1.1 Digital Communication block diagram

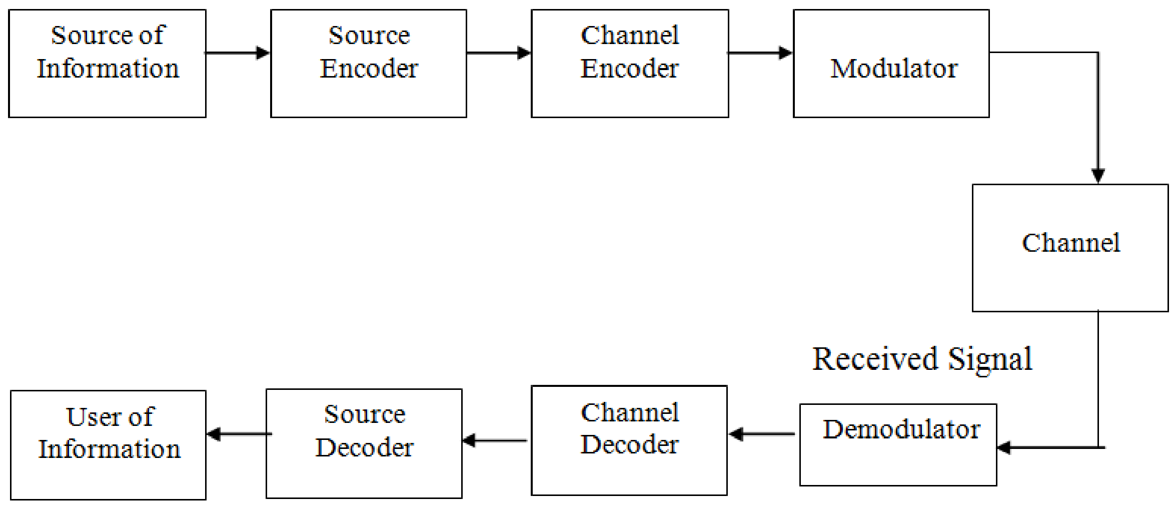


Fig. 1: Elements of digital communication system.

1.1.1 Information source:

The source of information can be analog or digital, e.g. analog: audio or video signal, digital: like teletype signal. In digital communication the signal produced by this source is converted into digital signal which consists of 1′s and 0′s.

1.1.2 Source Encoder

The Source encoder (or Source coder) converts the input i.e. symbol sequence into a binary sequence of 0‟s and 1‟s by assigning code words to the symbols in the input sequence. For eg. :-If a source set is having hundred symbols, then the number of bits used to represent each symbol will be 7 because 2^7=128 unique combinations are available. The important parameters of a source encoder are block size, code word lengths, average data rate and the efficiency of the coder (i.e. actual output data rate compared to the minimum achievable rate)

At the receiver, the source decoder converts the binary output of the channel decoder into a symbol sequence. The decoder for a system using fixed – length code words are quite simple, but the decoder for a system using variable – length code words will be very complex.

Aim of the source coding is to remove the redundancy in the transmitting information, so that bandwidth required for transmission is minimized. Based on the probability of the symbol code word is assigned. Higher the probability, shorter is the code word.

Ex: Huffman coding.

1.1.3 Channel Encoder

Error control is accomplished by the channel coding operation that consists of systematically adding extra bits to the output of the source coder. These extra bits do not convey any information but helps the receiver to detect and / or correct some of the errors in the information bearing bits.

There are two methods of channel coding:

- Block Coding: The encoder takes a block of "k" information bits from the source encoder and adds "r" error control bits, where "r" is dependent on "k" and error control capabilities desired.
- Convolution Coding: The information bearing message stream is encoded in a continuous fashion by continuously interleaving information bits and error control bits.

The Channel decoder recovers the information bearing bits from the coded binary stream. Error detection and possible correction is also performed by the channel decoder.

The important parameters of coder / decoder are: Method of coding, efficiency, error control capabilities and complexity of the circuit.

1.1.4 Modulator

It is performed for the efficient transmission of the signal over the channel. The modulator operates by keying shifts in the amplitude, frequency or phase of a sinusoidal carrier wave to the channel encoder output. The digital modulation techniques are referred to as amplitude-shift keying, frequency- shift keying or phase-shift keying respectively. The Modulator converts the input bit stream into an electrical waveform suitable for transmission over the communication channel. Modulator can be effectively used to minimize the effects of channel noise, to match the frequency spectrum of transmitted signal with channel characteristics, to provide the capability to multiplex many signals.

The detector performs demodulation, thereby producing a signal the follows the time variations in the channel encoder output. The modulator, channel and detector form a discrete channel (because both its input and output signals are in discrete form.

1.1.5 Channel

The Channel provides the electrical connection between the source and destination. The different channels are: Pair of wires, Coaxial cable, Optical fibre, Radio channel, Satellite channel or combination of any of these.

The communication channels have only finite Bandwidth, non-ideal frequency response, the signal often suffers amplitude and phase distortion as it travels over the channel. Also, the signal power decreases due to the attenuation of the channel. The signal is corrupted by unwanted, unpredictable electrical signals referred to as noise.

The important parameters of the channel are Signal to Noise power Ratio (SNR), usable bandwidth, amplitude and phase response and the statistical properties of noise.

1.2. Information

The output of a discrete information source is a message that consists of a sequence of symbols. The actual message that is emitted by the source during a message interval is selected at random from a set of possible messages. The communication system is designed to reproduce at the receiver either exactly or approximately the message emitted by the source.

To measure the information content of a message quantitatively, we are required to arrive at an intuitive concept of the amount of information.

Consider the examples: A trip to Miami, Florida from Minneapolis in the winter time,

- mild and sunny day,
- cold day,
- possible snow flurries.

The amount of information received is obviously different for these messages.

- The first message contains very little information since the weather in Miami is mild and sunny most of the time.
- The forecast of a cold day contains more information since it is not an event that occurs often.
- In contrast, the forecast of snow flurries conveys even more information since the occurrence of snow in Miami is a rare event.

Thus on intuitive basis the amount of information received from the knowledge of occurrence of an event is related to the probability or the likelihood of occurrence of the event. The message associated with an event least likely to occur contains most information.

The information content of a message can be expressed quantitatively in terms of probabilities as follows:

Suppose an information source emits one of 'q' possible messages m_1, m_2 …… m_q with p_1, p_2 …… p_q as their probs. of occurrence. Based on the above intusion, the information content of the k^{th} message, can be written as

$$\mathrm{I}\ (m_k) \alpha \frac{1}{p_k}$$

Also to satisfy the intuitive concept, of information.

$I(m_k)$ must $\rightarrow$ zero as $p_k \rightarrow 1$

Therefore,

$$\boxed{\begin{array}{lll} I(m_k) > I(m_j); & \text{if} & p_k < p_j \\ I(m_k) \rightarrow O(m_j); & \text{if} & p_k \rightarrow 1 \\ I(m_k) \geq O; & \text{when} & O < p_k < 1 \end{array}}$$

Another requirement is that when two independent messages are received, the total information content is – Sum of the information conveyed by each of the messages.

Thus the equation becomes

$$I(m_k \text{ and } m_j) \triangleq I(m_k m_j) = I(m_k) + I(m_j)$$

Whrere m_k and m_j are two independent messages.

A continuous function of p that satisfies the constraints specified in the above equations is the logarithmic function and we can define a measure of information as

$$I(m_k) = \log\left(\frac{1}{p_k}\right)$$

The base for the logarithmic in equation determines the unit assigned to the information content.

Natural logarithm base : 'nat'

Base - 10 : Hartley / decit

Base - 2 : bit

Using the binary digit as the unit of information is based on the fact that if two possible binary digits occur with equal proby ($p_1 = p_2 = ½$) then the correct identification of the binary digit conveys an amount of information. $I(m_1) = I(m_2) = -\log_2(½) = 1$ bit. Therefore one bit is the amount of information that we gain when one of two possible and equally likely events occurs.

Ex1: A source puts out one of five possible messages during each message interval. The probabilities of these messages are $P_1 = 1/2$, $P_2 = 1/4$, $P_3 = 1/4$, $P_4 = 1/16$, $P_5 = 1/16$. What is the information content of these messages?

Solution:

$$I(m_1) = log_2 \frac{1}{(1/2)} = 1\ bits$$

$$I(m_2) = log_2 \frac{1}{(1/4)} = 2\ bits$$

$$I(m_3) = log_2 \frac{1}{(1/4)} = 2\ bits$$

$$I(m_4) = log_2 \frac{1}{(1/16)} = 4\ bits$$

$$I(m_5) = log_2 \frac{1}{(1/16)} = 4\ bits$$

1.3 Entropy

Suppose a source that emits one of *M* possible symbols $s_1, s_2, \ldots\ldots s_M$ in a statistically independent sequence. Let $p_1, p_2 \ldots\ldots p_q$ be the probabilities of occurrence of the *M* symbols, respectively. In a long message containing *N* symbols, the symbol s_1 will occur on the average p_1N times, the symbol s_2 will occur p_2N times, and in general the symbol s_M will occur p_MN times. The information content of the i_{th} symbol is I (s_i)= $\log_2\left(\frac{1}{p_i}\right)$.

Therefore p_1N number of messages of type s_1 contains $p_1N\log_2\left(\frac{1}{p_1}\right)$bits. Similarly p_2N number of messages of type s_1 contains $p_2N\log_2\left(\frac{1}{p_2}\right)$bits.

∴ Total self-information contains all these messages as

$$I_{total} = p_1N\log_2\left(\frac{1}{p_1}\right) + p_2N\log_2\left(\frac{1}{p_2}\right) + \cdots \ldots \ldots \ldots \ldots . + p_MN\log_2\left(\frac{1}{p_M}\right)$$

$$I_{total} = N\sum_{i=1}^{M} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits}$$

The average information *per* symbol is obtained by dividing the total information content of the message by the number of symbols in the message, as

$$Entropy = H = \frac{T_{total}}{N} = \sum_{i=1}^{M} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits/symbol}$$

1.3.1 Average information rate

If the symbols are emitted by source at a fixed time rate $r_{s,}$ then the average information rate R_s is given by $R_s = r_s * H$ bits/sec

Eamples:

1. Consider a discrete memoryless source with a source alphabet A= ($s_{0,}s_{1,}s_2$) with respective probabilities $p_0 = \frac{1}{4}, p_1 = \frac{1}{4}, p_2 = \frac{1}{2}$. Find the entropy of the source.

Solution: By definition, the entropy of a source is given by

$$H = \sum_{i=1}^{M} p_i \log \frac{1}{p_i} \text{ bits/ symbol}$$

H for this example is

$$H(A) = \sum_{i=0}^{2} p_i \log \frac{1}{p_i}$$

Substituting the values given, we get

$$H(A) = p_o \log \frac{1}{P_o} + P_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2}$$

$$= ¼ \log_2 4 + ¼ \log_2 4 + ½ \log_2 2$$

$$= \left(\frac{3}{2}\right) = 1.5 \text{ bits}$$

if $r_s = 1$ per sec, then

$$\mathbf{H'(A) = r_s\, H(A) = 1.5 \text{ bits/sec}}$$

2. **An analog signal is band limited to B Hz, sampled at the Nyquist rate, and the samples are quantized into 4-levels. The quantization levels Q_1, Q_2, Q_3, and Q_4 (messages) are assumed independent and occur with probs.**

   $P_1 = P_2 = \frac{1}{8}$ and $P_2 = P_3 = \frac{3}{8}$. Find the information rate of the source.

Solution: By definition, the average information H is given by

$$H = p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2} + p_3 \log \frac{1}{p_3} + p_4 \log \frac{1}{p_4}$$

Substituting the values given, we get

$$H = \frac{1}{8} \log 8 + \frac{3}{8} \log \frac{8}{3} + \frac{3}{8} \log \frac{8}{3} + \frac{1}{8} \log 8$$

$$= 1.8 \text{ bits/ message.}$$

Information rate of the source by definition is

$$R = r_s\, H$$

$$R = 2B, (1.8) = (3.6\, B) \text{ bits/sec}$$

3. **Compute the values of H and R, if in the above example, the quantities levels are so chosen that they are equally likely to occur,**

Solution:

Average information per message is

$H = 4\ (\frac{1}{4} \log_2 4) = 2$ bits/message

and $R = r_s\ H = 2B\ (2) = (4B)$ bits/sec

PROPERTIES OF ENTROPY:

1. Entropy is average self information. It is given by

$$H(S) = \sum_{i=1}^{q} P_i \log (1/P_i)$$

$$H(S) = \sum_{i=1}^{q} P_i I_i \quad \text{bits/symbol}$$

where q represents number of symbols and also $\sum_{i=1}^{q} P_i = 1$.

2. For null event and sure event, the entropy vanishes.
3. The entropy is a symmetrical function of its arguments. The value of $H(S)$ remains the same irrespective of location of probabilities.
4. Extremal property: When all the source symbols become equiprobable, then the entropy attains the maximum value.

$$H(S)_{max} = \log_2 q.$$

5. When source symbols are not equiprobable, then entropy is less than maximum value.
6. The source efficiency, η_s is given by

$$\eta_s = \frac{H(S)}{H(S)_{max}}$$

7. The source redundancy $R_{\eta s}$ is given by

$$R_{\eta_s} = 1 - \eta_s$$

Usually efficiency and redundancy are represented in percentage.

Q) A binary source is emitting an independent sequence of 0's and 1's with probabilities P and (1-P) respectively. Plot the entropy of the source versus P.

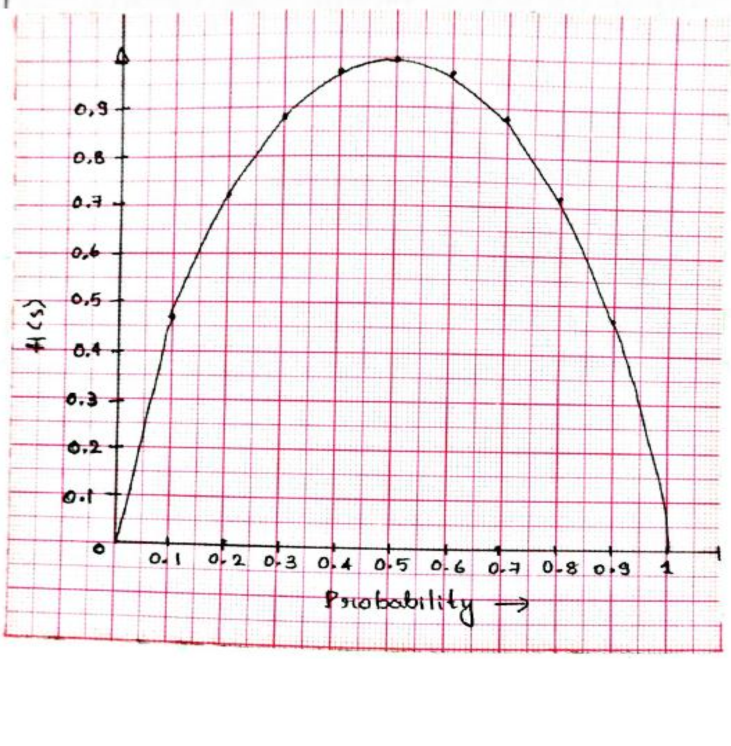
Sol^n: The entropy of the binary source is given by

$$H(S) = \sum_{i=1}^{2} P_i \log_2 \left(\frac{1}{P_i}\right)$$

$$H(S) = P \log_2 \left(\frac{1}{P}\right) + (1-P) \log_2 \left(\frac{1}{1-P}\right) \rightarrow (1)$$

Type eq^n (1) in calculator and 'calculate' the values of H(s) for assumed values of P. P should vary from 0.1 to 1

P	H(s)
0.1	0.469
0.2	0.722
0.3	0.881
0.4	0.971
0.5	1
0.6	0.971
0.7	0.881
0.8	0.722
0.9	0.469
1.0	—



Q5 A discrete message source 's' emits two independent symbols x and y with probability 0.55 and 0.45 respectively. Calculate the efficiency of the source and its redundancy.

Sol^n: $P(X) = P_x = 0.55$; $P(y) = P_y = 0.45$

$$H(s) = \sum_{i=1}^{2} P_i \log 1/P_i$$

$$= 0.55 \log \left(\frac{1}{0.55}\right) + 0.45 \log \left(\frac{1}{0.45}\right)$$

$$H(s) = 0.9928 \text{ bits/message symbol}$$

$$H(S)_{max} = \log_2 q \qquad | \text{ Here } q = 2$$

$$\therefore H(S)_{max} = \log_2 2 = 1$$

$$\eta_s = \frac{H(s)}{H(S)_{max}} = \frac{0.9928}{1} = \underline{\underline{0.9928}} \ / \ 99.28\%$$

$$R_{\eta_s} = 1 - \eta_s = \underline{\underline{0.0072}} \ / \ 0.72\%$$

1.3 Shannon's Encoding algorithm

Source encoding is the process by which the output of an information source is converted in to an r-array sequence. Coding is nothing but transformation of each of the source symbol S=$\{s_1, s_2, s_3 \dots \dots . . s_q\}$ using the symbols from the source alphabet X=$\{q_1, q_2, q_3 \dots \dots . . q_r)$.

In binary coding r represents number of different symbols used in the code alphabet. That is r=2 => X=(0,1). In general if $\{s_1, s_2, s_3 \dots \dots . . s_q)$ are to be transmitted, then q number of different states are required. In binary coding only 2 states are required. Hence the transmission process becomes much easier and efficiency of the system can be increased.

- Let the source symbols in the order of decreasing probabilities

  S=$\{s_1, s_2, s_3 \dots \dots . . s_q)$

  P=$\{p_1, p_2, p_3 \dots \dots . . p_q)$.

  $p_1 \geq p_2 \geq p_3 \dots \dots . \geq p_q$

- Compute the sequence

  α_1=0

  α_2=$p_1 = p_{1+}\alpha_1$

  $\alpha_3 = p_2 + p_1 = p_2 + \alpha_2$

  $\alpha_4 = p_3 + p_2 + p_1 = p_3 + \alpha_3$

  .

  .

  $\alpha_{q+1} = p_q + \alpha_q = 1$

- Determine the smallest integer for l_i (length of code word) using the inequality

  $2^{l^i} \geq \frac{1}{p_i}$ for all i=1 to q

- Expand the decimal numbers α_i in binary form up to l_i places neglecting the expansion beyond l_i places.
- Remove the decimal point to get the desired code.

Code efficiency: The average length 'L' of any code is given by $\boldsymbol{L = \sum_{i=1}^{q} p_i L_i}$ where $L_i = l_1 l_2 l_3 l_4 \ldots \ldots l_q$

Code efficiency, $\boldsymbol{\eta_c} = \frac{\boldsymbol{H(S)}}{\boldsymbol{L}} * \mathbf{100}$ for binary codes.

Ex: 1. Construct the Shannon's binary code for the following message symbols S={$s_{1,}s_{2,}s_3, s_4$) with probabilities P=(0.4, 0.3,0.2,0.1).

Solution:

- ***0.4*** > ***0.3*** > ***0.2*** > ***0.1***
- α*0* = ***0,***
  α*1* = ***0.4***
  α*2* = ***0.4+0.3=0.7***
  α*3*= ***0.7+0.2=0.9***
  α*4*= ***0.9*** + ***0.1*** = ***1.0***
- $2^{-l_1} \leq 0.4 \rightarrow l_1 = 2$

  $2^{-l_2} \leq 0.3 \rightarrow l_2 = 2$

  $2^{-l_3} \leq 0.2 \rightarrow l_3 = 3$

  $2^{-l_4} \leq 0.1 \rightarrow l_4 = 4$
-

$\alpha_0 = 0 = 0.00 \,|\underline{0}$

$\alpha_1 = 0.4 = 0.01 \,|\underline{10}$

$\alpha_2 = 0.7 = 0.101 \,|\underline{10}$

$\alpha_3 = 0.9 = 0.1110 \,|\underline{01}$

- The codes are
  $s_1 \rightarrow 00,\ s_2 \rightarrow 01,\ s_3 \rightarrow 101,\ s_4 \rightarrow 1110$

S_i	P_i	α_i	l_i	binary	Code
S_1	0.4	0	2	$(0.00000\cdots)_2$	00
S_2	0.3	0.4	2	$(0.01100\cdots)_2$	01
S_3	0.2	0.7	3	$(0.101100\cdots)_2$	101
S_4	0.1	0.9	4.	$(0.11100\cdots)_2$	1110

0.4×2
$0.8\times2 \rightarrow 0$
$1.6 \rightarrow 1$
0.6×2
$1.2 \rightarrow 1$
0.2×2
$0.4 \rightarrow 0$
$\Rightarrow 0.0110\cdots$

0.7×2
$1.4 \rightarrow 1$
0.4×2
$0.8\times2 \rightarrow 0$
$1.6 \rightarrow 1$
$0.6\times2 \rightarrow 1$
1.2
0.2×2
$0.4\times2 \rightarrow 0$
$0.8 \rightarrow 0$
$\Rightarrow 0.101100\cdots$

0.9×2
$1.8 \rightarrow 1$
0.8×2
$1.6 \rightarrow 1$
$0.6\times2 \rightarrow 1$
1.2
$0.2\times2 \rightarrow 0$
0.4×2
$0.8\times2 \rightarrow 0$
$\Rightarrow 0.11100\cdots$

The average length of this code is

$$L = 2 \times 0.4 + 2 \times 0.3 + 3\times 0.2 + 3 \times 0.1 = 2.4 \text{ Binits / message}$$

$$H(S) = 0.4 \log \frac{1}{0.4} + 0.3 \log \frac{1}{0.3} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} = 1.84644 \text{ bits / message;}$$

$$\%\eta_c = \frac{H(S)}{L} * 100 = \frac{1.8464}{2.4} * 100 = 76.93\%$$

Ex: 2: Apply Shannon’s binary encoding procedure to the following set of messages and obtain code efficiency and redundancy.

1/8, 1/16, 3/16, 1/4, 3/8

Solution:

$\alpha_1 = 0$

$\alpha_2 = P_1 = 3/8 = 0.375$

$\alpha_3 = P_2 + \alpha_2 = 1/4 + 3/8 = 0.625$

$\alpha_4 = P_3 + \alpha_3 = 3/16 + 0.625 = 0.8125$

$\alpha_5 = P_4 + \alpha_4 = 1/8 + 0.8125 = 0.9375$

S_i	P_i	α_i	l_i	binary	Code
S_1	3/8	0	2	0.00	00
S_2	1/4	0.375	2	0.01	01
S_3	3/16	0.625	3	0.101	101
S_4	1/8	0.8125	3	0.110	110
S_5	1/16	0.9375	4	0.1111	1111

0.375 × 2
0.75 × 2 → 0
1.5 → 1
0.5 × 2
1
⇒ $\boxed{0.01}$

0.625 × 2
1.25 → 1
0.25 × 2
0.5 × 2 → 0
1
$\boxed{0.101}$

0.8125 × 2
1.625 → 1
0.625 × 2
1.25 → 1
0.25 × 2
0.5 × 2 → 0
1
⇒ $\boxed{0.110}$

0.9375 × 2
1.875 → 1
0.875 × 2
1.75 → 1
0.75 × 2
1.5 → 1
0.5 × 2
1
⇒ $\boxed{0.1111}$

$$H(S) = \frac{1}{4}log_2 4 + \frac{3}{8}log_2\frac{8}{3} + \frac{1}{8}log_2 8 + \frac{3}{16}log_2\frac{16}{3} + \frac{1}{4}log_2 4 + \frac{1}{16}log_2 16$$

$H(S) = 2.1085\ bits/symbol$

$$L = \sum_{i=1}^{q} P_i l_i = \frac{1}{4}(2) + \frac{3}{8}(2) + \frac{1}{8}(3) + \frac{3}{16}(3) + \frac{1}{16}(4)$$

$L = 2.4375\ bits$/symbol

$$\%\eta = \frac{H(S)}{L} * 100 = 86.5\%$$

Redundancy=1- η=100-86.5=13.5%

Ex: 3: Repeat the above messages **(**$x_{1,}x_{2,}x_3$) with P= (1/2, 1/5, 3/10)

Solution:

x_i	P_i	α_i	l_i	binary α_i	Code
x_1	1/2 (0.5)	0	1	0	0
x_3	3/10 (0.3)	0.5	2	0.1	10
x_2	1/5 (0.2)	0.8	3	0.11001	110

$$H(S) = \frac{1}{2}log_2 2 + \frac{3}{10}log_2\frac{10}{3} + \frac{1}{5}log_2 5$$

$H(S) = 1.4855\ bits/symbol$

$$L = \sum_{i=1}^{3} P_i L_i = \frac{1}{2}(1) + \frac{3}{10}(2) + \frac{1}{5}(3)$$

$L = 1.7 bits/symbol$

$$\%\eta = \frac{H(S)}{L} * 100 = 87.38\%$$

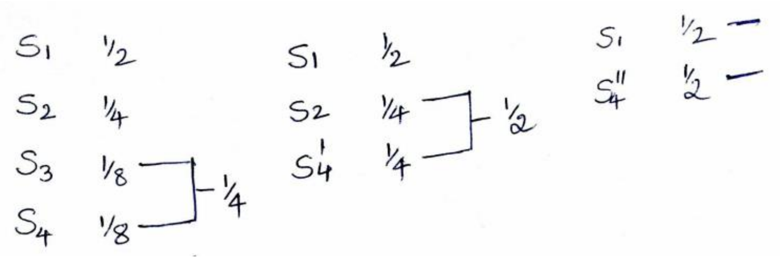
1.4 Huffman Coding

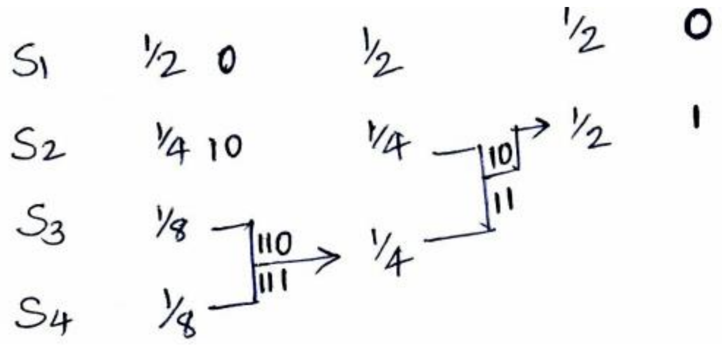
- The source symbols are listed in the decreasing order of probabilities.
- Check if $q = r + a(r\text{-}1)$ is satisfied and find the integer 'a', where q is number of source symbols and r is number of symbols used in code alphabets. 'a' values is calculated and it should be an integer, otherwise add suitable number of dummy symbols of zero probability of occurrence to satisfy the equation. This step is not required if we are to determine binary codes.
- Combine the last 'r' symbols into a single composite symbol whose probability of occurrence is equal to the sum of the probabilities of occurrence of the last ***r*** – symbols involved in the step.
- Repeat the above three steps respectively on the resulting set of symbols until in the final step exactly ***r***- symbols are left.
- The last source with 'r' symbols are encoded with 'r' different codes 0,1,2,3,….r-1
- In binary coding the last source are encoded with 0 and 1
- As we pass from source to source working backward, decomposition of one code word each time is done in order to form 2 new code words.
- This procedure is repeated till we assign the code words to all the source symbols of alphabet of source 's' discarding the dummy symbols.

Ex:1. Construct a Huffman Code for symbols having probabilities$\left\{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right\}$. Also find efficiency and redundancy.

$q = r + \alpha(r - 1)$

4=2+α(1) => α=2∈ Z





Symbols	Codes	Probabilities	Length
S_1	0	½	1
S_2	10	¼	2
S_3	110	1/8	3
S_4	111	1/8	3

$$H(S) = \frac{1}{2}log_2 2 + \frac{8}{8}log_2 8 + \frac{1}{4}log_2 4$$

$$H(S) = 1.75\ bits/symbol$$

$$L = \sum_{i=1}^{4} P_i L_i = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3)$$

$$L = 1.75\ bits/symbol$$

$$\%\eta = \frac{H(S)}{L} * 100 = 100\%$$

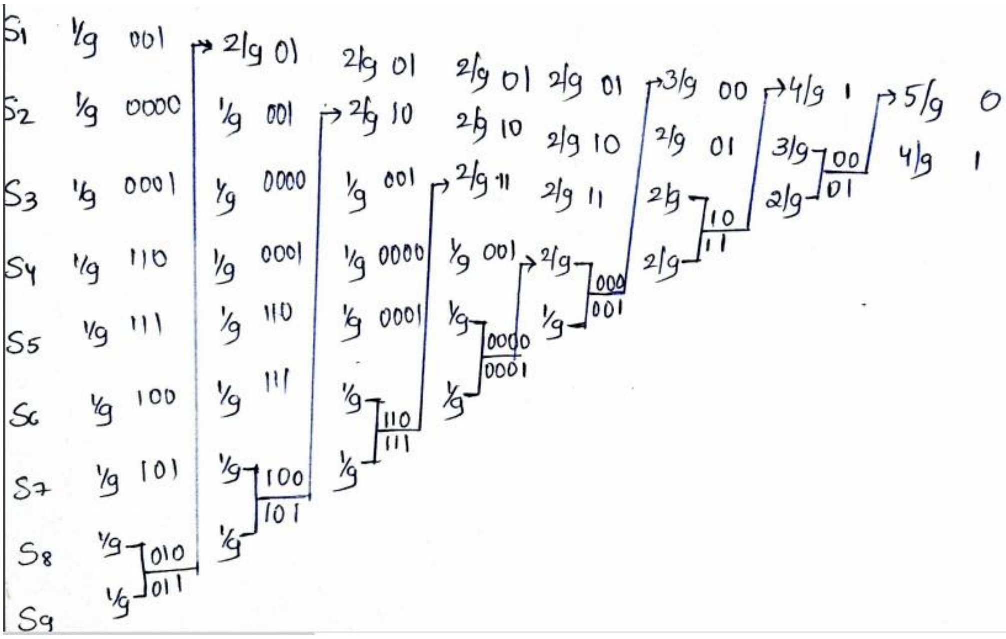
Redundancy=0%

Ex. 2: A source has 9 symbols and each occur with a probability of 1/9. Construct a binary Huffman code. Find efficiency and redundancy of coding.

Solution:

$$q = r + \alpha(r - 1)$$

9=2+α(1) => α=7∈ Z



Symbols	Codes	Probabilities	Length
S_1	001	1/9	3
S_2	0000	1/9	4
S_3	0001	1/9	4
S_4	110	1/9	3
S_5	111	1/9	3
S_6	100	1/9	3
S_7	101	1/9	3
S_8	010	1/9	3
S_9	011	1/9	3

$$H(S) = \frac{9}{9} log_2 9$$

$$H(S) = 3.17\ bits/symbol$$

$$L = \sum_{i=1}^{9} P_i L_i = \frac{1}{9}(3+4+4+3+3+3+3+3+3)$$

$$L = 3.22\ bits/symbol$$

$$\%\eta = \frac{H(S)}{L} * 100 = 98.45\%$$

Redundancy=100-% η=1.55%

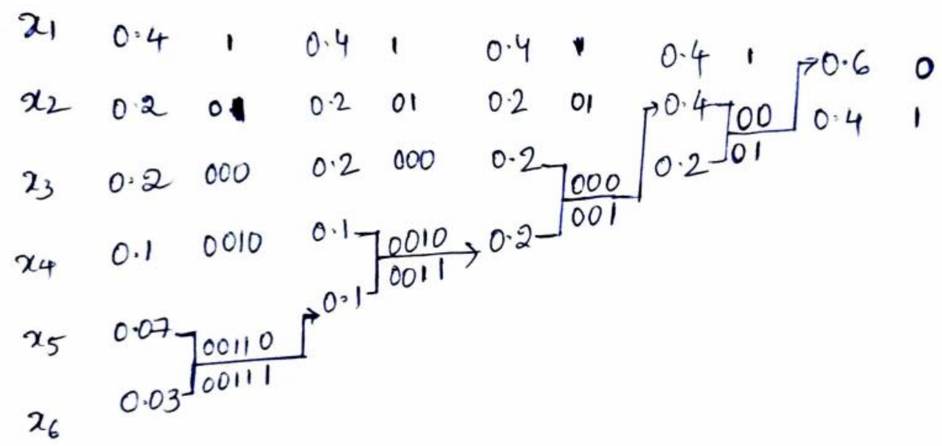
Ex. 3: Given the messages x_1, x_2, x_3, x_4, x_5 &x_6 with probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03. Construct binary and trinary code by applying Huffman encoding procedure. Also find efficiency and redundancy.

Solution:

(i) Binary

$q = r + \alpha(r-1)$

6=2+α(1) => α=4∈ Z



Symbols	Codes	Probabilities	Length
x_1	1	0.4	1
x_2	01	0.2	2
x_3	000	0.2	3
x_4	0010	0.1	4
x_5	00110	0.07	5
x_6	00111	0.03	5

$$H(S) = 0.4\, log_2\left(\frac{1}{0.4}\right) + 0.2\, log_2\left(\frac{1}{0.2}\right) + 0.2\, log_2\left(\frac{1}{0.2}\right) + 0.1\, log_2\left(\frac{1}{0.1}\right) + 0.07 log_2\left(\frac{1}{0.07}\right) + 0.03 log_2\left(\frac{1}{0.03}\right)$$

$H(S) = \quad bits/symbol$

$$L = \sum_{i=1}^{6} P_i L_i = 0.4(1) + 0.2(2) + 0.2(3) + 0.1(4) + 0.07(5) + 0.03(5)$$

$L = \quad bits/symbol$

$$\%\eta = \frac{H(S)}{L} * 100 =$$

Redundancy=

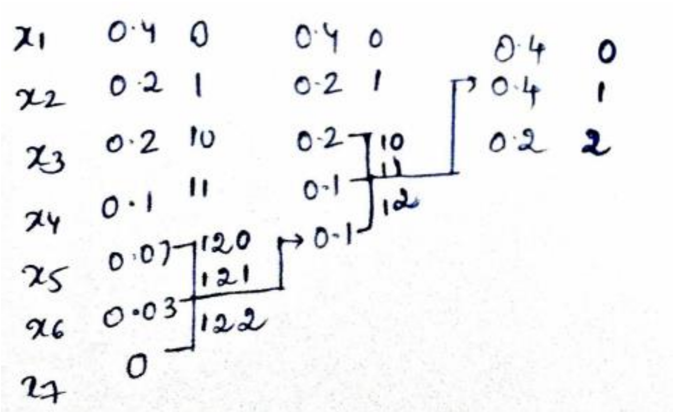
(ii) Trinary

$q = r + \alpha(r - 1)$

6=3+α(2) => α=3/2 ; Let α=2

If α=2 => $q = 3 + 2(2)$= 7

Hence add a symbol $x_7$with probability ‘0’.



Symbols	Codes	Probabilities	Length
x_1	0	0.4	1
x_2	1	0.2	1
x_3	10	0.2	2
x_4	11	0.1	2
x_5	120	0.07	3
x_6	121	0.03	3

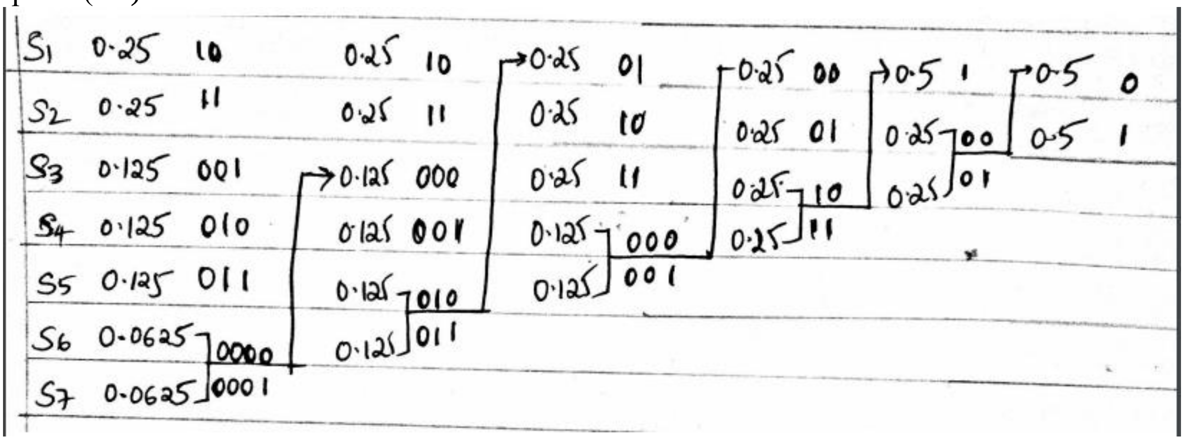
x_7 must be ignored as it is a dummy symbol.

$$H(S) = 0.4\, log_3\left(\frac{1}{0.4}\right) + 0.2\, log_3\left(\frac{1}{0.2}\right) + 0.2\, log_3\left(\frac{1}{0.2}\right) + 0.1\, log_3\left(\frac{1}{0.1}\right)$$
$$+ 0.07 log_3\left(\frac{1}{0.07}\right) + 0.03 log_3\left(\frac{1}{0.03}\right)$$

H(S)= bits/symbol

Ex. 4: Consider a zero memory source has an alphabet of 7 symbols whose probability of occurrence of (0.25, 0.25, 0.125, 0.125, 0.125, 0.0625, 0.0625). Compute the Huffman code

for this source moving a combined symbol **as high as possible**. Evaluate the code efficiency. And also construct code tree.
Solution:
q=r+α(r-1)   => α=5



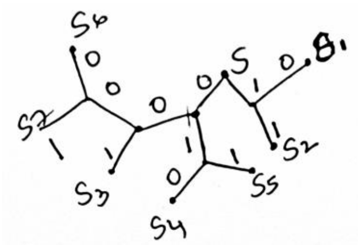
Symbols	Pi	Code	length
S1	0.25	10	2
S2	0.25	11	2
S3	0.125	001	3
S4	0.125	010	3
S5	0.125	011	3
S6	0.0625	0000	4
S7	0.0625	0001	4

H(S)=2.625 bits/symbol

L=2.625 bits/symbol

%η=H(s)/L=100%

<u>Code tree</u>



1.6 Discrete memoryless Channel:

A channel is defined as the medium through which the coded signals are generated by an information source are transmitted. In general, the input to the channel is a symbol belonging to an alphabet ‘A’ with ‘r’ symbols, the output of a channel is a symbol belonging to an alphabet ‘B’ with ‘s’ symbols.

Due to errors in the channel, the output symbols may differ from input symbols.

1.6.1 Representation of a channel:

A communication channel may be represented by a set of input alphabets A=$(a_{1,}a_{2,}a_3 \dots \dots .. a_r)$ consisting of ‘r’ symbols and set of output alphabets B=$(b_{1,}b_{2,}b_3 \dots \dots .. b_s)$ consisting of s symbols and a set of conditional probability P($^{b_j}/_{a_i}$) with i=1,2,….r and j=1,2,…….s

$$A \left.\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{matrix}\right\} \rightarrow \mathrm{P}(^{b_j}/_{a_i}) \rightarrow \left\{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{matrix}\right. B$$

The conditional probabilities come in to the existence due to the presence of noise in the channel. Because of noise there will be some amount of uncertainty in the reception of any symbols. For this reason there are ‘s’ number of symbols at the receiver from ‘r’ symbols at transmitter. Totally there are r * s conditional probabilities represented in a form of matrix which is called as Channel Matrix or Noise Matrix.

$$P(b_j/a_i) = \begin{array}{cc} & \begin{matrix} b_1 & \quad b_2 & \quad b_3 & \qquad\qquad b_s \end{matrix} \\ a_1 & \left[\begin{matrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) \dots\dots & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & P(b_3/a_2) \dots\dots & P(b_s/a_2) \\ P(b_1/a_3) & P(b_2/a_3) & P(b_3/a_3) \dots\dots & P(b_s/a_3) \\ P(b_1/b_4) & P(b_2/a_4) & P(b_3/a_4) \dots\dots & P(b_s/a_4) \\ \vdots & & & \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) \dots\dots & P(b_s/a_r) \end{matrix}\right] \end{array}$$

When a_1 is transmitted, it can be received as any one of the output symbols $(b_{1,}b_{2,}b_3 \dots \dots .. b_s)$

Therefore $P_{11} + P_{12} + P_{13} + \cdots \dots P_S$=1

=>P($^{b_1}/_{a_1}$)+ P($^{b_2}/_{a_1}$)+ P($^{b_3}/_{a_1}$)+…………….. P($^{b_s}/_{a_1}$)=1

In general, $\sum_{j=1}^{s} \mathrm{P}(^{b_j}/_{a_i}) = 1$ for i= 1 to r

Thus the sum of all the elements in any row of the channel matrix is equal to UNITY.

1.6.2 Joint Probability:

Joint probability between any input symbol a_i and any output symbol b_j is given by

$$\mathrm{P}(a_i \cap b_j) = \boldsymbol{P(a_i, b_j)} = \mathrm{P}(\boldsymbol{b_j}/\boldsymbol{a_i})\boldsymbol{P(a_i)}$$

$$\boldsymbol{P(a_i, b_j)} = \mathrm{P}(\boldsymbol{a_i}/\boldsymbol{b_j})\boldsymbol{P(b_j)}$$

Properties:

Consider the source alphabet A=$(a_{1,}a_{2,}a_3 \ldots \ldots .. a_r)$ and output alphabet B=$(b_{1,}b_{2,}b_3 \ldots \ldots .. b_s)$

- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$
- The entropy of the receiver or output is given by $H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$
- If all the symbols are equi-probable, then maximum source entropy is $H(A)_{max} = log_2 r$
- Conditional Entropy: The entropy of input symbols $a_{1,}a_{2,}a_3 \ldots \ldots .. a_r$ after the transmission and reception of particular output symbol b_j is defined as conditional entropy, denoted by $H(\boldsymbol{A}/\boldsymbol{b_j})$

$$H\left(A/_{b_j}\right) = \sum_{i=1}^{r} \mathrm{P}\left(a_i/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left(a_i/_{b_j}\right)}.$$

- If the average value of all the conditional probability is taken as j varies from 1 to s denoted by $H(A/_B) = \sum_{j=1}^{s} P(b_j)\ H\left(A/_{b_j}\right)$

$$=\sum_{j=1}^{s}\sum_{i=1}^{r} P(b_j)\ \mathrm{P}\left(a_i/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left(a_i/_{b_j}\right)}$$

$\boldsymbol{H(A/_B)} = \boldsymbol{\sum_{j=1}^{s}\sum_{i=1}^{r} P(a_i, b_j)\ log_2 \frac{1}{P(a_i/_{b_j})}}$ is conditional entropy of transmitter

Similarly $\boldsymbol{H(B/_A)} = \boldsymbol{\sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\ log_2 \frac{1}{P(b_j/_{a_i})}}$ is conditional entropy of receiver.

- $\boldsymbol{H(A, B)} = \boldsymbol{\sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\ log_2 \frac{1}{P(a_i,b_j)}}$ is joint conditional probability.

Problem : Consider a source with 8 alphabet A to H with respective probabilities of
0.22, 0.20, 0.18, 0.15, 0.10, 0.08, 0.05, 0.02.

i] Construct a binary compact code and determine the code efficiency.

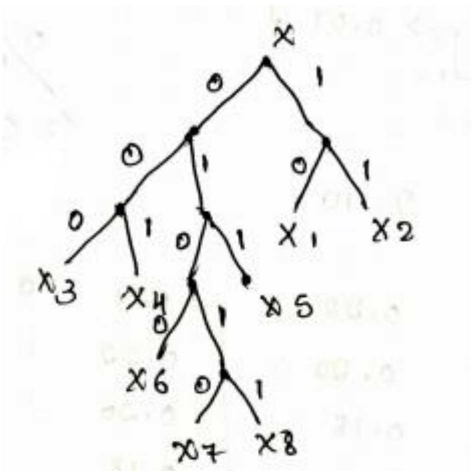
ii] Construct a ternary compact code and determine efficiency of the code

iii] Construct a quaternary compact code and determine the code efficiency. Compare and comment on the result. Draw code trees for all three cases.

iv] Decode the messages using appropriate code trees

a) 0101001000001101011001…
b) 12111011020012002 ….
c) 031132020300100231 …..

Solution:



0

(i) $q = r + \alpha(r-1)$

$q = 2 + \alpha(2-1)$

$q = 2$ ✗

$q = 8$, $r = 2$, $\alpha = 6$

Symbol	Prob	Code		Code		Code		Code		Code		Code		Code
x_1	0.22	10	0.22	10	0.22	10	0.25	01	0.33	00	0.42	1	0.58	0
x_2	0.20	11	0.20	11	0.20	11	0.22	10	0.25	01	0.33	00	0.42	1
x_3	0.18	000	0.18	000	0.18	000	0.20	11	0.22	10	0.25	01		
x_4	0.15	001	0.15	001	0.15	001	0.18	000	0.20	11				
x_5	0.10	011	0.10	011	0.15	010	0.15	001						
x_6	0.08	0100	0.08	0100	0.10	011								
x_7	0.05	01010	0.07	0101										
x_8	0.02	01011												

$H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$H(S) = 2.75$ bits/symbol

$L = \sum_{i=1}^{q} P_i l_i$

$L = 2.8$

$\eta_c = \frac{H(S)}{L}$

$\eta_c = 98.25\%$

$R\eta_c = 1.79\%$

iv a) 0101001000001101011001

$= X_7\ X_6\ X_3\ X_2\ X_8\ X_4$

(ii) $q = 3 + \alpha(2)$

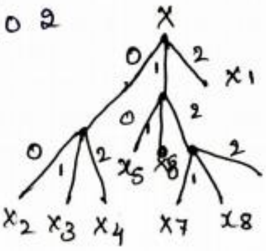
$\alpha = 2.5$

$\alpha = 3 \Rightarrow q = 9$

iv) b) 121 11 01 10 2 00 120 02

$x_8\ x_6\ x_3\ x_1\ x_2\ x_7\ x_4$

Symbol	Prob	Code		Code		Code		Code
x_1	0.22	2	0.22	2	0.25	1	0.53	0
x_2	0.20	00	0.20	00	0.22	2	0.25	1
x_3	0.18	01	0.18	01	0.20	00	0.22	2
x_4	0.15	02	0.15	02	0.18	01		
x_5	0.10	10	0.10	10	0.15	02		
x_6	0.08	11	0.08	11				
x_7	0.05	120	0.07	12				
x_8	0.02	121						
x_9	0	122						



$H_r(S) = \frac{H(S)}{\log_2 r}$

$H_r(S) = \frac{H(S)}{\log_2(3)} = 1.735$

$L = \sum_{i=1}^{q} P_i l_i$

$L = 1.85$

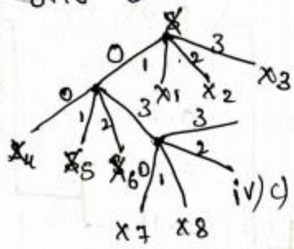
$\eta_c = \frac{H(S)}{L} = 93.73\%$

$R\eta_c = 1 - \eta_c = 6.21\%$

Decoding Alphabets: X8, X6, X3, X5, X1, X2, X7,X4

iii) $\alpha = 2$ $q = 10$

Symbol	Prob					
x_1	0.22	1	0.22	1	0.4	0
x_2	0.20	2	0.20	2	0.22	1
x_3	0.18	3	0.18	3	0.20	2
x_4	0.15	00	0.15	00	0.18	3
x_5	0.10	01	0.10	01		
x_6	0.08	02	0.08	02		
x_7	0.05	030	0.07	03		
x_8	0.02	031				
x_9	0	032				
x_{10}	0	033				



$H(S) = 1.375$

$L = \sum_{i=1}^{q} P_i l_i$

$L = 1.47$

$\eta_c = \frac{H(S)}{L} = 93.53\%$

$R_{\eta_c} = 1 - \eta_c = 6.47\%$

iv) c) 0311320203001002 3 1

$x_8 x_1 x_3 x_2 x_6 x_7 x_5 x_4 x_2 x_3 x_1$

Problem : Consider a Zero memory source with S=[S1,S2,S3,S4,S5,S6,S7] and Probabilities P=[0.4,0.2,0.1,0.1,0.1,0.05,0.05]

i. Construct a binary Huffman code by placing the composite symbol as low as possible.
ii. Repeat (i) by moving a composite symbol as high as possible.
iii. In each of the cases (i) and (ii) above,
   - Compute the variances of the word lengths and comment on the result.
   - Find Efficiency and Redundancy.
iv. Considering Case(ii) table,
   - Write the code tree and decode the message 0111011001100010O......
   - Determine probabilities of 0's and 1's.

Tips: Variance= $\sum_{i=1}^{i=q} P_i\ (l_i - L)^2$

Probability of 0's : P(0)= $\frac{1}{L}\sum_{i=1}^{i=q}(No.of 0'\text{s in } the\ \ code\ for\ Si)\ P_i$

(i)

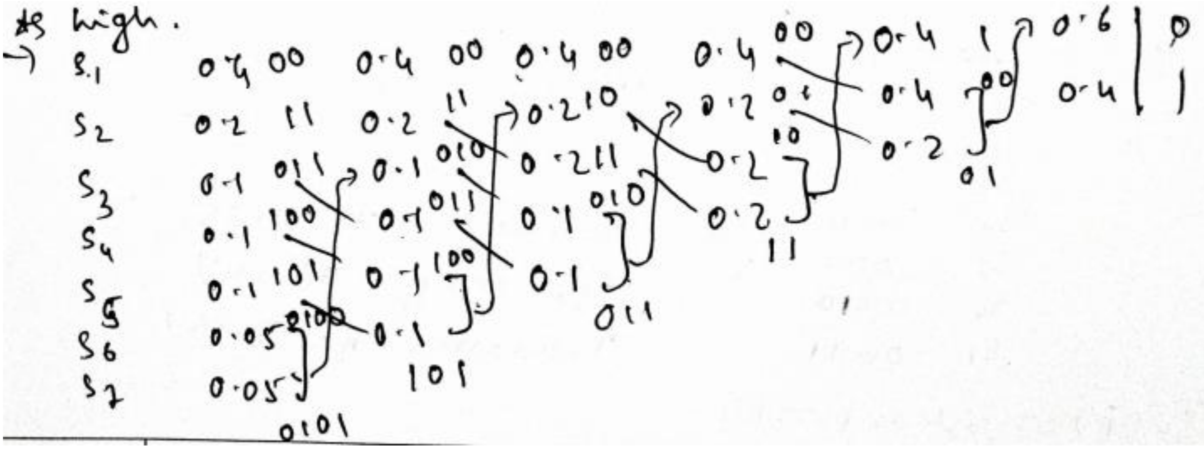
As low as possible.

s_1	0.4	1	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
s_2	0.2	01	0.2	01	0.2	01	0.2	01	0.4	00	0.4	1
s_3	0.1	0010	0.1	0010	0.2	000	0.2	000	0.2	01		
s_4	0.1	0011	0.1	0011	0.1	0010	0.2	001				
s_5	0.1	0000	0.1	0000	0.1	0011						
s_6	0.05	00010	0.1	0001								
s_7	0.05	00011										

(i) as low as possible

S_1	1	$H(S) = 2.421$
S_2	01	$L = 2.5$
S_3	0010	
S_4	0011	Variance = 2.25
S_5	0000	Efficiency η = 96.84%
S_6	00010	
S_7	00011	Redundancy R_η = 3.16%

(ii)

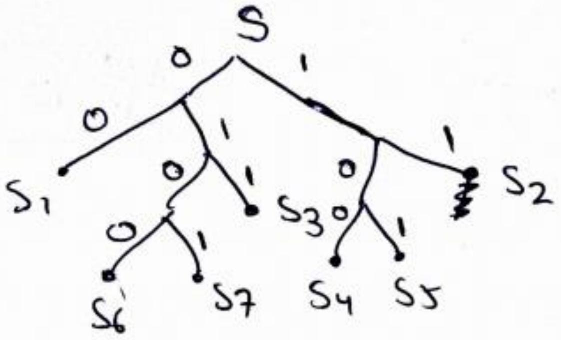


(ii) as high as possible

S_1	00	$H(S) = 2.421$
S_2	11	$L = 2.5$
S_3	011	Variance = 0.45
S_4	100	
S_5	101	Probability of 0's = 0.58
S_6	0100	Probability of 1's = 0.42
S_7	0101	Efficiency η = 96.84%
		Redundancy R_η = 3.16%

(iv)

Code tree using code in case (ii)



Sequence ⇒

$\underbrace{011}_{S_3}\,\underbrace{101}_{S_5}\,\underbrace{100}_{S_4}\,\underbrace{11}_{S_2}\,\underbrace{00}_{S_1}\,\underbrace{0100}_{S_6}$