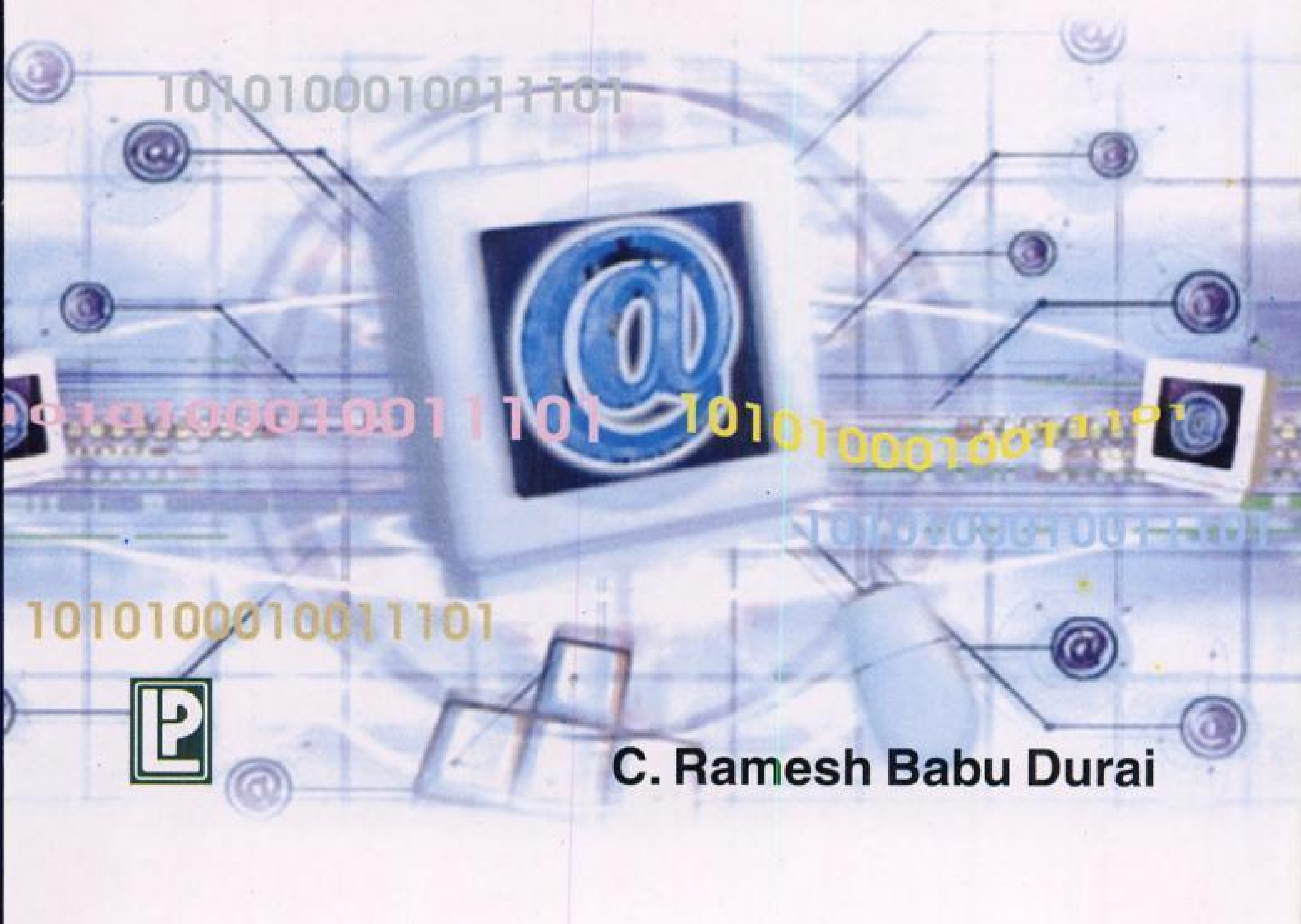
# DIGITAL SIGNAL PROCESSING

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DIGITAL SIGNAL PROCESSING

# Unit 1

Chapters :

# 1 Introduction

## Characterization and Classification of Signals

### Signal

A 'signal' is defined as any physical quantity that varies with time, space and any other independent variable or variables.

More precisely a signal is a function of a set of independent variables. The signal itself carries some kind of information available for observation.

### Processing

By 'processing' we mean operating in some fashion on signal to extract some useful information.

### Digital

The word 'digital' shall mean that the processing is done with a digital computer or special purpose digital hardware.

### Digital Signal Processing

Digital signal processing is concerned with the representation of signals by sequence of numbers or symbols and the processing of these sequence.

The purpose of such processing may be to estimate characteristic parameters or transform a signal into form which is in some sense more desirable.

### Application

Bio-medical engineering, acoustics, radar, speech communication, data communication, image processing, nuclear science and many others.

## 1.1 CLASSIFICATION OF SIGNALS

There are five methods of classifying signals based on different features :

(*a*) *Based on independent variable.*

(*b*) *Depending upon the number of independent variable.*

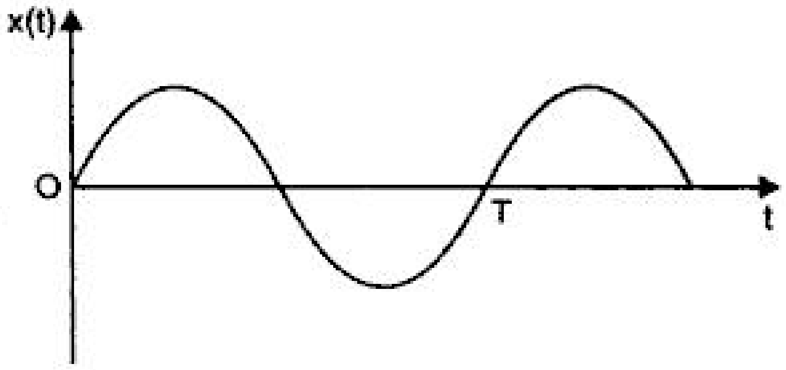
(*c*) *Depending upon the certainity by which the signal can be uniquely described.*

(*d*) *Based on repetition nature.*

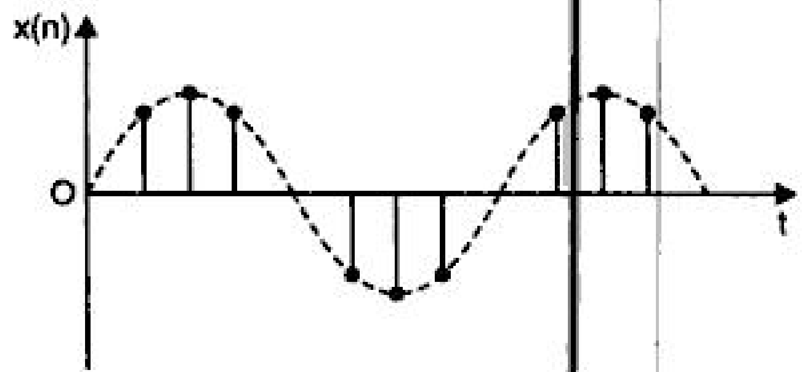
(*e*) *Based on reflection.*

(*a*) **Based on independent variable.** Independent variables can be continuous or discrete.

1. *Continuous Time Signal.* It is also referred as analog signal *i.e.*, the signal is represented continuously in time. In simple words, a signal $x(t)$ is said to be a continuous time signal if it is defined for all time.
2. *Discrete Time Signal.* Signals are represented as sequence at discrete time intervals. Thus, the independent variable has discrete values only.




(*a*) Continuous time signal




(*b*) Discrete time signal

Fig. 1.1

*e.g.* Speech signal is an example of analog signal.

A discrete time signal which discrete-valued represented by a finite number of digits is referred to as a "digital signal".

*e.g.* Digitized music signal stored in CD-ROM disk.

(*b*) **Depending upon the number of independent variable.**

(*i*) *1-D Signals.* It is a function of a single independent variable.

*e.g.* (*a*) speech signal–independent variable is time.

(*b*) music signal.

(*ii*) *2-D Signal.* It is a function of two independent variables.

*e.g.* Photographic image signal–two independent variables are the two spatial variables.

Each frame of a black and white video signal is a 2D-image signal that is a function of two discrete spatial variable, with each frame occurring squentially at discrete instants of time.

(*iii*) *M-D Signal.* It is a function of 'M' independent variable in time.

*e.g.* Video signal.

The black and white video signal can be considered an example of a 3D signal where the three independent variables are two spatial variables and time.

A colour video signal is a three-channel signal composed of three 3-D signals representing the three primary colours : red, green and blue (RGB).

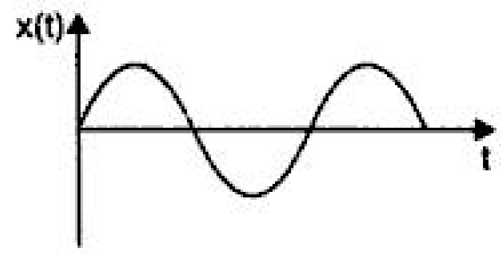
For transmission purpose, the RGB television signal is transformed into another type of 3-channel signal is composed of luminance component and two chrominance components.

(*c*) **Depending upon the certainity by which the signal can be uniquely described as**

(*i*) *Deterministic Signal.* A signal that can be uniquely determined by a well-defined process such as a mathematical expression or rule, or table look-up is called a deterministic signal.

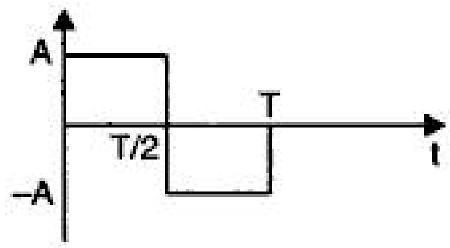
*e.g.* (*a*) A sinusoidal signal can be represented as,

$$v(t) = V_m \sin \omega t \quad \text{for } t \geq 0.$$



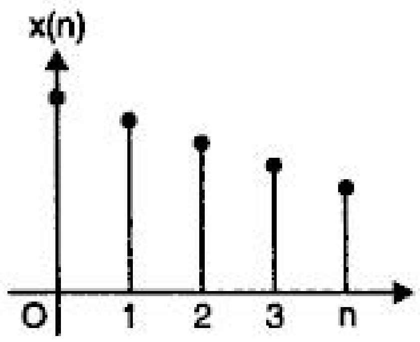

(*b*) A square signal can be defined as

$$x(t) = A \quad \text{for} \quad 0 < t < T/2$$
$$= -A \quad \text{for} \quad T/2 < t < T.$$

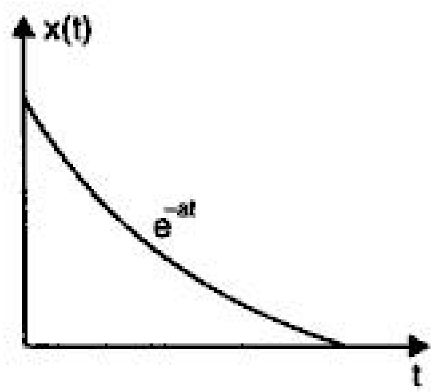



(*c*) An exponential,

discrete time signal $x(n) = e^{an}$ for $n \geq 0$




Continuous time signal $x(t) = e^{-at}$ for $n < 0$.




(*ii*) *Random Signal.* A signal that is generated in a random fashion and cannot be predicted ahead of time is called a "random signal".

*e.g.* Speech signal, ECG signal, EEG signals.

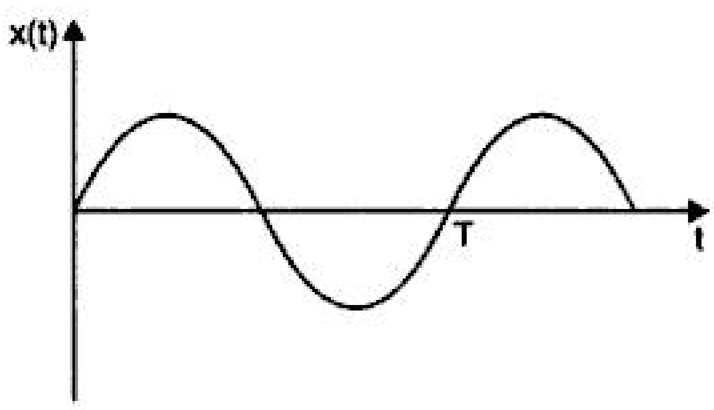
(*d*) **Based on repetition nature.** The signal can be classified into the two types.

(*i*) *Periodic Signal*

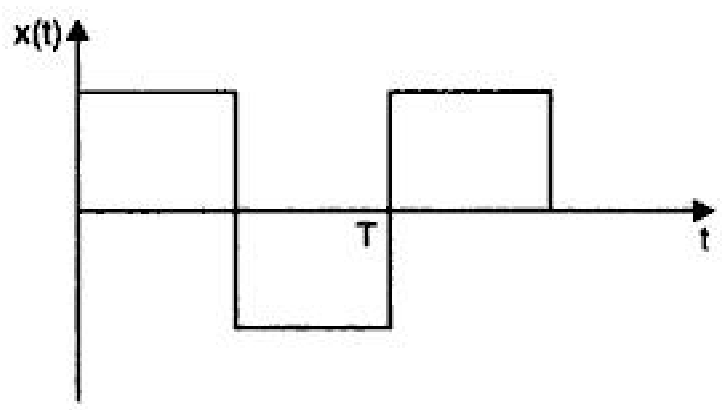
(*a*) *Continuous time.* The periodic Signal means, the signal which repeats every finite, interval of time or a continuous time signal $x(t)$ is a function that satisfies the condition.

$$\boxed{x(t) = x(t + T)} \quad \text{for all '}t\text{'} \qquad \ldots(1.1)$$

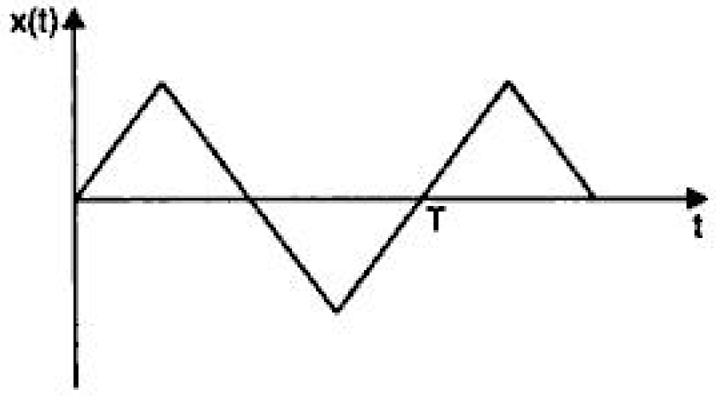
*e.g.* Sine wave, square wave, triangular wave etc.,



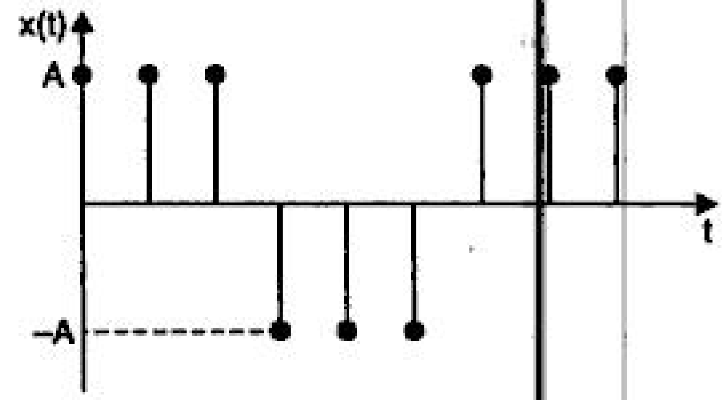

(*a*) Sine wave




(*b*) Square wave




(c) Triangular wave




(d) Discrete or impulse wave

Fig. 1.2. Periodic signal.

After every 'T' the signal repeats itself.

The smallest possible value of 'T' for which the equation (1) hold good is known as 'fundamental period', it defines the duration of one complete cycle of $x(t)$.

The reciprocal of fundamental period is known as fundamental frequency of the periodic signal $x(t)$.

*i.e.*, $$f = \frac{1}{T}.$$

$$\Omega = 2\pi f = \frac{2\pi}{T} \quad \Rightarrow \quad \text{Analog angular frequency.}$$

(*b*) *Discrete time periodic signal*. A discrete time signal is said to be periodic, if it satisfies the relation,

$$\boxed{x(n) = x(n + N)} \quad \text{for all integers of '}n\text{'} \qquad ...(1.2)$$

where, 'N' is a positive integer.

The fundamental period is the smallest positive value of 'N' for which $x(n) = x(n + N)$ hold good. The fundamental angular frequency $\omega = \frac{2\pi}{N}$.

where, ω–discrete angular frequency.

(*ii*) *Non-periodic Signal*. Any signal $x(t)$ for which there is no value of T to satisfy the relation given in equation $x(t) = x(t + T)$ is known as non-periodic signal.

*i.e.*, $x(t) \neq x(t + T)$, for all '$t$'

*e.g.*


Fig. 1.3. Non-periodic signal.

(*e*) **Based on reflection.** It can be defined as,

(*i*) *Even signal (symmetric).* A signal $x(t)$ or $x(n)$ is referred to as even signal, if it is identical to its time reversal counter part *i.e.*, with its about the origin.

$$x(t) = x(-t) \quad \text{for all 't'} \quad \text{CT even signal.}$$

and
$$x(n) = x(-n) \quad \text{for all 'n'} \quad \text{DT even signal.}$$

*Even signals are symmetric with respect to vertical axis.

*e.g.*

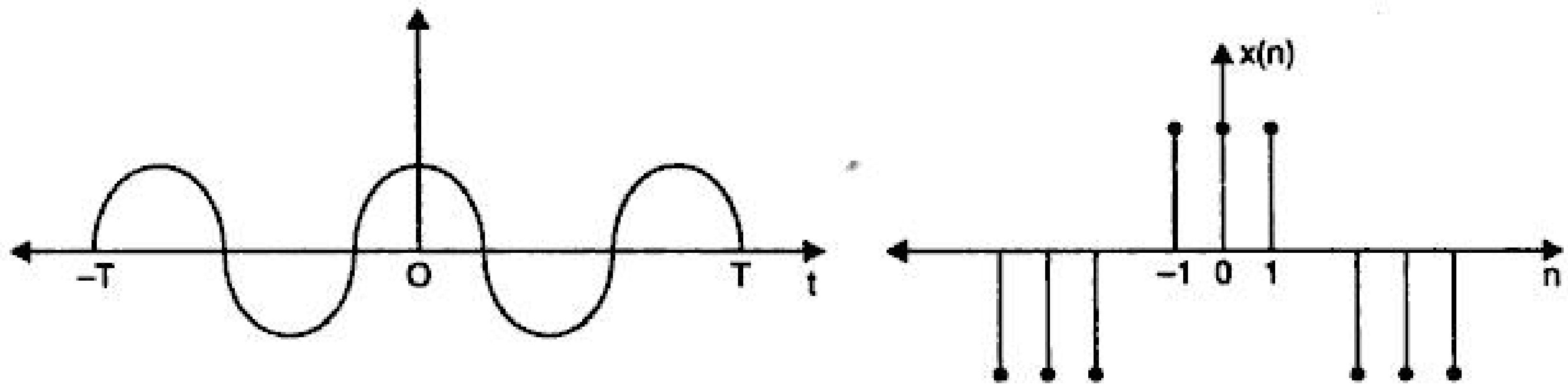



Fig. 1.4. Even signal.

(*ii*) *Odd signal (antisymmetric).* A signal is said to be odd signal if,

$$x(t) = -x(-t) \quad \text{for} \quad \text{CT signal.}$$
$$x(n) = -x(-n) \quad \text{for} \quad \text{DT signal.}$$

*e.g.*

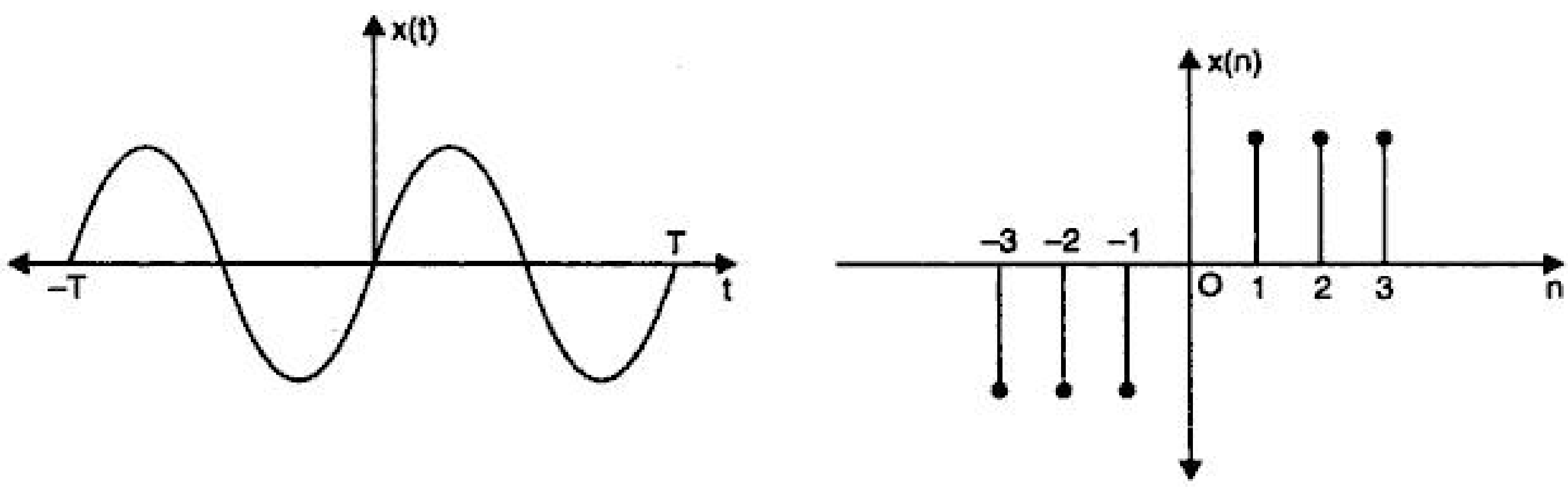



Fig. 1.5. Odd signal.

A signal can be broken into two parts one of which is even and the other is odd.

$$\begin{aligned}
\text{Even } [x(n)] &= 1/2 && \text{for } n < 0\\
&= 1 && \text{for } n = 0\\
&= 1/2 && \text{for } n > 0\\
\text{Odd } [x(n)] &= -1/2 && \text{for } n < 0\\
&= 0 && \text{for } n = 0\\
&= 1/2 && \text{for } n > 0.
\end{aligned}$$

Thus,
$$\text{Even } [x(n)] = \frac{1}{2}[x(n) + x(-n)] \qquad \text{...(1.3)}$$
$$\text{Odd } [x(n)] = \frac{1}{2}[x(n) - x(-n)] \qquad \text{...(1.4)}$$

**Properties of even and odd signal :**

1. The sum of two even signals are even signal.
2. The sum of two odd signals are odd.
3. The sum of an even signal and an odd signal is neither even nor odd signal.
4. The product of two even signal is even.
5. The product of two odd signal is even.
6. The product of even signal and an odd signal is odd.

## 1.2 MULTI CHANNEL

A signal can be generated by a single source or by multiple sources or multiple sensors. In the former case, it is a (single) scalar signal and in the later case it is a vector signal, often called a multichannel signal.

These type of signals can be represented in vector form as,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad ...(1.5)$$

Equation represents a 3-channel signal.

*e.g.* In electrocardiography [ECG] for example 3-lead and 12-lead electrocardiographs are often used in practice, which result in 3-channel and 12-channel signals.

## 1.3 MULTI DIMENSIONAL SIGNALS

If a signal is a function of a single independent variable, then it is called as one-dimensional signal. Similarly, if signal is a function of N-independent variables, it is called as N-dimensional signal.

*e.g.*

- Picture signal is a two dimensional signal, since the intensity $I(x, y)$ is a function of 'wo independent variables $x$ and $y$.
- Black and white television picture is an example of 3-dimensional signal because brightness $I(x, y, t)$ is a function of three independent variables $x$, $y$ and $t$ (time).
- It is also possible to have multichannel and multidimensional signals simultaneously. For example, a colour TV picture is described by three intensity functions of form $I_r(x, y, t)$ [red], $I_g(x, y, t)$ [green], and $I_b(x, y, t)$ [blue].

Hence colour TV picture is a three dimensional and three channel signal, which can be represented by the vector.

$$I(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix} \quad ...(1.6)$$

## 1.4 CONTINUOUS-TIME VERSUS DISCRETE-TIME SIGNALS

(1) Signals can be further classified into different categories depending on the characteristics of the time (independent) variables and the values they take.

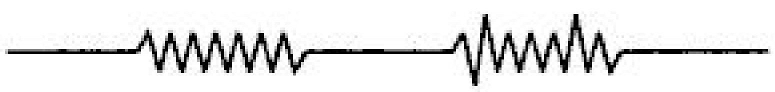
**Continuous**

- Continuous-time signals or analog signals are defined for every value of time and they take on values in the continuous interval $(a, b)$.

where $a$ can be $-\infty$

$b$ can be $+\infty$.

Mathematically, these signals can be described by functions of a continuous variable.



*e.g.* Speech signals $x_1(t) = \cos \pi t$

$$x_2(t) = e^{-|t|}, \quad -\infty < t < \infty.$$

**Discrete**

- Discrete time signals are defined only at certain specific values of time. These time instant need not be equidistant, but generally they are taken at equally spaced intervals for convenience.

  *e.g.*

$$x(t_n) = e^{-|t_n|}, n = 0, \pm 1, \pm 2 \ldots\ldots$$

  index '$n$' of the discrete-time instants as the independent variables.

**In applications, discrete-time signal may arise in two ways**

- In practical setting, such sequence $(n)$ can often arise from periodic sampling of an analog signal. In this case, the numeric value of the $n^{th}$ number in the sequence is equal to the value of analog signal $x_a(t)$ at time $n$T *i.e.*,

$$x(n) = x(n\text{T}).$$

  The quantity T is called sampling period and its reciprocal is the sampling frequency.
- By accumulating a variable over a period of time. For example, counting the number of cars in a given street every hour, or recording the value of gold every day, results in discrete-time signals.

(2) **Continuous-valued and Discrete-valued Signals.** A signal is said to be continuous valued signal if it takes on all possible values on a finite or infinite range. On the other hand, if the signal allowed to take on values from the given set, it is said to discrete-valued signal. Normally, these values are equidistance and hence can be expressed as an integer multiple of the distance between two successive values.

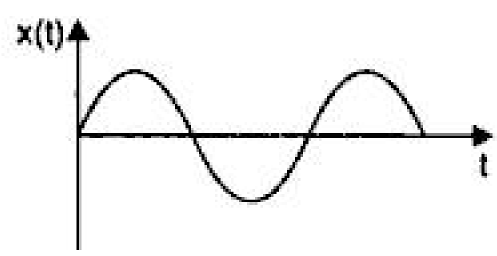
If the signal to be processed is in analog form, it is converted to a digital signal by sampling the analog signal at discrete instants in time, obtaining a discrete-time signal, and then by quantizing its values to a set of discrete values.

**Quantization.** The process of converting a continuous-valued signal into a discrete-valued signal, called quantization.

(3) **Deterministic Versus Random Signals**

Depending upon the certainity by which the signal can be uniquely described as

(*i*) *Deterministic Signal.* A signal that can be uniquely determined by a well-defined process such as a mathematical expression or rule, or table look-up is called a deterministic signal.



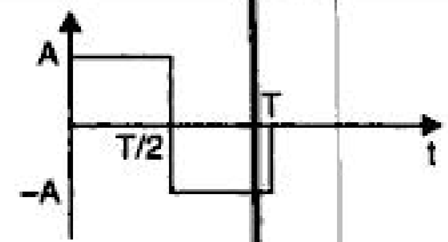

*e.g.*

(*a*) A sinusoidal signal can be represented as,

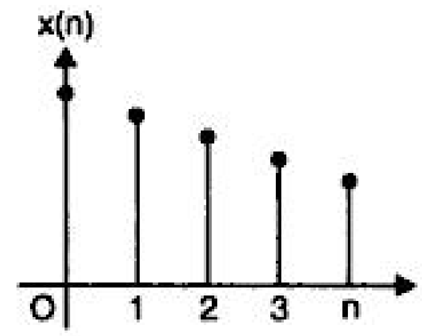
$$v(t) = \text{V}_m \sin \omega t \text{ for } t \geq 0.$$

(*b*) A square signal can be defined as

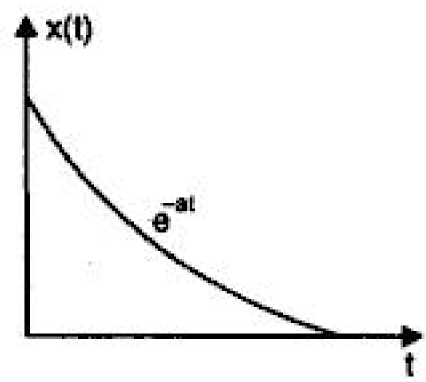
$$x(t) = A \quad \text{for} \quad 0 < t < T/2$$
$$= -A \quad \text{for} \quad T/2 < t < T.$$




(*c*) An exponential,
discrete time signal $x(n) = e^{an}$ for $n \geq 0$




Continuous time signal $x(t) = e^{-at}$ for $n < 0$.




(*ii*) *Random Signal.* A signal that is generated in a random fashion and cannot be predicted ahead of time is called a "random signal".
*e.g.* Speech signal, ECG signal, EEG signals.

## 1.5 FREQUENCY CONCEPT IS CONTINUOUS TIME AND DISCRETE-TIME SIGNALS

We know that the frequency is closely related to a periodic motion which is described by sinusoidal functions. As the frequency is directly related with time $\left(\text{Frequency} = \dfrac{1}{\text{Time period}}\right)$, therefore, we should expect that the nature of time (continuous or discrete) would affect the nature of frequency accordingly.

### 1.5.1 Continuous Time Sinusoidal Signals

A continuous time signal is mathematically described as,

$$x_a(t) = A\cos(\Omega t + \theta)\,; \quad -\alpha < t < +\alpha \qquad \ldots(1.7)$$

The above signal is shown in Fig. (1.6)

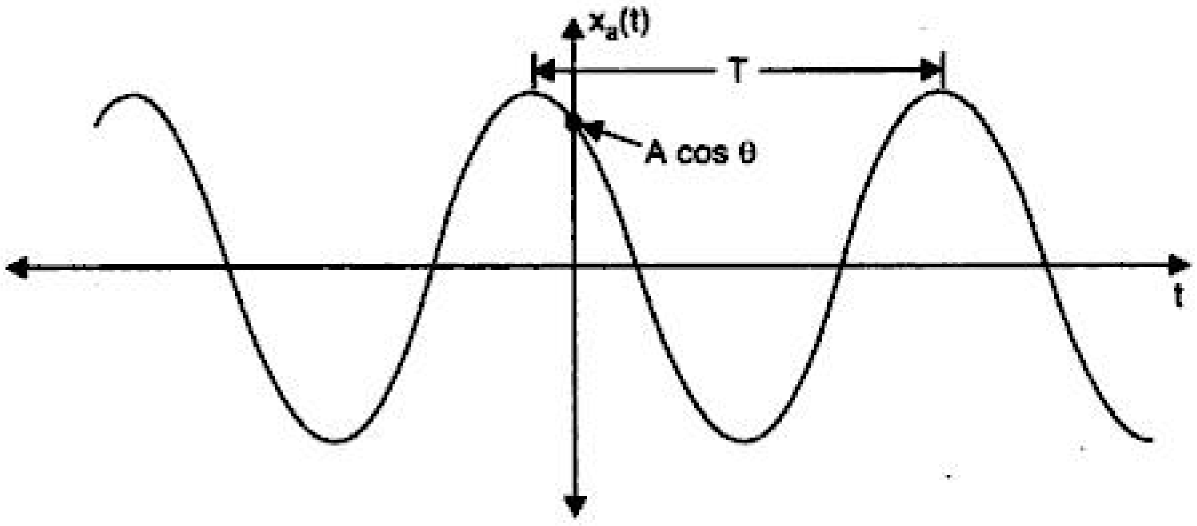



Fig. 1.6. Continuous time sinusoidal signal.

This signal is completely characterized by three parameters :

A is the amplitude of sinusoid

$\Omega$ is the frequency in radians per second (r/s) and $\theta$ is the phase in radians.

Eqn. 1.7 can be written as,

$$x_a(t) = A\cos(2\pi Ft + \theta)\,; \qquad -\alpha < t < +\alpha \qquad \text{...(1.8)}$$

where, $\Omega = 2\pi F$.

$$F = \frac{1}{T} = \text{cycles per second.}$$

T – period of sinusoid.

The analog signal given by eqn. (1.8) has the following properties :

(1) *If* $x_a(t+T) = x_a(t)$, then $x_a(t)$ *is periodic with period T, where* $T = \frac{1}{F}$ *is the fundamental period.*

**Proof.**

$$x_a(t) = A\cos(2\pi Ft + \theta)$$
$$x_a(t+T) = A\cos[2\pi F(t+T) + \theta]$$
$$x_a(t+T) = A\cos(2\pi ft + \theta)$$
$$x_a(t+T) = x_a(t).$$

(2) *Continuous-time sinusoidals with different (distinct) frequencies are themselves different.*

(3) *Increasing the frequency F results in an increase in the rate of oscillation of the signals.*

Now the sinusoidal signal may be expressed as,

$$x_a(t) = A\cos(\Omega t + \theta)$$
$$= \frac{A}{2}e^{j(\Omega t + \theta)} + \frac{A}{2}e^{-j(\Omega t + \theta)}.$$
$$(\because \quad e^{\pm j\phi} = \cos\phi \pm j\sin\phi)$$

Note that, a sinusoidal signal can be obtained by adding two-equal amplitude complex conjugate exponential signal, sometimes called 'phasors'.

*Respresentation of a cosine function by a pair of complex-conjugate exponential.*

As time progresses the phasors rotate in opposite directions with angular frequencies $\pm\,\Omega$ radians/second.

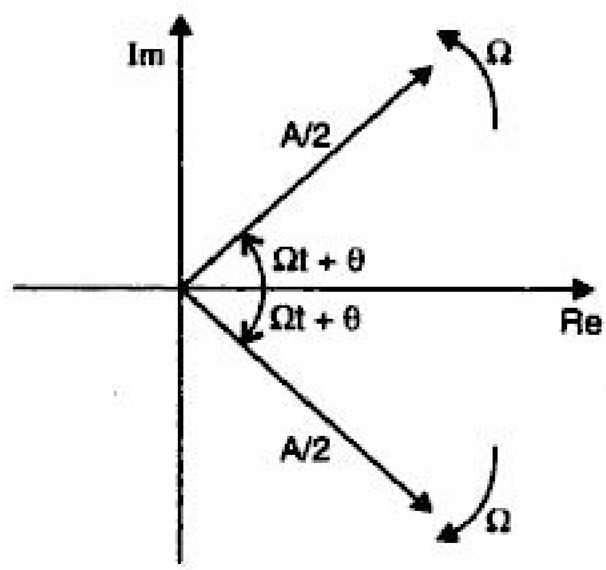



Fig. 1.7. Representation of cosine function.

Since, a ' positive frequency' corresponds to counter clockwise uniform angular motion, a 'negative frequency' simply corresponds to clockwise angular motion. The frequency range for analog sinusoids is $-\infty < F < \infty$.

### 1.5.2 Discrete-time Sinusoidal Signals

Mathematically, a discrete-time sinusoidal signal is represented as

$$x(n) = A\cos(\omega n + \theta) \qquad -\alpha < n < \alpha$$

$$x(n) = A\cos(2\pi fn + \theta) \quad -\alpha < n < \alpha$$

where, $n$ is an integer value, called the sample number

A is the amplitude

$\omega$ is the frequency is radians per sample.

and $\theta$ is the phase in radians.

Fig. 1.8 represents a discrete-time sinusoid with $\omega = \pi/3$ radians per sample $[2\pi f = \pi/2$ or $f = 1/12$ cycle per sample$]$ and $\theta = 0$.

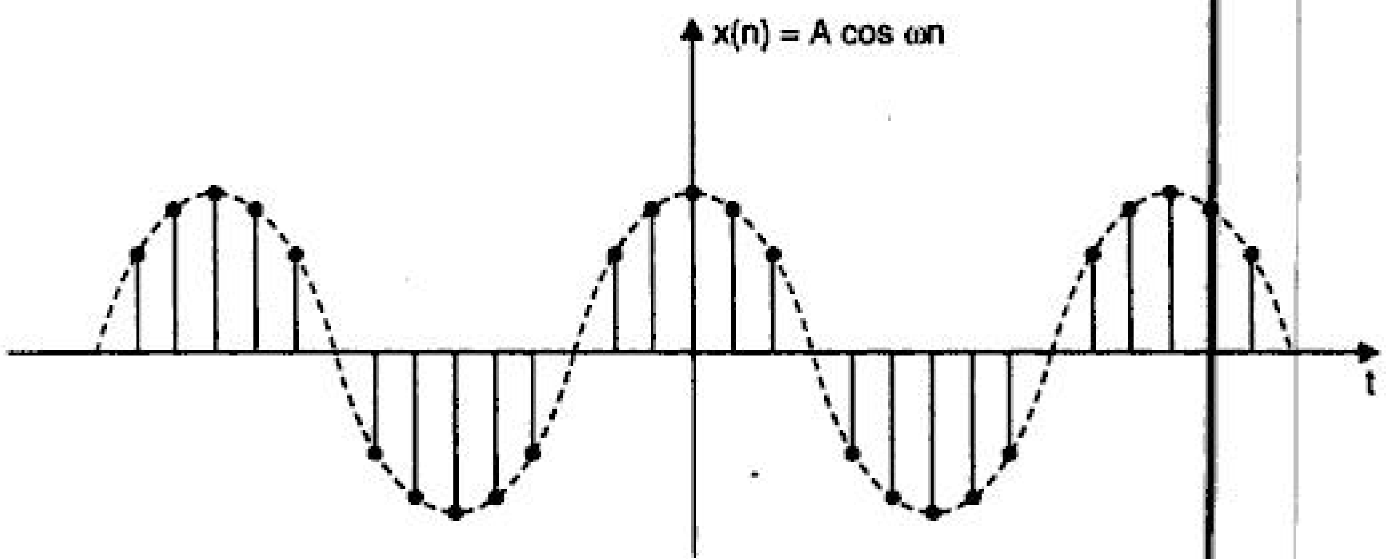



Fig. 1.8. Discrete time sinusoid signal.

The discrete-time signals are characterised by the following properties :

(1) *Discrete-time sinusoids are periodic only if its frequency $f_0$ is a rational number.*

The signal $x(n)$ is periodic with period N( N > 0) if and only if

$$x(n + N) = x(n) \qquad ...(1.9)$$

The smallest value of N is called the fundamental period.

**Proof :** For a sinusoid with frequency $f_0$ to be periodic, we should have,

$$x(n + N) = \cos[2\pi f_0 (N + n) + \theta]$$

$$= \cos[2\pi f_0 n + \theta]$$

The above relation is true if and only if, there exists an integer $k$ such that,

$$2\pi f_0 N = 2\pi k.$$

*i.e.,* $\therefore$

$$\boxed{f_0 = \frac{k}{N}} \qquad ...(1.10)$$

Therefore, the discrete-time sinusoids are periodic only if its frequency $f_0$ can be expressed as rational number (ratio of two integers).

To determine the fundamental period N of a periodic sinusoid, $k$ and N is eqn. 1.10 should be relatively prime. Then the fundamental period of sinusoid is equal to N. For example, *i.e.,* $$f_1 = \frac{21}{40}.$$

then fundamental period $N_1$ is 40 and if

$$f_2 = \frac{20}{40} = \frac{1}{2}$$

then fundamental period $N_2$ is 2. We observe that a small change in frequency may result in a large change in the period.

(2) *Discrete-time sinusoids whose frequencies are separated by an integer multiple of* $2\pi$ *are identical.*

**Proof :** Consider a sinusoid $\cos(\omega_0 n + \theta)$. If the frequencies are separated by $2\pi$, then,

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos[\omega_0 n + 2\pi n + \theta]$$
$$= \cos[\omega_0 n + \theta].$$

Therefore, all the sinusoid signals,

$$x_k(n) = A\cos(\omega_k n + \theta)\ ; \quad k = 0, 1, 2 \ldots\ldots$$

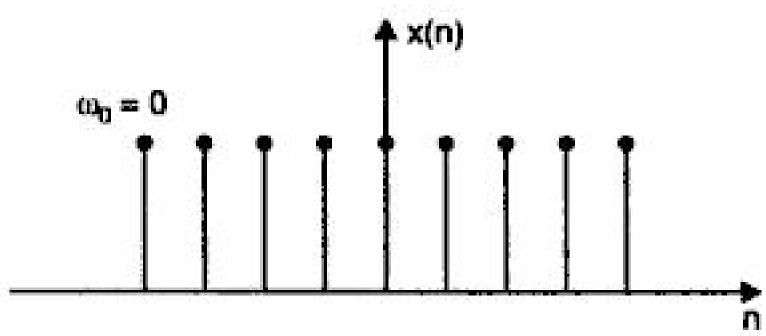
where, $w_k = w_0 + 2\pi k$, are identical (distinguishable).

**Conclusion.** The discrete-time sinusoids with frequencies $|\omega| \le \pi$ or $|f| \le 1/2$ are unique. The sequence resulting from a sinusoid with frequency $|\omega| > \pi$ or $|f| > 1/2$, are identical to the sequence obtained from the sinusoid with frequency $|\omega| < \pi$ or $|f| < 1/2$.

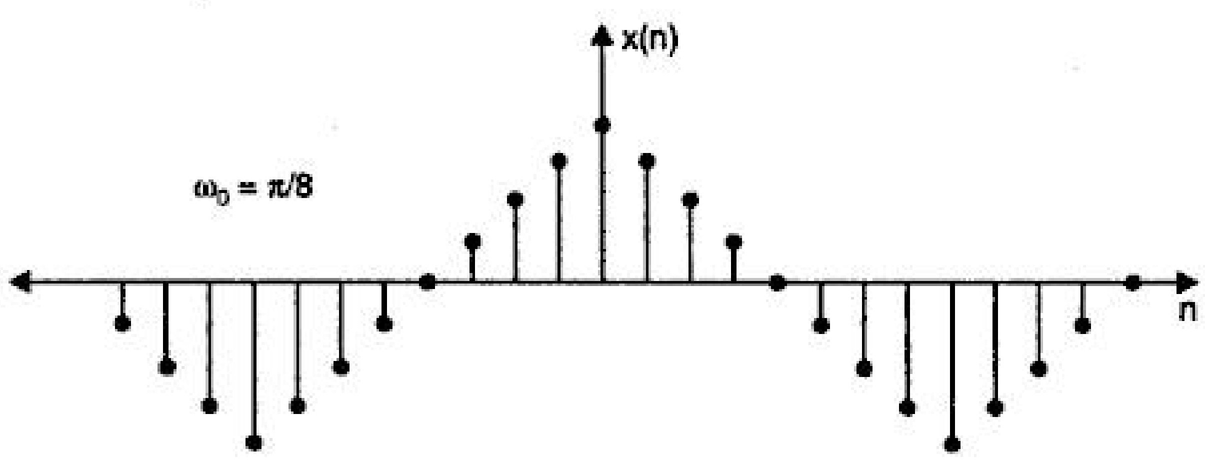
(3) *In discrete-time sinusoids, highest rate of oscillations is attained when* $\omega = \pi$ *(or* $-\pi$*) or equivalently* $f = 1/2$ *(or* $-1/2$*).*

To investigate the characteristic of the sinusoids, let us vary the frequency from 0 to $\pi$.

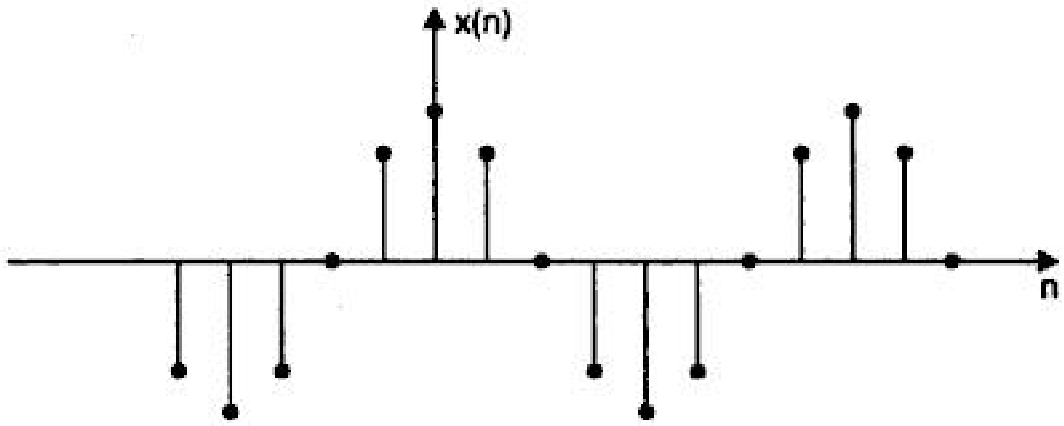
(*i*) $\omega_0 = 0$ ; $d_c$ *i.e.*, no oscillations ; $N = \alpha$



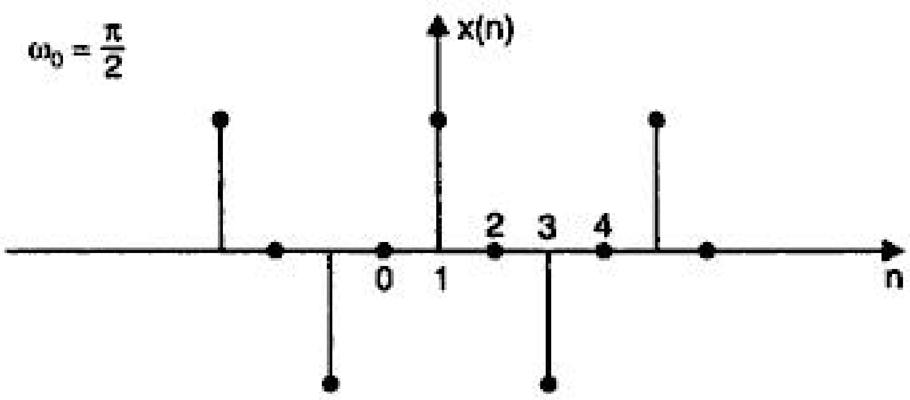

(*ii*) $\omega_0 = \pi/g$, or $f_0 = 1/16$ ; 16 samples in one cycle ; N = 16.



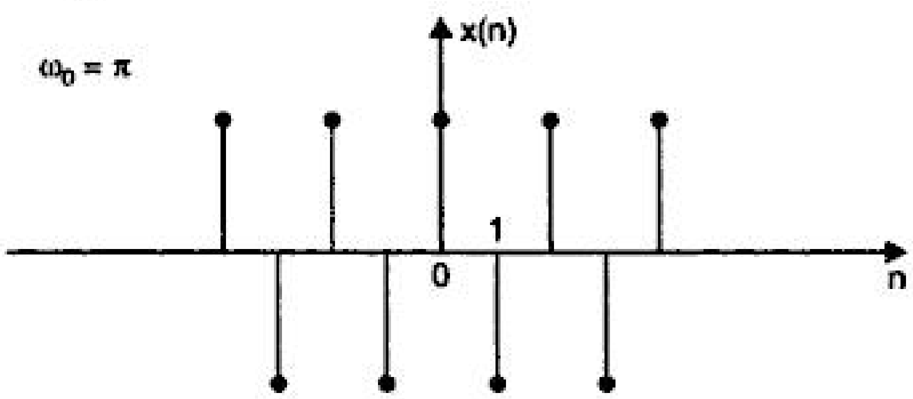
(*iii*) $\omega_0 = \pi/4$ or $f_0 = \frac{1}{8}$ ; 8 samples in one cycle ; N = 8.




(*iv*) $\omega_0 = \pi/2$ or $f_0 = 1/4$ ; 4 samples in one cycle ; N = 4.




(*v*) $\omega_0 = \pi$ or $f_0 = \frac{1}{2}$ ; 2-samples in one cycle ; N = 2.




### 1.5.3 Harmonically Related Complex Exponentials

Sinusoidal signals and complex exponentials play a major role in the analysis of signals and systems. In some cases we deal with sets of harmonically related complex exponentials (or sinusoidal). These are sets of periodic complex exponentials with fundamental frequencies that are multiples of a single positive frequency.

**Continuous-time exponentials.** The basic signals for continuous-time harmonically related exponentials are,

$$S_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t} \; ; k = 0, \pm 1, \pm 2 \; ..... \qquad ...(1.11)$$

we note that for each value of $k$, $S_k(t)$ is periodic with fundamental period $\frac{1}{kF_0} = \frac{T_p}{k}$ or fundamental frequency $kF_0$.

Since a signal that is periodic with period $T_p/k$ is also periodic with period $k\,[T_p/k] = T_p$ for any positive integer $k$, we see that all of the $S_k(t)$ have a common period of $T_p$.

From the basic signals in eqn. (1.11), we can construct a linear combination of harmonically related complex exponentials of the form,

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k S_k(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t} \qquad ...(1.12)$$

where, $C_k$, $k = 0, \pm, 1, \pm 2$ ....... are arbitrary complex constants.

The signal $x_a(t)$ is periodic with fundamental period $T_p = 1/F_0$ and its representation in terms of eqn. (1.12) is called the Fourier series expansion for $x_a(t)$.

### Discrete-Time exponentials

Since a discrete-time complex exponential is periodic if its relative frequency is a rational number, we choose $f_0 = 1/N$ and we define the sets of harmonically related complex exponentials by

$$S_k(n) = e^{j2\pi k f_0 n} \quad ; k = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots \qquad ...(1.13)$$

In contrast to the continuous-time case, we note that,

$$S_{K+N}(n) = e^{j2\pi n(k+N)/N} = e^{j2\pi n} S_k(n) = S_k(n).$$

This means that, consistent with eqn. (1.9) [*i.e.*, $x(N + n) = x(n)$] there are only N distinct periodic complex exponential is the set described by eqn. (1.13).

Furthermore, all members of the set have a common period of N samples.

Clearly, we can choose any consecutive N complex exponentials say from $k = n_0$ to $k = n_0 + N - 1$ to form a harmonically related set with fundamental frequency $f_0 = 1/N$.

For our convenience, we choose the set that corresponds to $n_0 = 0$, that is the set,

$$S_k(n) = e^{j2\pi kn/N} ; k = 0, 1, 2 \ldots.. N-1.$$

As in the case of continuous-time signals, it is obvious that the linear combination,

$$x(n) = \sum_{k=0}^{N-1} C_k S_k(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N},$$

results in a periodic signal with fundamental period 'N'. The sequence of $S_k(n)$ is called the $k^{th}$ harmonic of $x(n)$.

## 1.6 ENERGY AND POWER SIGNALS (CONTINUOUS TIME-INSTANTS)

Signals can also be classified as those having finite energy or finite average power. However, there are some signals which can neither be classified as energy signals nor power signals. Consider, a voltage source $v(t)$, across a unit resistance R, conducting, a current $i(t)$. The instantaneous power dissipated by the resistor is

$$P(t) = v(t)\, i(t) = \frac{v^2(t)}{R} = i^2(t)\, R.$$

Since $R = 1\ \Omega$, we have,

$$P(t) = v^2(t) = i^2(t).$$

The total energy and the average power are defined as the limits.

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\, dt \text{ joules} \qquad ...(1.14)$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt \text{ watts} \qquad ...(1.15)$$

The total energy and the average power normalised to unit resistance of any arbitrary signal $x(t)$ can be defined as,

$$\left.\begin{aligned} E &= \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt \text{ Joules.} \quad &\text{...(1.16)} \\ \text{and} \quad P &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \text{ watts} \quad &\text{...(1.17)} \end{aligned}\right\} \text{Continuous}$$

The energy signal is one which has finite energy and zero average power, *i.e.*, $x(t)$ is an energy signal if $0 < E < \infty$ and $P = 0$.

The power signal is one which has finite average power and infinite energy, *i.e.*, $0 < P < \infty$ and $E = \infty$.

If the signal does not satisfy any of these two conditions, then it is neither an energy nor a power signal.

## 1.7 SINGULARITY FUNCTIONS

Singularity functions are an important classification of non-periodic signals. They can be used to represents more complicated signals.

The unit impulse function, sometimes referred to as "delta function", is the basic singularity function and all other singularity functions can be derived by repeated integration or differentiation of the delta function. The other commonly used singularity functions are the unit-step and unit-ramp functions.

### 1.7.1 Unit-Impulse Function

The unit-impulse function is defined as,

$$\delta(t) = 0, \quad t \neq 0 \qquad \text{...(1.18)}$$

and

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \qquad \text{...(1.19)}$$

The eqn. (1.18) and (1.19) indicate that the area of the impulse function is unity and this area is confined to an infinitesimal interval on the $t$-axis and concentrated at $t = 0$.

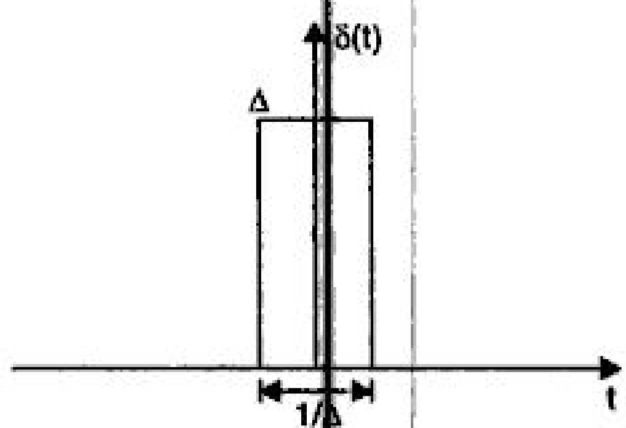


Fig. 1.9 (*a*). Continuous time impulse signal.

The unit impulse function is very useful in continuous-time analysis. It is used to generate the system response providing fundamental information about the system characteristics.

In discrete-time domain, the unit-impulse signal is called a "unit-sample signal".

It is defined as, $$\delta(n) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases}.$$

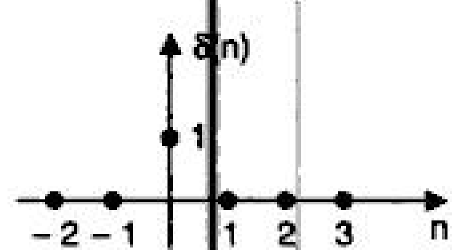


Fig. 1.9 (*b*). Discrete time impulse signal.

### 1.7.2 Unit-Step Function

The integral of the impulse function $\delta(t)$ gives,

$$\int_{-\infty}^{t} \delta(t) \, dt = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases},$$

Since, the area of the impulse function is all concentrated at $t = 0$, for any value of $t < 0$, the integral becomes zero and for $t > 0$, $\int_{-\infty}^{t} \delta(t)\, dt = 1$.

The integral of the impulse function is also a singularity function and called the unit-step function and is represented as,

$$u(t) = \begin{cases} 0 & , \ t < 0 \\ 1 & , \ t > 0 \end{cases}$$

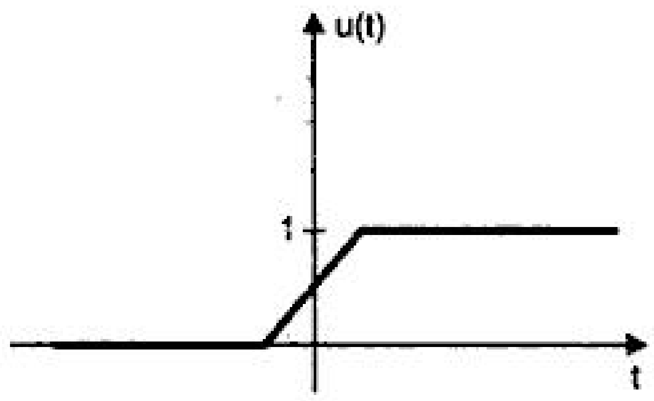



Fig. 1.10 (*a*). Continuous time unit step signal.

The value at $t = 0$ is taken to be finite and in most cases it is unspecified. The discrete-time unit-step signal is defined as

$$u(n) = \begin{cases} 0 & , \ n < 0 \\ 1 & , \ n \geq 0 \end{cases}.$$

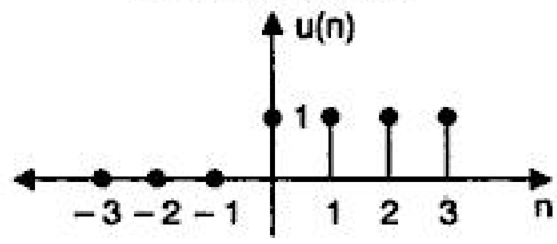



Fig. 1.10 (*b*). Discrete time unit step signal.

### 1.7.3 Unit-Ramp Function

The unit-ramp function, $r(t)$ can be obtained by integrating the unit-impulse function twice or integrating the unit-step function once,

*i.e.,*
$$r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \delta(\tau)\, d\tau\, d\alpha.$$

$$= \int_{-\infty}^{t} u(\alpha)\, d\alpha.$$

$$\boxed{r(t) = \int_{0^+}^{t} 1\, d\alpha}$$

That is,
$$r(t) = \begin{cases} 0 & , \ t < 0 \\ t & , \ t > 0 \end{cases}.$$

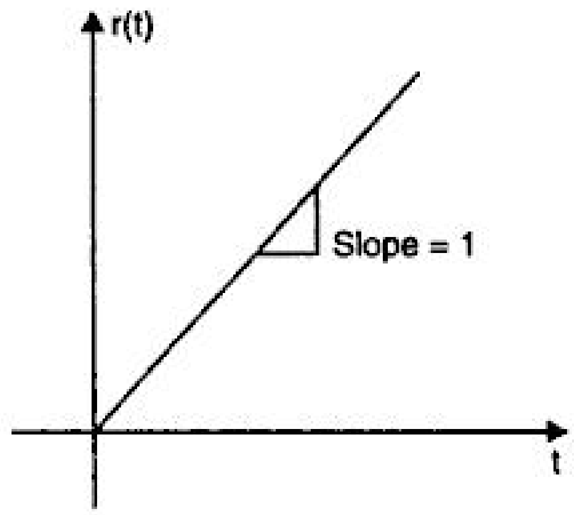



Fig. 1.11 (*a*). Continuous time ramp signal.

A ramp signal starts at $t = 0$ and increases linearly with time '$t$'.

In discrete-time domain, the unit-ramp signal is defined as,

$$r(n) = \begin{cases} 0 & , \ n < 0 \\ n & , \ n > 0 \end{cases}.$$

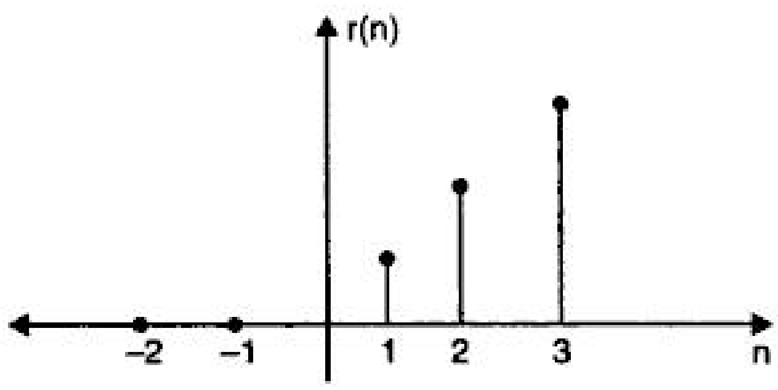



Fig. 1.11 (*b*). Discrete time ramp signal.

### 1.7.4 Unit-Pulse Function

An unit-pulse function, $\pi(t)$, is obtained from unit-step signal as shown below.

$$\pi(t) = u(t + 1/2) - u(t - 1/2)$$

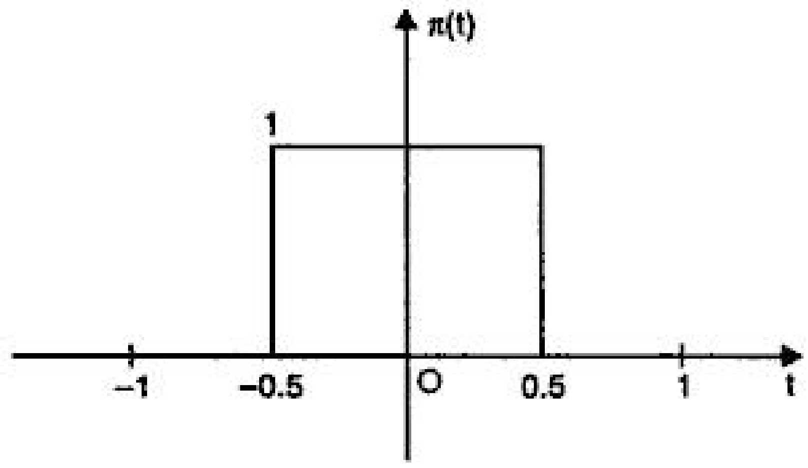



Fig. 1.12. Unit pulse signal.

The signal $u(t + 1/2)$ and $u(t - 1/2)$ are the unit-step signals shifted by 1/2 units in the time axis towards the left and right respectively.

**Advantage.** The advantage of the singularity function is that any arbitrary signal that is made up of straight line segments can be represented in terms of step and ramp functions.

**Properties of δ(t)**

(1) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

(2) $\int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0)$

**Proof for (2) :** $\int_{-\infty}^{\infty} x(t) \lim_{\Delta \to 0} \delta_\Delta(t)\, dt$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{-\infty}^{\infty} x(t)\, P_T(t) dt$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{-\infty}^{\infty} x(t)\, dt = \mathbf{x(0)}.$$

$\because \quad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} P_T(t) = \frac{1}{\Delta} \lim_{\Delta \to 0} P_T(t)$$

According to pulse function property,

$$P_T(t) = 1$$

(3) $\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$

(4) $\int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$

(5) $\delta(at) = \frac{1}{|a|} \delta(t)$

(6) $x(t)\, \delta(t - t_0) = x(t_0)$

(7) $x(t_0)\, \delta(t - t_0) = x(t_0)$

(8) $\int_{t_1}^{t_2} x(t) \delta^n (t - t_0)\, dt = (-1)^n x^n(t_0)$.

**Proof for (8):** $\frac{d}{dt}[x(t)\, \delta(t - t_0)] = x(t)\, \dot{\delta}\, (t - t_0) + \dot{x}\,(t)\, \delta(t - t_0)$

$$= x(t)\dot{\delta}(t - t_0) + \dot{x}(t_0)\, \delta(t - t_0),\ t_1 < t_0 < t_2$$

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**Sol.** The average power of the unit-step signal is,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \boxed{u^2(n)}$$

$$P = \lim_{N \to \infty} \frac{1+N}{2N+1} = \lim_{N \to \infty} \frac{1/N+1}{2+(1/N)} \qquad \boxed{P = 1/2.}$$

**Problem 2.** *Determine which of the following signals are periodic.*

(*a*) $x_1(t) = \sin 15\,\pi t$ (*b*) $x_2(t) = \sin 20\,\pi t$

(*c*) $x_3(t) = \sin \sqrt{2}\,\pi t$ (*d*) $x_4(t) = \sin 5\pi\, t$

(*e*) $x_5(t) = x_1(t) + x_2(t)$ (*f*) $x_6(t) = x_2(t) + x_4(t)$.

**Sol.** (*a*) $x_1(t) = \sin 15\pi t$ is periodic,

The fundamental period is,

$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = 0.1333 \text{ sec.}$$

(*b*) $x_2(t) = \sin 20\,\pi t$ is periodic,

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{20\pi} = 0.1$ sec.

(*c*) $x_3(t) = \sin \sqrt{2}\pi t$ is period,

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{2}\pi} = 1.41421$ sec.

(*d*) $x_4(t) = \sin 5\pi t$ is periodic

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5\pi} = 0.4$ sec.

(*e*) $x_5(t) = x_1(t) + x_2(t)$.

The fundamental period of $x_1(t) = T_{01} = 0.133$ sec. and the fundamental period of $x_2(t) = T_{02} = 0.1$ sec.

The ratio of fundamental frequencies,

$$\frac{T_{01}}{T_{02}} = \frac{0.1333}{0.1}, \text{ cannot be expressed as a ratio of integers.}$$

Hence, $x_5(t)$ is not periodic.

(*f*) $x_6(t) = x_2(t) + x_4(t)$.

The fundamental period of $x_2(t) = T_{02} = 0.1$ sec and the fundamental period of $x_4(t) = T_{04} = 0.4$ sec.

The ratio of fundamental frequencies,

$\dfrac{T_{02}}{T_{04}} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$, can be expressed as ratio of integers. Hence, $x_6(t)$ is periodic.

**Problem 3.** *Sketch of the following signal :*

(*a*) $x(t) = \pi(2t + 3)$ (*b*) $x(t) = 2\pi(t - 1/4)$

(*c*) $x(t) = \cos(20\,\pi t - 5\pi)$ and (*d*) $x(t) = r(-0.5t + 2)$.

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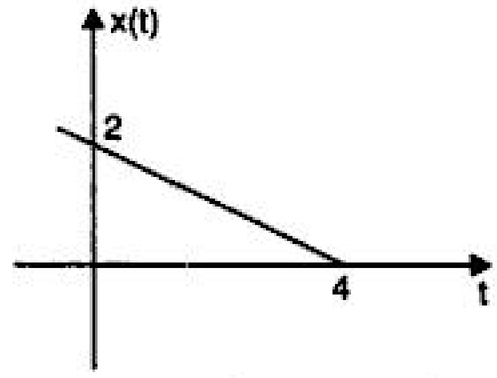
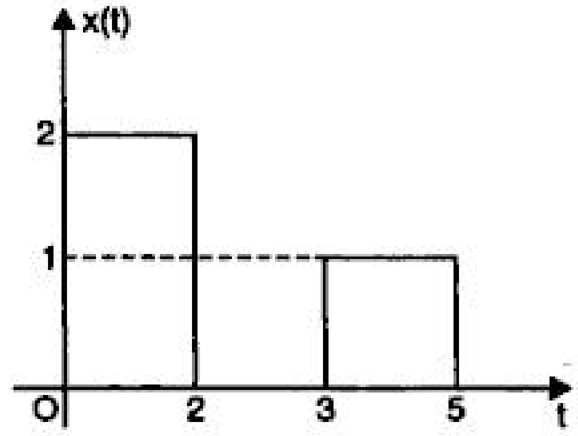
The signal is expanded by $\frac{1}{0.5} = 2$. When $t = 0$, the magnitude of the signal $x(t) = 2$.




**Problem 4.** *Write down the corresponding equation for the given signal,*




**Sol.** Representation through addition of two unit step functions, the signal $x(t)$ can be obtained by adding both the pulses, *i.e.*,

$$x(t) = 2[u(t) - u(t-2)] + [u(t-3) - u(t-5)]$$

Representation through multiplication of two unit step functions,

$$x(t) = 2[u(t)\,u(-t+2)] + [u(t-3)\,u(-t+5)]$$
$$= 2[u(t)\,u(2-t)] + [u(t-3)\,u(5-t)]$$

**Problem 5.** *Plot the following signals for the given $x(n) = (5-n)\,[u(n) - u(n-5)]$*

(*i*) $y_1(n) = x(4-n)$ (*ii*) $y_2(n) = (2n-3)$.

**Sol.** The given $x(n) = (5-n)\,[u(n) - u(n-5)]$ is plotted as shown below,

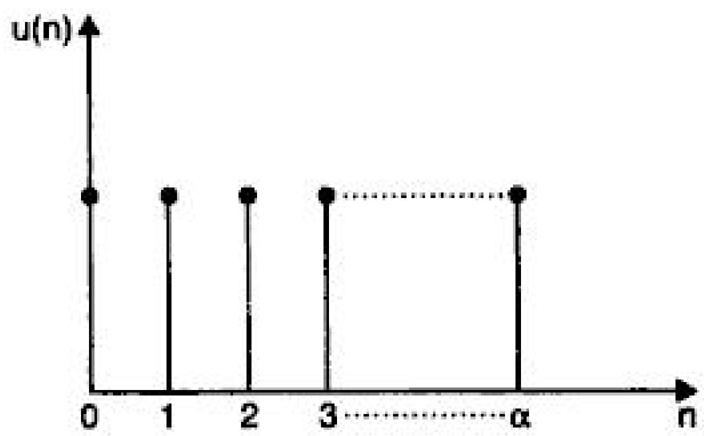



Fig. (*a*)

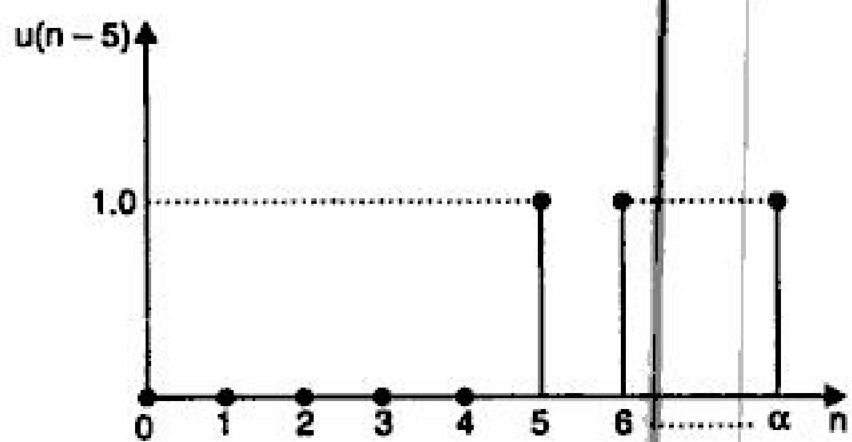



Fig. (*b*)

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## 1.10 ANALOG VERSUS DIGITAL SIGNAL PROCESSING

**Advantages :**

(1) **Flexibility.** Digital signal processing operations are flexible as the operations can be changed by changing the program.

(2) **Tolerance.** Unlike analog circuits the operation of the digital circuits does not depend on precise values of the digital signals. As a result, the digital circuits are less sensitive to tolerance component values.

(3) **Component drift with temperature and time.** Digital systems are fairly independent of temperature, aging (time), and most other external parameters, For example, due to change in temperature, the internal resistance R may change in analog systems. On the other hand, digital systems use logic 1 or logic 0 which are independent of temperature.

(4) **System Size.** Analog systems normally use L, C and R, therefore size of hardware is large as compared to digital system.

(5) **Storage.** Digital signals are easily stored on magnetic media (*e.g.* tape and disc) without deterioration or loss of signal fidelity, therefore the signal becomes transportable and can be processed off-line in a remote laboratory. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

(6) **Implementation.** It is very difficult to perform precise mathematical operations on signal in analog form but these same operations can be routinely implemented on the digital computer using hardware.

(7) **Cost.** Digital signal processing allows the sharing of a given processor among a number of signals by time sharing. Thus reducing the cost of processing per signal. This is done by "time-division multiplexing".

**Disadvantages :**

(1) **System Complexity.** Digital signal processing of analog signals is more complex because of the need for additional pre-and post processing devices such as A/D and D/A converters and their associated filters.

(2) **Band Width.** The second disadvantage associated with digital signal processing is the limited range of frequencies available for processing. This property limits its application particularly in the DSP of analog signals. The signals having extremely wide bandwidth require fast sampling rate A/D coverters. Hence, there are many analog signals with large bandwidth for which the digital signal processing approach is beyond the state of the art of the digital hardware.

(3) **Power.** The another disadvantage of DSP is that signal systems are constructed using active devices (transistor) that consumes power. On the other hand, a variety of analog processing algorithm can be implemented using passive circuits employing inductor, capacitor and resistor that do not need any power. Also active devices are less reliable than passive devices.

## REVIEW QUESTIONS

1. Write the major classification of signals.
2. Explain the difference between deterministic signal and random signal with suitable example.
3. Define periodic and aperiodic signals with the help of examples.
4. Explain even and odd signals with the help of examples.
5. Explain energy and power signal with the help of examples.
6. Define the following elementary signals
   (1) unit impulse signal. (2) unit step signal.
   (3) unit ramp signal.
7. Explain the following manipulations for independent variable of a signal
   (1) Time shifting (2) Time scaling
   (3) Time inversion or folding.
8. What are the advantages of Digital signal processing compared to Analog signal processing.
9. Briefly explain multichannel and multidimensional signals.
10. Define continuous time exponential and discrete time exponential signal.
11. Write the properties of impulse response signals.

## EXERCISES

1. Determine which of the following signals are periodic and determine the fundamental period also.
   (1) $x(t) = 20 \sin 25\,\pi t$ (2) $x(t) = 20 \sin \sqrt{5}\,\pi t$
   (3) $x(t) = 10 \cos 10\,\pi t$ (4) $x(t) = 3 \cos (5\,t + \pi/6)$
   (5) $x(n) = 3 \cos (5n + \pi/6)$ (6) $x(n) = 2 \exp (j\,(n/6 - \pi))$
   (7) $x(n) = \cos (n/8) \cos (\pi n/8)$.
2. Determine the even and odd components of each of the following signals :
   (1) $x(t) = \cos t + \sin t + \sin t \cos t$ (2) $x(t) = 1 + t + 2t^2 + 5t^3 + 8t^4$.
3. Consider the sinusoidal signal
   $$x(t) = A \cos (\omega t + \theta)$$
   Determine the average power of $x(t)$.
4. Sketch the waveforms of following signals :
   (1) $x(t) = u(t) - u(t-2)$ (2) $x(t) = u(t+1) - 2u(t) + u(t-1)$
   (3) $x(t) = r(t+1) - r(t) + r(t-2)$
5. Given a sinusoidal signal
   $$x(n) = 20 \cos \left[\frac{4\pi}{31} n + \frac{\pi}{5}\right]$$
   Determine the fundamental period of $x(n)$.
6. Given a complex valued exponential signal
   $$x(t) = Ae^{\alpha t + j\omega t} \text{ for } \alpha > 0$$
   Evaluate the real and imaginary components of $x(t)$.

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# 2 Applications of Digital Signal Processing

## 2.1 INTRODUCTION

Because of the availability of high resolution spectral analysis, DSP has various application areas, which requires high speed processors to implement the FFT algorithm. It is also popular due to availability of custom made DSP chip which is highly reliable. Speech processing, Audio processing, Radar signal processing and Image processing would be discussed in this chapter.

## 2.2 APPLICATION TO SPEECH PROCESSING

The signals of speech are one dimensional. DSP is applied to a wide range of problem in speech such as channel vocoders, spectrum analysis etc.

Problems in speech processing can generally be divided into three classes, first is the speech analysis. The speech analysis is performed to extract some desirable information of speech. This system starts with analysis of speech waveform and the desired result is used for speech recognization and speaker indentification. Second type of problem is speech synthesis. In it, input is in written text form and the output is a speech signal. For example, an automatic reading machine for which the input is written text and the output is speech. Finally the third type is speech compression which involves speech analysis followed by speech synthesis. If the speech is transmitted by simply sampling and digitizing, the data rate required is in the order of 90,000 bits per second of speech. Through the use of appropriate coding this can be reduced by factor of 50, depending on the type of system used.

### 2.2.1 Vocal Mechanism

**Production of speech.** The two important part responsible for human speech are (*a*) vocal cord and (*b*) vocal tract.

(*a*) *Vocal cord.* It has two bands of tough, elastic tissue, which is located at the opening of the larynx. It vibrates when the air from the lungs passes between them producing sound waves which are emitted from the lips and to some extent from the nose ; these are sound waves heard as speech.

(*b*) *Vocal tract.* It includes larynx, the pharnx and the nasal cavity.

**Kinds of Sounds**

(*i*) Voiced sound (*ii*) Unvoiced (fricative) sound.

Voiced sounds are produced by quasi-periodic pulses of air exciting the vocal tract. Unvoiced sounds are produced at some point along the vocal tract, usually towards the mouth.

There are some important speech technology areas. *viz.*, speech coding, speech enhancement, speech analysis and synthesis, speech recognition and speaker recognition.

### 2.2.2 Speech Technology

(*a*) **Speech coding.** "Speech Coding" is the process of capturing the speech of a person and processing it to transmit over a communication channel.

The application of "speech coding" is in the area of telephony, narrow-band cellular radio, military communication etc.

(*b*) **Speech enhancement.** This is the process of minimizing the derogatory effects of noise on the performance of speech communication, source coding etc.

The application of 'speech enhancement' is in the areas where the performance of equipment is improved in noisy atmosphere like factories etc.

(*c*) **Speech analysis and synthesis.** Analysing speech is done by studying its spectrum and extracting time-varying parameters from the signal for the production of speech.

Synthesizing speech lies in creating speech like waveforms from textual words or symbols, using a model for speech production and time-varying parameters.

The application of this are in voice alarms, reading machines for the dumb or blind, data-base enquiry services etc.

(*d*) **Speech recognition.** The process of deriving the meaning from a speech input whereby a request can be made for information or service from a machinery by conversing with it.

Application of "speech recognition" could be Banking from distant location, information retrieval systems etc.

(*e*) **Speaker recognition.** It means to recognize a particular person's identity with the sample speech dipping.

### 2.2.3 Parameters of Speech

(*i*) *Pitch :* Corresponds to frequency of sound (in Hz).

(*ii*) *Loudness :* This relates to intensity of sound (in dB).

(*iii*) *Quality :* This relates to harmonic constant of sound (in timbre).

'Phonemes' are the smallest unit of sound that are recognized by contrast with their environment, these are forming the basic units of speech. 'Dipones' are sounds that stretch from the middle of one phoneme to the centre of the next, there by spanning the transition region.

### 2.2.4 Speech Analysis

The most common methods of speech analysis are as follows :

(*a*) *Short-time fourier analysis*

(*b*) *Linear prediction.*

(*c*) *Homonorphic filtering.*

Let us discuss about these three methods of speech analysis.

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additive recombination of the set of sub-band signals, the original speech signal can be generated. Each band is separately quantized and coded using pulse code modulation and transmitted. The schematic is shown in Fig. 2.5.

## 2.3 APPLICATION TO IMAGE PROCESSING

Any function which bears two-dimensional information is called an image. Image can be represented by an array of real or complex (real and imaginary) numbers with finite number of bits with respect to speech signal (which are one-dimensional signals), image signals are two dimensional. Image can be divided into picture elements or pixels (smallest element of image). Manipulation of two-dimensional signal with the help of digital computer is called "Image Processing". Its purpose is to improve the visual appearance of Image.

A Digital Image is digitalization of picture. Normally two-dimensional image has resolution 128 × 128, 256 × 256, 512 × 512. So image can be processed using two-dimensional signal processing. The image processing including the following steps :

(*a*) Image Formation and Recording.

(*b*) Image Sampling and Quantization.

(*c*) Image Compression.

(*d*) Image Restoration.

(*e*) Image Enhancement.

All the operations are possible on advanced software artificial intelligence and high-tech digital computers. Let us discuss all the operations one by one.

### 2.3.1 Image Formation and Recording

The two-dimensional signal of image can be expressed by image function as

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - x_1, y - y_1)\, f(x_1, y_1)\, dx_1\, dy_1 \qquad \ldots(2.15)$$

Eqn. (2.15) governs a 2D linear time invariant system. Here system impulse response function $h(x - x_1, y - y_1)$ is commonly referred as point-spread function which is usually associated with optical image. The function $f(x, y)$ is the accumulation of energy from the objects radiant energy distribution.

Two major technologies are used for image sensing and recording, which are photo-chemical recording and photo-electronic recording. Both of the technologies are exemplified by readily available products which are photo-graphic films and television respectively. (Here "television" is used in generic sense not commercial broad casting television).

### 2.3.2 Image Sampling and Quantization

After formation and recording of an image, it is sampled and quantized for the suitability of digital processing.

In a system project, a spot of light with intensity $I_1$ incident on a film and intensity $I_2$ reflected from the film and collected by photo-multiplier. The transmittance is defined by

$$T = \frac{I_2}{I_1} \qquad \ldots(2.16)$$

This eqn. (2.16) can be used to compute optical density. The mathematical model can also be described as a spot of light, moves in a raster to sample the film is given by

$$g_1(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_a(x - x_1, y - y_1)\, g(x_1, y_1)\, dx_1\, dy_1 \qquad \ldots(2.17)$$

Here $h_a$ is the intensity profile of the spot of light projected an film. $g$ is the image on film and finally $g_1$ is actual sampled image. The sample matrix $g_1(k\,\Delta x, l\,\Delta y)$ is the sampled or digital image.

### 2.3.3 Image Compression

In a digital image $10^5$ to $10^6$ data are there. The processing of these higher value of image data is a very stupendous task. But a digital image has large number of redundancy which can be reduced by image compression. So we can say that image compression is a science of efficiently coding a digital image to reduce the number of bits, which required to represent it.

Uncompressed image consumes memory space in a large amount so it increases complexity in computational and need a very large transmission bandwidth. A compressed image reduces the redundancy in image. There are mainly three types of redundancy which are discussed one by one.

The first type redundancy is Spatial Redundancy which arises due to correlation between neighbouring pixels. Second type is Spectral Redundancy which is correlation between various colour plans. And finally is Temporal Redundancy is the correlation between different frames in an image sequence.

In an Image Compression System, the original continuous time image signal is fed to A/D (Analog to Digital) converter, which converts it into digital signal. Now a serial to parallel (S/P) converter decomposed signal into parallel channels which fed to a quantizer. The S/P converter is linear transformer or filter banks are used. This quantized output is coded by the use of lossless coding device, whose output is compressed digital image signal. A simple block diagram of image compression system is shown in Fig. 2.6.

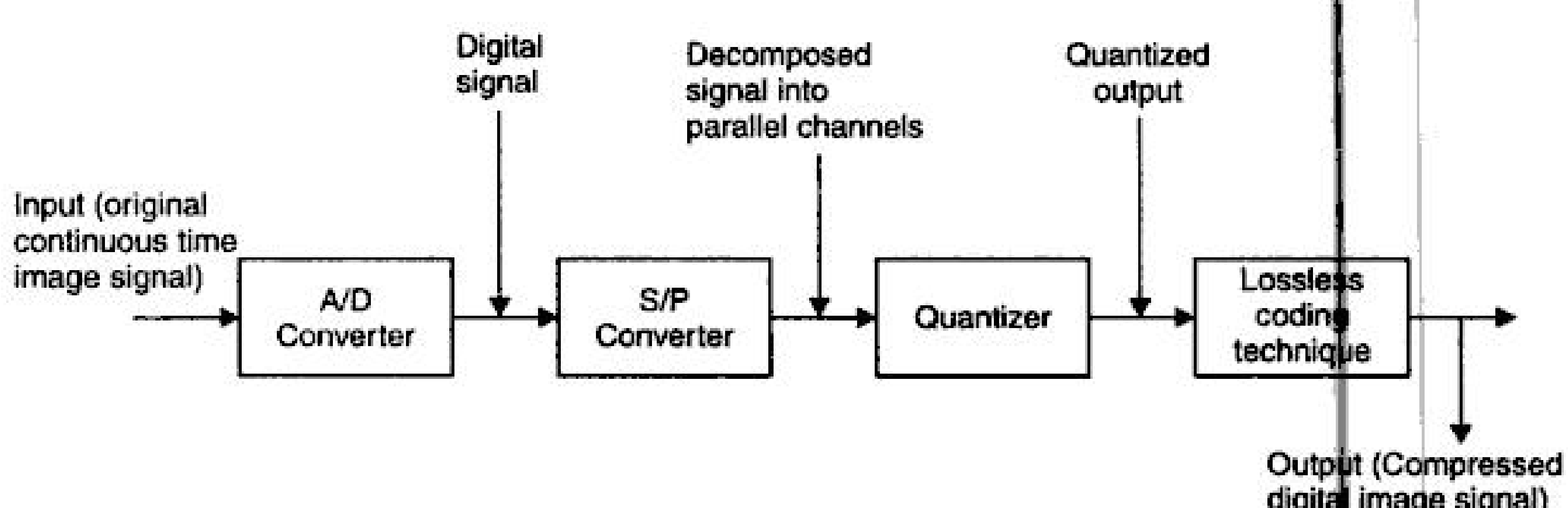



Fig. 2.6. Image compression system.

**Applications of digital signal processing image compression system**

There are mainly three types of compression technique based on the method of redundancy detection :

(*a*) Direct data compression method

(*b*) Transformation method

(*c*) Parametric extraction method.

### 2.3.4 Image Restoration

The process of image restoration is used for correcting imaging effect to recover an original signal. This type of effect (imaging effect) is due to variety of intermixing factors, which are defocusing imaging camera, relative motion between object and camera, noise in sensors etc., All types of imaging effects deteriorate image quality.

The process of image restoration is to attempt a image which should be sharp, clean and free from the degradation. The restoration process is also called Image Deblurring. The process of image formation and recording can be modelled as

$$g(x, y) = \mathrm{R}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - x_1, y - y_1)\, f(x_1, y_1)\, dx_1\, dy_1\right] + n(x, y) \qquad \text{...(2.18)}$$

Here $g(x, y)$ is the actual image, R is the response characteristic of the recording process and $n(x, y)$ is additive noise source.

In the restoration of digital image following equation can be expressed in discrete form :

$$g(p, q) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(i, j) h(p - i, q - j) \qquad \text{...(2.19)}$$

A large set of simultaneous linear equations can be solved by DSP techniques such as linear filters and FFT algorithms which are computationally efficient tools for solving these.

### 2.3.5 Image Enhancement

This technique improves the appearance of image for human perception by choosing some image features like edges or contrast etc. Its main application is in biomedical engineering field for computer aided mammographics studies.

In image enhancement spatial filtering is mainly used whose operation is done on image to reduce noise contamination of the image signal. Image enhancement is composed of a variety of methods whose suitability depends upon the goals at hand when enhancement is originally applied.

## REVIEW QUESTIONS

1. Give the areas in which signal processing find its application.
2. Explain the various stages in voice processing.
3. How is a speech signal generated ?
4. Give the model of speech production system ?
5. What is the need for short time spectral analysis ?
6. What is a vocoder ? Explain with a block diagram ?
7. Describe how targets can be detected using radar.
8. Give an expression for the following parameters related to radar

   (*a*) beam width, and

   (*b*) maximum unambiguous range.

9. Explain with the block diagram the modern radar system.
10. Give the various image processing applications.
11. Give the various coding techniques for images.
12. What is the need for image compression ?
13. Give the block diagram of basic restoration process.
14. What is sub-band coding ?
15. Explain the process of digital FM stereo signal generation.
16. Explain how privacy can be acheieved in telephone communications.

DIGITAL SIGNAL PROCESSING

# Unit 2

Chapters :

# 3 Discrete Time Systems

## 3.1 DISCRETE-TIME SIGNALS AND SYSTEMS

### 3.1.1 Definition

1. A discrete-time signal is a sequence, that is a function defined on the positive and negative integers.
2. A discrete-time system is a mapping from the set of acceptable discrete-time signals called the input set, to a set of discrete-time signals called output set.
3. A discrete-time signal whose values are from a finite set is called a digital signal.
4. A digital system is a mapping which assigns a digital output signal to every acceptable digital input signal.

### 3.1.2 Representations

1. **Graphical.** In digital signal processing, signals are represented as sequence of numbers called samples. A sampled value of typical discrete-time signal or sequence is denoted by $x(n)$ which is a function of independent variable that is an integer. It is graphically represented in Fig. 3.1.

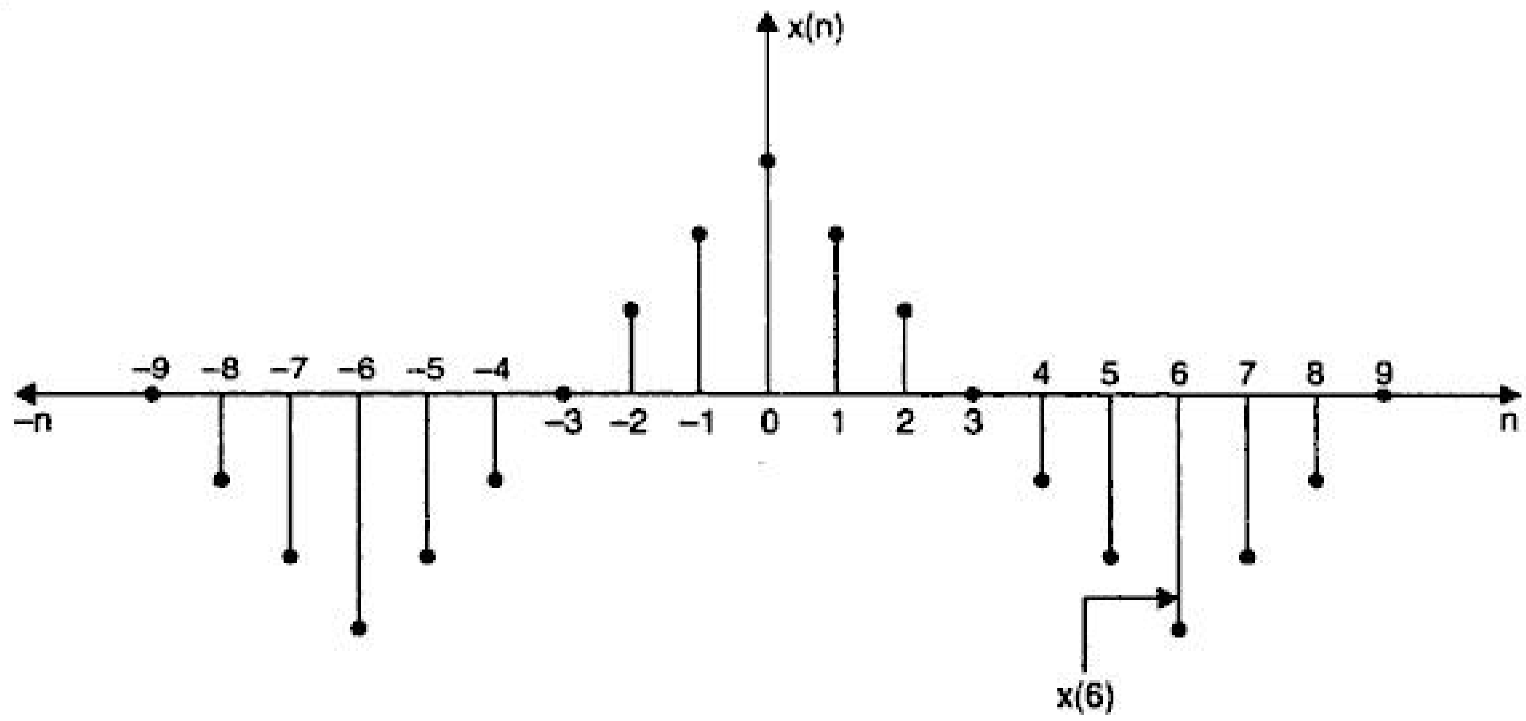



Fig. 3.1. Graphical representation.

It is important to note that $x(n)$ is defined only for integer values of $n$ and undefined for non-integer values of $n$.

In the signal we have assumed that a discrete-time sequence is defined for every integer value of $n$ for $-\infty < n < \infty$. The particular value of sequence at time $k$ is simply denoted by $x(k)$. Here, we primarily concerned with sequence at equally spaced intervals.

2. **Alternative representation.** The discrete-time sequence may be represented in number of ways. Some of the alternative representations that are often more convenient to use. These are,

(*i*) **Functional representation** such as,

$$x(n) = \begin{cases} 2 & \text{for} \quad n = 1, 3, 5 \\ 1 & \text{for} \quad n = -1, -2, 4, 7 \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

(*ii*) **Tabular representation**, such as

| $n$ | ....... | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | ...... |
|---|---|---|---|---|---|---|---|---|
| $x(n)$ | | $0$ | $1$ | $4$ | $3$ | $-5$ | $-1$ | |

(*iii*) **Sequence representation.** Representation based on length the discrete time signal may be finite length or infinite length sequence. This finite length also called finite duration sequence is only defined only for a finite time interval.

The finite duration sequence can be represented,

$$x(n) = \{-2, -1, 3, \underset{\uparrow}{1}, 0, 4, 2\}$$

where, the time origin (sign origin, $n = 0$) is indicated by symbol $\uparrow$.

An finite duration sequence can be represented as

$$x(n) = \{...... 0, 0, 1, -1, 2, 4, \underset{\uparrow}{1}, 0, 0, 1\}$$

A sequence which is zero for $n < 0$ can be represented as

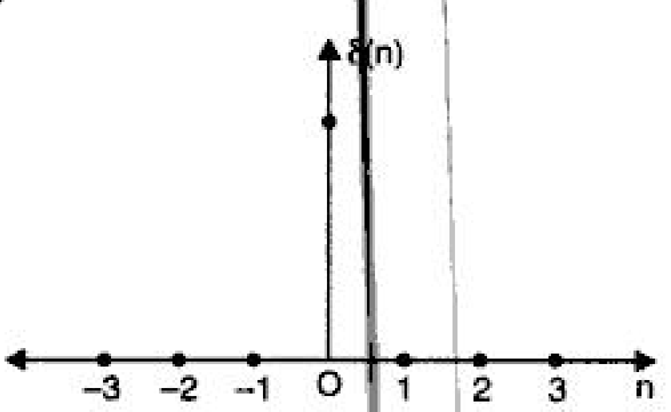
$$x(n) = \{\underset{\uparrow}{1}, 2, 0, 1, 4, 2, 0, 0\}.$$

### 3.1.3 Some Elementary Sequence

Any signal can be represented in terms of some basic sequences. Some of the basic sequences are defined below :

1. **Unit sample sequence (unit impulse sequence).** The unit sample sequence contains only one non-zero valued element and it is defined as,

$$\delta(n) = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

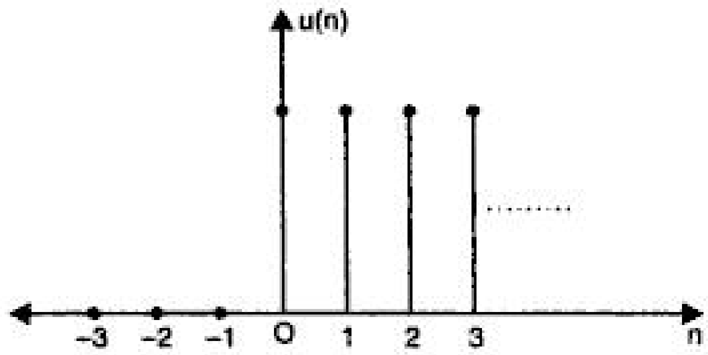



The delayed unit sampled sequence denoted by $\delta(n - k)$, has its non-zero value at sample time $k$, *i.e.*,

$$\delta(n - k) = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases}$$

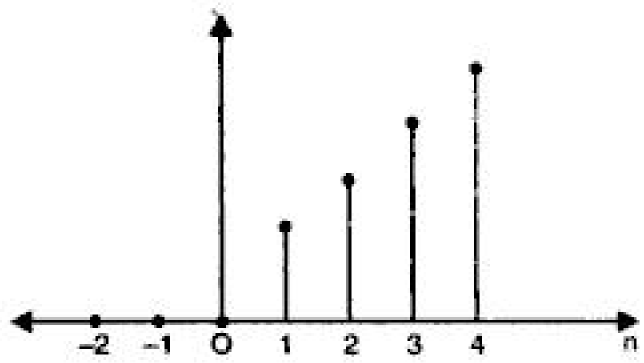
2. **Unit step sequence.** It is defined as,

$$u(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$




3. **Unit ramp sequence.**

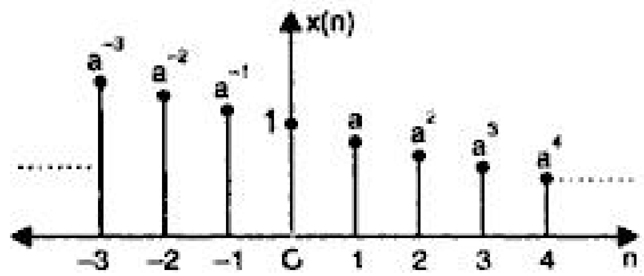
$$u_r(n) = \begin{cases} n & \text{for} \quad n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$




($a$ is real)

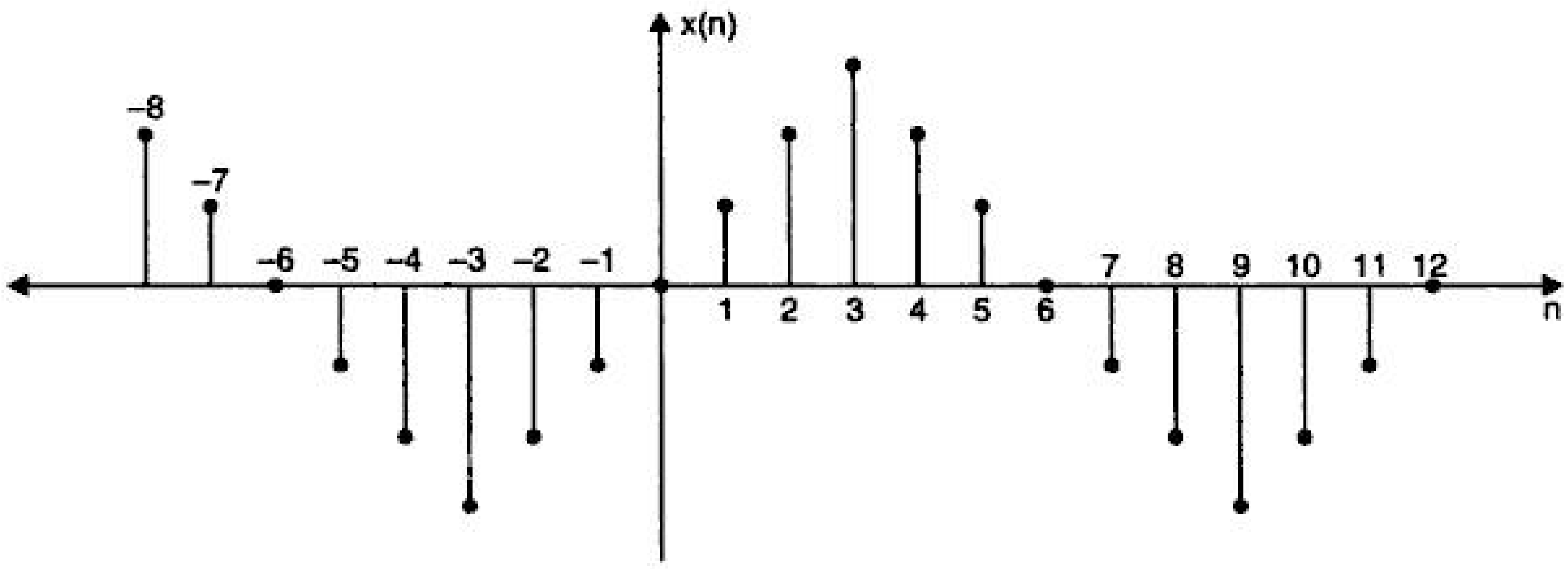
4. **Real exponential sequence.**

$x(n) = a^n$ for all '$n$' $0 < a < 1$.




5. **Sinusoidal sequence.**

$x(n) = \text{A} \sin \omega_0 n.$ for all '$n$'.




### 3.1.4 Representation of Arbitrary Sequence

An arbitrary sequence can be represented as a sum of scaled, delayed unit sample.

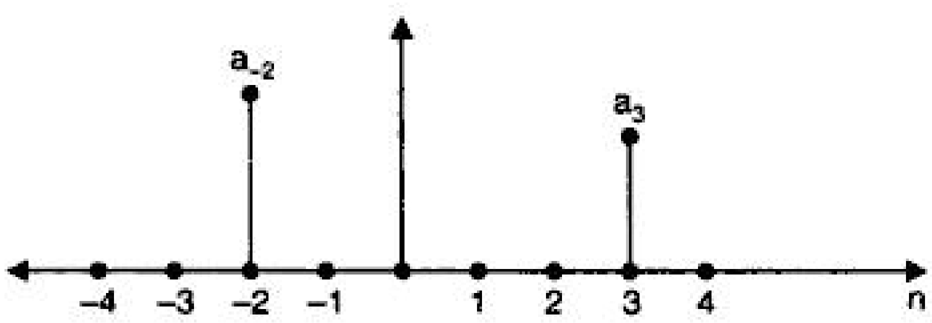



Fig. 3.2. Arbitrary sequence.

*e.g.* $x(n) = a_{-2}\,\delta(n+2) + a_3\,\delta(n-3)$

In general,

An arbitrary sequence can be expressed as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \ldots(3.1)$$

where, $x(k)$ represents the magnitude of the $k^{th}$ member of the sequence $x(n)$.

## 3.2 CLASSIFICATION OF DISCRETE-TIME SIGNAL

1. **Energy signals and power signals.** The energy signal $x(n)$ is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

We defined the signal energy of $x(n)$ over the finite interval $-N \leq n$

$$E_N = \sum_{n=-N}^{N} |x(n)|^2 \qquad \ldots(3.2)$$

We can express the signal energy E as,

$$E = \lim_{N\to\infty} E_N \qquad \ldots(3.3)$$

and the average power signal $x(n)$ as,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} E_N \qquad \ldots(3.4)$$

If $E_N$ is finite, $P = 0$

If E is infinite, the power P may be either finite or infinite.

2. **Periodic and aperiodic signals.** A signal $x(n)$ is periodic with period N if and only if,

$$x(n+N) = x(n) \quad \text{for all 'n'}. \qquad \ldots(3.5)$$

The smallest value of N, for which above condition holds is called (fundamental) period. If there is no value of N that satisfies this condition is called non-periodic or aperiodic.

*e.g.* The sinusoidal signal of form,

$$x(n) = A\cos\omega_0 n$$
$$x(n) = A\cos 2\pi f_0 n.$$

is periodic when it is a rational no. *i.e.*, $f_0$ can be expressed as,

$$f_0 = \frac{k}{N}.$$

where, $k$ and N are integer and relatively prime.

The energy of periodic signals over a single period $(0 \leq n \leq N-1)$ is finite if $x(n)$ takes on finite values over the period. However, energy of the periodic signal for $-\infty \leq n \leq \infty$ is infinite. On the other hand, the average power of periodic signal is finite.

If $x(n)$ is periodic with fundamental period N and its magnitude is finite, then its power is given by,

$$P_{av} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \qquad \ldots(3.6)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |A \sin 2\pi f_0 n|^2 = \lim_{N \to \infty} \frac{A}{N} \sum_{n=0}^{N-1} (1)^2$$

$$= \lim_{N \to \infty} \frac{A}{N} . N \qquad \boxed{P_{av} = A}$$

Therefore, periodic signals are power signals.

3. **Symmetric (even) and antisymmetric (odd) signals**

A real valued signal $x(n)$ is called symmetric

if $$x(n) = x(-n),$$

on the other hand, a signal $x(n)$ is called antisymmetric

if $$x(-n) = -x(n).$$

Even signal, $$X_{cs}(n) = X_e(n) = \frac{1}{2}[X(n) + X(-n)] \qquad \text{...(3.7)}$$

Odd signal, $$X_{ca}(n) = X_0(n) = \frac{1}{2}[X(n) - X(-n)] \qquad \text{...(3.8)}$$

$X(n)$ is defined as, $$\boxed{X(n) = X_e(n) + X_0(n)} \qquad \text{...(3.9)}$$

**Problem 1.** *Consider a sequence defined by,*

$$x(n) = \begin{cases} 4(-1)^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

*determine the power and energy of x(n).*

**Sol.** $$E = \sum_{n=0}^{\infty} |4(-1)^n|^2$$

$$= \sum_{n=0}^{\infty} |(4)^2(1)^n| = 16 \sum_{n=0}^{\infty} 1 = 16\left[\frac{1}{1-1}\right]$$

$$\boxed{E = \infty}$$

Power, $$P_{av} = \lim_{N \to \infty} \frac{16}{N+1} \sum_{n=0}^{\infty} (1)^n = \lim_{N \to \alpha} \frac{16(N+1)}{(N+1)}$$

$\boxed{P_{av} = 16}$ which is finite.

**Problem 2.** *Determine the response of the following systems to the input*

$$X(n) = \{\ldots\ldots, 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0 \ldots\ldots\}$$

*(a)* $y(n) = X(n+1)$ *(b)* $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

*(c)* $y(n) = \sum_{k=-\infty}^{n} X(k).$

**Sol.** (*a*) To find $y(n) = X(n+1)$

In this case the system 'advances' the input one sample into the future *i.e.*, at $n = 0$, $y(0) = x(1)$.

The response of this system to the given input is,

$$y(n) = \{........ 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0 ..........\}.$$

(*b*) To find $y(n) = 1/3\ [x(n+1) + x(n) + x(n-1)]$.

The output of the system at any time is the mean value of the present, the immediate past and the immediate future sample.

For example, the output at time $n = 0$ is,

$$y(0) = \frac{1}{3}\ [x(-1) + x(0) + x(1)]$$

$$= \frac{1}{3}\ [1 + 0 + 1]$$

$$\boxed{y(0) = 2/3}$$

Repeating this computation for every value of $n$, we obtain,

$$y(n) = \{\ ........ 0, 1, 5/3, 2, 1, \underset{\uparrow}{2/3}, 1, 2, 5/3, 1, 0......\}.$$

(*c*) $y(n) = \sum_{k=-\infty}^{n} x(k)$

This system is basically an accumulator that computes the running sum of all the past input values upto present time. The response of the system to the given input is,

$$y(n) = \{...... 0, 3, 5, 6, 6, 7, 9, 12, 12, ........\}.$$

## 3.3 SAMPLING

In some applications a discrete-time sequence $x(n)$ is generated by periodically sampling a continuous time signal $x_a(t)$ at uniform time intervals.

$$x(n) = x_a(t)\,|_{t=nT} = x_a(nT)\ ;\ n = .......\ -2, -1, 0, 1, 2\ ....... \quad ...(3.10)$$

The spacing T between two consecutive samples in eqn. (3.10) is called the sampling interval or sampling period.

The reciprocal of the sampling interval T, denoted as $F_T$ is called the sampling frequency.

$$F_T = \frac{1}{T}.\ \text{(cycle/sec or Hertz)}.$$

## 3.4 REAL AND COMPLEX SEQUENCE

**Real sequence.** If $X(n)$ is real for all values of $n$, then $\{x(n)\}$ is a real sequence.

**Complex sequence.** If the $n^{th}$ sample value is complex for one or more values of $n$, then it is complex sequence. It can be defined as,

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The device implementing the delay operation by one sample is called a **"Unit delay"** and its schematic representation is shown in Fig. 3.8

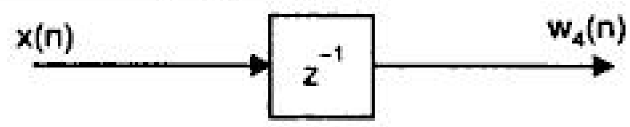


Fig. 3.8. Unit delay.

$$\boxed{w_4(n) = x[n-1]}$$

The schematic representation of the unit advance operation is shown in Fig. 3.9

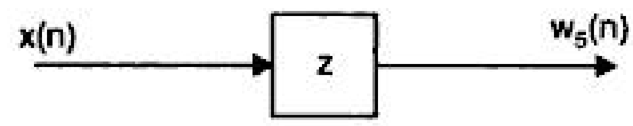


Fig. 3.9. Advance operation.

$$\boxed{w_5[n] = x[n+1]}$$

(*v*) *Time-reversed or Folding.* The time reversal operation, also called the folding operation, is another useful scheme to develop a new sequence.

$$\boxed{w_6(n) = x(-n)} \qquad \text{...(3.20)}$$

which is the time-reversed version of the sequence $x(n)$.

(*vi*) *Pick-off node.* It is used to provide multiple copies of a sequence.

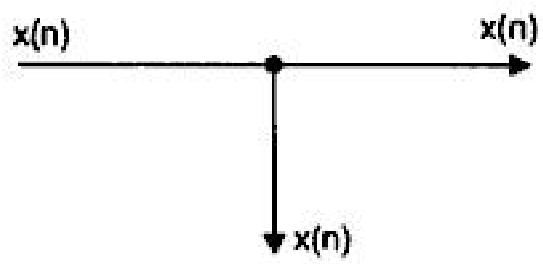


Fig. 3.10. Pick-off node.

**Problem 3.** *Consider the following two sequence of length 5 defined for $0 \le n \le 4$ :*

$$c(n) = \{3.2, 41, 36, -9.5, 0\}$$
$$d(n) = \{1.7, -0.5, 0, 0.8, 1\}.$$

*Determine $w_1(n)$, $w_2(n)$ and $w_3(n) = \frac{7}{2}c(n)$.*

**Sol.** (1) $w_1(n) = c(n) \,.\, d(n).$

$w_1(n) = \{5.44, -20.5, 0, -7.6, 0\}.$

(2) $w_2(n) = c(n) + d(n)$

$= \{4.9, 40.5, 36, -8.7, 1\}$

(3) $w_3(n) = \frac{7}{2}c(n)$

$w_3(n) = \{11.2, 143.5, 126, -33.25, 0\}.$

**Problem 4.** *Consider a sequence $[g(n)]$ of length 3 defined for $0 \le n \le 2$ given by, $[g(n)] = \{-21, 1.5, 3\}$, and a sequence of length 5 defined for $c(n) = \{3.2, 41, 36, -9.5, 0\}$ Find $c(n) \,.\, g(n)$ and $c(n) + g(n)$.*

**Sol.** We can develop another sequence $g(n)$ by operating on this sequence of length 5 and defined for $0 \le n \le 4$ by appending it with two zero-valued samples

$$g_e(n) = \{-21, 1.5, 3, 0, 0\}.$$

we can generate,

(1) $c(n) \,.\, g_e(n) = \{-67.2, 61.5, 108, 0, 0\}$

(2) $c(n) + g_e(n) = \{-17.8, 42.5, 39, -9.5, 0\}.$

**Problem 5.** *Find the $y(n)$ from the following sequence is given by,*

(*i*) $x(n) = \{1, 2, 1, -1\}$ *and* $a = 2$.

(*ii*) $x_1(n) = \{-1, 2, -3, -2\}$ *and* $x_2(n) = \{1, -1, -2, 1\}$

*find* $x_1(n) \,.\, x_2(n) = ?$

Whose time-reversed version is given by,

$$\{g^*(-n)\} = \{3, j2, -5+j6, \underset{\uparrow}{4}+j2, -2-j3, 1-j4, 0\}$$

$$g_{cs}(n) = \frac{1}{2}[x(n) + x^*(-n)].$$

$$\{g_{cs}(n)\} = \{1.5, 0.5+j3, -3.5+j4.5, \underset{\uparrow}{4}, -3.5-j4.5, 0.5-j3, 1.5\}$$

2. To determine conjugate anti-symmetric part $g_{ca}(n)$

$$\{g_{ca}(n)\} = \frac{1}{2}[x(n) - x^*(-n)].$$

$$\{g_{ca}(n)\} = \{-1.5, 0.5+j1, 1.5-j1.5, \underset{\uparrow}{-j2}, -1.5-j1.5, -0.5-j1, 1.5\}$$

It can be easily verified that,

$$g_{cs}(n) = g_{cs}^*(-n) \quad \text{and}$$

$$g_{ca}(n) = -g_{ca}^*(-n).$$

### 3.9.1 Periodic Conjugate-Symmetric Part and Periodic Conjugate Anti-symmetric Part

Periodic conjugate-symmetric part defined by,

$$x_{pcs}(n) = \frac{1}{2}[x(n) + x^*[(-n)_N]]. \quad , 0 \le n \le N-1 \qquad ...(3.24)$$

Periodic conjugate anti-symmetric part is defined by,

$$x_{pca}(n) = \frac{1}{2}[x(n) - x^*[(-n)_N]], \quad 0 \le n \le N-1 \qquad ...(3.25)$$

So that $\qquad x(n) = x_{pcs}(n) + x_{pca}(n), \quad 0 \le n \le N-1 \qquad ...(3.26)$

**Note.** A length N sequence $x[n]$ defined for $0 \le n \le N-1$ is said to be periodic conjugate symmetric if $x(n) = x^*[(-n)_N] = x^*[N-n]$, and is said to be periodic conjugate-antisymmetric if $x(n) = -x^*[(-n)_N] = -x^*[N-n]$.

**Problem 7.** *Consider finite-length sequence of length 4 defined for $0 \le n \le 3$ :*

$$\{u(n)\} = \{1+j4, -2+j3, 4-j2, -5-j6\}.$$

*To determine its periodic symmetric part $u_{pcs}(n)$ and its periodic conjugate antisymmetric part $u_{pca}(n)$.*

**Sol. To find $u_{pcs}(n)$ :**

$$u_{pcs}(n) = \frac{1}{2}[u(n) + u^*[(-n)_N]]$$

$$\{u^*(n)\} = \{1-j4, -2-j3, 4+j2, -5+j6\}$$

To compute modulo-4 time-reversed version $\{u^*(< -4>)\}$.

We observe that,

$$u^*[(-0)_4] = u^*(0) = 1-j4.$$

$$u^*[(-1)_4] = u^*(3) = -5+j6.$$

$$u^*[(-2)_4] = u^*(2) = 4+j2$$

$$u^*[(-3)_4] = u^*(1) = -2-j3.$$

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where, $$\boxed{\omega_0 = \frac{2\pi\Omega_0}{\Omega_T} = \Omega_0 T} \qquad ...(3.30)$$

It is the normalised angular frequency of the discrete-time signal $x(n)$.

**Units**

The unit of the normalised **digital** angular frequency $\omega_0$ is radians per sample.

While, the unit of the normalised **analog** angular frequency $\Omega_0$ is radians per sample and the unit analog frequency $f_0$ is hertz is the unit of the sampling period T is in seconds.

**Problem 8.** *Consider the three sequence generated by uniformly sampling the three cosine functions of frequencies 3Hz, 7Hz and 13Hz respectively : $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$ with sampling rate of 10 Hz. i.e., with $T = 0.1$ sec. Find the derived sequence or discrete sequence.*

**Sol.** $$g(t) = \cos(\Omega_0 t);$$
$$g(n) = \cos(\Omega_0 nT)$$
$$g_1(n) = \cos(6\pi n \times T)$$
$$g_1(n) = \cos(0.6\pi n)$$

Similarly, $$g_2(n) = \cos(1.4\pi n)$$

and $$g_3(n) = \cos(2.6\pi n).$$

**Problem 9.** *Determine the discrete time signal $v(n)$ obtained by uniformly sampling at sampling rate of 200 Hz., a continuous time signal $v_a(t)$ composed of a weighted sum of five sinusoidal of frequencies 30 Hz, 150 Hz, 170 Hz, 250 Hz and 330 Hz as given below :*

*$v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t)$.*

**Sol.** To find the sampling period (T) :

$$T = \frac{1}{F} = \frac{1}{200} = 0.005 \text{ sec.}$$

The generated discrete-time signal $v(n)$ is given by,

$$\begin{aligned} v(n) &= 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n) \\ &\qquad + 4\cos(2.5\pi n) + 10\sin(3.3\pi n). \\ &= 6\cos(0.3\pi n) + 3\sin[(2\pi - 0.5\pi)n] + 2\cos[(2\pi - 0.3\pi)n] \\ &\qquad + 4\cos[(2\pi + 0.5\pi)n] + 10\sin[(4\pi - 0.7\pi)n] \\ &= 6\cos[0.3\pi n] - 3\sin[0.5\pi n] + 2\cos[0.3\pi n] \\ &\qquad + 4\cos[0.5\pi n] - 10\sin[0.7\pi n] \\ v(n) &= [8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)] \end{aligned}$$

The discrete-time signal $v(n)$ is composed of a weighted sum of three-discrete-time sinusoidal signals of normalised angular frequencies : $0.3\pi$, $0.5\pi$ and $0.7\pi$].

## 3.11 CLASSIFICATION OF DISCRETE-TIME SYSTEMS

Discrete-time systems are classified according to their general properties and characteristics.

They are

(1) Static and Dynamic systems.

(2) Time-variant and time-invariant systems.

**Sol.** (*i*) $\mathbf{y(n) = x(n) + x(n - 1).}$

We know that, $y(n) = T[x(n)]. = x(n) + x(n - 1).$

If the input is delayed by $k$ units in time, we have,

$$y(n, k) = T[x(n - k)]$$

$$y(n, k) = x(n - k) + x(n - k - 1) \quad \ldots(1)$$

If we delay the output by $k$ units in time then,

$$y(n - k) = x(n - k) + x(n - k - 1) \quad \ldots(2)$$

(1) = (2)

Here, $y(n, k) = y(n - k)$

So, the system is time-invariant.

(*ii*) $\mathbf{y(n) = x(- n)}$

If the input is delayed by $k$ units in time and applied to the system, we have,

$$y(n, k) = T[x(n - k)] = x[- n - k] \quad \ldots(3)$$

If the output is delayed by $k$ samples,

$$y(n - k) = x[- (n - k)] = x[- n + k] \quad \ldots(4)$$

(3) ≠ (4)

Here, $y(n, k) \neq y(n - k)$

So, the system is time-variant.

(*iii*) $\mathbf{y(n) = x(2n)}$

The system is described by input output equation,

$$y(n) = T[x(n)]$$

$$y(n) = x(2n)$$

If the input is delayed by $k$ unit in time and applied to the system,

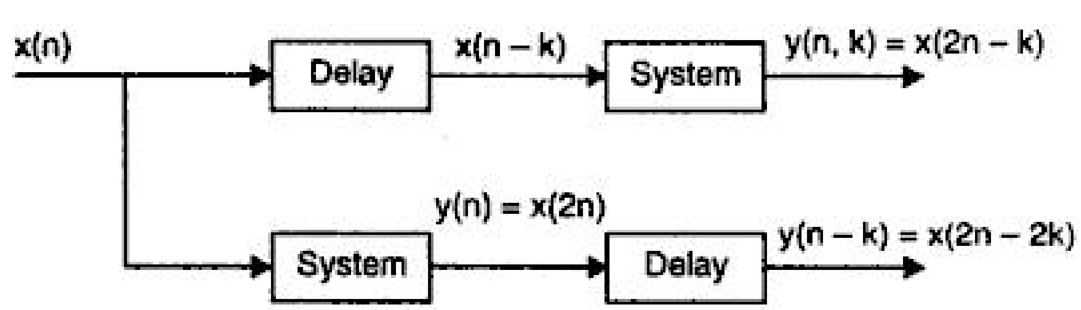
$$y(n, k) = T[x(n - k)]$$

$$y(n, k) = x(2n - k) \quad \ldots(1)$$

Now, if we delay the output $y(n)$ by $k$ unit in time, the result will be

$$y(n - k) = x[2(n - k)] = x[2n - 2k] \quad \ldots(2)$$

Since $y(n, k) \neq y(n - k)$, the system is time variant.




(*iv*) $\mathbf{y(n) = x(n) \sin \omega_0 n}$

If the *i/p* is delayed by $k$ unit in time and applied to the system,

$$y(n, k) = x(n - k) \sin \omega_0 n.$$

If we delay the output by $k$ unit in time, then $y(n - k) = x(n - k) \sin \omega_0 (n - k)$

Since $y(n - k) \neq y(n, k)$. So the system is time variant.

**3. Causal and Non-causal System**

In a discrete-time system the $n_0$th output sample $y[n_0]$ depends only on input samples $x[n]$ for $n \leq n_0$ and does not depend on input samples for $n > n_0$. This means that, a system is said to be causal if the output of the system at any time $n$ depends only on present and past inputs, but does not depend on future inputs. This can be expressed mathematically as,

$$y(n) = F[x(n), x(n-1), x(n-2) ......].$$

If a system depends not only on present and past inputs but also on future inputs, then it is said to be a non-causal system.

**4. Stable System**

There are various definitions of stability we define a discrete-time system to be stable if and only if, for every bounded input, the output is also bounded. This implies that, if the response to $x(n)$ is the sequence $y(n)$ and if,

$$[x(n)] < B_x \quad \text{or} \quad |\, x(n)\, | < M_x \qquad ...(3.34)$$

for all values of $n$, then

$$[y(n)] < B_y \quad \text{or} \quad |\, y(n)\, | < M_y \qquad ...(3.35)$$

for all values of $n$

where $B_x$ and $B_y$ are finite constants. This type of stability is usually referred to as bounded-input, bounded-output (BIBO) stability.

**Problem 11.** *Determine whether the following systems are*

*(i) Stable* *(ii) Causal.*

*(a)* $y(n) = e^{x(n)}$ *(b)* $y(n) = nx(n)$.

**Sol.** (*a*) $\mathbf{y(n) = e^{x(n)}}$

(*i*) If $x(n)$ is bounded, say $|\, x(n)\, | < M$, then $|\, y(n)\, | = e^{|M|} < \alpha$.

That is, this system produces bounded output for bounded input. So it is BIBO stable system.

(*ii*) Since the output of this system depends only on the present input $x(n)$, it is a causal system.

(*b*) $\mathbf{y(n) = nx(n)}$

(*i*) To show that the system is not BIBO stable requires the specification of bounded input that produces an unbounded output.

For example, $x(n) = u(n)$, the unit step function produces an output $y(n) = nu(n)$ that grows without bounded therefore the system is not BIBO stable.

(*ii*) To determine causality we can compute $y(n)$ at an arbitrary $n$, say $n_0$.

$$y(n) = n_0\, x(n_0)$$

The output at time $n_0$, depends upon the input at time $n_0$, and $n_0$ future input values, therefore, the system is causal.

**Problem 12.** *Determine if the system described by the following equations are causal or non-causal.*

*(i)* $y(n) = x(n) + \dfrac{1}{x(n-1)}$ *(ii)* $y(n) = x(n^2)$.

**Sol.** (*i*) $\mathbf{y(n) = x(n) + \dfrac{1}{x(n-1)}}$

For $n = -1$, $y(-1) = x(-1) + \dfrac{1}{x(-2)}$

For $n = 0$, $y(0) = x(0) + \dfrac{1}{x(-1)}$

For $n = 1$, $y(1) = x(1) + \dfrac{1}{x(0)}$.

For all the values of $n$ the output depends on present and past inputs. Therefore, the system is causal.

(*ii*) **y(n) = x(n²)**

For $n = -1, y(-1) = x(1)$

For $n = 0$, $y(0) = x(0)$

For $n = 1$, $y(1) = x(1)$.

For negative values of $n$, the system depends on future inputs. So, the system is non-causal.

5. **Linear and Non-linear System**

A system that satisfies the superposition principle is said to be a linear system. Super position principle states that, the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the outputs of the system to each of the individual input signals.

A system is linear if and only if

$$T[a_1x_1(n) + a_2x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1y_1(n) + a_2y_2(n) \quad ...(3.36)$$

for any arbitrary constants $a_1$ and $a_2$.

A relaxed system that does not satisfy the superposition principle is called non-linear.

**Problem 13.** *Determine whether the systems described by the following input-output equations are*

*(i) linear* *(ii) time-invariant*

*(a)* $y(n) = nx(n)$ *(b)* $y(n) = ax(n) + b$.

**Sol.** (*a*) **y(n) = nx(n)**

(*i*) We have to take two input sequence $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$y_1(n) = T[x_1(n)] = nx_1(n)$$

$$y_2(n) = T[x_2(n)] = nx_2(n).$$

A linear combination of the two *i/p* sequence result in the output,

$$T[x_1(n) + x_2(n)] = n[x_1(n) + x_2(n)] \quad ...(1)$$

On the other hand, a linear combination of the two outputs results in the output,

$$y_1(n) + y_2(n) = nx_1(n) + nx_2(n)$$

$$y_1(n) + y_2(n) = n[x_1(n) + x_2(n)] \quad ...(2)$$

(1) = (2)

$\therefore$ The system is linear.

(*ii*) $y(n) = T[x(n)] = nx(n)$

The response of this system to $x(n - n_0)$ is,

$$T[x(n - n_0)] = nx(n - n_0).$$

Now if we delay $y(n)$ by $n_0$ units in time,

$$y(n - n_0) = (n - n_0)\, x(n - n_0)$$

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$$\boxed{y(n) = x(n) * h(n)} \quad ...(3.43)$$

The operation of discrete-time convolution takes two sequences $x(n)$ and $h(n)$ and produces a third sequence $y(n)$.

## 3.13 THE CONVOLUTION PROCESS CAN BE SUMMARISED INTO THE FOLLOWING STEPS

**Step 1.** Choose an initial value of '$n$', the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$, is a good choice. Then express both sequence in terms of the index $k$.

**Step 2.** *Folding.* Fold the $h(k)$ about the origin and obtain $h(-k)$.

**Step 3.** *Time shifting.* Shift the $h(-k)$ by $n$ unit to right if $n$ is positive and left if $n$ is negative to obtain $h[-(k-n)] = h(n-k)$.

**Step 4.** *Multiplication.* Multiply $x(k)$ by $h(n-k)$ to obtain $w_n(k) = x(k)\,h(n-k)$.

**Step 5.** *Summation.* Sum all the values of the product $w(k)$ to obtain the value of output $y(n)$.

**Step 6.** Increment the index $n$, shift the sequence $h(n-k)$ to right by one sample and do step 4.

**Step 7.** Repeat step 6 until the sum of product is zero for all remaining values of $n$.

**Problem 15.** *Determine the output y(n) of a linear time invariant system with impulse response*

$$n(n) = \{\underset{\uparrow}{6}, 5, 4, 3, 2, 1\}$$

*when the input is*

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$$

**Sol.** $x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = 0$.

Therefore, the starting value of $n = n_1 + n_2 = 0$.

**Step 1.** *Folding.* The folding sequence $h(-k)$ is illustrated in Fig. 3.14.

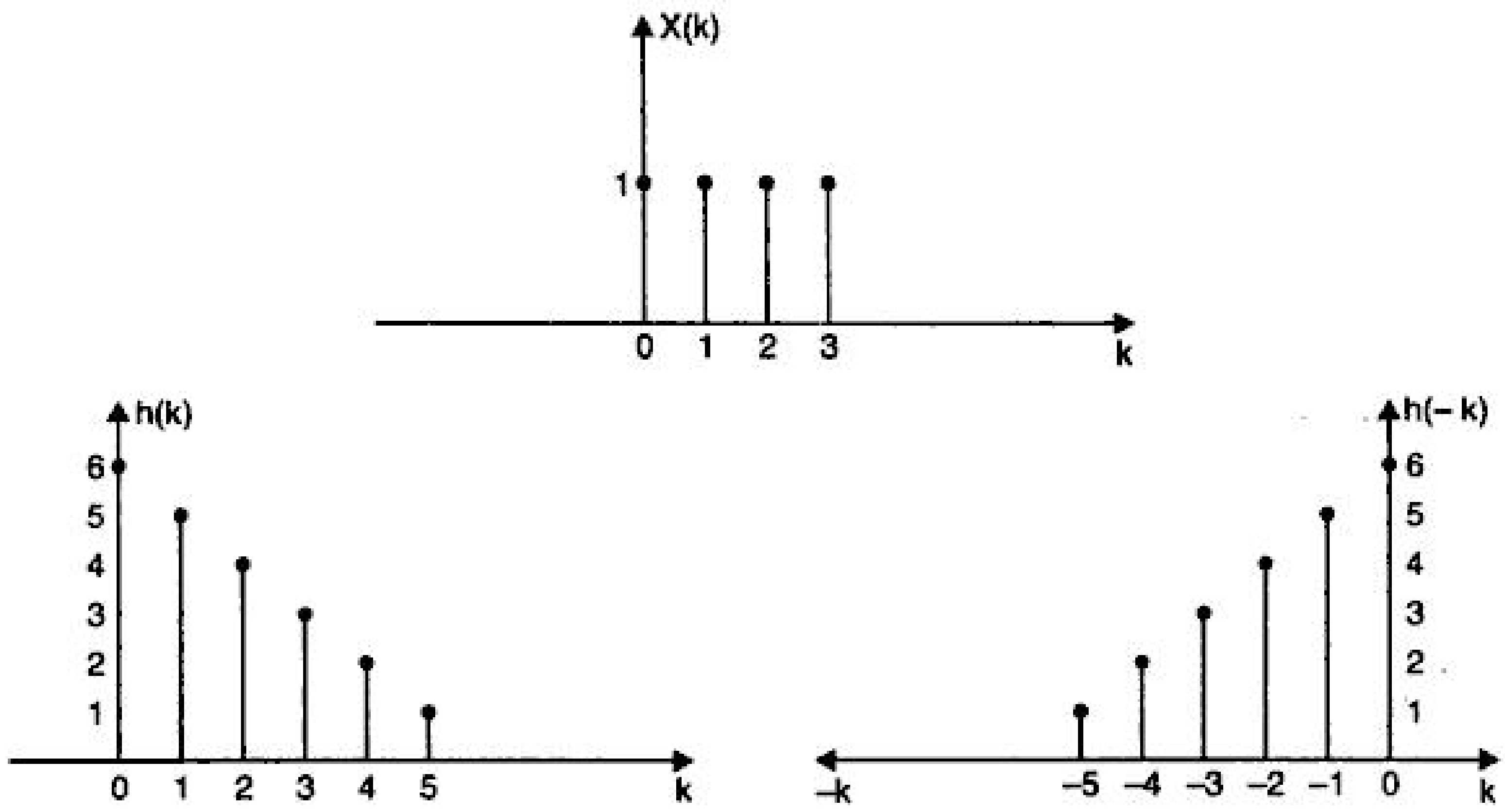


Fig. 3.14. Folding sequence.

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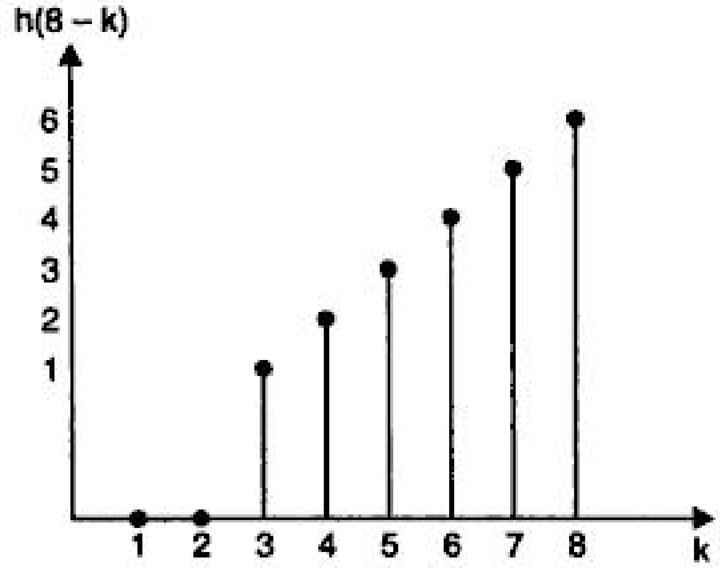
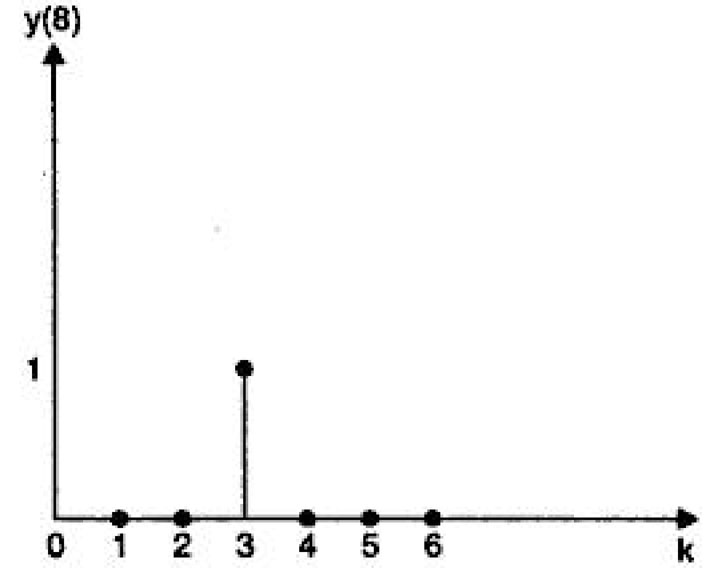
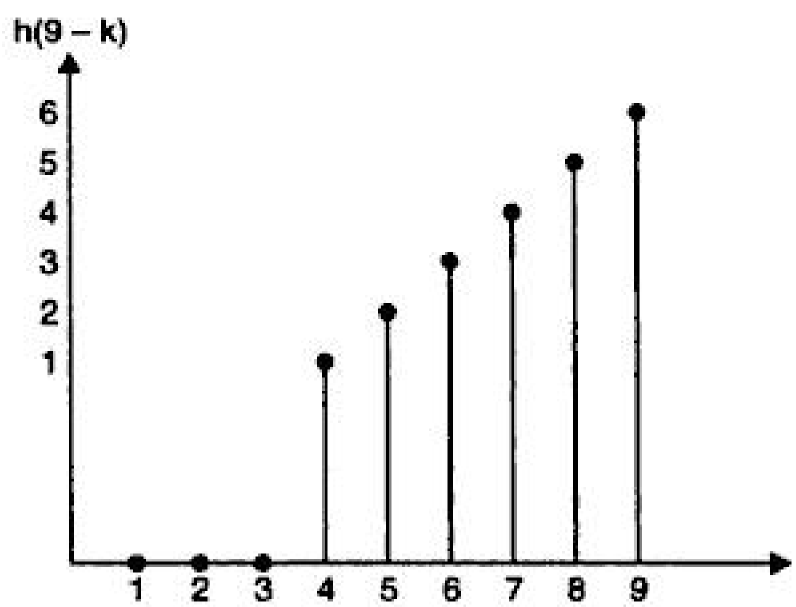
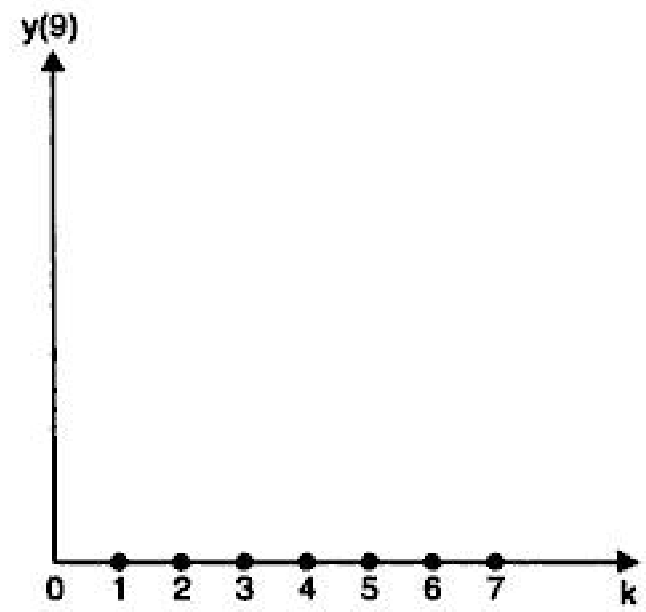



$$= 0 + 0 + 0 + (1 \times 1) + 0 + 0 + 0 + 0 + 0$$
$$= 1$$

**For n = 9,** $\quad y(9) = \sum_{k=-\infty}^{\infty} x(k)\, h(9-k)$




$$y(9) = x(0)\, h(9) + x(1)\, h(8) + x(2)\, h(7) + x(3)\, h(6)$$
$$+ x(4)\, h(5) + x(5)\, h(4) + x(6)\, h(3)$$
$$+ x(7)\, h(2) + x(8)\, h(1) + x(9)\, h(0).$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0.$$

Similarly, $\quad y(-1) = 0.$

Now we summarize the entire response for $-\infty < n < \infty$ as below :

$$y(n) = \{0, 0, \underset{\uparrow}{6}, 11, 15, 18, 19, 10, 6, 3, 1, 0, 0\}.$$

## EXERCISES

1. Determine the convolution $y(n)$ of the signals

$$x(n) = \begin{cases} 1 & 3 \le n \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

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**Problem 19.** *Determine the response of the relaxed system characterized by the impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ to the input signal $x(n) = 2^n u(n)$.*

**Sol.** Given $$x(n) = 2^n u(n)$$
$$h(n) = (1/2)^n u(n).$$

A causal signal is applied to a causal system.

Therefore, $$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$
$$= \sum_{k=0}^{n} 2^k (1/2)^{n-k} = (1/2)^n \sum_{k=0}^{n} 2^k (1/2)^{-k}$$
$$= (1/2)^n \sum_{k=0}^{n} 2^{2k}$$
$$= (1/2)^n [1 + 2^2 + 2^4 + 2^6 + \ldots\ldots + (n+1) \text{ terms}]$$
$$= (1/2)^n \left[\frac{(2^2)^{n+1} - 1}{(2)^2 - 1}\right] = (1/2)^n \left[\frac{4 . 4^n - 1}{3}\right]$$
$$\boxed{y(n) = (1/2)^n \left[\frac{4^{n+1} - 1}{3}\right].}$$

**Problem 20.** *Find the impulse response of the cascade systems, if*
$$h_1(n) = (-1/2)^n u(n)$$
$$h_2(n) = (1/2)^n u(n).$$
*Use the convolution sum to find the response to $x(n) = (1/4)^n u(n)$.*

**Sol.** (*i*) To find the impulse response of the system.
$$h(n) = h_1(n) * h_2(n) \qquad \rightarrow \text{cascade connection.}$$
$$= \sum_{k=0}^{n} h_1(k) h_2(n-k) = \left[\sum_{k=0}^{n} (-1/2)^k (1/2)^{n-k}\right] u(n)$$
$$= \left[(1/2)^n \sum_{k=0}^{n} (-1/2)^k (1/2)^{-k}\right]$$

Assume that $u(n) = 1$.
$$= \left[\left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \frac{(-1/2)^k}{(1/2)^k}\right]$$
$$= \left[\left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (-1)^k\right] = \left(\frac{1}{2}\right)^n \left[\frac{1-(-1)^{n+1}}{1-(-1)}\right]$$

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Let us consider an LTI system having an o/p at $n = n_0$

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)\,x(n_0 - k)$$

The above systems can be subdivided into the two terms, one having present and past values ($n \le n_0$) and other having future values ($n \ge n_0$). Thus

$$\begin{aligned} y(n_0) &= \left[\sum_{k=0}^{\infty} h(k)\,x(n_0 - k)\right] + \left[\sum_{k=-\infty}^{-1} h(k)\,x(n_0 - k)\right] \\ &= [h(0)\,x(n_0) + h(1)\,x(n_0 - 1) + \ldots\ldots] \\ &\quad + [h(-1)\,x(n_0 + 1) + h(-2)\,x(n_0 + 2) + \ldots\ldots] \qquad \ldots(3.50) \\ y(n_0) &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

The first term in the sum with $x(n_0)\,x(n_0 - 1)$ ... are the present and past input. Whereas the term (II) in the sum with $x(n_0 + 1)$, $x(n_0 + 2)$ ........ are the future values of input. Now if,

$$h(n) = 0 \quad \text{for } n < 0.$$

the output $y(n)$ depends on present and past values of inputs but does not depend on future values of inputs because the second term in sum becomes zero once $h(n) = 0$ for $n < 0$. Hence an LTI system is causal if and only if its impulse response is zero for $n < 0$.

The limit of summation of the convolution formula may be modified. Thus we have,

$$\begin{aligned} y(n) &= \sum_{k=0}^{\infty} h(k)\,x(n - k) \qquad \text{put } n - k = m \\ &= \sum_{m=-\infty}^{n} x(m)\,h(n - m) \end{aligned}$$

$$\boxed{y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n - k)} \qquad \ldots(3.51)$$

If the input to a causal linear time invariant system is causal *i.e.*, $x(n) = 0$ for $n < 0$, the limit on the convolution formula can be further modified. Thus we obtain,

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n - k) = \sum_{k=0}^{n} x(k)\,h(n - k).$$

$$y(n) = \sum_{k=0}^{n} h(k)\,x(n - k)$$

It is clear from above eqn. that the response of the causal system to a causal input sequence is causal because $y(n) = 0$ for $n < 0$.

**Stability.** A discrete time system is to be stable, bounded input, bounded output stable, if the output sequence $y(n)$ remains bounded for all bounded input sequence $x(n)$.

We now develop the stability condition for an LTI discrete-time system. We shall demonstrate that an LTI system is BIBO stable if and only if its impulse response is absolutely summable, *i.e.*,

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$$y(n) = \frac{1}{n+1}\,[ny(n-1) + x(n)]$$

$$y(n) = \frac{n}{n+1}\,y(n-1) + \frac{1}{n+1}\,x(n) \qquad \text{...(3.60)}$$

Eqn. (3.60) suggested that the computation of $y(n)$ requires two multiplication, one addition and one memory location.

The block diagram representation is shown in Fig. 3.18. This system is known as recursive system and $y(n-1)$ is called the initial condition of the system.

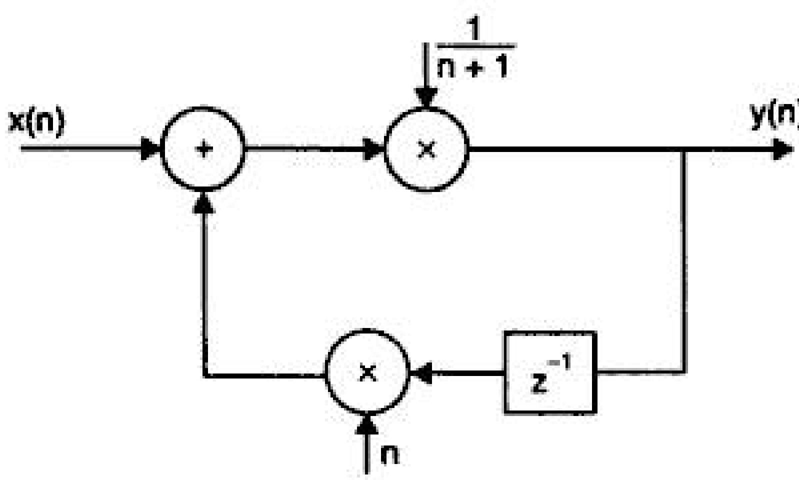


Fig. 3.18. Block diagram representation.

In general the o/p of a causal can be expressed as,

$$y(n) = f\,[y(n-1), y(n-2) \;.......\; y(n-\mathrm{N}), x(n), x(n-1) \;......\; x(n-\mathrm{M})] \qquad \text{...(3.61)}$$

The block diagram representation of causal system is shown in Fig. 3.19.

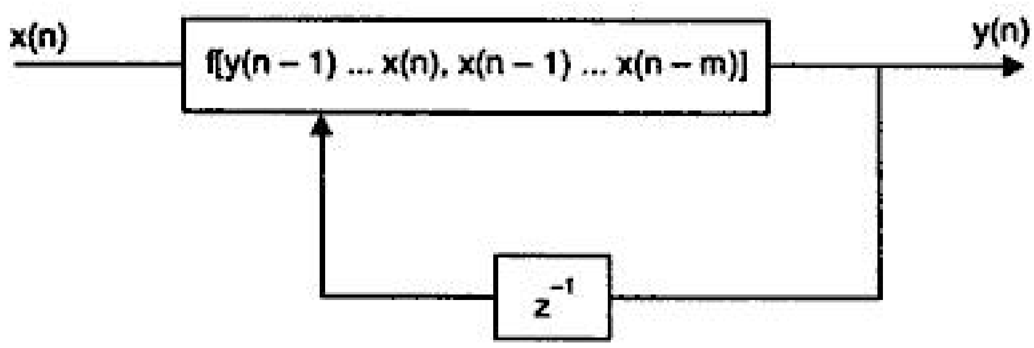


Fig. 3.19. Block diagram representation of causal system.

If $y(n)$ is only function of present and past inputs, then

$$y(n) = f\,[x(n), x(n-1) \;.......\; x(n-m)] \qquad \text{...(3.62)}$$

Such a system is known as non-recursive system.

The block diagram representation of causal is shown in Fig. 3.20

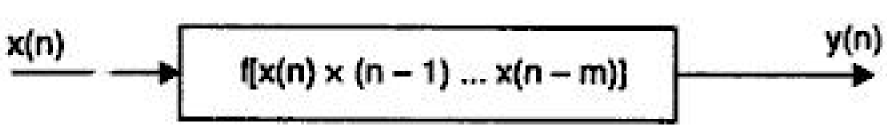


Fig. 3.20

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**Distinct roots.** For the case where eqn. (3.70) has N distinct roots $\alpha_1, \alpha_2, ......., \alpha_N$, then the most general solution is of the form,

$$y_h(n) = A_1 \alpha^n + A_2 \alpha_2^n + ........ + A_N \alpha_N^n \quad ...(3.71)$$

where, $A_1, A_2, A_N$ are weighting co-efficient.

Those co-efficients are determined from the initial condition specified by the term.

**Problem 26.** *Determine the homogeneous solution of the system described by the first order difference equation.*

$$y(n) + 3y(n-1) = x(n), \text{ with initial condition } y(-1) = 1.$$

**Sol.** For the homogeneous solution, $x(n) = 0$ thus, $y_h(n) + 3y_h(n-1) = 0$.

We assume solution of the form of

$$y_h(n) = \alpha^n$$

$$\therefore \quad \alpha^n + 3\alpha^{n-1} = 0.$$

$$\alpha^{n-1}[\alpha + 3] = 0$$

$$\boxed{\alpha = -3}$$

Thus, the general form of solution of homogeneous difference equation is,

$$y_h(n) = A\alpha^n$$
$$= A(-3)^n.$$

Using the initial condition $y(-1) = 1$. We have,

$$y_h(n) = -3y_h(n-1)$$

Put $n = 0$ $\quad y_h(0) = -3y_h(0-1) = -3\,;\, y_h(0) = A$

$$A = -3$$

Therefore the homogeneous solution is given by,

$$y_h(n) = 3(-3)^n = (-3)^{n+1}.$$

**Problem 27.** *Determine the homogeneous solution of 2nd order difference equation.*

$$y(n) - y(n-1) - y(n-2) = 0$$

*with initial condition $y(0) = 0$, $y(1) = 1$.*

**Sol.** Let us assume the solution of the form

$$y_h(n) = \alpha^n$$

$$\therefore \quad \alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0.$$

$$\alpha^{n-2}[\alpha^2 - \alpha - 1] = 0$$

Therefore, the roots are,

$$\alpha = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$$

The general solution to the homogeneous eqn. is.

$$y_h(n) = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using the initial condition

$$y(0) = 0, y(1) = 1, \text{ we have,}$$

$$0 = A_1 + A_2$$

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Therefore, $y_p(n) = \frac{4}{3} n\, 2^n u(n)$.

The total solution is,

$$\boxed{y(n) = A_1 2^n + A_2 (-1)^n + \frac{4}{3} n 2^n u(n)} \quad \ldots(1)$$

where the constant $A_1$ and $A_2$ are determined such that the initial conditions are satisfied. To accomplish this, evaluate the given eqn. at $n \geq 0, 1$.

$$y(0) - y(-1) - 2y(-2) = x(0) = 1.$$

$$\boxed{y(0) = 1}$$

and at $n = 1$.

$$y(1) - y(0) - 2y(-1) = x(1) + 2x(0) = 2 + 2 = 4.$$

$$\boxed{y(1) = 5}$$

Using the value of $y(0)$ and $y(1)$ in eqn. (1), we have

$$A_1 + A_2 = 1$$

$$2A_1 - A_2 + \frac{8}{3} = 5.$$

These two eqn. give $A_1 = \frac{1}{9}, A_2 = \frac{10}{9}$. Thus the final solution for $x(n) = 2^n u(n)$ is given by,

$$\boxed{y(n) = -\frac{1}{9} 2^n + \frac{10}{9}(-1)^n + \frac{4}{3} n 2^n.}$$

## 3.21 THE IMPULSE RESPONSE OF A LTI RECURSIVE SYSTEM

Impulse response of the linear time-invariant system was defined as a response of the system to a unit impulse *i.e.*, $x(n) = \delta(n)$.

Now consider the problem of determining the impulse response $h(n)$ given a linear constant co-efficient difference equation. In the proceeding subsection, we have described that the total response of the system to any input consists of solution to the homogeneous equation plus the particular solution. In case when the input is an impulse, then the particular solution is zero because $x(n) = 0$ for $n > 0$ *i.e.*,

$$y_p(n) = 0.$$

Therefore, the response of the system to an impulse consists of homogeneous solution.

**Problem 31.** *Determine the impulse response of the system described by,*

$$y(n) - ay(n-1) = x(n) \text{ with } y(-1) = 0.$$

**Sol.** For $x(n) = \delta(n)$ above equation reduced to,

$$y(n) - ay(n-1) = \delta(n),$$

for $n > 0$, this reduces the homogeneous eqn. *i.e.*,

$$y(n) - ay(n-1) = 0.$$

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Determine its values and sketch the signal $x(n)$

Determine $x(4-n)$

$$\left[\textbf{Ans.}\ (a)\ x(n) = \left\{.....0, \frac{1}{3}, \frac{2}{3}, \underset{\uparrow}{1}, 1, 1, 1, 0\right\}\quad (b)\ x(4-n) = \left\{0, \underset{\uparrow}{0}, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0\right\}\right]$$

**2.** From the two sequences $x(n) = \left(\frac{1}{2}\right)^n, y(n) = \left[-\frac{1}{2}\right]^n$

Prove that

(a) $x(n) + y(n) = 1 + (-1)^n\left(\frac{1}{2}\right)^n$ (b) $x(n)\,y(n) = \left[-\frac{1}{4}\right]^n$.

**3.** Express the sequence defined by

$$x(n) = \begin{cases} -2 & n = -1, 0, 1 \\ 4 & n = -2, 2 \\ 0 & \text{otherwise} \end{cases}$$

as a weighted sum of unit-sample sequences

[**Ans.** $x(n) = 4\delta(n+2) - 2\delta(n+1) - 2\delta(n) - 2\delta(n-1) + 4\delta(n-2)$]

**4.** Prove or disprove that $x(n) = 5\sin\left(2n + \frac{\pi}{4}\right)$

is period. If the sequence is periodic, determine its period. [**Ans.** Not periodic]

**5.** A discrete time system can be

(1) Static or dynamic

(2) Linear or non-linear

(3) Time invariant or time variant

(4) Causal or non-causal

(5) Stable or unstable with respect to above properties examine the following systems

(a) $x(n) = \cos[y(n)]$ [**Ans.** Static, non-linear, time-invariant causal, stable]

(b) $y(n) = x(n)\sin(\omega_0 n)$ [**Ans.** Static, linear, time variant, causal, stable]

(c) $y(n) = x(-n+3)$ [**Ans.** Dynamic, Linear, time-invariant, non-causal, stable]

(d) $y(n) = |\,x(n)\,|$ [**Ans.** Static, non-linear, time-invariant, causal, stable]

(e) $y(n) = x(n)\,u(n)$ [**Ans.** Static, linear, time-invariant, causal-stable]

(f) $y(n) = e^{x(n)}$. [**Ans.** Stable, causal, non-linear, time-invariant]

**6.** Compute the convolution $y(n) = x(n) * h(n)$ of the following signals :

(a) $x(n) = \{0, 1, -2, 3, -4\}$

$h(n) = \left\{\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right\}$ $\left[\textbf{Ans.}\left\{0, \frac{1}{2}, \frac{-1}{2}, \frac{3}{2}, -2, 0, \frac{-5}{2}, -2\right\}\right]$

(b) $x(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1\}$

$h(n) = \{1, \underset{\uparrow}{-2}, 3\}$ [**Ans.** $\{0, 0, 1, -1, 2, 2, 1, 3\}$]

(c) $x(n) = \begin{cases} \alpha^n & -3 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$

$h(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$ $\left[\textbf{Ans.}\ \sum_{k=0}^{4} x(n-k) \quad -3 \le n \le 9\right]$

$$e^{jk\omega n} = e^{jk\frac{2\pi}{N}n}, \qquad k = 0, 1, 2, \ldots\ldots N-1.$$

The fourier series representation of $x(n)$ consists of N harmonically related exponential functions.

$$\boxed{x(n) = \sum_{k=0}^{N-1} a_k\, e^{jk\frac{2\pi}{N}n}} \qquad \ldots(4.4)$$

Now multiply $e^{-j2\pi nm/N}$ to both sides,

$$e^{-jk2\pi mn/N}\, x(n) = \sum_{k=0}^{N-1} a_k\, e^{j2\pi kn/N} \,.\, e^{-j2\pi m/N}.$$

$$\sum_{n=0}^{N-1} e^{-jm2\pi n/N}\, x(n) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}(k-m)n}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} \qquad \ldots(4.5)$$

Here,
$$\sum_{n=0}^{N-1} a_n = \begin{cases} N & \text{if } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

$$\sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & k = 0, \pm N, \pm 2N\ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$e^{jk\frac{2\pi}{N}n} = \cos k\,\frac{2\pi}{N}\,n + j\sin k\,\frac{2\pi}{N}\,n.$$
$$= 1 + 0. \qquad (\text{for } k = 0, \pm N, \pm 2N \ldots\ldots)$$

$$\sum_{n=0}^{N} e^{j2\pi(k-m)n/N} = \begin{cases} N, & k-m = 0, \pm N_1, \pm 2N_1 \\ 0 & \text{otherwise} \end{cases}$$

Eqn. (4.5) is reduces to

$$\sum_{n=0}^{N-1} x(n)\, e^{-jm\frac{2\pi}{N}n} = a_m N.$$

$$\boxed{a_m = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jm\left(\frac{2\pi}{N}\right)n}}\;,\quad m = 0, 1, \ldots\ldots N-1 \qquad \ldots(4.6)$$

**Synthesis eqn.** $\quad x(n) = \sum_{k=0}^{N-1} a_k\, e^{jk\frac{2\pi}{N}n}$

**Analysis eqn.** $\quad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n}.$

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*Or*

*Plot the magnitude and phase response of a system whose impulse response $h(n) = a^n u(n)$ for $a = 0.5$.*

**Sol.** Given $$y(n) = ay(n-1) + x(n)$$

Take fourier transform on both sides,

$$y(e^{j\omega}) = ae^{-j\omega}Y(e^{j\omega}) + X(e^{j\omega})$$

$$y[e^{j\omega}][1 - ae^{-j\omega}] = X(e^{j\omega})$$

$$H = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-ae^{-j\omega}}.$$

**Another method :**

Consider the difference eqn. of a 1st order systems,

$$y(n) = ay(n-1) + x(n), \text{ with } X(n) = e^{j\omega n}.$$

The particular solution to this eqn. is in the form

$$y_p(n) = Ce^{j\omega n}.$$

Substituting this in original complete eqn.

$$y_p(n) - ay_p(n-1) = x(n)$$

$$Ce^{j\omega n} - ace^{j\omega(n-1)} = e^{j\omega n}.$$

$$Ce^{j\omega n}[1 - ae^{-j\omega}] = e^{j\omega n}.$$

Therefore, $$C = \frac{1}{1-ae^{-j\omega}}.$$

Thus, the steady state solution of the system is,

$$y_p(n) = \frac{1}{1-ae^{-j\omega}} . e^{j\omega n}.$$

This solution is of the form

$$H(e^{j\omega}) . e^{j\omega n}.$$

Therefore, the frequency response of this first order system is,

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}.$$

To plot $H(e^{j\omega})$, we find the magnitude and phase term as,

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}},$$

$$= \frac{1}{1-a[\cos\omega - j\sin\omega]} = \frac{1}{1-a\cos\omega + ja\sin\omega}$$

The magnitude response,

$$|H(e^{j\omega})| = \frac{1}{[(1-a\cos\omega)^2 + a^2\sin^2\omega]^{1/2}}$$

$$= \frac{1}{[1+a^2-2a\cos\omega]^{1/2}}$$

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$$= \sum_{n=-\infty}^{-1} |x(n)\, r^{-n}| + \sum_{n=0}^{\infty} |x(n)\, r^{-n}| = \sum_{n=1}^{\infty} |x(-n)\, r^{n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^{n}} \right| \qquad \ldots(4.12)$$

If X($z$) converges in some region of the complex plane, both summations in eqn. (4.12) must be finite.

If the first sum of eqn. (4.12) converges, there must exist values of $r$ small enough for $x(-n)r^n$ to be absolutely summable. Hence the ROC for the first sum consists of all points in a circle of radius $r_1$ as shown in Fig. (4.2) where $r_1 > r$.

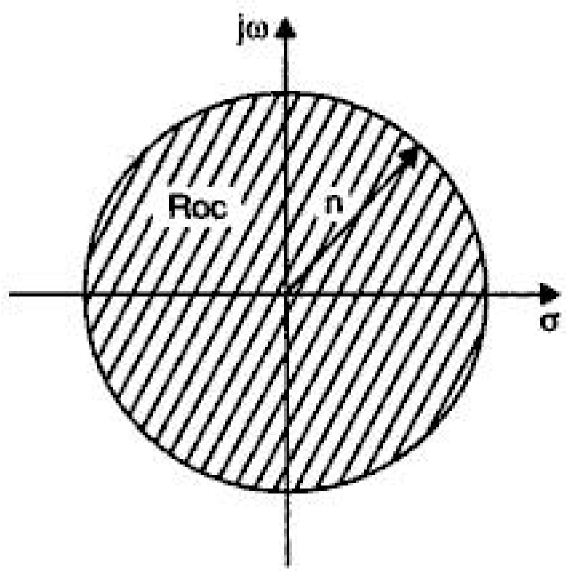



Fig. 4.2. ROC for $\sum_{n=1}^{\alpha} |x(-k)r^k|$.

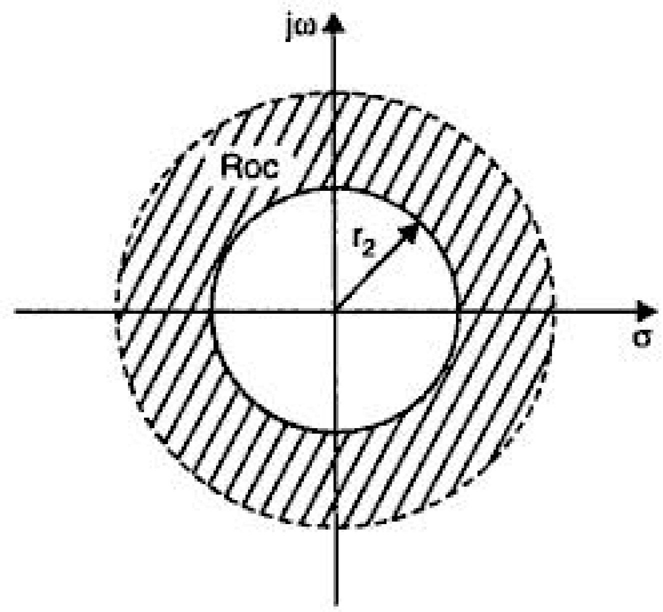



Fig. 4.3. ROC for $\sum_{n=0}^{\alpha} \left|\frac{x(n)}{r^n}\right|$.

If the second sum of eqn. (4.12) converges, there must exist large values of $r$ for which $x(n)/r^n$ is absolutely summable. Hence the ROC for the second sum consists of all points in a circle of radius, $r_2$ as shown in Fig. (4.3) where $r_2 < r$.

Therefore, the ROC of X(z) is the region in between two circles of radius $r_1$ and $r_2$ as shown in Fig. (4.4), where $r_2 < r < r_1$.

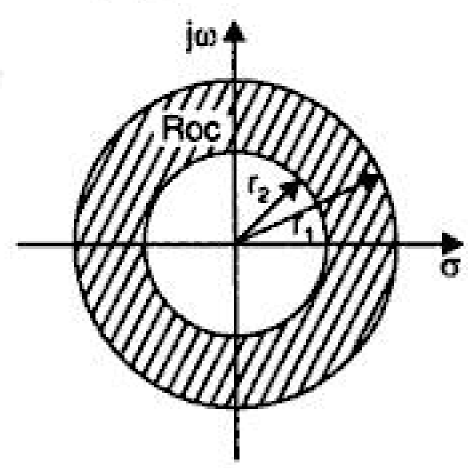
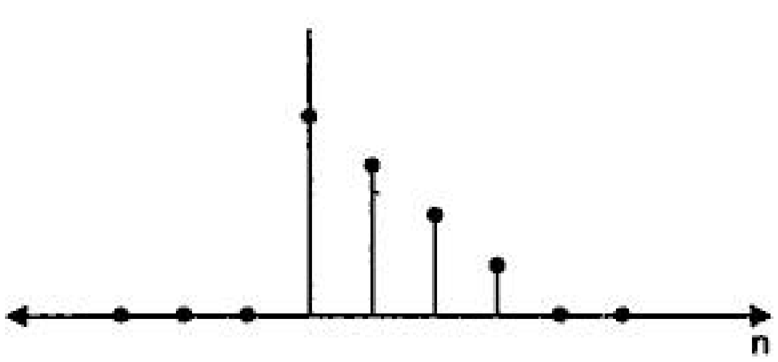


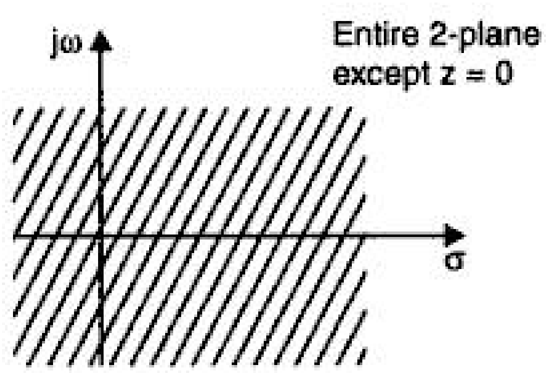

Fig. 4.4. $r_2 < r < r_1$.

**Signal** **ROC**

**Finite duration signal**

(1) Causal (or right sided)



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$$X(z) = \sum_{n=0}^{\alpha} \left(\frac{1}{4}z^{-1}\right)^n = \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{z}{z-\frac{1}{4}}$$

$$X(z) = \frac{z}{z-\frac{1}{4}}, \quad |z| > \frac{1}{4}.$$

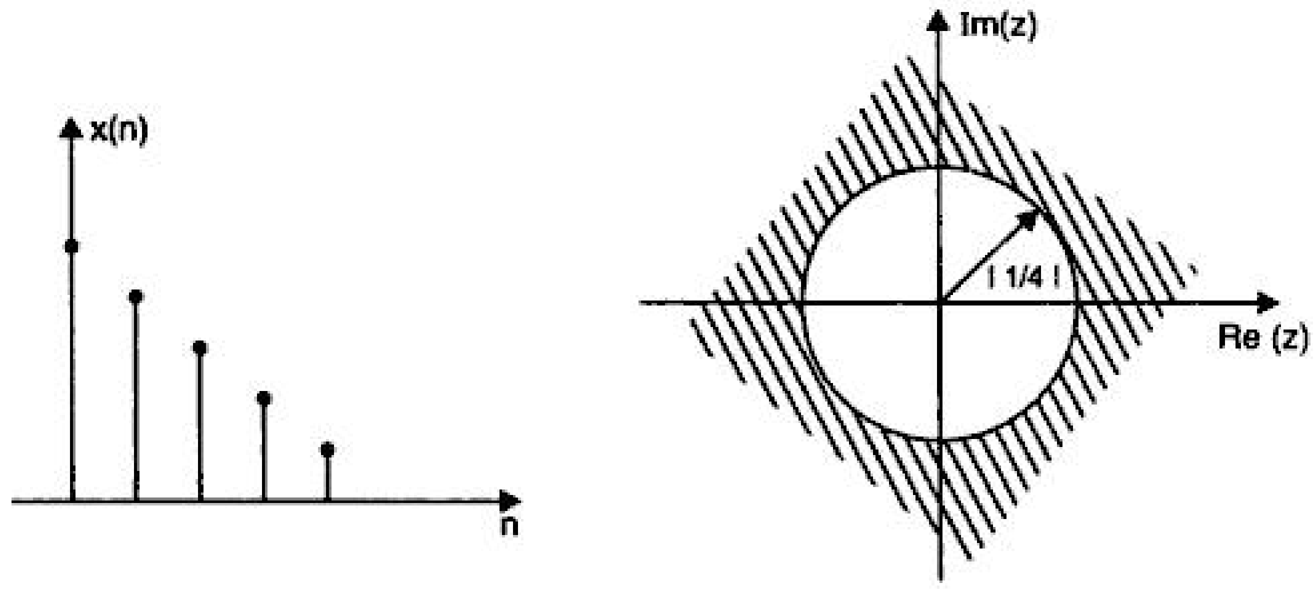


Fig. 4.5

**Problem 5.** *An anticausal signal x(n) is given by,*

$$x(n) = -a^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -a^n & n < 0 \end{cases}$$

*determine the z-transform and ROC.*

**Sol.**

$$X(z) = \sum_{n=-\alpha}^{\alpha} x(n) z^{-n}$$

$$= \sum_{n=-\alpha}^{-1} -a^n z^{-n} = -\sum_{n=-\alpha}^{-1} (az^{-1})^n$$

$$= -\sum_{n=1}^{-\alpha} (a^{-1} z)^n$$

$$X(z) = -\left[\sum_{n=0}^{\alpha} (a^{-1}z)^n - 1\right] \quad \ldots(5)$$

if $|a^{-1}z| < 1$ or

*i.e.*, $|z| < |a|$, the sum in eqn. (5) is converge for $|z| < |a|$ and

$$X(z) = -\left[\frac{1}{1-a^{-1}z} - 1\right] = -\frac{a^{-1}z}{1-a^{-1}z}.$$

$$X(z) = \frac{1}{1-az^{-1}}$$

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**Problem 7.** *Determine the z-transform of the following discrete sequence.*

*(a)* $x(n) = u(n)$ *(b)* $x(n) = \left(\frac{1}{2}\right)^n u(n)$ *(c)* $x(n) = \alpha^n u(n-1)$.

**Sol.** (*a*)

$$z[u(n)] = ?$$

$$u(n) = 1 \quad \text{for} \quad k \geq 0.$$
$$= 0 \quad \text{for} \quad k < 0.$$

$$z[u(n)] = \sum_{n=0}^{\alpha} u(n)\, z^{-n} = \sum_{n=0}^{\alpha} z^{-n} = \sum_{n=0}^{\alpha} (z^{-1})^n.$$

$$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

(*b*)

$$x(n) = \left(\frac{1}{2}\right)^n u(n).$$

$$u(n) = 1 \quad \text{for} \quad k \geq 0$$
$$= 0 \quad \text{for} \quad k < 0$$

$$z[x(n)] = \sum_{n=0}^{\alpha} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\alpha} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\boxed{X(z) = \frac{2z}{z-1}.}$$

(*c*)

$$x(n) = \alpha^n u(-n-1)$$

$u(-n-1)$ is a discrete unit step sequence, which is defined as,

$$u(-n-1) = 0 \quad \text{for} \quad k \geq 0.$$
$$= 1 \quad \text{for} \quad k \leq -1$$
$$x(n) = 0 \quad \text{for} \quad k \geq 0.$$
$$= \alpha^n \quad \text{for} \quad k \leq -1$$

$$z\,[x\,(n)] = X(z) = \sum_{k=-\alpha}^{\alpha} x(n)\, z^{-n}$$

$$= \sum_{k=-\alpha}^{-1} \alpha^n z^{-n} = \sum_{k=1}^{\alpha} \alpha^{-n} z^n$$

$$= \sum_{k=1}^{\alpha} (\alpha^{-1} z)^n = \sum_{k=0}^{\alpha} (\alpha^{-1} z)^n - 1$$

Using infinite geometric series sum, we get

$$X(z) = \frac{1}{1-\alpha^{-1}z} - 1 = \frac{1}{1-\frac{z}{\alpha}} - 1 = \frac{\alpha}{\alpha - z} - 1$$

$$\boxed{X(z) = \frac{z}{\alpha - z}.}$$

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**Sol.** (*a*) Since the ROC is exterior of a circle, we except $x(n)$ to be a causal sequence. Thus we divide so as to obtain a series in negative power of $z$. Carrying out the long division, we obtain,

$$
\begin{array}{r|l}
 & 1+4z^{-1}+7z^{-2}+10z^{-3}+\ldots\ldots \\
\hline
1-2z^{-1}+z^{-2} & 1+2z^{-1} \\
 & 1-2z^{-1}+z^{-2} \\
\cline{2-2}
 & \quad 4z^{-1}-z^{-2} \\
 & \quad 4z^{-1}-8z^{-2}+4z^{-3} \\
\cline{2-2}
 & \qquad 7z^{-2}-4z^{-3} \\
 & \qquad 4z^{-2}-14z^{-3}+7z^{-4} \\
\cline{2-2}
 & \qquad\qquad 10z^{-3}-7z^{-4} \\
 & \qquad\qquad 10z^{-3}-20z^{-4}+10z^{-5} \\
 & \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
 & \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Thus $X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + \ldots\ldots\ldots$

By comparing the relation, we have,

$$x(n) = \{\underset{\uparrow}{1}, 4, 7, 10, \ldots\ldots\ldots, 3n + 1 \ldots\ldots\ldots\ldots\}$$

(*b*) When the ROC is the interior of the circle, the signal $x(n)$ is anticausal signal. Thus we divide so as to obtain a series in power of $z$ as follows.

$$
\begin{array}{r|l}
 & 2z+5z^{2}+8z^{3}+11z^{4} \\
\hline
z^{-2}-2z^{-1}+1 & 2z^{-1}+1 \\
 & 2z^{-1}-4+2z \\
\cline{2-2}
 & \quad 5-2z \\
 & \quad 5-10z+5z^{2} \\
\cline{2-2}
 & \qquad 8z-5z^{2} \\
 & \qquad 8z-16z^{2}+8z^{3} \\
\cline{2-2}
 & \qquad\qquad 11z^{2}-8z^{3} \\
 & \qquad\qquad 11z^{2}-22z^{3}+11z^{4} \\
\cline{2-2}
 & \qquad\qquad\qquad 14z^{3}-11z^{4} \\
 & \qquad\qquad\qquad \vdots \qquad \vdots
\end{array}
$$

Thus, $X(z) = 2z + 5z^2 + 8z^3 + 11z^4 + \ldots\ldots\ldots$

In this case $x(n) = 0$ for $n \geq 0$, thus by comparing the result with eqn. $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$,

we get,

$$x(n) = \{\ldots\ldots\ldots (3n + 1) \ldots\ldots\ldots 11, 8, 5, 2, \underset{\uparrow}{0}\}.$$

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(*d*)
$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$
$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$
$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$
$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$
$$A_1 = \frac{X(z)}{z}(z+1)\bigg|_{z=-1} = 0.25$$
$$A_3 = \frac{X(z)}{z}(z-1)^2\bigg|_{z=1} = 0.5$$
$$A_2 = \frac{d}{dz}\left[\frac{X(z)}{z}(z-1)^2\right]\bigg|_{z=1} = \frac{d}{dz}\left[\frac{z^2}{(z+1)(z-1)^2}(z-1)^2\right]\bigg|_{z=1}$$
$$= \frac{d}{dz}\left[\frac{z^2}{z+1}\right]\bigg|_{z=1} = \frac{(z+1)2z - z^2}{(z+1)^2}\bigg|_{z=1} = \frac{(1+1)\times 2 - 1}{(1+1)^2} = \frac{3}{4} = 0.75$$
$$\frac{X(z)}{z} = \frac{0.25}{z+1} + \frac{0.75}{z-1} + \frac{0.5}{(z-1)^2}$$
$$X(z) = \frac{0.25z}{z+1} + \frac{0.75z}{z-1} + \frac{0.5z}{(z-1)^2}$$
$$x(n) = 0.25(-1)^n\ u(n) + 0.75\,u(n) + 0.5\,n\,(1)^n \quad \text{for} \quad n \geq 0.$$

**Problem 17.** *Determine the inverse z-transform of the following z-domain functions :*

(*a*) $X(z) = \dfrac{3z^2 + 2z + 1}{z^2 - 3z + 2}$ (*b*) $X(z) = \dfrac{z - 0.4}{z^2 + z + 2}$

(*c*) $X(z) = \dfrac{z-4}{(z-1)(z-2)^2}$.

**Sol.** (*a*)
$$X(z) = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2}$$

$$\begin{array}{r|l} & \quad 3 \\ z^2 - 3z + 2 & 3z^2 + 2z + 1 \\ & 3z^2 - 9z + 6 \\ \hline & \quad\ \ 11z - 5 \end{array}$$

$$= 3 + \frac{11z - 5}{z^2 - 3z + 2} = 3 + \frac{11z - 5}{(z-1)(z-2)}$$

By partial expansion, we get
$$X(z) = 3 + \frac{A_1}{z-1} + \frac{A_2}{z-2}$$
$$A_1 = \frac{11z - 5}{(z-1)(z-2)}(z-1)\bigg|_{z=1} = -6$$

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$$A = \left.\frac{z-1}{z+0.81}\right|_{z=0.31} = -0.616.$$

$$B = \left.\frac{z-1}{z-0.31}\right|_{z=-0.81} = 1.616.$$

$$Y(z) = \frac{-0.154}{1-0.31z^{-1}} + \frac{0.404}{1+0.81z^{-1}}$$

$$y(n) = [0.404\,(-0.81)^n - 0.154\,(0.31)^n]\,u(n).$$

**System function :**

Let us defined the function H(z) by,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad \ldots(4.19)$$

From the eqn. (4.19)

$$Y(z) = H(z)\,X(z)$$

The function H(z) is known as the transfer function of a system or system function.

When the input to the system is impulse signal $x(n) = \delta(n)$ then,

$$Y(z) = H(z)$$

$$H(z) = z[h(n)].$$

**Problem 20.** *Determine the system function H(z) of*

*(a)* $y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1).$

*(b)* $y(n) = \frac{3}{2}y(n-1) + 2x(n).$

**Sol.** (*a*) Taking *z*-transform of both sides,

$$Y(z) + \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}.$$

(*b*)

$$y(n) = \frac{3}{2}y(n-2) + 2x(n)$$

Taking *z*-transform of both sides,

$$Y(z) = \frac{3}{2}z^{-1}Y(z) + 2X(z)$$

$$H(z) = \frac{2}{1-\frac{3}{2}z^{-1}}.$$

Taking inverse *z*-transform,

$$h(n) = 2\left(-\frac{3}{2}\right)^n u(n).$$

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3. Find the fourier transform of $x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$.

[ **Hint.** $X(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} e^{-j\omega n}$

$= e^{-j\omega} + 1/2\, e^{-j2\omega} + \dots\dots\dots = e^{-j\omega}\ [1 + 1/2\, e^{-j\omega} + \dots\dots.]$

$= \frac{e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$ ].

### 5.11.4 Properties of Discrete-time Fourier Transform

1. **Linearity.** $F[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$.
2. **Periodicity.** The discrete-time FT $X(e^{j\omega})$ is periodic in $w$ with period $2\pi$.

$$X(e^{j\omega}) = X[e^{j(\omega + 2k\pi)}] \text{ for any integer } k.$$

3. **Time shifting.** If $F[x(n)] = X[e^{j\omega}]$, then

$$F[x(n-k)] = e^{-j\omega k} X[e^{j\omega}].$$

**Proof.** $F[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k)\, e^{-j\omega n}$ [Let $n - k = P$

$n = P + k$ ]

$$= \sum_{P=-\infty}^{\infty} x(P)\, e^{-j\omega(P+k)} = e^{-j\omega k} \sum_{P=-\infty}^{\infty} x(P)\, e^{-j\omega p}$$

$= e^{-j\omega k} X(e^{j\omega})$ [Amplitude spectrum does not change, The phase spectrum is changed by-$\omega k$]

4. **Frequency shifting property :**

If $X[e^{j\omega}] = F[x(n)]$, then

$$F[x(n)\, e^{j\omega_0 n}] = X[e^{j(\omega - \omega_0)}].$$

**Proof.** $$= \sum_{n=-\infty}^{\infty} x(n)\, e^{+j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega - \omega_0)n} = X[e^{j(\omega - \omega_0)}].$$

5. **Time reversal :**

If $F[x(n)] = X[e^{j\omega}]$, then

$$F[x(-n)] = X[e^{-j\omega}].$$

6. **Differentiation in frequency :**

If $x(n) \xleftrightarrow{F} X(\omega)$

$$nx(n) \xleftrightarrow{F} j\frac{dX(\omega)}{d\omega}.$$

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# 6 Discrete Fourier Transform

## 6.1 INTRODUCTION

Frequency analysis of a discrete time signal is usually performed on a digital signal processor. To perform a frequency analysis on a discrete time signal, we convert time domain sequence $x(n)$ to an equivalent frequency domain representation $X(\omega)$. We know that such a representation is given by the Fourier transform $X(\omega)$ of the signal $x(n)$.

A difficulty encounter with the direct application of Fourier transform to a discrete time signal is that the resulting representation $X(\omega)$ becomes continuous function of frequency. Hence, it is unsuitable for digital processing.

In this section we consider the representation of a discrete time signal $x(n)$ by samples of its spectrum $X(\omega)$. Such a frequency domain sampling leads to the discrete fourier transform, a second transform domain representation that is applicable only to a finite length sequences.

## 6.2 THE DISCRETE FOURIER TRANSFORM : (FOURIER REPRESENTATION OF PERIODIC SIGNALS)

Before defining the DFT, we consider the sampling of Fourier transform of an aperiodic discrete time sequence. Hence, we establish the relationship between the sampled Fourier transform and DFT. To start, let us consider an aperiodic discrete time sequence $x(n)$ with Fourier transform,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \ldots(6.1)$$

Let us sample $X(\omega)$ periodically in a frequency at a spacing $\Delta\omega$ radians between the successive samples.

As we know that $X(\omega)$ is periodic with period $2\pi$, we require the samples only in fundamental frequency range. We take N equidistance samples in the internal of $0 \le \omega \le 2\pi$ with spacing $\Delta\omega = \frac{2\pi}{N}$ which is shown in Fig. (6.1).

Let us evaluate eqn. (6.1) at $\omega = \frac{2\pi k}{N}$, we have,

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$$X(k) = \sum_{n=0}^{N/2-1} 1 . e^{\frac{-j2\pi k}{N}n} = \frac{1-e^{\frac{-j2\pi k}{N}(N/2)}}{1-e^{-j2\pi k/N}} = \frac{1-e^{-j\pi k}}{1-e^{-j2\pi k/N}}$$

$$X(k) = \begin{cases} N/2 & k = 0 \\ \dfrac{2}{1-e^{-j2\pi k/N}} & k = \text{odd} \\ 0 & k = \text{even} \quad 0 \le k \le N-1 \end{cases}$$

(*d*) Given that, $x(n) = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{\frac{-j2\pi k}{N}n} = \sum_{n=0}^{N-1} \left[ae^{\frac{-j2\pi k}{N}}\right]^n$$

$$= \frac{1-a^N e^{-j\frac{2\pi k}{N}\times N}}{1-ae^{-j\frac{2\pi k}{N}}} = \frac{1-a^N}{1-ae^{-j\frac{2\pi k}{N}}}.$$

**Problem 2.** *Compute 4-point DFT of causal three sample sequence given by,*

$$x(n) = \frac{1}{3}; \quad 0 \le n \le 2.$$
$$= 0 \ ; \quad else.$$

**Sol.** By the definition of N-point DFT, the $k^{\text{th}}$ complex co-efficient of $X(k)$, for $0 \le k \le N-1$, is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi k}{N}n}.$$

Here, $N = 4$, therefore the 4-point DFT is,

$$X(k) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi k}{4}n} = \sum_{n=0}^{3} x(n) e^{\frac{-j\pi kn}{2}}.$$

$$= x(0)e^0 + x(1)\, e^{-j\pi k/2} + x(2)\, e^{-j\pi k} + x(3)e^{-j3\pi k/2}$$

$$= \frac{1}{3} + \frac{1}{3}e^{-j\pi k/2} + \frac{1}{3}e^{-j\pi k} + 0 = \frac{1}{3}[1 + e^{-j\pi k/2} + e^{-j\pi k}]$$

$$= \frac{1}{3}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos \pi k - j \sin \pi k\right]$$

The values of $X(k)$ can be evaluated for $k = 0, 1, 2, 3$.

when $k = 0$, $X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j \sin 0] = \frac{1}{3}[1 + 1 + 1] = 1 \angle 0$

when $k = 1$, $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$

$$= \frac{1}{3}[1 + 0 - j - 1 - j0] = -j\frac{1}{3} = \frac{1}{3}\angle -\frac{\pi}{2}.$$

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**(2) Periodicity :**

If $$x(n) \xleftrightarrow[N]{\text{DFT}} X(k).$$

then $$x(n + N) = x(n)$$

then $$\boxed{X(k + N) = X(k)} \quad \text{...(6.15)}$$

**Proof.** $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N}n}$$

$$X(k + N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi(k+N)}{N}n}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} . e^{-j2\pi n} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$\boxed{X(k + N) = X(k).}$$

**(3) Circular shift of a sequence :**

This property is analogous to the time shifting property of the DTFT, but with some difference.

Let us consider a sequence $x(n)$ of length N which is defined for $0 \le n \le N - 1$. The sample value of such sequence is zero for $n < 0$ and $n \ge N$. For any arbitrary integer $k$, the shift sequence $x_1(n) = x(n - k)$ is no longer defined for the range $0 \le n \le N - 1$. Therefore we need to define another type of shift that will always keep the shifted sequence in the range $0 \le n \le N - 1$.

This shift is known as circular shift that can be represented as the index modulo N. Thus we can write,

$$\boxed{x_c(n) = x_p(n - k)} \quad \text{...(6.16)}$$

$$x_c(n) = x[(n - k), \text{ modulo N}] \qquad 0 \le n \le N - 1.$$

$$\boxed{x_c(n) = x[((n - k))_N]} \quad \text{...(6.17)}$$

where $x_c(n)$ is represented the circular shift of $x(n)$. Or more generally we can define

$$x_c(n) = \begin{cases} x_p(n - k) = x[((n - k))_N] & 0 \le n \le N - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{...(6.18)}$$

Eqn. (6.17) and (6.18) tell us how to construct $x_p(n)$.

**(4) Circular convolution and multiplication of two DFTs :**

Consider two finite duration sequences $x_1(n)$ and $x_2(n)$ both of length N, with their N-point DFTs $X_1(k)$ and $X_2(k)$ *i.e.*,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi k}{N}n} \ ; \quad k = 0, 1, 2, \ldots\ldots N - 1 \quad \text{...(6.19)}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j\frac{2\pi k}{N}n} \ ; \quad k = 0, 1, 2, \ldots\ldots N - 1 \quad \text{...(6.20)}$$

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**Proof.** $\text{DFT}[x((n-l))_N] = \sum_{n=0}^{N-1} x((n-l))_N \, e^{\frac{-j2\pi kn}{N}}$.

If we change the index from $n$ to $m = N + n - l$, then

$$= \sum_{n=0}^{N-1} x(m)\, e^{-j2\pi km/N},$$

$$= \sum_{n=0}^{N-1} x(N+n-l)\, e^{\frac{-j2\pi kn}{N}} \qquad \text{...(6.31)}$$

Eqn. (6.31) can be written as

$$= \sum_{n=0}^{l-1} x(N+n-l)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=l}^{N-1} x(N+n-l)\, e^{\frac{-j2\pi k(n)}{N}}$$

$$= \sum_{m=n-l}^{N-1} x(m)\, e^{\frac{-j2\pi k(m+l-N)}{N}} + \sum_{m=N}^{2N-l-1} x(m)\, e^{-j2\pi k\frac{(m+l-N)}{N}}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{\frac{-j2\pi k(m+l)}{N}} + \sum_{m=N}^{2N-l-1} x(m)\, e^{\frac{-j2\pi k(m+l)}{N}}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{\frac{-j2\pi k(m+l)}{N}} + \sum_{m=0}^{N-l-1} x(m)\, e^{\frac{-j2\pi k(m+l)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi kl/N}\, e^{-j2\pi km/N}$$

$$= e^{-j2\pi kl/N}\, X(k).$$

$$\boxed{\text{DFT}\{x((n-l))_N\} = e^{-j2\pi kl/N}\, X(k)} \qquad \text{...(6.32)}$$

(8) **Multiplication of two sequences :**

If $\qquad x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$

and $\qquad x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$

then $\qquad \boxed{x_1(n)\, x_2(n) \xleftrightarrow[N]{\text{DFT}} \frac{1}{N} X_1(k)\, X_2(k)} \qquad \text{...(6.33)}$

The proof of this property is similar to circular convolution.

(9) **Circular correlator :**

For complex valued sequence $x(n)$ and $y(n)$.

If $\qquad x(n) \xleftrightarrow[N]{\text{DFT}} X(k).$

and $\qquad y(n) \xleftrightarrow[N]{\text{DFT}} Y(k)$

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$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, e^{-j\frac{2\pi k}{N/2}n} \qquad 0 \le k \le 15.$$

$$Y(k) = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi kn}{8}}$$

$$Y(k) = X(k) \quad 0 \le k \le 15$$

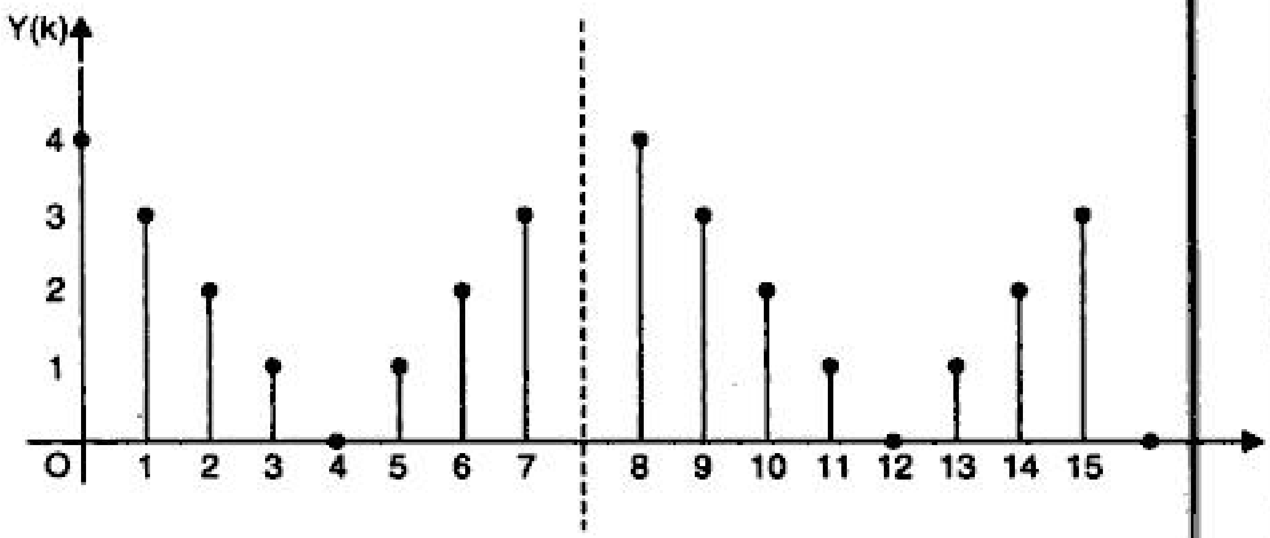


Fig. 6.9

Therefore 16-point DFT of interpolated signal $y(n)$ contains two copies of 8 point DFT of $x(n)$. Since for $Y(k)$, N = 8 therefore $Y(k)$ is periodic with period 8.

**Problem 6.** *Compute the DFT of sequence defined by $x(n) = (-1)^n$ for*
*(a) N = 3 (b) N = 4 (c) N = even (d) N = odd.*

**Sol.**
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn},$$

$$= \sum_{n=0}^{N-1} (-1)^n\, W_N^{kn} \qquad [\because \; W_N^N = 1]$$

$$= \sum_{n=0}^{N-1} [(-1) \times W_N^{k}]^n = \sum_{n=0}^{N-1} [-W_N^{k}]^n \qquad [\because \; W_N^k = 1]$$

$$= \frac{1-[-W_N^{k}]^N}{1+W_N^{k}} = \frac{1-(-1)^N}{1+W_N^{k}}.$$

(*a*) **for N = 3**
$$X(k) = \frac{1-(-1)}{1+W_3^{k}} = \frac{2}{1+W_3^{k}}$$

$$X(k) = \frac{2}{1+\cos\frac{2\pi k}{3} - j\sin\frac{2\pi k}{3}}.$$

(*b*) **for N = 4**
$$X(k) = \frac{1-(-1)^4}{1+W_4^{k}} = 0 \text{ for } W_4^{k} \ne -1 \quad \text{or} \quad k \ne 2$$

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**Problem 12.** *Perform circular convolution of the two sequences*

$$x_1(n) = \{2, 1, 2, 1\} \quad and \quad x_2(n) = \{1, 2, 3, 4\}.$$

**Sol.**

**Method 1 :**

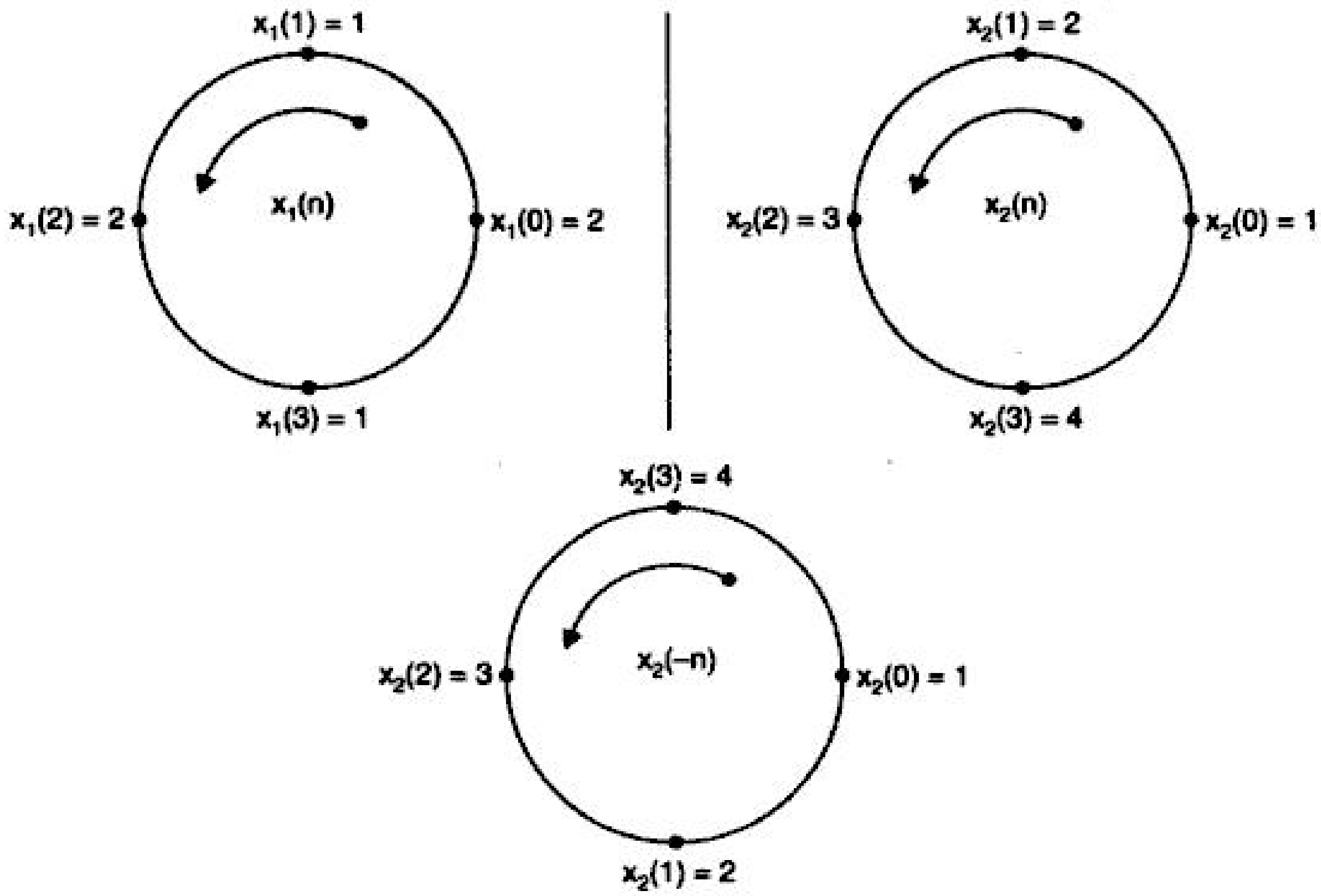
**Graphical method of computing circular convolution**

The circular convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n) = \sum_{n=0}^{N-1} x_1(n)\, x_{2,m}(n).$$

where $$x_{2,m}(n) = x_2(m-n).$$

The given sequences can be represented as points on a circle as shown below, the folded sequence $x_2(-n)$ is also represented on the circle.




**when m = 0**

$$x_3(0) = \sum_{n=0}^{N-1} x_1(n)\, x_{2,m}(-n) = \sum_{n=0}^{3} x_1(n)\, x_{2,0}(n)$$

**when m = 1**

$$x_3(1) = \sum_{n=0}^{N-1} x_1(n)\, x_2(1-n) = \sum_{n=0}^{3} x_1(n)\, x_{2,1}(n).$$

**when m = 2**

$$x_3(2) = \sum_{n=0}^{N-1} x_1(n)\, x_2(2-n) = \sum_{n=0}^{3} x_1(n)\, x_{2,2}(n)$$

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| $n$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_4(n)$ | | | | | | | | 4 | $-4$ | |
| $h(n)$ | | $-1$ | 1 | | | | | | | |
| $h(-n)$ | 1 | $-1$ | | | | | | | | |
| $h(6-n)=h_6(n)$ | | | | | | | 1 | $-1$ | | |
| $h(7-n)=h_7(n)$ | | | | | | | | 1 | $-1$ | |
| $h(8-n)=h_8(n)$ | | | | | | | | | 1 | $-1$ |

$$y_4(m) = \sum_{k=-\infty}^{\infty} x_4(n)\, h(m-n) = \sum_{n=-\infty}^{\infty} x_4(n)\, h_m(n) \text{ ; } m = 6, 7, 8$$

**when m = 6** $\quad y_4(6) = \sum_{k=5}^{7} x_4(n)\, h_6(n) = 0-4+0=-4$

**when m = 7** $\quad y_4(7) = \sum_{k=6}^{8} x_4(n)\, h_7(n) = 4+4+0=8$

**when m = 8** $\quad y_4(8) = \sum_{k=6}^{8} x_4(n)\, h_8(n) = 0-4+0=-4.$

**To combine the output of the convolution of each section.** It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In this method the overall output is obtained by combining the outputs of the convolution of each section. The overlapped portions (or samples) are added while combining the output.

The output of all sections can be represented in a table as shown below.

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $y_1(m)$ | $-1$ | 2 | $-1$ | | | | | | |
| $y_2(m)$ | | | $-2$ | 4 | $-2$ | | | | |
| $y_3(m)$ | | | | | $-3$ | 6 | $-3$ | | |
| $y_4(m)$ | | | | | | | $-4$ | 8 | $-4$ |
| $y(m)$ | $-1$ | 2 | $-3$ | 4 | $-5$ | 6 | $-7$ | 8 | $-4$ |

$$y(m) = x(n) * h(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}.$$

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**Method 2 :**

In method-II the overlapping samples are placed at the end of the section. Each section of longer sequence is converted to 3-sample sequence, using the samples of original longer sequence as shown below.

It can be observed that the last sample of $x_1(n)$ is placed as overlapping sample at the end of $x_2(n)$. The last sample of $x_2(n)$ is placed as overlapping sample at the end of $x_3(n)$. The last sample $x_3(n)$ is placed as overlapping sample at the end of $x_4(n)$. Since there is no previous section for $x_1(n)$, the overlapping sample of $x_1(n)$ is taken as zero.

| | | | |
|---|---|---|---|
| $x_1(n) = 1\ ;\ n = 0$ | $x_2(n) = 2\ ;\ n = 2$ | $x_3(n) = 3\ ;\ n = 4$ | $x_4(n) = 4\ ;\ n = 6$ |
| $= -1\ ; n = 1$ | $= -2\ ; n = 3$ | $= -3\ ; n = 5$ | $= -4\ ; n = 7$ |
| $= 0\ ;\ n = 2$ | $= -1\ ; n = 4$ | $= -2\ ; n = 6$ | $= -3\ ; n = 8$ |

Now perform circular convolution of each section with $h(n)$. The output sequence obtained from circular convolution will have three samples.

The circular convolution of each section is performed by tabular method as shown below.

Here $h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $y_1(m)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $y_2(m)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $y_3(m)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $y_4(m)$ will start at $n = n_4 + n_h = 6 + 0 = 6$.

**Convolution of section 1 :**

| $n$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x_1(n)$ | | | 1 | $-1$ | 0 |
| $h(n)$ | | | $-1$ | 1 | 0 |
| $h(-n) = h_0(n)$ | 0 | 1 | $-1$ | 0 | 1 |
| $h(1-n) = h_1(n)$ | | 0 | 1 | $-1$ | 0 |
| $h(2-n) = h_2(n)$ | | | 0 | 1 | $-1$ |

$$y_1(m) = x_1(n) \circledast h(n)$$

$$= \sum_{n=0}^{N-1} x_1(n)\, h(m-n)$$

$$y_1(m) = \sum_{n=0}^{N-1} x_1(n)\, h_m(n)\ ,\ m = 0, 1, 2$$

**when m = 0** $\quad y_1(0) = \sum x_1(n)\, h_0(n) = -1 + 0 + 0 = -1.$

**when m = 1** $\quad y_1(1) = \sum x_1(n)\, h_1(n) = 1 + 1 + 0 = 2$

**when m = 2** $\quad y_1(2) = \sum x_1(n)\, h_2(n) = 0 - 1 + 0 = -1,$

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# 6.8 FAST FOURIER TRANSFORMS ALGORITHMS

## 6.8.1 Introduction

The Discrete Fourier transform plays an important role in many applications of digital signal processing like linear filtering, correlation analysis and spectrum analysis. A major reason for its application is the existence of efficient algorithm for computation of DFT. The term Fast Fourier Transform(FFT) usually refers to a class of algorithms for efficiently computing the DFT.

The basic idea behind all fast algorithms for computing the FFT is to decompose successively the N-point DFT computation into computation of smaller size DFTs and to take the advantage of periodicity and symmetry of the complex number $W_N^{kn}$. This basic approach leads to a family of an efficient computational algorithms known collectively as FFT algorithms.

Consider the DFT of a finite length of sequence

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(6.40)}$$

Similarly, the IDFT becomes,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_n^{-kn}, \quad n = 0, 1, \ldots\ldots\ldots\ldots N-1 \qquad \text{...(6.41)}$$

The sequence $x(n)$ is also assumed to be a complex value.

We observe from eqn. (6.40), that direct computation of $X(k)$ requires following number of arithmetic operations.

(1) N complex multiplications for each value of $k$.

(2) $(N - 1)$ complex additions for each value of $k$.

(3) $N^2$ complex multiplications, for N values of $k$.

(4) $N(N - 1)$ complex additions, for N values of $k$.

For a complex valued sequence $x(n)$ of N-point, the DFT can be expressed as,

$$X_R(k) = \sum_{n=0}^{N-1} \left[ x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N} \right] \qquad \text{...(6.42)}$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[ x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N} \right] \qquad \text{...(6.43)}$$

$$\therefore \quad X(k) = X_R(k) + jX_I(k). \qquad \text{...(6.44)}$$

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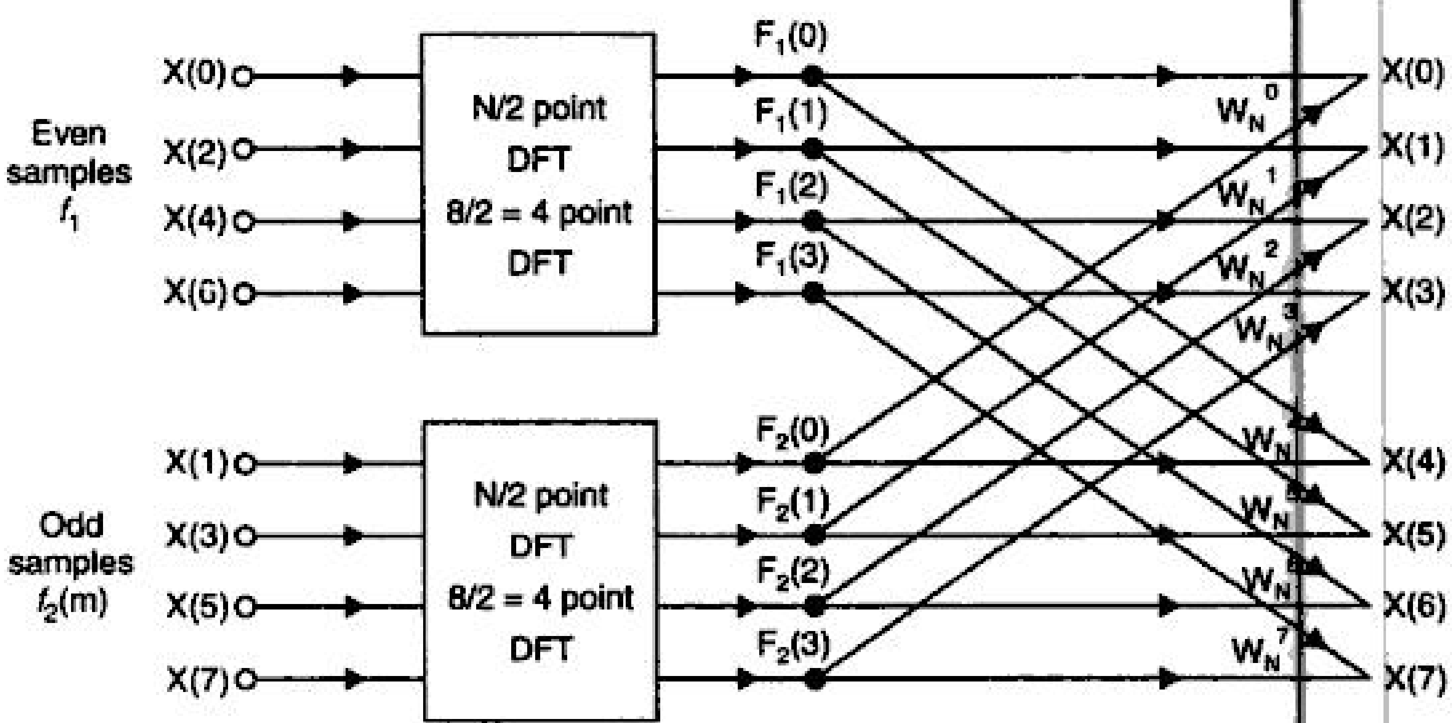
Thus, we have

**Table 2**

$$
\begin{aligned}
X(0) &= F_1(0) + W_N^0\, F_2(0)\\
X(1) &= F_1(1) + W_N^1\, F_2(1)\\
X(2) &= F_1(2) + W_N^2\, F_2(2)\\
X(3) &= F_1(3) + W_N^3\, F_2(3)\\
X(4) &= F_1(0) + W_N^4\, F_2(0)\\
X(5) &= F_1(1) + W_N^5\, F_2(1)\\
X(6) &= F_1(2) + W_N^6\, F_2(2)\\
X(7) &= F_1(3) + W_N^7\, F_2(3).
\end{aligned}
$$

Figure shows this computation




In addition if we use,

$$
\begin{aligned}
W_N(k + N/2) &= e^{-j\frac{2\pi}{N}(k+N/2)}\\
&= e^{-j2\pi k/N} .\ e^{-j\pi} = -\,e^{-j2\pi k/N}
\end{aligned}
$$

$$\boxed{W_N^{(k+N/2)} = -\,W_N^k} \qquad \ldots(6.55)$$

We can have **Table 3**

$$
\begin{aligned}
W_N^4 &= -\,W_N^0\\
W_N^5 &= -\,W_N^1\\
W_N^6 &= -\,W_N^2\\
W_N^7 &= -\,W_N^3
\end{aligned}
$$

Using table 3 we can write,

$$X(k) = F_1(k) + W_N^k\, F_2(k). \qquad k = 0, 1, \ldots\ldots\ldots N-1 \qquad \ldots(6.56)$$

$$X(k + N/2) = F_1(k) - W_N^k\, F_2(k) \qquad k = 0\ , 1, \ldots\ldots\ldots N-1. \qquad \ldots(6.57)$$

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The relation between the samples of various sequences are given below :

**even**

| | |
|---|---|
| $V_{11}(0) = f_1(0) = x(0)$ | $V_{21}(0) = f_2(0) = x(1)$ |
| $V_{11}(1) = f_1(2) = x(4)$ | $V_{21}(1) = f_2(2) = x(5)$ |
| $V_{11}(n) = f_1(2n) = x(4n)$ | |

**odd**

| | |
|---|---|
| $V_{12}(0) = f_1(1) = x(2)$ | $V_{22}(0) = f_2(1) = x(3)$ |
| $V_{12}(1) = f_1(3) = x(6)$ | $V_{22}(1) = f_2(3) = x(7)$ |
| $V_{12}(n) = f_1(2n+1) = x(4n+2).$ | |

Similarly, $V_{21}(n) = f_2(2n) = x(4n+1)$

$V_{22}(n) = f_2(2n+1) = x(4n+3).$

**The first stage of computation :**

In the first stage of computation the two point DFTs of the 2-point sequences are computed.

Let, $V_{11}(k) = \text{DFT}\{V_{11}(n)\}.$

$$V_{11}(k) = \sum_{n=0,1} v_{11}(n)\, W_{N/4}^{nk} \quad \text{for} \quad k = 0, 1.$$

**when k = 0**

$$V_{11}(0) = v_{11}(0)\, W_{N/4}^{0} + v_{11}(1)\, W_{N/4}^{0}$$
$$= v_{11}(0) + v_{11}(1)$$

$$\boxed{V_{11}(0) = x(0) + x(4)}$$

**when k = 1**

$$V_{11}(k) = V_{11}(1) = v_{11}(0)\, W_{N/4}^{0} + v_{11}(1)\, W_{N/4}^{1}$$
$$= v_{11}(0) - v_{11}(1)$$

$$\boxed{V_{11}(1) = x(0) - x(4)}$$

Let $V_{12}(k) = \text{DFT}\{v_{12}(n)\}.$

$$V_{12}(k) = \sum_{n=0,1} v_{12}(n)\, W_{N/4}^{nk}, \text{ for } k = 0, 1.$$

**when k = 0**

$$V_{12}(k) = V_{12}(0) = v_{12}(0)\, W_{N/4}^{0} + v_{12}(1)\, W_{N/4}^{0}$$
$$= v_{12}(0) + v_{12}(1)$$

$$\boxed{V_{12}(0) = x(2) + x(6)}$$

**when k = 1**

$$V_{12}(1) = v_{12}(0)\, W_{N/4}^{0} + v_{12}(1)\, W_{N/4}^{1}$$
$$= v_{12}(0) - v_{12}(1)$$

$$\boxed{V_{12}(1) = x(2) - x(6)}$$

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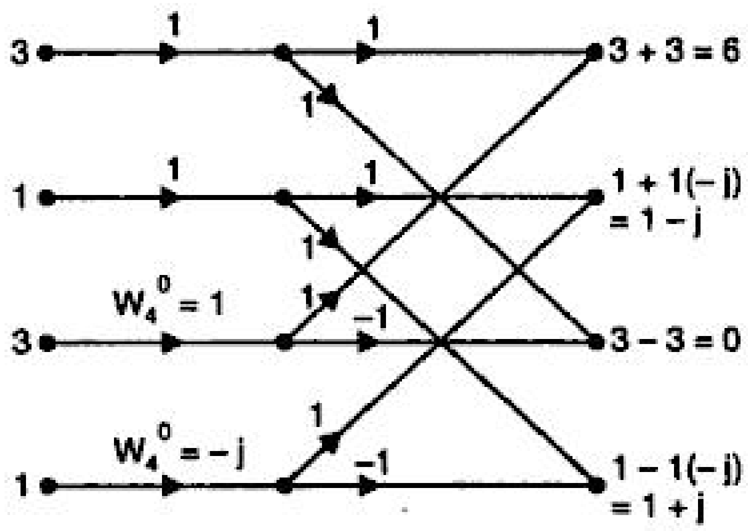
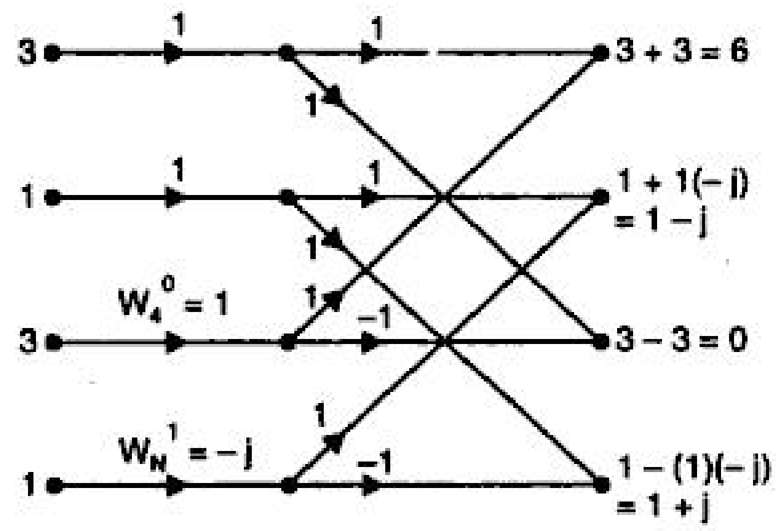
The butterfly computations are shown below :




Output DFT sequence = {6, 1 – $j$, 0, 1 + $j$, 6, 1 – $j$, 0, 1 + $j$}.

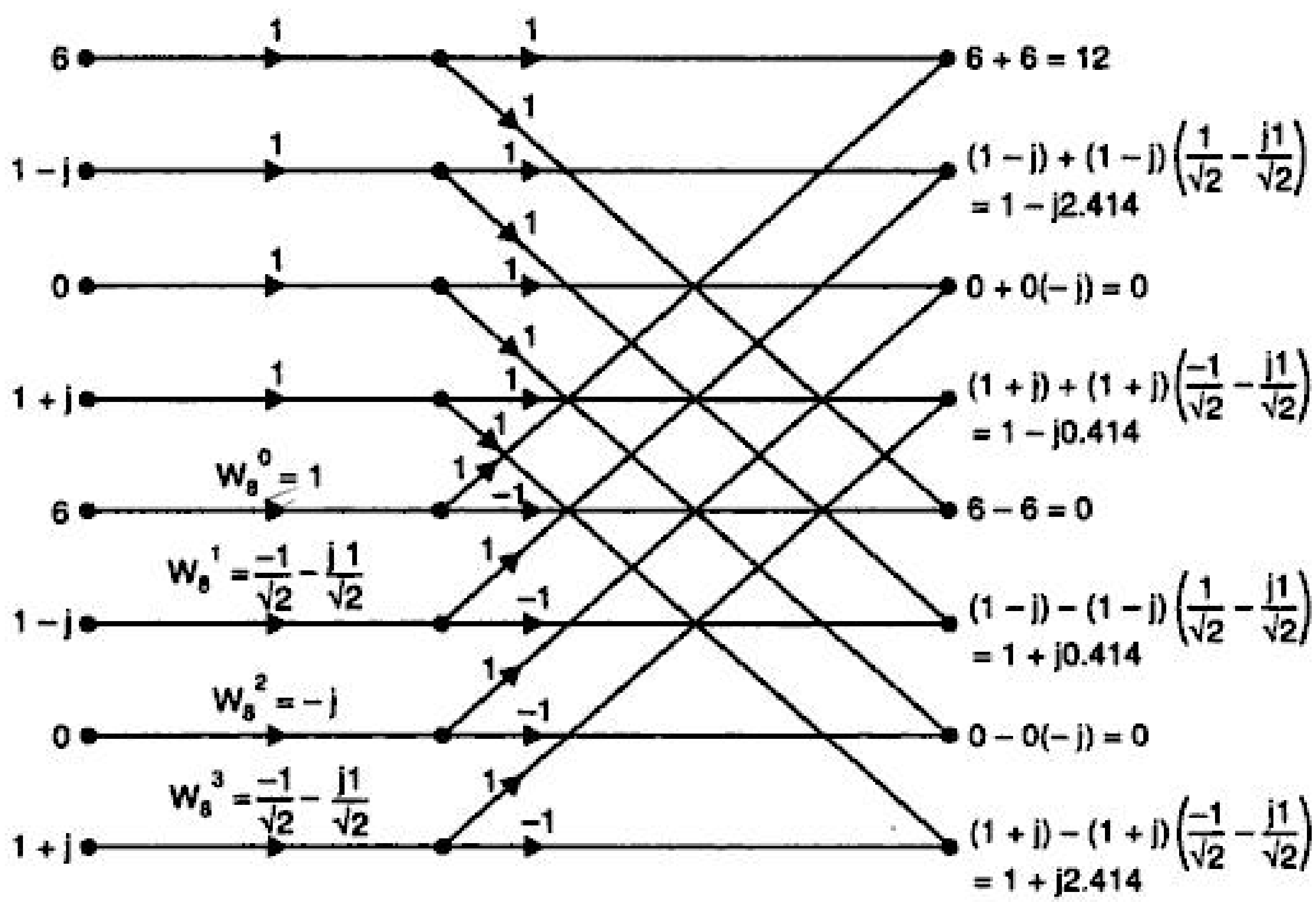
**Third stage computation :**

Input DFT sequence = {6, 1 – $j$, 0, 1 + $j$, 6, 1 – $j$, 0, 1 + $j$ }.

The phase factors involved in third stage computation are $W_8^0$, $W_8^1$, $W_8^2$ and $W_8^3$.

$$W_8^0 = 1,\ W_8^1 = j\frac{1}{\sqrt{2}},\ W_8^2 = -j,\ W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}.$$

The butterfly computations of third stage are shown below :




The o/p DFT sequence = {12, 1 – $j$2.414, 0, 1 – $j$0.414, 0, 1 + $j$0.414, 0, 1 + $j$2.414}

$\therefore$ DFT $x(n) = X(k)$ = {12, 1 – $j$2.414, 0, 1 – $j$0.414, 0, 1 + $j$0.414, 0, 1 + $j$2. 414}.

$$|X(k)| = \{12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61\}.$$

$$\angle X(k) = \{0, -0.37\pi, 0, -0.12\pi, 0, 0.12\pi, 0, 0.37\pi\}$$

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Using equation (6.80) the eqns. (6.75) and (6.77) can be written as,

$$\therefore \qquad G_1(2k) = \sum_{n=0}^{N/4-1} d_{11}(n)\, W_{N/4}^{kn} = D_{11}(k). \qquad \ldots(6.82)$$

$$G_1(2k+1) = \sum_{n=0}^{N/4-1} d_{12}(n)\, W_{N/4}^{kn} = D_{12}(k) \qquad \ldots(6.83)$$

Using eqn. (6.81) the eqns. (6.78) and (6.79) can be written as,

$$G_2(2k) = \sum_{n=0}^{N/4-1} d_{21}(n)\, W_{N/4}^{kn} = D_{21}(k) \qquad \ldots(6.84)$$

$$G_2(2k+1) = \sum_{n=0}^{N/4-1} d_{22}(n)\, W_{N/4}^{kn} = D_{22}(k) \qquad \ldots(6.85)$$

where, $D_{11}(k)$, $D_{12}(k)$, $D_{21}(k)$ and $D_{22}(k)$ are N/4 point DFTs of $d_{11}(n)$, $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ respectively.

**Butterfly computation involves the following operations**

(*i*) In each computation two complex numbers $a$ and $b$ are considered.

(*ii*) The sum of the two complex number is computed which forms a new complex number A.

(*iii*) Then subtract the complex number $b$ from $a$ to get the term $(a - b)$. The difference term $(a - b)$ is multiplied with the phase factor $W_N^n$ to form a new complex number B.

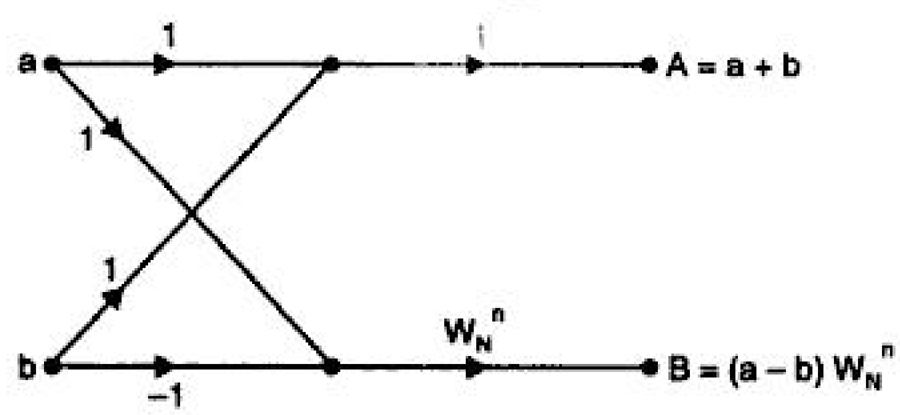


Fig. 6.19

### 6.11.1 The 8-Point DFT Using Radix-2 DIF FFT

Let $x(n)$ be an 8-point sequence.

**First stage of computation.** In the first stage of computation, two numbers of 4-point sequences $g_1(n)$ and $g_2(n)$ are obtained using equations (6.69) and (6.70) respectively.

$$g_1(n) = x(n) + x(n + N/2)$$

$$g_1(n) = x(n) + x(n + 4) \quad \text{for } n = 0, 1, 2, 3.$$

when $n = 0$, $\quad g_1(0) = x(0) + x(4)$

when $n = 1$, $\quad g_1(1) = x(1) + x(5)$

when $n = 2$, $\quad g_1(2) = x(2) + x(6)$

when $n = 3$, $\quad g_1(3) = x(3) + x(7)$

$$g_2(n) = [x(n) - x(n + N/2)]\, N_N^n$$

$$g_2(n) = [x(n) - x(n + 4)]\, W_8^n \quad \text{for } n = 0, 1, 2, 3$$

when $n = 0$, $\quad g_2(0) = [x(0) - x(4)]W_8^0$

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**Problem 19.** *An 8-point sequence is given by x(n) = {2, 2, 2, 2, 1, 1, 1, 1}. Compute 8-point DFT of x(n) by radix-2 DIF FFT.*

**Sol.** For 8-point DFT by radix-2 FFT we require 3-stages of computation with 4-butterfly computation in each stage.

The given sequence is the i/p to the first stage. For other stages of computations, the o/p of the previous stage will be the i/p for the current stage.

**First stage of computation :**

The input sequence = { 2, 2, 2, 2, 1, 1, 1, 1}

The phase factors involved in first stage of computations are $W_8^0$, $W_8^1$, $W_8^2$ and $W_8^3$,

$$W_8^0 = 1,\ W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}},\ W_8^2 = -j,\ W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}.$$

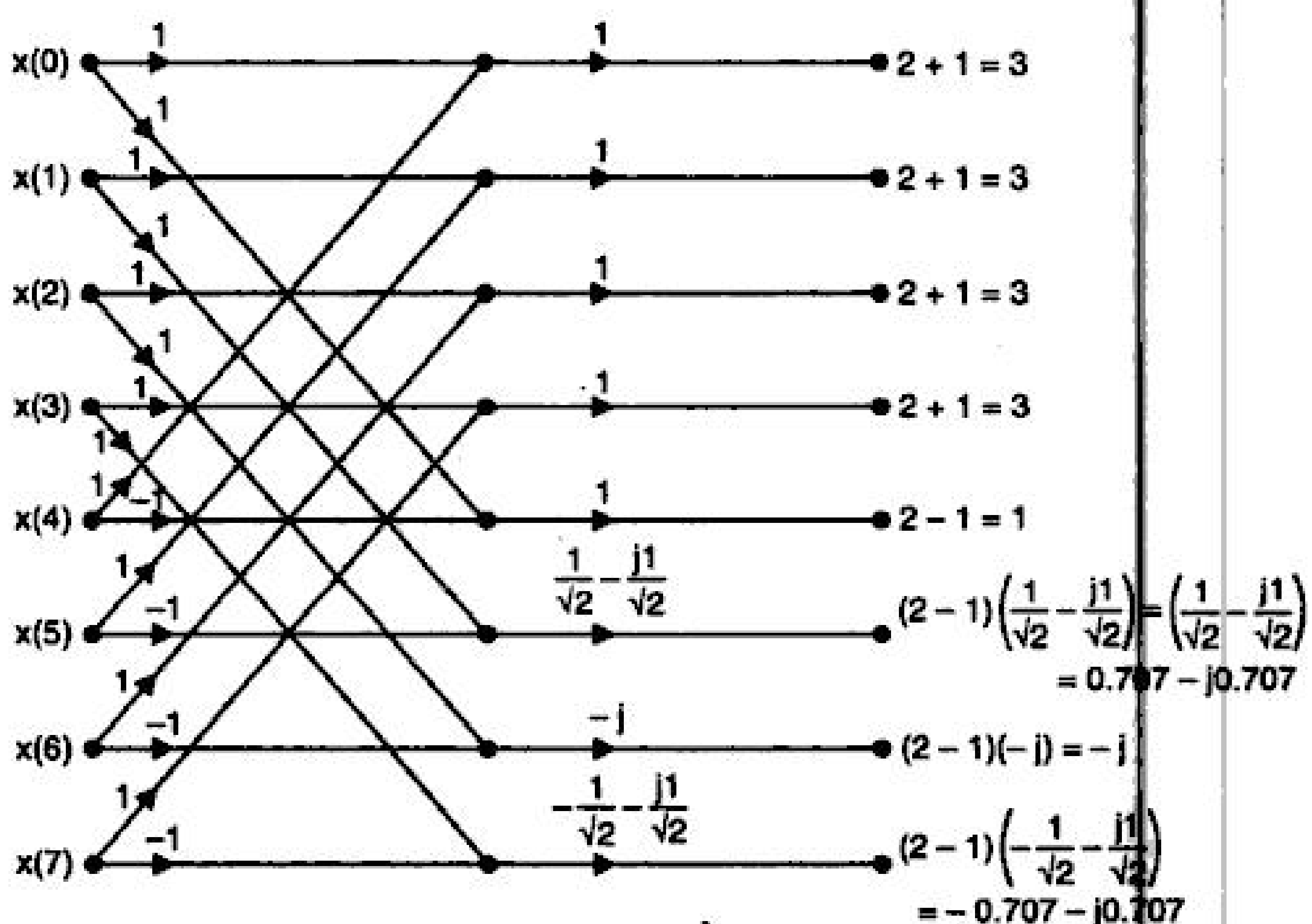
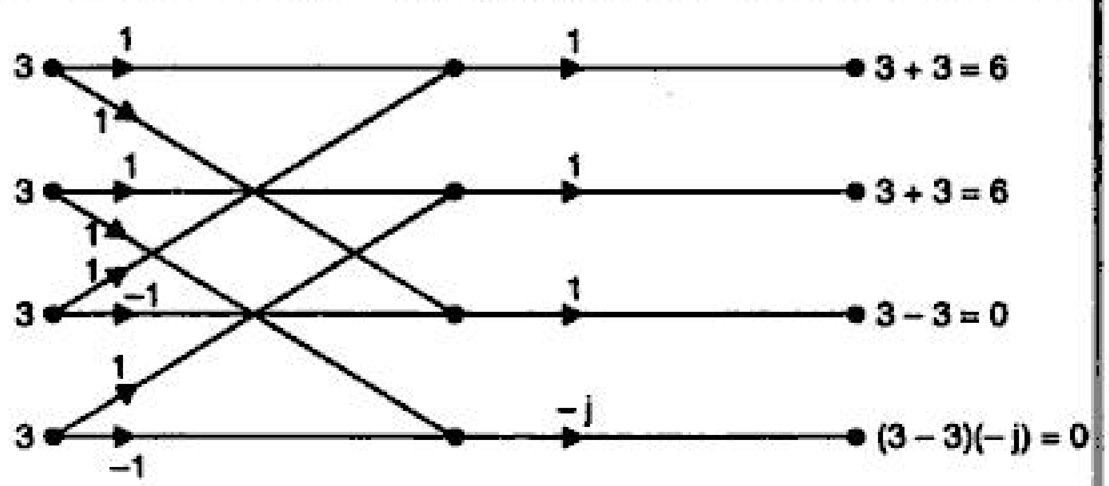



Fig. 6.24

$$\left.\begin{array}{r}\text{The output sequence of first}\\ \text{stage of computation}\end{array}\right\} = \{3, 3, 3, 3, 0.707, -j\,0.707, -j, -0.707 -j\,0.707\}$$

**Second stage of computation :**

The i/p sequence for 2nd stage = {3, 3, 3, 3, 0.707, $-j$ 0.707, $-j$, $-$ 0.707, $-j$ 0.707}



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Substituting eqn. (7.9) in eqn. (7.8) yields,

$$F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X[j(\Omega - m\Omega_0)].$$

$$\boxed{F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left[j\left(\Omega - \frac{2\pi m}{T}\right)\right]} \qquad \text{...(7.10)}$$

Thus the fourier transform of the sampled signal is given by an infinite sum of shifted replicas of the Fourier transform of the original signal.

Now consider the signal $x(t)$ is band limited to $f_m$. That is the highest frequency component of $x(t)$ is $f_m$. Then,

$$x(j\Omega) = 0 \qquad \text{for } |\Omega| > \Omega_m \ (\Omega_m = 2\pi f_m)$$

In eqn. (7.10) the term $X\left(j\left(\Omega - \frac{2\pi m}{T}\right)\right)$ is the shifting of $X(j\Omega)$ from $\Omega = 0$ to $\Omega = \frac{2\pi m}{T}$.

Hence $X_s(j\Omega)$ is the sum of shifted replicas of $\frac{X(j\Omega)}{T}$ centering at $\frac{2\pi m}{T}$, $m = 0, \pm 1, \pm 2$.

The Fig. [7.5(*a*)] shows the plot of $\frac{X(j\Omega)}{T}$ for different value of $\frac{\pi}{T}$.

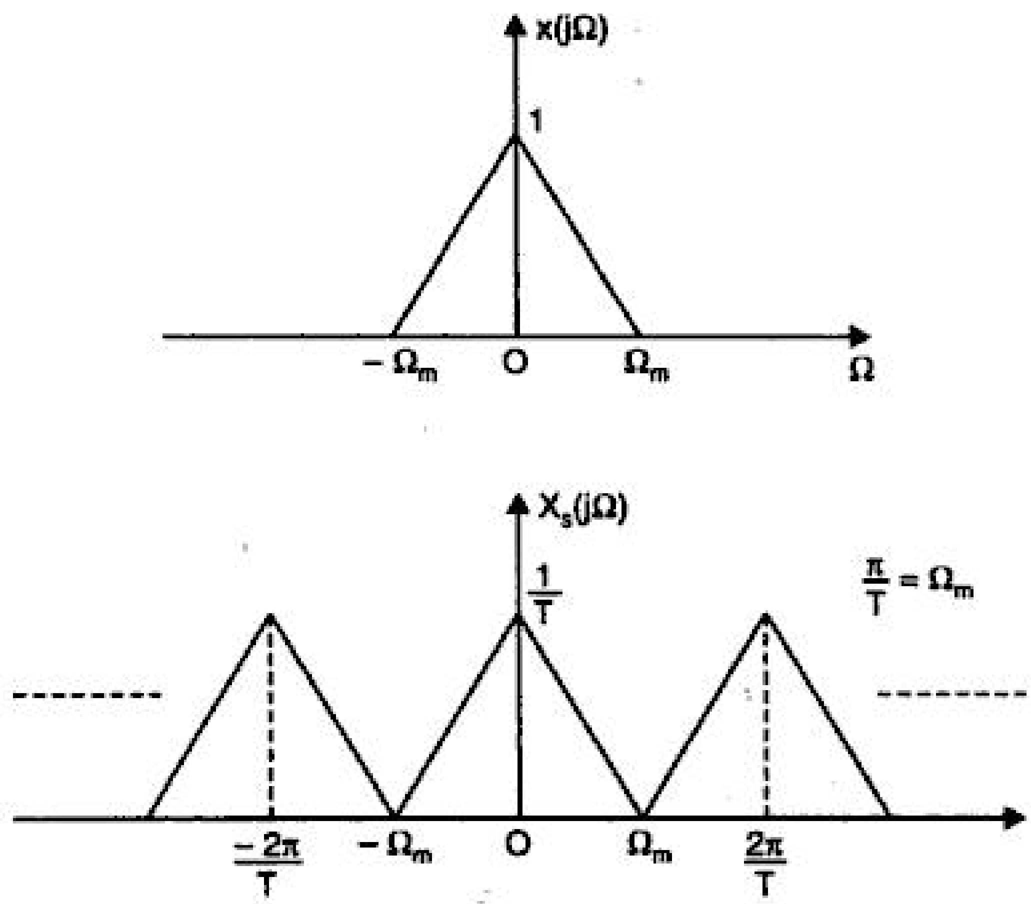


Fig. 7.5 (*a*)

We see that if $\frac{\pi}{T} > \Omega_m$, the replicas will not overlap as in Fig. [7.5(*b*)] and as a result the frequency spectrum of $TX_s(j\Omega)$ in frequency range $\left(-\frac{\pi}{T}, \frac{\pi}{T}\right)$ is identical to $X(j\Omega)$. The frequency spectrum $X(j\Omega)$ can be recovered from $X_s(j\Omega)$ by using a low pass filter which has sharp cut off at $\Omega = \frac{\pi}{T}$. The same explanation holds for $\frac{\pi}{T} = \Omega_m$.

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The sampling rate, $\quad f_s = 8\text{Hz},$

$$\Omega_s = 2\pi f_s = 2\pi(8) = 16\pi$$

$$\Omega_m = 2\pi f_m = 10\pi$$

$$f_m = 5\text{Hz}.$$

Nyquist rate $\quad = 2f_m = 10 \text{ Hz}.$

The sampling rate is less than Nyquist rate. So, the original signal cannot be recovered from the samples.

The frequency spectra of sampled $x(t)$ is given by,

$$X_s(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left[j\left(\Omega - \frac{2\pi n}{T}\right)\right] \qquad \left[\because \Omega_s = \frac{2\pi}{T}\right]$$

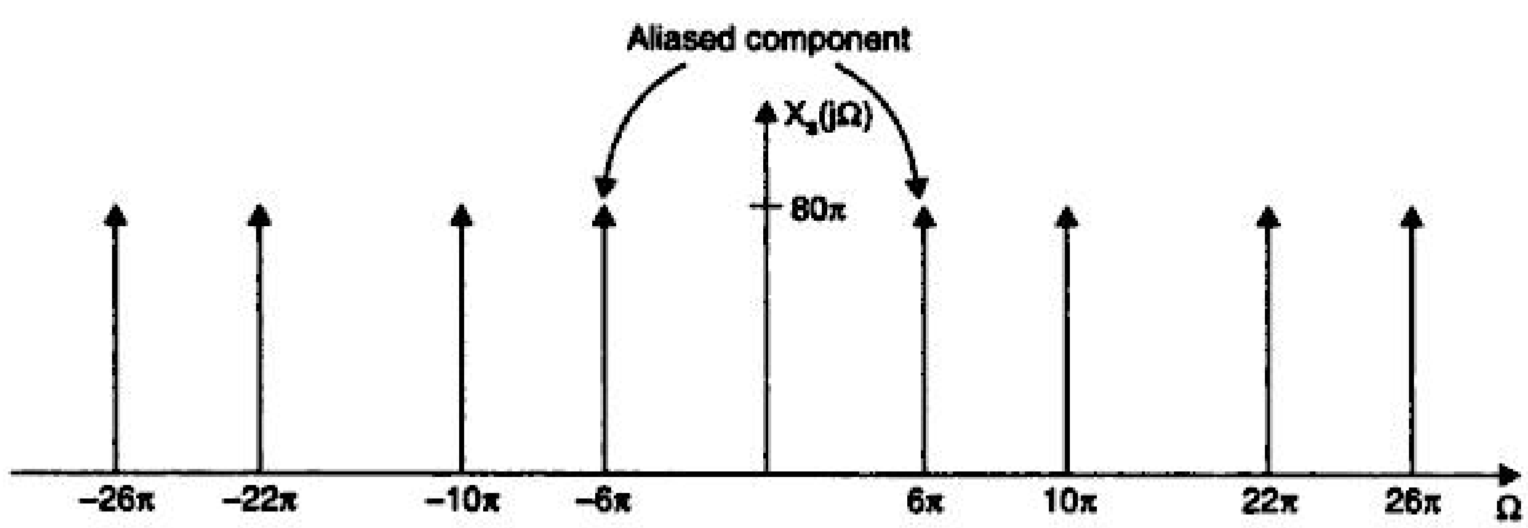
$$\therefore \quad X_s(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} 10\{\pi[\delta(\Omega + 10\pi - n\Omega_s)] + \pi[\delta(\Omega - 10\pi - n\Omega_s)]\}$$

where, $\quad \Omega_s = \frac{2\pi}{T} \quad \text{and} \quad \frac{1}{T} = f_s = 8$

$$\therefore \quad X_s(j\Omega) = 8\sum_{n=-\infty}^{\infty} 10\pi\{\delta(\Omega + 10\pi - 16n\pi) + \delta(\Omega - 10\pi - 16\pi n)\}$$

$$= 80\pi \sum_{n=-\infty}^{\infty} [\delta(\Omega + 10\pi - 16n\pi) + \delta(\Omega - 10\pi - 16\pi n)]$$

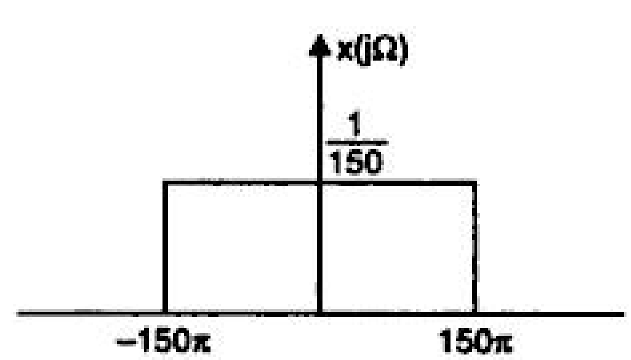
The plot of amplitude spectrum for $|\Omega| \leq 30\pi$ is shown in figure.




**Problem 3.** *A signal $x(t) = \text{sinc}(150\pi t)$ is sampled at a rate of (a) 100 Hz (b) 200 Hz (c) 300 Hz. For each of these three cases, explain if you can recover the signal from the sampled signal.*

**Sol.** Given $\quad x(t) = \text{sinc}(150\,\pi t).$

The spectrum of the signal $x(t)$ is a rectangular pulse with a band width (maximum frequency component) of $150\pi$ rad/sec is shown in figure.



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$$H(s)\,H(-s) = \frac{1}{1+(-1)^N (s)^{2N}} = \frac{1}{1+(-s^2)^N} \qquad \text{...(7.38)}$$

The above relations tell us that this function has poles in the LHP as well as in the RHP, because of the presence of two factors H($s$) and H($-s$).

If H($s$) has roots in the LHP then H($-s$) has the corresponding roots in the RHP. These roots we can set by equating the denominator to zero.

*i.e.,* $$1 + (-s^2)^N = 0 \qquad \text{...(7.39)}$$

**For N odd,**

Eqn. (7.39) can be written as,

$$s^{2N} = 1 = e^{j2\pi k}. \qquad \text{...(7.40)}$$

Now the roots of eqn. (7.39) can be found as,

$$s_k = e^{j\pi k/N} \quad , K = 1, 2, \ldots\ldots\ldots 2N.$$

**For N even,**

Eqn. (7.39) reduces to, $s^{2N} = -1 = e^{j(2K-1)\pi}$, which gives,

$$s_k = e^{j(2k-1)\pi/2N} \quad \text{for } K = 1, 2, \ldots\ldots, 2N.$$

**For N = 3,**

Eqn. (7.39) becomes,

$$s^6 = 1\ldots..$$

$$\therefore \quad K = 1, 2, \ldots\ldots\ldots 2 \times 3.$$

$$= 1, 2, \ldots\ldots\ldots 6.$$

Now the roots of eqn. (7.39) can be found as,

$$s_1 = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j\,0.866$$

$$s_2 = e^{j2\pi/3} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j\,0.866$$

$$s_3 = e^{j3\pi/3} = \cos\pi + j\sin\pi = -1$$

$$s_4 = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j\,0.866$$

$$s_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j\,0.866$$

$$s_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1.$$

All the above poles in the $s$-plane as shown in Fig. (7.18). It is found that the angular separation between the poles is given by $\frac{360^\circ}{2N}$ ; which in this case is equal to 60° and all the poles lie on a unit circle.

**Stability.** To ensure stability, considering only the poles that lie in the left half of the $s$-plane we can write the denominator of the transfer function H($s$) as,

$$(s + 1)\{(s + 0.5)^2 + (0.866)^2\} = (s + 1)(s^2 + s + 1).$$

Therefore, the transfer function of a 3rd order Butterworth filter for cut off frequency $\Omega_c$ = 1 r/s is,

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}.$$

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**Problem 6.** *Given the specifications $\alpha_p = 1\ dB$ ; $\alpha_s = 30\ dB$, $\Omega_p = 200\ r/s$, $\Omega_s = 600\ r/s$. Determine the order of the filter.*

**Sol.**
$$N \geq \frac{\log A}{\log (1/k)}.$$

**To find A :**
$$A = \frac{\lambda}{\varepsilon} = \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{0.5}$$
$$= \left[\frac{10^3 - 1}{10^{0.1} - 1}\right]^{0.5} = 62.115$$

**To find k :**
$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

$\therefore$
$$N \geq \frac{\log A}{\log 1/k}$$
$$\geq \frac{\log (62.115)}{\log (3)} = 3.758.$$

Rounding off N to the next integer, we get,

$$\boxed{N = 4}$$

**Problem 7.** Prove that $\Omega_c = \dfrac{\Omega_p}{[10^{0.1\alpha_p} - 1]^{1/2N}}$.

**Sol.** The magnitude square function of Butterworth analog lowpass filter is given by,

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad ...(1)$$

We know,
$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}} \qquad ...(2)$$

Comparing in eqn. (1) and (2), we get

$$1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N}.$$

Simplifying the above eqn. by substituting

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}, \quad \text{we obtain}$$
$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p} - 1.$$

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**Low pass-to-band pass transformation.** Consider a band pass filter with lower band edge frequency $\Omega_c$ and upper band edge frequency $\Omega_u$. The transformation for converting lowpass analog filter, with band edge frequency $\Omega_p$, into bandpass filter can be accomplished by first converting the lowpass filter into another lowpass filter with band edge frequency $\Omega_p' = 1$ and then performing transformation,

$$S \to \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} \qquad \text{...(7.87)}$$

we can also obtain the same result in a single step by means of the transformation

$$S \to \Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} \qquad \text{...(7.88)}$$

Thus we have
$$H_{bp}(s) = H_p\left(\Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}\right) \qquad \text{...(7.89)}$$

**Low pass-to-band stop transformation.** To convert a lowpass analog filter with band edge frequency $\Omega_p$ , into the band stop filter, the tansformation is simply the inverse of eqn. (7.87) with addition factor $\Omega_p$ serving to normalised for the band edge frequency of the lowpass filter.

Thus we have
$$S \to \Omega_p \frac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_u \Omega_L} \qquad \text{...(7.90)}$$

which gives
$$H_{bs}(S) = H_p\left(\Omega_p \frac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_u \Omega_L}\right) \qquad \text{...(7.91)}$$

All the above four transformations are summarised in Table (7.2)

**Table 7.2. Frequency transformations for analog filter**

| *Filter type* | *Transformation* | *Band edge frequency of new filter* |
|---|---|---|
| Low pass | $S \to \frac{\Omega_p}{\Omega_p'} s$ | $\Omega_p'$ |
| High pass | $S \to \frac{\Omega_p \Omega_p'}{s}$ | $\Omega_p'$ |
| Band pass | $S \to \Omega_p \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}$ | $\Omega_L, \Omega_u$ |
| Band stop | $S \to \Omega_p \frac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_L \Omega_u}$ | $\Omega_L, \Omega_u$. |

**Problem 9.** *Transform the single pole lowpass Butterworth filter with system function,*

$$H_a(s) = \frac{\Omega_p}{s + \Omega_p}$$

*into a high pass filter with band edge frequency $\Omega_p'$ and band stop filter with upper and lower band edge frequencies $\Omega_u$ and $\Omega_L$ respectively.*

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We know, $$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_p}{\varepsilon^{1/N}}$$

$$\Omega_c = \frac{0.1\pi}{(0.882)^{1/2}} = \frac{0.1\pi}{0.939} = 0.106\pi$$

H($s$) for $\Omega_c = 0.106\,\pi$ can be obtained by substituting $S \to \frac{s}{0.106\,\pi}$ in H($s$).

$$H(s) = \frac{1}{\left(\frac{s}{0.106\,\pi}\right)^2 + \sqrt{2}\left(\frac{s}{0.106\,\pi}\right)+1}$$

$$= \frac{1}{\frac{s^2}{0.1107} + 4.248s + 1}$$

$$H(s) = \frac{1}{9s^2 + 4.248s + 1}.$$

**Problem 13.** *Obtain an analog Chebyshev filter transfer fuction that satisfies the constraints*

$$\frac{1}{\sqrt{2}} \le |\,H(j\Omega)\,| \le 1 \quad ; 0 \le \Omega \le 2$$

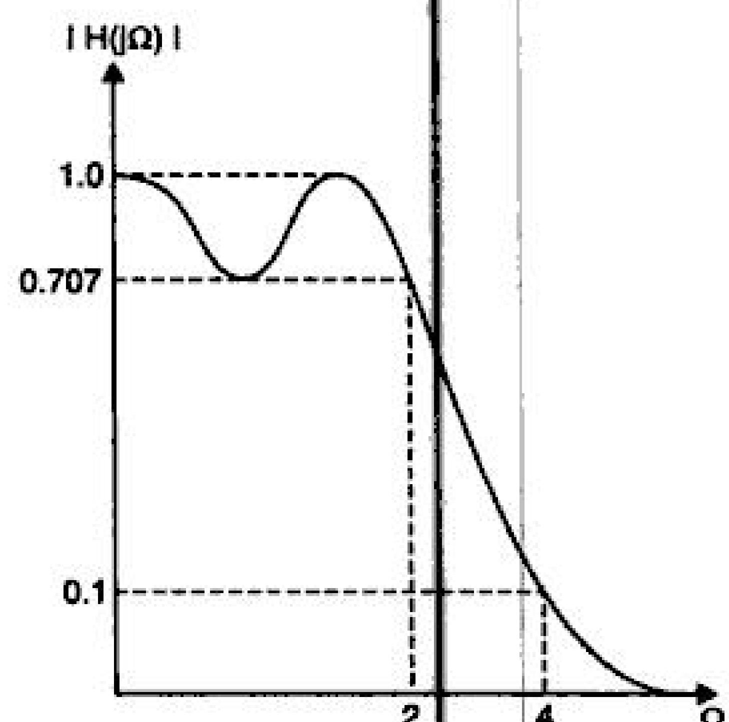
$$|\,H(j\Omega)\,| < 0.1 \quad ; \Omega \ge 4.$$

**Sol.** From the given data we can find that

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}.$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1$$

$$\Omega_p = 2 \quad \text{and} \quad \Omega_s = 4.$$




from which we can obtain $\varepsilon = 1$ and $\lambda = 9.95$.

We know,

$$N \ge \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1} 9.95}{\cosh^{-1} 2} = 2.269 \simeq 3.$$

Finding the values of $a$ and $b$,

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414.$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} - (2.414)^{-1/3}}{2}\right]$$

$$\boxed{a = 0.596}$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} + (2.414)^{-1/3}}{2}\right]$$

$$b = 2.087$$

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# 8 Digital Filter Structures

## 8.1 INTRODUCTION

In this chapter, we consider the realization problem of causal IIR and FIR transfer functions and outline realization methods based on both the time-domain and the transform-domain representations. Here, we describe the most commonly employed methods to implement the digital filter structure from either its difference equation, its unit sample response, or its $z$-transform. A structural representation using interconnected basic building blocks is the first step in the hardware or software implementation of an LTI digital filter. The structural representation provides the relations between some pertinent internal variable with the input and the output that in turn provide the keys to the implementation. There are various forms of the structural representations of a digital filter. We review in this chapter two such representations, and then describe some popular schemes for the realization of real causal IIR and FIR digital filters.

The digital filter structure determine directly from either the difference equation or the system function is called the direct form-I. An alternative view of the same equation results in the memory efficient structure, called the Direct Form-II. Digital filter structures as cascade, parallel and lattice structures are shown to have the advantages in terms of hardware implementation.

## 8.2 SYSTEM DESCRIBING EQUATIONS

The equation that describe the input and output relationship, in the time and $z$-transform domain, has been defined in the previous chapters. They are repeated here for reference.

The linear time invariant system is described by the difference equation of form,

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad \text{...(8.1)}$$

or equivalently,

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \qquad \text{...(8.2)}$$

where $a_k$ and $b_k$ are constants with $a_0 = 1$.

As we have seen by mean of $z$-transform such a class of LTI discrete-time systems are also characterized by rational system function.

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**Problem 4.** *Using first order section, obtain a cascade realization for*

$$H(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{10}z^{-1}\right)}.$$

**Sol.** The H(z) can be decomposed into the three section as $H(z) = H_1(z)\,H_2(z)\,H_3(z)$.

where, $$H_1(z) = \frac{1+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$H_2(z) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_3(z) = \frac{1}{1-\frac{1}{10}z^{-1}}.$$

The cascade form structure is shown in Fig. (8.13).

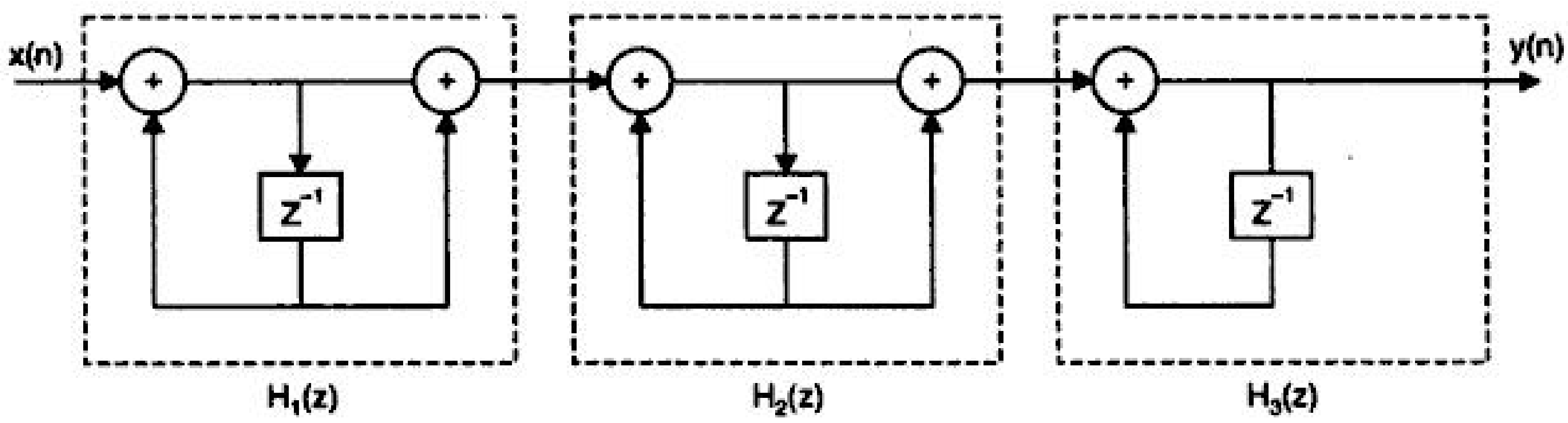


Fig. 8.13

**Problem 5.** *Obtain the cascade form structure for the system characterized by*

$$y(n) = \frac{5}{6}\,y(n-1) - \frac{1}{6}\,y(n-2) + x(n) + \frac{1}{8}x(n-1).$$

**Sol.** The system function of above system is given by

$$H(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}.$$

The above equation can be decomposed into two parts as

$$H(z) = \frac{1+\frac{1}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$H(z) = H_1(z)\,H_2(z)$$

where, $$H_1(z) = \frac{1+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-1}}\ ;\ H_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}.$$

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The resulting parallel from structure with first order sections is shown in Fig. 8.17 (*b*).

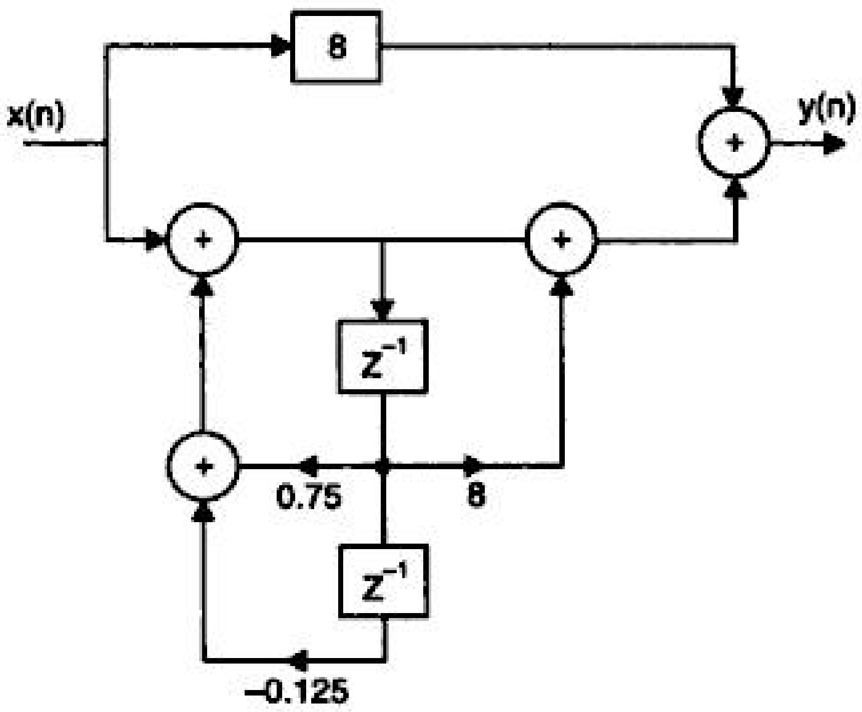


Fig. 8.17 (*a*)

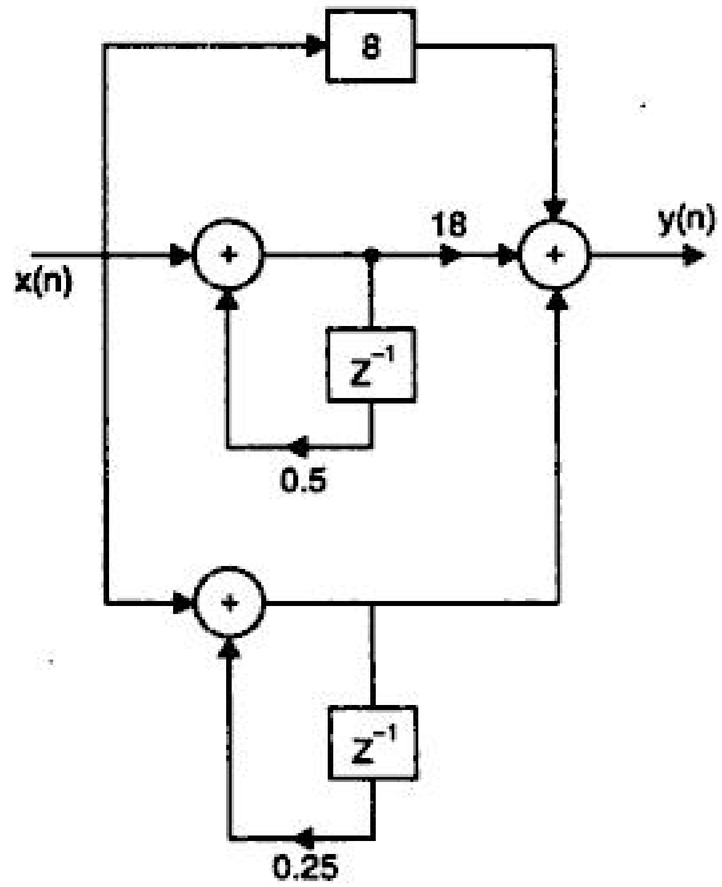


Fig. 8.17 (*b*)

**Problem 8.** *Obtain direct form I, direct form II, cascade and parallel structure for the system described by*

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252\,x(n-2).$$

**Sol.** We take *z*-transform both sides of difference eqn., we have

$$Y(z) = -0.1\,z^{-1}\,Y(z) + 0.72\,z^{-2}\,Y(2) + 0.7X(z) - 0.252z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.7(1-0.36z^{-2})}{1+0.1z^{-1}-0.72z^{-2}} = H_1(z)\,H_2(z)$$

$$H_1(z) = (1-0.36\,z^{-2}) \quad \text{and} \quad H_2(z) = \frac{0.7}{1+0.1z^{-1}-0.72z^{-2}}.$$

(*a*) **Direct form I**

Fig. (8.18) illustrates direct form I realization.

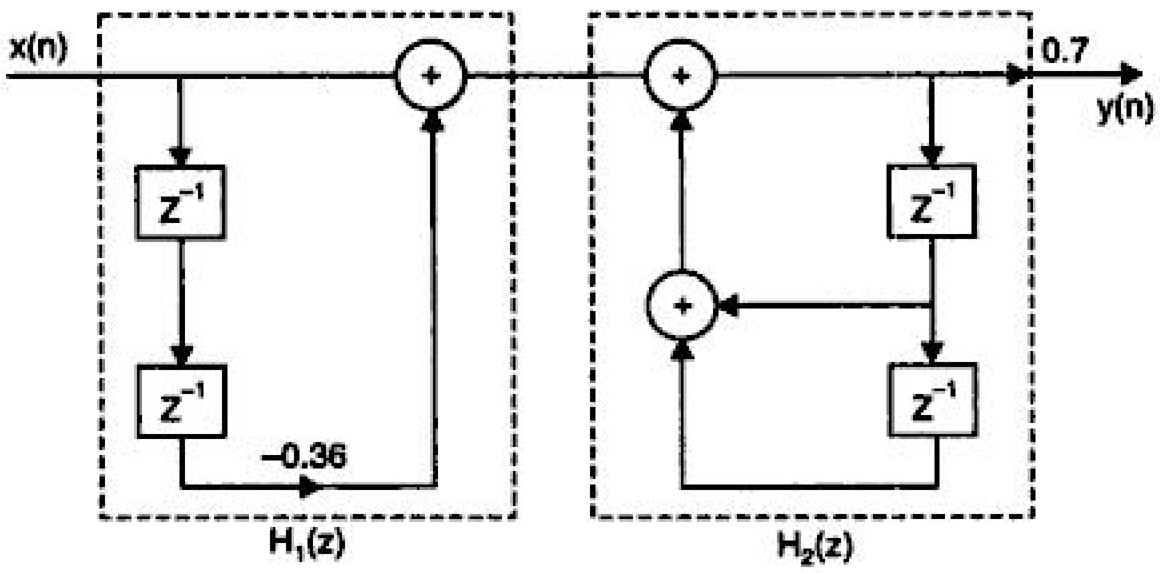


Fig. 8.18

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**Problem 10.** *Obtain cascade form realization of the following system function :*

$$H(z) = \left[1 + \frac{1}{4}z^{-1} + \frac{z^{-2}}{2}\right]\left[1 + \frac{1}{8}z^{-1} + \frac{z^{-2}}{2}\right].$$

**Sol.** The system function H(z) can be written as,

$$H(z) = H_1(z)\,H_2(z).$$

where, $$H_1(z) = 1 + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}.$$

$$H_2(z) = 1 + \frac{1}{8}z^{-1} + \frac{1}{2}z^{-2}.$$

The cascade form realization is shown in Fig. (8.26).

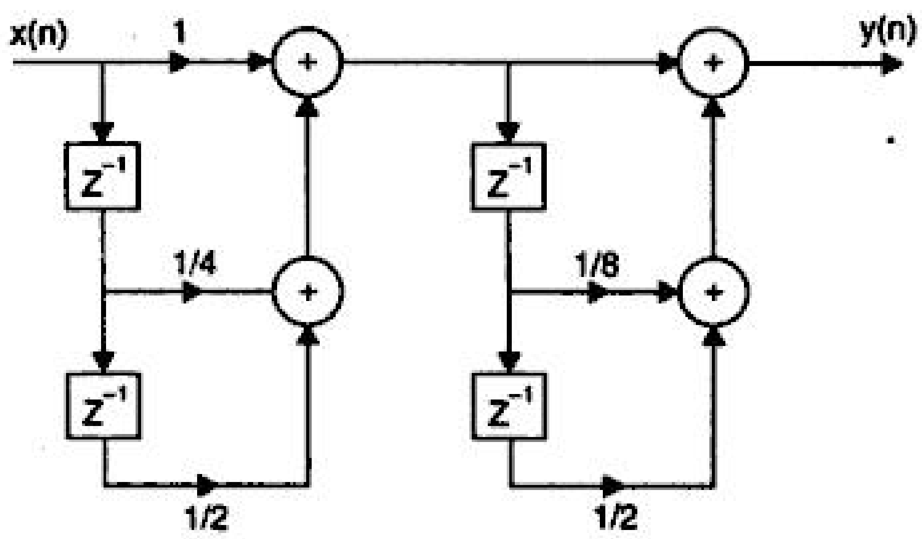


Fig. 8.26. Cascade form realization.

### 8.6.3 Linear Phase FIR Structure

When the FIR system has linear phase, the unit sample response of the system satisfies the symmetric condition is given by

$$h(n) = h(N - 1 - n) \qquad \text{...(8.32)}$$

or antisymmetric condition.

$$h(n) = -\,h(N - 1 - n) \qquad \text{...(8.33)}$$

Using the symmetry condition, we can write

$$H(z) = \sum_{n=0}^{N-1} h(n)\,z^{-n}.$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n)\left[z^{-n} + z^{-(N-1-n)}\right] \qquad \text{...(8.34)}$$

for N is even, and

$$H(z) = h\left(\frac{N-1}{2}\right)z^{-(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\,[z^{-n} + z^{-(N-1-n)}] \qquad \text{...(8.35)}$$

for N is odd.

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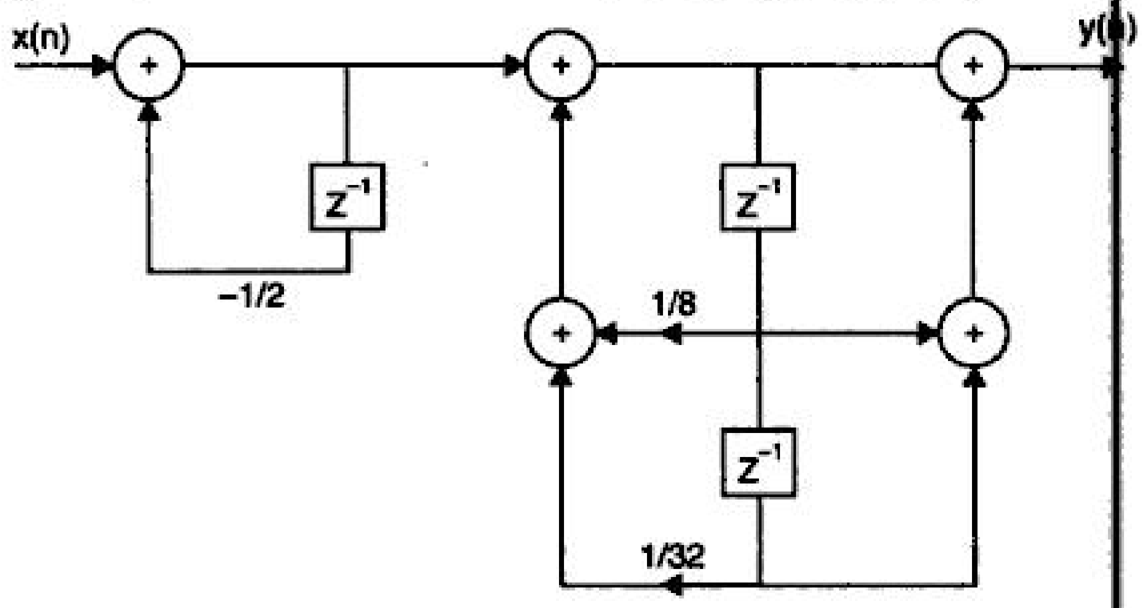
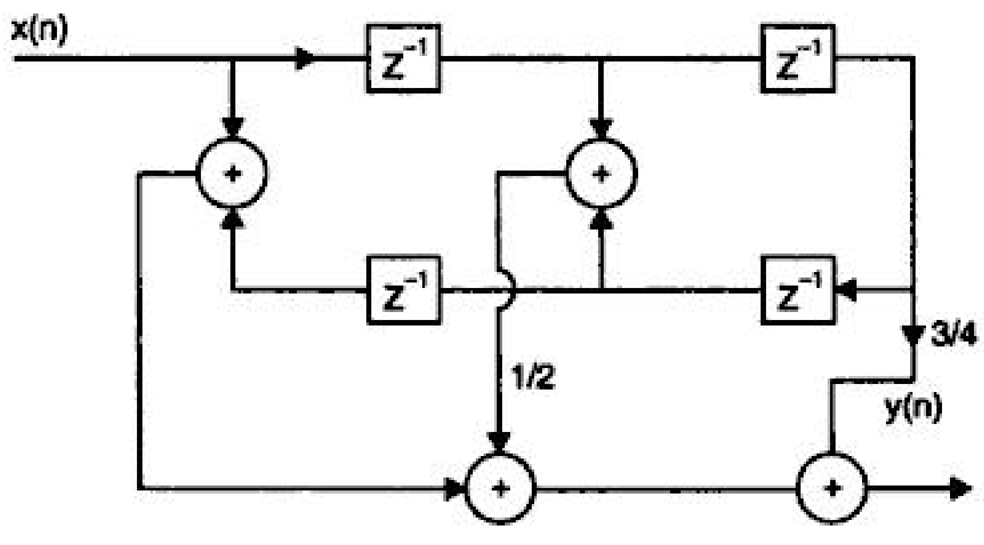



$$\left[\textbf{Ans. } H(z) = \frac{1}{\left(1+\frac{z^{-1}}{2}\right)} \frac{(2+z^{-1}+z^{-2})}{\left(1-\frac{z^{-1}}{8}-\frac{z^{-2}}{32}\right)}\right]$$

**4. Realize the following system functions using a minimum number of multipliers**

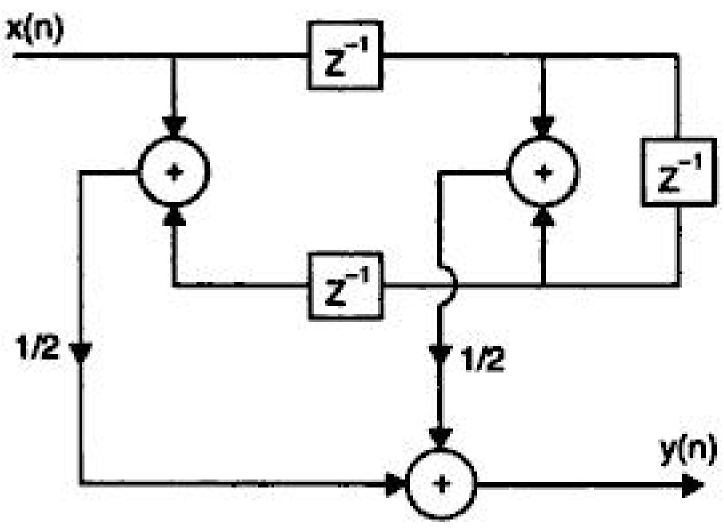
(*a*) $H(z) = 1 + \frac{1}{2} z^{-1} + \frac{3}{4} z^{-2} + \frac{1}{2} z^{-3} + z^{-4}$.

**Ans.**




(*b*) $H(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} + z^{-3}$.

**Ans.**



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(*d*) $$H(z) = \frac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$$

(*e*) $$H(z) = \frac{\left(1+\frac{1}{4}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$

(*f*) $$H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{8}z^{-1}\right)}$$

(*g*) $$H(z) = \frac{1+\frac{1}{2}z^{-1}}{\left(1-z^{-1}+\frac{1}{4}z^{-2}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}.$$

**14.** Find the digital network in direct and transposed form and get the transfer function of the system described by $y(n) = x(n) + 0.5\,x(n-1) + 0.4\,x(n-2) - 0.6\,y(n-1)$.

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## About the Book

This book deals with the analysis of Digital Signal Processing in a lucid and precise style. There are about 200 solved problems apart from exercises.

This book covers the latest syllabus prescribed by the Anna University for Electrical and Electronics engineering students. Exercise problems and review questions are included at the end of each chapter. All the above aspects should make this book extremely valuable for engineering students preparing for Anna University examinations as well as for practicing engineers.

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