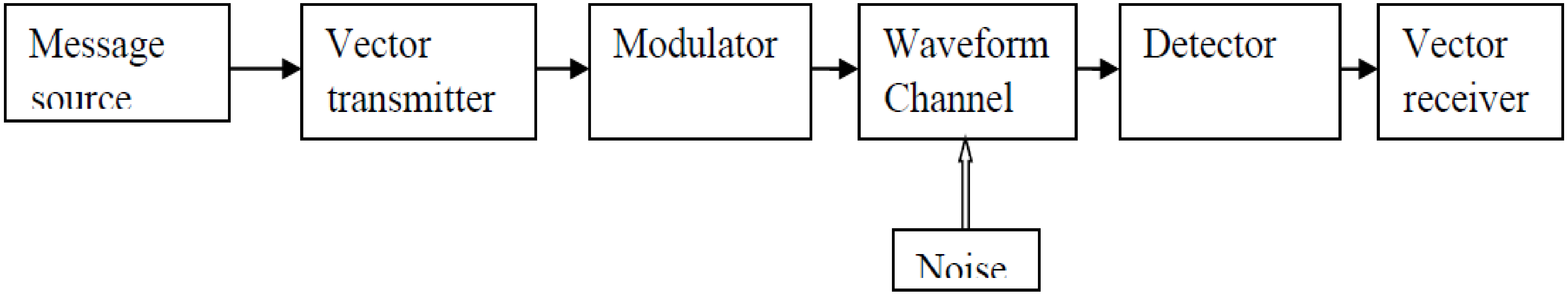
Module -3

Detection concepts and ISI

- **Detection concepts**:
  - Digital Communication block diagram, Model of DCS
  - Gram-Schmidt Orthogonalization procedure
  - Geometric interpretation of signals
  - Optimum receivers-Matched and correlator receivers
- **ISI:**
  - Inter Symbol Interference
  - Eye pattern
  - Adaptive equalization for data transmission

Model of DCS



A message source emits one symbol every T seconds, with the symbols belonging to an alphabet of M symbols denoted by $m_1, m_2, \ldots, m_M$

- A probabilities $p_1, p_2, \ldots, p_M$ specify the message source output probabilities.

- If the M symbols of the alphabet are equally likely, we may express the probability that symbol m, is emitted by the source as:

$$P_i = P(m_i) = 1/M \text{ for } i = 1,2,3 \ldots M$$

Introduction

- consider the remote connection of two digital computers, with one computer acting as the information source by calculating digital outputs based on observations and inputs fed into it; the other computer acts as the recipient of the information.
- The source output consists of a sequence of 1s and 0s, with each *binary symbol* being emitted every T seconds.
- The transmitting part of the digital communication system takes the 1's and 0's emitted by the source computer and encodes them into distinct signals denoted by $s1(t)$ and $s2(t)$, respectively, which are suitable for transmission over the analog channel.

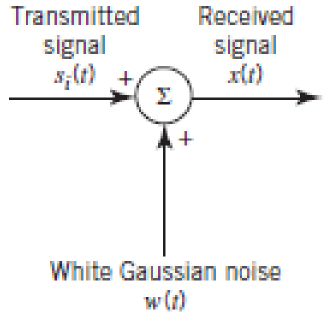
Digital Communication block diagram, Model of DCS

- The transmitter takes the message source output *m*, and codes it into a distinct signal *si(t)* suitable for transmission over the channel.
- The signal *si(t)* occupies the full duration T allotted to symbol m.
- Most important, *si(t)* is a real-valued energy signal (i.e., a signal with finite energy),

$$E_i = \int_0^{T_b} s_i^2(t)\,dt, \qquad i = 1, 2$$

- With the analog channel represented by an AWGN model, the *received signal* is defined by x(t) where *w*(*t*) is the *channel noise*. The receiver has the task of observing the received signal *x*(*t*) for a duration of *Tb* seconds and then making an *estimate* of the transmitted signal

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T_b \\ i = 1, 2 \end{cases}$$



AWGN model of a channel.

Geometric Representation of Signals

- The essence of geometric representation of signals is to represent any set of M energy signals {***si(t)***} as linear combinations of N orthonormal basis functions, where $N \leq M$. That is to say, given a set of real-valued energy signals ***s1(t), s2(t), ..., sM(t),***each of duration T seconds, we write

$$s_i(t) = \sum_{j=i}^{N} S_{ij}\phi_j(t), \qquad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, ..M \end{cases}$$

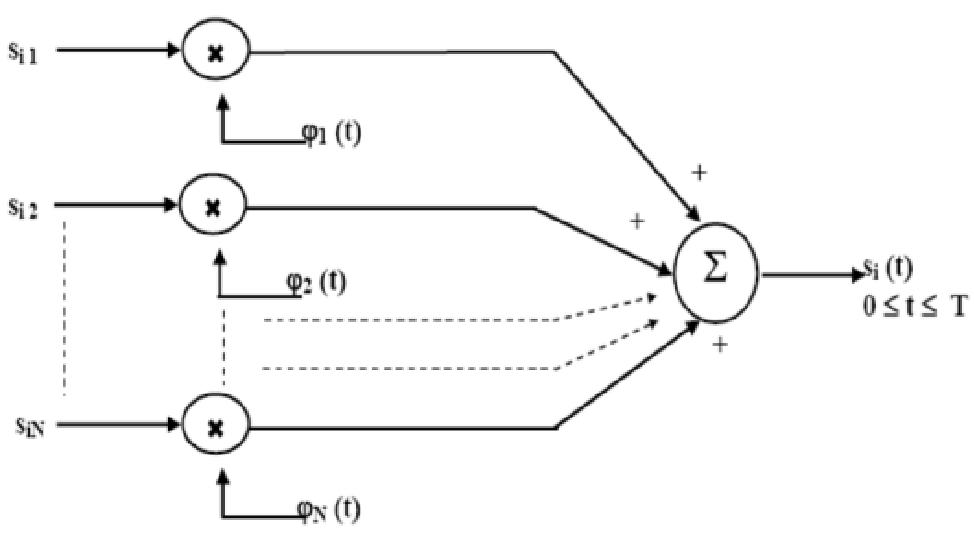
Where the coefficients of the expansion are defined by:

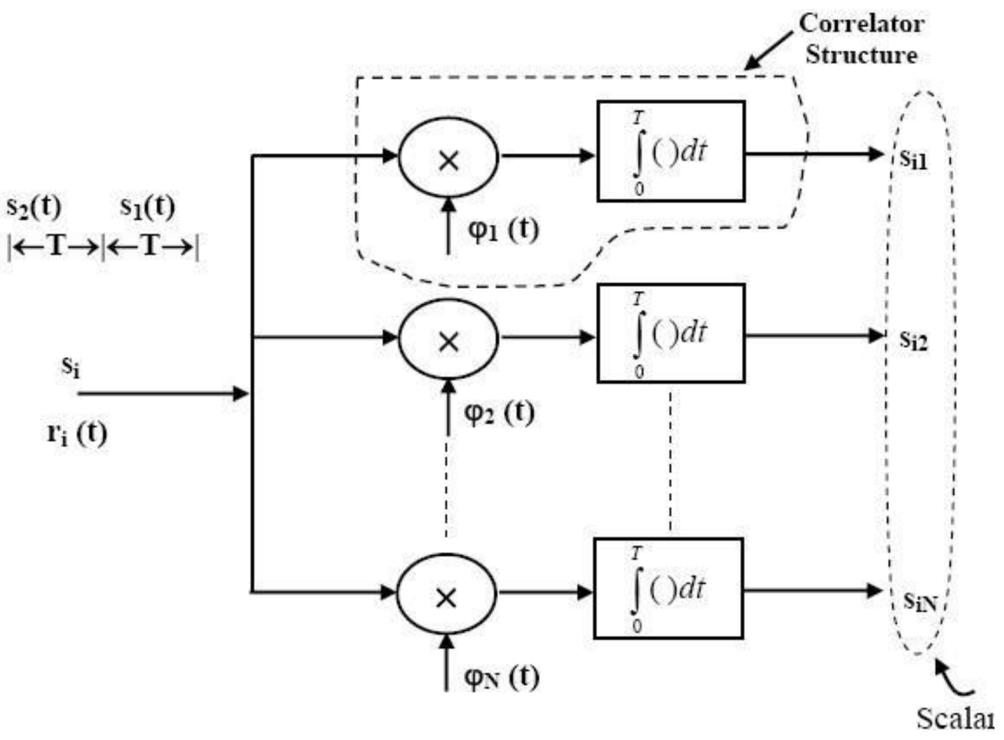
$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \qquad \begin{cases} i = 1, 2, ... M \\ j = `1, 2, ... N \end{cases}$$

The real-valued basis functions are orthonormal which means

$$s_{ij} = \int_0^T \phi_i(t)\phi_j(t)dt \qquad \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

- The set of coefficients may naturally be viewed as an N dimensional vector, denoted by ***si***. The important point to note here is that the vector ***si*** bears a one to one relationship with the Transmitted signal (a) Construction of signal from basis functions, (b) Getting coefficients from Basis functions





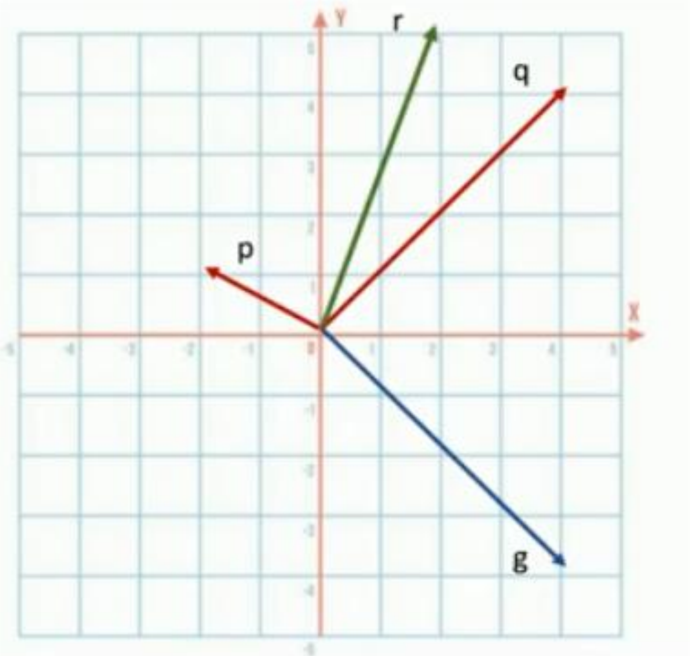
Signal Vector

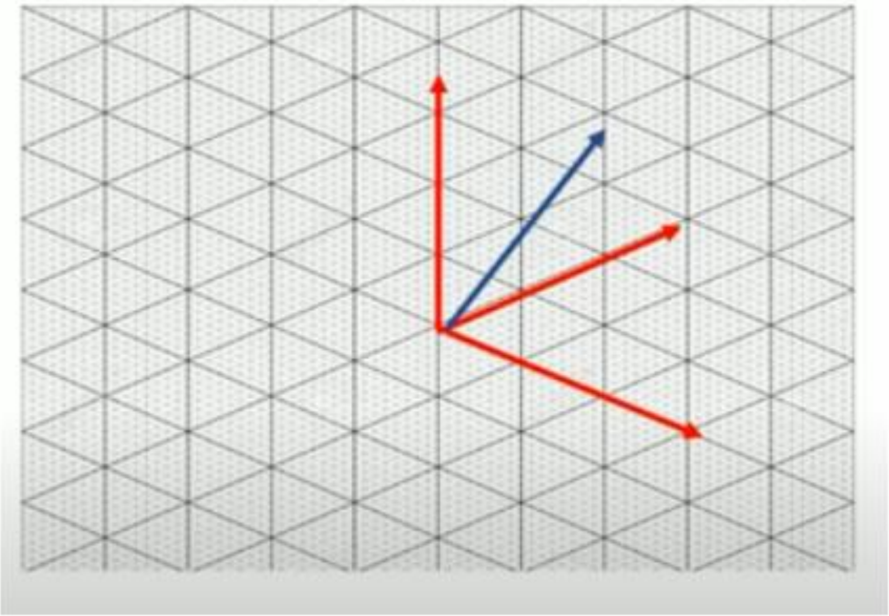
- We may state that each signal is completely determined by the vector of its coefficients

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ . \\ . \\ s_{iN} \end{bmatrix}, \qquad i = 1,2,\ldots.M$$

Definitions

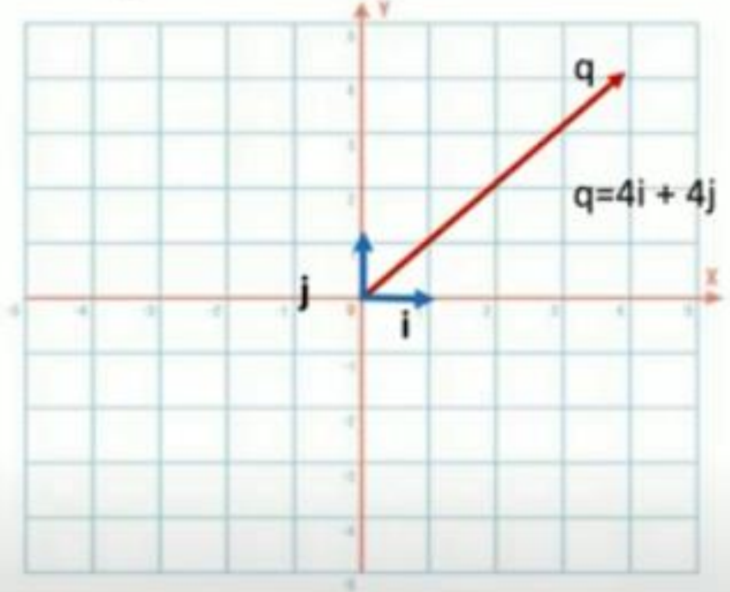
- **Signal Space**: The N-Dimensional Euclidean space constructed using basis functions as mutually perpendicular axis is called the signal space.
- **Length**: In an N-dimensional Euclidean space, it is customary to denote the length (also called the **absolute value or norm**) of a signal vector ***si*** by the symbol $\|s_i\|$
- Space consisting of vectors together with the associative and commutative operation of addition of vectors and the associative and distributive operation of multiplication of vectors by scalars

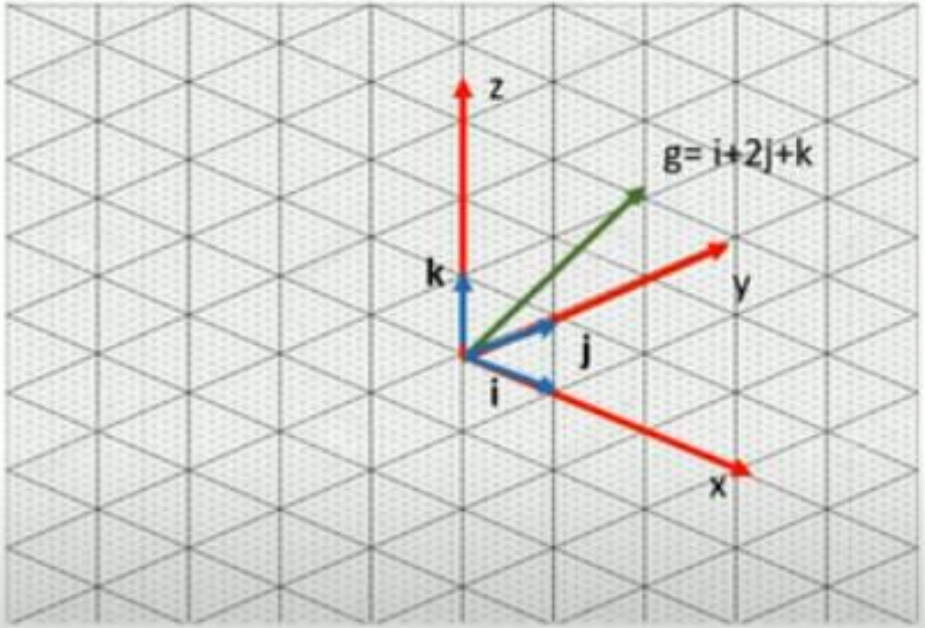




Basis function

$$<\emptyset_i, \emptyset_j> = \begin{cases} 0 \quad when \quad i \neq j; \quad orthogonality \\ 1 \quad when \quad i = j; \; orthonormality \end{cases}$$





Hence in an n dimensional vector space, any vector can be conveniently defined as the linear combination of basis functions

$$X = x_1\emptyset_1 + x_2\emptyset_2 + x_3\emptyset_3 + \ldots\ldots\ldots + x_n\emptyset_n = \sum_{j=1}^{n} x_j\emptyset_j$$

Geometric Representation of signals

Consider a N-dimensional signal space represented by N-orthonormal basis functions, consisting of M signals.
Then each of the signal in this signal space is represented by the summation of product of co-ordinates and respective basis functions

i.e. $s_i(t) = s_{i1}\emptyset_1(t) + s_{i2}\emptyset_2(t) + \ldots\ldots\ldots\ldots + s_{MN}\emptyset_N(t)$..............(1)

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\emptyset_j(t) \text{..............(2)}; \begin{cases} 1 \le i \le M \\ 0 \le t \le T \end{cases}$$

for a set of N basis functions that are linearly independent to each other then,

$$\int_0^T \emptyset_i(t)\,\emptyset_j(t)dt = \begin{cases} 0 \quad when \quad i \neq j; \quad orthogonality \\ 1 \quad when \quad i = j; \; orthonormality \end{cases} \text{..............(3)}$$

Keeping i=1 in equation 1, yields

. $s_1(t) = s_{11}\emptyset_1(t) + s_{12}\emptyset_2(t) + \ldots\ldots\ldots\ldots + s_{1N}\emptyset_N(t)$..............(4)

Out of n basis function, take $\emptyset_1(t)$ and multiply and integrate both side to equation (4), yields

$$\int_0^T s_1(t)\emptyset_1(t)dt = \int_0^T s_{11}\emptyset_1(t)\emptyset_1(t)dt + \int_0^T s_{12}\emptyset_2(t)\emptyset_1(t)dt \ldots . + \int_0^T s_{1N}\emptyset_N(t)\emptyset_1(t)dt \ldots\ldots\ldots\ldots (5)$$

from equation (3) and equation (5)

$$\int_0^T s_1(t)\emptyset_1(t)dt = s_{11} \ldots\ldots\ldots\ldots (6)$$

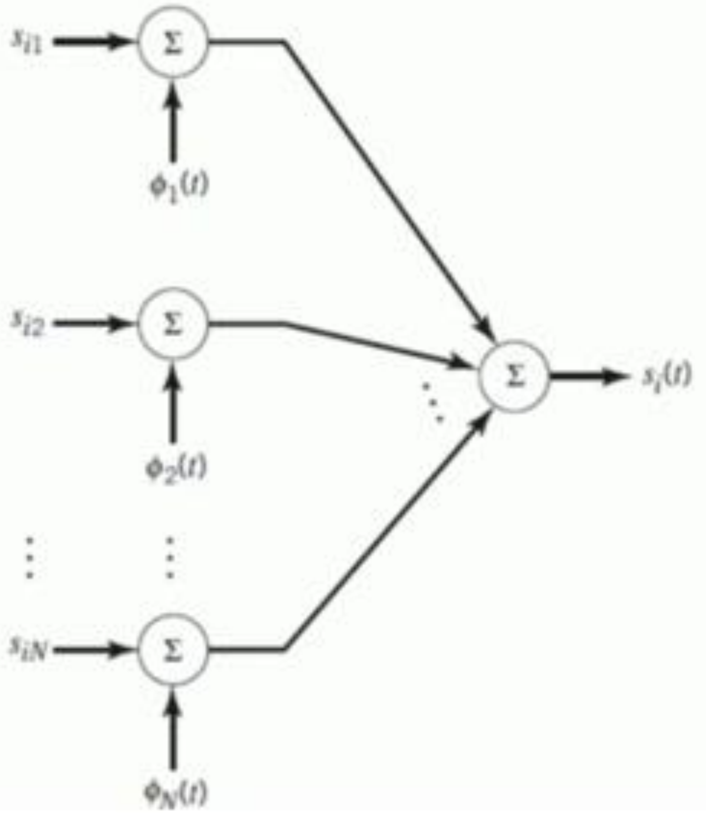
$$\int_0^T s_i(t)\emptyset_j(t)dt = s_{ij} \ldots\ldots\ldots\ldots (7)$$

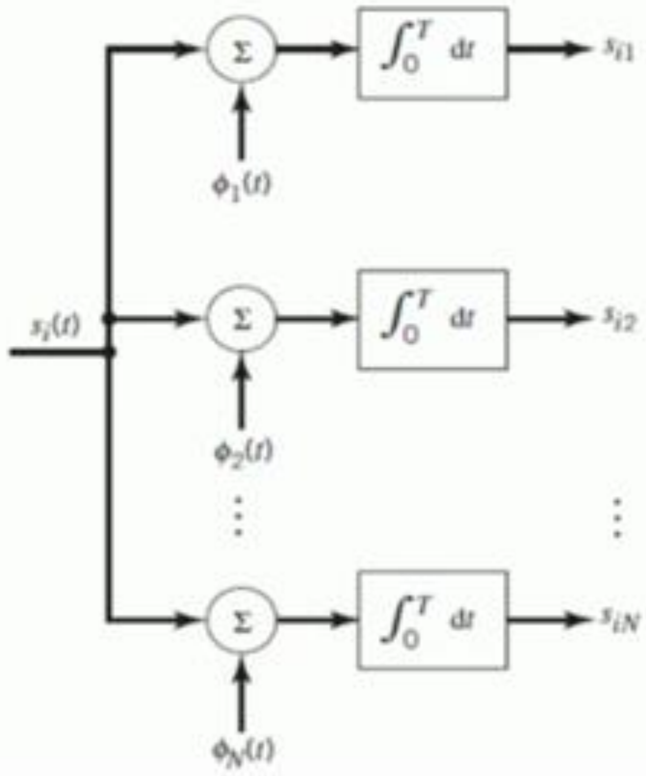
Synthesizer for generating the signal using equation (2)

$$s_i(t) = \sum_{j=1}^{N} s_{ij} \emptyset_j(t)$$

Analyzer for reconstructing the signal vector using equation (7)

$$s_{ij} = \int_0^T s_i(t)\, \emptyset_j(t) dt$$





Energy of the signal $x(t)$ in terms of its coordinates $x_1, x_2 \dots x_n$

Consider a real valued signal $x(t)$

By synthesizer equation it is known that any signal in the signal space can be represented in terms of its coordinates and basis function

i.e.

$$x(t) = x_1\phi_1(t) + x_2\phi_2(t) + \dots\dots + x_n\phi_n(t) \quad \dots\dots(1)$$

$$x(t) = \sum_{i=1}^{N} x_i\phi_i(t) \dots\dots\dots(2); \qquad (0 \le t \le T)$$

where, $\phi_1(t), \phi_2(t) \dots\dots \phi_N(t)$ are orthonormal basis functions

and $x_1, x_2 \dots x_n$ are coordinates

Energy of the signal is given by

$$E = \int_0^T x^2(t)dt \dots\dots\dots(3)$$

$$E = \int_0^T \sum_{i=1}^{N} x_i\phi_i(t) \sum_{i=1}^{N} x_i\phi_i(t)\, dt$$

$$\left\{ \because \int_0^T \phi_i(t)\,\phi_j(t)dt = \begin{cases} 0 & when \;\; i \neq j; \;\; orthogonality \\ 1 & when \;\; i = j; \; orthonormality \end{cases} \right\}$$

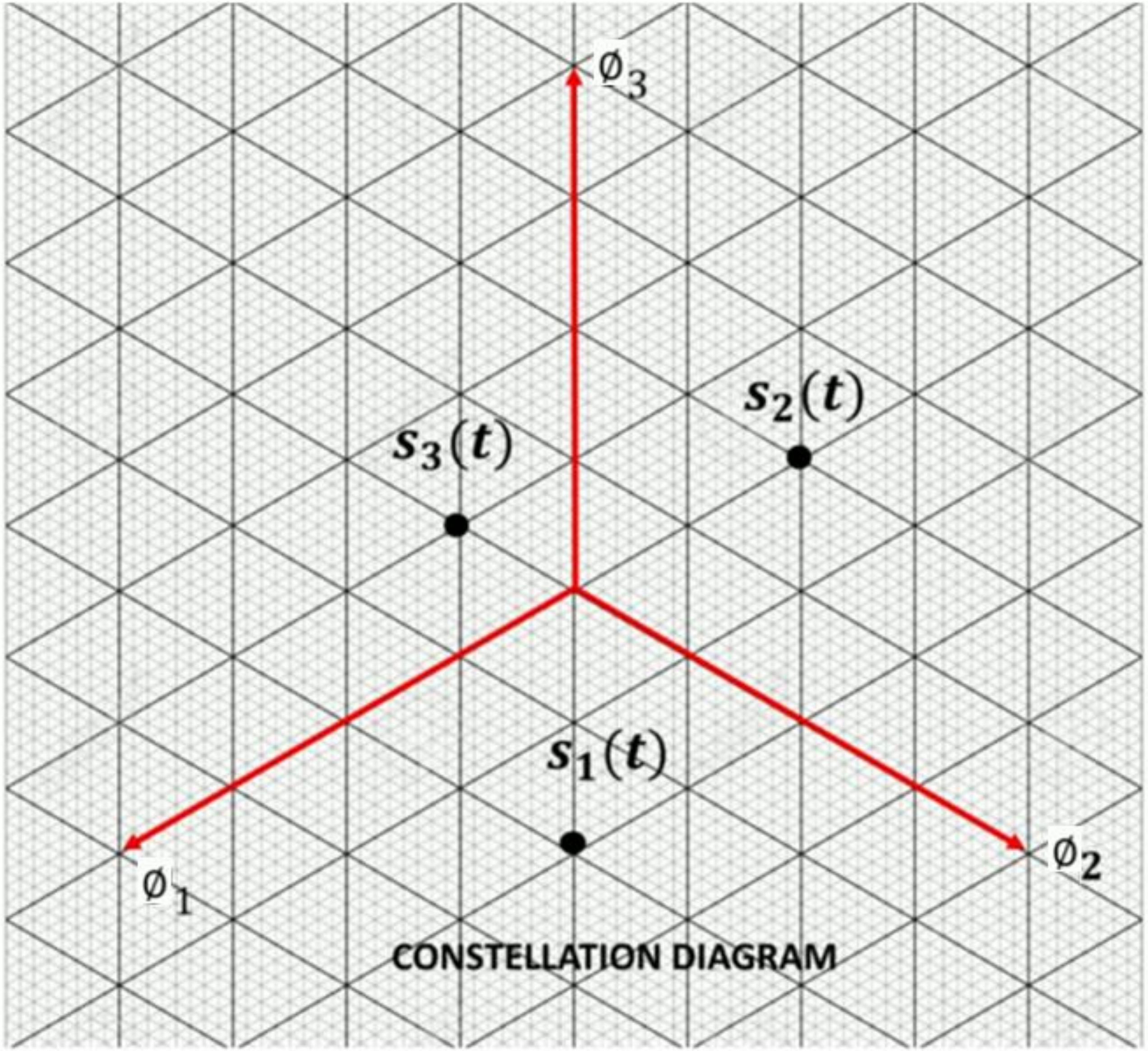
$$E = \sum_{i=1}^{N} {x_i}^2 \dots\dots\dots(4)$$

Important points

1.Signals can be described as vectors
2.Like vector space there is a concept called signal space.
3.The dimension of this signal space is represented by basis functions.
4.Basis functions are orthogonal and Orthonormal

$$\int_0^T \emptyset_i(t)\,\emptyset_j(t)dt = \begin{cases} 0 \quad when \quad i \neq j; \quad orthogonality \\ 1 \quad when \quad i = j; \quad orthonormality \end{cases}$$

5.Signal in signal space is represented by points in the space.
6. Signal space diagram is also known as constellation diagram



Some defnitions

$$s_i(t) = s_{i1}\emptyset_1(t) + s_{i2}\emptyset_2(t) + \ \dots\dots\dots\dots. + s_{MN}\emptyset_N(t)$$

1.Signal Vector

Every signal in a signal space can easily be represented by the knowledge of its coordinates (s_{ij}), this set of co-ordinates is known as signal vector and represented in terms of a column vector

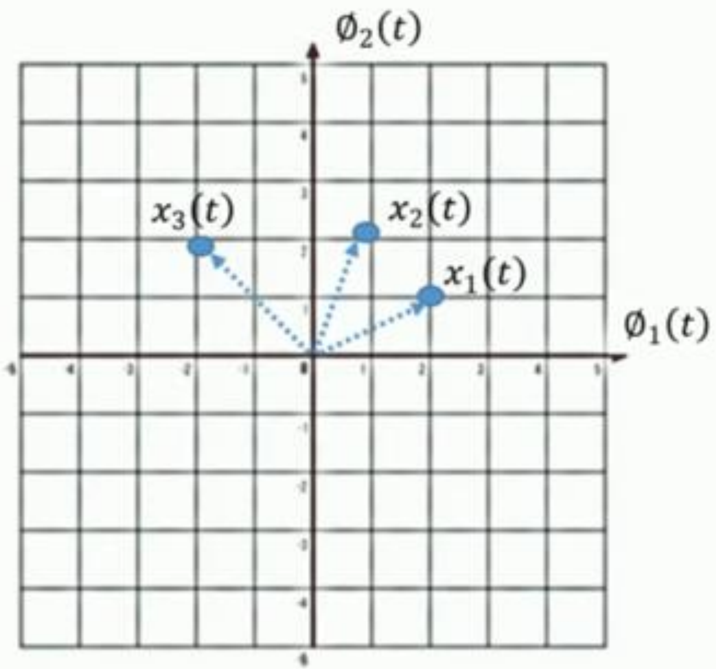
$$\text{signal vector}=\overline{s_i} = \begin{bmatrix} s_{i1} \\ s_{i2} \\ s_{i3} \\ \vdots \\ s_{ij} \end{bmatrix} \quad \text{where, } i\text{=1,2......}M$$

2.Signal Space

From the equation, we may visualise each signal as a point in an N-dimensional Euclidean space, with mutually perpendicular axes labelled as $\emptyset_1(t)$, $\emptyset_2(t)$..... $\emptyset_N(t)$, this N dimensional Euclidean space is called signal space. It is generally referred as signal space diagram or constellation diagram.

1. An example of two dimensional signal space with three signals

$$\overline{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \overline{x_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \overline{x_3} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Solution:

Two dimensional signal space means there is going to be two basis functions $\emptyset_1(t)$ and $\emptyset_2(t)$

Using synthesizer equation

$$x_i(t) = \sum_{j=1}^{N} x_{ij}\emptyset_j(t)\,;\, 0 \le t \le T$$

$x_1(t)$=$x_{11}\emptyset_1(t) + x_{12}\emptyset_2(t) \Rightarrow x_1(t)$=$2\emptyset_1(t) + 1\emptyset_2(t)$

$x_2(t)$=$x_{21}\emptyset_1(t) + x_{22}\emptyset_2(t) \Rightarrow x_2(t)$=$1\emptyset_1(t) + 2\emptyset_2(t)$

$x_3(t)$=$x_{31}\emptyset_1(t) + x_{32}\emptyset_2(t) \Rightarrow x_3(t)$=-$2\emptyset_1(t) + 2\emptyset_2(t)$

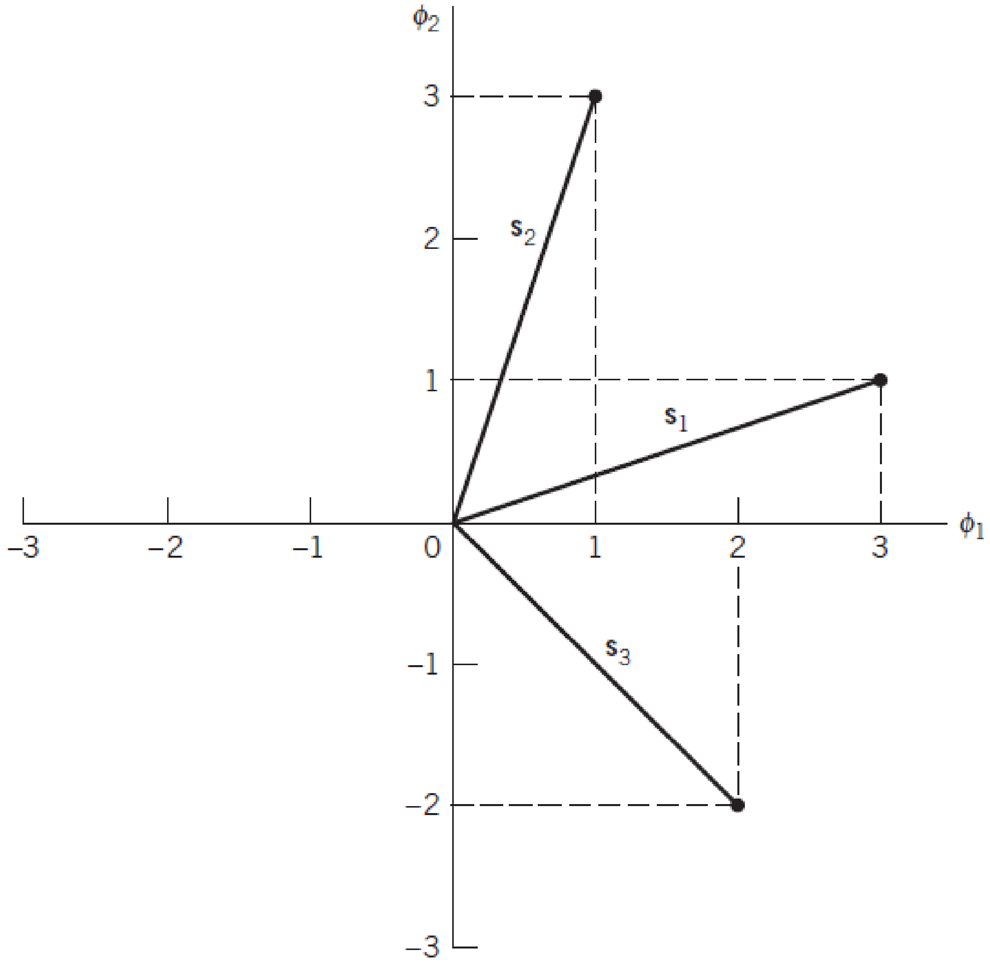
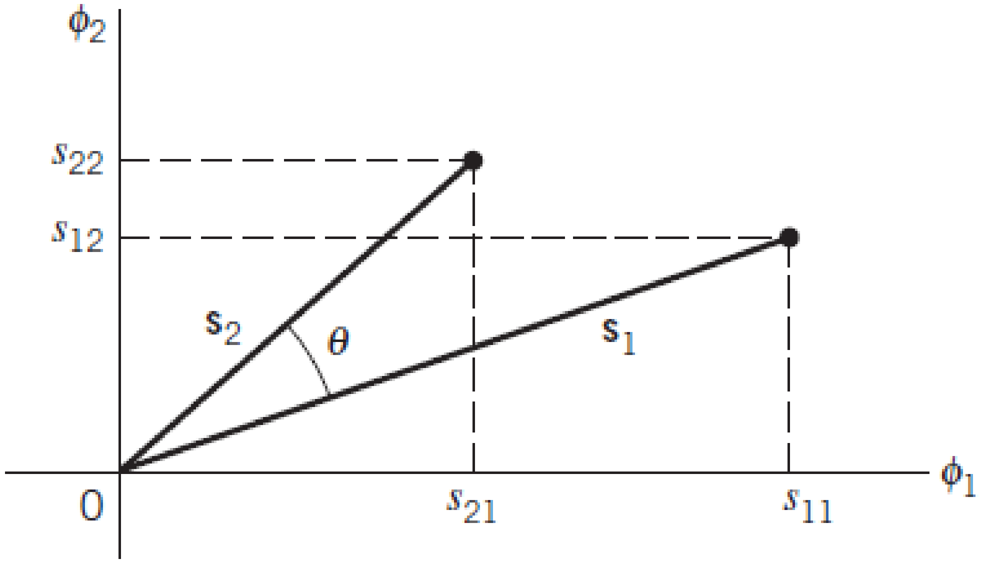


Figure 7.3 Illustrating the geometric representation of signals for the case when $N = 2$ and $M = 3$.

$$\mathbf{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}$$

$$\mathbf{s}_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) + \cdots + s_{1N}\phi_N(t)$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) + \cdots + s_{2N}\phi_N(t)$$

$$s_3(t) = s_{31}\phi_1(t) + s_{32}\phi_2(t) + \cdots + s_{3N}\phi_N(t)$$
$$\vdots$$
$$s_M(t) = s_{M1}\phi_1(t) + s_{M2}\phi_2(t) + \cdots + s_{MN}\phi_N(t)$$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad 0 \leqslant t \leqslant T, \quad i = 1, 2, \ldots, M$$

$$\int_0^T \phi_i(t)\phi_j(t)\,dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Gram-Schmidt method proceeds as follows.

Step 1. In equation (4.10a) set all coefficients $s_{ij} = 0$ except s_{11}. Then we ha
$s_1(t) = s_{11}\phi_1(t)$ and $\phi_1(t) = s_1(t)/s_{11}$; but $\phi_1(t)$ is normalized. Hence

$$\int_0^T \phi_1^2(t)\,dt = 1 = \int_0^T \frac{s_1^2(t)}{s_{11}^2}\,dt \qquad (4.1$$

Solving for s_{11}, we have

$$s_{11} = \left[\int_0^T s_1^2(t)\,dt\right]^{1/2} \qquad (4.1$$

In this step s_{11} and $\phi_1(t) = s_1(t)/s_{11}$ are determined.

Step 2. In equation (4.10b) we set all coefficients except s_{21} and s_{22} to zero. We then ha

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) \tag{4.1}$$

Multiply both sides of equation (4.14) by $\phi_1(t)$ and integrating over the interval T a noting that $\phi_1(t)$ and $\phi_2(t)$ are orthonormal, we have

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt \tag{4.1}$$

Thus s_{21} is known. s_{22} is evaluated as follows. Rewriting equation (4.14) we get

$$s_2(t) - s_{21}\phi_1(t) = s_{22}\phi_2(t) \tag{4.1}$$

Squaring and integrating equation (4.16), we have

$$\int_0^T [s_2(t) - s_{21}\phi_1(t)]^2\,dt = \int_0^T s_{22}^2\phi_2^2(t)\,dt = s_{22}^2$$

Therefore

$$s_{22} = \left[\int_0^T [s_2(t) - s_{21}\phi_1(t)]^2 \, dt \right]^{1/2}$$

Since now both s_{21} and s_{22} are known, from equation (4.14), $\phi_2(t)$ can be written as

$$\begin{aligned} \phi_2(t) &= \frac{1}{s_{22}} [s_2(t) - s_{21}\phi_1(t)] \\ &= \frac{1}{s_{22}} \left[s_2(t) - \frac{s_{21}s_1(t)}{s_{11}} \right] \end{aligned}$$

Step 3. From equation (4.10c), we write $s_3(t)$ as

$$s_3(t) = s_{31}\phi_1(t) + s_{32}\phi_2(t) + s_{33}\phi_3(t)$$

Setting s_{34}, s_{35}, etc to zero, we have

$$s_{31} = \int_0^T s_3(t)\phi_1(t)\,dt$$

$$s_{32} = \int_0^T s_3(t)\phi_2(t)\,dt$$

$$s_{33}^2 = \int_0^T [s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(\mathrm{t})]^2 dt$$

and

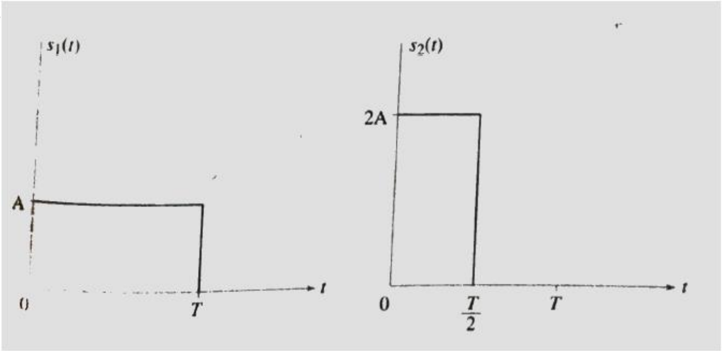
$$s_{33} = \left[\int_0^T [s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(\mathrm{t})]^2\,dt\right]^{1/2}$$

$$s_{33} = \left[\int_0^T [s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)]^2\, dt\right]^{1/2}$$

$$\phi_3(t) = \frac{s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)}{s_{33}}$$

Step 4. The procedure mentioned in the above steps is continued till we find N orthonormal functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ and the coefficients s_{ij}. Using the N orthonormal functions $\phi_1(t)$, $\phi_2(t), \ldots, \phi_N(t)$, and the coefficients s_{ij}, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, the signals $s_1(t)$ $s_2(t) \ldots s_M(t)$ can be expressed as a linear combination of the orthonormal functions. In general, $N \leqslant M$. Note that N is the number of linearly independent functions in the set $s_1(t), s_2(t), \ldots, s_M(t)$ of M functions. We wish to inform the reader that $\phi_1(t)$ depends on the chosen $s_1(t)$, $\phi_2(t)$ depends on the chosen $s_1(t)$ and $s_2(t)$, and so on. In brief, the set of N orthonormal basic functions that constitute the basis functions is not unique, for a given set of M signals $\{s_i(t)\}$, $i = 1, 2, \ldots, M$.

Two functions $s_1(t)$ and $s_2(t)$ are given in Figure DP4.1(a). The interv
$0 \leqslant t \leqslant T$ seconds. Using Gram-Schmidt procedure, express these functions in ter
of orthonormal functions. Also Sketch $\phi_1(t)$ and $\phi_2(t)$.



Solution: From equation (4.13), we have

$$s_{11} = \left[\int_0^T s_1^2(t)\,dt\right]^{1/2} = \left[\int_0^T A^2\,dt\right]^{1/2} = A\sqrt{T}$$

and

$$\phi_1(t) = \frac{s_1(t)}{s_{11}} = \frac{A}{A\sqrt{T}} = \frac{1}{\sqrt{T}} \qquad \text{for } 0 \leqslant t \leqslant T$$

From equation (4.15), we have

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt = \int_0^{T/2} 2A\left(\frac{1}{\sqrt{T}}\right)dt = \frac{2A}{\sqrt{T}}\frac{T}{2} = A\sqrt{T}$$

$$s_{22} = \left[\int_0^T [s_2(t) - s_{21}\phi_1(t)]^2\,dt\right]^{1/2}$$

$$= \left[\int_0^{T/2} (2A - A)^2\,dt + \int_{T/2}^{T} (0 - A)^2\,dt\right]^{1/2}$$

$$= [A^2 T]^{1/2} = A\sqrt{T}$$

$$\phi_2(t) = \frac{1}{s_{22}}[s_2(t) - s_{21}\phi_1(t)]$$

$$= \frac{1}{s_{22}}\left[s_2(t) - s_{21}\frac{s_1(t)}{s_{11}}\right]$$

it $s_{21} = s_{11}$, therefore

$$\phi_2(t) = \frac{1}{A\sqrt{T}}[s_2(t) - s_1(t)] \quad \text{for } 0 \leqslant t \leqslant T, \qquad s_1(t) = s_{11}\phi_1(t)$$

and $s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$; where $s_{11} = s_{21} = s_{22} = A\sqrt{T}$.

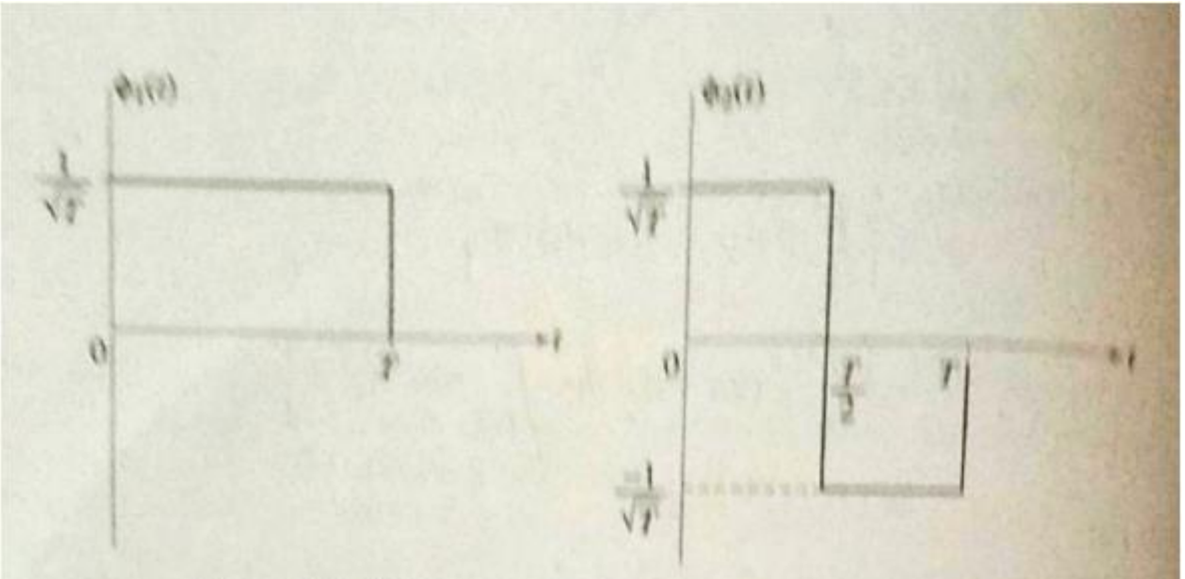
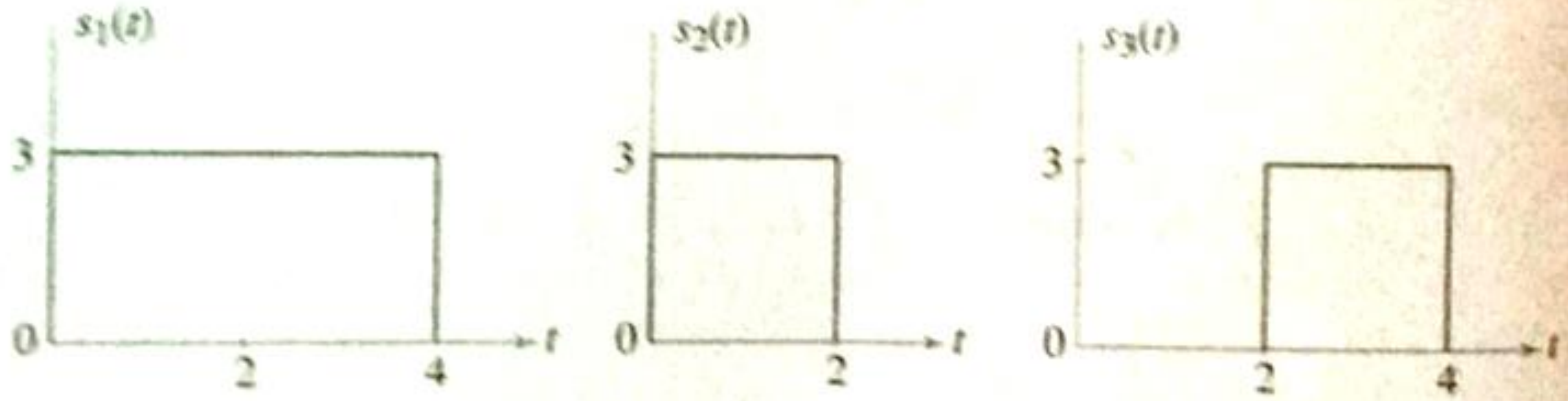


Figure DP4.1(b) Sketch of functions $\phi_1(t)$ and $\phi_2(t)$.

DP4.11. Three signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ are as shown in Figure DP4.11(a). Apply Gram-Schmidt procedure to obtain an orthonormal basis for the signals. Express the signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ in terms of orthonormal basis functions. Also give the signal constellation diagram.



Solution: We have $s_1(t) = s_{11}\phi_1(t)$.

Computation of s_{11}:

$$\int_0^4 s_1^2(t)dt = 9 \times 4 = 36 = E_1 = s_{11}^2$$

Choose

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{6}$$

$$\phi_1(t) = \begin{cases} \frac{1}{2}, & 0 \leqslant t \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

ave $s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$.

ation of s_{21}:

$$\int_0^T s_2(t)\phi_1(t)dt = \int_0^2 3 \times \frac{1}{2}dt = \frac{3}{2} \times 2 = 3 = s_{21}$$

$$s_{22}\phi_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$\int_0^T s_{22}^2\phi_2^2(t)dt = s_{22}^2 = \int_0^T (s_2(t) - 3\phi_1(t))^2 dt$$

$$\Longrightarrow s_{22}^2 = \int_0^2 s_2^2(t)dt + \int_0^4 9\phi_1^2(t)dt - 2\int_0^2 3s_2(t)\phi_1(t)dt = 9$$

$$\Longrightarrow s_{22} = 3$$

$$\phi_2(t) = \frac{1}{3}[s_2(t) - 3\phi_1(t)]$$

Substituting the values of $s_2(t)$ and $\phi_1(t)$, we have

$$\phi_2(t) = \begin{cases} \frac{1}{2}, & 0 \leqslant t \leqslant 2 \\ -\frac{1}{2}, & 2 \leqslant t \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

From Figure DP4.11(a), we find that $s_3(t) = s_1(t) - s_2(t)$.
Hence, a third basis function is not needed.
Summarizing:

$$s_1(t) = 6\phi_1(t)$$
$$s_2(t) = 3\phi_1(t) + 3\phi_2(t)$$
$$s_3(t) = 3\phi_1(t) - 3\phi_2(t)$$

The final constellation diagram is as shown in Figure DP4.11(b).

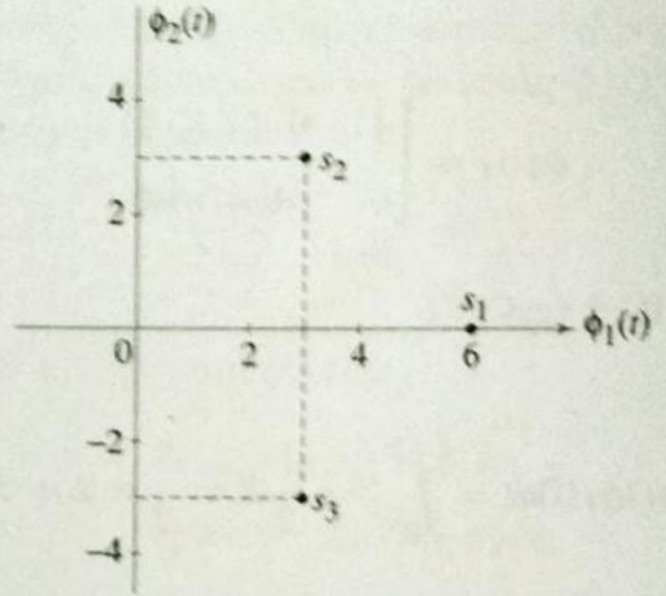
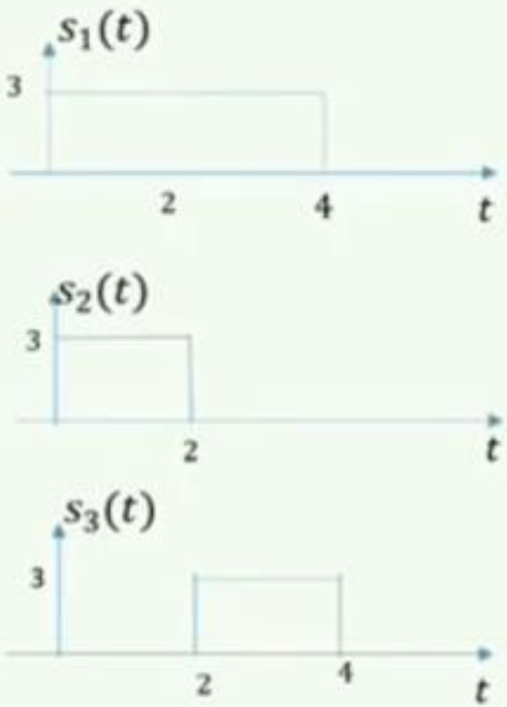


Figure DP4.11(b) Signal constellation diagram.

2.Applying GSO procedure, find a set of orthonormal basis function from a given set of signals as a linear combination of basis functions. Draw the constellation diagram



Solution:

Step 1) <u>Get a reduced set of N linearly independent signal from any given set of M signals</u>

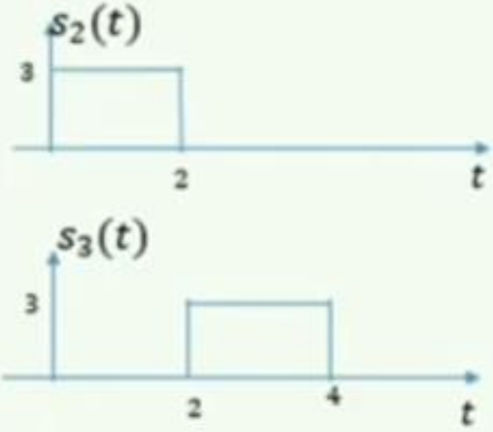
M=3

$s_1(t)$=$s_2(t) + s_3(t)$

$s_1(t)$ is linearly dependent on signals $s_1(t)$ and $s_3(t)$ whereas signals $s_2(t)$and $s_3(t)$ are linearly independent

So, N=2 Signals

So, it is required to construct 2 orthonormal basis function $\emptyset_1(t)$ & $\emptyset_2(t)$



$s_2(t)$=$s_{22}\emptyset_1(t)$....(1)
$s_3(t)$=$s_{33}\emptyset_2(t)$....(2)
From equation (1)&(2)

$\emptyset_1(t) = \frac{s_2(t)}{s_{22}}$ ....(3)

$\emptyset_2(t) = \frac{s_3(t)}{s_{33}}$....(4)

$s_{22} = \sqrt{E_2}$....(5)

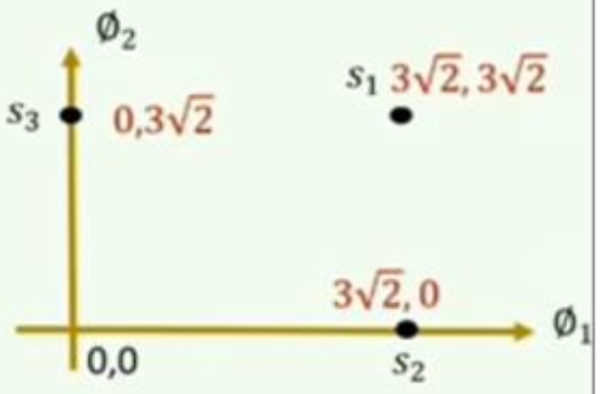
$s_{33} = \sqrt{E_3}$....(6)

$$E_2 = \int_0^2 |s_2(t)|^2 dt \Rightarrow E_2 = \int_0^2 |3|^2 dt$$

$s_{22} = \sqrt{E_2} = 3\sqrt{2}$ ....(7)

$$E_3 = \int_2^4 |3|^2 dt \Rightarrow E_2 = \int_2^4 |3|^2 dt$$

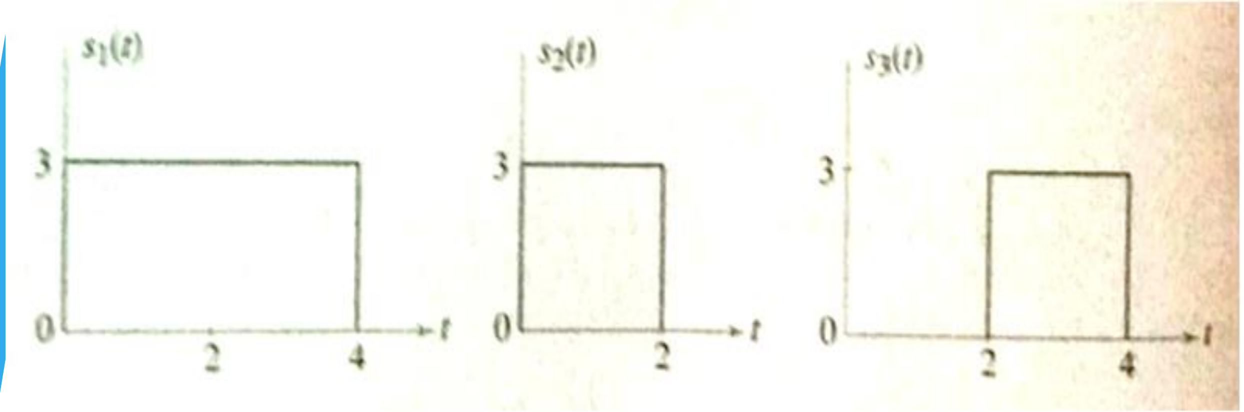
$s_{33} = \sqrt{E_3} = 3\sqrt{2}$ ....(8)

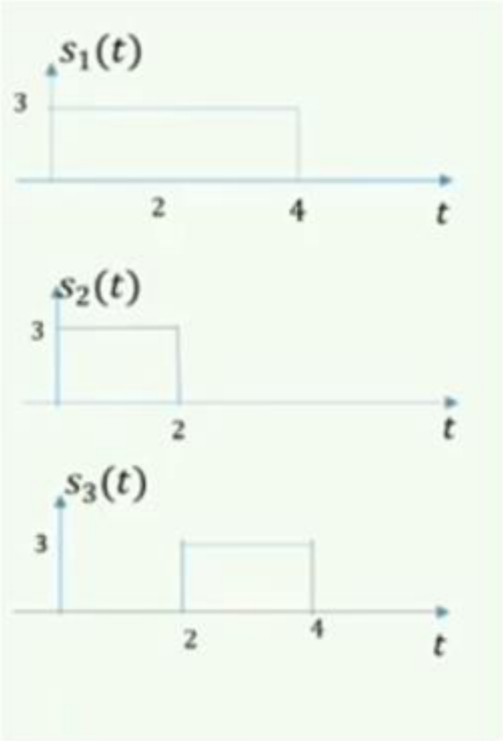


$s_2(t)$=$s_{22}\emptyset_1(t) \Rightarrow s_2(t)$=$s_{22}\ \emptyset_1(t) + 0.\emptyset_2(t) \Rightarrow s_2(t)$=$3\sqrt{2}\emptyset_1(t) + 0.\emptyset_2(t)$

$s_3(t)$=$s_{33}\emptyset_2(t) \Rightarrow s_3\ (t)$=$0.\emptyset_1(t) + s_{33}\emptyset_2(t) \Rightarrow s_3(t)$=$0.\emptyset_1(t) + 3\sqrt{2}\emptyset_2(t)$

$s_1(t)$=$s_2(t) + s_3(t) \quad \Rightarrow s_1(t)$=$3\sqrt{2}\emptyset_1(t) + 3\sqrt{2}\emptyset_2(t)$



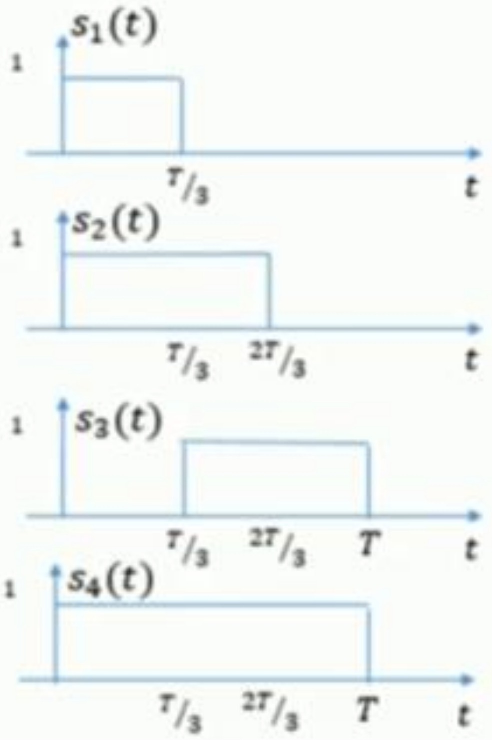


$$s_1(t) = 6\phi_1(t)$$
$$s_2(t) = 3\phi_1(t) + 3\phi_2(t)$$
$$s_3(t) = 3\phi_1(t) - 3\phi_2(t)$$

$$s_2(t)=3\sqrt{2}\emptyset_1(t) + 0.\emptyset_2(t)$$
$$s_3(t)=0.\emptyset_1(t) + 3\sqrt{2}\emptyset_2(t)$$
$$s_1(t)=3\sqrt{2}\emptyset_1(t) + 3\sqrt{2}\emptyset_2(t)$$

Gram-Schmidt Orthogonalization (GSO) Prodedure

- In Digital communication, we apply input as binary bits which are then converted into symbols and waveforms by a digital modulator.
- These **waveforms should be unique** and different from each other so that receiver can easily identify what symbol/bit is transmitted.
- To **make them unique, Gram-Schmidt Orthogonalization procedure** can be applied.



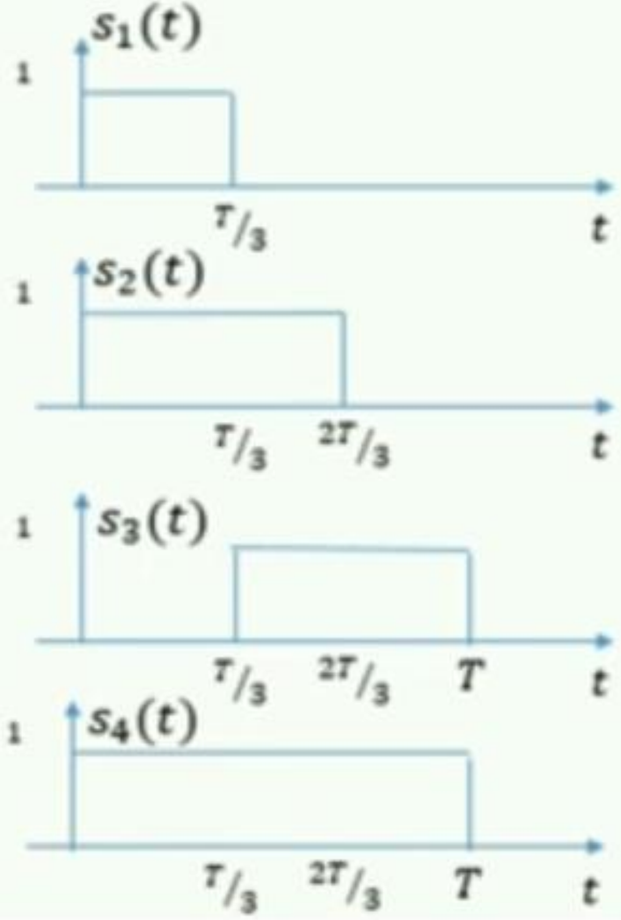
There are two steps to be applied in GSO procedure

1)Get a reduced set of N linearly independent signal from any given set of M signals

2)Construct a set of N orthonormal basis function from N linearly independent signal

Given set of signals M=4
Reduced set of linearly independent signals N=3
So, it is required to construct N orthonormal basis function

1.Using GSO procedure find set of orthonormal basis function from the given set of signals and construct the corresponding signal space diagram



Solution:

Step 1) Get a reduced set of N linearly independent signal from any given set of M signals

M=4 (signals)

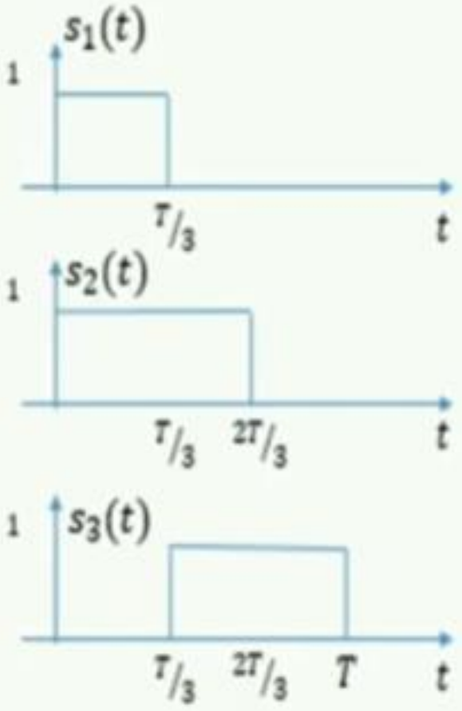
$s_4(t) = s_1(t) + s_3(t)$

$s_4(t)$ is linearly dependent on signals $s_1(t)$ and $s_3(t)$ whereas signals $s_1(t)$, $s_2(t)$and $s_3(t)$ are linearly independent

Hence, N=3 (Linearly Independent Signals)

So, it is required to construct 3 orthonormal basis function $\emptyset_1(t)$,$\emptyset_2(t)$ and $\emptyset_3(t)$

Step 2) Construct a set of N(=3) orthonormal basis function from N(=3) linearly independent signal



Using synthesizer equation,

$s_i(t) = \sum_{j=1}^{N} s_{ij}\emptyset_j(t)\,;\ 0 \le t \le T$............(1)

We can write three equations as

$s_1(t)$=$s_{11}\emptyset_1(t)$; (i=1;N=1) ....(2)

$s_2(t)$=$s_{21}\emptyset_1(t) + s_{22}\emptyset_2(t)$; (i=2;N=2) ....(3)

$s_3(t)$=$s_{31}\emptyset_1(t) + s_{32}\emptyset_2(t) + s_{33}\emptyset_3(t)$; (i=3;N=3) ....(4)

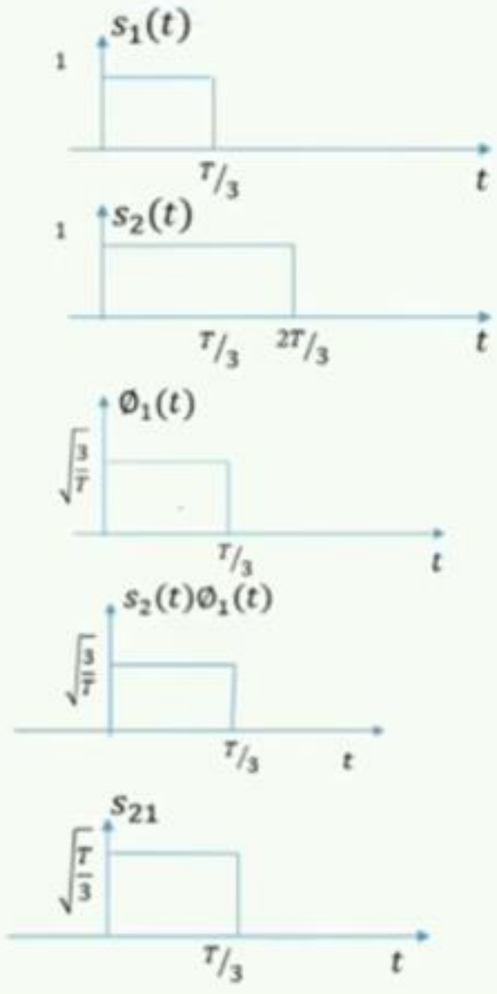
Signals $s_2(t)$, $s_1(t)$ are non orthogonal, so we need to find the projection of signal $s_2(t)$ on basis function $\emptyset_1(t)$ i.e s_{21}

Similarly, Signals $s_3(t)$, $s_2(t)$ are non orthogonal, so we need to find the projection of signal $s_3(t)$ on basis function $\emptyset_2(t)$ i.e s_{32}

while, Signal $s_3(t)$, $s_1(t)$ are **orthogonal**, so need not to find the projection of signal $s_3(t)$ on basis function $\emptyset_1(t)$ i.e $s_{31} = 0$

Projection of a signal onto basis function can be obtained using Analyzer equation

$s_{ij} = \int_0^T s_i(t)\emptyset_j(t)dt$ ....(5) ; $i = 1,2 \ldots . M$ & $j = 1,2 \ldots . N$



ii) Determination of $\emptyset_2(t)$

we have

$s_2(t)=s_{21}\emptyset_1(t) + s_{22}\emptyset_2(t)$ (i=2;N=2)

$$\emptyset_2(t) = \frac{s_2(t)-s_{21}\emptyset_1(t)}{s_{22}} = \frac{g_2(t)}{\sqrt{E_2}} \text{......(1)}$$

So, $g_2(t) = s_2(t) - s_{21}\emptyset_1(t)$.......(2)

$s_{22} = \sqrt{E_2}$.......(3)

and, E_2 = energy of the signal $g_2(t)$

s_{21} is projection of signal $s_2(t)$ on $\emptyset_1(t)$

So, $s_{21} = \int_0^T s_2(t)\,\emptyset_1(t)dt$ .....(4)

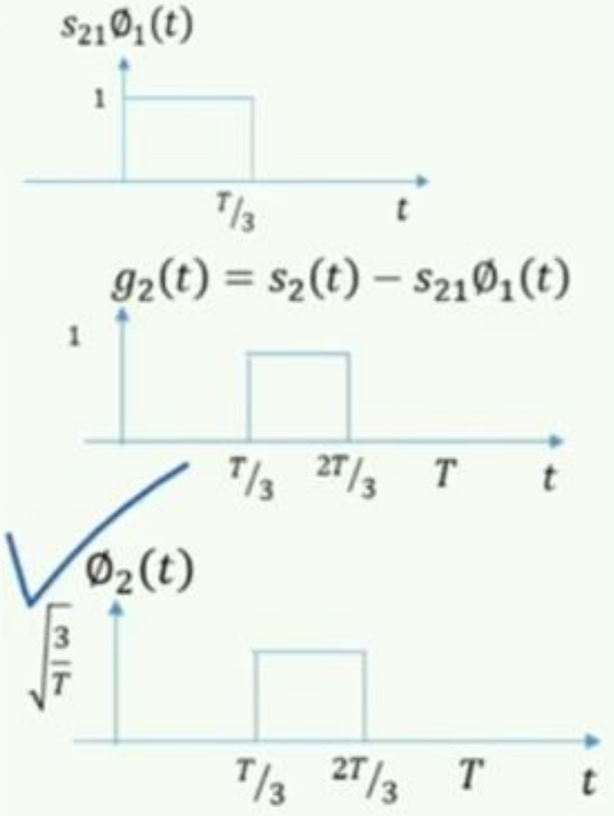
$$\Rightarrow s_{21} = \left(\sqrt{\frac{3}{T}}\right)(T/3) = \sqrt{\frac{T}{3}} \text{.....(5)}$$

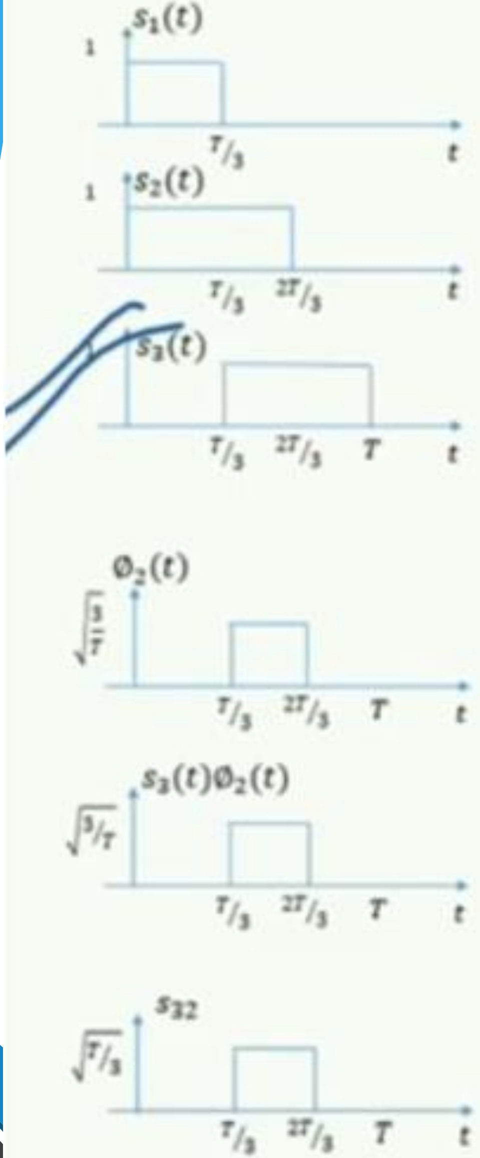
$\Rightarrow E_2 = (1)(T/3) = (T/3)$......(6)

From equation (10) & (15)

$$\emptyset_2(t) = \frac{g_2(t)}{\sqrt{E_2}} = \frac{g_2(t)}{\sqrt{\frac{T}{3}}} \text{.........(7)}$$

$$\emptyset_2(t) = \sqrt{\frac{3}{T}}g_2(t) \text{.........(8)}$$





iii) Determination of $\emptyset_3(t)$

we have

$s_3(t)=s_{31}\emptyset_1(t) + s_{32}\emptyset_2(t) + s_{33}\emptyset_3(t)$;(i=3 & N=3)

$$\emptyset_3(t) = \frac{s_3(t)-s_{31}\emptyset_1(t)-s_{32}\emptyset_2(t)}{s_{33}} = \frac{g_3(t)}{\sqrt{E_3}} \text{ ....1)}$$

here, $s_{33} = \sqrt{E_3}$.......(2)

and, E_3 = energy of the signal $g_3(t)$

$g_3(t) = s_3(t) - s_{31}\emptyset_1(t) - s_{32}\emptyset_2(t)$.......(3)

∵ signal $s_3(t)$ and $s_1(t)$ are orthogonal

hence, $s_{31} = 0$

So, $g_3(t) = s_3(t) - s_{32}\emptyset_2(t)$.......(4)

s_{32} is projection of signal $s_3(t)$ on $\emptyset_2(t)$

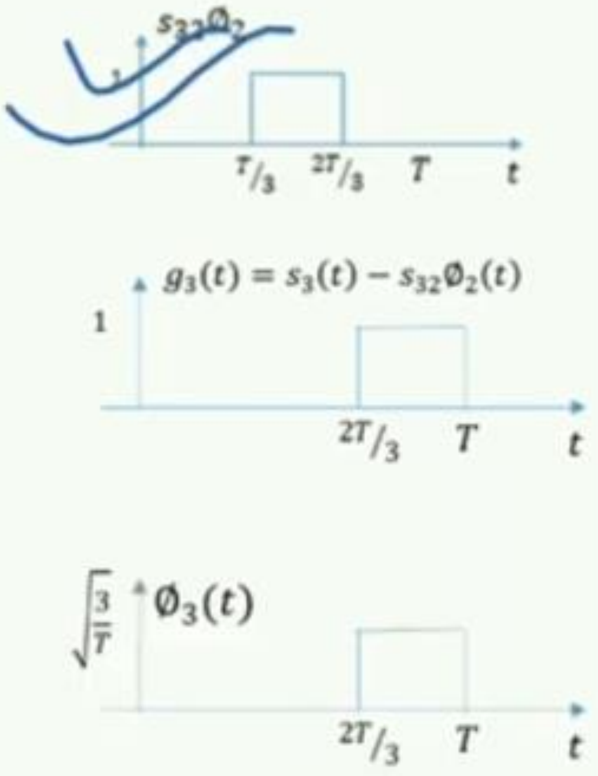
$s_{32} = \int_0^T s_3(t)\,\emptyset_2(t)dt$ .....(5)

$$\Rightarrow s_{32} = \left(\sqrt{\frac{3}{T}}\right)(T/3) = \sqrt{\frac{T}{3}} \text{.....(6)}$$

$E_3 = (1)(T/3) = (T/3)$......(7)

$$\emptyset_3(t) = \frac{g_3(t)}{\sqrt{E_2}} = \frac{g_3(t)}{\sqrt{\frac{T}{3}}}$$

$$\boxed{\Rightarrow \emptyset_3(t) = \sqrt{\frac{3}{T}}\,g_3(t) \text{ ... (8)}}$$



$\emptyset_1(t)$	$\emptyset_2(t)$	$\emptyset_3(t)$
$s_{11} = \sqrt{E_1} = \sqrt{\frac{T}{3}}$	$s_{22} = \sqrt{E_2} = \sqrt{\frac{T}{3}}$	$s_{33} = \sqrt{E_3} = \sqrt{\frac{T}{3}}$
	$s_{21} = \sqrt{\frac{T}{3}}$	$s_{32} = \sqrt{\frac{T}{3}}$
		$s_{31} = 0$

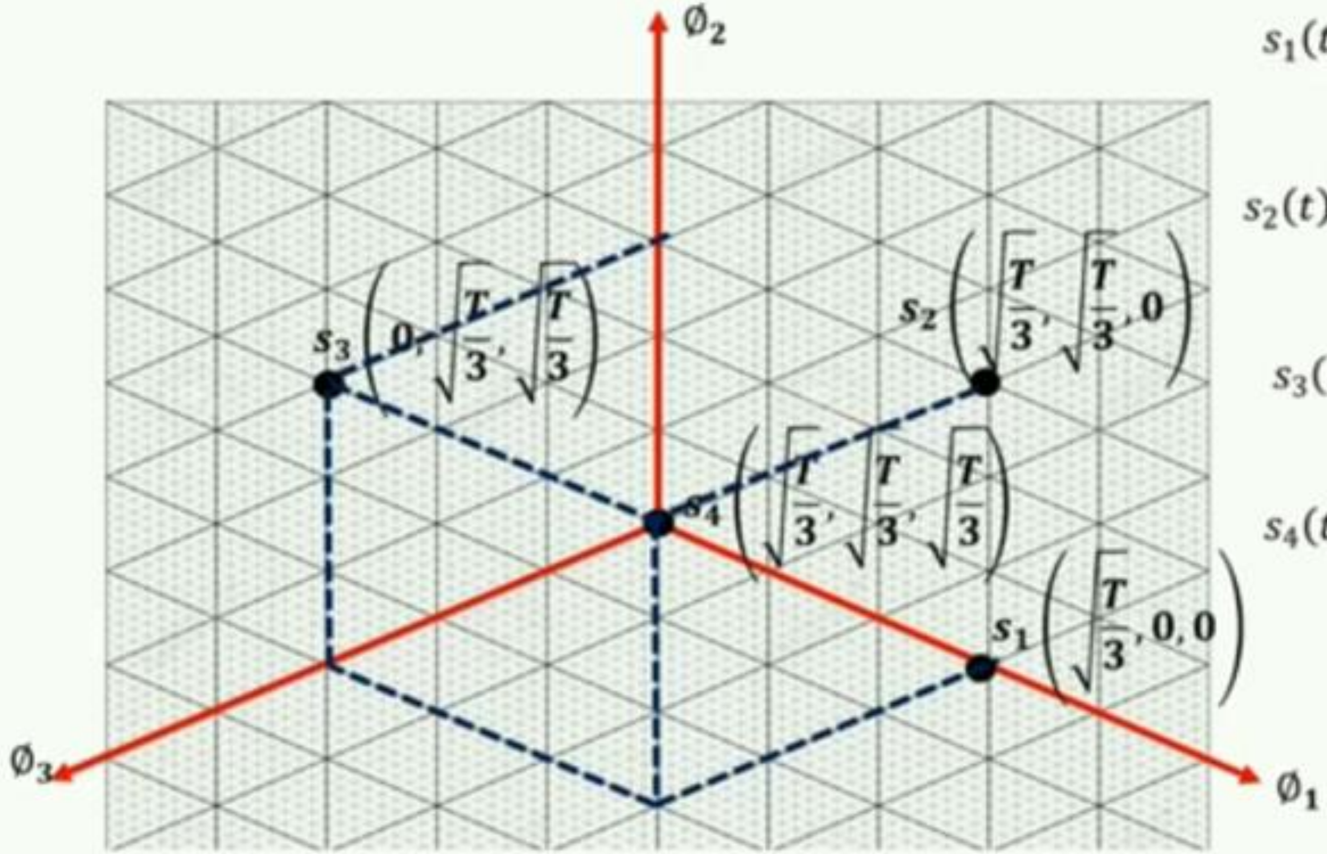
$s_1(t) = s_{11}\emptyset_1(t) \Rightarrow s_1(t) = s_{11}\,\emptyset_1(t) + 0\emptyset_2(t) + 0\emptyset_3(t) \Rightarrow s_1(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + 0\emptyset_2(t) + 0\emptyset_3(t)$

$s_2(t) = s_{21}\emptyset_1(t) + s_{22}\emptyset_2(t) \Rightarrow s_2(t) = s_{21}\,\emptyset_1(t) + s_{22}\emptyset_2(t) + 0\emptyset_3(t) \Rightarrow s_2(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + 0\emptyset_3(t)$

$s_3(t) = s_{31}\emptyset_1(t) + s_{32}\emptyset_2(t) + s_{33}\emptyset_3(t) \Rightarrow s_3(t) = 0\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + \sqrt{\frac{T}{3}}\emptyset_3(t)$

$s_4(t) = s_1(t) + s_3(t) \Rightarrow s_4(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + \sqrt{\frac{T}{3}}\emptyset_3(t)$

SIGNAL SPACE DIAGRAM / CONSTELLATION DIAGRAM



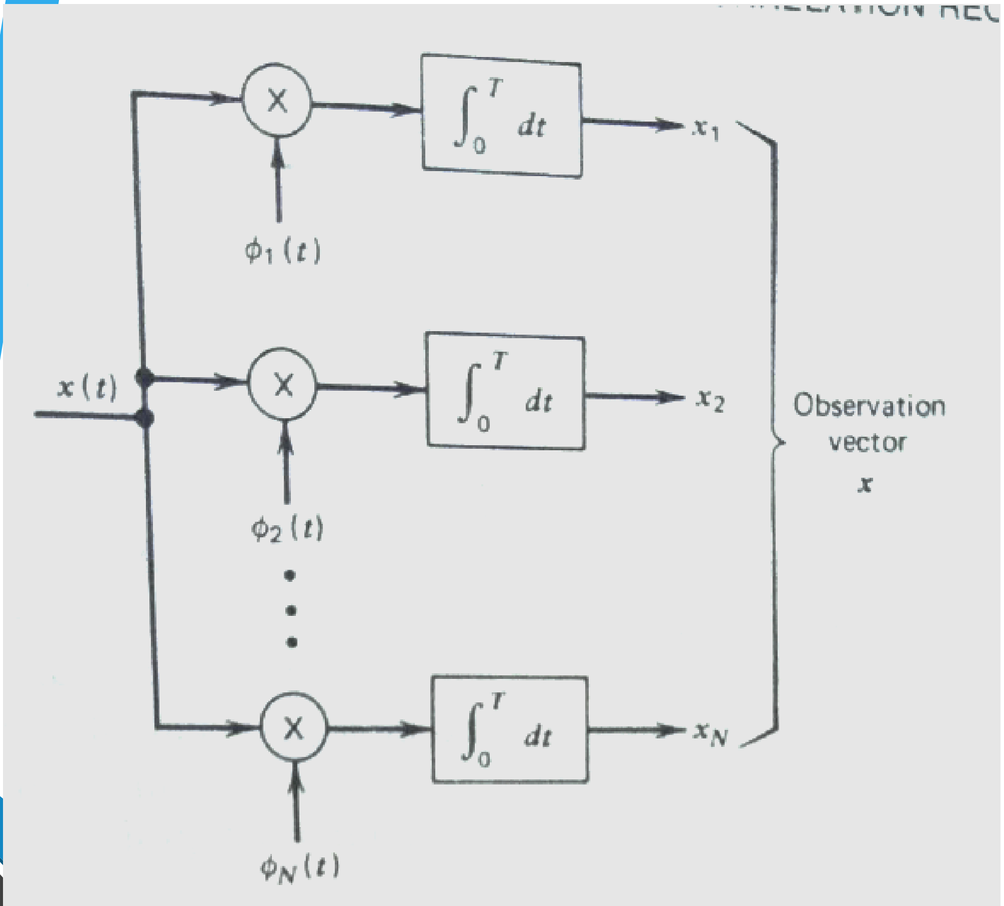
$$s_1(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + 0\emptyset_2(t) + 0\emptyset_3(t)$$

$$s_2(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + 0\emptyset_3(t)$$

$$s_3(t) = 0\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + \sqrt{\frac{T}{3}}\emptyset_3(t)$$

$$s_4(t) = \sqrt{\frac{T}{3}}\emptyset_1(t) + \sqrt{\frac{T}{3}}\emptyset_2(t) + \sqrt{\frac{T}{3}}\emptyset_3(t)$$

Optimum receivers-Matched and correlator receivers



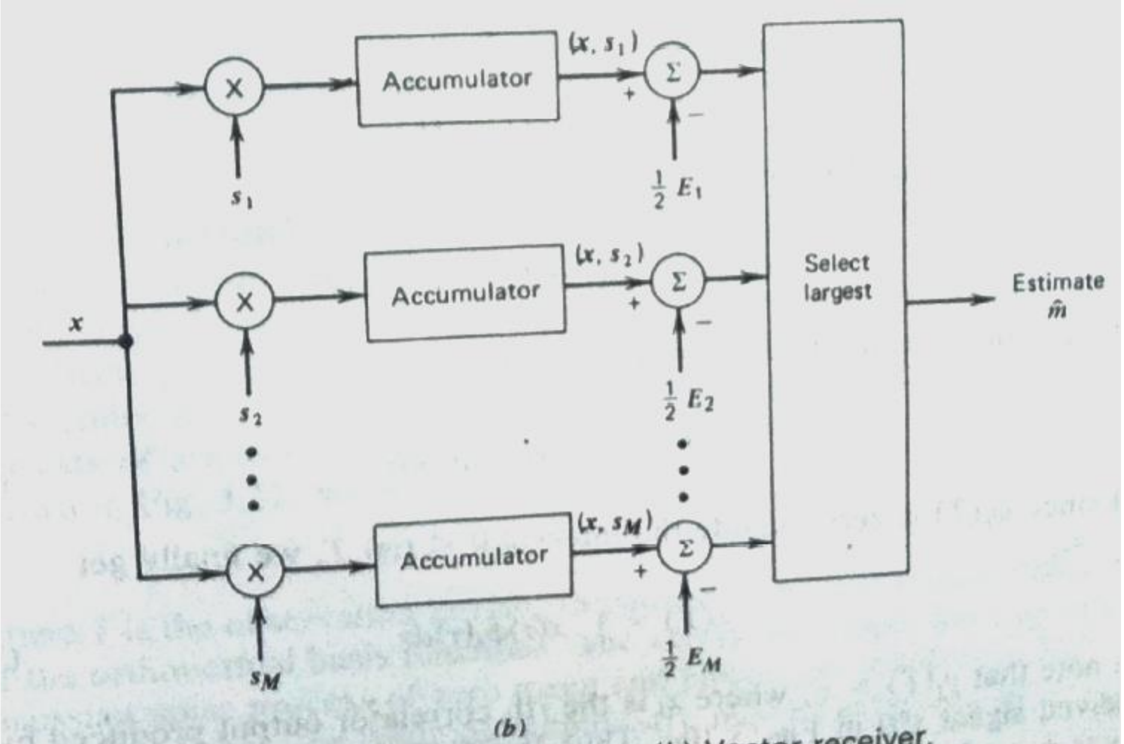


Figure 3.10 (a) Detector. (b) Vector receiver.

Matched filter

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)h_j(t - \tau)d\tau$$

Suppose we now set the impulse response

$$h_j(t) = \phi_j(T - t)$$

Then the resulting filter output is

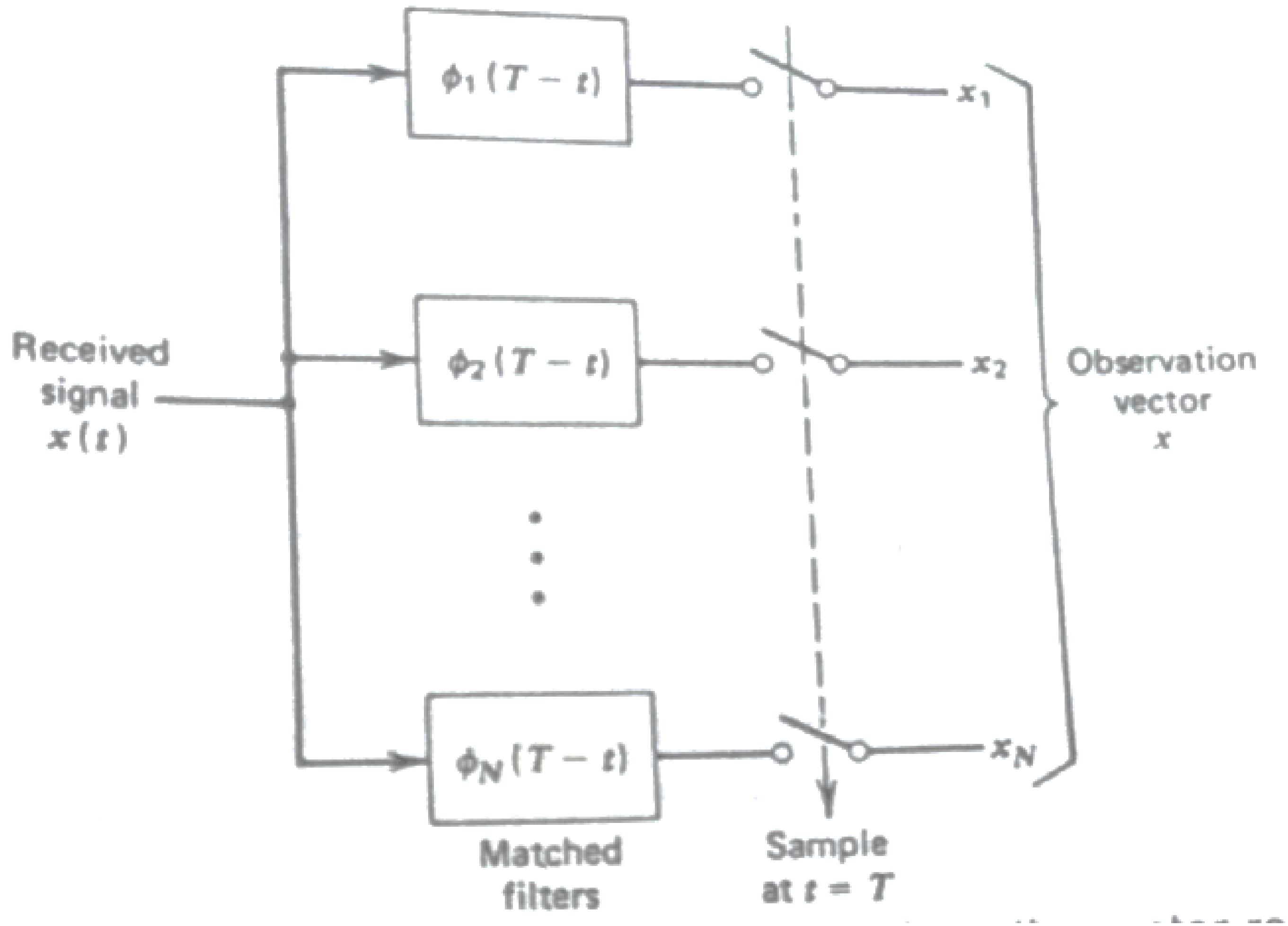
$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\phi_j(T - t + \tau)d\tau$$

Sampling this output at time $t = T$, we get

$$y_j(T) = \int_{-\infty}^{\infty} x(\tau)\phi_j(\tau)d\tau$$

and since $\phi_j(T)$ is zero outside the interval $0 \leqslant t \leqslant T$, we finally get

$$y_j(T) = \int_{0}^{T} x(\tau)\phi_j(\tau)d\tau$$



Received
signal
x(t)
φ1(T − t)
φ2(T − t)
φN(T − t)
x1
x2
xN
Observation
vector
x
Matched
filters
Sample
at t = T

(2) Properties of Matched Filters

We note that a filter, which is matched to a known signal $\phi(t)$ of duration T seconds, is characterized by an impulse response that is a time-reversed and delayed version of the input $\phi(t)$, as shown by

$$h_{opt}(t) = \phi(T - t) \tag{3.98}$$

In the frequency domain, the matched filter is characterized by a transfer function that is, except for a delay factor, the complex conjugate of the Fourier transform of the input $\phi(t)$, as shown by

$$H_{opt}(f) = \Phi^*(f)\exp(-j2\pi fT) \tag{3.99}$$

Based on this fundamental pair of relations, we may derive some important properties of matched filters, which should help the reader develop an intuitive grasp of how a matched filter operates.

PROPERTY 1

The spectrum of the output signal of a matched filter with the matched signal as input is, except for a time delay factor, proportional to the energy spectral density of the input signal.

Let $\Phi_o(f)$ denote the Fourier transform of the filter output $\phi_o(t)$. Then

$$\begin{aligned}\Phi_o(f) &= H_{opt}(f)\Phi(f)\\ &= \Phi^*(f)\Phi(f)\exp(-j2\pi fT)\\ &= |\Phi(f)|^2 \exp(-j2\pi fT)\end{aligned} \tag{3.100}$$

which is the desired result.

PROPERTY 2

The output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the input signal to which the filter is matched.

This property follows directly from Property 1, recognizing that the autocorrelation function and energy spectral density of a signal form a Fourier transform pair. Thus, taking the inverse Fourier transform of Eq. 3.100, we may express the matched-filter output as

$$\phi_o(t) = R_\phi(t - T) \tag{3.101}$$

where $R_\phi(\tau)$ is the autocorrelation function of the input $\phi(t)$ for lag τ. Equation 3.101 is the desired result. Note that at time $t = T$, we have

$$\phi_o(T) = R_\phi(0) = E \tag{3.102}$$

where E is the signal energy. That is, in the absence of noise, the maximum value of the matched-filter output, attained at time $t = T$, is proportional to the signal energy.

PROPERTY 3

The output signal-to-noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input.

To demonstrate this property, consider a filter matched to an input signal $\phi(t)$. From Property 2, the maximum value of the filter output, at time $t = T$, is proportional to the signal energy E. Substituting Eq. 3.92 in Eq. 3.87 gives the average output noise power as

$$E[n^2(t)] = \frac{N_0}{2} \int_{-\infty}^{\infty} |\Phi(f)|^2 df = \frac{N_0}{2} E \tag{3.103}$$

where we have made use of Rayleigh's energy theorem. Therefore, the output signal-to-noise ratio has the maximum value

$$(\mathrm{SNR})_{O,\max} = \frac{E^2}{N_0 E/2} = \frac{2E}{N_0} \tag{3.104}$$

PROPERTY 4

The matched-filtering operation may be separated into two matching conditions; namely, spectral phase matching that produces the desired output peak at time T, and spectral amplitude matching that gives this peak value its optimum signal-to-noise density ratio.

In polar form, the spectrum of the signal $\phi(t)$ being matched may be expressed as

$$\Phi(f) = |\Phi(f)|\exp[j\theta(f)]$$

where $|\Phi(f)|$ is the amplitude spectrum and $\theta(f)$ is the phase spectrum of the signal. The filter is said to be *spectral phase matched* to the signal $\phi(t)$ if the transfer function of the filter is defined by*

$$H(f) = |H(f)|\exp[-j\theta(f) - j2\pi fT]$$

where $|H(f)|$ is real and nonnegative and T is a positive constant. The output of such a filter is

$$\begin{aligned}\phi_o'(t) &= \int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi ft)df \\ &= \int_{-\infty}^{\infty} |H(f)||\Phi(f)|\exp[j2\pi f(t-T)]df\end{aligned}$$

where the product $|H(f)||\Phi(f)|$ is real and nonnegative. The spectral phase matching ensures that all the spectral components of the output $\phi_o'(t)$ add constructively at time $t = T$, thereby causing the output to attain its maximum value, as shown by

$$\phi_o'(t) \leq \phi_o'(T) = \int_{-\infty}^{\infty} |\Phi(f)||H(f)|df$$

For *spectral amplitude matching*, we choose the amplitude response $|H(f)|$ of the filter to shape the output for best signal-to-noise ratio at $t = T$ by using

$$|H(f)| = |\Phi(f)|$$

Inter symbol interference

$$x(t) = \sum_{k=-\infty}^{\infty} A_k g(t - kT_b)$$

$$A_k = \begin{cases} +a, & \text{symbol 1} \\ -a, & \text{symbol 0} \end{cases}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \mu A_k p(t - kT_b)$$

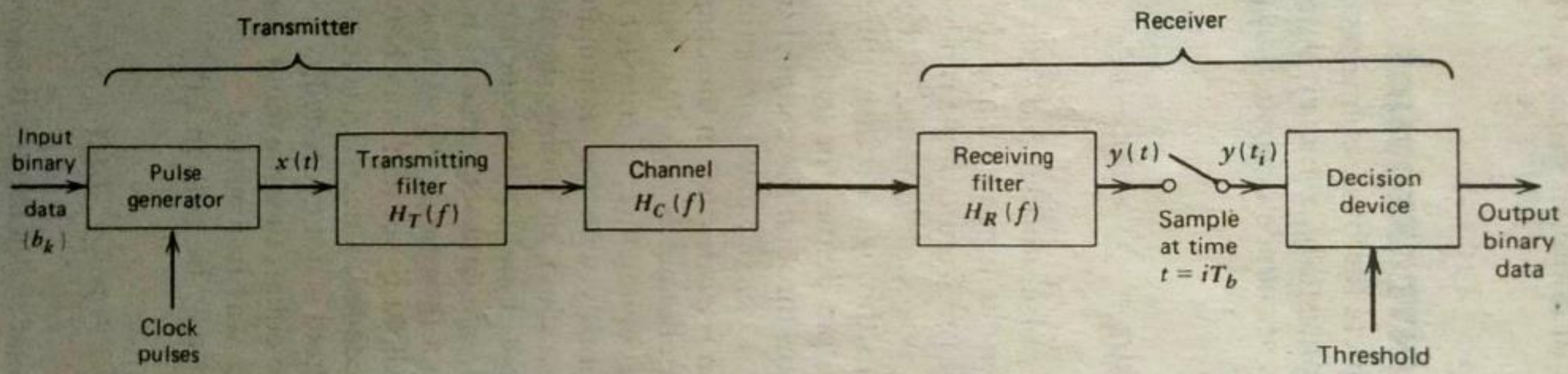


Figure 6.5 Baseband binary data transmission system.

$$x(t) = \sum_{k=-\infty}^{\infty} A_k g(t - kT_b)$$

$$g(t) \xleftrightarrow{\text{FT}} G(f),$$

$$g(t - t_0) \xleftrightarrow{\text{FT}} e^{-j2\pi f t_0} G(f)$$

$$X(f) = \sum_{k=-\infty}^{\infty} A_k G(f) e^{-j2\pi f k T_b}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \mu A_k p(t - kT_b)$$

$$y(t) = \sum_{k=-\infty}^{\infty} \mu A_k p(t - kT_b)$$

$$Y(f) = \sum_{k=-\infty}^{\infty} \mu A_k P(f) e^{-j2\pi f k T_b}$$

$t = iT_b$, where $i = 0, \pm 1, \pm 2, \ldots$, we get

$$\begin{aligned} y(iT_b) &= \sum_{k=-\infty}^{\infty} \mu A_k p(iT_b - kT_b) \\ &= \sum_{k=-\infty}^{\infty} \mu A_k p[(i-k)T_b] \\ \Rightarrow y(iT_b) &= \mu A_i p(0) + \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} \mu A_k p[(i-k)T_b] \end{aligned}$$

$$Y(f) = X(f) H_T(f) H_C(f) H_R(f)$$

$$\sum_{k=-\infty}^{\infty} \mu A_k P(f) e^{-j2\pi f k T_b} = \sum_{k=-\infty}^{\infty} A_k G(f) e^{-j2\pi f k T_b} H_T(f) H_C(f) H_R(f)$$

$$\Rightarrow \mu P(f) = G(f) H_T(f) H_C(f) H_R(f)$$

$$y(iT_b) = \mu A_i + \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} \mu A_k p[(i-k)T_b]$$

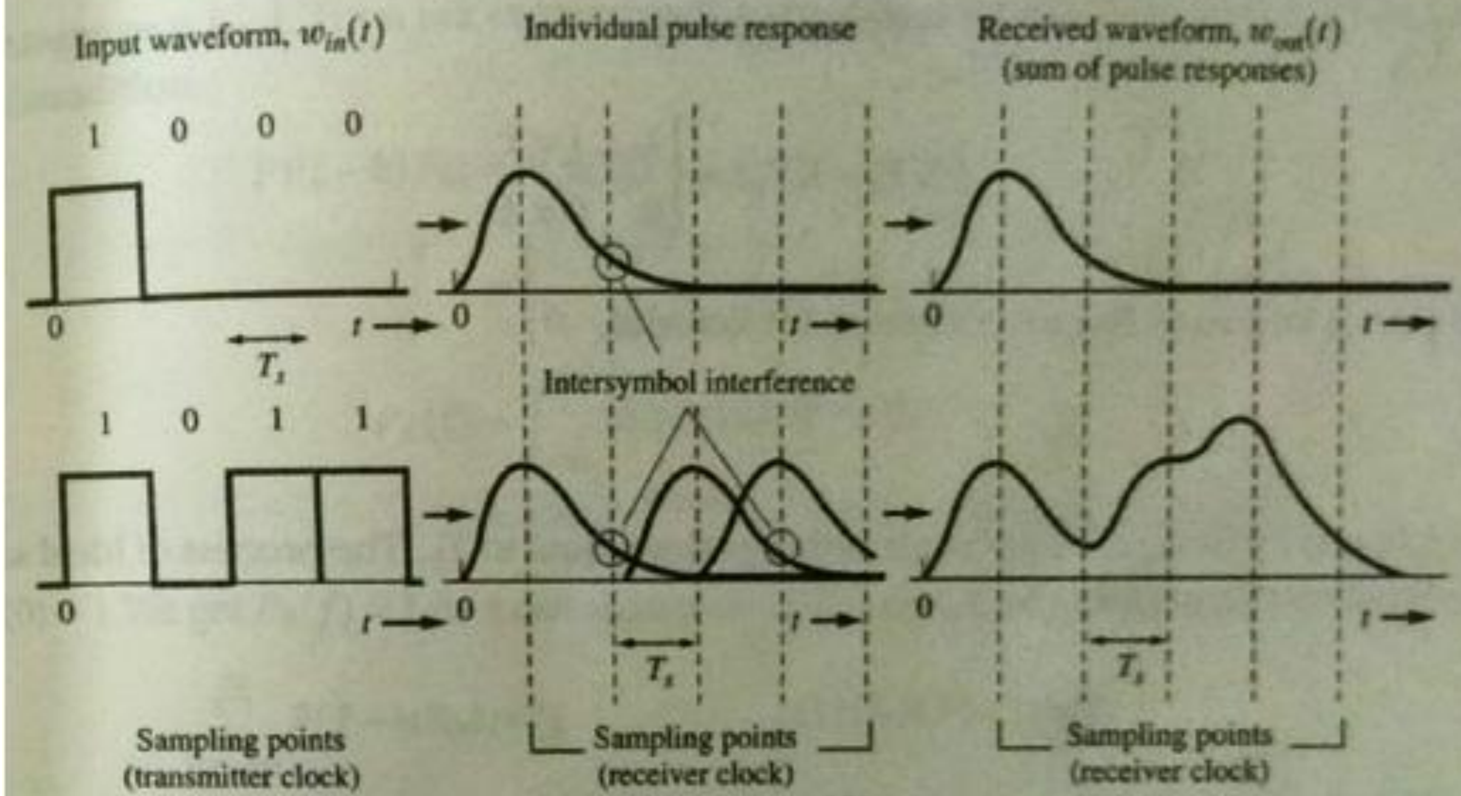


Figure 4.3: ISI

$$y(t_i) = \mu A_i + \sum_{\substack{K=-\infty \\ K\neq i}}^{\infty} A_K P[(i-K)T_b]$$

$$= \mu A_i \qquad \text{If} \sum_{\substack{K=-\infty \\ K\neq i}}^{\infty} A_K P[i-KT_b] = 0$$

Eye diagram

The ISI and other signal degradations can be studied through eye diagram.
The eye pattern is contains all the information concerning the degradation of quality

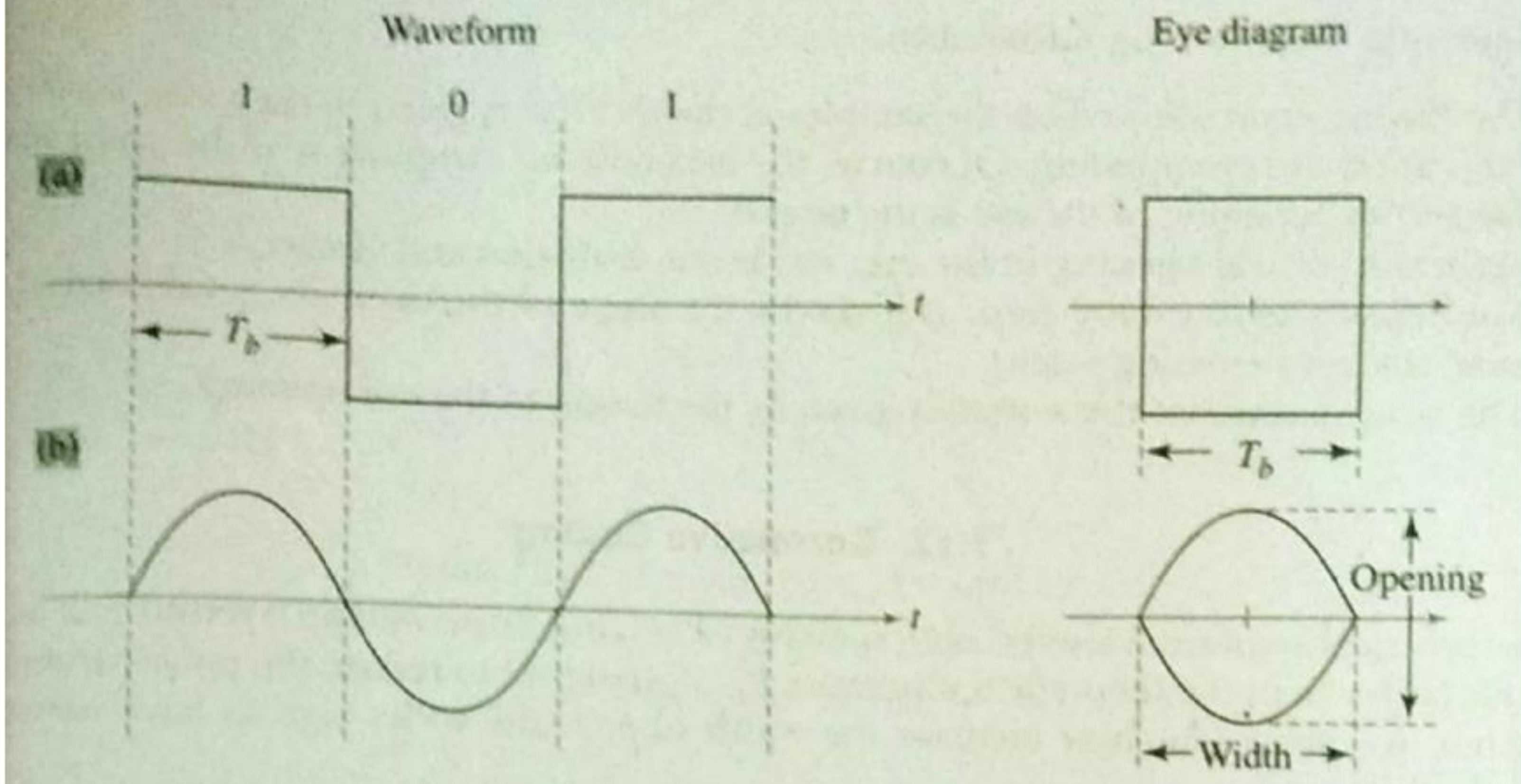


Figure 7.20 (a) and (b) Received waveforms and their eye diagrams.

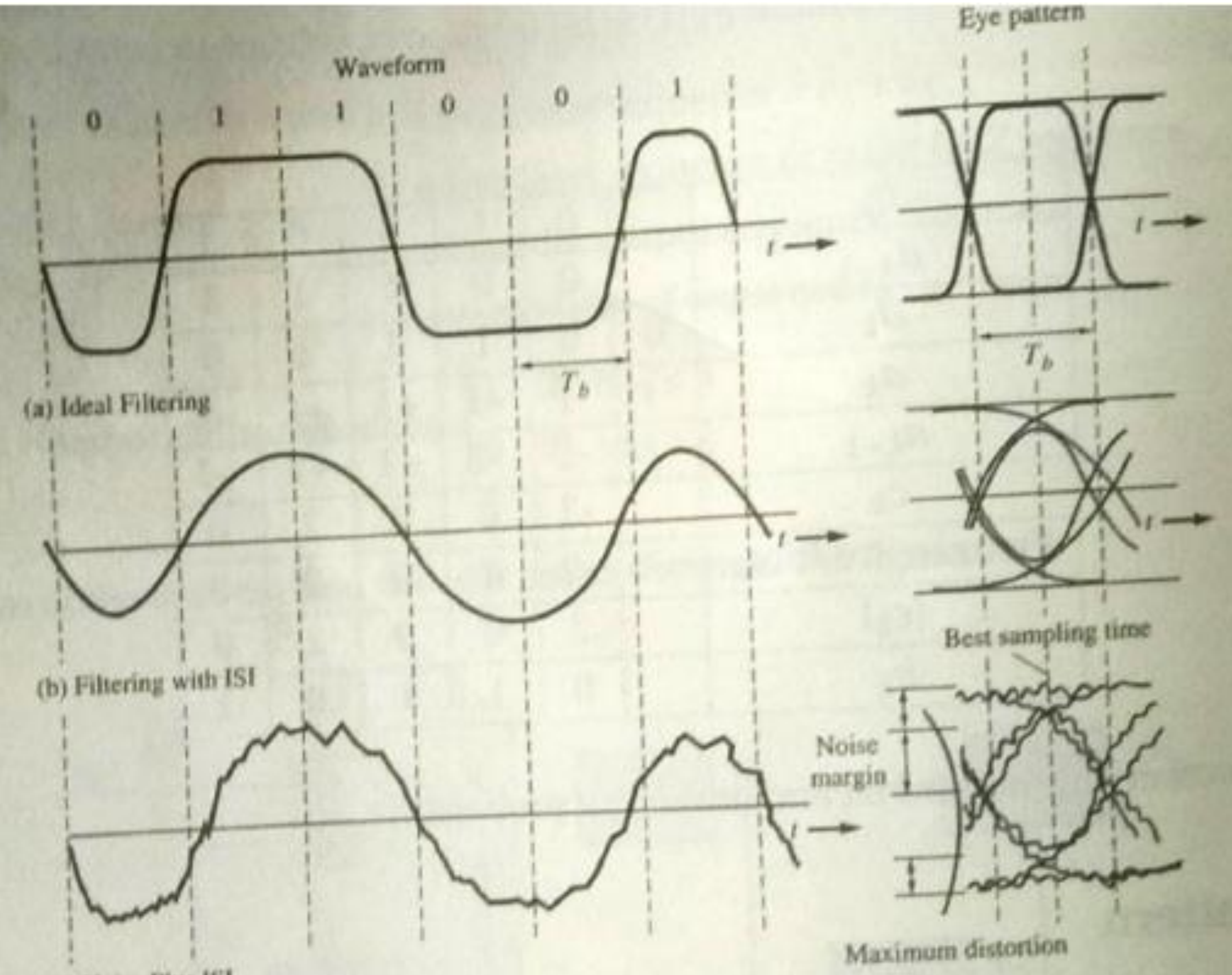


Figure 4.12: Eye pattern

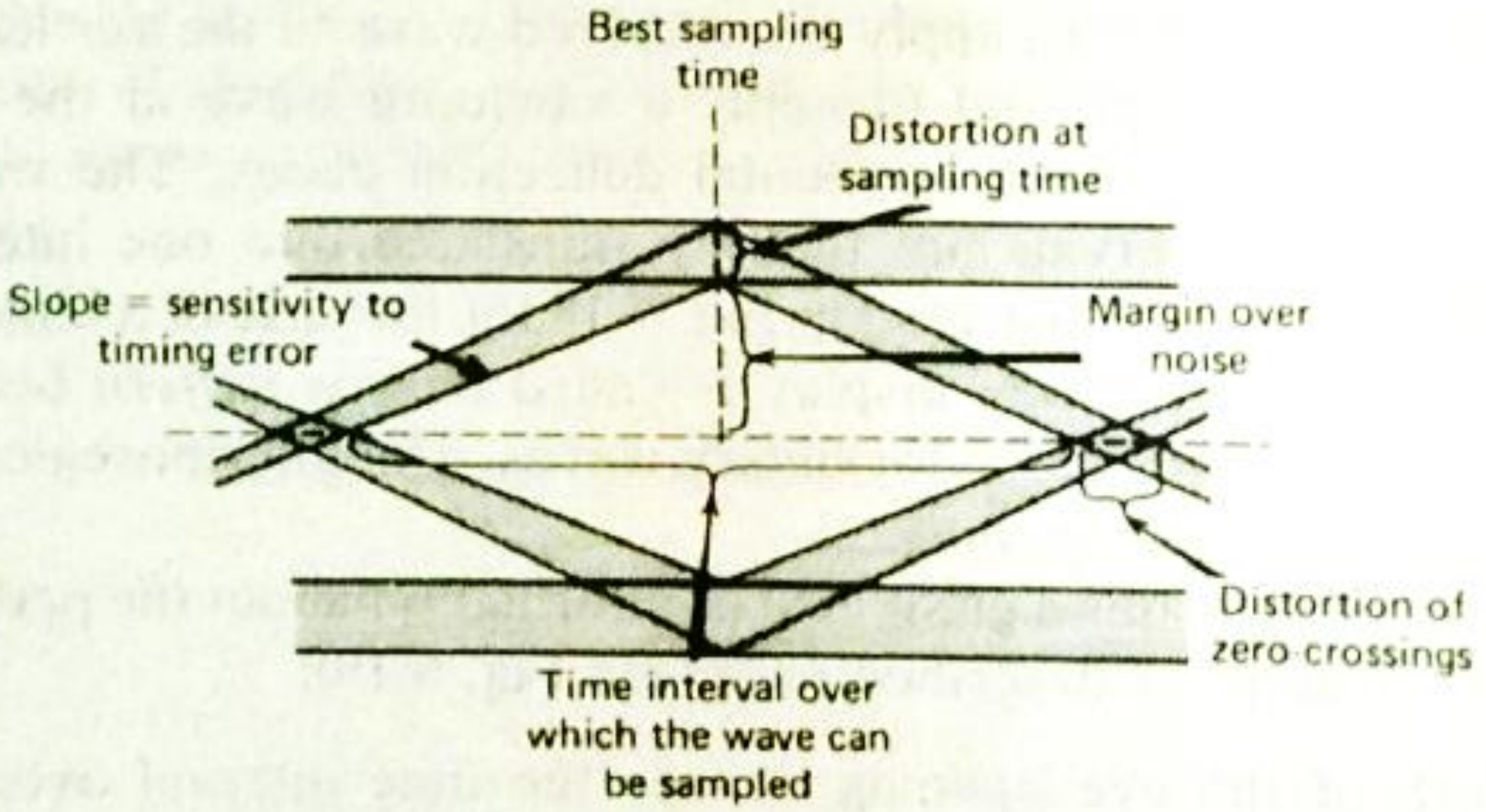


Figure 6.19 Interpretation of eye pattern.

Adaptive Equalisation for data transmission

- At the receiving end of the system the received wave is demodulated and then sampled and quantized.
- Due to dispersion of pulse shape by the channel, number of detectable amplitude levels is limited by ISI than noise.
- To realise the full transmission capability of a telephone channel, there is need for adaptive equalisation, the process is said to be adaptive when it adjusts itself continuously during data transmission by operating on the input signal

Adaptive Equalisation for data transmission

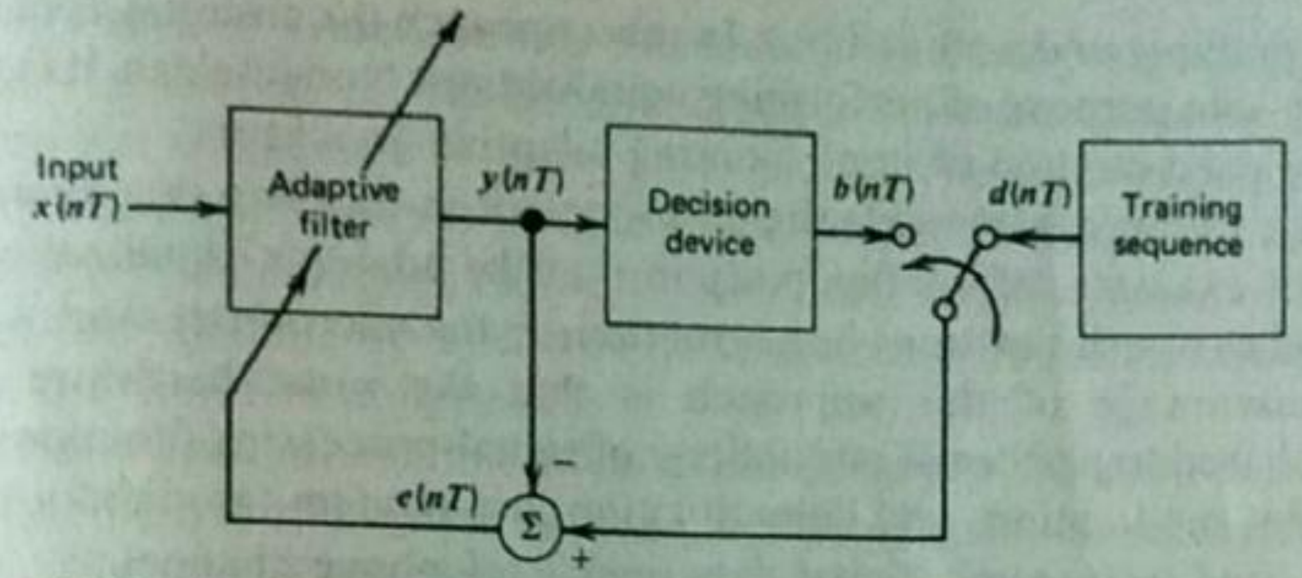


Figure 6.20 Illustrating the modes of operation of an adaptive equalizer.

$$y(nT) = \sum_{i=0}^{M-1} W_i x(nT - iT)$$

$$\{e(nT)\} = \{d(nT)\} - \{y(nT)\}, \quad n = 0, 1, \ldots, N-1$$

$$E = \sum_{n=0}^{N-1} e^2(nT)$$

$$\hat{W}_i(nT + T) = \hat{W}_i(nT) + \mu e(nT) x(nT - iT)$$