Example 5.11 For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

$$\boxed{\begin{aligned} A &= (s+1)\frac{2}{(s+1)(s+2)}\Big|_{s=-1} \\ &= 2 \\ B &= (s+2)\frac{2}{(s+1)(s+2)}\Big|_{s=-2} \\ &= -2 \end{aligned}}$$

Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

i.e., $(s - p_k)$ is transformed to $1 - e^{p_k T} z^{-1}$.

There are two poles $p_1 = -1$ and $p_2 = -2$. So

$$H(z) = \frac{2}{1 - e^{-T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

For $T = 1$ sec

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.04976 z^{-2}}$$

Example 5.12 Using impulse invariance with $T = 1$ sec determine $H(z)$ if $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

Solution

Given $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2 + \sqrt{2}s + 1}\right]$$

$$= L^{-1}\left[\frac{1}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= L^{-1}\left[\sqrt{2} \cdot \frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$(x+a)^2$

$x = \frac{1}{\sqrt{2}} - j/\sqrt{2}$

$$= \sqrt{2}L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2})$$

Let $t = nT$

$$h(nT) = \sqrt{2}\, e^{-nT/\sqrt{2}} \sin \frac{nT}{\sqrt{2}}$$

If $T = 1$ sec

$$h(n) = \sqrt{2}\, e^{-n/\sqrt{2}} \sin \frac{n}{\sqrt{2}}$$

$$H(z) = Z[h(n)] = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}}z^{-1} \sin 1/\sqrt{2}}{1 - 2e^{-1/\sqrt{2}}z^{-1} \cos \frac{1}{\sqrt{2}} + e^{-\sqrt{2}}z^{-2}}\right]$$

$$= \frac{0.453z^{-1}}{1 - 0.7497z^{-1} + 0.2432z^{-2}}$$

Example 5.13 Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1$ sec.

Solution

From the table 5.1, for $N = 3$, the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2 + s + 1)}$$

$$= \frac{A}{s+1} + \frac{B}{s + 0.5 + j0.866} + \frac{C}{s + 0.5 - j0.866}$$

$$A = (s+1)\frac{1}{(s+1)(s^2+s+1)}\bigg|_{s=-1} = \frac{1}{(-1)^2-1+1} = 1$$

$$B = (s+0.5+j0.866)\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\bigg|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-j0.866-j0.866)}$$

$$= \frac{1}{-j1.732(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$H(s) = \frac{1}{s+1} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5+j0.866)}$$

In impulse invariant technique

$$\text{if} \quad H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}, \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T}z^{-1}}$$

Therefore,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5}e^{-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Example 5.14 Apply impulse invariant method and find $H(z)$ for $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$.

Solution The inverse Laplace transform of given function is

$$h(t) = \begin{cases} e^{-at}\cos(bt) & \text{for} \quad t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the function produces

$$h(nT) = \begin{cases} e^{-anT}\cos(bnT) & \text{for} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)z^{-n} \\
&= \sum_{n=0}^{\infty}\left[e^{-anT}z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[(e^{-(a-jb)T}z^{-1})^{n} + (e^{-(a+jb)T}z^{-1})^{n}\right] \\
&= \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right] \\
&= \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}
\end{aligned}$$

Example 5.15 An analog filter has a transfer function $H(s) = \dfrac{10}{s^2+7s+10}$. Design a digital filter equivalent to this using impulse invariant method for $T = 0.2$ sec.

Solution

Given

$$\begin{aligned}
H(s) &= \frac{10}{s^2+7s+10} \\
&= \frac{-3.33}{s+5} + \frac{3.33}{s+2} = \frac{-3.33}{s-(-5)} + \frac{3.33}{s-(-2)}
\end{aligned}$$

Using Eq. (5.81b) we have

$$\begin{aligned}
H(z) = T\left[\frac{-3.33}{1-e^{-5T}z^{-1}} + \frac{3.33}{1-e^{-2T}z^{-1}}\right] &= 0.2\left[\frac{-3.33}{1-e^{-1}z^{-1}} + \frac{3.33}{e^{-0.4}z^{-1}}\right] \\
&= \left[\frac{-0.666}{1-0.3678z^{-1}} + \frac{0.666}{1-0.67z^{-1}}\right] \\
&= \frac{0.2012z^{-1}}{1-1.0378z^{-1}+0.247z^{-2}}
\end{aligned}$$

Practice Problem 5.7 An analog filter has a transfer function

$$H(s) = \frac{5}{s^3+6s^2+11s+6}$$

Design a digital filter equivalent to this using impulse invariant method for $T = 1$ sec.

---

where $\{z_k\}$ are the zeros and $\{p_k\}$ are the poles of the filter, then the system function of the digital filter is

$$H(z) = \frac{\prod_{k=1}^{M}(1 - e^{z_k T} z^{-1})}{\prod_{k=1}^{N}(1 - e^{p_k T} z^{-1})} \tag{5.97}$$

where T is the sampling interval. Thus each factor of the form $(s - a)$ in $H(s)$ is mapped into the factor $1 - e^{aT} z^{-1}$. This mapping is called the matched z-transform.

Example 5.16 Apply bilinear transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Substitute $s = \dfrac{2}{T}\left[\dfrac{1 - z^{-1}}{1 + z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$
$$= \frac{2}{(s+1)(s+2)}\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 1\right\}\left\{2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 2\right\}}$$
$$= \frac{\cancel{2}(1 + z^{-1})^2}{(3 - z^{-1})(\cancel{4})2}$$
$$= \frac{(1 + z^{-1})^2}{6 - 2z^{-1}}$$
$$= \frac{0.166(1 + z^{-1})^2}{(1 - 0.33z^{-1})}$$

Example 5.17 Using the bilinear transform, design a highpass filter, monotonic in passband with cutoff frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.

Solution

Given $\alpha_p = 3\,\text{dB}$; $\omega_c = \omega_p = 2\times\pi\times 1000 = 2000\pi$ rad/sec

$$\alpha_s = 10\,\text{dB};\quad \omega_s = 2\times\pi\times 350 = 700\pi \text{ rad/sec}$$

$$T = \frac{1}{f} = \frac{1}{5000} = 2\times 10^{-4} sec$$

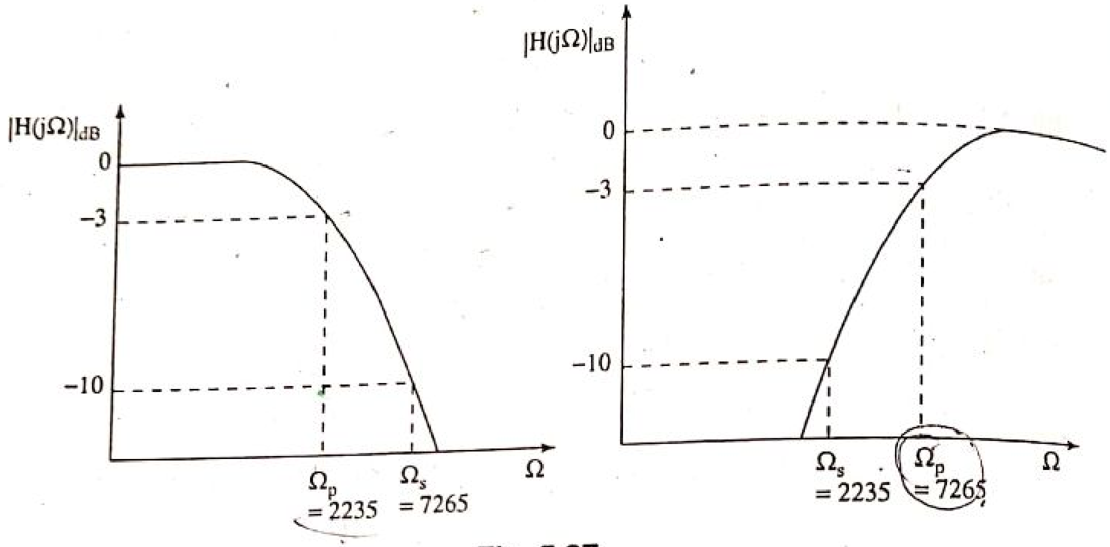

Fig. 5.27

The characteristics are monotonic in both passband and stopband. Therefore, the filter is Butterworth filter.

Prewarping the digital frequencies we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p T}{2} = \frac{2}{2\times 10^{-4}}\tan\frac{(2000\pi\times 2\times 10^{-4})}{2}$$
$$= 10^4\tan(0.2\pi) = 7265 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2} = \frac{2}{2\times 10^{-4}}\tan\frac{(700\pi\times 2\times 10^{-4})}{2}$$
$$= 10^4\tan(0.07\pi) = 2235\text{rad/sec}$$

First we design a lowpass filter for the given specifications and use suitable transformation to obtain transfer function of highpass filter.

The order of the filter

$$N = \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{\log\sqrt{\dfrac{10^{0.1(10)}-1}{10^{0.1(3)}-1}}}{\log\dfrac{7265}{2235}} = \frac{\log 3}{\log 3.25} = \frac{0.4771}{0.5118} = 0.932$$

Therefore, we take $N = 1$.

The first-order Butterworth filter for $\Omega_c = 1$ rad/sec is $H(s) = \dfrac{1}{1+s}$

The highpass filter for $\Omega_c = \Omega_p = 7265$ rad/sec can be obtained by using the transformation

$$s \to \frac{\Omega_c}{s}$$

$$i.e., \ s \to \frac{(7265)}{s}$$

The transfer function of highpass filter

$$H(s) = \left.\frac{1}{s+1}\right|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \left.\frac{s}{s+7265}\right|_{s=\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+7265}$$

$$= \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}$$

Example 5.18 Determine $H(z)$ that results when the bilinear transformation is applied to $H_a(s) = \dfrac{s^2+4.525}{s^2+0.692s+0.504}$

Solution

In bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Assume $T = 1$ sec.

Then

$$H(z) = \frac{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2+4.525}{4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2+0.692\times2\times\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+0.504}$$

$$= \frac{1.4479+0.1783z^{-1}+1.4479z^{-2}}{1-1.18752z^{-1}+0.5299z^{-2}}$$

$$\begin{aligned} c_0 &= b_0 - \sum_{i=1}^{3} c_i a_i(i-m) \\ &= b_0 - [c_1 a_1(1) + c_2 a_2(2) + c_3 a_3(3)] \\ &= 1 - \left[0.8281\left(\frac{1}{4}\right) + 1.4583\left(\frac{1}{2}\right) + \frac{1}{3}\right] = -0.2695. \end{aligned}$$

The lattice ladder structure for the given pole-zero filter is shown in Fig. 5.61.

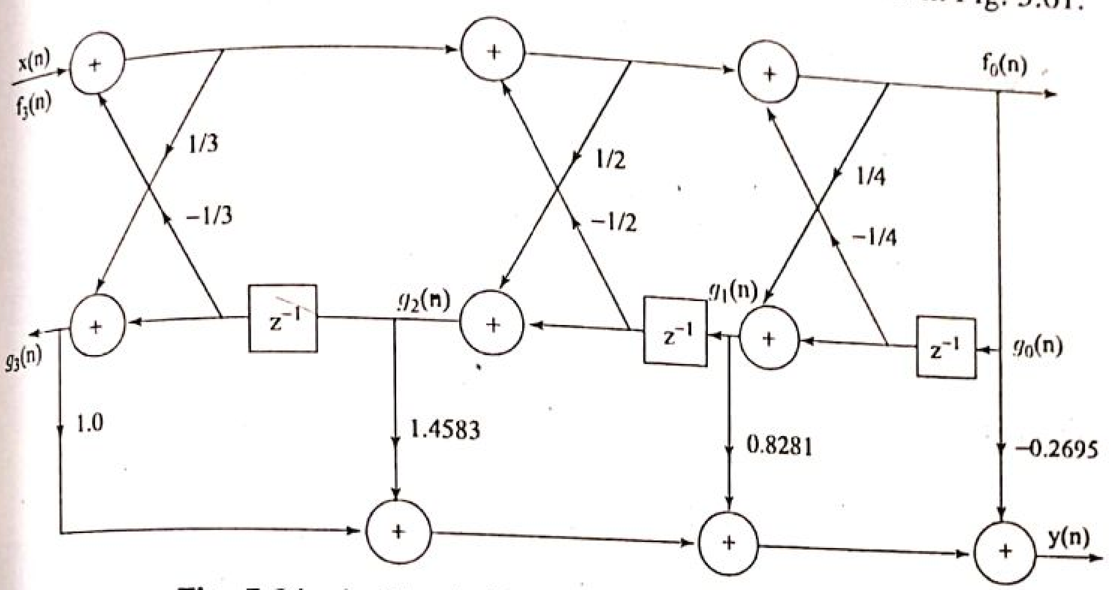

Fig. 5.61 Lattice-ladder form for the example 5.29.

To convert a lattice-ladder form into a direct form, we first use Eq. (5.159) to determine $\{a_N(k)\}$ and then use Eq. (5.168) recursively to obtain $b_M(k)$.

Practice Problem 5.16 Convert the following pole-zero IIR filter into a lattice-ladder structure

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

Additional Examples

Example 5.30 Design a digital Butterworth filter satisfying the constraints

$$0.707 \le |H(e^{j\omega})| \le 1 \qquad \text{for } 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \qquad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ sec using (a) The bilinear transformation (b) Impulse invariance. Realize the filter in each case using the most convenient realization form.

(AU ECE May'07) (AU ECE Nov'06)

Solution

(a) Bilinear transformation

Given data $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707;\quad \frac{1}{\sqrt{1+\lambda^2}} = 0.2; \omega_p = \frac{\pi}{2}; \omega_s = \frac{3\pi}{4}$.

The analog frequency ratio is

$$\frac{\Omega_s}{\Omega_p} = \frac{\frac{2}{T}\tan\frac{\omega_s}{2}}{\frac{2}{T}\tan\frac{\omega_p}{2}} = \frac{\tan\frac{3\pi}{8}}{\tan\frac{\pi}{4}} = 2.414$$

The order of the filter $N \geq \frac{\log\frac{\lambda}{\varepsilon}}{\log\frac{\Omega_s}{\Omega_p}}$.

From the given data $\lambda = 4.898 \quad \varepsilon = 1$.

So $N \geq \frac{\log 4.898}{\log 2.414} = 1.803$.

Rounding N to nearest higher value we get $N = 2$. We know

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \Omega_p \quad (\because \varepsilon = 1)$$

$$= \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{\pi}{4} = 2\,\text{rad/sec}$$

The transfer function of second order normalized Butterworth filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$H_a(s)$ for $\Omega_c = 2$ rad/sec can be obtained by substituting $s \to s/2$ in $H(s)$

$$\text{i.e.,}\quad H_a(s) = \frac{1}{(s/2)^2 + \sqrt{2}\cdot(s/2) + 1}$$

$$= \frac{4}{s^2 + 2.828s + 4}$$

By using bilinear transformation $H(z)$ can be obtained as

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Thus $H(z) = \frac{4}{s^2 + 2.828s + 4}\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad (\because T = 1\,\text{sec})$

$$= \frac{4(1+z^{-1})^2}{4(1-z^{-1})^2 + 5.656(1-z^{-2}) + 4(1+z^{-1})^2}$$

$$= \frac{0.2929(1+z^{-1})^2}{1 + 0.1716z^{-2}}$$

The above system function can be realized in direct form II as shown in Fig. 5.62.

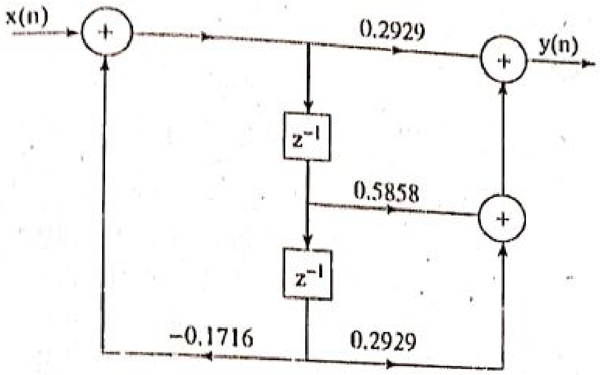

Fig. 5.62

(b) Impulse Invariant Method

Solution

The relationship between analog & digital frequencies in Impulse invariant method is $\omega = \Omega T$.

From the given data $T = 1$ sec i.e., $\omega = \Omega$

$$\Rightarrow \quad \Omega_p = \omega_p; \quad \Omega_s = \omega_s$$

We know $\lambda = 4.898; \quad \varepsilon = 1.$

The order of the filter

$$N \geq \frac{\log \dfrac{\lambda}{\varepsilon}}{\log \dfrac{\Omega_s}{\Omega_p}} = \frac{\log 4.989}{\log \dfrac{3\pi/4}{\pi/2}}$$

$$N \geq 3.924$$

i.e., $N = 4$

The transfer function of a fourth order normalized Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.76537 + 1)(s^2 + 1.8477s + 1)}$$

As $\varepsilon = 1$: $\quad \Omega_p = \Omega_c = 0.5\pi = 1.57$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{1.57}}$$

$$= \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

$H_a(s)$ in the partial fraction form is given by

$$H_a(s) = \frac{A}{(s + 1.45 + j0.6)} + \frac{A^*}{(s + 1.45 - j0.6)} + \frac{B}{(s + 0.6 + j1.45)} + \frac{B^*}{(s + 0.6 - j1.45)}$$

$$A = (s + 1.45 + j0.6)\frac{(1.57)^4}{(s + 1.45 + j0.6)(s + 1.45 - j0.6)(s^2 + 1.202s + 2.465)}\Bigg|_{s=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(-j0.6 - 0.6)\left[(-1.45 - j0.6)^2 + 1.202(-1.45 - j0.6) + 2.465\right]}$$

$$= \frac{(1.57)^4}{-j(1.2)\left[1.7425 + 1.74j - 1.7429 - j0.7212 + 2.465\right]}$$

$$= \frac{(1.57)^4}{-j(1.2)(2.465 + j1.0188)}$$

$$= \frac{5.063}{1.0188 - j2.465} = \frac{5.063(1.0188 + j2.465)}{7.114}$$

$$= 0.7116(1.0188 + j2.465) = 0.7253 + j1.754$$

$$B = (s + 0.6 + j1.45)\frac{(1.57)^4}{(s + 0.6 + j1.45)(s + 0.6 - j1.45)(s^2 + 2.902s + 2.465)}\Bigg|_{s=-0.6-j1.45}$$

$$= \frac{(1.57)^4}{-j(2.9)\left[(-0.6 - j1.45)^2 + 2.902(-0.6 - j1.45) + 2.465\right]}$$

$$= \frac{(1.57)^4}{-j(2.9)\left[-1.7425 + j1.74 - 1.7412 - j4.208 + 2.465\right]}$$

$$= \frac{2.095}{-j\left[-1.0187 - j2.468\right]}$$

$$= \frac{2.095}{-2.468 + j1.0187} = \frac{2.095\left[-2.468 - j1.0187\right]}{7.1287}$$

$$= 0.29388\left[-2.468 - j1.0187\right] = -0.7253 - 0.3j$$

$$H_a(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)} + \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

We know for $T = 1$ sec

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k} z^{-1}}$$

Therefore

$$H_a(s) = \frac{0.7253 + j1.754}{1 - e^{-1.45}e^{-j0.6}z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45}e^{j0.6}z^{-1}}$$
$$+ \frac{-(0.7253 + 0.3j)}{1 - e^{-0.6}e^{-j1.45}z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6}e^{j1.45}z^{-1}}$$
$$= \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}$$

This can be realized using parallel form as shown in Fig. 5.63.

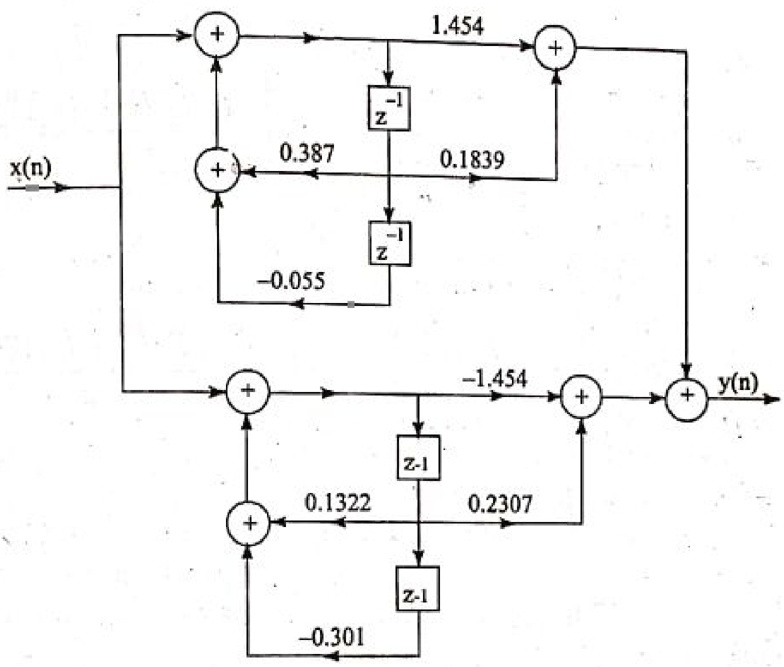

Fig. 5.63

Example 5.31 Design a Chebyshev lowpass filter with the specifications $\alpha_p = 1$ dB ripple in the passband $0 \leq \omega \leq 0.2\pi$, $\alpha_s = 15$ dB ripple in the stopband $0.3\pi \leq \omega \leq \pi$, using (a) bilinear transformation (b) Impulse invariance.

Solution

Given data $\alpha_p = 1\,\text{dB}; \omega_p = 0.2\pi; \alpha_s = 15\,\text{dB}; \omega_s = 0.3\pi$.

Prewarped frequency values: Since we intend to employ the bilinear transformation method, we must prewarp these frequencies. The prewarped values are given by (Assume $T = 1$ sec);

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{0.2\pi}{2} = 0.65$$
$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\frac{0.3\pi}{2} = 1.02$$

Value of N

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{1.5}-1}{10^{0.1}-1}}}{\cosh^{-1}\dfrac{1.02}{0.65}} = 3.01$$

Let us take $N = 4$.

Axis of the ellipse

We know $\varepsilon = \sqrt{10^{0.1\alpha_p}-1} = 0.508$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 4.17$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(4.17)^{1/4}-(4.17)^{-1/4}}{2}\right]$$

$$= 0.237$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(4.17)^{1/4}+(4.17)^{-1/4}}{2}\right]$$

$$= 0.6918$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} k = 1,2,3,4$$

$\phi_1 = 112.5°; \phi_2 = 157.5°, \phi_3 = 202.5°; \phi_4 = 247.5°$.

The poles are

$$s_k = a\cos\phi_k + jb\sin\phi_k; \quad k = 1,2,3,4$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = 0.237\cos 112.5° + j0.6918\sin 112.5°$$
$$= -0.0907 + j0.639$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = 0.237\cos 157.5° + j0.6918\sin 157.5°$$
$$= -0.2189 + j0.2647$$

$$s_3 = a\cos\phi_3 + jb\sin\phi_3 = 0.237\cos 202.5° + j0.6918\sin 202.5°$$
$$= -0.2189 - j0.2647$$

$$s_4 = a\cos\phi_4 + jb\sin\phi_4 = 0.237\cos 247.5° + j0.6918\sin 247.5°$$
$$= -0.0907 - j0.639$$

The denominator polynomial of

$$H(s) = [(s+0.0907)^2 + (0.639)^2][(s+0.2189)^2 + (0.2647)^2]$$
$$= (s^2 + 0.1814s + 0.4165)(s^2 + 0.4378s + 0.118)$$

As N is even, the numerator of $H(s) = \frac{(0.4165)(0.118)}{\sqrt{1+\varepsilon^2}} = 0.04381.$

The transfer function $H(s) = \frac{0.04381}{(s^2+0.1814s+0.4165)(s^2+0.4378s+0.1180)}.$

The z-transform of the digital filter

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{0.04381}{(s^2+0.1814s+0.4165)(s^2+0.4378s+0.1180)}\Bigg|_{s=2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$\because T = 1\,\text{sec}$

$$= \frac{0.04381(1+z^{-1})^4}{\{4(1-z^{-1})^2+0.1814\times 2(1-z^{-2})+0.4165(1+z^{-1})^2\}\{4(1-z^{-1})^2+0.4378\times 2(1-z^{-2})+0.1180(1+z^{-1})^2\}}$$

$$= \frac{0.04381(1+z^{-1})^4}{(4.7794-7.1668z^{-1}+4.0538z^{-2})(4.9936-7.764z^{-1}+3.2424z^{-2})}$$

$$= \frac{0.001836(1+z^{-1})^4}{(1-1.499z^{-1}+0.8482z^{-2})(1-1.5548z^{-1}+0.6493z^{-2})}$$

(b) Impulse Invariance Method

Given data $\omega_p = 0.2\pi;\ \omega_s = 0.3\pi;\ \alpha_p = 1\,\text{dB};\ \alpha_s = 15\,\text{dB}.$

The Analog frequency ratio $\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p} = \frac{0.3\pi}{0.2\pi} = 1.5$

$\boxed{(\because \omega = \Omega T \text{ and } T = 1\,\text{sec})}$

Value of *N*

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \cosh^{-1}\frac{\sqrt{\frac{10^{1.5}-1}{10^{0.1}-1}}}{\cosh^{-1}(0.5)} = 3.2$$

Rounding the value of N to a higher value, we get $N = 4$.

Axis of ellipse

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N};\quad k = 1, 2, \ldots N$$

$$\phi_1 = 112.5^\circ; \phi_2 = 157.5^\circ; \phi_3 = 202.5^\circ$$

$$\phi_4 = 247.5^\circ$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1} - 1} = 0.508$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 4.17$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.2\pi \left[\frac{4.17^{1/4} - 4.17^{-1/4}}{2}\right] = 0.229$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.2\pi \left[\frac{4.17^{1/4} + 4.17^{-1/4}}{2}\right] = 0.67$$

The poles of the filter

$$s_k = a\cos\phi_k + jb\sin\phi_k; \quad k = 1, 2, \ldots 4$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = -0.0876 + j0.619$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = -0.2115 + j0.2564$$

$$s_3 = a\cos\phi_3 + jb\sin\phi_3 = -0.2115 - j0.2564$$

$$s_4 = a\cos\phi_4 + jb\sin\phi_4 = -0.0876 - j0.619$$

The denominator polynomial of

$$\begin{aligned} H(s) &= \{(s+0.0876)^2 + (0.619)^2\}\{(s+0.215)^2 + (0.2564)^2\} \\ &= (s^2 + 0.175s + 0.391)(s^2 + 0.423s + 0.11) \end{aligned}$$

For N even

The numerator of $H(s) = \dfrac{(0.391)(0.11)}{\sqrt{1+\varepsilon^2}} = 0.03834.$

$$\begin{aligned} H(s) &= \frac{0.03834}{(s^2 + 0.175s + 0.391)(s^2 + 0.423s + 0.11)} \\ &= \frac{A}{s - (-0.0876 + j0.619)} + \frac{A^*}{s - (-0.0876 - j0.619)} \\ &\quad + \frac{B}{s - (-0.2115 + j0.2564)} + \frac{B^*}{s - (-0.2115 - j0.2564)} \end{aligned}$$

Solving for A, A^*, B, B^* and using

$$\boxed{\begin{aligned} A &= -0.0413 + j0.0814 \\ B &= 0.0413 - j0.2166 \end{aligned}}$$

Impulse invariant transform

i.e., $$\sum_{k=1}^{N} \frac{c_k}{s - p_k} = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}.$$

we can obtain

$$H(z) = \frac{-0.083 - 0.0245z^{-1}}{1 - 1.49z^{-1} + 0.839z^{-2}} + \frac{0.083 + 0.0238z^{-1}}{1 - 1.56z^{-1} + 0.655z^{-2}}$$

Example 5.32 Design a Butterworth filter using the impulse variance method for the following specifications

$$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

Solution

Given $\dfrac{1}{\sqrt{1+\varepsilon^2}} = 0.8$ from which $\varepsilon = 0.75$, $\dfrac{1}{\sqrt{1+\lambda^2}} = 0.2$ from which $\lambda = 4.899$

$$\omega_s = 0.6\pi \text{ rad}; \quad \omega_p = 0.2\pi \text{ rad}$$
$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$
$$N = \frac{\log \lambda/\varepsilon}{\log 1/k} = \frac{\log \dfrac{4.899}{0.75}}{\log 3} = 1.71$$

Approximating to nearest higher values we have $N = 2$.

For $N = 2$ the transfer function of normalized Butterworth filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$
$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.231\pi$$
$$H_a(s) = H(s)\Big|_{s \to s/0.231\pi}$$
$$= \frac{0.5266}{s^2 + 1.03s + 0.5266}$$

$$= \frac{0.516j}{s + 0.51 + j0.51} - \frac{0.516j}{s + 0.51 - j0.51}$$
$$= \frac{0.516j}{s - (-0.51 - j0.51)} - \frac{0.516j}{s - (-0.51 + j0.51)}$$
$$H(z) = \frac{0.516j}{1 - e^{-0.51T}e^{-j0.51T}z^{-1}} - \frac{0.516j}{1 - e^{-0.51T}e^{j0.51T}z^{-1}} \qquad (\because T = 1 \text{ sec})$$
$$= \frac{0.3019z^{-1}}{1 - 1.048z^{-1} + 0.36z^{-2}}$$

Practice Problem 5.17 Repeat example 5.32 using bilinear transformation

Ans: $\dfrac{0.084[1 + z^{-1}]^2}{1 - 1.028z^{-1} + 0.3651z^{-2}}$

Practice Problem 5.18 Design a Chebyshev lowpass filter with the specifications $\alpha_p = 0.5\,\text{dB}$ ripple in the passband $0 \le \omega \le 0.25\pi$, $\alpha_s = 20\,\text{dB}$ ripple in the stopband $0.4\pi \le \omega \le \pi$ using (a) bilinear transformation (b) Impulse invariance.

Practice Problem 5.19 Repeat practice problem 5.18 to design a Butterworth lowpass filter.

Example 5.33 Determine the system function $H(z)$ of the lowest order Chebyshev and Butterworth digital filter with the following specification

(a) $3db$ ripple in pass band $0 \le \omega \le 0.2\pi$

(b) $25db$ attenuation in stop band $0.45\pi \le \omega \le \pi$

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Butterworth filter

Using bilinear transformation

$$\omega_p = 0.2\pi; \quad \omega_s = 0.45\pi; \quad \alpha_p = 3db; \quad \alpha_s = 25db; \quad T = 1$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \frac{2}{T}\tan(0.1\pi) = 0.65$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = \frac{2}{T}\tan\left(\frac{0.45\pi}{2}\right) = 1.71$$

$$N \ge \frac{\log\sqrt{\dfrac{10^{2.5}-1}{10^{0.3}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = 2.97$$

$$N = 3$$

$$\Omega_p = \Omega_c = 0.65$$

For $N = 3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{0.65}} = \frac{(0.65)^3}{(s+0.65)(s^2+0.65s+0.4225)}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{[2(1-z^{-1})+0.65(1+z^{-1})]\,[4(1-z^{-1})^2+0.65(1-z^{-2})+0.4225(1+z^{-1})^2]}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{(2.65-1.35z^{-1})\,(4-8z^{-1}+4z^{-2}+0.65-0.65z^{-2}+0.4225+0.4225z^{-2}+0.845z^{-1})}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{(2.65-1.35z^{-1})(5.0725-7.155z^{-1}+3.7725z^{-2})}$$

$$= \frac{0.02066(1+z^{-1})^3}{(1-0.51z^{-1})(1-1.41z^{-1}+0.751z^{-2})}$$

Chebyshev filter

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{2.5}-1}{10^{0.3}-1}}}{\cos h^{-1}\frac{\Omega_s}{\Omega_p}}$$

$$\Rightarrow \quad N = 3$$

$$\epsilon = \sqrt{10^{0.3}-1} = 1$$

$$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(2.414)^{1/3}-(2.414)^{-1/3}}{2}\right] = 0.1935$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(2.414)^{1/3}+(2.414)^{-1/3}}{2}\right] = 0.678$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, 3$$

$$\phi_1 = 120°; \quad \phi_2 = 180°; \quad \phi_3 = 240°$$

$$\begin{aligned} s_1 &= a\cos\phi_1 + jb\sin\phi_1 \\ &= 0.1935\cos(120°) + j0.678\sin(120°) \\ &= -0.09675 + j0.587 \\ s_2 &= a\cos\phi_2 + jb\sin\phi_2 \\ &= 0.1935\cos(180°) + j0.678\sin(180°) \\ &= -0.1935 \\ s_3 &= a\cos\phi_3 + jb\sin\phi_3 \\ &= 0.1935\cos(240°) + j0.678\sin(240°) \\ &= -0.09675 - j0.587 \end{aligned}$$

The denominator polynomial of $H(s) = (s+0.1935)\left[(s+0.09675)^2+0.587^2\right]$
$= (s+0.1935)\left[(s^2+0.1935s+0.354)\right]$

The transfer of $H(s) = (0.1935)(0.354) = 0.0685$

The transfer function $H(s) = \dfrac{0.0685}{(s+0.1935)(s^2+0.1935s+0.354)}$

$$H(z) = H(s)\Big|_{s=\frac{2(1-z^{-1})}{1+z^{-1}}}$$

$$= \frac{0.0685(1+z^{-1})^3}{(2.1935-1.8065z^{-1})(4.5475-7.292z^{-1}+4.1605z^{-2})}$$

$$= \frac{0.00687(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.6z^{-1}+0.915z^{-2})}$$

Example 5.34 Design a Chebyshev filter for the following specification using (a) bilinear transformation (b) impulse invariance Method.

$$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

Solution

(a) Given $\omega_s = 0.6\pi$, $\omega_p = 0.2\pi$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 0.6498 \quad (\because T = 1 \text{ sec})$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2.752$$

$$N = \frac{\cosh^{-1}\lambda/\varepsilon}{\cosh^{-1}1/k} = 1.208$$

$$\Rightarrow N = 2$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 0.3752$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 0.75$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2$$

$$s_k = a\cos\phi_k + jb\sin\phi_k$$

$$s_1 = -0.2653 + j0.53$$

$$s_2 = -0.2653 - j0.53$$

Denominator of

$$H(s) = (s + 0.2653)^2 + (0.53)^2$$
$$= s^2 + 0.5306s + 0.3516$$

For N even, Numerator of $H(s)$ is $\dfrac{0.3516}{[1 + (0.75)^2]^{1/2}} = 0.28$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Using bilinear transformation

$$H(z) = H(z)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \qquad \boxed{\because T = 1 \text{ sec}}$$

$$H(z) = \frac{0.28(1 + z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}}$$
$$= \frac{0.052(1 + z^{-1})^2}{1 - 1.3480z^{-1} + 0.608z^{-2}}$$

(b) By using impulse invariance method

$$\omega = \Omega T \Rightarrow \omega_p = \Omega_p T \quad \text{and} \quad \omega_s = \Omega_s T$$

For $T = 1$ sec

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\cosh^{-1}\frac{\lambda}{\varepsilon}}{\cosh^{-1}\frac{1}{k}} = \frac{\cosh^{-1}\frac{4.899}{0.75}}{\cosh^{-1} 3} = 1.45$$

Approximating N to next higher integer, we get $N = 2$. We know $\mu = 3$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.3627$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.7255$$

$$\phi_1 = 135°; \phi_2 = 225°$$
$$s_1 = -0.2564 + j0.513$$
$$s_2 = -0.2564 - j0.513$$

Numerator of $H(s) = 0.264$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33} = \frac{(0.5146)(0.513)}{(s + 0.2564)^2 + (0.513)^2}$$

Taking inverse Laplace transform we obtain

$$h(t) = 0.5146e^{-0.2564t} \sin 0.513t$$

Let $t = nT$. Then $h(nT) = 0.5146e^{-0.2564nT} \sin 0.513nT$.
The z-Transform

$$H(z) = \frac{0.5146e^{-0.2564T} z^{-1} \sin 0.513T}{1 - 2e^{-0.2564T} z^{-1} \cos 0.513T + e^{-0.513T} z^{-2}}$$

Assume $T = 1$ sec

$$H(z) = \frac{0.1954z^{-1}}{1 - 1.3483z^{-1} + 0.5987z^{-2}}$$

(or)

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33} = \frac{0.257j}{s + 0.256 - j0.513} - \frac{0.257j}{s + 0.256 + j0.513}$$

$$H(z) = \frac{0.257j}{1 - e^{-0.256T} e^{j0.513T} z^{-1}} - \frac{0.257j}{1 - e^{-0.256T} e^{-j0.513T} z^{-1}}$$

$$= \frac{0.1954z^{-1}}{1 - 1.3483z^{-1} + 0.5987z^{-2}}$$

Example 5.35 Design a bandstop Butterworth and Chebyshev type-I filter to meet the following specifications

(a) Stopband 100 to 600 Hz.
(b) 20 dB attenuation at 200 and 400 Hz.
(c) The gain at $\omega = 0$ is unity.
(d) The passband ripple for the chebyshev filter is 1.1 dB.
(e) The passband attenuation for Butterworth filter is 3 dB.

Solution

Given
$f_l = 100$ Hz; $f_1 = 200$ Hz; $f_2 = 400$ Hz; $f_u = 600$; Hz

Then

$$\Omega_l = 2 \times \pi \times 100 = 200\pi \quad rad/sec$$
$$\Omega_1 = 2 \times \pi \times 200 = 400\pi \quad rad/sec$$
$$\Omega_2 = 2 \times \pi \times 400 = 800\pi \quad rad/sec$$
$$\Omega_u = 2 \times \pi \times 600 = 1200\pi \quad rad/sec$$

First we design a prototype normalized lowpass filter and then use suitable transformation to obtain the transfer function of bandstop filter.

1/A
(s − a)
(s · b)

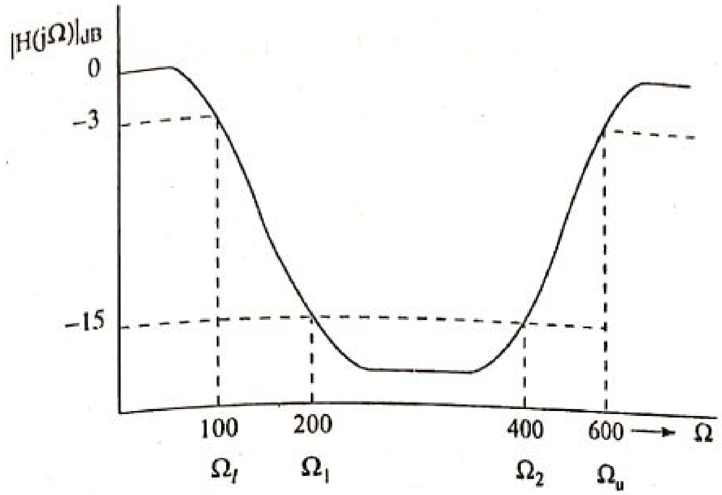

Fig. 5.64

For the normalized lowpass filter

$$\Omega_r = \min\{|A|, |B|\}$$

where

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} = \frac{400\pi[1200\pi - 200\pi]}{-(400\pi)^2 + (200\pi)(1200\pi)}$$

$$= 5$$

$$B = \frac{\Omega_2(\Omega_l - \Omega_u)}{-\Omega_2^2 + \Omega_l\Omega_u} = \frac{800\pi[1200\pi - 200\pi]}{-(800\pi)^2 + (200\pi)(1200\pi)}$$

$$= -2$$

$\Omega_r = \min \ \{|5|, |-2|\} = 2.$

(a) Butterworth filter

The order of normalized Butterworth filter is

$$N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}}$$

Given

$\alpha_p = 3\,\text{dB}, \alpha_s = 20\,\text{dB}$

$$\frac{\Omega_s}{\Omega_p} = \Omega_r = 2$$

$$= \frac{\log\sqrt{\frac{10^2-1}{10^{0.3}-1}}}{\log 2} = \frac{0.9978}{0.3010} = 3.32$$

Take $N = 4$.

For $N = 4$ the transfer function of Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

To get the transfer function of bandstop filter, use the transformation

$$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

$$i.e., s \to \frac{s(1000\pi)}{s^2 + 24 \times 10^4 \pi^2}$$

$$H(s)\Big|_{s \to \frac{1000\pi s}{s^2 + 24 \times 10^4 \pi^2}}$$

$$= \frac{(s^2 + 24 \times 10^4 \pi^2)^4}{[(1000\pi)^2 s^2 + (0.765 \times 1000\pi s)(s^2 + 24 \times 10^4 \pi^2) + (s^2 + 24 \times 10^4 \pi^2)^2] \\ [(1000\pi)^2 s^2 + 1.848 \times 1000\pi s(s^2 + 24 \times 10^4 \pi^2) + (s^2 + 24 \times 10^4 \pi^2)]}$$

$$= \frac{(s + 24 \times 10^4 \pi^2)^4}{[s^4 + 2403.32 s^3 + 1.4607 \times 10^7 s^2 + 5.69 \times 10^9 s + 5.61 \times 10^{12}] \\ [s^4 + 5.805 \times 10^3 s^3 + 1.4607 \times 10^7 s^2 + 1.375 \times 10^{10} s + 5.61 \times 10^{12}]}$$

(b) Chebyshev filter

The order of the Chebyshev filter

$$N = \frac{\cos h^{-1} \sqrt{\frac{10^2 - 1}{10^{0.3} - 1}}}{\cos h^{-1} 2}$$

$$= 2.75$$

Take $N = 3$

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = (10^{0.11} - 1)^{0.5} = 0.5368$$

$$\lambda = (10^{0.1\alpha_s} - 1)^{0.5} = (10^2 - 1)^{0.5} = 9.95$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 3.97$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] \quad ; \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 1\left[\frac{(3.97)^{1/3} - (3.97)^{-1/3}}{2}\right] \qquad = 1\left[\frac{(3.97)^{1/3} + (3.97)^{-1/3}}{2}\right]$$

$$= 0.476 \qquad\qquad = 1.107$$

($\Omega_p = 1$for normalized Chebyshev filter)

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, 3$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = 120°$$

$$\phi_2 = \frac{\pi}{2} + \frac{\pi}{2} = 180°$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 240^\circ$$

$$\begin{aligned}
s_1 &= a\cos\phi_1 + jb\sin\phi_1 \\
&= 0.476\cos 120^\circ + j1.107\sin 120^\circ \\
&= -0.238 + j0.9586 \\
s_2 &= a\cos\phi_2 + jb\sin\phi_2 \\
&= (0.476)\cos 180^\circ + j1.107\sin 180^\circ \\
&= -0.476 \\
s_3 &= a\cos\phi_3 + jb\sin\phi_3 \\
&= 0.476\cos 240^\circ + j1.107\sin 240^\circ \\
&= -0.238 - j0.9586
\end{aligned}$$

Denominator of the transfer function

$$\begin{aligned}
&= (s + 0.476)\{(s + 0.238)^2 + (0.9586)^2\} \\
&= (s + 0.476)(s^2 + 0.476s + 0.975)
\end{aligned}$$

Numerator of the transfer function

$$= (0.476)(0.9755) = 0.46463$$

$$H(s) = \frac{0.4643}{(s + 0.476)(s^2 + 0.476s + 0.9755)}$$

The transfer function of Bandstop filter can be obtained by using the following transformation

$$H(s)\Big|_{s \to \frac{s(1000\pi)}{s^2 + 24\times 10^4\pi^2}}$$

$$= \frac{0.4643(s^2 + 24\times 10^4\pi^2)^3}{[1000\pi s + 0.476(s^2 + 24\times 10^4\pi^2)]\,[s^2(1000\pi)^2 + 0.476(1000\pi s)(s^2 + 24\times 10^4\pi^2) + 0.9755(s^2 + 24\times 10^4\pi^2)^2]}$$

$$= \frac{0.4643(s^2 + 24\times 10^4\pi^2)^3}{(0.476s^2 + 1000\pi s + 1.1275\times 10^6)(0.9755s^4 + 4.621\times 10^6 + 5.47\times 10^{12} + 9.369\times 10^6\pi^2 s^2 + 1495.4s^3 + 3.5\times 10^9 s)}$$

$$= \frac{(s^2 + 24\times 10^4\pi^2)^3}{(s^2 + 6600s + 2.3687\times 10^6)(s^4 + 1533s^3 + 1.45\times 10^7 s^2 + 3.589\times 10^9 s + 5.6\times 10^{12})}$$

Example 5.36 Using bilinear transformation design a digital bandpass Butterworth filter with the following specifications

Sampling frequency F = 8 KHz f_l f_u

$\alpha_p = 2\,\text{dB}$ in the passband $800\text{Hz} \le f \le 1000$ Hz

$\alpha_s = 20\,\text{dB}$ in the stopband $0 \le f \le 400$ Hz and 2000 Hz $\le f \le \infty$

f_1 f_2

Solution

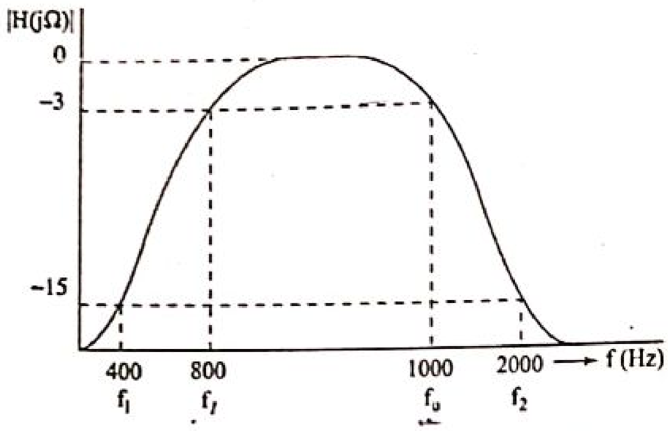

Fig. 5.65

$$\frac{\omega_1 T}{2} = \frac{2 \times \pi \times 400}{2 \times 8000} = \frac{\pi}{20}$$
$$\frac{\omega_l T}{2} = \frac{2\pi \times 800}{2 \times 8000} = \frac{\pi}{10}$$
$$\frac{\omega_u T}{2} = \frac{2 \times \pi \times 1600}{2 \times 8000} = \frac{\pi}{5}$$
$$\frac{\omega_2 T}{2} = \frac{2 \times \pi \times 2000}{2 \times 8000} = \frac{\pi}{4}$$

Prewarped analog frequencies are given by

$$\frac{\Omega_1 T}{2} = \tan\frac{\omega_1 T}{2} = \tan\frac{\pi}{20} = 0.1584$$
$$\frac{\Omega_l T}{2} = \tan\frac{\omega_l T}{2} = \tan\frac{\pi}{10} = 0.325$$
$$\frac{\Omega_u T}{2} = \tan\frac{\omega_u T}{2} = \tan\frac{\pi}{5} = 0.7265$$
$$\frac{\Omega_2 T}{2} = \tan\frac{\omega_2 T}{2} = \tan\frac{\pi}{4} = 1$$

First we design a prototype normalized lowpass filter and then use suitable transformation to obtain the transfer function of bandpass filter.

To reduce computational complexity we use above values to find Ω_r and substitute $s = \frac{1-z^{-1}}{1+z^{-1}}$ for bilinear transformation ($\because$ all the above frequencies contains the term $\frac{T}{2}$).

We have

$$A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)};$$
$$= \frac{-(0.1584)^2 + (0.325)(0.7265)}{0.1584(0.7265 - 0.325)}$$
$$= 3.318$$

$$B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$
$$= \frac{1 - (0.7265)(0.325)}{1(0.7265 - 0.325)}$$
$$= 1.90258$$

$\Omega_r = \min\{|A|, |B|\} = 1.90258$

$$N = \frac{\log_{10}\sqrt{\frac{10^{2}-1}{10^{0.2}-1}}}{\log_{10}(1.90258)} = 3.9889$$

Let us choose $N = 4$.

The Fourth order normalized Butterworth lowpass filter transfer function is given by

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)}$$

The transformation for the bandpass filter is

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 0.236}{s(0.402)}$$

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)}\Bigg|_{s \rightarrow \frac{s^2 + 0.236}{s(0.402)}}$$

$$= \frac{1}{\left[\left(\frac{s^2 + 0.236}{0.402s}\right)^2 + 0.76537\left(\frac{s^2 + 0.236}{0.402s}\right) + 1\right]\left[\left(\frac{s + 0.236}{0.402s}\right)^2 + 1.84776\left(\frac{s^2 + 0.236}{0.402s}\right) + 1\right]}$$

$$= \frac{0.0261s^4}{(s^4 + 0.30768s^3 + 0.6336s^2 + 0.0726s + 0.055696)(s^4 + 0.7428s^3 + 0.6336s^2 + 0.1753s + 0.055696)}$$

$$H(z) = H(s)\Big|_{s = \frac{1 - z^{-1}}{1 + z^{-1}}}$$

$$H(z) = \frac{0.0261(1 - z^{-1})^4}{5.3962 - 21.2398z^{-1} + 44.566z^{-2} - 60.512z^{-3} + 58.1635z^{-4} - 39.86z^{-5} + 19.28z^{-6} - 6.0087z^{-7} + 1.009z^{-8}}$$

$$= \frac{0.004837(1 - z^{-1})^4}{1 - 3.936z^{-1} + 8.2587z^{-2} - 11.214z^{-3} + 10.778z^{-4} - 7.3866z^{-5} + 3.573z^{-6} - 1.1135z^{-7} + 0.187z^{-8}}$$

Example 5.37 Design a Chebyshev type-I bandreject filter with the following specifications

passband d.c. to 275 Hz and 2 KHz to ∞

stopband 550 Hz to 1000 Hz

$\alpha_p = 1\,\text{dB}$; $\alpha_s = 15\,\text{dB}$; F = 8KHz

Solution

The digital frequencies are given by

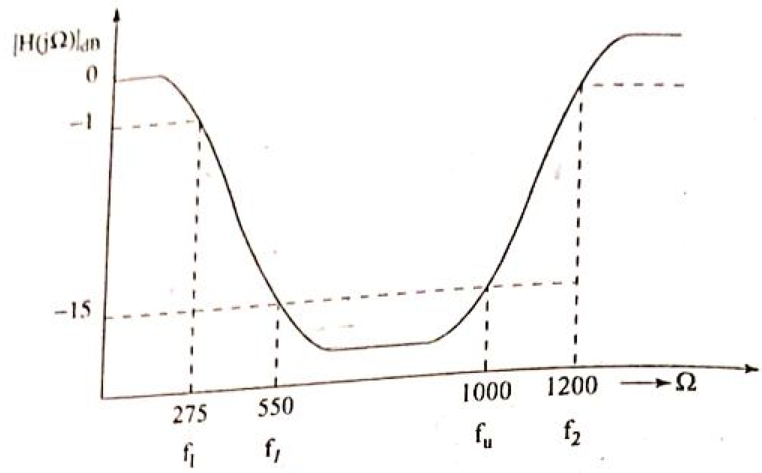

Fig. 5.66

$$\frac{\omega_l T}{2} = \frac{2\pi \times 275}{2(8000)} = 0.034375\pi$$
$$\frac{\omega_1 T}{2} = \frac{2\pi \times 550}{2(8000)} = 0.06875\pi$$
$$\frac{\omega_2 T}{2} = \frac{2\pi \times 1000}{2(8000)} = 0.125\pi$$
$$\frac{\omega_u T}{2} = \frac{2\pi \times 2000}{2(8000)} = 0.25\pi$$

Prewarped analog frequencies are

$$\frac{\Omega_l T}{2} = \tan\frac{\omega_l T}{2} = 0.1084$$
$$\frac{\Omega_1 T}{2} = \tan\frac{\omega_1 T}{2} = 0.2194$$
$$\frac{\Omega_2 T}{2} = \tan\frac{\omega_2 T}{2} = 0.4141$$
$$\frac{\Omega_u T}{2} = \tan\frac{\omega_u T}{2} = 1$$

First we design a prototype normalized lowpass filter and then use suitable transformation to obtain the transfer function of bandreject filter.

$$\Omega_r = \min\{|A|, |B|\}$$
$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} = \frac{0.2194(1 - 0.1084)}{-(0.2194)^2 + (1)(0.1084)} = 3.246$$
$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u} = \frac{0.4142(1 - 0.1084)}{-(0.4142)^2 + (1)(0.1084)} = -5.847$$

$$\Omega_r = 3.246$$

$$N = \frac{\cos h^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cos h^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cos h^{-1}\sqrt{\frac{10^{1.5}-1}{10^{0.1}-1}}}{\cos h^{-1}3.246} = 1.666$$

$$\boxed{\because \frac{\Omega_s}{\Omega_p} = \Omega_r; \quad \Omega_p = 1}$$

Choose $N = 2$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1} - 1} = 0.508$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 4.17$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = \left[\frac{(4.17)^{1/2} - (4.17)^{-1/2}}{2}\right] = 0.776$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = \left[\frac{(4.17)^{1/2} + (4.17)^{-1/2}}{2}\right] = 1.266$$

For normalized chebyshev filter $\Omega_p = 1$ rad/sec

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, \dots N$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = -0.5487 + j0.895$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = -0.5487 - j0.895$$

The transfer function of lowpass filter is given by

$$H_L(s) = \frac{0.9825}{s^2 + 1.0974s + 1.102}$$

To get the transfer function of bandreject filter use the transformation

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}$$

$$\Longrightarrow s \rightarrow \frac{0.891\check{c}s}{s^2 + 0.1084}$$

$$H(s) = \frac{0.9825}{\left(\frac{0.8916s}{s^2 + 0.1084}\right)^2 + 1.0974\left(\frac{0.8916s}{s^2 + 0.1084}\right) + 1.102}$$

$$= \frac{0.89156(s^4 + 0.2168s^2 + 0.01175)}{s^4 + 0.8878s^3 + 0.9382s^2 + 0.09618s + 0.01174}$$

The transfer function of digital bandreject filter using bilinear transformation is given by

$$H(z) = H(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.3732(1 - 3.2176z^{-1} + 4.588z^{-2} - 3.2176z^{-3} + z^{-4})}{1 - 1.8869z^{-1} + 1.429z^{-2} - 0.8077z^{-3} + 0.3292z^{-4}}$$

Example 5.38 Find pole and zero locations of an analog Chebyshev type II filter for the following digital filter specifications. Use bilinear transformation.

$$-1 \leq |H(e^{j\omega})|_{\text{dB}} \leq 0 \qquad 0 \leq |\omega| \leq 0.2\pi$$
$$|H(e^{j\omega})|_{\text{dB}} \leq -20 \quad |\omega| \geq 0.3\pi$$

$\alpha_p = -1 \quad \alpha_s = -20,$

Solution

The prewarped analog frequencies are given by

$$\frac{\Omega_p T}{2} = \tan\frac{\omega_p T}{2} = \tan\frac{0.2\pi}{2} = 0.32492$$
$$\frac{\Omega_s T}{2} = \tan\frac{\omega_s T}{2} = \tan\frac{0.3\pi}{2} = 0.50953$$

The order of the filter is given by

$$N = \frac{\cos h^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cos h^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cos h^{-1}\sqrt{\dfrac{10^2-1}{10^{0.1}-1}}}{\cos h^{-1}\dfrac{0.50953}{0.32492}} = \frac{3.66}{1.021} = 3.59$$

Choose $N = 4$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \ldots 4$$
$$\phi_1 = 112.5°; \quad \phi_2 = 157.5°; \quad \phi_3 = 202.5°; \quad \phi_4 = 247.5°$$

The zeros are located on the imaginary axis at the points

$$s_k = \frac{j\Omega_s}{\sin\phi_k} \qquad k = 1, 2, \ldots 4$$
$$s_1 = \frac{j0.50953}{\sin 112.5°} = j0.55151$$
$$s_2 = \frac{j0.50953}{\sin 157.5°} = j1.3314$$
$$s_3 = \frac{j0.50953}{\sin 202.5°} = -j1.3314$$
$$s_4 = \frac{j0.50953}{\sin 247.5°} = -j0.55151$$

$$\mu = \lambda + \sqrt{1+\lambda^2}$$
$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{99}$$
$$\mu = 9.9498 + 10 = 19.9498 = 19.95$$
$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.3249\left[\frac{(19.94)^{1/4} - (19.94)^{-1/4}}{2}\right]$$
$$= 0.2664$$
$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.3249\left[\frac{(19.94)^{1/4} + (19.94)^{-1/4}}{2}\right] = 0.4202$$

$$\sigma_1 = a\cos\phi_1 = -0.102; \qquad \Omega_1 = b\sin\phi_1 = 0.3882$$
$$\sigma_2 = a\cos\phi_2 = -0.2461; \qquad \Omega_2 = b\sin\phi_2 = 0.1608$$
$$\sigma_3 = a\cos\phi_3 = -0.2481; \qquad \Omega_3 = b\sin\phi_3 = -0.1608$$
$$\sigma_4 = a\cos\phi_4 = -0.102; \qquad \Omega_4 = b\sin\phi_4 = -0.3882$$
$$x_k = \frac{\Omega_s\sigma_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1,2,3,4; \qquad y_k = \frac{\Omega_s\Omega_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1,2,3,4$$
$$x_1 = \frac{\Omega_s\sigma_1}{\sigma_1^2 + \Omega_1^2} = -0.3226 \qquad y_1 = \frac{\Omega_s\Omega_1}{\sigma_1^2 + \Omega_1^2} = -1.22775$$
$$x_2 = \frac{\Omega_s\sigma_2}{\sigma_2^2 + \Omega_2^2} = -1.45 \qquad y_2 = \frac{\Omega_s\Omega_2}{\sigma_2^2 + \Omega_2^2} = -0.948$$
$$x_3 = \frac{\Omega_s\sigma_3}{\sigma_3^2 + \Omega_3^2} = -1.45 \qquad y_3 = \frac{\Omega_s\Omega_3}{\sigma_3^2 + \Omega_3^2} = 0.948$$
$$x_4 = \frac{\Omega_s\sigma_4}{\sigma_4^2 + \Omega_4^2} = -0.3226 \qquad y_4 = \frac{\Omega_s\Omega_4}{\sigma_4^2 + \Omega_4^2} = 1.22775$$

Therefore, the zeros are at $\pm j0.55151, \pm j1.33141$.
The poles are at $-0.3226 \pm j1.22775$ and $-1.45 \pm j0.948$.

Example 5.39 Design a digital Chebyshev filter to meet the constraints

$$\frac{1}{\sqrt{2}} \leq H(e^{j\omega}) \leq 1 \qquad \text{for } 0 \leq \omega \leq 0.2\pi$$
$$0 \leq |H(e^{j\omega})| \leq 0.1 \qquad \text{for } 0.5\pi \leq \omega \leq \pi$$

by using bilinear transformation and assume sampling period $T = 1$ sec.
(AU ECE May'05)

Solution Given $\omega_s = 0.5\pi$; $\omega_p = 0.2\pi$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1 \Rightarrow \lambda = 9.95$$
$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}} \Rightarrow \varepsilon = 1$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan(0.1\pi) = 0.65$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\left(\frac{\pi}{4}\right) = 2$$

$$N = \frac{\cos h^{-1}\frac{\lambda}{\varepsilon}}{\cos h^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cos h^{-1} 9.95}{\cos h^{-1}\left(\frac{2}{0.65}\right)} = 1.669$$

approximate $N = 2$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1 + \sqrt{2} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.65\left[\frac{2.414^{1/2} - 2.414^{-1/2}}{2}\right] = 0.295$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.65\left[\frac{2.414^{1/2} + 2.414^{-1/2}}{2}\right] = 0.717$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1,2; \quad \phi_1 = 135°; \ \phi_2 = 225°$$

$$s_k = a\ \cos\phi_k + j\,b\ \sin\phi_k$$

$$s_1 = 0.295\cos 135° + j\,0.717\sin 135° = -0.2086 + j0.507$$

$$s_2 = 0.295\cos 225° + j\,0.717\sin 225° = -0.2086 - j0.507$$

Denominator of

$$H(s) = (s + 0.2086)^2 + (0.507)^2 = s^2 + 0.4172s + 0.3$$

For N even, numerator of $H(s)$ is

$$= \frac{0.3}{\sqrt{1+\varepsilon^2}} = 0.212$$

$$H(s) = \frac{0.212}{s^2 + 0.4172s + 0.3}$$

Using bilinear transformation

$$H(z) = H(s)\big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Since $T = 1$

$$H(z) = \frac{0.212(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.8344(1-z^{-2}) + 0.3(1+z^{-1})^2}$$

$$= \frac{0.212(1+2z^{-1}+z^{-2})}{4 - 8z^{-1} + 4z^{-2} + 0.8344 - 0.8344z^{-2} + 0.3 + 0.6z^{-1} + 0.3z^{-2}}$$

$$= \frac{0.212(1+2z^{-1}+z^{-2})}{5.1344 - 7.40z^{-1} + 3.4656z^{-2}}$$

$$= \frac{0.0413(1+z^{-1})^2}{1 - 1.44z^{-1} + 0.675z^{-2}}$$