DIGITAL IIR FILTERS

1 What are the conditions to be satisfied while transforming transfer function of an analog filter into transfer function of a digital filter?

1. Left half of s-plane should map into region inside unit circle in z-plane so that a stable analog filter will be converted into a stable digital filter.

2. The $j\Omega$ axis in s-plane should map into the unit circle in z-plane so that there is a direct relationship between the two frequency variables in the two domains.

2 Explain the impulse invariant transformation method of transforming an analog filter to digital filter. (2)

Let $H(s)$ be the system function of the analog filter.

Let $$H(s) = \sum_{K=1}^{N} \frac{b_K}{s - P_K} \quad \text{---} \quad (1)$$

where $\{P_K\}$ are the poles of the analog filter and $\{b_K\}$ are the coefficients in the partial fraction expansion.

Taking inverse Laplace transform of (1), we get,

$$h(t) = \sum_{K=1}^{N} b_K e^{P_K t}, \quad t \geq 0 \quad \text{---} \quad (2)$$

If we sample h(t) periodically at $t = nT_s$, we get

$$h(n) = h(t)\Big|_{t=nT_s}$$

$$= \sum_{K=1}^{N} b_K e^{P_K n T_s} \quad \text{---} \quad (3)$$

Taking z-transform of $h(n)$, we get

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sum_{K=1}^{N} b_K e^{P_K n T_s} z^{-n}$$

$$= \sum_{K=1}^{N} b_K \cdot \sum_{n=0}^{\infty} e^{P_K n T_s} z^{-n}$$

$$= \sum_{K=1}^{N} b_K \cdot \sum_{n=0}^{\infty} \left(e^{P_K T_s} z^{-1}\right)^n$$

$$= \sum_{K=1}^{N} b_K \cdot \frac{1}{1 - e^{P_K T_s} z^{-1}} \quad \text{---(4)}$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

for $|\alpha| < 1$

Comparing (1) & (4), we get

$$\frac{1}{S - P_K} \longrightarrow \frac{1}{1 - e^{P_K T_s} z^{-1}}$$

From (1), the poles of analog filter are at $S = P_K$. --- (5)

From (4), the poles of digital filter are at

$$z = e^{P_K T_s} \quad \text{--- (6)}$$

$$\therefore \quad z = e^{S T_s} \quad \text{--- (7)} \quad \text{using (5)}$$

But, $z = r e^{jw}$ and $S = \sigma + j\Omega$

$$\therefore \quad r e^{jw} = e^{\sigma T_s} e^{j\Omega T_s} \quad \text{from (7)}$$

$$\therefore \quad r = e^{\sigma T_s} \quad \text{--- (8)}$$

$\omega = \Omega T_s \quad \cdots \quad (9)$

From (8),

when $\sigma < 0$, $r < 1$.

ie, left half of s-plane is mapped inside the unit circle in the z-plane.

when $\sigma = 0$, $r = 1$.

ie, imaginary axis in s-plane is mapped onto the unit circle in z-plane.

when $\sigma > 0$, $r > 1$.

ie, right half of s-plane is mapped outside the unit circle.

These are the desirable properties of impulse invariance method.

Now, consider (9).

(9) gives the relationship between digital frequency 'ω' and analog frequency 'Ω'.

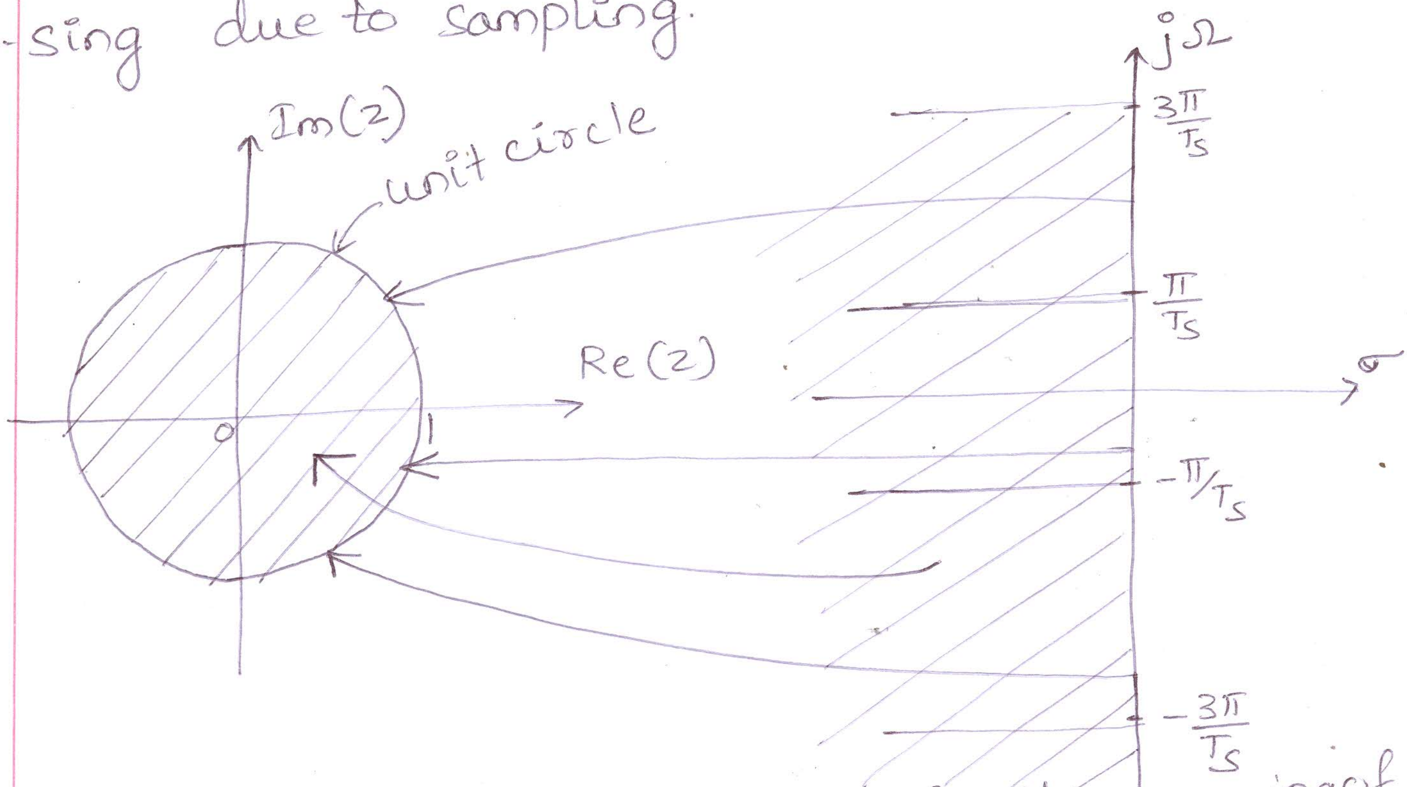
Since ω is unique in the range $(-\pi, \pi)$, the mapping $\omega = \Omega T_s$ implies that, the interval $\frac{-\pi}{T_s} \le \Omega \le \pi/T_s$ maps into the range, $-\pi \le \omega \le \pi$.

Further, $\frac{\pi}{T_s} \le \Omega \le \frac{3\pi}{T_s}$ also maps into the interval, $-\pi \le \omega \le \pi$. and, in general, (6)

so does the interval $(2k-1)\frac{\pi}{T_s} \le \Omega \le (2k+1)\frac{\pi}{T_s}$

where k is an integer.

Thus, the mapping from analog frequency 'Ω' to digital frequency 'ω' is many to one, which reflects the effect of aliasing due to sampling.



This method is suitable only for the mapping of LPFs and low band pass filters.

3 Explain Bilinear Transformation method. Derive an expression showing mapping from s-plane to z-plane. Show that there is no aliasing effect in BLT.

Consider an analog filter with system

function,

$$H(s) = \frac{b}{s+a} \quad \text{--- (1)}$$

$$\therefore \quad \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$(s+a)\,Y(s) = b\,X(s)$$

Taking inverse Laplace transform, we get,

$$\frac{d}{dt}y(t) + a\,y(t) = b\,x(t) \quad \text{--- (2)}$$

Let us integrate (2) from $nT_s - T_s$ to nT_s,

$$\int_{nT_s-T_s}^{nT_s} \frac{d}{dt}y(t)\,dt + a\int_{nT_s-T_s}^{nT_s} y(t)\,dt = b\int_{nT_s-T_s}^{nT_s} x(t)\,dt \quad \text{--- (3)}$$

According to Trapezoidal rule of integration,

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

Using this, equation (3) can be simplified as,

$$y(t)\Big|_{nT_s-T_s}^{nT_s} + a\cdot\frac{T_s}{2}\left[y(nT_s)+y(nT_s-T_s)\right] = \frac{bT_s}{2}\left[x(nT_s)+x(nT_s-T_s)\right]$$

$$y(n)-y(n-1) + \frac{aT_s}{2}\left[y(n)+y(n-1)\right] = \frac{bT_s}{2}\left[x(n)+x(n-1)\right]$$

$$\left(1+\frac{aT_s}{2}\right)y(n) - \left(1-\frac{aT_s}{2}\right)y(n-1)$$

$$= \frac{bT_s}{2}\left[x(n)+x(n-1)\right]$$

Taking z- transform, we get,

$$\left(1+\frac{aT_s}{2}\right)Y(z) - \left(1-\frac{aT_s}{2}\right)z^{-1}Y(z) = \frac{bT_s}{2}\left[1+z^{-1}\right]X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{\frac{bT_s}{2}\left(1+z^{-1}\right)}{1+a\frac{T_s}{2} - \left(1-\frac{aT_s}{2}\right)z^{-1}}$$

$$= \frac{b\frac{T_s}{2}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)+a\frac{T_s}{2}\left(1+z^{-1}\right)}$$

Dividing both numerator and denominator by $\frac{T_s}{2}\left(1+z^{-1}\right)$, we get,

$$H(z) = \frac{b}{\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}+a} \quad \cdots\cdots (4)$$

Comparing (1) and (4), we get the mapping from s to z as,

$$s = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_s}\frac{z-1}{z+1} \quad \cdots (5)$$

Substituting $z = re^{j\omega}$ and $s = \sigma + j\Omega$,

we get,

$$\sigma + j\Omega = \frac{2}{T_s} \frac{re^{j\omega} - 1}{re^{j\omega} + 1}$$

$$= \frac{2}{T_s} \frac{re^{j\omega} - 1}{re^{j\omega} + 1} \times \frac{re^{-j\omega} + 1}{re^{-j\omega} + 1}$$

$$= \frac{2}{T_s} \cdot \frac{r^2 + re^{j\omega} - re^{-j\omega} - 1}{r^2 + re^{j\omega} + re^{-j\omega} + 1}$$

$$= \frac{2}{T_s} \frac{r^2 - 1 + j\,2r\sin(\omega)}{r^2 + 1 + 2r\cos(\omega)}$$

$$= \frac{2}{T_s} \left[\frac{r^2 - 1}{r^2 + 1 + 2r\cos(\omega)} + j \frac{2r\sin(\omega)}{r^2 + 1 + 2r\cos(\omega)} \right] \quad \text{----} (6)$$

$$\therefore \sigma = \frac{2}{T_s} \frac{r^2 - 1}{r^2 + 1 + 2r\cos(\omega)} \quad \text{---} (7)$$

$$\Omega = \frac{2}{T_s} \frac{2r\sin(\omega)}{r^2 + 1 + 2r\cos(\omega)} \quad \text{---} (8)$$

Consider (7)

when $r < 1$, $\sigma < 0$.

ie, left half of s-plane is mapped inside the unit circle.

when $r>1$, $\sigma>0$.
ie, right half of s-plane is mapped outside the unit circle.

when $r=1$, $\sigma=0$.
ie, $j\Omega$ axis in s-plane is mapped onto the unit circle in z-plane.

When $r=1$ and $\sigma=0$, (8) becomes

$$\Omega = \frac{2}{T_s} \frac{2\sin(\omega)}{2+2\cos(\omega)}$$

$$= \frac{2}{T_s} \cdot \frac{\sin(\omega)}{1+\cos(\omega)}$$

$$= \frac{2}{T_s} \frac{2\sin\left(\frac{\omega}{2}\right)\cos\left(\frac{\omega}{2}\right)}{2\cos^2\left(\frac{\omega}{2}\right)}$$

$$= \frac{2}{T_s} \tan\left(\frac{\omega}{2}\right) \quad \text{--- (9)}$$

OR

$$\omega = 2\tan^{-1}\left(\frac{\Omega T_s}{2}\right) \quad \text{--- (10)}$$

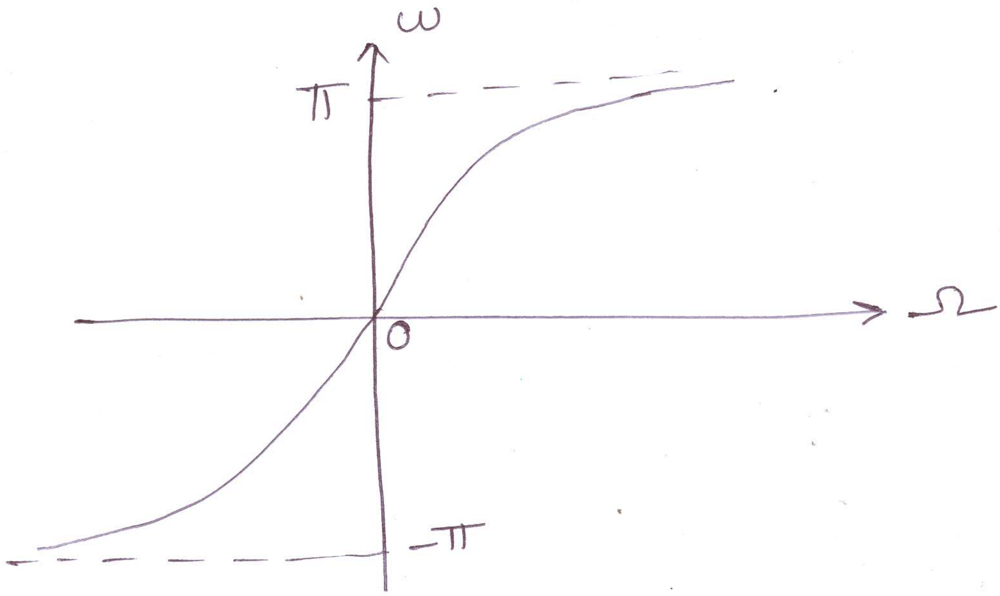
we observe, from (10), that the entire range of Ω is mapped only once into the range $-\pi \le \omega \le \pi$

Hence, there is no aliasing.

However, the mapping is highly non-linear.

We observe a frequency compression due to non-linearity of $\tan^{-1}$ function.

This is called frequency warping.



4 Given $H(s) = \dfrac{2}{(s+1)(s+2)}$, determine $H(z)$ using impulse invariant method. Take $T_s = 1$ sec

Let $\quad H(s) = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$$\therefore \quad A = \left. \frac{2}{s+2} \right|_{s=-1} = 2$$

$$B = \left. \frac{2}{s+1} \right|_{s=-2} = -2$$

$$\therefore \; H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

poles : $P_1 = -1$, $P_2 = -2$

$$\therefore \; H(z) = \frac{2}{1 - e^{P_1 T_s} z^{-1}} - \frac{2}{1 - e^{P_2 T_s} z^{-1}}$$

$$= \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$= \frac{2z}{z - 0.3678} - \frac{2z}{z - 0.1353}$$

$$= \frac{2z^2 - 0.2706z - 2z^2 + 0.7356z}{z^2 - 0.5031z + 0.0498}$$

$$= \frac{0.465z}{z^2 - 0.5031z + 0.0498}$$

5 Given $H(s) = \frac{s+2}{(s+1)(s+3)}$, find $H(z)$ using impul-se invariance method. Take $T_s = 1$ sec.

Let $H(s) = \frac{A}{s+1} + \frac{B}{s+3}$

$$A = \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2}$$

$$B = \left.\frac{s+2}{s+1}\right|_{s=-3} = \frac{1}{2}$$

$$\therefore\ H(s) = \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+3)}.$$

$$\therefore\ H(z) = \frac{1}{2}\frac{1}{1-e^{-1T_s}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-3T_s}z^{-1}}$$

$$= \frac{1}{2}\frac{1}{1-e^{-1}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-3}z^{-1}}$$

$$= \frac{1}{2}\left[\frac{1}{1-0.3678z^{-1}} + \frac{1}{1-0.0498z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-0.3678} + \frac{z}{z-0.0498}\right]$$

$$= \frac{1}{2}\left[\frac{z^2-0.0498z+z^2-0.3678z}{z^2-0.4176z+0.0183}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2-0.4176z}{z^2-0.4176z+0.0183}\right]$$

6 Convert the analog filter system function

$$H(s) = \frac{(s+a)}{(s+a)^2 + b^2}$$ into a digital

filter transfer function by impulse invariance method.

$$\frac{s}{s^2+b^2} \xrightarrow[\text{Transform}]{\text{Inverse Laplace}} \cos(bt)\,u(t)$$

$$\frac{(s+a)}{(s+a)^2+b^2} \xrightarrow[\text{Transform}]{\text{Inverse Laplace}} e^{-at}\cos(bt)u(t)$$

$$\therefore \quad h(t) = e^{-at}\cos(bt)\,u(t)$$

$$h(n) = h(t)\Big|_{t=nT_s}$$

$$= e^{-anT_s}\cos(nbT_s)\;u(nT_s)$$

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT_s}\cos(bnT_s)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT_s}\left[\frac{e^{jbnT_s} + e^{-jbnT_s}}{2}\right] z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} e^{-anT_s} \cdot e^{jbnT_s} z^{-n}$$

$$+ \frac{1}{2} \sum_{n=0}^{\infty} e^{-anT_s} e^{-jbnT_s} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-(aT_s - jbT_s)} z^{-1} \right)^n +$$

$$\frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-(aT_s + jbT_s)} z^{-1} \right)^n$$

$$= \frac{1}{2} \frac{1}{1 - e^{-(aT_s - jbT_s)} z^{-1}}$$

$$+ \frac{1}{2} \frac{1}{1 - e^{-(aT_s + jbT_s)} z^{-1}}$$

$$= \frac{1}{2} \left[\frac{1 - e^{-(aT_s + jbT_s)} z^{-1} + 1 - e^{-(aT_s - jbT_s)} z^{-1}}{\left(1 - e^{-(aT_s - jbT_s)} z^{-1}\right)\left(1 - e^{-(aT_s + jbT_s)} z^{-1}\right)} \right]$$

$$= \frac{1}{2}\left[\frac{2 - e^{-aT_s} z^{-1}\left(e^{-jbT_s} + e^{jbT_s}\right)}{1 - e^{-(aT_s+jbT_s)} z^{-1} - e^{-(aT_s-jbT_s)} z^{-1} + e^{-2aT_s} z^{-2}}\right]$$

$$= \frac{1}{2} \frac{2 - 2\cos(bT_s)\, e^{-aT_s} z^{-1}}{1 - e^{-aT_s} z^{-1}\left(e^{-jbT_s} + e^{jbT_s}\right) + e^{-2aT_s} z^{-2}}$$

$$= \frac{1 - \cos(bT_s)\, e^{-aT_s} z^{-1}}{1 - 2\cos(bT_s)\, e^{-aT_s} z^{-1} + e^{-2aT_s} z^{-2}}$$

7 Convert the analog filter transfer function $H(s) = \frac{b}{(s+a)^2 + b^2}$ into $H(z)$ using impulse invariance method.

$$H(s) = \frac{b}{(s+a)^2 + b^2}$$

Taking inverse Laplace transform, we get;

$$h(t) = e^{-at} \sin(bt)\, u(t)$$

$$h(n) = h(t)\Big|_{t=nT_s}$$

$$= e^{-anT_s} \sin(bnT_s)\, u(nT_s)$$

$$\therefore H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT_s} \left[\frac{e^{jbnT_s} - e^{-jbnT_s}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} e^{-(anT_s - jbnT_s)} z^{-n} - \frac{1}{2j} \sum_{n=0}^{\infty} e^{-(anT_s + jbnT_s)} z^{-n}$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{-(aT_s - jbT_s)} z^{-1}} - \frac{1}{1 - e^{-(aT_s + jbT_s)} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{1 - e^{-(aT_s + jbT_s)} z^{-1} - 1 + e^{-(aT_s - jbT_s)} z^{-1}}{1 - e^{-(aT_s + jbT_s)} z^{-1} - e^{-(aT_s - jbT_s)} z^{-1} + e^{-2aT_s} z^{-2}} \right]$$

$$= \frac{1}{2j} \left[\frac{\left(-e^{-jbT_s} + e^{jbT_s} \right) z^{-1} e^{-aT_s}}{1 - z^{-1} e^{-aT_s}\, 2\cos(bT_s) + z^{-2} e^{-2aT_s}} \right]$$

$$= \frac{1}{2j}\left[\frac{2j\sin(bT_s)z^{-1}e^{-aT_s}}{1-2e^{-aT_s}\cos(bT_s)z^{-1}+e^{-2aT_s}z^{-2}}\right]$$

$$= \frac{\sin(bT_s)\,z^{-1}\,e^{-aT_s}}{1-2e^{-aT_s}\cos(bT_s)z^{-1}+e^{-2aT_s}z^{-2}}$$

8. Convert $H(S) = \frac{S+0.1}{(S+0.1)^2+9}$ into $H(z)$ using impulse invariance method.

Comparing given $H(s)$ with $H(s)$ given in problem #13, we get

$a = 0.1, \quad b = 3.$

$$H(z) = \frac{1-\left[e^{-0.1T_s}+\cos(3T_s)\right]z^{-1}}{1-2\left[e^{-0.1T_s}+\cos(3T_s)\right]z^{-1}+e^{-0.2T_s}z^{-2}}$$

9. Convert the system function,

$$H(s) = \frac{2}{(S+1)(S+3)}$$

into $H(z)$ using Bilinear Transformation with $T_s = 0.1$ sec.

$$H(z) = H(s)\Big|_{s=\frac{2}{T_s}\frac{z-1}{z+1}}$$

(Equation (5), question #8)

$$\therefore \quad H(z) = \frac{2}{\left(\frac{2}{0.1}\frac{z-1}{z+1}+1\right)\left(\frac{2}{0.1}\frac{z-1}{z+1}+3\right)}$$

$$= \frac{2}{\left(20\,\frac{z-1}{z+1}+1\right)\left(20\,\frac{z-1}{z+1}+3\right)}$$

$$= \frac{2\,(z+1)^2}{\left(20(z-1)+z+1\right)\left(20(z-1)+3(z+1)\right)}$$

$$= \frac{2\,(z+1)^2}{(20z+z-19)\,(20z-20+3z+3)}$$

$$= \frac{2\,(z+1)^2}{(21z-19)\,(23z-17)}$$

10 The system function of an analog filter is given by $H(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$. Using BLT, obtain the system function of the digital filter which is resonant at $\frac{\pi}{2}$ rad.

Let us first find the resonant frequency of the analog filter.

Equating denominator of $H(s)$ to zero, we get

$$(s+0.1)^2+16=0$$

$$s+0.1=\pm j4$$

$$\sigma+j\Omega+0.1=\pm j4$$

$$\therefore \Omega_r = 4 \text{ rad/s}$$

(resonant frequency of analog filter)

given: $\omega_r = \frac{\pi}{2}$ rad.

According to BLT,

$$\omega_r = 2\tan^{-1}\left(\frac{\Omega_r T_s}{2}\right)$$

(equation (10), question #8)

$$\therefore \frac{\pi}{2} = 2\tan^{-1}\left(\frac{4}{2}T_s\right)$$

$$\frac{\pi}{4} = \tan^{-1}(2T_s)$$

$$\therefore \tan\left(\frac{\pi}{4}\right) = 2T_s$$

$$\therefore T_s = \frac{1}{2}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T_s}\frac{z-1}{z+1}}$$

$$= H(s)\Big|_{s = 4\frac{z-1}{z+1}}$$

$$= \frac{4\left(\frac{z-1}{z+1}\right) + 0.1}{\left[4\left(\frac{z-1}{z+1}\right) + 0.1\right]^2 + 16}$$

$$= \frac{4\left(\frac{z-1}{z+1}\right) + 0.1}{\left[\frac{4(z-1) + 0.1(z+1)}{(z+1)}\right]^2 + 16}$$

$$= \frac{4\left(\frac{z-1}{z+1}\right) + 0.1}{\left[\frac{4.1z - 3.9}{z+1}\right]^2 + 16}$$

$$= \frac{\frac{4z - 4 + 0.1(z+1)}{z+1}}{\frac{16.81z^2 + 15.21 - 31.98z}{(z+1)^2} + 16}$$

$$= \frac{4.1z - 3.9}{\frac{16.81z^2 - 31.98z + 15.21}{(z+1)} + 16}$$

$$= \frac{(4.1z - 3.9)(z+1)}{16.81z^2 - 31.98z + 15.2 + 16z + 16}$$

$$= \frac{4.1z^2 - 3.9z + 4.1z - 3.9}{16.81z^2 - 15.98z + 31.2}$$

$$= \frac{4.1z^2 - 0.2z - 3.9}{16.81z^2 - 15.98z + 31.2}$$

11 Discuss the four types of analog to analog frequency transformations.

Low-pass to low-pass

Suppose that we have a low-pass filter with passband edge frequency Ω_p and we wish to convert it into another low-pass filter with passband edge frequency Ω_p'.
The transformation that accomplishes this is

$$s \longrightarrow \frac{\Omega_p}{\Omega_p'} s.$$

Low-pass to high-pass

Suppose that we have a low-pass filter with passband edge frequency Ω_p and we wish to convert it into a high-pass filter with passband edge frequency Ω_p'.

The transformation that accomplishes this is

$$s \longrightarrow \frac{\Omega_p \Omega_p'}{s}$$

low-pass to band-pass

Suppose that we have a low pass filter with passband edge frequency Ω_p and we wish to convert it to a bandpass filter having lower and upper band edge frequencies Ω_l and Ω_u respectively,

The transformation that accomplishes this is

$$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

low-pass to band-reject filter

Suppose that we have a lowpass filter with passband edge frequency Ω_p and we wish to convert it to a band-reject filter having lower and upper band edge frequencies Ω_l and Ω_u, respectively.

The transformation that accomplishes this is

$$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

12 $H(s) = \dfrac{1}{s^2 + \sqrt{2}\, s + 1}$ represents the transfer function of a low pass filter with ~~pa~~ cut-off frequency of 1 rad/s. Using frequency transformation, find the transfer function of the

following analog filters.

a) a low-pass filter with cutoff frequency 10 rad/s.

$\Omega_c = 1$ rad/s

$\Omega_c' = 10$ rad/s

Transformation : $s \longrightarrow \frac{\Omega_c}{\Omega_c'} s$

ie, $s \longrightarrow \frac{s}{10}$

$\therefore$ Required $H(s) = \dfrac{1}{\left(\frac{s}{10}\right)^2 + \sqrt{2}\left(\frac{s}{10}\right) + 1}$

$$= \frac{1}{\frac{s^2}{100} + \frac{\sqrt{2}}{10}s + 1}$$

$$= \frac{100}{s^2 + 10\sqrt{2}s + 100}$$

b) a highpass filter with cut-off frequency of 100 rad/s

Transformation: $s \rightarrow \dfrac{\Omega_c \Omega_c'}{s} = \dfrac{100}{s}$

$\Omega_c = 1$ rad/s, $\Omega_c' = 100$ rad/s

$\therefore$ Required $H(s) = \dfrac{1}{\left(\frac{100}{s}\right)^2 + \sqrt{2}\left(\frac{100}{s}\right) + 1}$

$$= \frac{1}{\frac{100^2}{s^2} + \sqrt{2}\frac{100}{s} + 1}$$

$$= \frac{s^2}{100^2 + \sqrt{2}\,100 s + s^2}$$

c) a bandpass filter with centre frequency $\Omega_0 = 100$ rad/s and bandwidth $B_0 = 10$ rad/s

$\Omega_l = 95$, $\Omega_u = 105$

Tansformation : $s \to \Omega_c \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$

$$= \frac{s^2 + 9975}{s \cdot 10}$$

$$\therefore \text{Required } H(s) = \frac{1}{\left(\frac{s^2+9975}{10s}\right)^2 + \sqrt{2}\left(\frac{s^2+9975}{10s}\right) + 1}$$

$$= \frac{1}{\frac{s^4 + 9975^2 + 19950s}{100s^2} + \sqrt{2}\,\frac{s^2+9975}{10s} + 1}$$

$$= \frac{100s^2}{s^4 + 19950s + 9975^2 + \sqrt{2}(s^2+9975)10s + 100s^2}$$

$$= \frac{100s^2}{s^4 + 100s^2 + 10\sqrt{2}s^3 + (19950 + 99750\sqrt{2})s + 9975^2}$$

13 Use bilinear transformation to obtain digital LPF from the standard normalized Butterworth filter $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$. (26)

Assume cut-off frequency as 100Hz and sampling frequency as 1KHz.

Note: standard, normalized Butterworth filter means, cutoff frequency $= \Omega_c = 1$ rad/s

$$f_c' = 100\text{Hz}$$

$$f_s = 1000\text{Hz}$$

$$\omega_c' = \frac{2\pi f_c'}{f_s} = \frac{\pi}{5} \text{ rad.}$$

$$\therefore \Omega_c' = \frac{2}{T}\tan\left(\frac{\omega_c'}{2}\right) \quad \text{[Prewarping]}$$

$$= 0.6498 \text{ rad/s} \quad [T=1] \text{ [Assumption]}$$

$$\therefore \text{Required } H(s) = H(s)\Big|_{s=\frac{\Omega_c s}{\Omega_c'}} = \frac{s}{0.6498}$$

1 rad/s

$$= \frac{1}{\frac{s^2}{0.6498^2} + \sqrt{2}\frac{s}{0.6498} + 1}$$

$$= \frac{0.6498^2}{s^2 + \sqrt{2}\cdot 0.6498 s + 0.6498^2}$$

$$= \frac{0.4222}{s^2 + 0.9189s + 0.4222}$$

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\cdot\frac{z-1}{z+1}} \quad (\text{BLT})$$

$$= 2\,\frac{z-1}{z+1}$$

$$= \frac{0.6498^2}{4\cdot\left(\frac{z-1}{z+1}\right)^2 + 0.9189\left(\frac{z-1}{z+1}\right) + 0.4222}$$

14 Using BLT, design a digital high pass filter with cut off frequency 100 Hz. sampling frequency 1 KHz.

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$f_c' = 100\ Hz$

$\omega_c' = 2\pi \frac{f_c}{f_s} = \frac{\pi}{5}$ rad.

$\Omega_c' = 0.6498$ rad/s.

Required $H(s) = H(s)\Big|_{s = \frac{\Omega_c \Omega_c'}{s}} = \frac{0.6498}{s}$

1 rad/s

$$= \frac{1}{\left(\frac{0.6498}{s}\right)^2 + \left(\frac{0.6498}{s}\right)\sqrt{2} + 1}$$

$$= \frac{1}{\frac{0.4222}{s^2} + \frac{0.6498}{s}\sqrt{2} + 1}$$

$$= \frac{s^2}{0.4222 + 0.6498\sqrt{2}s + s^2}$$

Required $H(z) = H(s)$ at $s = \frac{2}{T_s}\frac{z-1}{z+1}$, $T_s = \frac{1}{f_s}$

15 Use BLT to obtain digital BPF with the Butterworth characteristics to meet the following specifications.

$N = 2$, $f_l = 200Hz$, $f_u = 300Hz$, $f_s = 1KHz$

$$\omega_l = \frac{2\pi f_l}{f_s} = \frac{2\pi}{5} \text{ rad}$$

$$\omega_u = \frac{2\pi f_u}{f_s} = \frac{2\pi\, 300}{1000} = \frac{3\pi}{5} \text{ rad}$$

$$\Omega_l = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = 1.453 \text{ rad/s}$$

$$\Omega_u = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = 2.753 \text{ rad/s}$$

Required $H(s) = H(s)\Big|_{s = \Omega_c \cdot \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}}$

$$= \frac{1}{s+1}\Bigg|_{s = \frac{s^2+4}{s \cdot 1.3}}$$

$$= \frac{1}{\frac{s^2+4}{1.3s} + 1} = \frac{1.3s}{s^2 + 1.3s + 4}$$

Required $H(z) = H(s)\Big|_{s = \frac{2}{T_s}\frac{z-1}{z+1}}$

16 Design a digital BP filter from a 2nd order analog low pass Butterworth prototype filter using BLT. The lower and upper cut off frequencies are $\frac{5\pi}{12}$ rad and $\frac{7\pi}{12}$ rad. Assume $T = 2$ sec.

$$\omega_l = \frac{5\pi}{12}, \qquad \omega_u = \frac{7\pi}{12}$$

$$\Omega_l = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = 0.7673 \text{ rad/s}$$

$$\Omega_u = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = 1.3032 \text{ rad/s}$$

Transformation : $$s \longrightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$= \frac{s^2+1}{0.5359s}$$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \quad \text{(Prototype)}$$

$$\therefore \text{Required } H(s) = \frac{1}{\left(\frac{s^2+1}{0.5359s}\right)^2 + \sqrt{2}\left(\frac{s^2+1}{0.5359s}\right) + 1}$$

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{z-1}{z+1} = \frac{z-1}{z+1}}$$

17 Compare IIR and FIR filters.

1. All the poles of causal FIR filters lie at $z=0$, in the z-plane.
∴ FIR filters are always stable.

Poles of IIR filters may lie anywhere in the z-plane.
∴ Stability is not always guaranteed.

2. FIR filters can be designed for linear phase.
Causal and stable IIR filters cannot have linear phase.

3. For the given transition width, order of the FIR filter will be greater than the order of the IIR filter.

4. For the given order, transition width of FIR filter will be greater than that of IIR filter.

5. For the given specifications, computational complexity in FIR filters is more than that in IIR filters.

18 Compare Butterworth and Chebyshev filters.

1. Low pass Butterworth filters are all pole filters

Lowpass Chebyshev Type-I filters are all pole filters, but Type-II Chebyshev filters contain both poles and zeros.

2. Frequency response of Butterworth filters is monotonic in both passband and stopband.

Type-I Chebyshev filters exhibit equiripple behaviour in the passband and a monotonic characteristic in the stopband.

Type-II Chebyshev filters exhibit a monotonic behavior in the passband and an equiripple behavior in the stopband.

3. For the given frequency response characteristics, order of the Chebyshev filter is less than the order of the Butterworth filter.

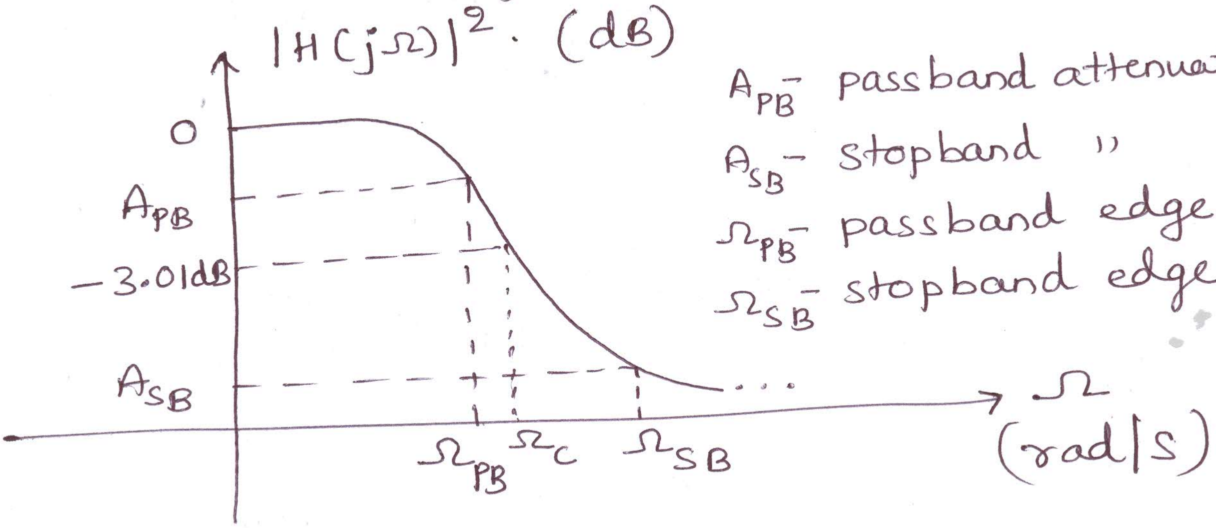
4. Poles of a Butterworth filter lie on a circle.

Poles of a Chebyshev filter lie on an ellipse.

19 Derive the expression for order and cut-off frequency of Butterworth lowpass filter.

The magnitude squared response of low

pass Butterworth filter is as shown below.



A_{PB} - passband attenuation
A_{SB} - stopband "
Ω_{PB} - passband edge
Ω_{SB} - stopband edge

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \text{---} \quad (1)$$

i) To find the order of the filter.

$$A_{PB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{PB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N}}\right)$$

$$= 10\log_{10}\left(1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N}\right)$$

$$\therefore \quad 1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{PB}}{10}}$$

$$\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{PB}}{10}} - 1 \quad \text{----} \quad (2)$$

$$A_{SB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{SB}}$$

$$= -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{SB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N}}\right)$$

$$= 10\log_{10}\left(1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N}\right)$$

$$1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{SB}}{10}}$$

$$\therefore \left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{SB}}{10}} - 1 \quad \text{----} \quad (3)$$

(2) ÷ (3) gives,

$$\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)^{2N} = \frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}$$

$$\therefore 2N\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right) = \log_{10}\left(\frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}\right)$$

$$\therefore N = \frac{\log_{10}\left(\frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}\right)}{2\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)}$$

$$\therefore N = \frac{\log_{10}\left(\sqrt{\frac{10^{\frac{A_{PB}}{10}}-1}{10^{\frac{A_{SB}}{10}}-1}}\right)}{\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)} \quad \text{---} \quad (4)$$

$$N = \lceil N \rceil$$

(Note: To get the order of Chebyshev Type I filter, replace 'log' with '$\cosh^{-1}$')

ii) To find the cut-off frequency

From (2),

$$\left(\frac{\Omega_{PB}}{\Omega_C}\right)^{2N} = 10^{\frac{A_{PB}}{10}} - 1$$

$$\therefore \frac{\Omega_{PB}}{\Omega_C} = \left[10^{\frac{A_{PB}}{10}} - 1\right]^{\frac{1}{2N}}$$

$$\therefore \Omega_C = \frac{\Omega_{PB}}{\left[10^{\frac{A_{PB}}{10}} - 1\right]^{\frac{1}{2N}}} \quad \text{---} \quad (5)$$

iii) To find the location of poles.

From (1), $$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}}$$

$$\therefore H(j\Omega)\,H^*(j\Omega) = \frac{1}{1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}}$$

$$\therefore H(j\Omega)\,H(-j\Omega) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$\therefore H(s)\,H(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} \quad \text{--- (6)}$$

To find location of poles,

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

$$\left(\frac{s}{j\Omega_c}\right)^{2N} = -1 = e^{j(2k+1)\pi} \qquad k = 0,1,2,\ldots 2N-1$$

$$\therefore \frac{s_k}{j\Omega_c} = e^{j(2k+1)\frac{\pi}{2N}}, \quad k = 0,1,2,\ldots 2N-1$$

$$s_k = j\Omega_c\, e^{j(2k+1)\frac{\pi}{2N}} \quad \ldots (7)$$

$$k = 0,1,\ldots 2N-1$$

There are 2N roots. out of which N-roots correspond to H(s) and N correspond to H(-s).

$\therefore$ Poles of H(s) are,

$$s_k = j\Omega_c\, e^{j(2k+1)\frac{\pi}{2N}}, \quad k = 0,1,2,\ldots N-1$$

$$= j\Omega_c\left[\cos\left\{(2k+1)\frac{\pi}{2N}\right\} + j\sin\left\{(2k+1)\frac{\pi}{2N}\right\}\right]$$

$$= -\Omega_c \sin\left\{\frac{(2K+1)\pi}{2N}\right\} + j\Omega_c \cos\left\{\frac{(2K+1)\pi}{2N}\right\} \quad \text{---(8)}$$

$K = 0, 1, 2, 3, \ldots N-1$

Equation (8) gives location of poles.

$$H(s) = \frac{(-1)^N \prod_{K=0}^{N-1} S_K}{\prod_{K=0}^{N-1} (s - S_K)} \quad \text{---} \quad (9)$$

20 Define Chebyshev polynomial and list its properties.

Chebyshev polynomial:

$$C_N(\Omega) = \cos\left[N\cos^{-1}(\Omega)\right] \quad \text{----} \quad (1)$$

$$C_{N+1}(\Omega) = \cos\left[(N+1)\cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega) + \cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega)\right]\Omega - \sin\left[N\cos^{-1}(\Omega)\right] \sin\left[\cos^{-1}(\Omega)\right] \quad \text{---(2)}$$

$$C_{N-1}(\Omega) = \cos\left[(N-1)\cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega)\right]\Omega + \sin\left[N\cos^{-1}(\Omega)\right] \sin\left[\cos^{-1}(\Omega)\right] \quad \text{---(3)}$$

(2)+(3) gives,

$$C_{N+1}(\Omega)+C_{N-1}(\Omega) = 2\Omega C_N(\Omega)$$

$$\therefore C_{N+1}(\Omega) = 2\Omega C_N(\Omega) - C_{N-1}(\Omega) \quad \cdots (4)$$

From (1)

$$C_0(\Omega) = 1.$$

$$C_1(\Omega) = \Omega$$

$$C_2(\Omega) = 2C_1(\Omega)\Omega - C_0(\Omega)$$
$$= 2\Omega^2 - 1$$

$$C_3(\Omega) = 2C_2(\Omega)\Omega - C_1(\Omega)$$
$$= 4\Omega^3 - 3\Omega$$

$$C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$$

Properties of Chebyshev polynomial

1. $C_N(\Omega) = -C_N(-\Omega)$ for odd N

   $C_N(\Omega) = C_N(-\Omega)$ for even N

2. $C_N(0) = (-1)^{\frac{N}{2}}$ for even N

   $C_N(0) = 0$ for odd N

3. $C_N(1) = 1$ for all N

4. $C_N(\Omega)$ oscillates with equal ripple between ± 1 for $|\Omega| < 1$

$$0 \le |C_N(\Omega)| \le 1 \quad \text{for} \quad |\Omega| \le 1$$

5. $C_N(\Omega)$ monotonically increases for $|\Omega|>1$

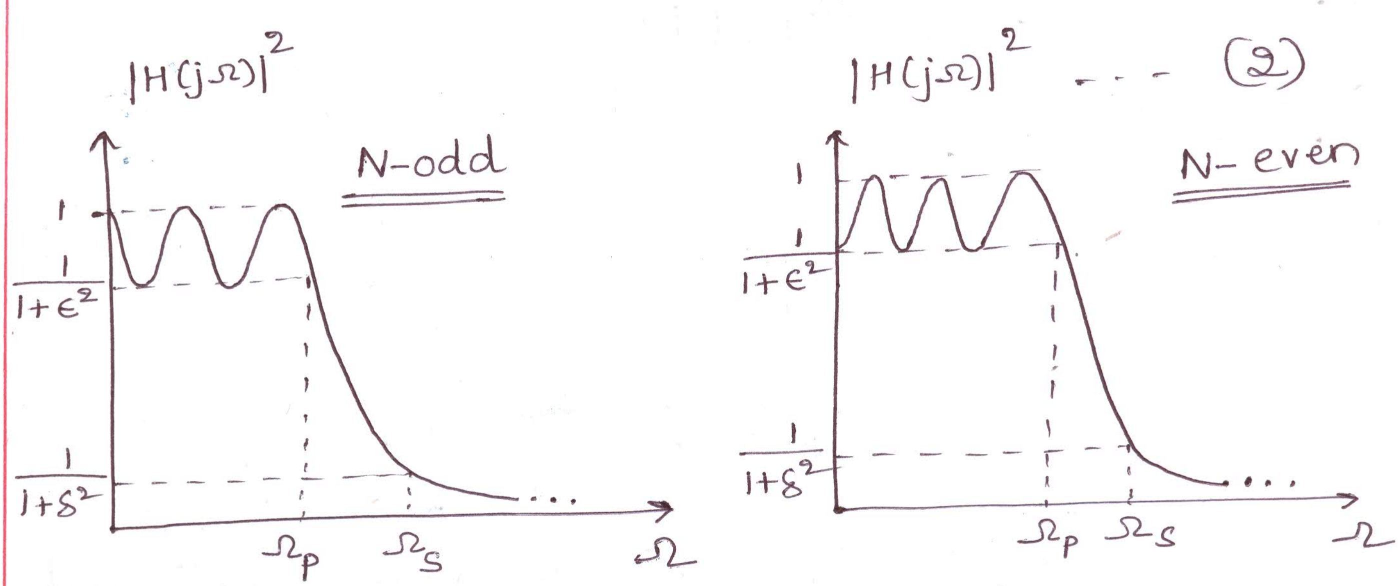
$$|C_N(\Omega)| > 1 \quad \text{for} \quad |\Omega| > 1$$

2) Derive the expression for order and cut-off frequency of Chebyshev lowpass filter (Type-I)

The magnitude squared response of Type-I low pass chebyshev filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_{PB}}\right)} \quad \cdots (1)$$

$$C_N\left(\frac{\Omega}{\Omega_{PB}}\right) = \begin{cases} \cos\left(N \cos^{-1}\left(\frac{\Omega}{\Omega_{PB}}\right)\right), & \text{for } \Omega \le \Omega_{PB} \\ \cosh\left(N \cosh^{-1}\left(\frac{\Omega}{\Omega_{PB}}\right)\right), & \text{for } \Omega > \Omega_{PB} \end{cases}$$



$$A_{PB} = -10 \log_{10} \left(|H(j\Omega)|^2 \right) \Big|_{\Omega = \Omega_{PB}}$$

$$= -10 \log_{10} \left(\frac{1}{1+\epsilon^2} \right)$$

$$= 10 \log_{10} (1+\epsilon^2)$$

$$\therefore 1+\epsilon^2 = 10^{\frac{A_{PB}}{10}}$$

$$\therefore \epsilon = \sqrt{10^{\frac{A_{PB}}{10}} - 1} \quad \text{---} (3)$$

i) To find the order of the filter,

$$A_{SB} = -10 \log_{10} \left(|H(j\Omega)|^2 \right) \Big|_{\Omega = \Omega_{SB}}$$

$$= -10 \log_{10} \left(\frac{1}{1+\epsilon^2 C_N^2 \left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)} \right)$$

$$= 10 \log_{10} \left(1+\epsilon^2 C_N^2 \left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) \right)$$

$$= 10 \log_{10} \left(1+\epsilon^2 \cosh^2 \left(N \cosh^{-1} \left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) \right) \right)$$

$$\therefore 1+\epsilon^2 \cosh^2 \left(N \cosh^{-1} \left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) \right) = 10^{\frac{A_{SB}}{10}}$$

$$\therefore \epsilon \cosh \left(N \cosh^{-1} \left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) \right) = \sqrt{10^{\frac{A_{SB}}{10}} - 1}$$

$$\cosh\left(N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right) = \frac{\sqrt{10^{\frac{A_{SB}}{10}}-1}}{\epsilon}$$

$$= \sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}$$

$$\therefore N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) = \cosh^{-1}\left(\sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}\right)$$

$$\therefore N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}\right)}{\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)} \quad \text{---} (4)$$

$$N = \lceil N \rceil \quad \text{---} (5)$$

ii) To find Ω_c,

$$|H(j\Omega)|^2\Big|_{\Omega=\Omega_c} = \frac{1}{2}$$

$\therefore$ from (1), we can write,

$$\epsilon^2 C_N^2\left(\frac{\Omega_c}{\Omega_{PB}}\right) = 1$$

$$\epsilon^2\cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right)\right) = 1$$

$$\therefore \quad \epsilon \cosh\left(N \cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right)\right) = 1$$

$$N \cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right) = \cosh^{-1}\left(\frac{1}{\epsilon}\right)$$

$$\therefore \quad \Omega_c = \Omega_{PB} \cosh\left[\frac{1}{N} \cosh^{-1}\left(\frac{1}{\epsilon}\right)\right] \quad \text{---(6)}$$

22 Design an analog Butterworth filter that has a passband ripple of 2 dB at 1 rad/s and atleast 30 dB attenuation at 3 rad/s.

$A_{PB} = 2\ dB$ $\qquad \Omega_{PB} = 1\ rad/s$

$A_{SB} = 30\ dB$ $\qquad \Omega_{SB} = 3\ rad/s.$

1. To find the order of the filter,

$$N = \frac{\log_{10}\left[\frac{10^{0.1A_{PB}} - 1}{10^{0.1A_{SB}} - 1}\right]}{2\log\left[\frac{\Omega_{PB}}{\Omega_{SB}}\right]}$$

$$= \left[\frac{-3.53}{2 \times -0.477}\right]$$

$$= 4.$$

2. To find the cut-off frequency

$$\Omega_c = \frac{\Omega_{PB}}{\left[10^{0.1A_{PB}} - 1\right]^{\frac{1}{2N}}} = \frac{1}{\left(10^{0.2} - 1\right)^{\frac{1}{8}}}$$

$$= 1.0693\ rad$$

3. To find pole-locations:

$$S_k = \Omega_c \left[-\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\cos\left(\frac{(2k+1)\pi}{2N}\right) \right]$$

$k = 0, 1, \ldots N-1$

$$S_0 = -0.4092 + j0.9879$$

$$S_1 = -0.9879 + j0.4092$$

$$S_2 = -0.9879 - j0.4092$$

$$S_3 = -0.4092 - j0.9879$$

4. Transfer function,

$$H(s) = \frac{(-1)^N \prod_{k=0}^{N-1} S_k}{\prod_{k=0}^{N-1} (S - S_k)}$$

$$= \frac{\Omega_c^N}{\prod_{k=0}^{N-1} (S - S_k)}$$

$$= \frac{1.3074}{(S-S_0)(S-S_3)(S-S_1)(S-S_2)}$$

$$= \frac{1.3074}{(S^2 + 0.8184S + 1.1435)(S^2 + 1.9759S + 1.1435)}$$

23 For the specifications given below, design an analog Butterworth filter.

$$0.9 \le |H(j\Omega)| \le 1 \quad \text{for} \quad 0 \le \Omega \le 0.2\pi$$

$$|H(j\Omega)| \le 0.2 \quad \text{for} \quad \Omega \ge 0.4\pi$$

Given: $A_{PB} = -20\log_{10}(0.9) = 0.915 \text{ dB}$

$A_{SB} = -20\log_{10}(0.2) = 14 \text{ dB}.$

$\Omega_{PB} = 0.2\pi \text{ rad/s}$

$\Omega_{SB} = 0.4\pi \text{ rad/s}.$

Note: If magnitude squared response is given, ~~then~~ for example,

$0.9 \le |H(j\Omega)|^2 \le 1$, then,

$A_{PB} = -10\log(0.9) = 0.458 \text{ dB}.$

1. Order of the filter,

$$N = \frac{\log_{10}\left[\dfrac{10^{0.1A_{PB}} - 1}{10^{0.1A_{SB}} - 1}\right]}{2\log_{10}\left(\dfrac{\Omega_{PB}}{\Omega_{SB}}\right)} = \frac{-2.0122}{2 \times -0.301}$$

$$= 3.34$$

$\therefore N = 4$

2. Cut-off frequency

$$\Omega_c = \frac{\Omega_{PB}}{\left[10^{0.1A_{PB}} - 1\right]^{\frac{1}{2N}}} = 0.754 \text{ rad/s.}$$

3. Pole locations:

$$S_k = \Omega_c\left[-\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\cos\left(\frac{(2k+1)\pi}{2N}\right)\right]$$

$K = 0, 1, 2, 3$

$$S_0 = -0.2885 + j0.6966$$

$$S_1 = \cancel{+0.15332+j} \; -0.6966 + j0.2885$$

$$S_2 = -0.6966 - j0.2885$$

$$S_3 = -0.2885 - j0.6966$$

4. Transfer function:

$$H(S) = \frac{(-1)^N \prod_{K=0}^{N-1} S_K}{\prod_{K=0}^{N-1}(S - S_K)} = \frac{\Omega_c^N}{\prod_{K=0}^{N-1}(S - S_K)}$$

$$= \frac{0.3232}{(S-S_0)(S-S_3)(S-S_1)(S-S_2)}$$

$$= \frac{0.3232}{(S^2 + 0.577S + 0.5685)(S^2 + 1.3932S + 0.5684)}$$

24 Design a digital low pass Butterworth filter using BLT to meet the following specifications.

Passband ripple $\leq$ 1 dB

Passband edge = 100π rad/s

Stopband attenuation $\geq$ 35 dB

Stopband edge = 1000π rad/s

Sampling frequency = 2 KHz.

$A_{PB} = 1$ dB $\qquad \Omega_{PB} = 100\pi$ rad/s

$A_{SB} = 35$ dB $\qquad \Omega_{SB} = 1000\pi$ rad/s

1. $\omega_{PB} = \dfrac{\Omega_{PB}}{f_s} = 0.05\pi$ rad

2. $\omega_{SB} = \dfrac{\Omega_{SB}}{f_s} = 0.5\pi$ rad

3. $\Omega_{PB}^{new} = \dfrac{2}{T} \tan\left(\dfrac{\omega_{PB}}{2}\right) = 0.0787$ rad/s

4. $\Omega_{SB}^{new} = \dfrac{2}{T} \tan\left(\dfrac{\omega_{SB}}{2}\right) = 1$ rad/s

5. To find the order of the filter.

$$N = \frac{\log_{10}\left(\dfrac{10^{0.1A_{PB}} - 1}{10^{0.1A_{SB}} - 1}\right)}{2\log_{10}\left(\dfrac{\Omega_{PB}^{new}}{\Omega_{SB}^{new}}\right)}$$

$$= \left\lceil \frac{-4.0867}{2 \times -1.1040} \right\rceil$$

$$= \lceil 1.85 \rceil$$

$$= 2.$$

6. To find cut-off frequency,

$$\Omega_C = \frac{\Omega_{PB}^{new}}{\left[10^{\frac{A_{PB}}{10}} - 1\right]^{\frac{1}{2N}}} = 0.1103 \text{ rad/s}$$

7. To find pole-locations

$$S_K = \Omega_C \left[-\sin\left(\frac{(2K+1)\pi}{2N}\right) + j\cos\left(\frac{(2K+1)\pi}{2N}\right) \right]$$

$$K = 0, 1, \ldots N-1$$

$$S_0 = -0.0780 + 0.0780j$$

$$S_1 = -0.0780 - 0.0780j$$

8. Transfer function of analog filter,

$$H(S) = \frac{(-1)^N \prod_{K=0}^{N-1} S_K}{\prod_{K=0}^{N-1} (S - S_K)} = \frac{\Omega_C^N}{\prod_{K=0}^{N-1} (S - S_K)}$$

$$= \frac{0.0122}{s^2 + 0.156s + 0.0122}$$

9. system function of digital filter.

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{z-1}{z+1} = \frac{z-1}{z+1}}$$

$\because$ we have taken $T=2$

$$= \frac{0.0122}{\left(\frac{z-1}{z+1}\right)^2 + 0.156\left(\frac{z-1}{z+1}\right) + 0.0122}$$

$$= \frac{0.0122\,(z+1)^2}{(z-1)^2 + 0.156(z-1)(z+1) + 0.0122\,(z+1)^2}$$

$$= \frac{0.0122\,(z^2+2z+1)}{(z^2-2z+1) + 0.156\,(z^2-1) + 0.0122\,(z^2+2z+1)}$$

$$= \frac{0.0122\,z^2 + 0.0244\,z + 0.0122}{1.156\,z^2 - 1.9756\,z + 0.8562}$$

25 Design a Butterworth filter using impulse invariance method for the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1 \quad , \text{ for } \quad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad , \text{ for } \quad 0.6\pi \le \omega \le \pi$$

Assume $T_s = 1$ sec for conversion of $H(s)$ to $H(z)$

$$A_{PB} = -20 \log_{10}(0.8) = 1.9382 \text{ dB}$$

$$A_{SB} = -20 \log_{10}(0.2) = 14 \text{ dB}$$

$$\omega_{PB} = 0.2\pi \text{ rad.}$$

$$\omega_{SB} = 0.6\pi \text{ rad.}$$

1. $\Omega_{PB} = \frac{\omega_{PB}}{T_s} = 0.2\pi$ rad/s

2. $\Omega_{SB} = \frac{\omega_{SB}}{T_s} = 0.6\pi$ rad/s

[Note: $\omega = \frac{\Omega}{f_s} = \Omega T_s$

This is the mapping between digital frequency 'ω' and analog frequency 'Ω' according to impulse invariance transformation.

For derivation, see question #7, Assign-ment #7]

3. To find 'N'

$$N = \frac{\log_{10}\left[\frac{10^{0.1A_{PB}}-1}{10^{0.1A_{SB}}-1}\right]}{2\log\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)} = \frac{-1.6322}{2\times -0.4771}$$

$$= 1.71$$

$\therefore$ $N = 2$

4. To find Ω_c

$$\Omega_c = \frac{\Omega_{PB}}{\left[10^{0.1A_{PB}}-1\right]^{\frac{1}{2N}}} = 0.7255 \text{ rad/s}$$

5. Transfer function.

$$H(S) = \frac{(-1)^N \Omega_c^N}{\prod_{K=0}^{1}(S-S_K)}$$

$$S_K = \Omega_c\left[-\sin\left(\frac{(2K+1)\pi}{2N}\right)+j\cos\left(\frac{(2K+1)\pi}{2N}\right)\right] \quad K=0,1$$

$$S_0 = 0.7255\left[-0.707+j\,0.707\right] = -0.5129+j0.5129$$

$$S_1 = -0.5129-j0.5129$$

$$\therefore H(S) = \frac{0.5264}{(S-S_0)(S-S_1)}$$

$$= \frac{0.5264}{S^2+1.03S+0.526}$$

6. To find $H(z)$,

Take partial fractions and write $H(s)$ as,

$$H(s) = \frac{0.516j}{s-(-0.51-j0.51)} - \frac{0.516j}{s-(-0.51+j0.51)}$$

$$\therefore H(z) = \frac{0.516j}{1-e^{(-0.51-j0.51)T}z^{-1}} - \frac{0.516j}{1-e^{(-0.51+j0.51)T}z^{-1}}$$

$$= \frac{0.3019z^{-1}}{1-1.048z^{-1}+0.36z^{-2}}$$ (Taking $T=1s$)

26 Design an ~~dig~~ analog, low pass, Type-I Chebyshev filter to meet the following specifications. (52)

Acceptable passband ripple = 2 dB

Stopband attenuation = 30 dB

Passband edge frequency = 1 rad/s

Stopband edge frequency = 2 rad/s

$A_{PB} = 2\,dB$ $A_{SB} = 30\,dB$

$\Omega_{PB} = 1\,rad/s$ $\Omega_{SB} = 2\,rad/s$

1. To find the order of the filter,

$$N = \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{0.1A_{SB}}-1}{10^{0.1A_{PB}}-1}}\right)}{\cosh^{-1}\left(\dfrac{\Omega_{SB}}{\Omega_{PB}}\right)} = \left[\frac{4.41}{\cancel{3.348}\ 1.316}\right] = 4$$

2. $$\varepsilon = \sqrt{10^{0.1A_{PB}}-1} = 0.7648$$

3. $$k = \left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 1.311$$

4. $$\sigma_k = -\Omega_{PB}\left(\frac{k^2-1}{2k}\right)\sin\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1, \ldots N-1$

$\sigma_0 = -0.1049$

$\sigma_1 = -0.2532$

$\sigma_2 = -0.2532$

$\sigma_3 = -0.1049$

5. $$\Omega_k = \Omega_{PB}\left(\frac{R^2+1}{2R}\right)\cos\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1, \ldots N-1$

$\Omega_0 = 0.9580$

$\Omega_1 = 0.3968$

$\Omega_2 = -0.3968$

$\Omega_3 = -0.9580$

6. Pole locations:

$S_k = \sigma_k + j\Omega_k$

$S_0 = -0.1049 + 0.9580j$

$S_1 = -0.2532 + 0.3968j$

$S_2 = -0.2532 - 0.3968j$

$S_3 = -0.1049 - 0.9580j$

7. Transfer function:

$$H(S) = \frac{(-1)^N b_0 \prod_{k=0}^{N-1} S_k}{\prod_{k=0}^{N-1}(S - S_k)}$$

$$= \frac{b_0 \cdot \prod_{k=0}^{3} s_k}{(s-s_0)(s-s_3)(s-s_1)(s-s_2)}$$

$$= \frac{0.7943 \times 0.2058}{(s^2+0.2098s+0.9287)(s^2+0.5064s+0.2216)}$$

$$b_0 = \frac{1}{\sqrt{1+\epsilon^2}} = 0.7943$$

$\because$ N is even

$$\therefore H(s) = \frac{0.1634}{(s^2+0.2098s+0.9287)(s^2+0.5064s+0.2216)}$$

27 Obtain an analog chebyshev filter of type I that meets the following requirements

$$\frac{1}{\sqrt{2}} \leq |H(j\Omega)| \leq 1, \quad 0 \leq \Omega \leq 2$$

$$|H(j\Omega)| < 0.1, \quad \Omega \geq 4$$

$$A_{PB} = -20\log\left(\frac{1}{\sqrt{2}}\right) = 3.010 \text{ dB}$$

$$A_{SB} = -20\log(0.1) = 20 \text{ dB}$$

$$\Omega_{PB} = 2 \text{ rad/s}$$

$$\Omega_{SB} = 4 \text{ rad/s}$$

1. Order of the filter

$$N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{0.1A_{SB}}-1}{10^{0.1A_{PB}}-1}}\right)}{\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)}$$

$$= \frac{\cosh^{-1}(9.95)}{\cosh^{-1}(2)}$$

$$= \frac{2.988}{1.317}$$

$$= 2.269$$

$$\therefore N = 3$$

2. $\varepsilon = \sqrt{10^{0.1A_{PB}}-1} = 1$

3. $\beta = \left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 1.3415$

4. $\sigma_k = -\Omega_{PB}\left(\frac{\beta^2-1}{2\beta}\right)\sin\left(\frac{(2k+1)\pi}{2N}\right)$

$k=0,1,2$

$$\sigma_0 = -2 \times 0.298 \times 0.5 = -0.298$$

$$\sigma_1 = -2 \times 0.298 \times 1 = -0.596$$

$$\sigma_2 = -2 \times 0.298 \times 0.5 = -0.298$$

5. $\Omega_k = \Omega_{PB}\left(\frac{\beta^2+1}{2\beta}\right)\cos\left(\frac{(2k+1)\pi}{2N}\right)$ $k=0,1,2$

$$\Omega_0 = 2 \times 1.0435 \times 0.866 = 1.807$$

$$\Omega_1 = 0$$

$\Omega_2 = -1.807$

6. Pole locations:

$S_0 = -0.298 + j1.807$

$S_1 = -0.596$

$S_2 = -0.298 - j1.807$

7. Transfer function,

$$H(s) = \frac{(-1)^N b_0 \prod_{k=0}^{2} S_K}{\prod_{k=0}^{2} (s - S_K)}$$

$b_0 = 1 \quad \because N \text{ is odd}$

$$\therefore H(s) = \frac{2}{(s+0.596)(s^2+0.596s+3.354)}$$

Transfer function of standard normalized Butterworth filter: (Cut-off frequency = 1 rad/sec)

1. $N=1$

$$H(s) = \frac{1}{s+1}$$

2. $N=2$

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

3. $N=3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

4. $N=4$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

28 Design a digital low pass Chebyshev filter of type I, using bilinear transformation with the following magnitude spectrum characteristics.

Acceptable passband ripple of 1dB

Passband edge frequency of 200Hz

Stopband attenuation of 20 dB or greater beyond 600Hz.

Sampling frequency of 2KHz.

$A_{PB} = 1dB$ $\quad A_{SB} = 20\,dB$

$f_{PB} = 200Hz$ $\quad f_{SB} = 600Hz$ $\quad f_s = 2000Hz$

1. $\omega_{PB} = \dfrac{2\pi f_{PB}}{f_s} = 0.2\pi$ rad.

2. $\omega_{SB} = \dfrac{2\pi f_{SB}}{f_s} = 0.6\pi$ rad

3. $\Omega_{PB} = \dfrac{2}{T}\tan\left(\dfrac{\omega_{PB}}{2}\right) = 0.3249$ rad/s (Taking T=2sec)

4. $\Omega_{SB} = \dfrac{2}{T}\tan\left(\dfrac{\omega_{SB}}{2}\right) = 1.376$ rad/s. (Taking T=2 sec.)

5. To find the order of the filter,

$$N = \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{0.1A_{SB}}-1}{10^{0.1A_{PB}}-1}}\right)}{\cosh^{-1}\left(\dfrac{\Omega_{SB}}{\Omega_{PB}}\right)} = \left\lceil \frac{3.66}{2.12} \right\rceil = 2$$

6. $$\varepsilon = \sqrt{10^{0.1A_{PB}}-1} = 0.5088$$

7. $$\beta = \left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 2.042$$

8. $$\sigma_k = -\Omega_{PB}\left(\frac{\beta^2-1}{2\beta}\right)\sin\left(\frac{(2k+1)\pi}{2N}\right),$$

$k=0,1,\dots N-1$

$$\sigma_0 = -0.1783$$

$$\sigma_1 = -0.1783$$

9. $$\Omega_k = \Omega_{PB}\left(\frac{\beta^2+1}{2\beta}\right)\cos\left(\frac{(2k+1)\pi}{2N}\right)$$

$k=0,1,\dots N-1$

$$\Omega_0 = 0.2908$$

$$\Omega_1 = -0.2908$$

10. Pole locations:

$$s_0 = \sigma_0 + j\Omega_0 = -0.1783 + j0.2908$$

$$s_1 = \sigma_1 + j\Omega_1 = -0.1783 - j0.2908$$

11. Transfer function of analog filter

$$H(s) = \frac{b_0 (-1)^N \prod_{k=0}^{N-1} s_k}{\prod_{k=0}^{N-1} (s - s_k)}$$

$$b_0 = \begin{cases} 1 & \text{for } N\text{-odd} \\ \dfrac{1}{\sqrt{1+\epsilon^2}} & \text{for } N\text{-even} \end{cases}$$

$$\therefore b_0 = \frac{1}{\sqrt{1+\epsilon^2}} = 0.89127$$

(since $N=2$)

$$\therefore H(s) = \frac{0.89127 \times 0.1164}{(s^2 + 0.3566s + 0.1164)}$$

$$= \frac{0.1037}{s^2 + 0.3566s + 0.1164}$$

12. Transfer function of digital filter,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{z-1}{z+1} = \frac{z-1}{z+1}} \quad (\text{Taking } T = 2\,\text{sec})$$

$$= \frac{0.1037}{\left(\frac{z-1}{z+1}\right)^2 + 0.3566\left(\frac{z-1}{z+1}\right) + 0.1164}$$

$$= \frac{0.1037\,(z+1)^2}{(z-1)^2 + 0.3566\,(z-1)(z+1) + 0.1164\,(z+1)^2}$$

$$= \frac{0.1037\,(z+1)^2}{z^2+1-2z+0.3566\,(z^2-1) + 0.1164\,(z^2+2z+1)}$$

$$= \frac{0.1037\,(z+1)^2}{1.473z^2 - 1.7672z + 0.7598}$$

$$= 0.1037z^2 + 0.2074z + 0.1037$$

29 For the following speuifications, design a highpass, analog Butterworth filter.

$A_{PB} = 3\,dB$ $\qquad \Omega_{PB} = 1000$ rad/s

$A_{SB} = 15\,dB$ $\qquad \Omega_{SB} = 500$ rad/s

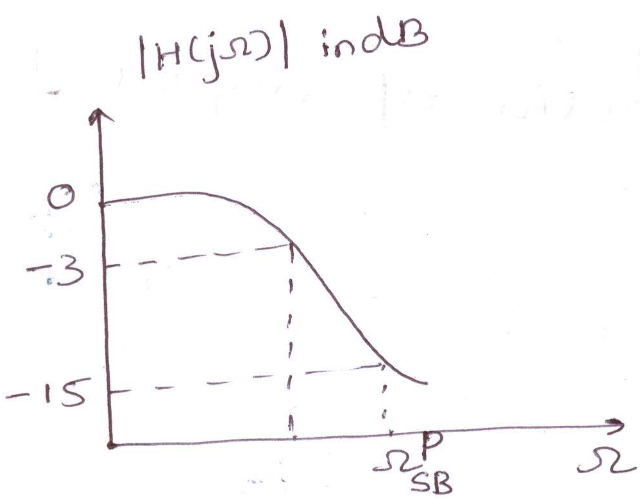


fig. (1)

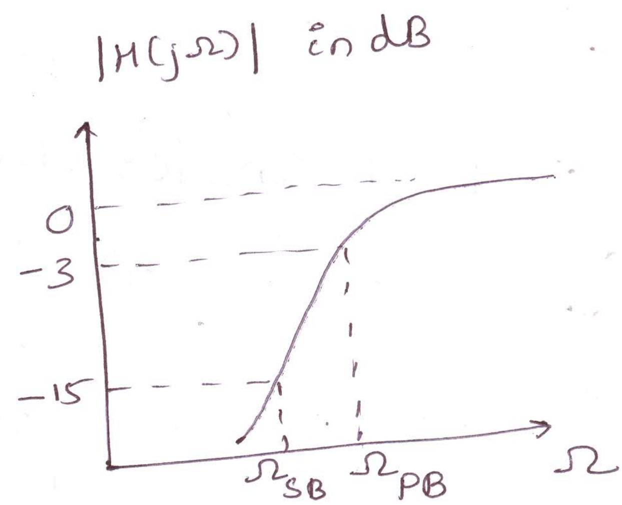


fig (2).

To design a high pass Butterworth filter, we first design a low pass prototype filter and then apply frequency transformation.

Frequency response of prototype filter is as shown in fig. (a).

Here, $\Omega^{P}_{PB} = 1$ rad/s.

To get Ω_{SB} of prototype filter, ie, Ω^{P}_{SB}

$$\cancel{\Omega^{P}_{SB} = \frac{\Omega_{SB}}{\Omega_{PB}}}$$

$$\Omega^{P}_{SB} = \frac{\Omega_{PB}}{\Omega_{SB}} = \frac{1000}{500} = 2$$

To design prototype lpf,

$$N = \frac{\log\left(\sqrt{\dfrac{10^{0.1A_{SB}}-1}{10^{0.1A_{PB}}-1}}\right)}{\log\left(\dfrac{\Omega^{P}_{SB}}{\Omega^{P}_{PB}}\right)} = \frac{0.744}{0.301} = 2.47$$

$\therefore N = 3$

for $N=3$, transfer function of prototype filter is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$\therefore$ Transfer function of the required high pass filter,

$$H^{HP}(s) = H(s)\Big|_{s \Rightarrow \frac{1000}{s}}$$

$$= \frac{1}{\left(\frac{1000}{s}+1\right)\left(\left(\frac{1000}{s}\right)^2 + \frac{1000}{s}+1\right)}$$

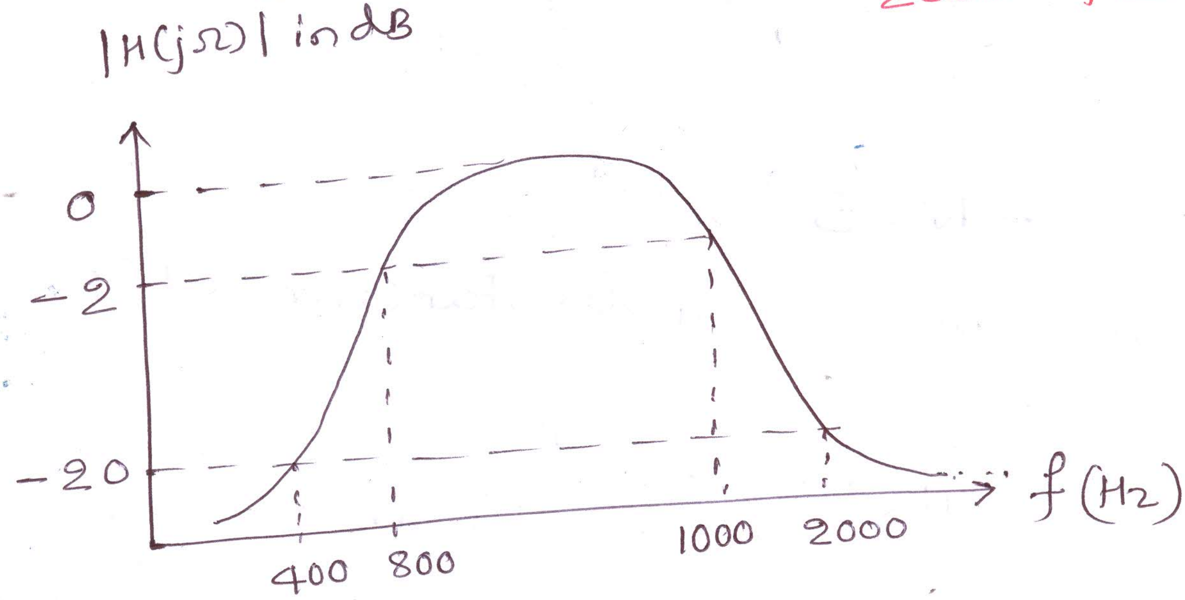
$$= \frac{s^3}{(s+1000)(s^2+1000s+1000^2)}$$

30 Using BLT, design a digital bandpass Butterworth filter with the following specifications.

Sampling frequency = 8 KHz.

Passband ripple = 2 dB for $800\text{Hz} \le f \le 1600\text{Hz}$

Stopband attenuation = 20 dB for $0 \le f \le 400$ and $2000 \le f \le \infty$



$f_1 = 400\text{ Hz}$. $\therefore\ \omega_1 = \frac{2\pi f_1}{f_s} = 0.1\pi$ rad

$f_2 = 2000\text{ Hz}$ $\therefore\ \omega_2 = \frac{2\pi f_2}{f_s} = 0.5\pi$ rad

$f_l = 800\,Hz \quad \therefore \omega_l = 2\pi \frac{f_l}{f_s} = 0.2\pi$ rad

$f_u = 1600\,Hz \quad \therefore \omega_u = 2\pi \frac{f_u}{f_s} = 0.4\pi$ rad

Prewarped analog frequencies are

$$\Omega_1 = \frac{2}{T} \tan\left(\frac{\omega_1}{2}\right) = 0.1584 \text{ rad/s}$$

$$\Omega_2 = \frac{2}{T} \tan\left(\frac{\omega_2}{2}\right) = 1 \text{ rad/s}$$

$$\Omega_l = \frac{2}{T} \tan\left(\frac{\omega_l}{2}\right) = 0.325 \text{ rad/s}$$

$$\Omega_u = \frac{2}{T} \tan\left(\frac{\omega_u}{2}\right) = 0.7265 \text{ rad/s}$$

(Assuming $T = 2$ sec)

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1 (\Omega_u - \Omega_l)} = 3.318$$

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2 (\Omega_u - \Omega_l)} = 1.9026$$

$$\Omega_r = \min(|A|, |B|) = 1.9026.$$

We consider Ω_r as the Ω_{SB} for the low pass prototype filter.

For the low-pass prototype filter

$$N = \frac{\log_{10} \sqrt{\frac{10^{0.1A_{SB}} - 1}{10^{0.1A_{PB}} - 1}}}{\log_{10}\left(\frac{\Omega_{SB}}{1}\right)} = 3.9889$$

$\therefore \; N = 4$

Transfer function of 4^{th} order, normalized Butterworth filter is given by

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)}$$

To design the required bandpass filter, we use the transformation,

$$s \longrightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 0.236}{0.402s}$$

$$\therefore H(s) = \left. \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.8477s + 1)} \right|_{s \to \frac{s^2 + 0.236}{0.402s}}$$

$$= \frac{0.0261 s^4}{(s^4 + 0.3076s^3 + 0.6336s^2 + 0.0726s + 0.0557)(s^4 + 0.743s^3 + 0.634s^2 + 0.175s + 0.0557)}$$

$$H(z) = \left. H(s) \right|_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{1 - z^{-1}}{1 + z^{-1}}} \qquad (T = 2 \text{ sec})$$

$$= \frac{0.00484(1 - z^{-1})^4}{(1 - 3.936z^{-1} + 8.259z^{-2} - 11.21z^{-3} + 10.78z^{-4} - 7.39z^{-5} + 3.57z^{-6} - 1.11z^{-7} + 0.187z^{-8})}$$

31 Design a Chebyshev, type I, digital bandreject filter with the following specifications.

passband : $0 \le f \le 275\ Hz$ and

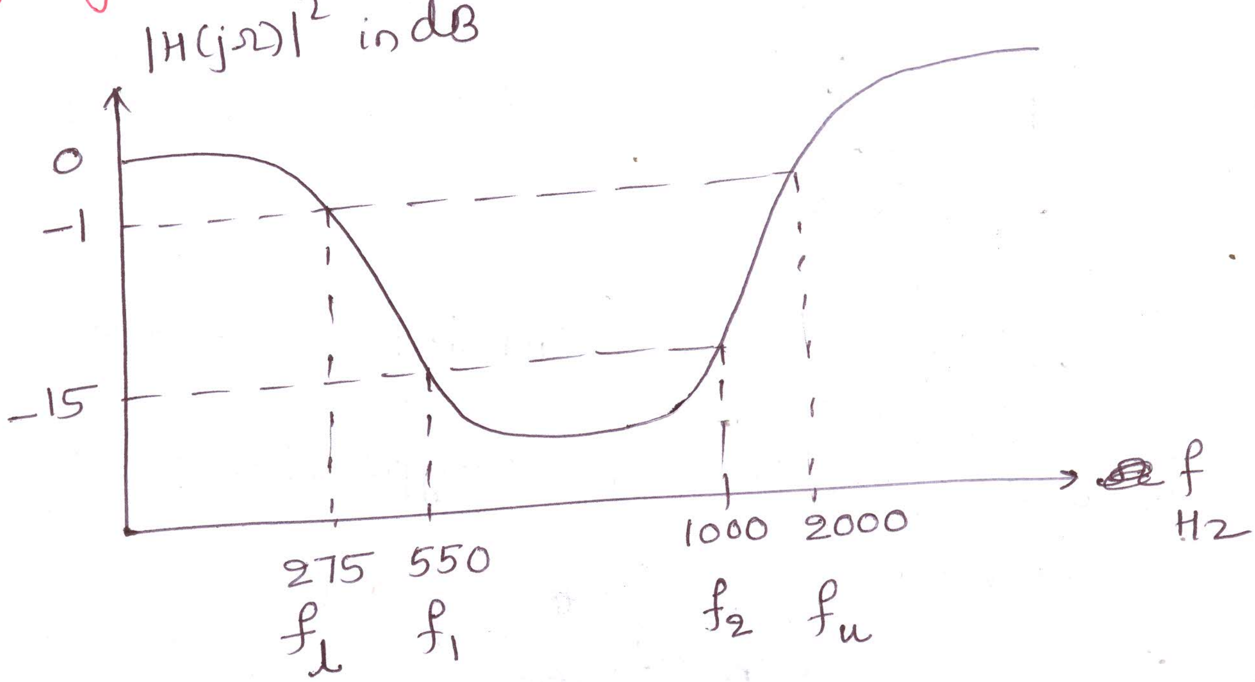
$2\ KHz \le f \le \infty$

stopband : $550\ Hz \le f \le 1000\ Hz$

$A_{PB} = 1\ dB$

$A_{SB} = 15\ dB$

Sampling frequency $= 8\ KHz$. Use BLT.



$f_1 = 550\ Hz \quad \therefore \omega_1 = \dfrac{2\pi f_1}{f_s} = 0.1375\pi$ rad

$f_2 = 1000\ Hz \quad \therefore \omega_2 = 2\pi \dfrac{f_2}{f_s} = 0.25\pi$ rad

$f_l = 275\ Hz \quad \therefore \omega_l = 2\pi \dfrac{f_l}{f_s} = 0.06875\pi$ rad

$f_u = 2000\ Hz \quad \therefore \omega_u = 2\pi \dfrac{f_u}{8000} = 0.5\pi$ rad.

Prewarped frequencies are

$$\Omega_1 = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = 0.2194 \text{ rad/s}$$

$$\Omega_2 = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = 0.4142 \text{ rad/s}$$

$$\Omega_l = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = 0.1084 \text{ rad/s}$$

$$\Omega_u = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = 1 \text{ rad/s}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} = 3.246$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u} = -5.847$$

$$\therefore \Omega_r = \min(|A|, |B|) = 3.246 \text{ rad/s}$$

Order of the prototype, lowpass Chebyshev filter,

$$N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{0.1A_{SB}} - 1}{10^{0.1A_{PB}} - 1}}\right)}{\cosh^{-1}\left(\frac{\Omega_r}{1}\right)} = \frac{3.0775}{1.8459} = 1.667$$

$$\therefore N = 2$$

$$\epsilon = \sqrt{10^{0.1A_{PB}} - 1} = 0.5088$$

$$k = \left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 2.0422$$

$$\sigma_K = -\underset{\substack{\uparrow \\ 1\,rad/s}}{\Omega_{PB}} \left(\frac{\beta^2 - 1}{2\beta}\right) \sin\left(\frac{(2K+1)\pi}{2N}\right)$$

$K = 0, 1$

$$\sigma_0 = -0.5489$$

$$\sigma_1 = -0.5489$$

$$\Omega_K = \underset{\substack{\uparrow \\ 1\,rad/s}}{\Omega_{PB}} \left(\frac{\beta^2 + 1}{2\beta}\right) \cos\left(\frac{(2K+1)\pi}{2N}\right)$$

$K = 0, 1$

$$\Omega_0 = 0.895$$
$$\Omega_1 = -0.895$$

$\therefore$ Pole locations,

$$S_0 = -0.5489 + 0.895j$$
$$S_1 = -0.5489 - 0.895j$$

$H(s)$ of prototype, lpf,

$$H_{LPF}(S) = \frac{b_0 (-1)^N \prod_{K=0}^{1} S_K}{\prod_{K=0}^{1} (S - S_K)}$$

$$b_0 = \frac{1}{\sqrt{1+\epsilon^2}} = 0.8913$$

$$S_0 S_1 = 1.1023$$

$$H_{LPF}(S) = \frac{0.9825}{S^2 + 1.097S + 1.102}$$

To get the transfer function of the bandreject filter, we use the transformation

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} = \frac{0.8916s}{s^2 + 0.1084}$$

$$\therefore \quad H(s) = \frac{0.9825}{\left(\frac{0.8916s}{s^2+0.1084}\right)^2 + 1.097\left(\frac{0.8916s}{s^2+0.1084}\right) + 1.102}$$

$$= \frac{0.8916\left(s^4 + 0.2168s^2 + 0.0118\right)}{s^4 + 0.8878s^3 + 0.9382s^2 + 0.09618s + 0.01174}$$

$$H(z) = H(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}} \quad (\text{Taking } T = 2\,\text{sec})$$

$$= \frac{0.3732\left(1 - 3.2176z^{-1} + 4.588z^{-2} - 3.218z^{-3} + z^{-4}\right)}{1 - 1.887z^{-1} + 1.43z^{-2} - 0.808z^{-3} + 0.329z^{-4}}$$

32 Given that $|H(j\Omega)|^2 = \frac{1}{1+16\Omega^4}$, determine the analog filter transfer function $H(s)$. (70)

Magnitude squared response of Butterworth filter is given by.

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \ldots (1)$$

Magnitude squared response given in the problem is,

$$|H(j\Omega)|^2 = \frac{1}{1+16\Omega^4}$$

$$= \frac{1}{1+\left(\frac{\Omega}{\frac{1}{2}}\right)^{2\times 2}} \quad \ldots (2)$$

Comparing (1) and (2), we get,

$\Omega_c = \frac{1}{2}$ rad/s

$N = 2$

$\therefore$ Pole locations are,

$$S_K = -\Omega_c\left[\sin\left(\frac{(2K+1)\pi}{2N}\right) + j\cos\left(\frac{(2K+1)\pi}{2N}\right)\right]$$

$K = 0, 1.$

$$S_0 = -\Omega_c\left[\sin\left(\frac{\pi}{4}\right) + j\cos\left(\frac{\pi}{4}\right)\right] \quad \text{(putting } K=0\text{)}$$

$= -0.3536 + j0.3536$

$S_1 = S_0^*$

$= -0.3536 - j0.3536$

$\therefore H(s) = \dfrac{\Omega_c^N}{(s-s_0)(s-s_1)}$

$= \dfrac{0.25}{s^2 + 0.7072s + 0.25}$

Note:

1. Maximally flat analog filter means Butterworth filter
2. Analog filter with monotonic frequency response means Butterworth filter
3. The filter which has ripples in passband and monotonic response in stopband is Chebyshev Type-I filter.

33 Obtain the DF-I and DF-II structure of the filter

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - z^2 + \frac{1}{2}z - \frac{1}{4}z^2 + \frac{1}{4}z - \frac{1}{8}}$$

$$= \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - \frac{5}{4}z^2 + \frac{3}{4}z - \frac{1}{8}}$$

$$\frac{Y(z)}{X(z)} = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

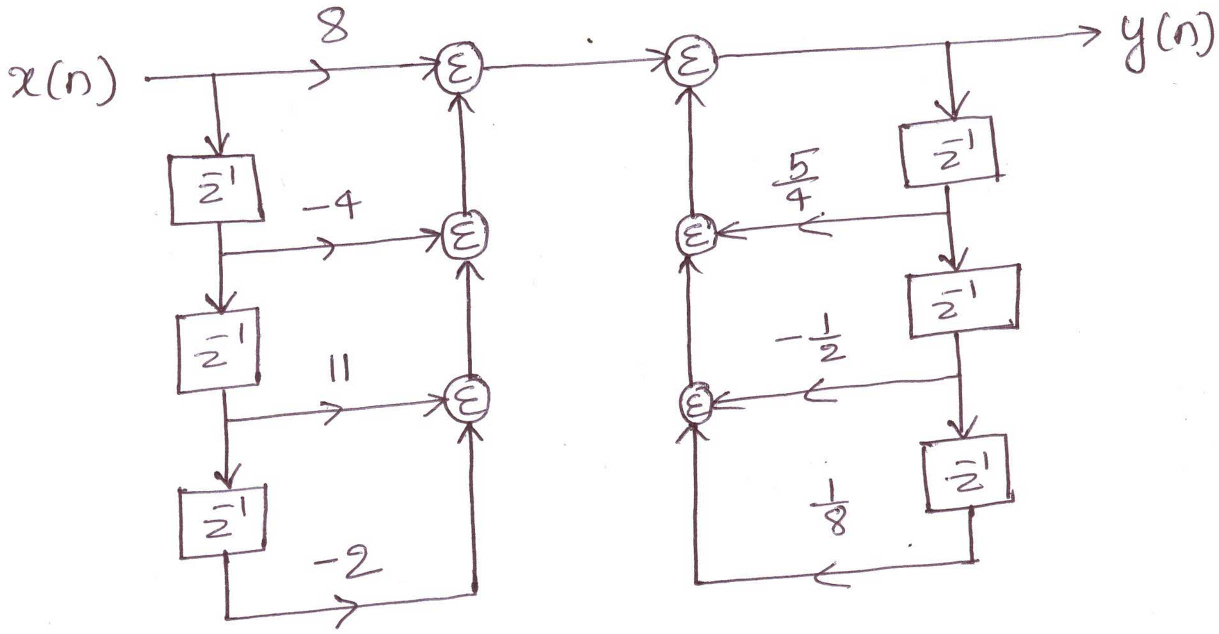
$$\therefore Y(z)\left[1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}\right] = X(z)\left[8 - 4z^{-1} + 11z^{-2} - 2z^{-3}\right]$$

Taking inverse z-transform, we get,

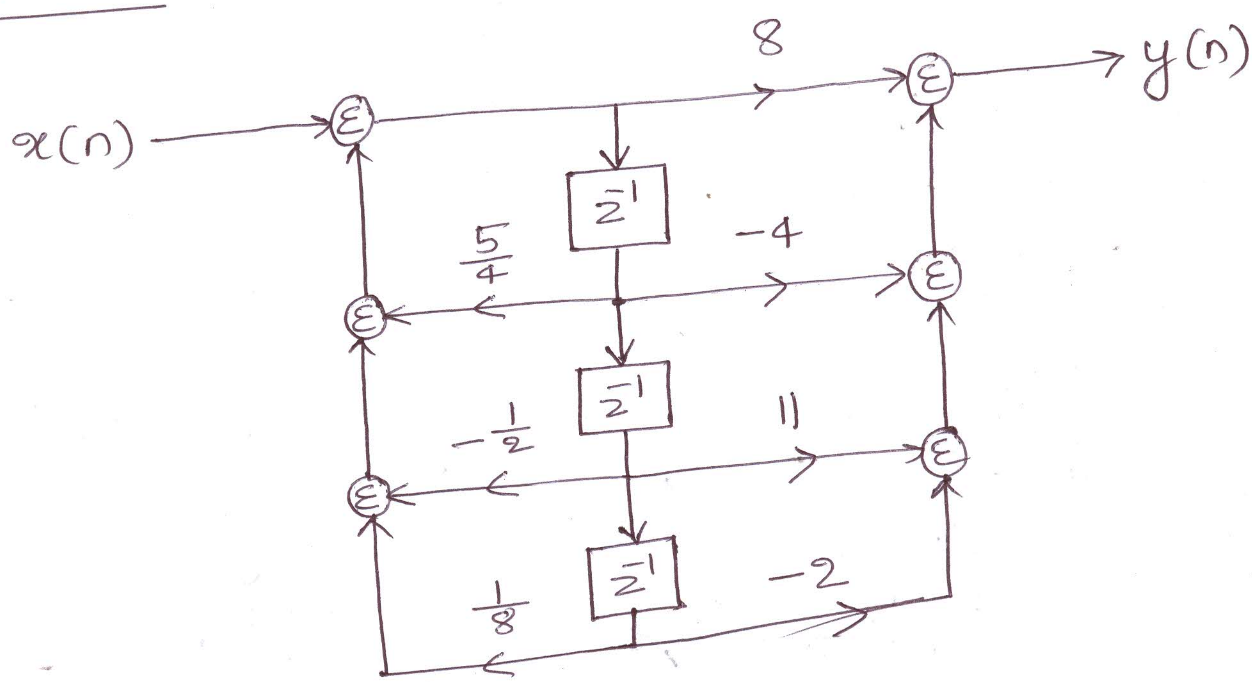
$$y(n) - \frac{5}{4}y(n-1) + \frac{3}{4}y(n-2) - \frac{1}{8}y(n-3) = 8x(n) - 4x(n-1) + 11x(n-2) - 2x(n-3)$$

$$\therefore y(n) = 8x(n) - 4x(n-1) + 11x(n-2) - 2x(n-3) + \frac{5}{4}y(n-1) - \frac{3}{4}y(n-2) + \frac{1}{8}y(n-3)$$

DF-I



DF-II



34 Obtain the cascade realization of

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

$$H(z) = \frac{(z^2+5z+6)(z^2-4z+3)}{(z^2+6z+5)(z^2-6z+8)}$$

Let $H_1(z) = \dfrac{z^2+5z+6}{z^2+6z+5}$

and $H_2(z) = \dfrac{z^2-4z+3}{z^2-6z+8}$ and $H(z) = H_1(z)H_2(z)$

$$H_1(z) = \frac{1+5z^{-1}+6z^{-2}}{1+6z^{-1}+5z^{-2}}$$

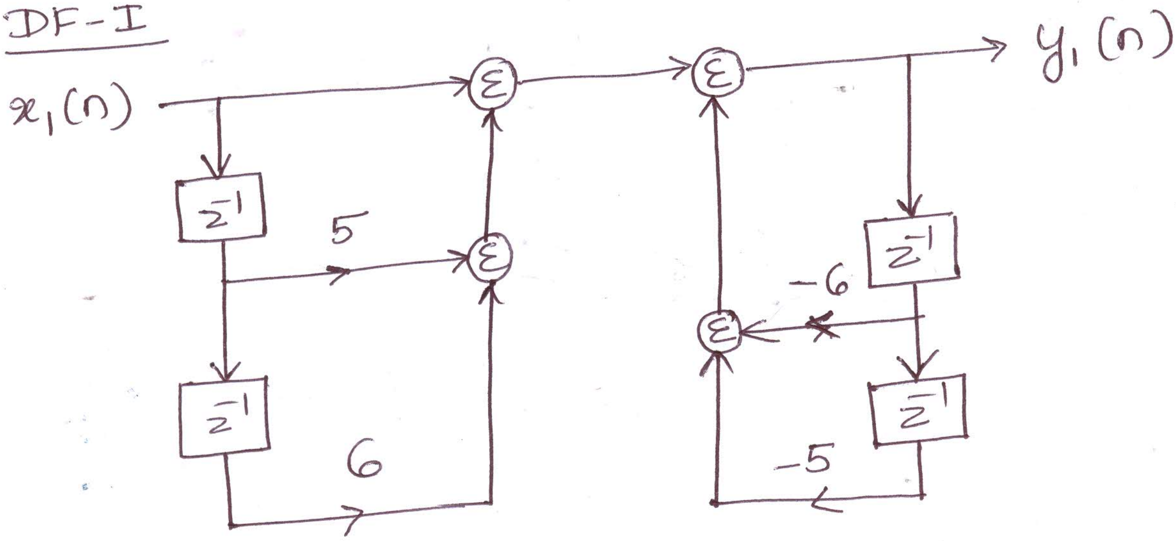
$$\therefore \frac{Y_1(z)}{X_1(z)} = \frac{1+5z^{-1}+6z^{-2}}{1+6z^{-1}+5z^{-2}}$$

$$Y_1(z)\left[1+6z^{-1}+5z^{-2}\right] = X_1(z)\left[1+5z^{-1}+6z^{-2}\right]$$

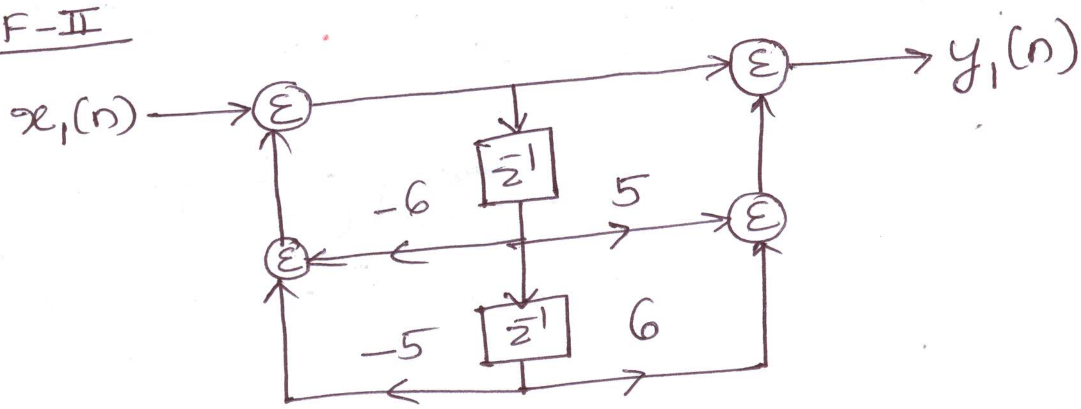
$$y_1(n)+6y_1(n-1)+5y_1(n-2) = x_1(n)+5x_1(n-1)+6x_1(n-2)$$

$$y_1(n) = x_1(n)+5x_1(n-1)+6x_1(n-2)-6y_1(n-1)-5y_1(n-2)$$

DF-I



DF-II



$$H_2(z) = \frac{z^2 - 4z + 3}{z^2 - 6z + 8.}$$

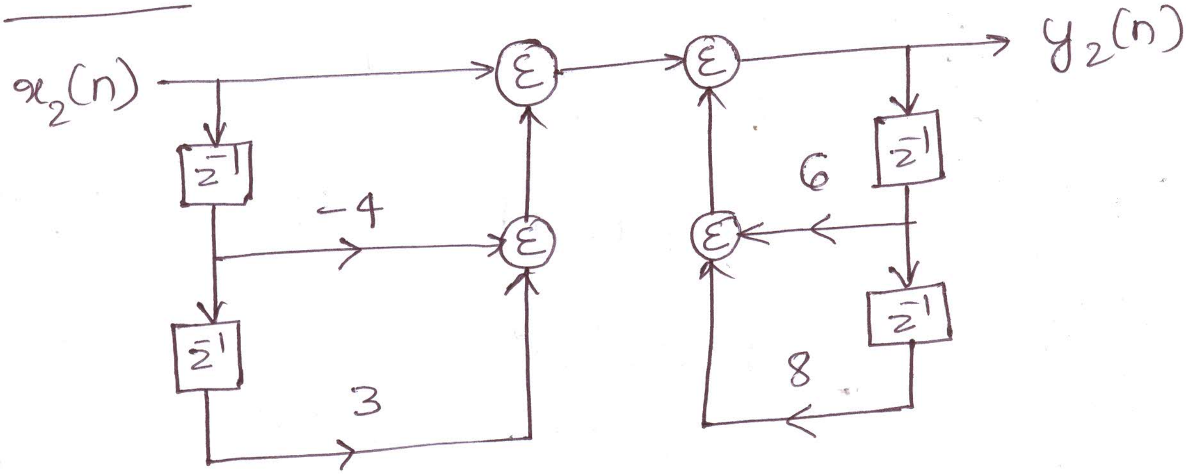
$$\frac{Y_2(z)}{X_2(z)} = \frac{1 - 4z^{-1} + 3z^{-2}}{1 - 6z^{-1} + 8z^{-2}}$$

$$Y_2(z)\left[1 - 6z^{-1} + 8z^{-2}\right] = X_2(z)\left[1 - 4z^{-1} + 3z^{-2}\right]$$

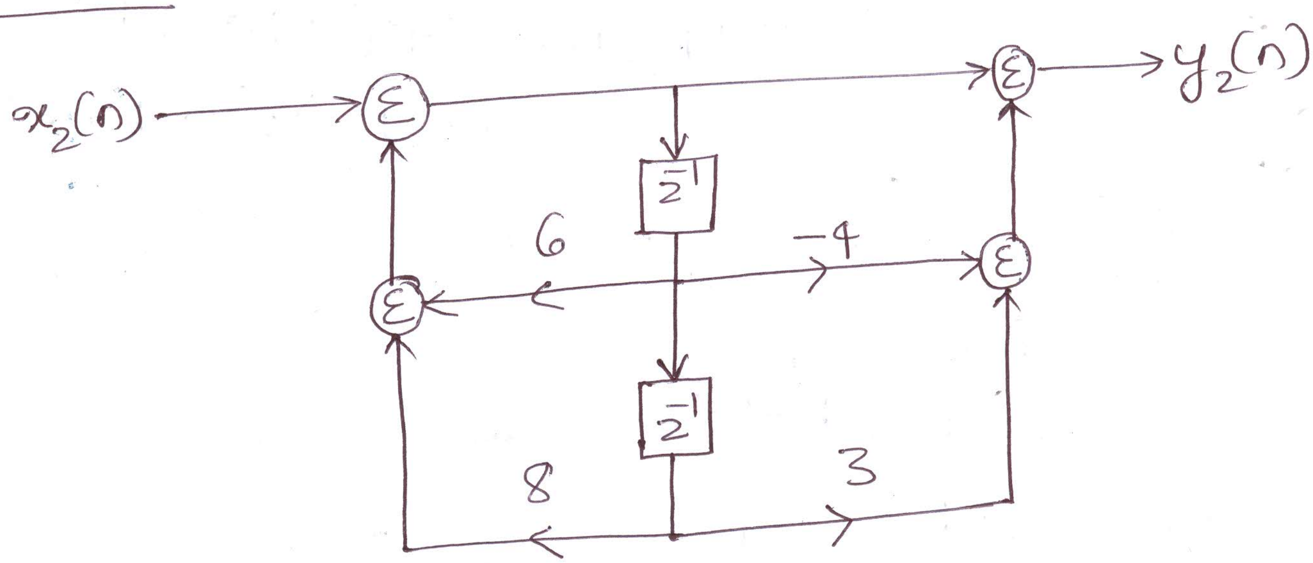
$$y_2(n) - 6y_2(n-1) + 8y_2(n-2) = x_2(n) - 4x_2(n-1) + 3x_2(n-2)$$

$$y_2(n) = x_2(n) - 4x_2(n-1) + 3x_2(n-2) + 6y_2(n-1) + 8y_2(n-2)$$

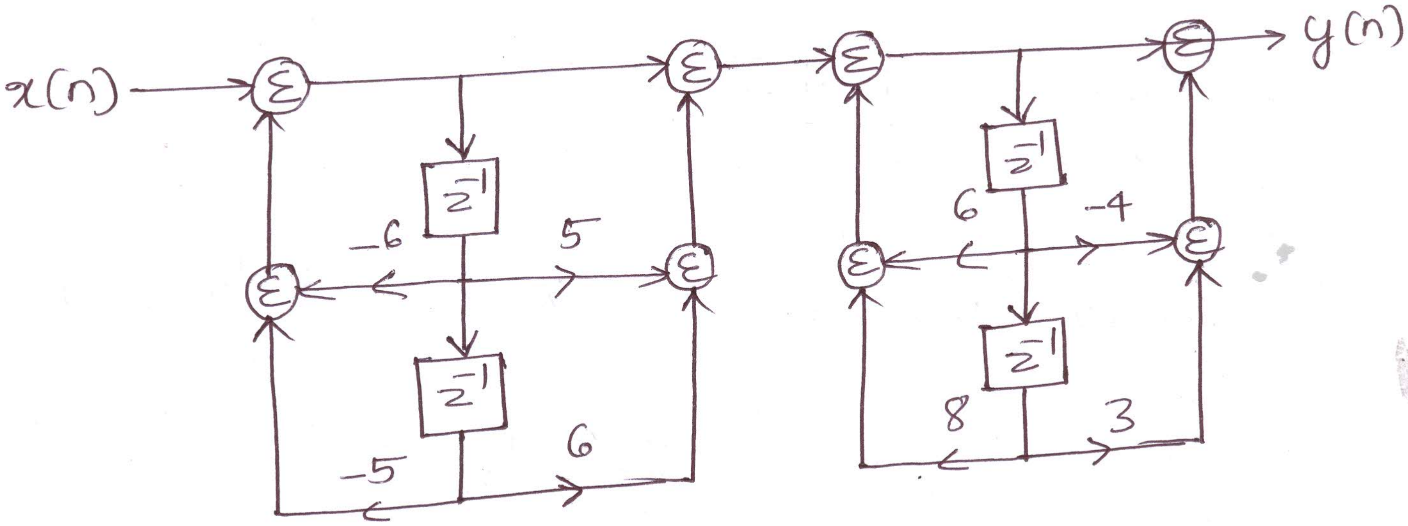
DF–I



DF–II



Cascade realization of $H(z)$.



35 Obtain the cascade realization of

$$H(z) = 1 + \frac{3}{4}z^{-1} + \frac{17}{8}z^{-2} + \frac{3}{4}z^{-3} + z^{-4}$$

Let $1 + \frac{3}{4}z^{-1} + \frac{17}{8}z^{-2} + \frac{3}{4}z^{-3} + z^{-4}$

$$= (1 + az^{-1} + bz^{-2})(1 + cz^{-1} + dz^{-2})$$

$$= 1 + cz^{-1} + dz^{-2} + az^{-1} + acz^{-2} + adz^{-3} + bz^{-2} + bcz^{-3} + bdz^{-4}$$

$$= 1 + (a+c)z^{-1} + (d+ac+b)z^{-2} + (ad+bc)z^{-3} + bdz^{-4}$$

Comparing the coefficients, we get,

$$a + c = \frac{3}{4}$$

$$d + ac + b = \frac{17}{8}$$

$$ad + bc = \frac{3}{4}$$

$$bd = 1$$

$d = 1$ (Assumption)

$\Rightarrow b = 1$

$d + ac + b = \frac{17}{8}$

$1 + ac + 1 = \frac{17}{8}$

$ac = \frac{17}{8} - 2 = \frac{1}{8}$

$a + c = \frac{3}{4}$

$c = \frac{3}{4} - a$

$a\left(\frac{3}{4} - a\right) = \frac{1}{8}$

$a^2 - \frac{3}{4}a + \frac{1}{8} = 0$

$\Rightarrow a = \frac{1}{2}$

$\therefore c = \frac{1}{4}$

$\therefore 1 + \frac{3}{4}z^{-1} + \frac{17}{8}z^{-2} + \frac{3}{4}z^{-3} + z^{-4}$

$= \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)$

Let $H_1(z) = 1 + \frac{1}{2}z^{-1} + z^{-2}$ and

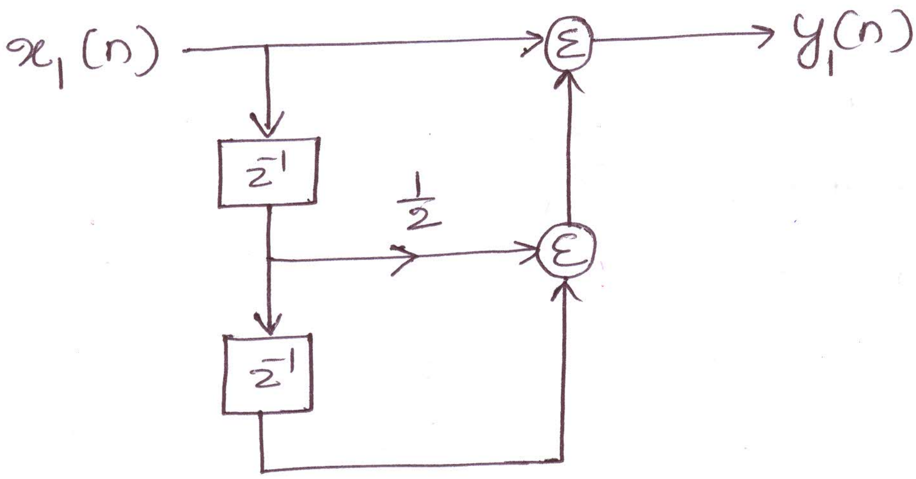
$H_2(z) = 1 + \frac{1}{4}z^{-1} + z^{-2}$

then $H(z) = H_1(z)\,H_2(z)$

Consider $H_1(z) = 1 + \frac{1}{2}z^{-1} + z^{-2} = \frac{Y_1(z)}{X_1(z)}$

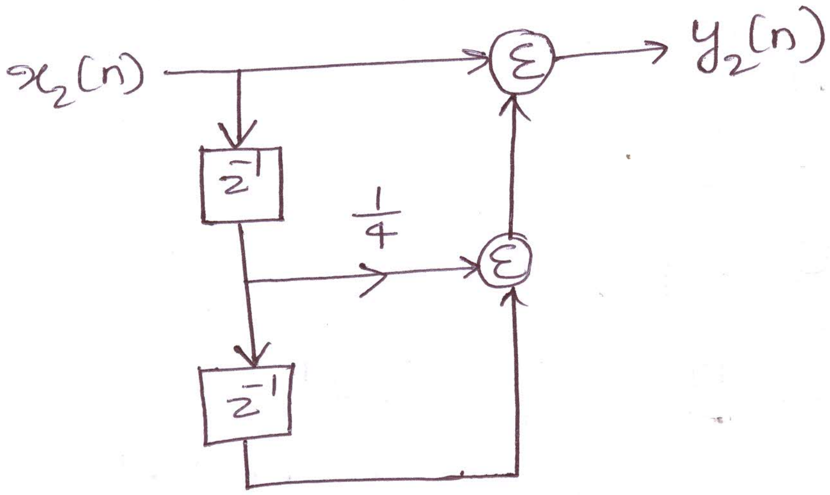
$\therefore Y_1(z) = X_1(z)\left[1 + \frac{1}{2}z^{-1} + z^{-2}\right]$

$y_1(n) = x_1(n) + \frac{1}{2}x_1(n-1) + x_1(n-2)$

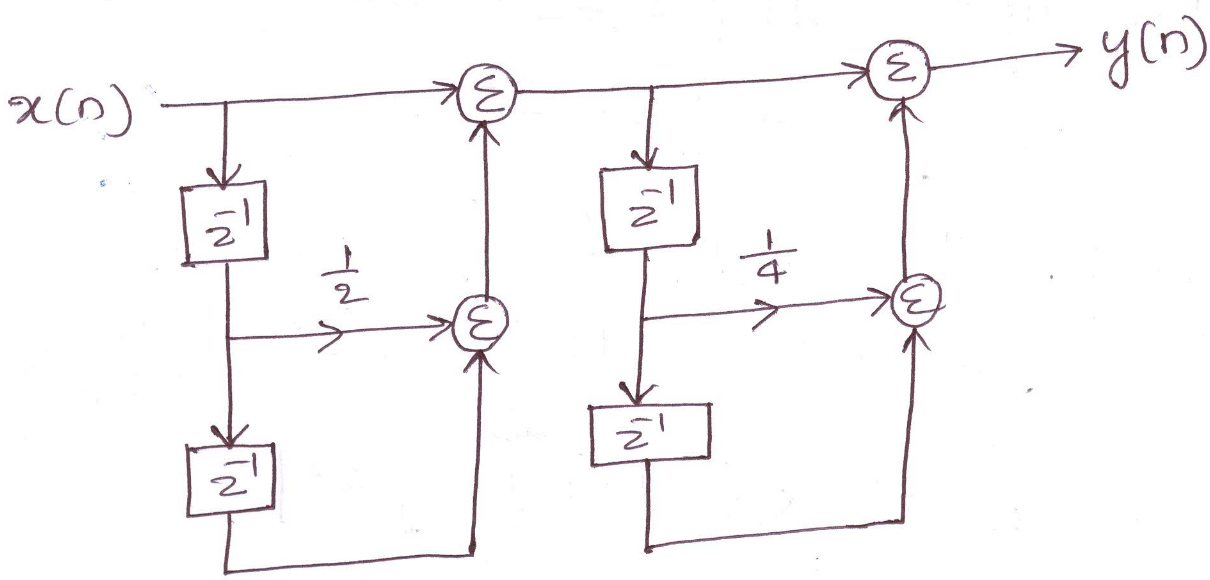


Consider $H_2(z) = 1 + \frac{1}{4} z^{-1} + z^{-2} = \dfrac{Y_2(z)}{X_2(z)}$

$$y_2(n) = x_2(n) + \frac{1}{4} x_2(n-1) + x_2(n-2)$$



Cascade realization of $H(z)$



36 Obtain the parallel realization of the system given by

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)}$$

$$H(z) = \frac{\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{A}{1-\frac{1}{5}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}}$$

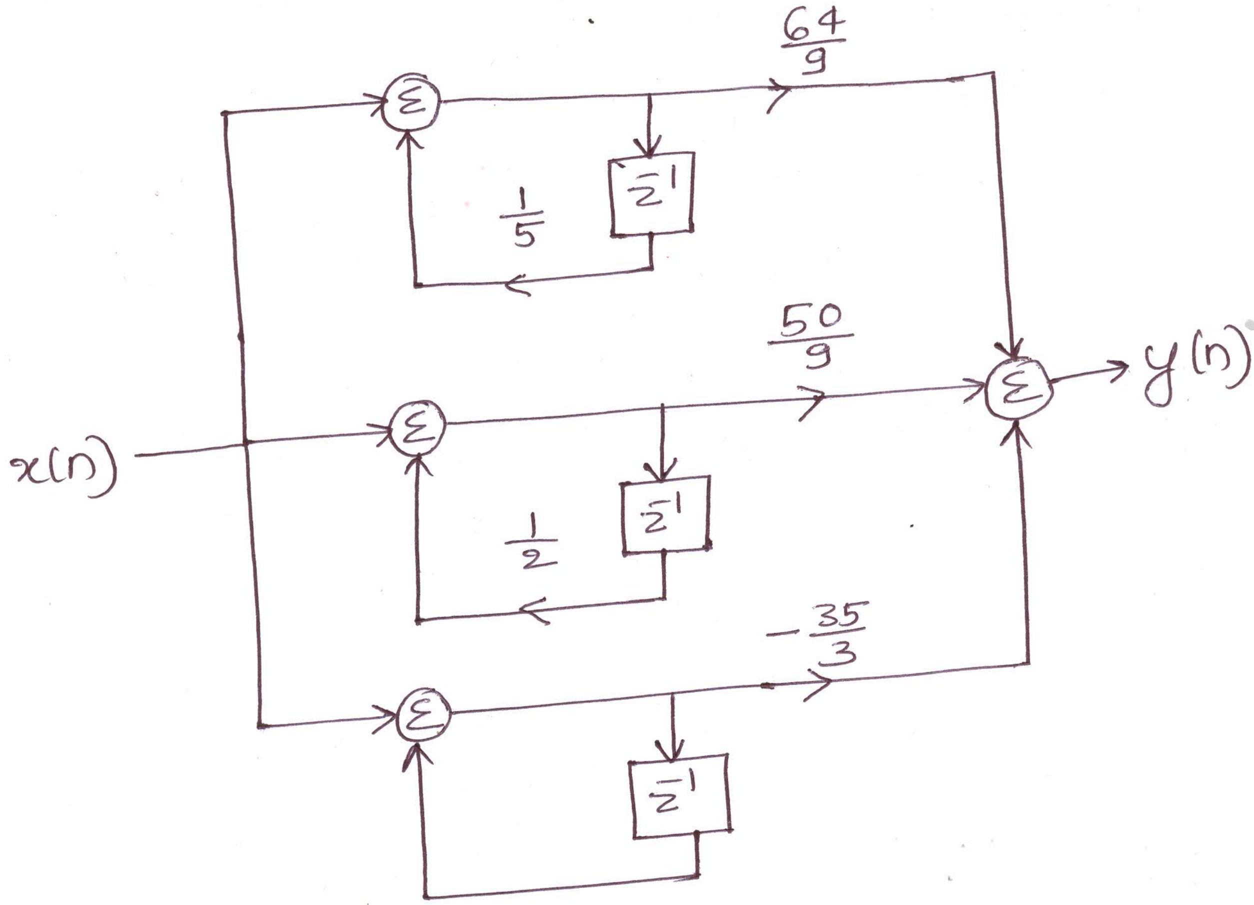
$$A = \left.\frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=5} = \frac{64}{9}$$

$$B = \left.\frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=2} = \frac{50}{9}$$

$$C = \left.\frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=4} = -\frac{35}{3}$$

$$\therefore H(z) = \frac{(64/9)}{1-\frac{1}{5}z^{-1}} + \frac{(50/9)}{1-\frac{1}{2}z^{-1}} + \frac{(-35/3)}{1-\frac{1}{4}z^{-1}}$$

Parallel realization of the system



37. Determine the parallel realization

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)}$$

Let

$$H(z) = \frac{A}{1-\frac{3}{4}z^{-1}} + \frac{B}{1-\frac{1}{8}z^{-1}} + \frac{C}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}} + \frac{D}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

$$A = \left.\frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)}\right|_{z^{-1}=\frac{4}{3}}$$

$$= 2.933$$

$$B = \left. \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]} \right|_{z^{-1}=8}$$

$$= -17.68$$

$$C = \left. \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right)} \right|_{z^{-1}=\frac{1}{\frac{1}{2}+j\frac{1}{2}}}$$

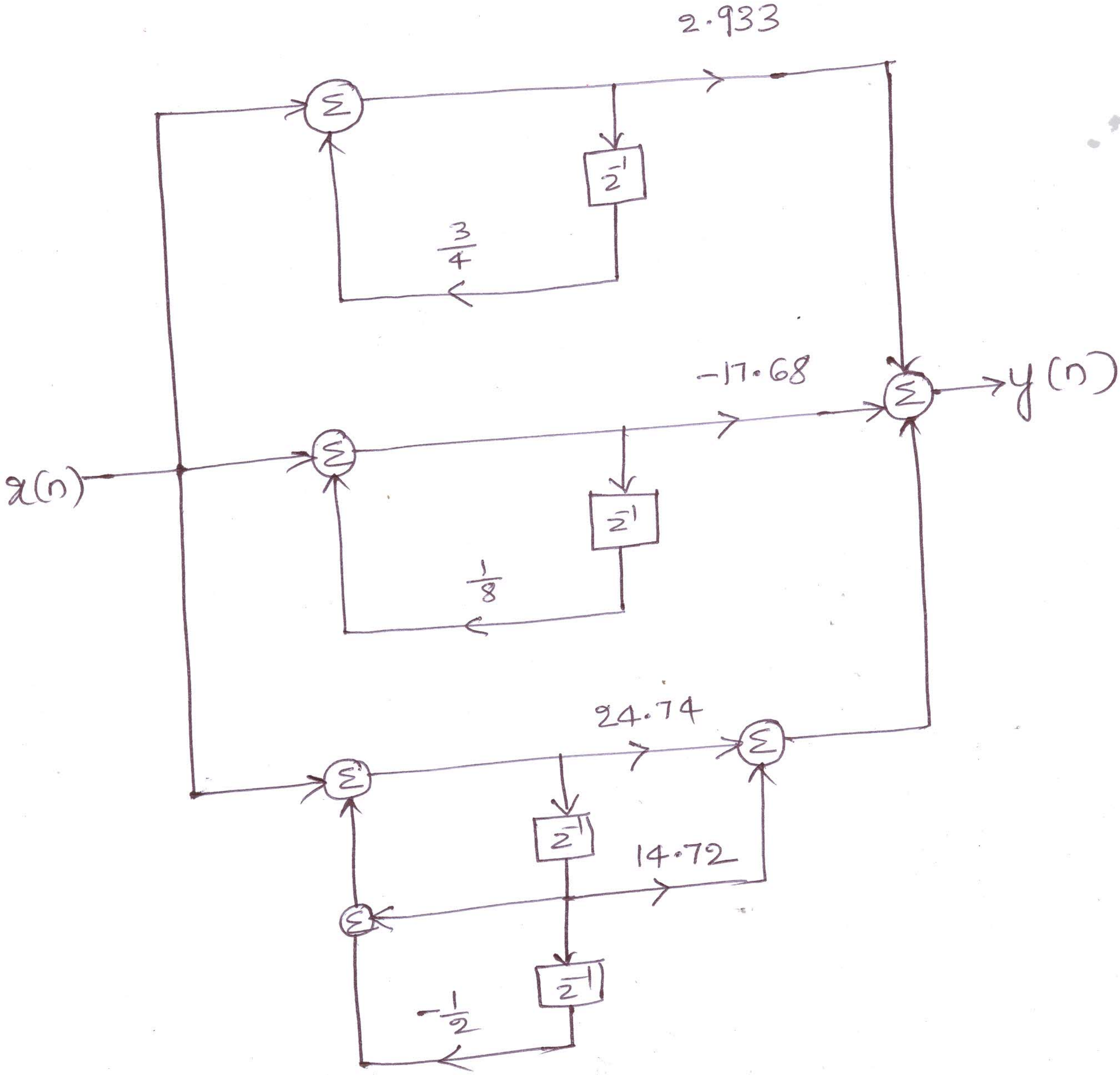
$$= 12.37 - 14.72j$$

$$D = 12.37 + 14.72j$$

$$\therefore H(z) = \frac{2.933}{1-\frac{3}{4}z^{-1}} - \frac{17.68}{1-\frac{1}{8}z^{-1}} + \frac{12.37-14.72j}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}} + \frac{12.37+14.72j}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

$$= \frac{2.933}{1-\frac{3}{4}z^{-1}} - \frac{17.68}{1-\frac{1}{8}z^{-1}} + \frac{24.74+14.72z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

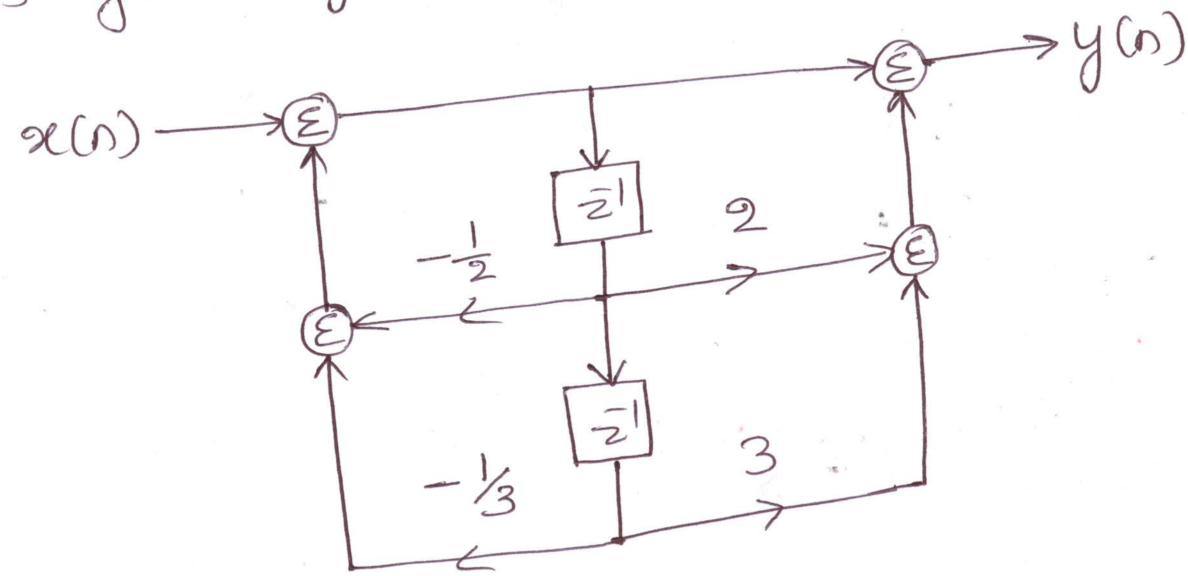
Parallel realization of the system:


2.933
z^{-1}
$\frac{3}{4}$
-17.68
$y(n)$
$x(n)$
z^{-1}
$\frac{1}{8}$
24.74
z^{-1}
14.72
z^{-1}
$-\frac{1}{2}$

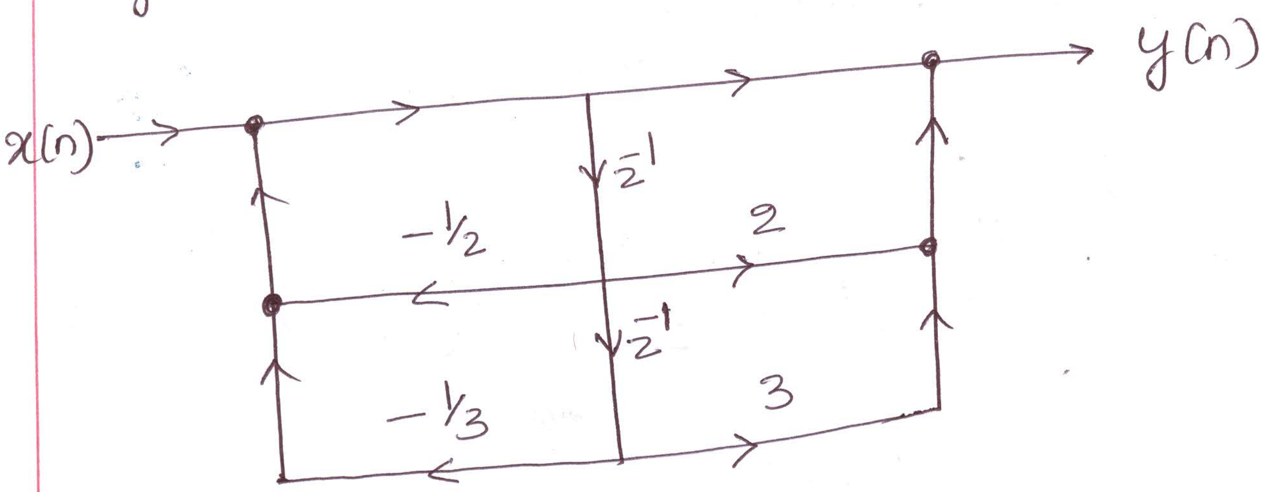
38 For the system function $H(z) = \dfrac{1+2z^{-1}+3z^{-2}}{1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$ draw the signal flow graph and the transposed structure.

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+3z^{-2}}{1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$

Direct form II realization of the system is given by,

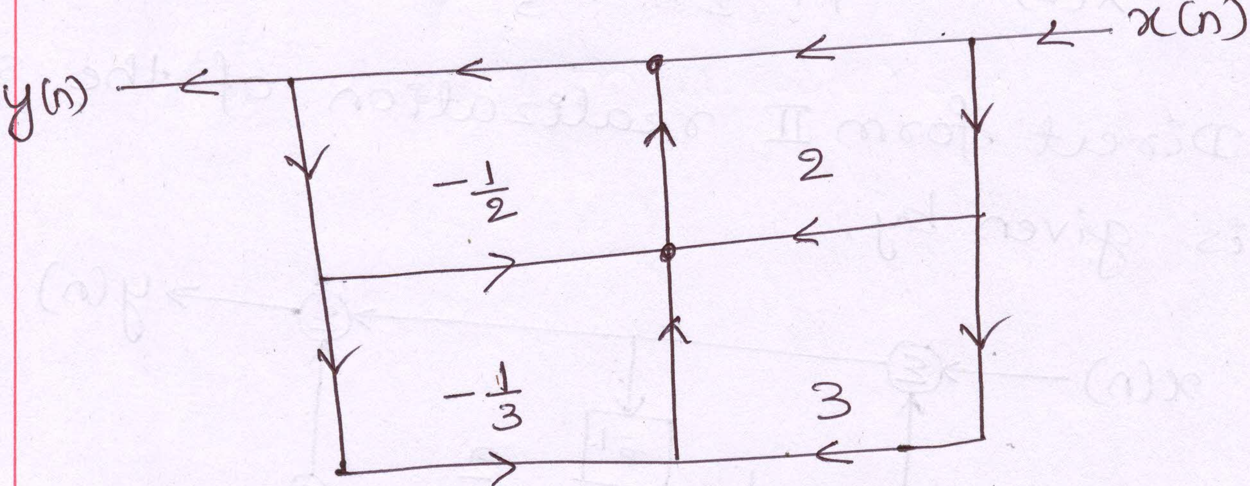


Signal flow graph:



According to transposition theorem, if the diretions of all the branches are reversed and positions of input and output are interchanged, the system function remains unchanged.

Transposed structure is given by



39. Obtain the parallel realization of

$$H(z) = \frac{6z^2+7z+1}{z^2+0.75z+0.125}$$

Note that $H(z)$ is not a proper fraction.

∴ We consider

$$\frac{H(z)}{z} = \frac{6z^2+7z+1}{z\,(z^2+0.75z+0.125)}$$

$$= \frac{A}{z} + \frac{B}{(z+0.25)} + \frac{C}{(z+0.5)}$$

$$A = \left.\frac{6z^2+7z+1}{(z+0.25)(z+0.5)}\right|_{z=0} = 8$$

$$B = \left. \frac{6z^2+7z+1}{z(z+0.5)} \right|_{z=-0.25} = 6$$

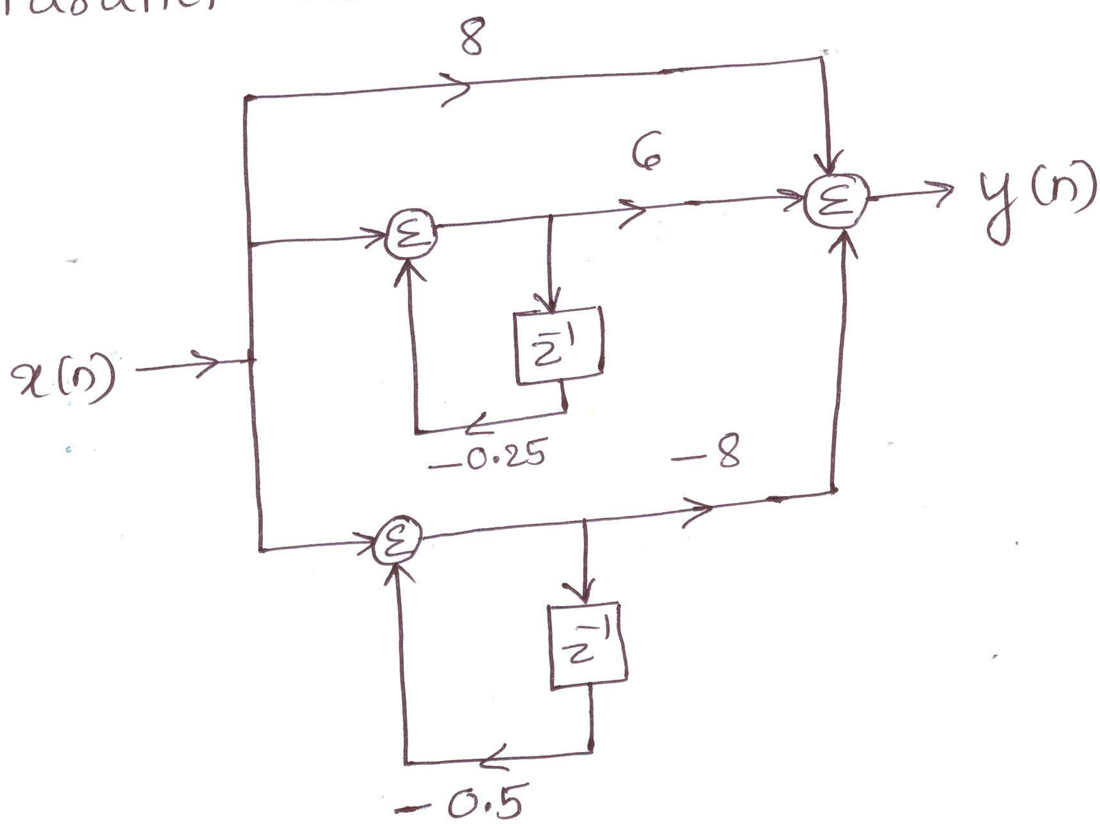
$$C = \left. \frac{6z^2+7z+1}{z(z+0.25)} \right|_{z=-0.5} = -8$$

$$\therefore \frac{H(z)}{z} = \frac{8}{z} + \frac{6}{z+0.25} - \frac{8}{(z+0.5)}$$

$$\therefore H(z) = 8 + \frac{6z}{z+0.25} - \frac{8z}{z+0.5}$$

$$= 8 + \frac{6}{1+0.25z^{-1}} - \frac{8}{1+0.5z^{-1}}$$

Parallel realization:



some questions from VTU old question papers follow.

1 What are the orders of Butterworth and Chebyshev filters necessary to meet the following specifications?

magnitude of ripples in the passband : 0.01

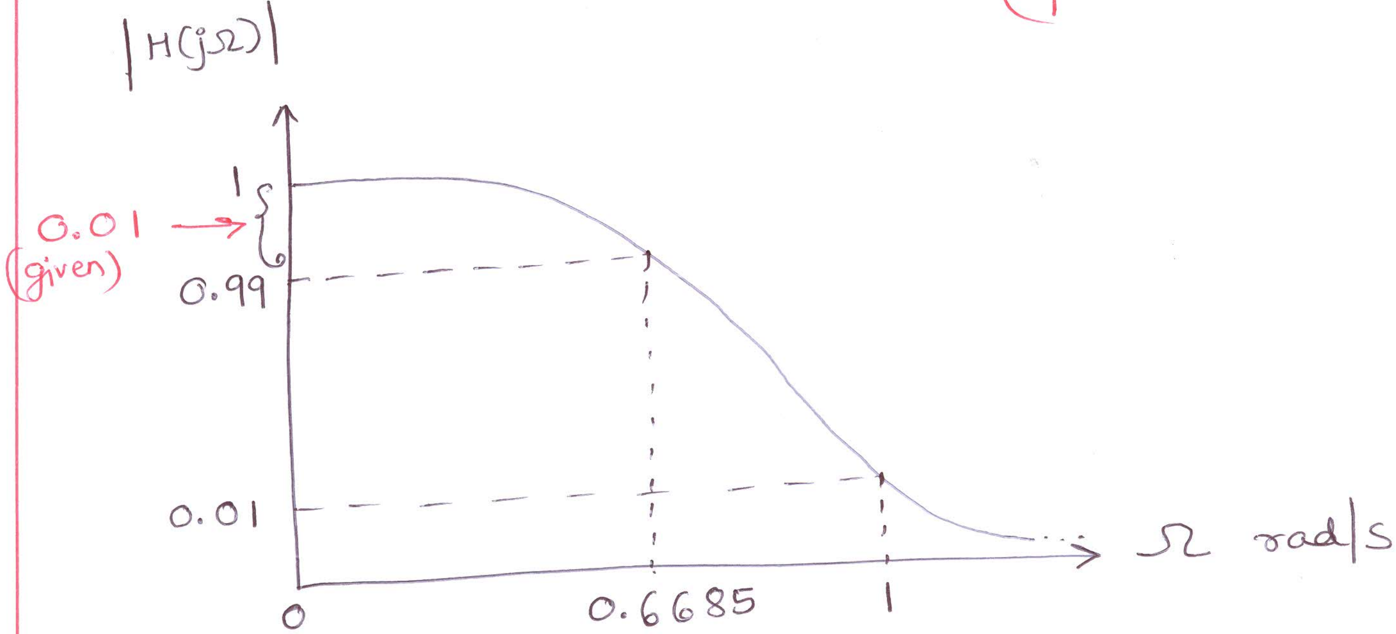
" " " " stopband : 0.01

passband edge frequency = 0.6685 rad/s

stopband edge frequency = 1 rad/s

(VTU, TCE, Dec. 2015)

(4 Marks)



$$A_{PB} = -20 \log (0.99)$$

$$= 0.0873 \text{ dB}$$

$$A_{SB} = -20 \log (0.01)$$

$$= 40 \text{ dB}$$

$$\Omega_{PB} = 0.6685 \text{ rad/s}$$

$$\Omega_{SB} = 1 \text{ rad/s}$$

Order of the Butterworth filter,

$$N = \frac{\log_{10}\left(\frac{10^{0.1 A_{PB}} - 1}{10^{0.1 A_{SB}} - 1}\right)}{2 \log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)}$$

$$= \frac{-5.69235}{2 \times (-0.174899)}$$

$$= 16.2732$$

We take $N = 17$.

Order of the Chebyshev filter,

$$N = \frac{\cosh^{-1}\sqrt{\left(\frac{10^{0.1 A_{SB}} - 1}{10^{0.1 A_{PB}} - 1}\right)}}{\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)}$$

$$= \frac{7.2467}{0.9587}$$

$$= 7.5586.$$

We take $N = 8$.

2 Design a high pass Butterworth filter of third order for cut-off frequency of 50 Hz by explicitly finding $H_n(s)$ of lowpass filter. (6 Marks)

(VTU, TCE, Dec. 2015)

Note: To design a highpass filter, we first design a normalized lowpass filter and then use frequency transformation to obtain the system function of high pass filter. Normalized lowpass filter means that the filter with cut-off frequency of 1 rad/s.

To design a normalized lowpass filter:

Given : $N = 3$, $\Omega_c = 1$ rad/s.

location of poles:

$$S_K = \Omega_c \left[-\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\cos\left(\frac{(2k+1)\pi}{2N}\right) \right]$$

$k = 0, 1, 2.$

$$S_0 = -\sin\left(\frac{\pi}{6}\right) + j\cos\left(\frac{\pi}{6}\right)$$

$$= -0.5 + j\,0.866$$

$$S_1 = -\Omega_c$$
$$= -1$$
$$S_2 = S_0^*$$
$$= -0.5 - j0.866$$

$$\therefore H_n(S) = \frac{(-1)^N \Omega_c^N}{(S-S_0)(S-S_1)(S-S_2)}$$

$$= \frac{1}{(S+0.5-j0.866)(S+1)(S+0.5+j0.866)}$$

$$= \frac{1}{(S+1)\left[(S+0.5)^2 + 0.866^2\right]}$$

$$= \frac{1}{(S+1)\left[S^2 + 2\times0.5\times S + 0.5^2 + 0.866^2\right]}$$

$$= \frac{1}{(S+1)\left[S^2 + S + 1\right]}$$

This is the transfer function of the 3rd order, normalized Butterworth filter.

To find the transfer function of highpass filter, of cut-off frequency 50Hz, ie,

$$\Omega_c = 2\pi f_c \text{ rad/s}$$

$$= 2\pi \times 100 \text{ rad/s}$$

$$= 200\pi \text{ rad/s.}$$

Use the lowpass to highpass frequency transformation.

Replace 's' in $H_n(s)$ with $\frac{200\pi}{s}$.

∴ Required transfer function,

$$H(s) = H_n(s)\Big|_{s \to \frac{200\pi}{s}}$$

$$= \frac{1}{\left(\frac{200\pi}{s}+1\right)\left[\left(\frac{200\pi}{s}\right)^2+\left(\frac{200\pi}{s}\right)+1\right]}$$

$$= \frac{1}{\left(\frac{200\pi+s}{s}\right)\left[\frac{40000\pi^2}{s^2}+\frac{200\pi}{s}+1\right]}$$

$$= \frac{1}{\left(\frac{200\pi+s}{s}\right)\left[\frac{40000\pi^2+200\pi s+s^2}{s^2}\right]}$$

$$= \frac{s^3}{(s+200\pi)(s^2+200\pi s+40000\pi^2)}$$

3 Find the transfer function of 2nd order normalized analog filter with equiripple characteristic in passband and monotonic fall-off characteristics in stopband having 1 dB passband ripples.

(6 Marks)

(VTU, TCE, Dec. 2015)

Note: Equiripple characteristic in passband and monotonic characteristic in stopband indicates that we have to design Chebyshev filter of Type-I.

Monotonic characteristic in both passband and stopband indicates that we have to design Butterworth filter.

Given: $N = 2$, $A_{PB} = 1$ dB

$$\varepsilon = \sqrt{10^{0.1A_{PB}} - 1} = 0.5088$$

$$\beta = \left[\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right]^{\frac{1}{N}} = 2.0422$$

$$\sigma_k = -\Omega_{PB} \cdot \left(\frac{\beta^2-1}{2\beta}\right) \sin\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1, \ldots N-1$

$$\sigma_0 = -\left(\frac{\beta^2-1}{2\beta}\right) \sin\left(\frac{\pi}{4}\right)$$

$$= -0.7763 \times 0.7071$$

$$= -0.5489.$$

Note: Consider $\Omega_{PB} = 1$ rad/s, since we have to design a normalized filter.

$$\sigma_1 = -\left(\frac{\beta^2-1}{2\beta}\right) \sin\left(\frac{3\pi}{4}\right)$$

$$= -0.5489.$$

$$\Omega_k = \Omega_{PB}\left(\frac{\beta^2+1}{2\beta}\right) \cos\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1$

$$\Omega_0 = \frac{\beta^2+1}{2\beta} \cos\left(\frac{\pi}{4}\right)$$

$$= \cancel{+0.5489} \quad 0.8952$$

$\Omega_1 = -0.8952$

$\therefore$ Pole locations are,

$s_K = \sigma_K + j\Omega_K, \quad K = 0, 1$

$s_0 = -0.5489 + j0.8952$

$s_1 = -0.5489 - j0.8952$

$$b_0 = \begin{cases} \dfrac{1}{\sqrt{1+\epsilon^2}}, & \text{if } N \text{ is even} \\ 1, & \text{if } N \text{ is odd.} \end{cases}$$

Since, $N = 2$,

$$b_0 = \frac{1}{\sqrt{1+\epsilon^2}} = 0.8913$$

$$\therefore H(s) = \frac{(-1)^N b_0 \prod_{K=0}^{N-1} s_K}{\prod_{K=0}^{N-1} (s - s_K)}$$

$$= \frac{(-1)^2 \times 0.8913 \times s_0 \times s_1}{(s - s_0)(s - s_1)}$$

$$= \frac{0.8913 \times 1.10267}{s^2 - (s_0 + s_1)s + s_0 s_1}$$

$$= \frac{0.9828}{s^2 + 1.0978s + 1.10267}$$

4 A linear time invariant digital IIR filter is specified by the following transfer function.

$$H(z) = \frac{(z-1)(z-2)(z+1)z}{\left[z-\left(\frac{1}{2}+\frac{1}{2}j\right)\right]\left[z-\left(\frac{1}{2}-\frac{1}{2}j\right)\right]\left[z-j\frac{1}{4}\right]\left[z+j\frac{1}{4}\right]}$$

Realize the system in the following forms

i) direct form I

ii) direct form II

(VTU, Dec. 2017, 12 Marks)

$$H(z) = \frac{(z^2-2z-z+2)(z^2+z)}{\left[\left(z-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]\left[z^2+\left(\frac{1}{4}\right)^2\right]}$$

$$= \frac{z^4-2z^3-z^3+2z^2+z^3-2z^2-z^2+2z}{\left[z^2+\frac{1}{4}-z+\frac{1}{4}\right]\left[z^2+\frac{1}{16}\right]}$$

$$= \frac{z^4-2z^3-z^2+2z}{z^4-z^3+\frac{1}{2}z^2+\frac{1}{16}z^2-\frac{1}{16}z+\frac{1}{32}}$$

$$= \frac{z^4-2z^3-z^2+2z}{z^4-z^3+\frac{9}{16}z^2-\frac{1}{16}z+\frac{1}{32}}$$

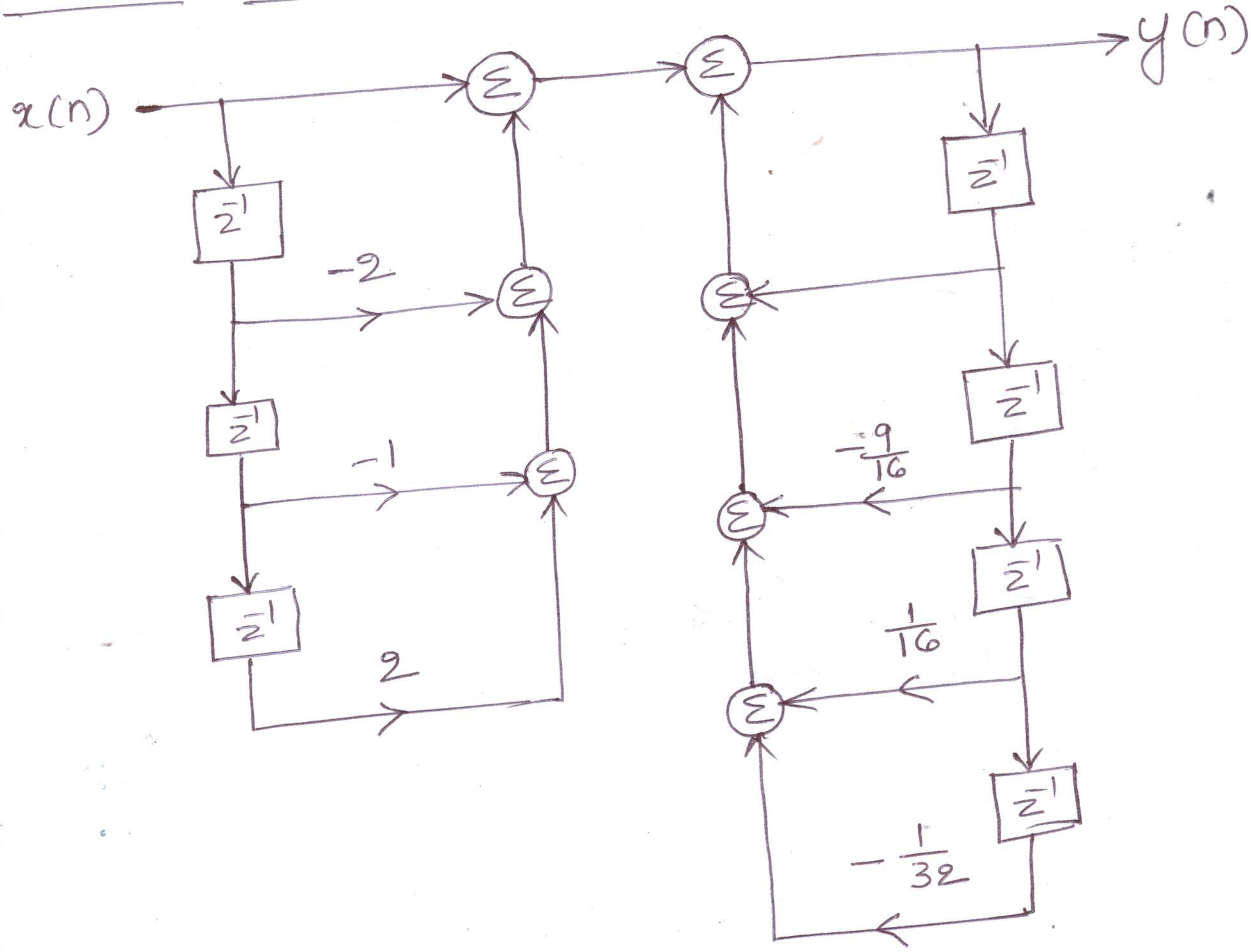
$$= \frac{1-2z^{-1}-z^{-2}+2z^{-3}}{1-z^{-1}+\frac{9}{16}z^{-2}-\frac{1}{16}z^{-3}+\frac{1}{32}z^{-4}}$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}-z^{-2}+2z^{-3}}{1-z^{-1}+\frac{9}{16}z^{-2}-\frac{1}{16}z^{-3}+\frac{1}{32}z^{-4}}$$

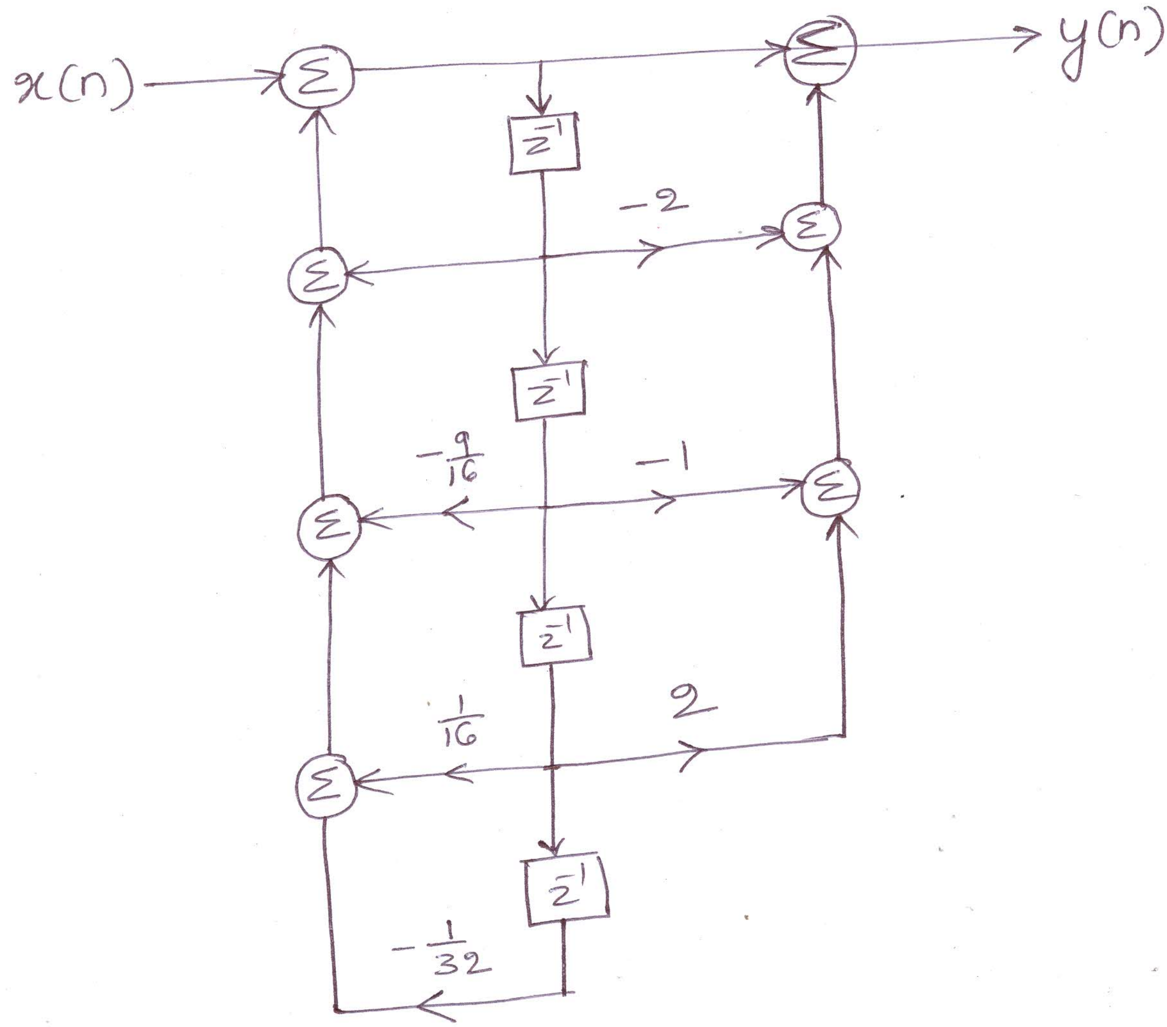
$$\therefore y(n)-y(n-1)+\frac{9}{16}y(n-2)-\frac{1}{16}y(n-3)+\frac{1}{32}y(n-4)$$
$$= x(n)-2x(n-1)-x(n-2)+2x(n-3)$$

$$\therefore y(n) = x(n)-2x(n-1)-x(n-2)+2x(n-3)+y(n-1)$$
$$-\frac{9}{16}y(n-2)+\frac{1}{16}y(n-3)-\frac{1}{32}y(n-4)$$

Direct form-I



Direct form II



5 Obtain a cascade realization of system function given below.

$$H(z) = \frac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$$

(Dec 2017, 04 Marks, VTU)

Let $H_1(z) = \dfrac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$ and $H_2(z) = \dfrac{(1+z^{-1})^2}{1-z^{-1}+\frac{1}{2}z^{-2}}$

Note that $H(z) = H_1(z)\,H_2(z)$

$$H_1(z) = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}.$$

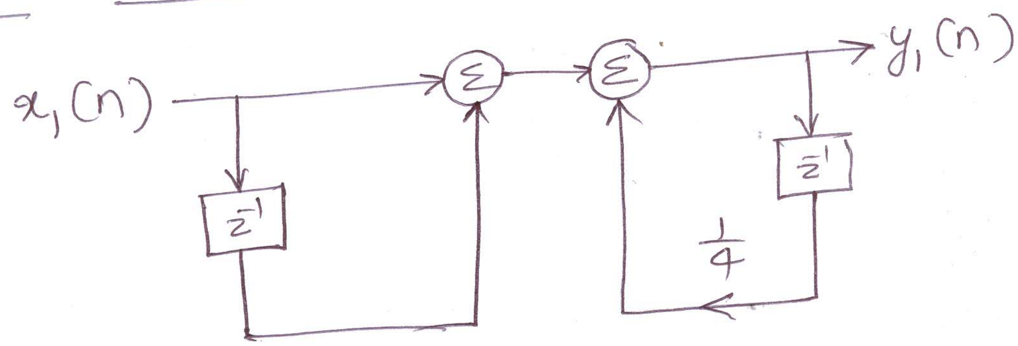
$$\therefore \frac{Y_1(z)}{X_1(z)} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$$

$$Y_1(z)\left[1-\frac{1}{4}z^{-1}\right] = X_1(z)\left[1+z^{-1}\right]$$

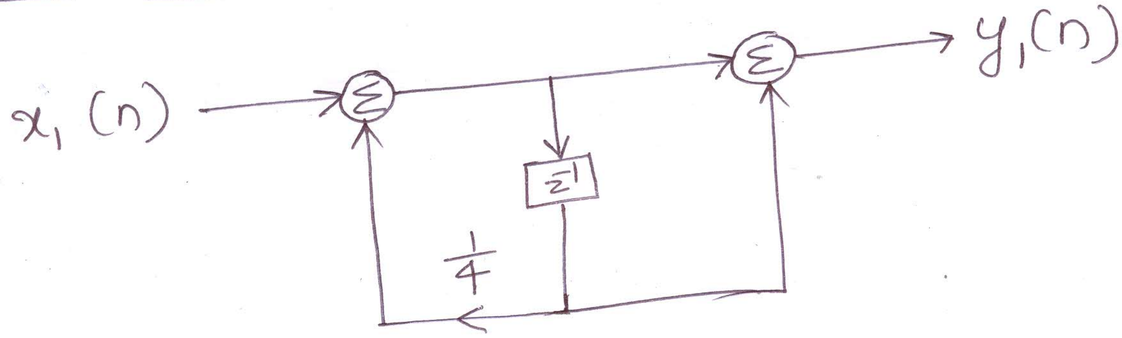
$$y_1(n) - \frac{1}{4}y_1(n-1) = x_1(n) + x_1(n-1)$$

$$\therefore y_1(n) = x_1(n) + x_1(n-1) + \frac{1}{4}y_1(n-1)$$

DF-I realization



DF-II realization



$$H_2(z) = \frac{(1+z^{-1})^2}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

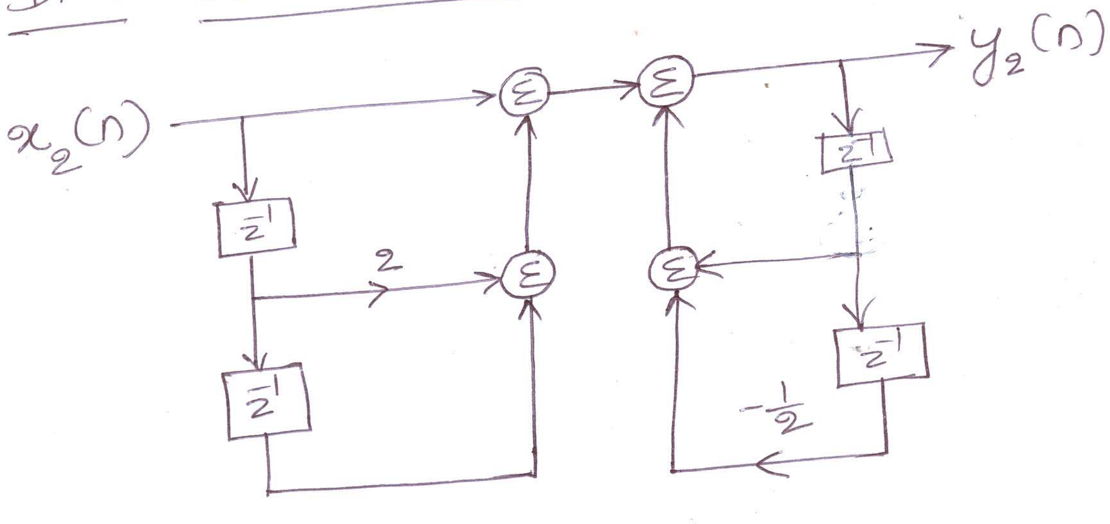
$$= \frac{1 + z^{-2} + 2z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

$$\therefore \frac{Y_2(z)}{X_2(z)} = \frac{1 + z^{-2} + 2z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

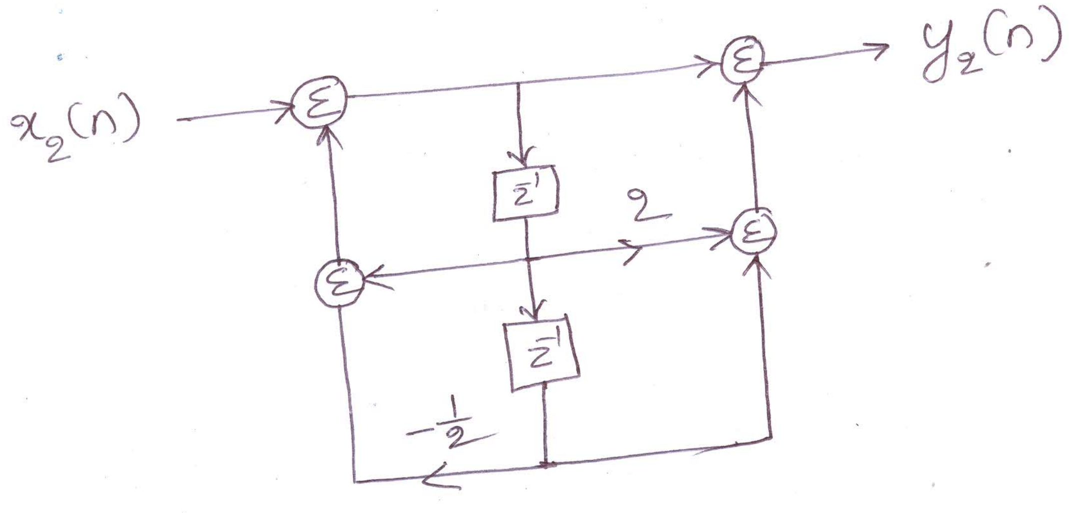
$$\therefore y_2(n) - y_2(n-1) + \frac{1}{2} y_2(n-2) = x_2(n) + 2x_2(n-1) + x_2(n-2)$$

$$\therefore y_2(n) = x_2(n) + 2x_2(n-1) + x_2(n-2) + y_2(n-1) - \frac{1}{2} y_2(n-2)$$

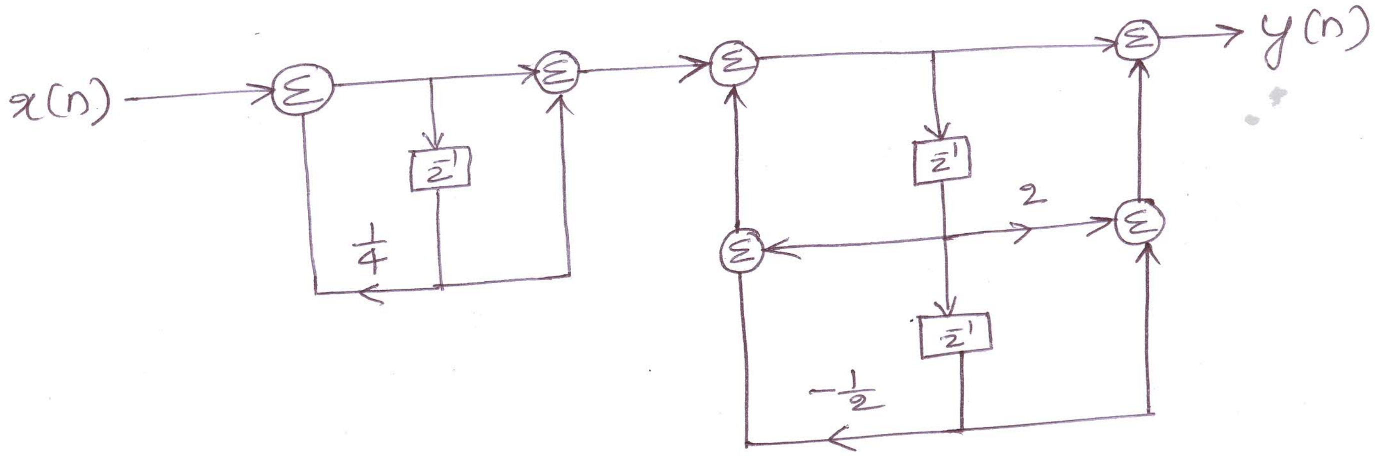
<u>DF-I</u> <u>realization</u>



<u>DF-II</u> <u>realization</u>



Cascade realization of $H(z)$ is ~~given by~~ obtained by cascading DF-II realization of $H_1(z)$ and $H_2(z)$.



6 Obtain the parallel realization of

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

(VTU, June 2018, 5 Marks)

$H(z)$ is an improper fraction.

$\therefore$ Let us consider $\frac{H(z)}{z}$.

$$\frac{H(z)}{z} = \frac{8z^3 - 4z^2 + 11z - 2}{z\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

Let $$\frac{8z^3 - 4z^2 + 11z - 2}{z\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)} = \frac{A}{z} + \frac{B}{z - \frac{1}{4}} + \frac{Cz + D}{z^2 - z + \frac{1}{2}}$$

$$A = \left.\frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}\right|_{z=0} = \frac{-2}{\left(-\frac{1}{4}\right)\left(\frac{1}{2}\right)} = 16$$

$$B = \left. \frac{8z^3 - 4z^2 + 11z - 2}{z\left(z^2 - z + \frac{1}{2}\right)} \right|_{z=\frac{1}{4}} = 8$$

$$8z^3 - 4z^2 + 11z - 2 = A\left(z - \tfrac{1}{4}\right)\left(z^2 - z + \tfrac{1}{2}\right) +$$
$$B(z)\left(z^2 - z + \tfrac{1}{2}\right) +$$
$$(Cz + d)(z)\left(z - \tfrac{1}{4}\right)$$

Comparing coefficients of z^3, we get

$$8 = A + B + C$$

$$\therefore \quad C = 8 - A - B = -16$$

Comparing coefficients of z^2, we get

$$-4 = -A - \tfrac{1}{4}A + d - \tfrac{1}{4}C$$

$$\therefore \quad d = -4 + 1.25A + 0.25C$$
$$= 12$$

$$\therefore \frac{H(z)}{z} = \frac{16}{z} + \frac{8}{z - \frac{1}{4}} + \frac{-16z + 12}{z^2 - z + \frac{1}{2}}$$

$$\therefore H(z) = 16 + \frac{8z}{z - \frac{1}{4}} + \frac{-16z^2 + 12z}{z^2 - z + \frac{1}{2}}$$

Let $\quad H_1(z) = 16$

$$H_2(z) = \frac{8z}{z-\frac{1}{4}}$$

$$H_3(z) = \frac{-16z^2+12z}{z^2-z+\frac{1}{2}}$$

Then, $H(z) = H_1(z) + H_2(z) + H_3(z)$

Consider $H_1(z) = 16$

$$\therefore \frac{Y_1(z)}{X_1(z)} = 16$$

$$\therefore y_1(n) = 16x_1(n)$$

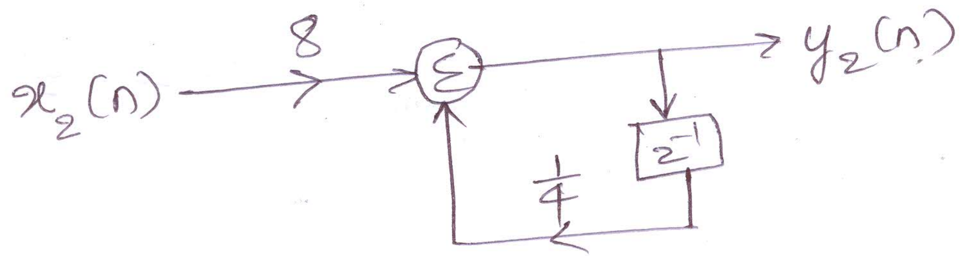
$$x_1(n) \xrightarrow{16} y_1(n)$$

Consider $H_2(z) = \dfrac{8z}{z-\frac{1}{4}}$

$$\frac{Y_2(z)}{X_2(z)} = \frac{8}{1-\frac{1}{4}z^{-1}}$$

$$y_2(n) - \frac{1}{4}y_2(n-1) = 8x_2(n)$$

$$\therefore y_2(n) = 8x_2(n) + \frac{1}{4}y_2(n-1)$$



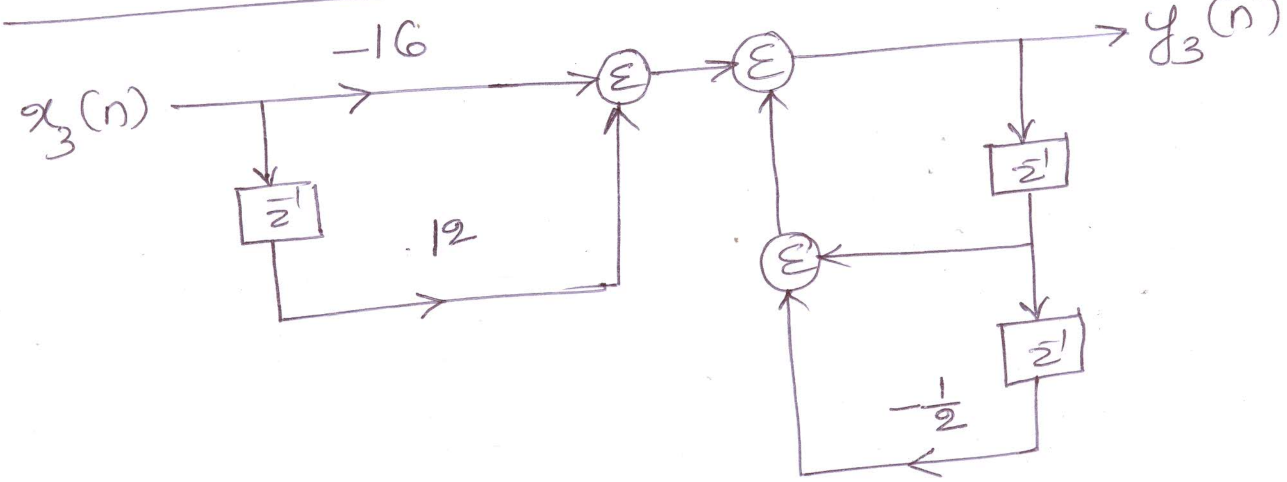
Consider $H_3(z) = \dfrac{-16z^2 + 12z}{z^2 - z + \frac{1}{2}}$

$$\therefore \frac{Y_3(z)}{X_3(z)} = \frac{-16 + 12z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

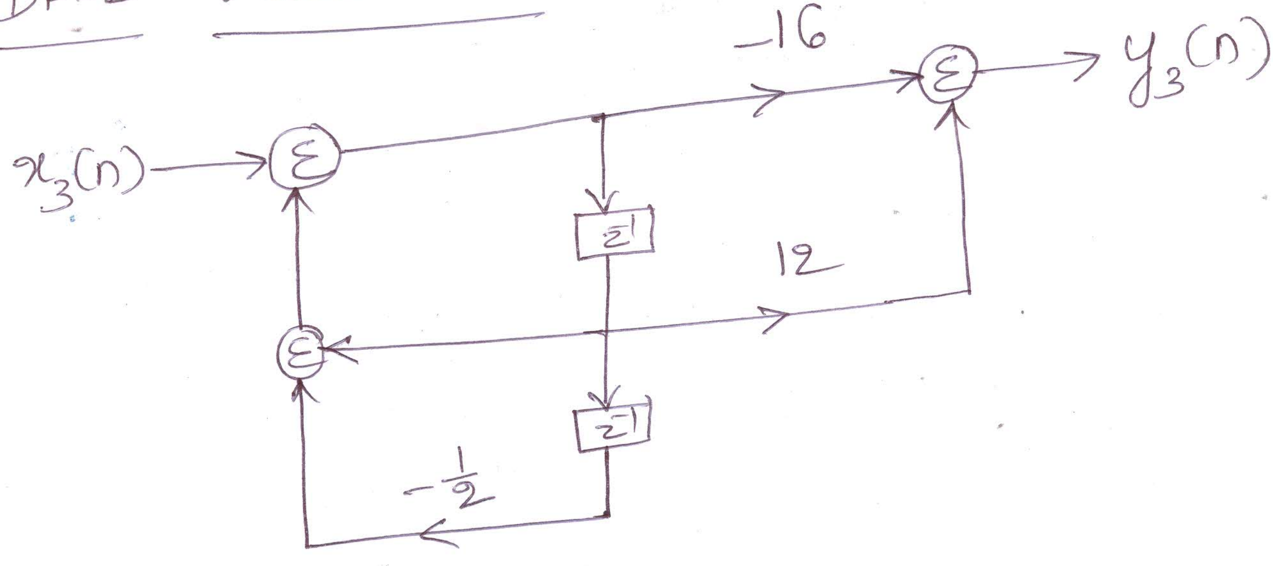
$$y_3(n) - y_3(n-1) + \frac{1}{2}y_3(n-2) = -16x_3(n) + 12x_3(n-1)$$

$$y_3(n) = -16x_3(n) + 12x_3(n-1) + y_3(n-1) - \frac{1}{2}y_3(n-2)$$

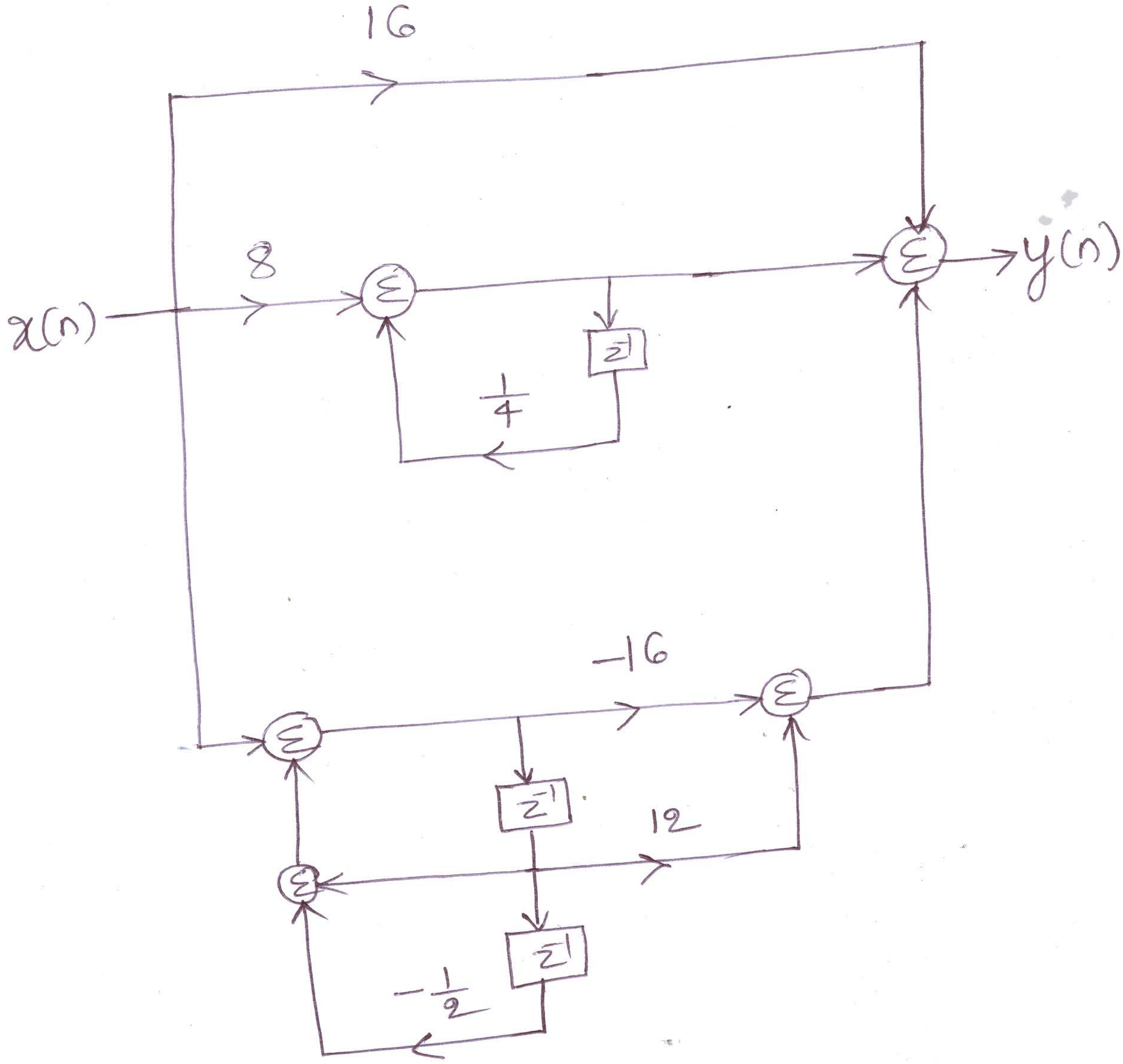
DF-I realization



DF-II realization



Parallel realization



7 Obtain the parallel realization of the system function

$$H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})(1+\frac{1}{8}z^{-1})}$$

(VTU Model Question Paper, 6 Marks)

Let $H(z) = \dfrac{A}{1+\frac{1}{2}z^{-1}} + \dfrac{B}{1-\frac{1}{2}z^{-1}} + \dfrac{C}{1+\frac{1}{8}z^{-1}}$

$$A = \left.\frac{(1+z^{-1})(1+2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{8}z^{-1})}\right|_{z^{-1}=(-2)} = 2$$

$$B = \left.\frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{8}z^{-1})}\right|_{z^{-1}=2} = 6$$

$$C = \left.\frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})}\right|_{z^{-1}=-8} = -7$$

$$\therefore\ H(z) = \frac{2}{1+\frac{1}{2}z^{-1}} + \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{7}{1+\frac{1}{8}z^{-1}}$$