MODULE 2

ERROR CONTROL CODE

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Introduction

The purpose of error control coding is to enable the receiver to detect or even correct the errors by introducing some redundancies in to the data to be transmitted.

There are basically two mechanisms for adding redundancy:

1. Block coding
2. Convolutional coding

Types of codes

i) Block Codes:

Block code consists of **(n-k)** number of check bits(<u>redundant bits)</u> being added to **k** number of information bits to form 'n' bit code-words.

i) Convolutional code:

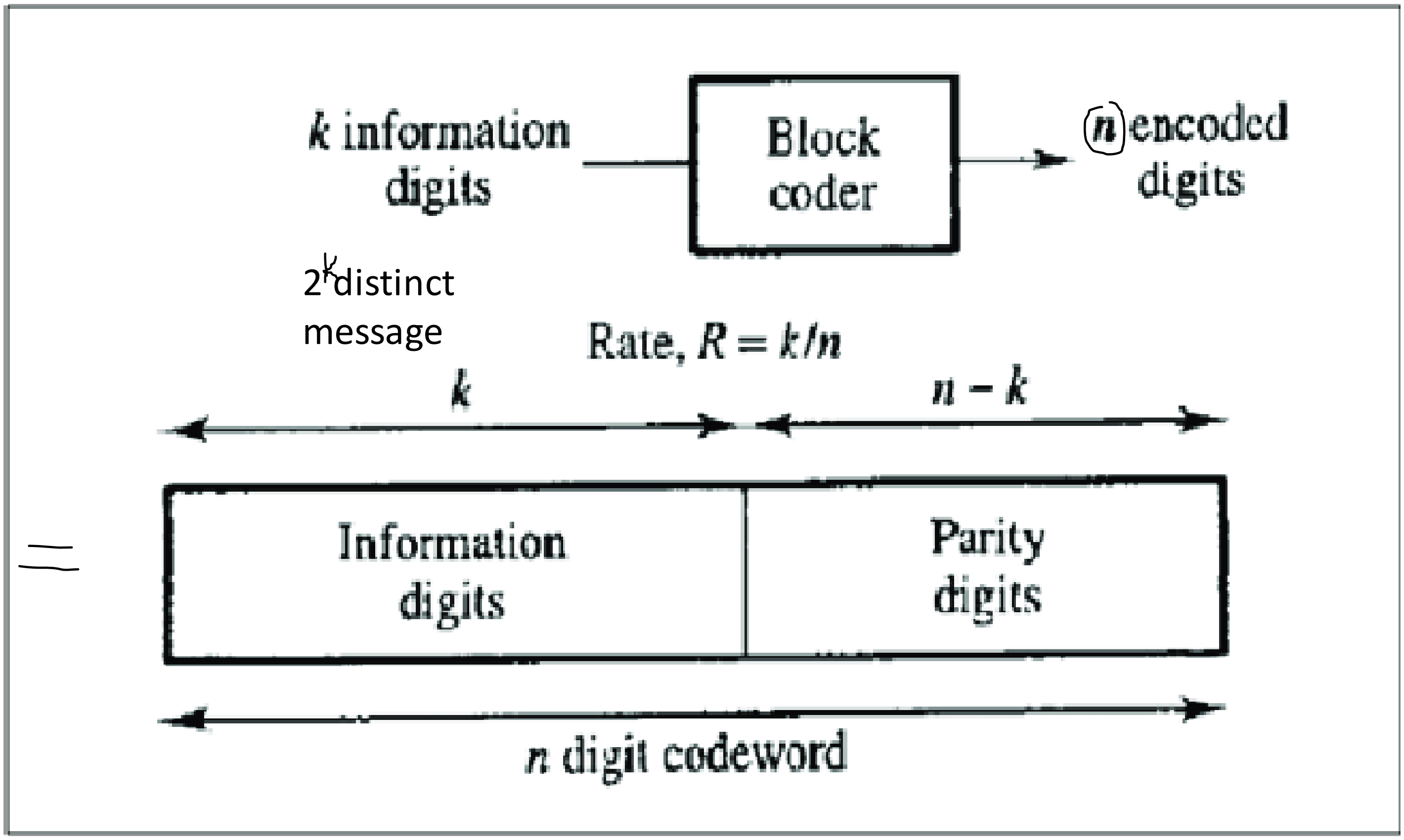
In this code, input databits are fed as streams of data bits which convolve to output bits based upon the logic function of the encoder.

Linear Block codes

- Let C_1 and C_2 be any two code words(n-bits) belonging to a set of (n, k) block code

- If $C_1 \oplus C_2$, is also a n-bit code word belonging to the same set of(n,k) block code, such a block code is called (n,k)linear block code.

Illustrating the formation of linear block codes



Matrix description of linear block code

- Let the message block of k-bits(code-words) be represented as a "row-vector" or "k-tuple" called "message vector" is given by

  $[D]=\{ d_1, d_2, ...d_k\}$

- 2^k code-vectors can be represented by

  $C=\{c_1,c_2,.......c_n\}$

- Also $c_i=d_i$ for all i=1,2,......k
- $[C]= \{c_1,c_2,........,c_k, \; c_{k+1}, c_{k+2},.......c_n \}$

- (n-k) number of check bits $c_{k+1}, c_{k+2},.......c_n$ are derived from 'k' message bits using a predetermined rule as below

$c_{k+1} = p_{11}d_1 + p_{21}d_2 +...............+ p_{k1}d_k$

$c_{k+2} = p_{12}d_1 + p_{22}d_2 +...............+ p_{k2}d_k$

:

:

$c_{k+1} = p_{1.\,n-k}d_1 + p_{2.n-k}d_2 +...............+ p_{k.n-k}d_k$

- In matrix form,

$$[c_1,c_2,..,c_k,\ c_{k+1}, c_{k+2},...c_n]=[d_1, d_2, ...d_k] \begin{bmatrix} 1\ 0\ 0...0 & p_{11}\ p_{12}\ ....\ p_{1.\,n-k} \\ 0\ 1\ 0...0 & p_{21}\ p_{22}\ ....\ p_{2.\,n-k} \\ & \\ 0\ 0\ 0... & p_{k1}\ p_{k2}\ ....\ p_{k.\,n-k} \end{bmatrix}$$

[C]= [D] [G]

- [G] is called as ***generator matrix*** of order (k x n)
- $[G] = [I_k \mid P]_{(k \times n)}$ where I_k unit matrix of order 'k'

  [P]= Parity matrix of order k x (n-k)
- Also $[G] = [P \mid I_k]$

PARITY CHECK MATRIX [H]

- $[H] = [P^T \mid I_{n-k}]$

$$\therefore [H] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{k1} & 1\ 0\ 0 \ldots\ldots. 0 \\ p_{12} & p_{22} & \cdots & p_{k2} & 0\ 1\ 0 \ldots\ldots. 0 \\ & . & & & \\ & . & & & \\ p_{1,\ n-k} & p_{2,\ n-k} & \cdots & P_{k,n-k} & 0\ 0\ 0\ \ldots..1 \end{bmatrix}$$

[H] matrix is a (n-k) x (n) matrix.

For a systematic (6,3) linear block code, the parity matrix P is given by

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find all possible code-vectors

- Solution:

Given n=6, k=3,

Since k=3, 2^k =8 message vectors given by (000)

(001), (010), (001), (011), (100), (101), (111)

- [C]= [D] [G]

where $[G] = [I_k \mid P]$

$[I_3 \mid P]$

$$[G]= \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 1 \\ 0 & 1 & 0 & : & 0 & 1 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 0 \end{bmatrix}$$

- [C]=[D] [G]

$$=[d_1\ d_2\ d_3\ ] \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [\ d_1,\ d_2, d_3,\ (d_1+d_3),\ (d_2+d_3), (d_1+d_2)]$$

C=[d1, d2,d3, (d1+d3), (d2+d3),(d1+d2)]

Code name	**Message-vector** $d_1 d_2 d_3$	**code-vector for (6,3) linear block code**
C_a	000	000000
C_b	001	001110
C_c	010	010011
C_d	011	011101
C_e	100	100101
C_f	101	101011
C_g	110	110110
C_h	111	111000

- $C_c + C_g = (010011) + (110110)$

  $= (100101) = c_e$

For a systematic (7,4) linear block code, generated by

$$G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Find all possible code vectors.

Solution: $n=7$, $k=4$; $(n-k)= 3$

$\therefore$ $2^k = 2^4 = 16$ message vec

$[C]= [D]\ [G] =$ $\mathbf{[d_1\ \ d_2\ \ d_3\ \ d_4]}$ $\left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$

$$\mathbf{=[d_1\ ,\ d_2\ ,\ d_3\ ,\ d_4\ , (d_1 + d_2 + d_3), (d_1 + d_2 + d_4), (d_1 + d_3 + d_4)\ ]}$$

Message Vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	0	1	0	1
0	0	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	1	1	0	1
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	1	0	0	1	0
1	0	1	0	1	0	1	1	0	0	1
1	0	1	1	1	1	0	0	0	0	1
1	1	0	0	1	1	0	1	0	1	0
1	1	0	1	1	1	1	0	1	0	0
1	1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1

If C is a valid code vector, namely C=[D G]. Then prove that $CH^T = 0$ where H^T is the transpose of the parity check matrix H.

Wkt

$$\begin{bmatrix} 1\ 0\ 0 \ldots\ldots 0 & p_{11}\ p_{12} \ldots p_{1.\,n-k} \\ 0\ 1\ 0 \ldots\ldots 0 & p_{21}\ p_{22} \ldots p_{2.\,n-k} \\ 0\ 0\ 0 \ldots\ldots 1 & p_{k1}\ p_{k2} \ldots p_{k.\,n-k} \end{bmatrix}$$

i^{th} row of [G] matrix is given by

$$\mathbf{g_i} = [0\ \ 0\ 0 \ldots\ldots \underset{\substack{\uparrow \\ i^{th}\text{ element}}}{1} \ldots\ldots 0\ \ p_{i1}\ \ p_{i2} \ldots \underset{\substack{\uparrow \\ (k+j)^{th}\text{ element}}}{p_{ij}} \ldots\ldots p_{k.\,n-k}\ ]$$

J^{th} row of [H] matrix is given by

$$h_j = [p_{1j}\quad p_{2j} \ldots\ldots p_{ij} \quad \ldots p_{kj} \quad 0\ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ ]$$

$g_i.h_j^T = [0\ \ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ \ p_{i1}\ \ p_{i2}\ \ldots.\ p_{ij}\ \ldots\ldots p_{k.\,n-k}\ ] . [p_{1j}\ \ p_{2j} \ldots\ldots\ p_{ij}\ \ldots.\ p_{kj}\ \ 0\ 0\ 0\ \ldots\ldots 1 \ldots\ldots 0\ ]^T$

$$= [0\ \ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ \ p_{i1}\ \ p_{i2}\ \ldots.\ p_{ij}\ \ldots\ldots p_{k.\,n-k}\ ] \begin{bmatrix} p_{1j} \\ p_{2j} \\ : \\ p_{ij} \\ : \\ p_{kj} \\ 0 \\ 0 \\ : \\ 1 \\ : \\ 0 \end{bmatrix}$$

ith element

(k+j)th element

Modulo-2 multiplication yields ; $g_i.h_j^T$

$$= p_{ij} + p_{ij} = p_{ij}\,(1+1) = p_{ij}\ .\ 0 = 0$$

hence proved.

Error correction and Syndrome

- Let us suppose that

$C=(c_1\ c_2\ ....c_n)$ be a valid code-vector transmitted over a noisy communication channel belonging to a (n,k) linear block code.

- Let $R=\{r_1\ r_2........r_n\ \}$ be the received vector.
- Error-vector or error pattern E is defined as difference between R and C.

E=R-C ....**(1)**

$E=(\ e_1\ \ e_2\ \ ....e_n\ )$ ....**(2)**

$e_i=1$ if $R\neq C$

$e_i=0$ if $R=C$

(the 1's present in the error-vector 'E' represent the errors caused by noise in the channel)

- Receiver does the decoding operation by determining an (n-k) vector S defined as $S=RH^T$ ....**(3)**

$= (s_1\ \ s_2\ \ \ ....s_{n-k}\ )$

The (n-k) vector S is called **'error syndrome'** of R .

From eqn (1) ; $R=C+E$

$\therefore\ S=(C+E)\ H^T$

$= C\ H^T + E\ H^T$

$S = E\ H^T$

note: when $R\neq C$ then $S \neq 0$.

1.For a systematic (6,3) code, find all the transmitted code vector , draw the encoding circuit. If received vector R=[110010], detect and correct the error that has occurred due to noise. Given $P=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution:

(Refer notes)

(problem has been solved during the class also)

Video link

- https://youtu.be/qI1M6UzdyQw

- (The above video can be watched for linear block code concept)

2.For the systematic (6,3) code , parity matrix P is given by $[P]= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
The received vector $R=[r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ]$. Construct the corresponding syndrome calculation circuit.

Solution:(refer previous example for steps to find H^T)

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S=[s_1\ s_2\ s_3]= RH^T= =[r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(r_1+r_3+r_4), (r_2+r_3+r_5), (r_1+r_2+r_6)]$$

$\therefore$ The syndrome bits are

$S_1 = r_1 + r_3 + r_4$

$S_2 = r_2 + r_3 + r_5$

$S_3 = r_1 + r_2 + r_6$

Syndrome calculation circuit is

(refer notes)

3]For a systematic (7,4) linear block code, the parity matrix P is given by

$$[P]=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

i)Find all possible valid-code vectors

ii)Draw the corresponding encoding circuit.

iii)A single error has occurred in each of these received vectors. Detect and correct those errors.

a) R_A=[0111110] b)R_B = [1011100] c)R_C =[1010000]

iv) Draw the syndrome calculation circuit.

Solution:

i) Generator matrix $[G] = [I_k \mid P] = [I_4 \mid P]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}$$

- C= [D] [G] =

$$[d_1\ d_2\ d_3\ d_4\ ] \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}$$

$$[d_1,\ d_2,\ d_3,\ d_4,\ (d_1+d_2+d_3)\ ,\ (d_1+d_2+d_4)\ ,\ (d_1+d_3+d_4)\ ]$$

Message Vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	0	1	0	1
0	0	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	1	1	0	1
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	0	1	0
1	0	1	1	1	0	1	1	0	0	1
1	1	0	0	1	1	0	0	0	0	1
1	1	0	1	1	1	0	1	0	1	0
1	1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1

ii)encoding circuit (refer notes)

iii)Given R_A= [0 1 1 1 1 1 0]

Parity check matrix H is given by $[H] = [P^T \mid I_{n-k}] = [P^T \mid I_3]$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & : & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & : & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

Syndrome $S_A = R_A H^T$

$$= [0\ 1\ 1\ 1\ 1\ 1\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [1\ 1\ 0]$$

b)Given R_B= [1 0 1 1 1 0 0]

Syndrome $S_B = R_B H^T$

$$= [1\ 0\ 1\ 1\ 1\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= [1 01], which is located in 3rd row of H^T matrix. Hence the 3rd bit counting from left is in error.

∴ Corresponding error vector is given by

E_B=[0 0 1 0 0 0 0]

∴ Corrected code-vector which is the transmitted vector is given by

$C_B = R_B + E_B$=[1 0 1 1 1 0 0] +[0 0 1 0 0 0 0]

=[1 0 0 1 1 0 0]

This is the valid code vector corresponding to message vector 1001 (refer table)

c)Given R_C= [1 0 1 0 0 0 0]

Syndrome $S_C = R_C H^T$

$$= [1\ 0\ 1\ 0\ 0\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= [0 1 0], which is located in 6^{th} row of H^T matrix. Hence the 6^{th} bit counting from left is in error.

∴ Corresponding error vector is given by

E_C=[0 0 0 0 0 1 0]

∴ Corrected code-vector which is the transmitted vector is given by

$C_C = R_C + E_C$=[1 0 1 0 0 0 0] +[0 0 0 0 0 1 0]

=[1 0 1 0 0 1 0]

This is the valid code vector corresponding to message vector 1010 (refer table)

iv) Syndrome calculation circuit :

Let R= [r_1 r_2 r_3 r_4 r_5 r_6]

Syndrome

$$S=[s_1\ s_2\ s_3]= RH^T= =[r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(r_1+r_2+r_3 + r_5), (r_1+r_2+r_4 + r_6), (r_1+r_3+r_4 + r_7)]$$

(Refer notes to write syndrome calculation circuit)

This syndrome is located in second row of H^T matrix. Hence the 2^{nd} bit counting from left is in error.

∴ Corresponding error vector is given by

$E_A=[0\ \ 1\ 0\ 0\ 0\ 0\ 0\ ]$

∴ Corrected code-vector which is the transmitted vector is given by

$C_A=R_A+E_A=[\ 0\ 1\ 1\ 1\ 1\ 1\ 0]+[0\ 1\ 0\ 0\ 0\ 0\ 0]$

$=[0\ 0\ 1\ 1\ 1\ 1\ 0]$

This is the valid code vector corresponding to message vector 0011 (refer table)

4]The generator matrix of a (5,1) repetition code(represent simplest type of linear block code) is given by

$[G]= [\ 1\ 1\ 1\ 1 \ ¦\ 1\ ]$

i) Write its parity check matrix.

ii) Evaluate the syndrome for all five possible single error patterns and also for all ten possible double error patters.

Solution: Given n=5, k=1

$[I_k]= [I_1]=[1]$

And $[G]= [P: I_k] =[P \ ¦\ I_1]= [1\ 1\ 1\ 1 \ ¦\ 1]$

$\therefore [P]=[1\ 1\ 1\ 1]$

$\therefore [H]= [I_{n-k} \ ¦\ P^T\ ] = [I_4 \ ¦\ P^T\ ]$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & ¦ & 1 \\ 0 & 1 & 0 & 0 & ¦ & 1 \\ 0 & 0 & 1 & 0 & ¦ & 1 \\ 0 & 0 & 0 & 1 & ¦ & 1 \end{bmatrix}$$

ii) Since k=1, message vector [D] can be either [0] or [1]

Wkt [C]=[D] [G]

When [D]=[0], [C]= [0] [1 1 1 1 ¦ 1] = [0 0 0 0 0]

When [D]=[1], [C]= [1] [1 1 1 1 ¦ 1] = [1 1 1 1 1]

Let the transmitted vector be [0 0 0 0 0];

Then there are 5 single –error patterns given by

[1 0 0 0 0], [0 1 0 0 0], [0 0 1 0 0], [0 0 0 1 0], [0 0 0 0 1].

Syndrome for all these 5 single error received vectors can be found using equation

$[S]=[R]\,[H^T]$

$\therefore$ For [1 0 0 0 0], S_A= [1 0 0 0 0] $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

= [1 0 0 0]

For [0 1 0 0 0], S_B= [0 1 0 0 0] $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

= [0 1 0 0]

For [0 0 1 0 0], S_C= [0 0 1 0]

For [0 0 0 1 0], S_D=[0 0 0 1]

For [0 0 0 0 1], S_E=[1 1 1 1]

- There are 10 double error patterns given by

[1 1 0 0 0], [1 0 1 0 0], [1 0 0 1 0], [1 0 0 0 1], [0 1 1 0 0], [0 1 0 1 0], [0 1 0 0 1], [0 0 1 1 0], [0 0 1 0 1] and [0 0 0 1 1]

For [1 1 0 0 0], S_F = [1 1 0 0]

For [1 0 1 0 0], S_G = [1 0 1 0]

For [1 0 0 1 0], S_H = [1 0 0 1]

For [1 0 0 0 1], S_I = [0 1 1 1]

For [0 1 1 0 0], S_J = [0 1 1 0]

For [0 1 0 1 0], S_K = [0 1 0 1]

For [0 1 0 0 1], S_L = [1 0 1 1]

For [0 0 1 1 0], S_M = [0 0 1 1]

For [0 0 1 0 1], S_N = [1 1 0 1]

For [0 0 0 1 1], S_0 = [1 1 1 0]

Binary Cyclic codes

- It form a subclass of linear block codes.

- Definition of cyclic codes:

A (n,k) linear block code C is said to be a cyclic code if every cyclic shifts of the code is also a code-vector of C

- Ex: let C_1=0111001 be a code-vector of C.

  If C_2=1011100 (the last '1' of C_1 has moved into the first position) is also a code-vector of C, then its called as 'Cyclic Code'

  Similarly, C_3=0101110,

  C_4=0010111 etc.. Will also be code vectors of C.

Algebraic structure of Cyclic Codes

- Let the n-tuple be represented by

$$V=(v_0 \quad v_1 \quad v_2 \ldots\ldots\ldots\ldots v_{n-1}) \qquad \ldots\ldots\ldots(1)$$

If v belongs to cyclic code, then

$$\left.\begin{array}{l} V^{(1)} = (v_{n-1} \quad v_0 \quad v_1 \quad v_2 \ \ldots\ldots\ \ldots\ldots v_{n-2}) \\ V^{(2)} =(v_{n-2} \quad v_{n-1} \quad v_0 \quad v_1 \quad v_2 \ \ldots\ldots\ v_{n-3}) \\ : \\ : \\ V^{(i)} = (v_{n-i} \quad v_{n-i+j} \ \ldots\ldots \quad v_0 \quad v_1 \quad v_2 \ \ldots\ldots\ V_{n-i-1}) \end{array}\right\} \qquad \ldots\ldots(2)$$

$$\therefore \quad \left.\begin{array}{l} V(x)= v_0 +v_1x +v_2x^2+\ldots\ldots\ldots\ldots+v_{n-1}x^{n-1} \\ V^{(1)}(x)= v_{n-1}+v_0x +v_1x^2+\ldots\ldots\ldots\ldots+v_{n-2}x^{n-1} \\ V^{(2)}(x)= v_{n-2}+v_{n-1}x +v_0x^2+\ldots\ldots\ldots\ldots+v_{n-3}x^{n-1} \\ : \\ V^{(i)}(x)= v_{n-i}+v_{n-i+1}x +v_{n-i+2}x^2+\ldots\ldots\ldots\ldots+v_{n-i-1}x^{n-1} \end{array}\right\} \qquad \ldots\ldots\ldots\ldots(3)$$

Cont...

- The coefficients v_0 v_1 ......v_{n-1} belong to a binary field with the following rules of addition and multiplication[modulo-2 arithmetic]

Modulo-2 addition	Modulo-2 multiplication
0+0=0	0.0=0
0+1=1	0.1=0
1+0=1	1.0=0
1+1=0	1.1=1

Modulo-2 algebra

- <u>Addition:</u>
  x+x can be written as
  $x+x = x(1+1)=x.0=0$
  Similarly $x^2+x^2=x^2(1+1)=0$
  $x^3+x^3=x^3(1+1)=0$
  and so on.

- <u>Substraction</u>:
  Substraction of polynomials is same as addition in modulo-2 algebra.

- <u>Multiplication:</u>
  The quantity $x.x=x^2$
  $x^2.x=x^3$ and so on.

Find the product of polynomials $f_1(x)=(x+1)$ and $f_2(x)=x^3+x+1$ using modulo-2 algebra

Solution: $f_1(x) . f_2(x) = (x+1)\ (x^3+x+1)$

$$= x^4+x^2+x+x^3+x+1$$

$$= x^4+x^3+x^2+x+x+1$$

$$= x^4+x^3+x^2+x(1+1)+1$$

$$= x^4+x^3+x^2+1$$

2. Multiply $f_1(x) = 1+x+x^3$ and $f_2(x)=(1+x+x^2+x^4)$ using modulo-2 algebra

Solution: $f_1(x) \cdot f_2(x)$

$$=(1+x+x^3)(1+x+x^2+x^4)$$

$$= 1+x+x^2+x^4 + x+x^2+x^3+x^5+x^3 +x^4+x^5+x^7$$

$$= 1+x^7$$

3. Divide $f_2(x)= x^6+x^5+x^2$ by $f_1(x)=x^3+x+1$ using modulo-2 algebra

Solution:

x^3+x+1) $\mathbf{x^6+x^5+x^2}$ (x^3+x^2+x Q(x)

$\underline{x^6+x^4+x^3}$

$x^5+x^4+x^3+x^2$

$\underline{x^5+x^3+x^2}$

x^4

$\underline{x^4+x^2+x}$

x^2+x R(x)

$x+x^2+x^3$ is called "QUOTIENT POLYNOMIAL"

R(x) = $x+x^2$ is called "REMAINDER POLYNOMIAL"

4. If $f(x)= x^4+x+1$, then show that $[f(x)]^2 =f(x^2)$ in modulo-2 algebra

Solution:

Consider $[f(x)]^2 = [x^4+x+1]^2$

$= (x^4+x+1)(x^4+x+1)$

$= x^8+x^5+x^4 + x^5+x^2+x + x^4+x+1$

$= x^8+x^2 +1$

Consider $f(x^2) = (x^2)^4 + x^2 +1$

$= x^8 + x^2$

Comparing , we can conclude that $[f(x)]^2 =f(x^2)$

This equation is true for all polynomials f(x) of any degree

Properties of cyclic codes

(refer notes)

1] For the (7,4) single-error correcting cyclic code, $D(x)=d_0+d_1x+d_2x^2+d_3x^3$ and $x^n+1=x^7+1= (1+x+x^3)(1+x+x^2+x^4)$. Using the generator polynomial $g(x)=1+x+x^3$, find all the 16 code-vectors of the cyclic code both in non-systematic and systematic form.

- Non-systematic Cyclic code:
  $V(x)= D(x)\, g(x)$

  Consider a message –vector
  $D=(d_0\ d_1\ d_2\ d_3) =(0001)$

Message polynomial is $D(x)=d_0+d_1x+d_2x^2+d_3x^3$

$=(0)+(0)x+(0)x^2 +(1)x^3$

$= x^3$

$\therefore V(x)= D(x)\, g(x)= (x^3)(1+x+x^3)$

$= x^3 + x^4 + x^6$

$=0+(0)x+(0)\ x^2 + (1)\ x^3 +(1)\ x^4 + (0)\ x^5 + (1)\ x^6$

$\therefore$ [V] = [0001101]

Cont…

Consider a message –vector
$D=(d_0\ d_1\ d_2\ d_3) =(1101)$

Message polynomial is $D(x)=d_0+d_1x+d_2x^2+d_3x^3$

$$=(1)+(1)x+(0)x^2 +(1)x^3$$
$$= 1+x+ x^3$$

$\therefore V(x)= D(x)\ g(x)= (1+x+ x^3)\ (1+x+x^3)$

$$= 1+x^2 + x^6$$
$$= (1)+ (0)x+(1)\ x^2 + (0)\ x^3 + (0)\ x^4 +(0)\ x^5 + x^6$$

$\therefore\ [V] = [1010001]$

Message(D)	Code-vector(V)	Message(D)	Code-vector(V)
0000	0000000	1000	1101000
0001	0001101	1001	1100101
0010	0011010	1010	1110010
0011	0010111	1011	1111111
0100	0110100	1100	1011100
0101	0111001	1101	1010001
0110	0101110	1110	1000110
0111	0100011	1111	1001011

Table 1

Systematic Cyclic code:

$$R(x) = \frac{x^{n-k}\ D(x)}{g(x)} + Q(x)$$

Let D=[1001]

$\therefore$ $D(x) = 1+x^3$

$\therefore$ $x^{n-k}\ D(x) = x^{7-4}\ (1+x^3) = x^3+x^6$

$$
\begin{array}{rll}
x^3+x+1\ ) & x^6+x^3 & (x^3+x \\
 & \underline{x^6+x^4+x^3} & \\
 & \quad x^4 & \\
 & \quad \underline{x^6+x^2+x} & \\
 & \qquad x^2+x &
\end{array}
$$

$R(x) = x+x^2$

$= (0)+(1)(x)+(1)\,x^2$

[R]= [011]

Code-vector is given by [V]= [011 1001]

Message(D)	Code-vector(V)	Message(D)	Code-vector(V)
0000	0000000	1000	1101000
0001	1010001	1001	0111001
0010	1110010	1010	0011010
0011	0100011	1011	1001011
0100	0110100	1100	1011100
0101	1100101	1101	0001101
0110	1000110	1110	0101110
0111	001011	1111	1111111

Table 2

Generator and parity Check matrices (7,4) cyclic codes

- Wkt $x^n+1=g(x).\ h(x)$
- The generator matrix [G] of the order k x n = 4 x 7 can be constructed starting with the generator polynomial g(x)
- From property 4, polynomials g(x), x g(x), x^2 g(x) and x^3 g(x) also represent the code vector polynomials of same cyclic code.
- $g(x)=(1+x+x^3)$
- Refer table2:

$\therefore g(x)= (1)+ (1)(x) + (0)(x^2) + (1)(x^3) + (0)(x^4) + (0)(x^5) + (0)(x^6)$

  - Code vector corresponding to g(x) is 1101000 (which is the code vector for message 1000)

$x\,g(x)= x(1+x+x^3) = x+x^2+x^4$

code is 0110100

$x^2 g(x)= x^2 ((1+x+x^3)$

code is 0011010

$x^3 g(x)= x^3 ((1+x+x^3)$

code is 0001101

Cont...

- [G]= $\begin{bmatrix} 1\ 1\ 0\ 1\ 0\ 0\ 0 \\ 0\ 1\ 1\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 1\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1\ 1\ 0\ 1 \end{bmatrix}$ Its not in systematic form
- Add 1^{st} row to 3^{rd} row and place result in 3^{rd} row

  [G]= $\begin{bmatrix} 1\ 1\ 0\ 1\ 0\ 0\ 0 \\ 0\ 1\ 1\ 0\ 1\ 0\ 0 \\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1\ 1\ 0\ 1 \end{bmatrix}$
- Now add 1^{st}, 2^{nd} and 4^{th} rows and place result in 4^{th} row ;

[G]= $\begin{bmatrix} 1\ 1\ 0\ 1\ 0\ 0\ 0 \\ 0\ 1\ 1\ 0\ 1\ 0\ 0 \\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \\ 1\ 0\ 1\ 0\ 0\ 0\ 1 \end{bmatrix}$ $= [\ P \,\vdots\, I_4\ ]$ form

Cont...

- [V]= [D] [G]

$$=[d_0\ d_1\ d_2\ d_3\ ] \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

- This method also generates the same code-vectors as given in table 2.

(Refer slide no.47 for table2)

Parity Check Matrix H

- Wkt $x^n+1=g(x).\ h(x)$
- For a (7,4) cyclic code, we have n=7,

$\therefore$ $x^7+1=g(x).\ h(x)$

The parity check polynomial $h(x)= \dfrac{x^7+1}{g(x)} = \dfrac{x^7+1}{x^3+x+1}$

$$
\begin{array}{l}
x^3+x+1)\ x^7+1 \qquad (x^4+x^2+x+1 \\
\qquad\qquad \underline{x^7+x^5+x^4} \\
\qquad\qquad x^5+x^4+1 \\
\qquad\qquad \underline{x^5+x^3+x^2} \\
\qquad\qquad x^4+x^3+x^2+1 \\
\qquad\qquad \underline{x^4+x^2+x} \\
\qquad\qquad\quad x^3+x+1 \\
\qquad\qquad\quad \underline{x^3+x+1} \\
\qquad\qquad\qquad 0
\end{array}
$$

$\therefore$ The parity check polynomial $h(x) =1+x+ x^2 +x^4$

The "reciprocal of h(x)" is defined as $x^k\ h(x^{-1})$.

This polynomial is also a factor of $1+x^n$.

To get Parity Check Matrix H:

- Let us consider $x^4\ h(x^{-1})$ for (7, 4) cyclic code
- We have $h(x) = 1+x+x^2+x^4$
- $\therefore\quad h(x^{-1}) = 1 + (1/x) + (1/x^2) + (1/x^4)$
- $\therefore x^4\ h(x^{-1}) = 1+x+x^2+x^4$; code is 1011100
- $\therefore x^5\ h(x^{-1}) = x+x^3+x^4+x^5$; code is 0101110
- $\therefore x^6\ h(x^{-1}) = x^2+x^4+x^5+x^6$; code is 0010111
- The parity check matrix H, which is a (n-k) x n =3 x 7 matrix, is given by

$$[H]=\begin{bmatrix} 1&0&1&1&1&0&0 \\ 0&1&0&1&1&1&0 \\ 0&0&1&0&1&1&1 \end{bmatrix}$$

This is not of the standard form $[I_{n-k} \mid P^T] = [I_3 \mid P^T]$

By adding 1st & 3rd row and placing the result in 1st row ,

$$[H]=\begin{bmatrix} 1&0&0&1&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&0&1&0&1&1&1 \end{bmatrix}$$

Problem2]

The generator polynomial for a (15,7) cyclic code is

$g(x)= 1+x^4+x^6+x^7+x^8$

i) find the code-vector in systematic form for the message $D(x)= x^2+x^3+x^4$

Solution: From property (5) of cyclic codes, the systematic-code vector is found by dividing $x^{n-k} D(x)$ by $g(x)$ to get remainder polynomial $R(x)$

$\therefore x^{n-k} D(x)= x^{15-7} D(x)= x^8 (x^2 + x^3 + x^4)= x^{10} + x^{11} + x^{12} = x^{12} + x^{11} + x^{10}$

- $g(x)= x^8 +x^7+x^6+x^4 +1$

$$
\begin{array}{l|l l}
x^8 +x^7+x^6+x^4 +1) & x^{12} + x^{11} + x^{10} & (x^4 +1 \\
 & \underline{x^{12} + x^{11} + x^{10} +x^8 + x^4} & \\
 & x^8+x^4 & \\
 & \underline{x^8 +x^7 + x^6 + x^4 +1} & \\
 & x^7 +x^6+1 &
\end{array}
$$

$\therefore \quad R(X)= x^7 +x^6+1 = 1+x^6 +x^7$

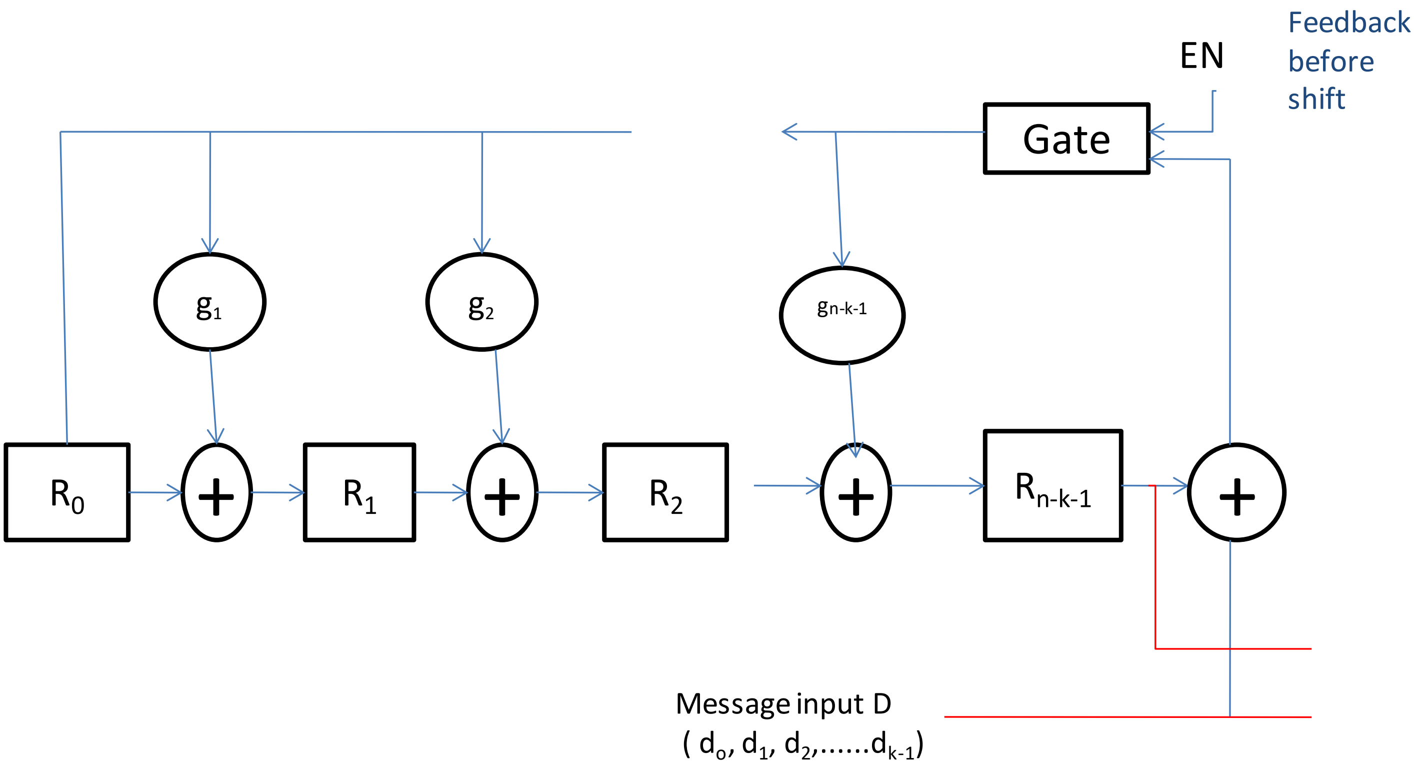
$=1 + (0)(x) +(0) x^2 +(0) x^3 +(0) x^4 +(0) x^5 +(1) x^6 +(1) x^7$

R = [1 0 0 0 0 0 1 1]

And $D(x)= x^2+x^3+x^4$ = [0 0 1 1 1 0 0]

V = [1 0 0 0 0 0 1 1 0 0 1 1 1 0 0]

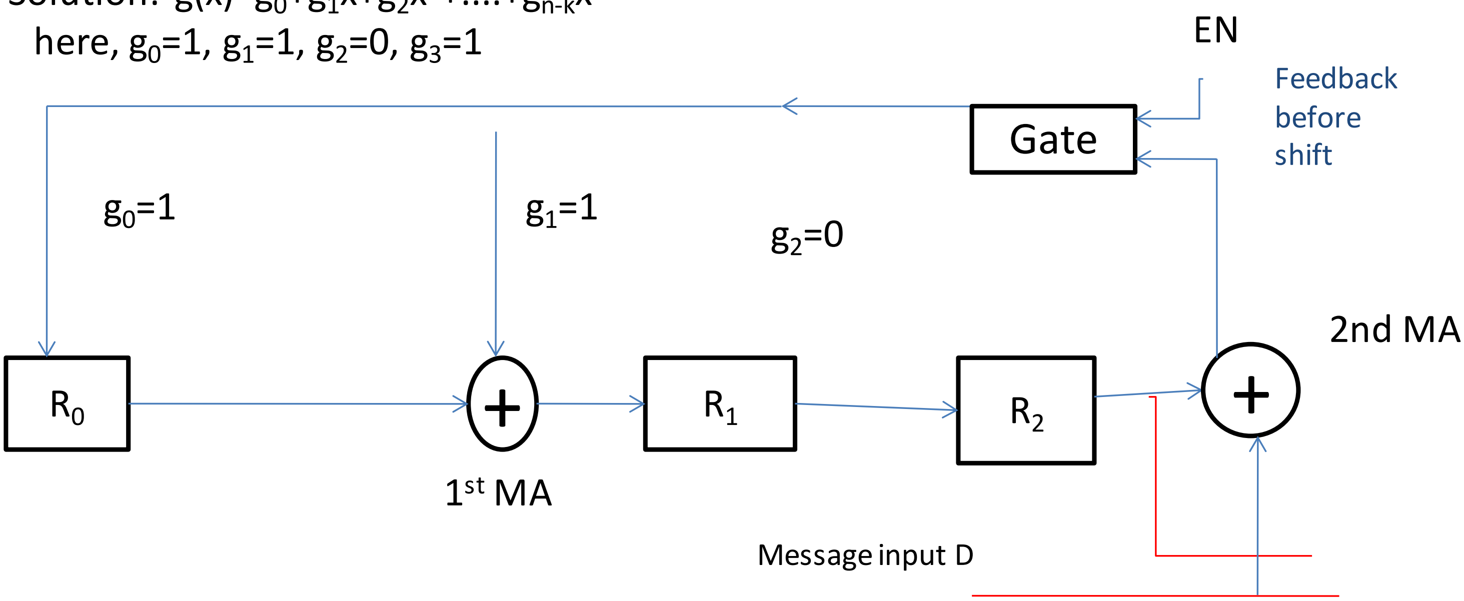
Encoding using (n-k) bit shift register



Design an encoder for the (7,4) binary cyclic code generated by $g(x)=1+x+x^3$ & verify its operation using the message vectors (1 0 0 1) and (1 1 0 1)

Solution: $g(x)=g_0+g_1x+g_2x^2+....+g_{n-k}x^{n-k}$

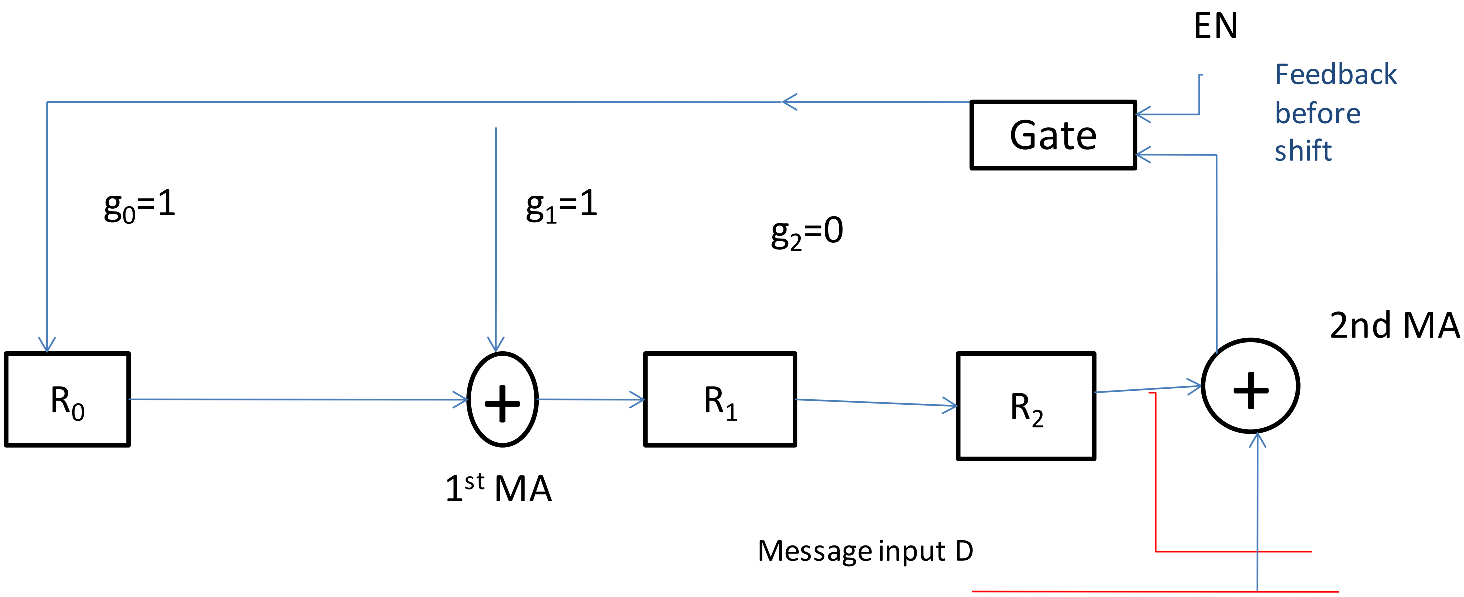
here, $g_0=1$, $g_1=1$, $g_2=0$, $g_3=1$



Input D	R_0	R_1	R_2
	0	0	0
1	1	1	0
0	0	1	1
0	1	1	1
1	**0**	**1**	**1**

V=[**011** 1001]

For (1 0 1 1)



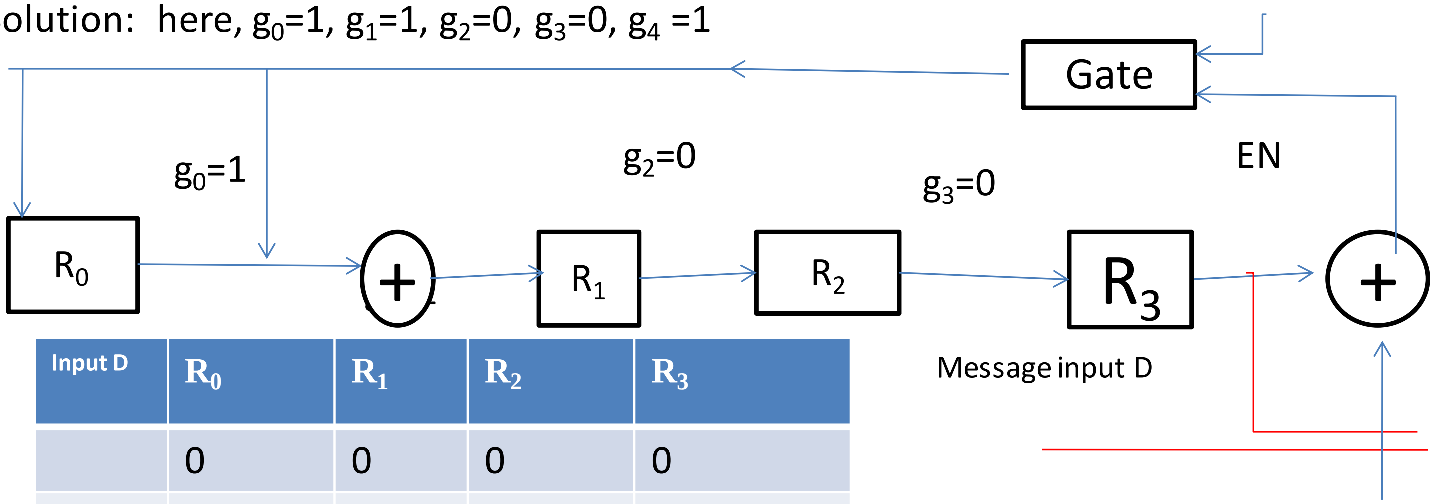
Input D	R_0	R_1	R_2
	0	0	0
1	1	1	0
1	1	0	1
0	1	0	0
1	**1**	**0**	**0**

V=[**100** 1011]

- 2] Consider a (15,11) cyclic code generated by $g(x)=1+x+X^{4}$. Devise a feedback shift register encoder circuit. Illustrate the encoding procedure with the message vector

  1 0 0 1 0 1 1 0 1 1 1 by listing the state of the registers

Solution: here, g_0=1, g_1=1, g_2=0, g_3=0, g_4 =1



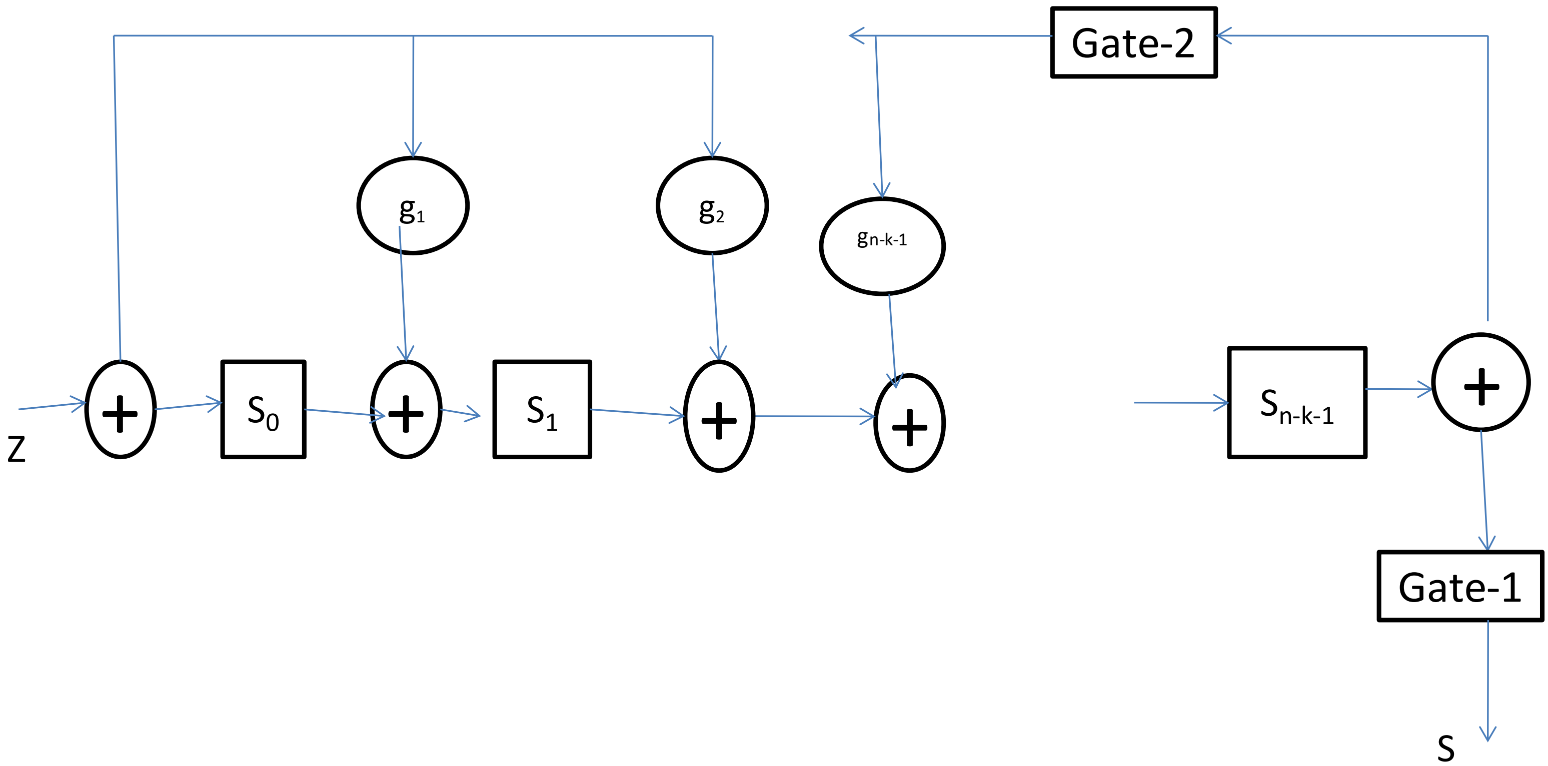
Input D	R_0	R_1	R_2	R_3
	0	0	0	0
1	1	1	0	0
1	1	0	1	0
1	1	0	0	1
0	1	0	0	0
1	0	0	0	0
1	1	0	0	0
0	0	1	0	0
1	1	1	1	0
0	0	1	1	1
0	1	1	1	1
1	**0**	**1**	**1**	**1**

V=[**0 1 1 1** 1 0 0 1 0 1 1 0 1 1 1

Syndrome Calculation-Error Detection and Correction:

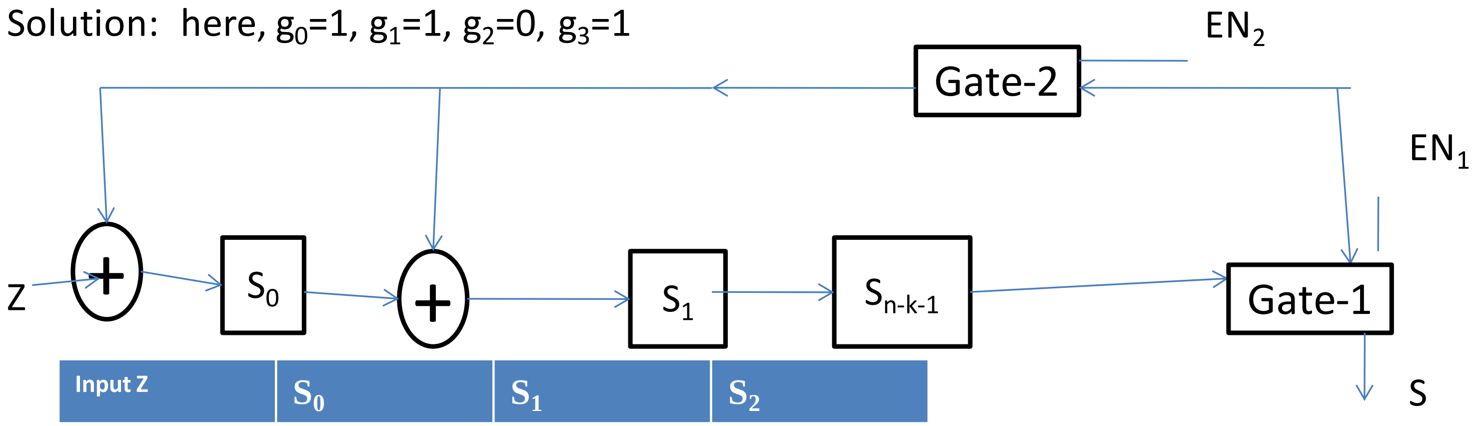
- If $V(x)$ is the transmitted code vector &
- $R(x)$ is received code vector
- If $V(x)=R(x)$, then the syndrome polynomial $S(x)=0$
- If $V(x) \neq R(x)$, then the syndrome polynomial $S(x) \neq 0$
- To calculate $S(x)$, the received code vector is divided by generator polynomial. If remainder of division is '0', then there is no error in the received code vector.
- The remainder of division gives the error syndrome.
- The error polynomial depends upon syndrome polynomial.
- To determine the co-efficients of syndrome polynomial, the dividing circuit for a (n-k) cyclic code is shown below.

A (n-k) syndrome calculation circuit for a (n,k) cyclic code.



For a (7,4) cyclic code, the received vector Z(x) is 1110101 and the generator polynomial is $g(x)= 1+x+x^3$. Draw the syndrome calculation circuit and correct the single error in the received vector.

Solution: here, $g_0=1$, $g_1=1$, $g_2=0$, $g_3=1$



Input Z	S_0	S_1	S_2
	0	0	0
1	1	0	0
0	0	1	0
1	1	0	1
0	1	0	0
1	1	1	0
1	1	1	1
1	**0**	**0**	**1**

Since, Syndrome is non-zero , there is an error

Cont..

- To correct the error,

[from the given problem, Find [G] matrix and [H] Matrix]

$$[H]=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad [H^T]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- The syndrome s_0,s_1,s_2 = 0 0 1 is located in 3rd row of H^T matrix. Hence the third bit is in error.

Error vector E= 0 0 1 0 0 0 0

$\therefore$ The corrected vector = Z + E

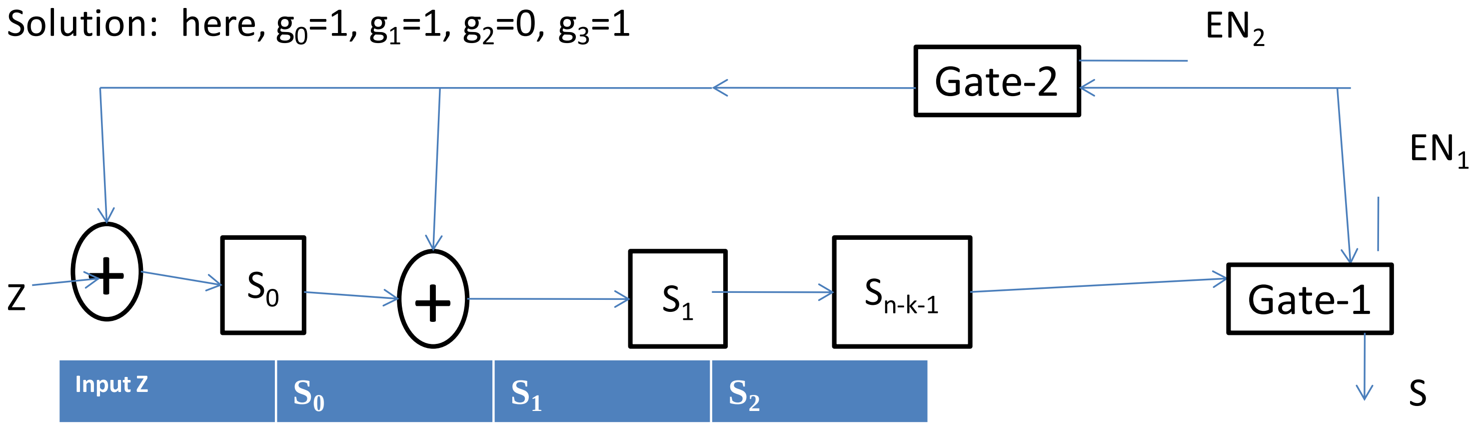
= 1110101 + 0 0 1 0 0 0 0

= 1 1 0 0 1 0 1

(1 1 0 0 1 0 1 is a valid code-vector as seen from table2)

For a (7,4) cyclic code, the received vector Z(x) is 0 1 0 0 1 0 1 and the generator polynomial is $g(x)= 1+x+x^3$. Draw the syndrome calculation circuit and correct the single error in the received vector.

Solution: here, $g_0=1$, $g_1=1$, $g_2=0$, $g_3=1$



Input Z	S_0	S_1	S_2
	0	0	0
1	1	0	0
0	0	1	0
1	1	0	1
0	1	0	0
0	0	1	0
1	1	0	1
0	**1**	**0**	**0**

Since, Syndrome is non-zero , there is an error

Cont..

- To correct the error,

[from the given problem, Find [G] matrix and [H] Matrix]

$$[H]=\begin{bmatrix} 1&0&0&1&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&0&1&0&1&1&1 \end{bmatrix} \qquad [H^T]=\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 1&1&0 \\ 0&1&1 \\ 1&1&1 \\ 1&0&1 \end{bmatrix}$$

- The syndrome s_0,s_1,s_2 = 0 0 1 is located in 1st row of H^T matrix. Hence the third bit is in error.

Error vector E= 1 0 0 0 0 0 0

$\therefore$ The corrected vector = Z + E

= 0100101 + 1000000

= 1 1 0 0 1 0 1

(1 1 0 0 1 0 1 is a valid code-vector as seen from table2)

- A(15,5) linear cyclic code has a generator polynomial $g(x)=1+x+x^2+x^4+x^5+x^8+x^{10}$

a) Draw the block diagram of an encoder and syndrome calculator for this code.

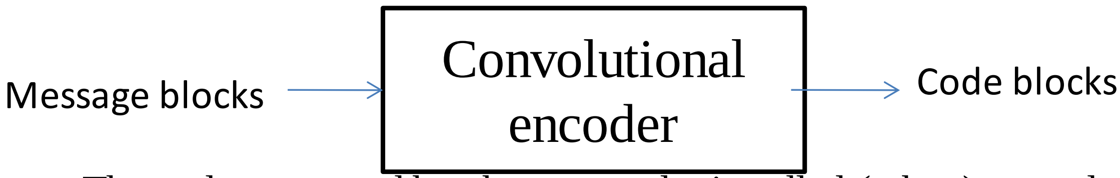
b) Find the code polynomial for the message polynomial.

$D(x)=1+x^2+x^4$ in systematic form.

c) Is $V(x)= 1+ x^4+x^6+x^8+x^{14}$ a code polynomial?

Convolutional codes

- In convolutional codes, a block of ‘n’ code digits generated by the encoder in a time unit depends not only on the block of ‘k’ message digits within that time unit, but also on the preceding (m-1) blocks of message digits (m>1).
- General convolutional encoder:



  - The code generated by above encoder is called (n,k,m) convolutional code of constraint length “nm” digits and “rate efficiency k/n” .
  - the encoder consists of shift registers and modulo-2 adders.

    n = number of outputs=number of modulo-2 adders ,

    k = number of input bits entering at any time

    m = number of **stages** of the flip-flop
- In convolutional encoder, the message stream continuously runs through the encoder whereas in block coding schemes, the message stream is first divided into long blocks and then encoded.
- In general, there are two methods of generating convolutional codes
  - Time domain approach
  - Transfer domain approach

Encoding of convolutional codes using Time-domain approach:

Let us consider a (n , k , m 2,1,3) convolutional encoder.

<u>Matrix method:</u>

The generator sequences $g_1^{(1)}$ $g_2^{(1)}$ $g_3^{(1)}$ .... $g_{m+1}{}^{(1)}$ for top adder.

$g_1^{(2)}$ $g_2^{(2)}$ $g_3^{(2)}$ .... $g_{m+1}{}^{(2)}$ for bottom adder.

can be interlaced and arranged in a matrix form.

No. of rows equal to number of digits in the message sequence= L rows

Number of columns=n(L+m).

Such a matrix of order [L] x [n(L+m)] is called "generator matrix" of convolutional encoder.

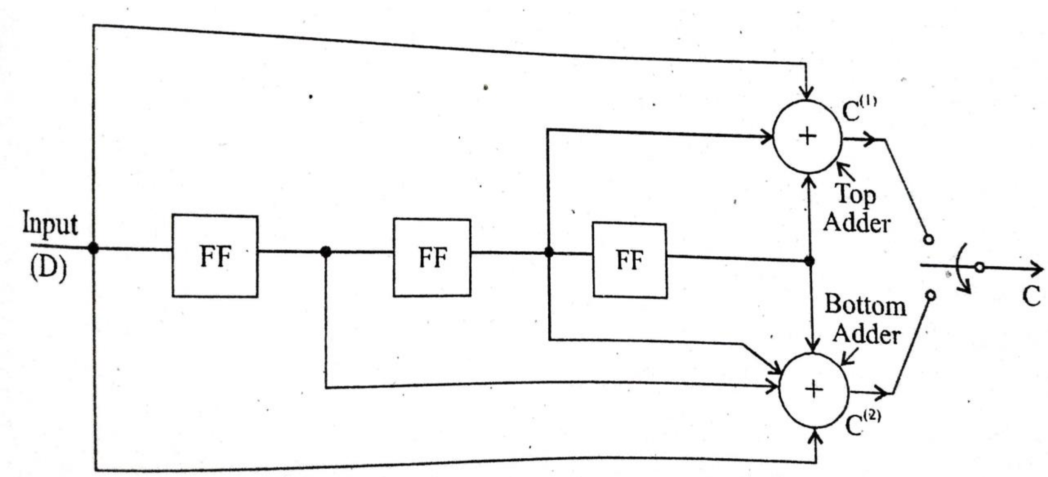


Figure1

- In general, for a two modulo-2 adder convolutional encoder, the generator matrix G is given by

$$
G = \begin{array}{cccccccccccccccc}
g_1^{(1)} & g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & g_3^{(1)} & g_3^{(2)} & \ldots\ldots & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\
0 & 0 & g_1^{(1)} & g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & \ldots\ldots g_m^{(1)} & g_m^{(2)} & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0 & 0 & \ldots\ldots & 0 & 0 & \\
0 & 0 & 0 & 0 & g_1^{(1)} & g_1^{(2)} & \ldots\ldots & \ldots\ldots & g_m^{(1)} & g_m^{(2)} & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0 & 0 & & \\
: & & & & & & & & & & & & & & & \\
: & & & & & & & & & & & & & & & \\
: & : & : & : & : & : & & & & & & & & & g_{m+1}^{(1)} & g_{m+1}^{(2)}
\end{array}
$$

- The encoder output C=dG
- Let the message sequence be $d_1\ d_2\ d_3\ d_4\ d_5$=1 0 1 1 1
- $g^{(1)} = [\ g_1^{(1)}\ \ g_2^{(1)}\ \ g_3^{(1)}\ \ g_4^{(1)}\ ] = [1\ 0\ 1\ 1\ ]$
- $g^{(2)} = [\ g_1^{(2)}\ \ g_2^{(2)}\ \ g_3^{(3)}\ \ g_4^{(4)}\ ] = [1\ 1\ 1\ 1]$
- L=5 rows and n(L+m)=2(5+3)=16 columns given by
-

$$
G = \begin{array}{cccccccc}
11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\
00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\
00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\
00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\
00 & 00 & 00 & 00 & 11 & 01 & 11 & 11
\end{array}
$$

- The encoder output

C= dG

$$= [\,1\ 0\ 1\ 1\ 1\,] \begin{array}{cccccccc} 1\,1 & 0\,1 & 1\,1 & 1\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 1\,1 & 0\,1 & 1\,1 & 1\,1 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 1\,1 & 0\,1 & 1\,1 & 1\,1 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 1\,1 & 0\,1 & 1\,1 & 1\,1 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 1\,1 & 0\,1 & 1\,1 & 1\,1 \end{array}$$

C= [11, 01, 00, 01, 01, 01, 00, 11]

Encoding of convolutional codes using Transform-domain approach:

- For “j” number of modulo-2 adders(**‘j’ varies from 1 to n**).
- Generator polynomial is defined as
  $g^{(j)}(x)= g_1^{(j)} + g_2^{(j)} x + g_3^{(j)} x^2 + \ldots\ldots + g_{m+1}^{(j)} x^m$

- The corresponding output of each of the adders is given by
  $C^{(j)}(x)= d(x)\, g^{(j)}(x)$

- The final encoder output polynomial is obtained from
  $C(x)= C^{(1)}(x^n) + x\, C^{(2)}(x^n) + x^2 C^{(3)}(x^n) + \ldots\ldots + x^{n-1} C^{(n)}(x^n)$

1]Obtain the output of the convolutional encoder of figure 1, using transform domain approach.

- Solution:

The generator sequence for the top adder is given by

$g^{(1)} = [\ g_1^{(1)}\ \ g_2^{(1)}\ \ g_3^{(1)}\ g_4^{(1)}\ ] = [\ 1\ 0\ 1\ 1]$

- Generator polynomial corresponding to the top adder is given as

$g^{(1)}(x) = g_1^{(1)} + g_2^{(1)}\ x + g_3^{(1)}\ x^2 + ... + g_4^{(j)}\ x^3$

$= 1+0+ x^2 + x^3$

$= 1+ x^2 + x^3$

The generator sequence for the bottom adder is given by

$g^{(2)} = [\ g_1^{(2)}\ \ g_2^{(2)}\ \ g_3^{(2)}\ g_4^{(2)}\ ] = [\ 1\ 1\ 1\ 1]$

- Generator polynomial corresponding to the bottom adder is given as

$g^{(2)}(x) = g_1^{(2)} + g_2^{(2)}\ x + g_3^{(2)}\ x^2 + ... + g_4^{(2)}\ x^3$

$= 1+x+ x^2 + x^3$

The top-adder output polynomial is given by $C^{(1)}(x) = d(x)\ g^{(1)}\ (x)$

- For the message d=1 0 1 1 1

Message polynomial is given by $d(x) = 1+x+ x^3+x^4$

$C^{(1)}(x) = (1+x+ x^3+x^4\ )\ (1+ x^2 + x^3\ )$

$= 1+ x^2 + x^3\ + x^4\ + x^2 + x^4\ + x^5\ + x^6 + x^3 + x^5\ + x^6\ + x^7$

$= 1 + x^7$

The bottom -adder output polynomial is given by

$C^{(2)}(x)= d(x)\, g^{(2)}(x)$

- For the message d=1 0 1 1 1

Message polynomial is given by $d(x)= 1+x^2+ x^3+x^4$

$C^{(2)}(x)= (1+x^2+ x^3+x^4) (1+x+ x^2+x^3)$

$= 1+x+ x^2 +x^3 +x^2 + x^3 +x^4 +x^5 + x^3 +x^4 +x^5 +x^6 +x^7 + x^4 +x^5 +x^6 +x^7$

$=1 + x + x^3+x^4 +x^5 +x^7$

The final encoder output polynomial is given by

$C(x)= C^{(1)}(x^n)+ C^{(2)}(x^n)$ here, n=2

$C(x)= C^{(1)}(x^2)+ C^{(2)}(x^2)$

we have $C^{(1)}(x) = 1+ x^7$

$C^{(1)}(x^2) = 1+ x^{14}$

and $C^{(2)}(x)= 1 + x + x^3+x^4 +x^5 +x^7$

$C^{(2)}(x^2)= 1 + x^2+x^6+x^8 +x^{10} + x^{14}$

$C(x)= 1+ x^7 +x(1 + x^2+x^6+x^8 +x^{10} + x^{14})$

$= 1+ x + x^3+x^7+x^9 +x^{11} + x^{14} + x^{15}$

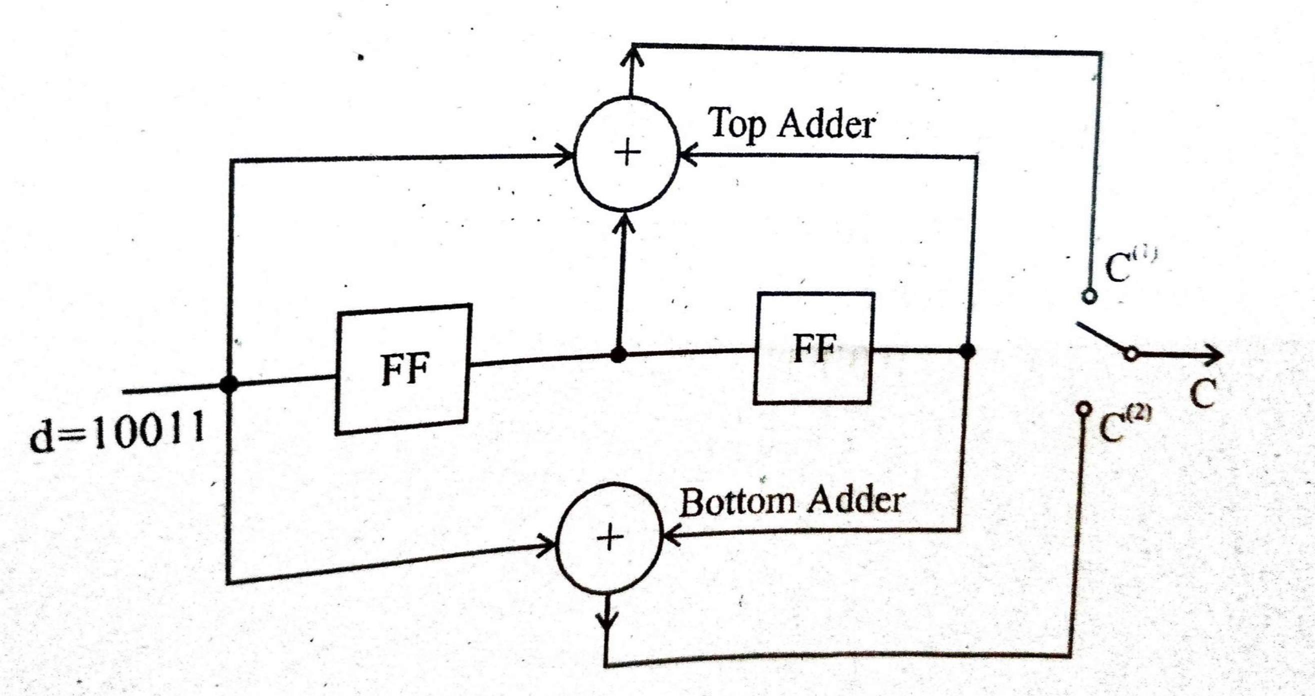
The code-word corresponding to this polynomial is

C=[11, 01, 00, 01, 01, 01, 00, 11]

2]For the convolutional encoder shown in figure. , the information sequence is d=1 0 0 1 1. Find the output sequence using the following two approaches

a) Time-domain approach

b) Transfer-domain approach



Solution:

- Time-domain approach

The generator sequence for the top adder is given by

$g^{(1)} = [\ g_1^{(1)} \quad g_2^{(1)} \quad g_3^{(1))}\ ] = [\ 1\ \ 1\ 1]$

The generator sequence for the bottom adder is given by

$g^{(2)} = [\ g_1^{(2)} \quad g_2^{(2)} \quad g_3^{(2)}] = [\ 1\ 0\ \ 1]$

The generator matrix is of the order Lx [n(L+m)]= 5x[2(5+2)]= 5x 14

- G=

11	10	11	00	00	00	00
00	11	10	11	00	00	00
00	00	11	10	11	00	00
00	00	00	11	10	11	00
00	00	00	00	11	10	11

- [C]= [d] [G]

=[1 0 0 1 1]

11	10	11	00	00	00	00
00	11	10	11	00	00	00
00	00	11	10	11	00	00
00	00	00	11	10	11	00
00	00	00	00	11	10	11

[C]= [11, 10, 11, 11, 01, 01, 11]

ii) Transform-domain approach

The message [d]= [1 0 0 1 1] hence, d(x) = =1 + x^3+x^4

- The generator sequence for the top adder is given by

$g^{(1)} = [\ g_1^{(1)}\ \ g_2^{(1)}\ \ g_3^{(1))}\ ] = [\ 1\ \ 1\ 1]$ hence $g^{(1)}(x)= 1+ x +x^2$

- The generator sequence for the bottom adder is given by

  $g^{(2)} = [\ g_1^{(2)}\ \ g_2^{(2)}\ \ g_3^{(2)}] = [\ 1\ 0\ \ 1]$ hence $g^{(2)(}x)= 1+ x^2$

- The top-adder output polynomial is given by $C^{(1)(}x) = d(x)\, g^{(1)}\ (x)$

  $= (1+x^3+x^4\ )\ (1+ x +x^2\ )$

  $= 1+ x+ x^2 +x^3\ +x^6$

- The bottom-adder output polynomial is given by $C^{(2)(}x)= d(x)\, g^{(2)}\ (x)$

  $=(1+x^3+x^4\ )\ (1+ x^2\ )$

  $= 1+ x^3\ +x^4\ + x^2 +x^5\ +x^6$

  $= 1+ x^2\ +x^3 + x^4 + x^5\ +x^6$

The final encoder output polynomial is given by

$C^{(}x)= C^{(1)}\ (x^n\ )+ xC^{(2)}\ (x^n\ )$ here, n=2

$C^{(}x)= C^{(1)}\ (x^2)+ xC^{(2)}\ (x^2\ )$

$= 1+ x^2+ x^4 +x^6\ +x^{12} + x(1+ x^4+ x^6 +x^8\ +x^{10} +x^{12}\ ) = 1+x+ x^2+ x^4 +x^5\ +x^6 +\ x^7+ x^9 +x^{11} +x^{12} +x^{13}$

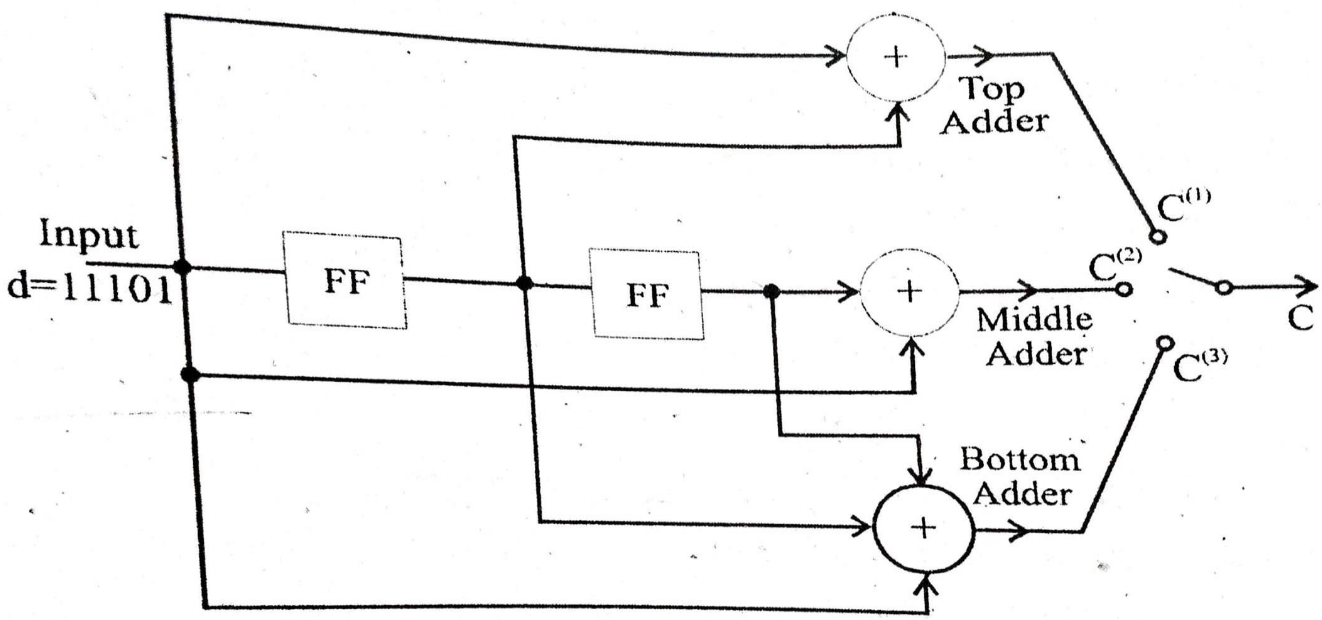
The code-word C=[11, 10, 11, 11, 01, 01, 11]

3]Consider the (3,1,2) convolutional code with $g^{(1)} = (1, 1, 0)$, $g^{(2)} = (1, 0, 1)$ and $g^{(3)} = (1, 1, 1)$

i) Draw the encoder block diagram.

ii)Find the generator matrix.

Iii) Find the code-word corresponding to information sequence (1 1 1 0 1) using time-domain and transform-domain approach.



Solution:

i) The encoder with n=3, k=1, m=2

ii) The generator matrix is of the order Lx [n(L+m)]= 5x[3(5+2)]= 5x 21

$$G=\begin{matrix} g_1^{(1)} & g_1^{(2)} & g_1^{(3)} & g_2^{(1)} & g_2^{(2)} & g_2^{(3)} & g_3^{(1)} & g_3^{(2)} & g_3^{(3)} \ldots\ldots & g_{m+1}^{(1)} & g_{m+1}^{(2)} & g_{m+1}^{(3)} & 0 & 0 & 0 & \ldots\ldots & 000 \\ 0 & 0 & 0 & g_1^{(1)} & g_1^{(2)} & g_1^{(3)} & g_2^{(1)} & g_2^{(2)} & g_2^{(3)} & g_m^{(1)} & g_m^{(2)} & g_m^{(3)} & g_{m+1}^{(1)} & g_{m+1}^{(2)} & g_{m+1}^{(3)} & \ldots 000 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_1^{(1)} & g_1^{(2)} & g_1^{(3)} & : & & & ; & & & ; & : \\ & : & & & : \end{matrix}$$

- [G]=

111	101	011	000	000	000	000
000	111	101	011	000	000	000
000	000	111	101	011	000	000
000	000	000	111	101	011	000
000	000	000	000	111	101	011

- [C]=[d] [G]

= [1 1 1 0 1]

111	101	011	000	000	000	000
000	111	101	011	000	000	000
000	000	111	101	011	000	000
000	000	000	111	101	011	000
000	000	000	000	111	101	011

= [111 , 010, 001, 110 , 100, 101 , 011]

ii) Transform-domain approach

Message [d] = [1 1 1 0 1] hence $d(x) = 1+x^3+x^4$

$g^{(1)} = [1\ 1\ 0]$, hence $g^{(1)}(x) = 1+x$; $g^{(2)} = [1\ 01]$, hence $g^{(2)}(x) = 1+x^2$; $g^{(3)} = [1\ 1\ 1]$ hence, $g^{(3)}(x) = 1+x+x^2$

- The output polynomial for the three adders is given by

$C^{(j)}(x) = d(x)\, g^{(j)}(x)$ for j= 1, 2, 3

- For top adder(j=1)

$C^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1+x+x^2+x^4)(1+x)$

$= 1+x^3+x^4+x^5$

- For middle -adder (j=2)

$C^{(2)}(x) = d(x)\, g^{(2)}(x)$

$= (1+x+x^2+x^4)(1+x^2)$

$= 1+x+x^3+x^6$

- For bottom -adder (j=3)

$C^{(3)}(x) = d(x)\, g^{(3)}(x)$

$C^{(2)}(x) = (1+x+x^2+x^4)(1+x+x^2)$

$= 1+x^2+x^5+x^6$

The final encoder output polynomial is given by

$C(x) = C^{(1)}(x^3) + x\, C^{(2)}(x^3) + x^2\, C^{(3)}(x^3)$ (here, n=3)

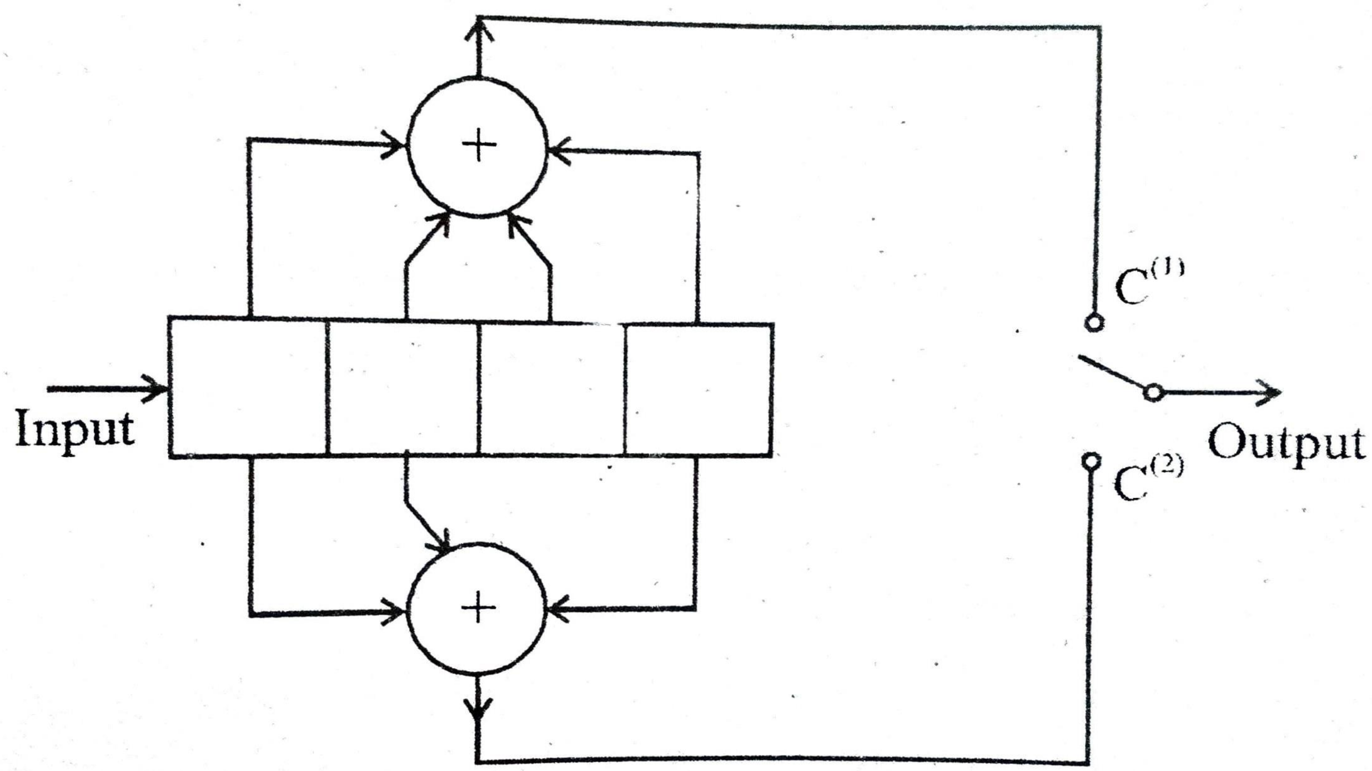
$= 1+x+x^2+x^4+x^8+x^9+x^{10}+x^{12}+x^{15}+x^{17}+x^{19}+x^{20}$

Hence, The output of encoder is [111, 010, 001, 110, 100, 101, 011]

4]For the convolution encoder shown in fig.

i)Find the impulse response and hence calculate the output produced by the information sequence 10111.

ii) Write generator polynomials of the encoder and recompute the output for the input of (i) and compare with that of (i)



Solution:

i) The generator sequence(impulse responses) are given by

$g^{(1)} = [0\ 1\ 1\ 1\ 1\ ]$ fo top adder and $g^{(2)} = [1\ 01\ 1\ 1]$ for bottom adder.

The parameters of encoder are L=5, n=2, k=1, m=4

The generator matrix is of the order Lx [n(L+m)]= 5x[2(5+4)]= 5x18

$$[G]= \begin{matrix} 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 00 & 11 & 11 & 10 & 00 & 00 & 00 & 00 \\ 00 & 00 & 00 & 11 & 11 & 10 & 00 & 00 & 00 \\ 00 & 00 & 00 & 00 & 11 & 11 & 10 & 00 & 00 \\ 00 & 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 \end{matrix}$$

[C]= [d] [G]

$$= [1\ 0\ 1\ 1\ 1] \quad \begin{matrix} 00 & 11 & 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 00 & 11 & 11 & 10 & 00 & 00 & 00 & 00 \\ 00 & 00 & 00 & 11 & 11 & 10 & 00 & 00 & 00 \\ 00 & 00 & 00 & 00 & 11 & 11 & 10 & 00 & 00 \\ 00 & 00 & 00 & 00 & 00 & 11 & 11 & 10 & 11 \end{matrix}$$

$$= [00,\ 11,\ 11,\ 01,\ 11,\ 10,\ 10,\ 01,\ 11\ ]$$

ii) Message [d] = [1 0 1 1 1] hence $d(x) = 1+x^2 + x^3 + x^4$

$g^{(1)} = [0\ 1\ 1\ 1\ 1]$, hence $g^{(1)}(x) = x + x^2 + x^3 + x^4$ for top adder.

$g^{(2)} = [1\ 01\ 1\ 1]$, hence $g^{(2)}(x) = x + x^2 + x^4$ for bottom adder.

- The output polynomial for the top adder is given by

$C^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1+x^2 + x^3 + x^4)(x + x^2 + x^3 + x^4)$

$= x + x^2 + x^4 + x^5 + x^6 + x^8$

- The output polynomial for the bottom adder is given by

$C^{(2)}(x) = d(x)\, g^{(2)}(x)$

$= (1+x+x^2+x^4)(1 + x^2)$

$= x + x^2 + x^3 + x^4 + x^7 + x^8$

- The final encoder output polynomial is given by

$C(x) = C^{(1)}(x^2) + x\, C^{(2)}(x^2)$ (here, n=2)

$= x^2 + x^4 + x^8 + x^{10} + x^{12} + x^{16} + x(x^2 + x^4 + x^6 + x^8 + x^{14} + x^{16})$

$= x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + x^9 + x^{10} + x^{12} + x^{15} + x^{16} + x^{17}$

Hence, The output of encoder is [00, 11, 11, 01, 11, 10, 10, 01, 11]

5]For the (3,2,1) encoder shown in fig, find code-word C for input sequences of $d^{(1)} = 101$ and $d^{(2)} = 110$ using

a) time –domain approach (using generator matrix)

b)Transform domain approach by constructing transfer function matrix.

Solution:

i) The encoder consists of two m=1 stage shift registers with n=3 modulo-2 adders and two commutators. , one at the input and other at the output. The number of input bits entering through input commutator is k=2.

ii) For k=2, input message sequences can be written as

$d^{(1)} = [d_1^{(1)} \quad d_2^{(1)} \quad d_3^{(1)} \; .... \;]$

and $d^{(2)} = [d_1^{(2)} \quad d_2^{(2)} \quad d_3^{(2)} \; .... \;]$

- The generator sequences corresponding to “i” number of segments in the input commutator and “j” number of segments in the output commutator is given by

$$g_i^{(j)} = [g_{i,1}^{(j)} \quad g_{i,2}^{(j)} \; .... \; g_{i,\, m+1}^{(j)} \;]$$

- There are 2 segments in the input commutator and 3 output commutator.

∴ ‘i’ takes values 1 and 2 and j, 1,2 and 3.

- For i=1, j=1, $g_1^{(1)} = [1,1]$
- For i=1, j=2, $g_1^{(2)} = [1,0]$
- For i=1, j=3, $g_1^{(3)} = [1,1]$
- For i=2, j=1, $g_2^{(1)} = [0,1]$
- For i=2, j=2, $g_2^{(2)} = [1,1]$
- For i=2, j=3, $g_2^{(3)} = [0,0]$

The generator matrix for a (3,2,m) convolutional code can be written as

$$G=\begin{matrix} g_{11}^{(1)} & g_{11}^{(2)} & g_{11}^{(3)} & g_{12}^{(1)}\, g_{12}^{(2)}\, g_{12}^{(3)} \ldots\ldots & g_{1,\,m+1}^{(1)}\, g_{1,\,m+1}^{(2)}\, g_{1,m+1}^{(3)} & 0 \quad 0 & 0 \;\ldots\ldots & 0 \\ g_{21}^{(1)} & g_{21}^{(2)} & g_{21}^{(3)} & g_{22}^{(1)}\, g_{22}^{(2)}\, g_{22}^{(3)} \ldots\ldots & g_{2,\,m+1}^{(1)}\, g_{2,\,m+1}^{(2)}\, g_{2,m+1}^{(3)} & 0 \quad 0 & 0 \;\ldots\ldots & 0 \\ 0 & 0 & 0 & g_{11}^{(1)}\; g_{11}^{(2)}\; g_{11}^{(3)} \ldots\ldots & g_{1.m}^{(1)}\, g_{1.m}^{(2)}\, g_{1.m}^{(3)} \ldots\ldots & & g_{1,\,m+1}^{(1)}\, g_{1,\,m+1}^{(2)}\, g_{1,m+1}^{(3)} & 0 \\ 0 & 0 & 0 & g_{21}^{(1)}\; g_{21}^{(2)}\; g_{21}^{(3)} \ldots\ldots & g_{2.m}^{(1)}\, g_{2.m}^{(2)}\, g_{2.m}^{(3)} \ldots\ldots & & g_{2,\,m+1}^{(1)}\, g_{2,\,m+1}^{(2)}\, g_{2,m+1}^{(3)} & 0 \end{matrix}$$

- The generator sequences for the encoder is given by
  $g_1^{(1)} = [g_{11}^{(1)} \;\; g_{12}^{(1)}] = [1, 1]$, $\quad g_2^{(1)} = [g_{21}^{(1)} \;\; g_{22}^{(1)}] = [0, 1]$
  $g_1^{(2)} = [g_{11}^{(2)} \;\; g_{12}^{(2)}] = [1, 0]$, $\quad g_2^{(2)} = [g_{21}^{(2)} \;\; g_{22}^{(2)}] = [1, 1]$
  $g_1^{(3)} = [g_{11}^{(3)} \;\; g_{12}^{(3)}] = [1, 1]$, $\quad g_2^{(3)} = [g_{21}^{(3)} \;\; g_{22}^{(3)}] = [0, 0]$
- The generator matrix will now have L+L=2L number of rows and n(L+m) number of columns.
- Here, L=3, n=3, m=1
- Hence 6 rows and 12 columns
- $$G=\begin{matrix} 1\,1\,1 & 1\,0\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,1\,0 & 1\,1\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 1\,1\,1 & 1\,0\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,1\,0 & 1\,1\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 1\,1\,1 & 1\,0\,1 \\ 0\,0\,0 & 0\,0\,0 & 0\,1\,0 & 1\,1\,0 \end{matrix}$$

ii) Message [d] = [1 0 1 1 1] hence $d(x) = 1+x^2 + x^3 + x^4$

$g^{(1)}$ = [0 1 1 1 1] , hence $g^{(1)}(x) = x + x^2 + x^3 + x^4$ for top adder.

$g^{(2)}$ = [1 01 1 1], hence $g^{(2)}(x) = x + x^2 + x^4$ for bottom adder.

- The output polynomial for the top adder is given by

$C^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1+x^2 + x^3 + x^4)\ (x + x^2 + x^3 + x^4)$

$= x + x^2 + x^4 + x^5 + x^6 + x^8$

- The output polynomial for the bottom adder is given by

$C^{(2)}(x) = d(x)\, g^{(2)}(x)$

$= (1+x+x^2+x^4)\ (1+ x^2)$

$= x + x^2 + x^3 + x^4 + x^7 + x^8$

- The final encoder output polynomial is given by

$C(x) = C^{(1)}(x^2) + x\, C^{(2)}(x^2)$ (here, n=2)

$= x^2 + x^4 + x^8 + x^{10} + x^{12} + x^{16} + x(x^2 + x^4 + x^6 + x^8 + x^{14} + x^{16})$

$= x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + x^9 + x^{10} + x^{12} + x^{15} + x^{16} + x^{17}$

Hence, The output of encoder is [00, 11, 11, 01, 11, 10, 10, 01, 11]

Consider the binary convolutional encoder shown in the figure. Draw the state table , state transition table, State diagram, corresponding code tree. Using the code tree, find the encoded sequence for the message 10111

- Part-1: State Table

There are 2 flip-flops in the shift register and hence there are $2^2 = 4$ states.

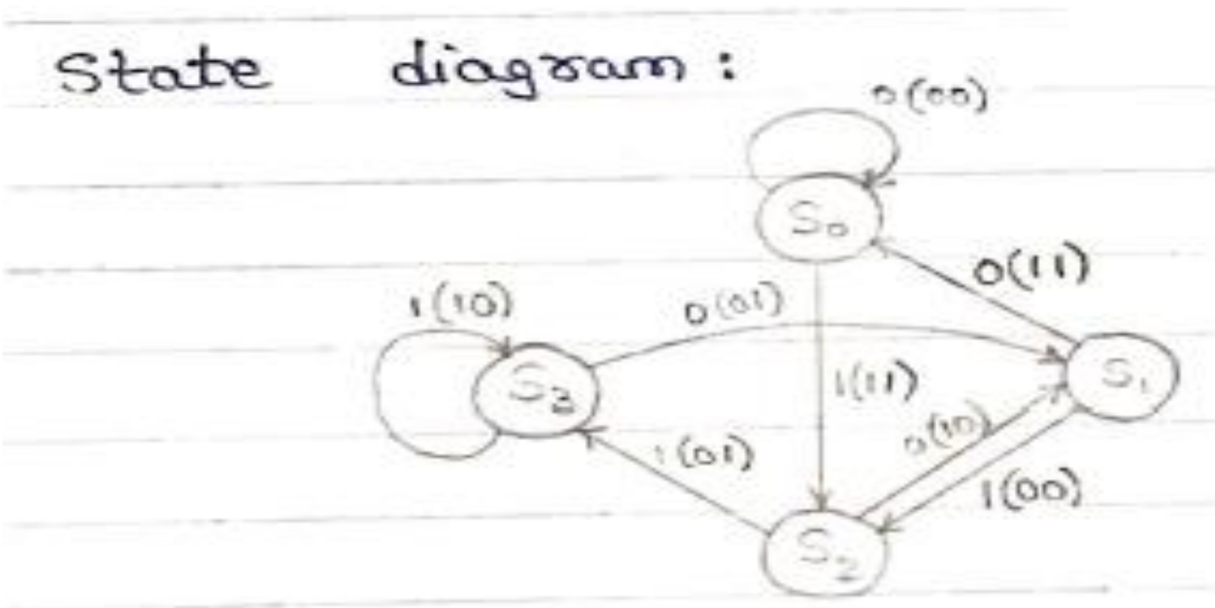
State	s_0	s_1	s_2	s_3
Binary Description	00	01	10	11

- Part-2: State Transition Table

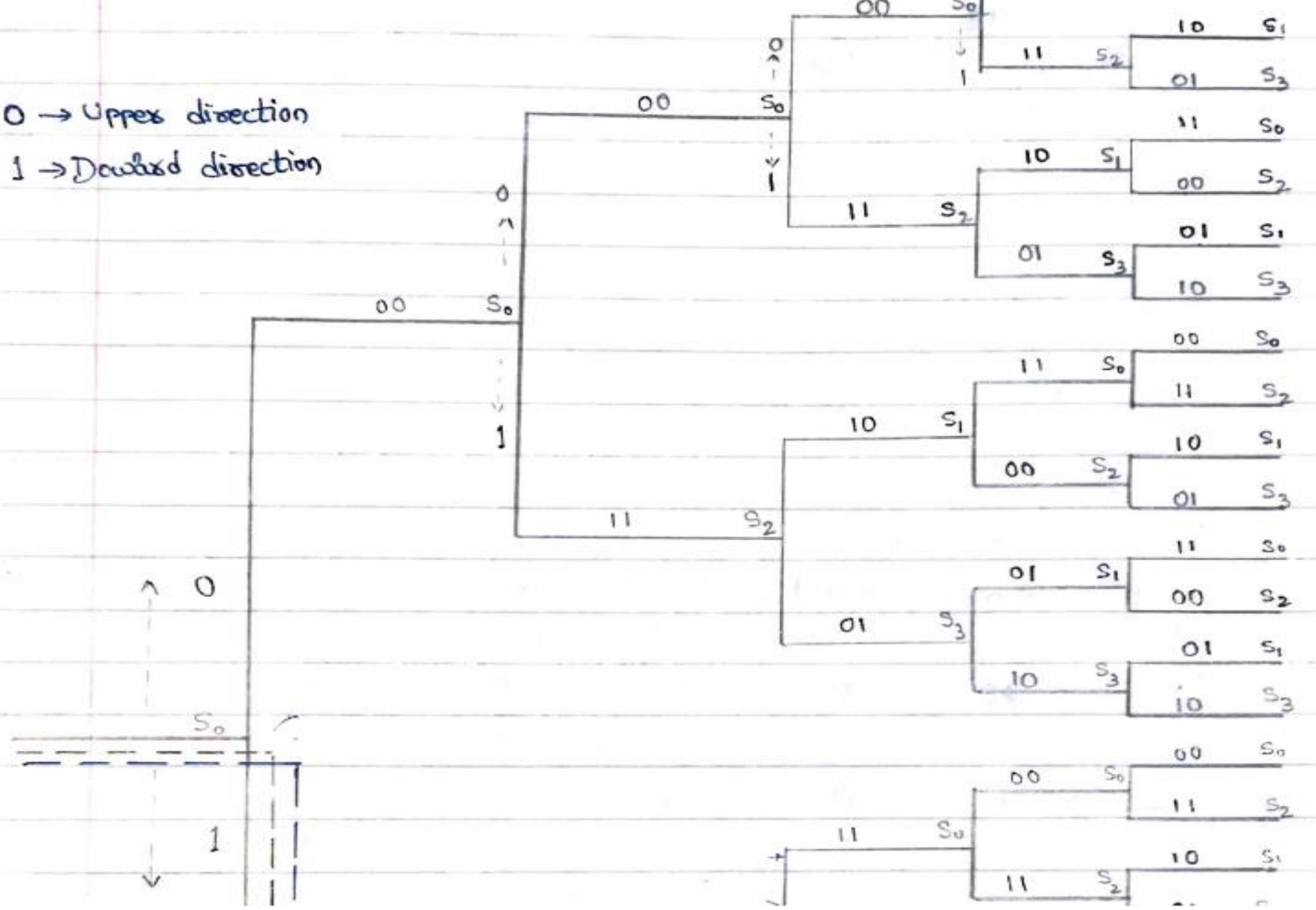
  From fig; $C^{(1)} = d_l + d_{l-1} + d_{l-2}$ $\qquad C^{(2)} = d_l + d_{l-2}$

Present State	Binary Description	Input	Next state	Binary Description	d_l d_{l-1} d_{l-2}	Output $C^{(1)}$ $C^{(2)}$
s_0	00	0 1	s_0 s_2	00 10	000 100	0 0 1 1
s_1	01	0 1	s_0 s_2	00 10	001 101	1 1 0 0
s_2	10	0 1	s_1 s_3	01 11	010 110	1 0 0 1
s_3	11	0 1	s_1 s_3	01 11	011 111	0 1 1 0

Part-3:
State Diagram



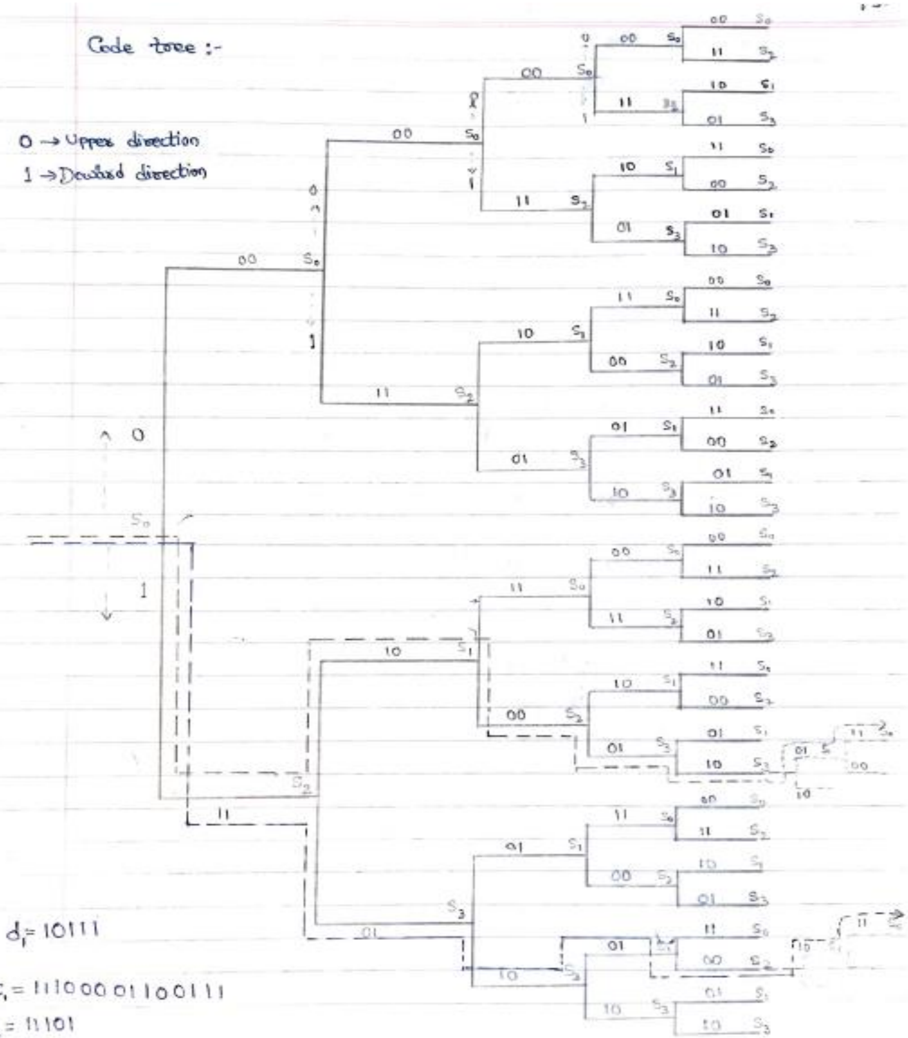
0 → Upper direction
1 → Dowbrd direction



Code tree :-

0 → Upper direction

1 → Downward direction



$d_1 = 10111$

$c_1 = 11100001100111$

$d_2 = 11101$

$c_2 = 11011001001011$