$-2 - j2$	$-4 + j4$
3	$-0.41 - j0.41$	$-0.41 + j1.59$	$0.82 - j0.49$
4	0	2	0
5	$-0.41 + j0.41$	$-0.41 - j1.59$	$0.82 + j0.49$
6	$j2$	$-2 + j2$	$-4 - j4$
7	$2.41 + j2.41$	$2.41 + j4.41$	$-4.82 + j16.49$

To find $y(n)$:

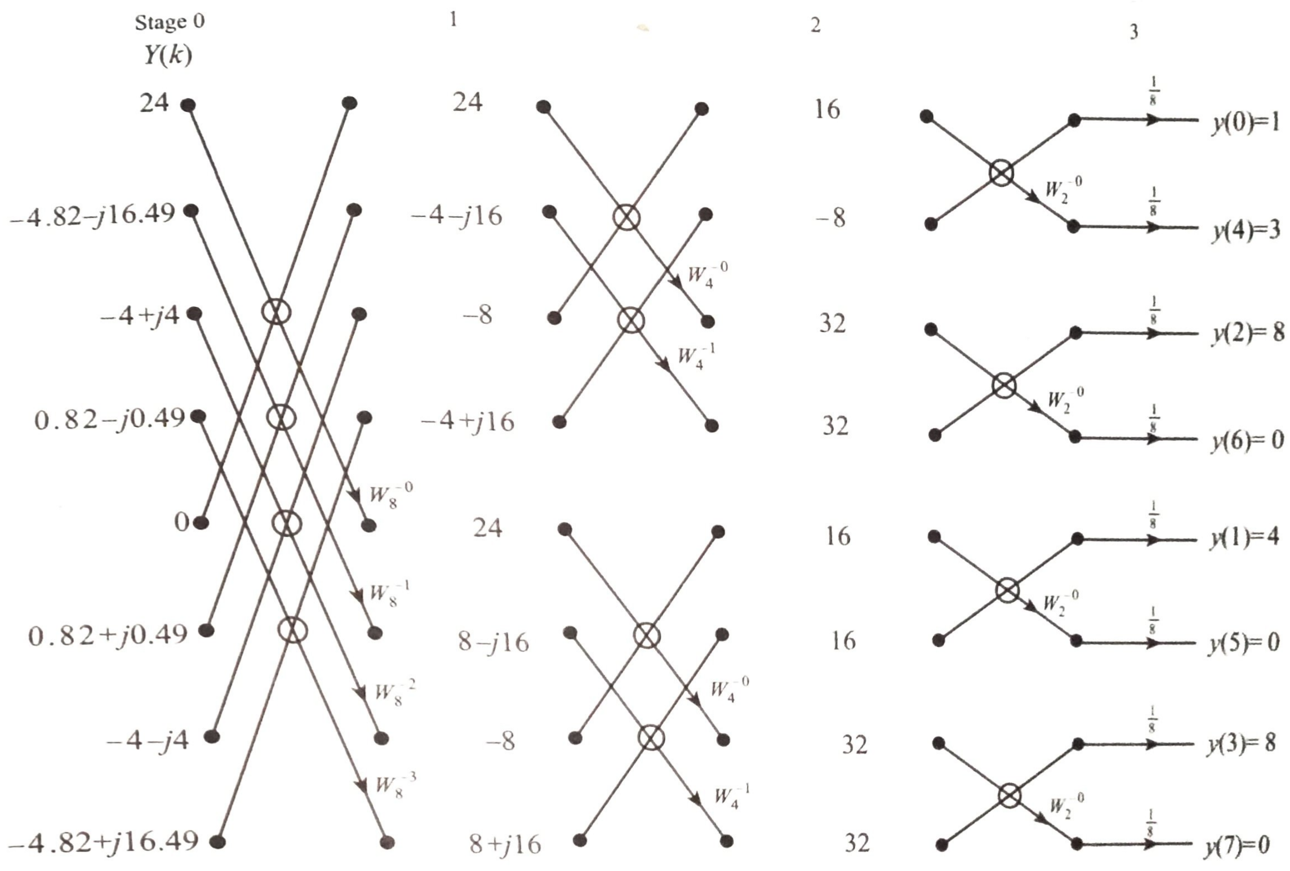


Fig. Ex.3.56(d) Flow diagram for computing 8-point IDFT.

Let

$$y(n) = x_2(n) \circledast h_2(n)$$

$$\Rightarrow \quad y(n) = \text{IDFT}\{X_2(k)\ H_2(k)\}$$

Let us employ DIF-FFT algorithm to find $y(n)$. The scaling factors are as follows:

$$\begin{aligned}
W_8^{-0} &= 1 \\
W_8^{-1} &= (W_8^1)^* = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
W_8^{-2} &= (W_8^2)^* = j \\
W_8^{-3} &= (W_8^3)^* = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}
\end{aligned}$$

Also,

$$W_4^{-1} = W_8^{-2}$$

Refer Fig. Ex.3.56(d), we get

Output of stage 1

$$\begin{aligned}
y_1(0) &= 24 + 0 = 24 \\
y_1(4) &= (-4.82 - j16.49) + 0.82 + j0.49 = -4 - j16 \\
y_1(2) &= (-4 + j4) + (-4 - j4) = -8 \\
y_1(6) &= (0.82 - j0.49) + (-4.82 + j16.49) = -4 + j16 \\
y_1(1) &= (24 - 0)W_8^{-0} = 24 \\
y_1(5) &= [(-4.82 - j16.49) - (0.82 + j0.49)]\, W_8^{-1} = 8 - j16 \\
y_1(3) &= [(-4 + j4) - (-4 - j4)]\, W_8^{-2} = -8 \\
y_1(7) &= [(0.82 - j0.49) - (-4.82 + j16.49)]\, W_8^{-3} = 8 + j16
\end{aligned}$$

Output of stage 2 The final result only is given. The calculations are left as an exercise to the reader.

$$\begin{aligned}
y_2(0) &= 16 & y_2(1) &= 16 \\
y_2(4) &= -8 & y_2(5) &= 16 \\
y_2(2) &= 32 & y_2(3) &= 32 \\
y_2(6) &= 32 & y_2(7) &= 32
\end{aligned}$$

Output of stage 3

$$\begin{aligned}
y(0) &= y_3(0) = (16 - 8)\frac{1}{8} = 1 \\
y(4) &= y_3(4) = (16 + 8)W_2^{-0} \times \frac{1}{8} = 3 \\
y(2) &= y_3(2) = (32 + 32)\frac{1}{8} = 8 \\
y(6) &= y_3(6) = (32 - 32)W_2^{-0} \times \frac{1}{8} = 0 \\
y(1) &= y_3(1) = (16 + 16)\frac{1}{8} = 4 \\
y(5) &= y_3(5) = (16 - 16)W_2^{-0} \times \frac{1}{8} = 0 \\
y(3) &= y_3(3) = (32 + 32)\frac{1}{8} = 8 \\
y(7) &= y_3(7) = (32 - 32)W_2^{-0} \times \frac{1}{8} = 0
\end{aligned}$$

That is,

$$y(n) = (1, 4, 8, 8, 3, 0, 0, 0)$$

Note that there are three trailing zeros.

Example 3.57 Find the 4-point circular convolution of $x(n)$ and $h(n)$ given in Fig. Ex.3.57 using radix-2 DIF-FFT algorithm.

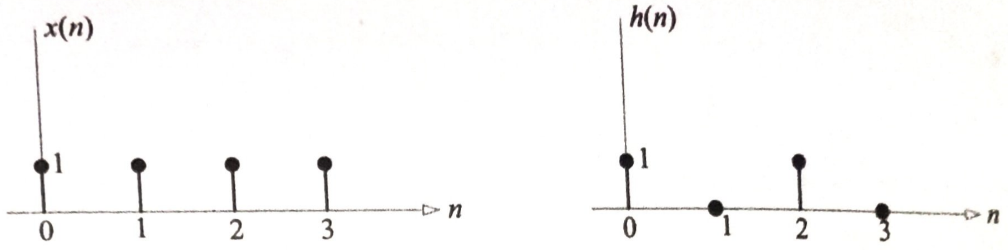


Fig. Ex.3.57 Sequences $x(n)$ and $h(n)$ for Example.3.57.

Solution

Let

$$y(n) = x(n) \circledast_4 h(n)$$
$$= \text{IFFT}\{X(k)H(k)\}$$

To find $X(k)$:

Let the computational block or the butterfly used be of the form as shown in Fig. Ex.3.57(a).

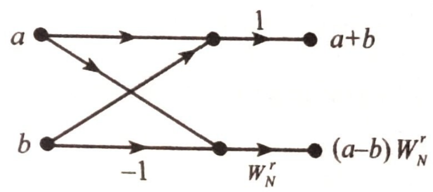


Fig. Ex.3.57(a) Preferred butterfly.

The flow diagram for $N = 4$, DIF-FFT algorithm is shown in Fig. Ex.3.57(b).

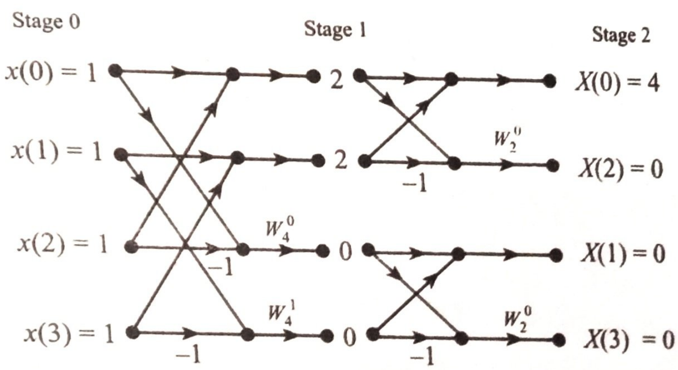


Fig. Ex.3.57(b) Flow diagram for computing 4-point DFT.

Recall: $W_4^0 = 1$, $W_4^1 = -j$ and $W_2^0 = 1$.

The result is tabulated in the normal order as given below:

k	$X(k)$	k	$X(k)$
0	4	2	0
1	0	3	0

or equivalently $X(k) = (4, 0, 0, 0)$.

To find $H(k)$

The flow diagram for $N = 4$ to compute $H(k)$, using DIF-FFT algorithm is shown in Fig. Ex.3.57(c).

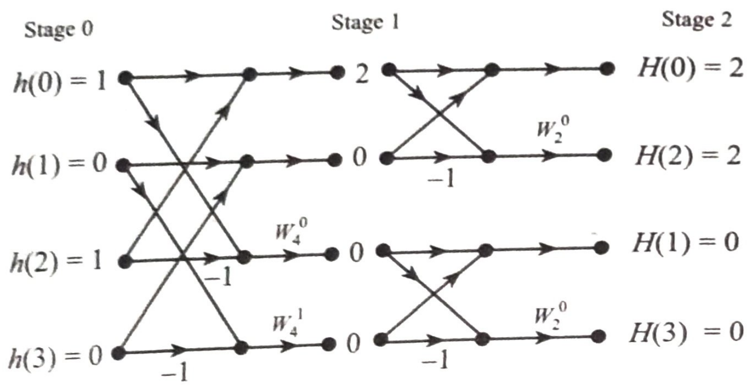


Fig. Ex.3.57(c) Flow diagram for computing 4-point DFT.

The result is tabulated in the normal order as given below.

k	$H(k)$	k	$H(k)$
0	2	2	2
1	0	3	0

or equivalently $H(k) = (2, 0, 2, 0)$.

To find $y(n)$

$$Y(k) = X(k)\, H(k) = (8, 0, 0, 0)$$

IFFT of $Y(k)$ is found using the flow diagram given in Fig. Ex.3.57(d).

Recall: $W_4^{-0} = (W_4^0)^* = 1$

$W_4^{-1} = (W_4^1)^* = j$

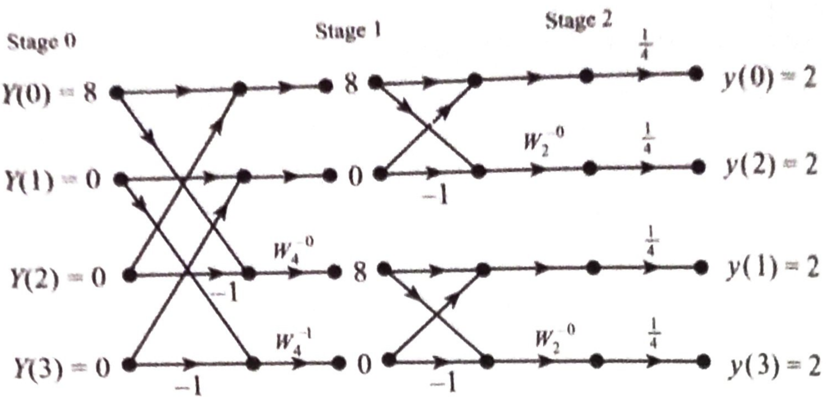


Fig. Ex.3.57(d) Flow diagram for computing 4-point IDFT.

Thus,
$$y(n) = (2, 2, 2, 2)$$
$$\underset{n=0}{\uparrow}$$

Example 3.58 Find the periodic convolution of $x(n)$ and $h(n)$ shown in Fig. Ex.3.58, using

a. the time-domain convolution operation,

b. the DFT operation. Is this result same as that of part (a)? and

c. the radix-2 FFT and zero-padding. Is this result same as that of part (a)? should it be?

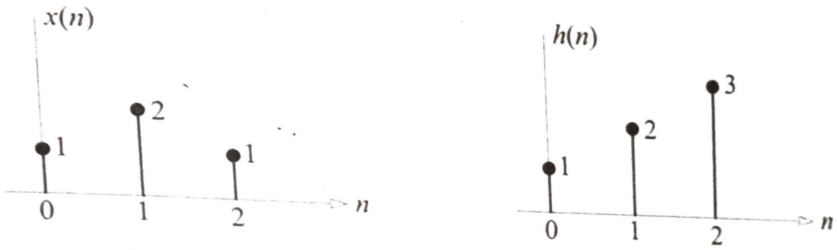


Fig. Ex.3.58 Sequences $x(n)$ and $h(n)$ for Example.3.58.

☐ Solution

a. Let
$$y_l(n) = x(n) * h(n)$$
$$\Rightarrow \quad y_l(n) = [\delta(n) + 2\delta(n-1) + \delta(n-2)] * [\delta(n) + 2\delta(n-1) + 3\delta(n-2)]$$

Recall the property:
$$\delta(n-\alpha) * \delta(n-\beta) = \delta[n-(\alpha+\beta)]$$

Hence,
$$y_l(n) = \delta(n) + 4\delta(n-1) + 8\delta(n-2) + 8\delta(n-3) + 3\delta(n-4)$$
$$\Rightarrow \quad y_l(n) = (1, 4, 8, 8, 3)$$

Since, the circular convolution of $x(n)$ and $h(n)$, namely $y_c(n)$ should have 3 samples, the aliasing takes place in $y_l(n)$. That is, 5 samples of $y_l(n)$ get reduced to 3 samples in $y_c(n)$. By wraparound, we have

$$\begin{aligned} y_c(0) &= y_l(0) + y_l(3) = 1 + 8 = 9 \\ y_c(1) &= y_l(1) + y_l(4) = 4 + 3 = 7 \\ y_c(2) &= y_l(2) = 8 \end{aligned}$$

Thus,
$$y_c(n) = x(n) \circledast_3 y(n) = (9, 7, 8)$$

b. Let
$$\begin{aligned} y_c(n) &= x(n) \circledast_3 h(n) \\ \Rightarrow \quad Y_c(k) &= X(k)\, H(k), \qquad 0 \leqslant k \leqslant 2 \\ &= \left[1 + 2W_3^k + W_3^{2k}\right]\left[1 + 2W_3^k + 3W_3^{2k}\right] \\ &= 1 + 2W_3^k + 3W_3^{2k} + 2W_3^k + 4W_3^{2k} + 6W_3^{3k} \\ &\quad + W_3^{2k} + 2W_3^{3k} + 3W_3^{4k} \end{aligned}$$

Recall the property: $\quad W_3^{3k} = W_3^{0k} \quad$ and $\quad W_3^{4k} = W_3^k$

Then, $\quad Y_c(k) = 9 + 7W_3^k + 8W_3^{2k}$

Taking 3-point IDFT, we get
$$y_c(n) = (9, 7, 8)$$

c. To compute $X(k)$ and $H(k)$ using DIF-FFT algorithm, let
$$\begin{aligned} x(n) &= (1, 2, 1, 0) \\ h(n) &= (1, 2, 3, 0) \end{aligned}$$
and

FACTS:

- Since $h(n)$ and $x(n)$ are both real, their DFTs will observe the symmetry condition:
$$\begin{aligned} H(k) &= H^*(N-k) \\ X(k) &= X^*(N-k) \end{aligned}$$
and
- Since $x(n)$ and $h(n)$ are neither odd nor even their DFTs will neither be purely imaginary nor purely real.

To compute $X(k)$:
The butterfly diagram used is as shown in Fig. Ex.3.58(a).

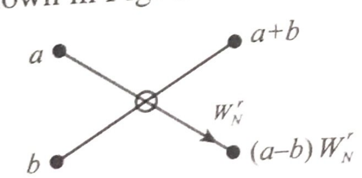


Fig. Ex.3.58(a) Preferred butterfly.

The flow diagram for 4-point DIF-FFT is shown in Fig. Ex.5.58(b).

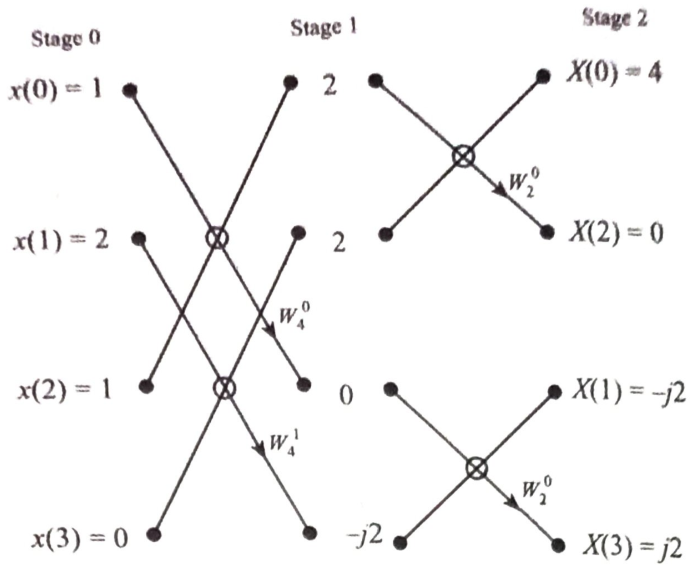


Fig. Ex.3.58(b) Flow diagram for computing 4-point DFT.

Hence, $$X(k) = (4, -2j, 0, 2j)$$

To find $H(k)$:

The 4-point DFT of $h(n)$ is found using the flow diagram as shown in Fig. Ex.3.58(c).

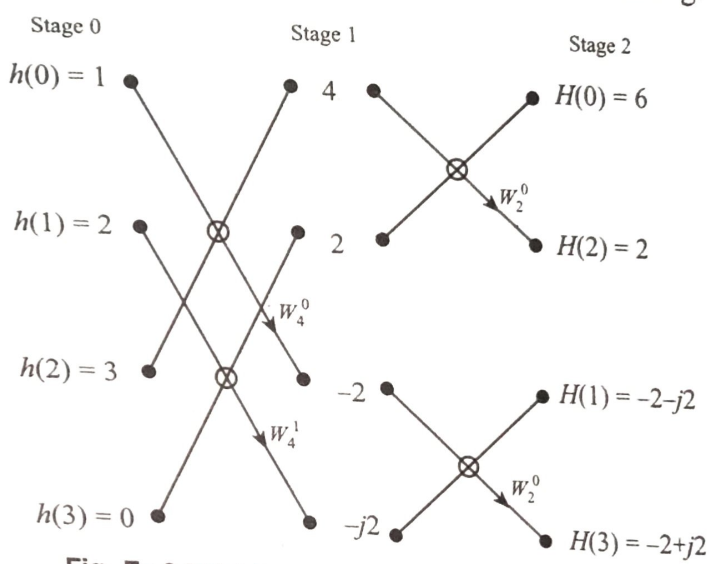


Fig. Ex.3.58(c) Flow diagram for computing 4-point DFT.

Thus, $$H(k) = (6, -2-2j, 2, -2+2j)$$

To find $y_c(n)$:

$$y_c(n) = x(n) \circledast_4 h(n)$$

In this case, the 4-point circular convolution of $x(n)$ and $h(n)$ is found using the Stockham's method. That is,

$$y_c(n) = \text{IFFT}\{X(k)\,H(k)\} = \text{IFFT}\{Y_c(k)\}$$

where

$$\begin{aligned} Y_c(k) &= X(k)\,H(k) \\ &= (24, -4+j4, 0, -4-j4) \end{aligned}$$

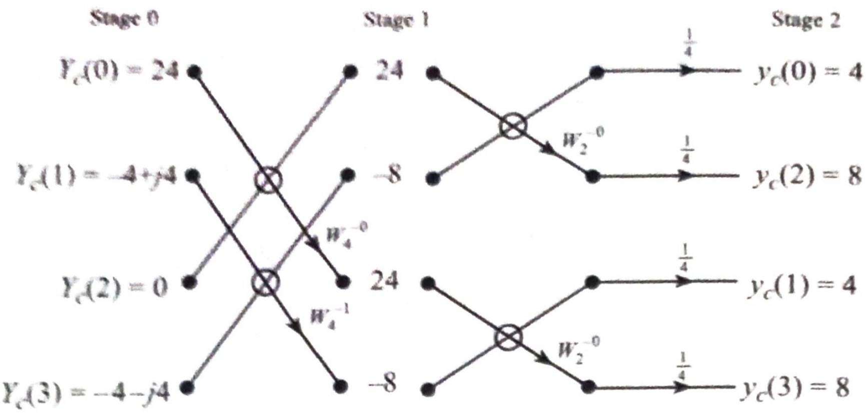


Fig. Ex.3.58(d) Flow diagram for computing 4-point IDFT.

Hence,
$$y_c(n) = (\underset{\substack{\uparrow\\ n=0}}{4}, 4, 8, 8)$$

This result does not match with part (a) and part (b). The reason is that the sequences $x(n)$ and $h(n)$ are assumed to be periodic with a period 4 (not 3).

Example 3.59 Find the 4-point DFT of the sequence, $x(n) = \cos\left(\frac{\pi}{4}n\right)$ using DIF-FFT algorithm.

☐ Solution

The scale factors are $W_4^0 = 1$ and $W_4^1 = -j$.

We find that
$$x(n) = \left(\underset{\substack{\uparrow\\ n=0}}{1}, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

FACTS:

- Since $x(n)$ is a real sequence, the symmetry condition: $X(k) = X^*(N-k)$ will be observed.
- Also, $X(k)$ will not be purely real or purely imaginary.

Let the butterfly diagram be as shown in Fig. Ex.3.59(a).

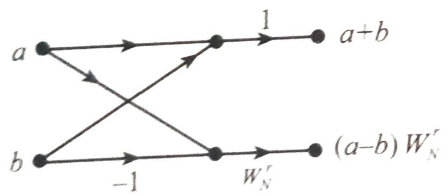


Fig. Ex.3.59(a) Preferred butterfly.

The flow diagram for DIF-FFT algorithm is shown in Fig. Ex.3.59(b).

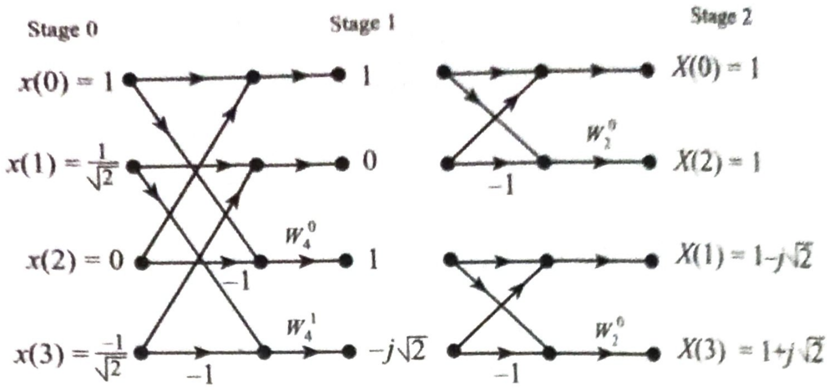


Fig. Ex.3.59(b) Flow diagram for computing 4-point DFT.

Thus, $$X(k) = \left(1, 1 - j\sqrt{2}, 1, 1 + j\sqrt{2}\right)$$

Example 3.60 Find the 4-point real sequence $x(n)$ if its 4-point DFT samples are $X(0) = 6$, $X(1) = -2 + j2$, $X(2) = -2$. Use DIF-FFT algorithm.

☐ Solution

Since $x(n)$ is a real sequence, it has to satisfy the following symmetry condition:

$$X(k) = X^*(4-k)$$

$$\Rightarrow \quad X(3) = X^*(1) = -2 - j2$$

Hence, $$X(k) = (6, -2 + j2, -2, -2 - j2)$$

Since $X(k)$ is neither purely real nor purely imaginary, its inverse DFT, $x(n)$ will neither be even nor odd.

The sample butterfly used in the flow diagram is shown in Fig. Ex.3.60(a).

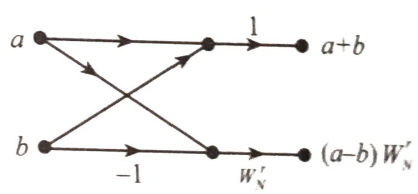


Fig. Ex.3.60(a) Preferred butterfly.

The flow diagram for the computation of 4-point IFFT of $X(k)$ is shown in Fig. Ex.3.60(b).

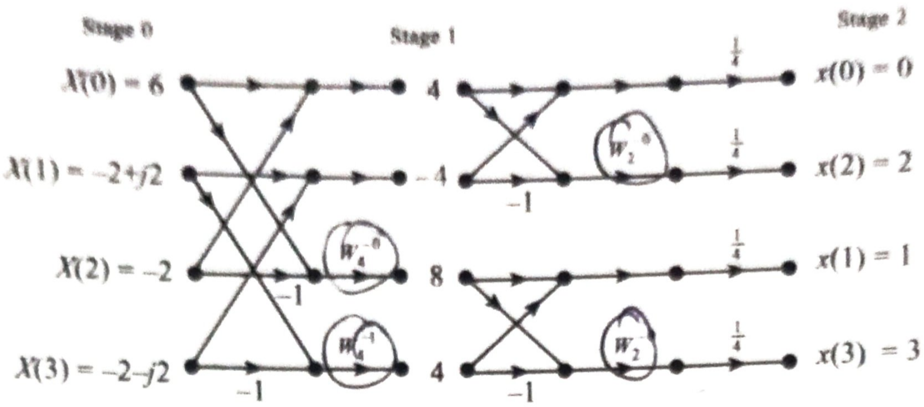


Fig. Ex.3.60(b) Flow diagram for computing 4-point IDFT.

Hence,

$$x(n) = (0, 1, 2, 3)$$
$$\uparrow_{n=0}$$

3.13 Signal Segmentation (Linear filtering of long data sequence using DFT/FFT)

In this section, we deal with a practical difficulty which often arises, when linear convolution is performed digitally by fast convolution. Our discussion till now assumes that $x(n)$ and $h(n)$ contain roughly the same number of samples, and that they are zero-filled up to the same length of transform. However, in practice we often encounter a situation in which we need to convolve a long input signal with a relatively short impulse response. For example, we may have a signal with several hundreds of samples, but an impulse response with less than fifty samples. In such cases, it is uneconomical in terms of computing time (and storage) to use the same length of transform for both $x(n)$ and $h(n)$. In addition, in real-time applications the use of very long transform lengths for $x(n)$ may give rise to an unacceptable processing delay.

The above mentioned problems may be overcome by segmenting the input signal into sections of manageable length and then performing fast convolution on each section and finally combining the outputs. Two well-known techniques are commonly used: the *overlap-add method* and the *overlap- save method.*

3.13.1 Overlap-add method

Let us consider two sequences, $x(n)$ and $h(n)$ having lengths $K \times N$ and M respectively with K being a very large integer. The sequence $x(n)$ is subdivided into non-overlapping sections each of length N as shown in Fig. 3.20. The objective is to find $y(n) = x(n) * h(n)$.

Procedure for overlap-add method

Step 1: The number of samples M of the system's impulse response $h(n)$ is known, so we decide upon L, the number of points for which the DFTs are to be computed. We choose $L = 2^{m}$ (a power of 2), so that a radix-2 FFT can be used. The sequence $h(n)$ is zero-padded so that last $(N - 1)$ values are zeros. The zero-padded impulse response is denoted as $h'(n)$ in the analysis to follow.

Step 2: The input sequence $x(n)$ is sectioned into blocks $x_1(n), x_2(n) \cdots$ of lengths N such that $M + N - 1 = L$, so that the effect of linear convolution will be achieved. Notice that the