1 DISCRETE FOURIER TRANSFORM

To perform frequency analysis on a DTS $x(n)$, we convert time domine sequence to an equivalent frequency domine Represen-tation. if we apply F.T on $x(n)$. we get $X(W)$ .which is continuous and periodic function of frequency. it is not a computational convenient Representation of the sequence $x(n)$.

"Representation of a sequence $x(n)$ by samples of its spec-trum $X(W)$ is known as Discrete Fourier Transform (DFT)."

FREQUENCY DOMINE SAMPLING

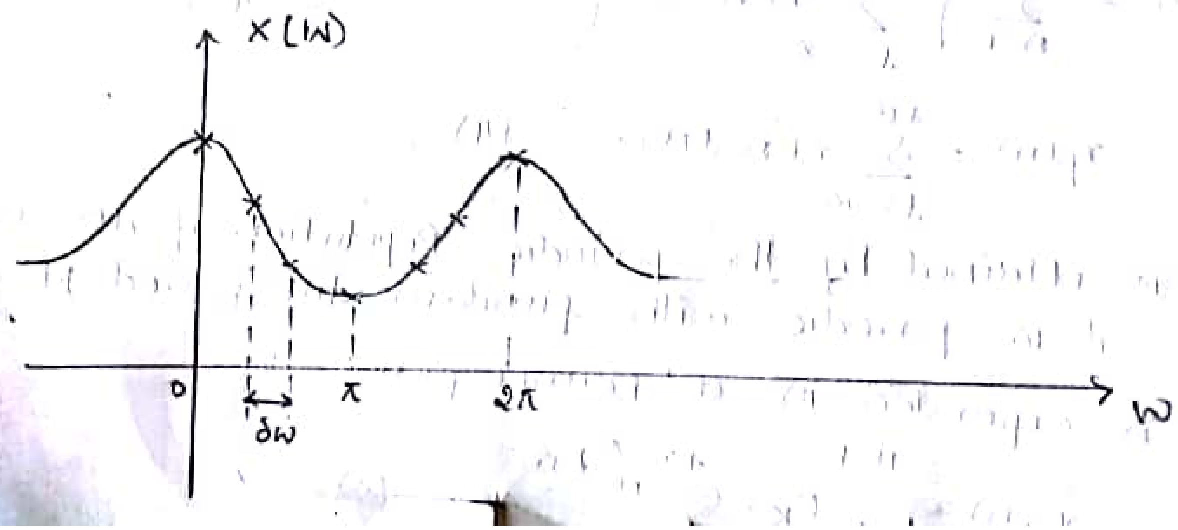
We know that aperiodic finite energy signals have conti-nuous spectra.

F.T of aperiodic DTS $x(n)$ is given by

$$X(W) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-jwn} \quad \text{——(1)}$$

Since $X(W)$ is periodic with period 2π, only samples in the fundamental frequency Range are obtained by sampling periodi-cally. in frequency at a spacing of δW Radians betn successive samples.

We take N equidistant samples in the interval $0 \le W \le 2\pi$ with spacing $\delta W = \frac{2\pi}{N}$



(2) Substituting $w = \frac{2\pi}{N} k$ in eqn (1)

$$x\left(\frac{2\pi}{N} k\right) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad \text{——— (2)}$$

where $K = 0, 1, 2 \ldots \ldots (N-1)$

Summation in eqn (2) can be sub-divided into infinite number of summations, where each sum contains N-terms

$$x\left(\frac{2\pi}{N}k\right) = \ldots\ldots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$+ \sum_{n=N}^{2N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn} + \ldots\ldots$$

$$x\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{+\infty} \sum_{n=lN}^{lN+N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn}$$

if we change the index in the ~~linner~~ inner summation from n to (n-lN) and interchanging the order of the summation, we get :

$$x\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{+\infty} x(n-lN)\, e^{-j\frac{2\pi}{N}k(n-lN)}$$

$$e^{-j\frac{2\pi}{N}k(n-lN)} = e^{-j\frac{2\pi}{N}kn} \cdot e^{+j\frac{2\pi}{N}k \cdot lN}$$

W.K.T $\quad e^{-j\frac{2\pi}{N} \cdot kLN} = e^{-j2\pi kl} = 1$

$$x\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left\{ \sum_{l=-\infty}^{+\infty} x(n-lN) \right\} e^{-j\frac{2\pi}{N}kn} \qquad \text{——— (3)}$$

Let
$$x_p(n) = \sum_{l=-\infty}^{+\infty} x(n-lN) \qquad \text{——— (4)}$$

$x_p(n)$ is obtained by the periodic Repetation of $x(n)$ every N samples. it is periodic with fundamental period N

$x_p(n)$ is expanded in a Fourier series as

$$x_p(n) = \sum_{K=0}^{N-1} c_K \cdot e^{+j\frac{2\pi}{N}kn} \qquad \text{——— (5)}$$

$n = 0, 1, 2 \ldots\ldots (N-1)$.

with Fourier co-efficients.

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{———}(6)$$

$k = 0, 1, 2 \ldots\ldots (N-1)$.

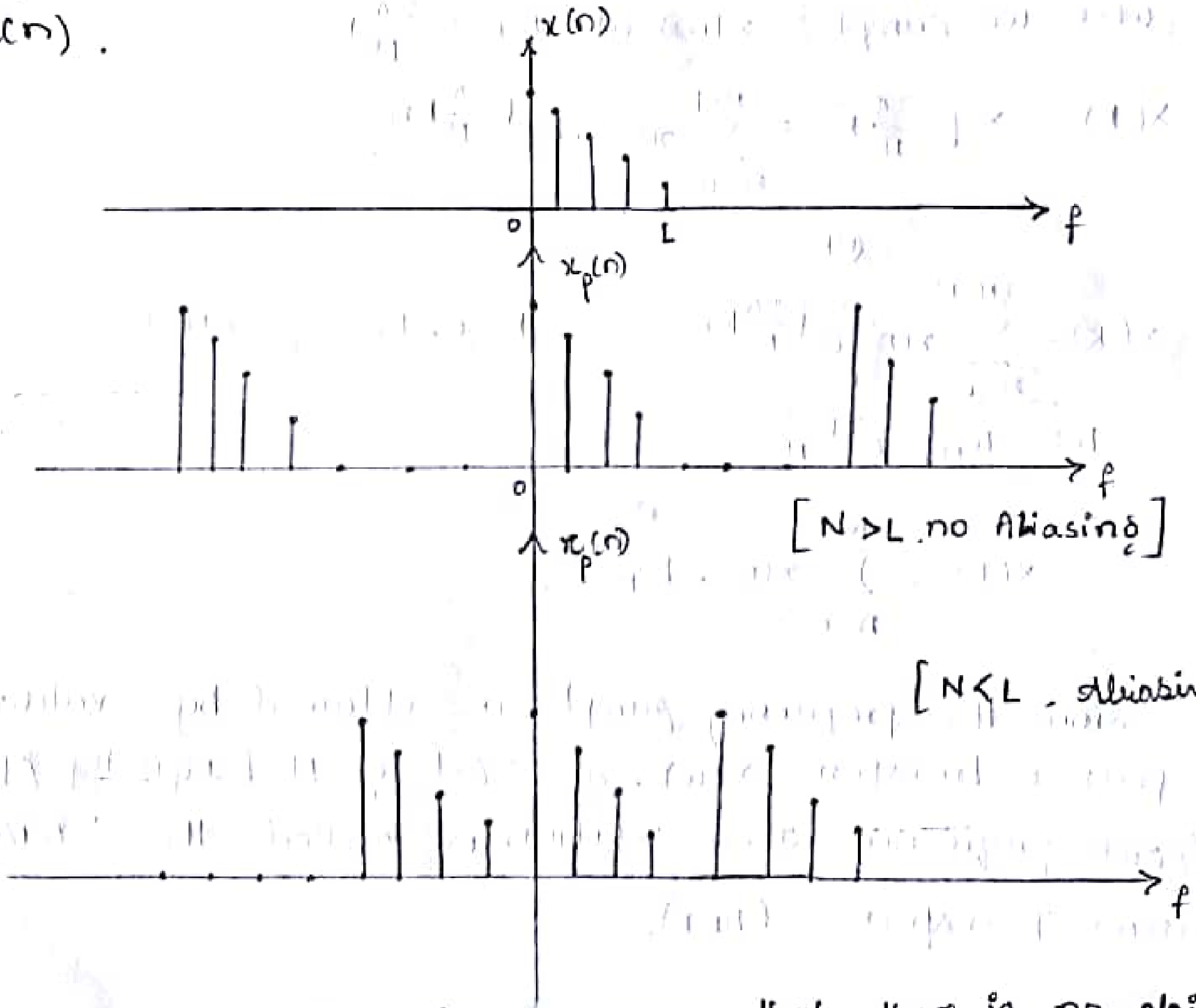
comparing eqn (3) & (6):

$$C_k = \frac{1}{N} \cdot X\left(\frac{2\pi}{N}k\right) \quad \text{———}(7)$$

Substituting eq (7) in eqn (5)

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} \quad \text{———}(8)$$

Eqn (8) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of $X(\omega)$. To reconstruct $x(n)$ or $X(\omega)$ from the samples, we require a Relationship b/w $x_p(n)$ & $x(n)$.



from the above fig. we observe that there is no aliasing if $N \geq L$ & $x(n)$ can be recovered from $x_p(n)$.

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i.e. $x(n) = x_p(n)$ $\quad 0 \le n \le (N-1)$.

* If $N < L$, $x(n)$ cannot be Recovered from $x_p(n)$ because of time domain aliasing.

DISCRETE FOURIER TRANSFORM (DFT)

if $x(n)$ has a finite duration of length $L \le N$, then $x_p(n)$ is periodic Repetition of $x(n)$

where $x_p(n) = \begin{cases} x(n), & 0 \le n \le (L-1) \\ 0, & L \le n \le (N-1) \end{cases}$

F.T of a finite duration sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{N-1} x(n) . e^{-j\omega n}$$

when we sample $X(\omega)$ at $\omega_k = \frac{2\pi}{N} k$

$$X(K) = X\left(\frac{2\pi}{N} k\right) = \sum_{n=0}^{N-1} x(n) . e^{-j\frac{2\pi}{N}kn}$$

OR

$$X(K) = \sum_{n=0}^{(N-1)} x(n) e^{-j\frac{2\pi}{N}kn} \qquad K = 0, 1, 2, \ldots (N-1)$$

let $W_N = e^{-j\frac{2\pi}{N}}$

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) . W_N^{kn}}$$

Since the frequency samples are obtained by evaluating the fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequency, above relation is called the "Discrete Fourier Transform" (DFT).

* We can get back $x(n)$ from $X(K)$ by using a relation known as IDFT.

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) \cdot e^{j\frac{2\pi}{N}kn}$$

$n = 0, 1, 2, \ldots (N-1)$.

or

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}}$$

DFT as a Linear Transformation

W.K.T

$$\text{DFT}\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$\text{IDFT}\{X(K)\} = x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) \cdot W_N^{-kn}.$$

$$x(n) \xleftrightarrow[\text{DFT}]{N\text{-pt}} X(K)$$

from the above eq^n, we observe that the computa-tion of each point of the DFT Required N-complex multiplications and $(N-1)$ complex additions.

*** hence N-point DFT Requires

* N^2 complex multiplications &
* $N(N-1)$ complex additions.

DFT & IDFT can be viewed as linear transformations on sequences $x(n)$ & $X(K)$.

Let us define N-point i/p vector as x_N

—"— o/p vector as X_N

—"— N x N matrix W_N

i.e.

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

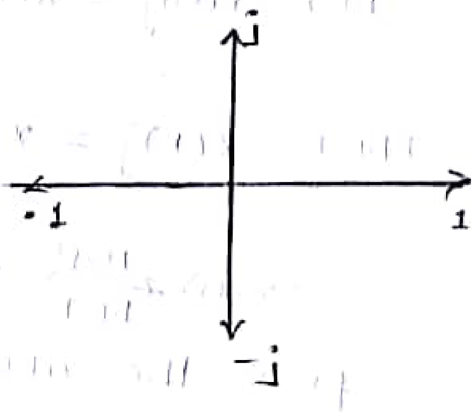
$$W_N = \begin{bmatrix} W_N^{0.0} & W_N^{0.1} & \cdots & W_N^{0(N-1)} \\ W_N^{0.0} & W_N^{1.1} & \cdots & W_N^{1(N-1)} \\ \vdots & & & \\ W_N^{0.0} & W_N^{1(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

from the above definations, N- point DFT can be expressed in matrix form as:

$$\boxed{X_N = W_N \cdot x_N}$$

or

$$\boxed{x_N = W_N^{-1} X_N}$$



we observe that W_N is a symmetric matrix, we assum that inverse of W_N exists then, IDFT formula can be expressed in matrix form as:

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N}$$

where W_N^{*} is the complex conjugate of W_N.

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: PROBLEMS :

1. compute 4-pt DFT of a sequence

$x(n) = \{1, 2, 3, 4\}$

Solution : <u>I Method</u>.

From the definition of DFT, WKT

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$k = 0, 1, 2, \ldots (N-1)$.

given <u>N=4</u>

$$X(K) = \sum_{n=0}^{4-1} x(n) \cdot W_4^{kn} = \sum_{n=0}^{3} x(n) \cdot W_4^{kn}.$$

$$X(K) = x(0) + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k}$$

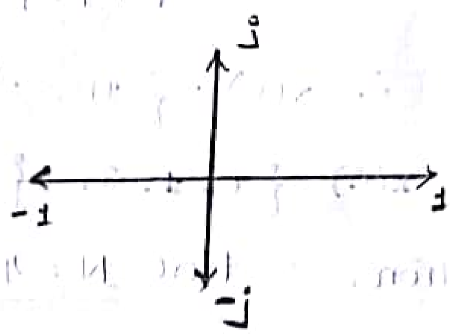
$$\boxed{X(K) = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}}$$

$k = 0, 1, 2, 3.$

Twiddle Factor

$W_4^0 = 1$ $\quad W_4^2 = -1$

$W_4^1 = -j$ $\quad W_4^3 = j$



At $k=0$: $X(0) = 1+2+3+4 = 10$

At $k=1$: $X(1) = 1 + 2W_4^1 + 3W_4^2 + 4W_4^3$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$X(1) = -2 + 2j$

At $k=2$: $X(2) = 1 + 2W_4^2 + 3W_4^4 + 4W_4^6$

$= 1 + 2W_4^2 + 3W_4^0 + 4W_4^2$

$X(2) = 1 + 2(-1) + 3(1) + 4(-1)$

$= 1 - 2 + 3 - 4 = -2$

(8)

At $K=3$; $X(3) = 1 + 2\omega_4^3 + 3\omega_4^6 + 4\omega_4^9$

$= 1 + 2(j) + 3(-1) + 4(-j)$

$X(3) = -2-2j$

$\therefore X(K) = \{10, -2+2j, -2, -2-2j\}$.

II Method :-

$X_4 = W_4 x_4$

$$X(K) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\begin{bmatrix} 1+2+3+4 = 10 \\ 1-2j-3+4j = -2+2j \\ 1-2+3-4 = -2 \\ 1+2j-3-4j = -2-2j \end{bmatrix}$$

$\therefore X(K) = \{10, -2+2j, -2, -2-2j\}$

2. $x(n) = \{0, 1, 2, 3\}$ find $X(K)$

Solution: Here $N = 4$

$X(K) = x(n) \cdot W_N$

$$X(K) = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 0+1+2+3 = 6 \\ 0-j-2+3j = -2+2j \\ 0-1+2-3 = -2 \\ 0+j-2-3j = -2-2j \end{bmatrix}$$

$\therefore\ X(K) = \{6, -2+2j, -2, -2-2j\}$

3. If $x(n) = \{1, 2, 2, 1\}$ find $X(K)$.

Solution :- Here $N = 4$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad \text{where } K = 0, 1, 2, 3$$

$$X(K) = \sum_{n=0}^{3} x(n) \cdot W_4^{kn}$$

$$X(K) = x(0) + x(1) W_4^{K} + x(2) W_4^{2K} + x(3) W_4^{3K}$$

$$X(K) = 1 + 2 W_4^{K} + 2 W_4^{2K} + W_4^{3K}$$

At $K=0$; $X(0) = 1+2+2+1 = 6$

At $K=1$; $X(1) = 1 + 2W_4^1 + 2W_4^2 + W_4^3$

$= 1 + 2(-j) + 2(-1) + (j)$

$X(1) = -1-j$

At $K=2$; $X(2) = 1 + 2W_4^2 + 2W_4^4 + W_4^6$

$= 1 + 2(-1) + 2(1) + (-1)$

$X(2) = 0$

At $K=3$; $X(3) = 1 + 2W_4^3 + 2W_4^6 + W_4^9$

$= 1 + 2j + 2(-1) + (-j)$

$X(3) = -1+j$

$\therefore\ X(K) = \{6, -1-j, 0, -1+j\}$

4. If $x(n) = \{1, 1, -1, -1\}$ find $X(K)$.

Solution :- Here $N=4$

$X(K) = x(n) \cdot W_N$

(10) $X(K) = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

$$X(K) = \begin{bmatrix} 1+1-1-1 = 0 \\ 1-j+1-j = 2-2j \\ 1-1-1+1 = 0 \\ 1+j+1+j = 2+2j \end{bmatrix}$$

$X(K) = \{0, 2-2j, 0, 2+2j\}$

5. $x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$ find $X(K)$.

Solution :- $x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$

(08) $x(n) = \{1, 2, 3, 4\}$

here $N=4$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad \text{where } k = 0, 1, 2, 3$$

$$X(K) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 1+2+3+4 = 10 \\ 1-2j-3+4j = -2+2j \\ 1-2+3-4 = -2 \\ 1+2j-3-4j = -2-2j \end{bmatrix}$$

$\therefore X(K) = \{10, -2+2j, -2, -2-2j\}$

6. Find 4-pt IDFT of $X(K) = \{10, -2+2j, -2, -2-2j\}$

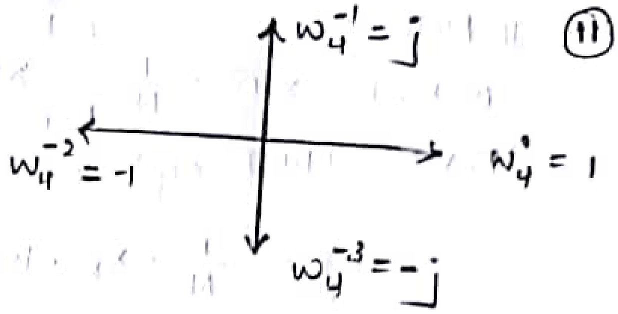
Solution :- From the defination of IDFT

I Method :- W.K.T

$$IDFT\{X(K)\} = x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-KN}$$

$n = 0, 1, 2, 3$

$W_4^0 = 1 \quad W_4^{-1} = j \quad W_4^{-2} = -1 \quad W_4^{-3} = -j$



Given: $\underline{N=4}$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) . W_4^{-kn}$$

$$x(n) = \frac{1}{4}\{ X(0) + X(1)\, W_4^{-n} + X(2)\, W_4^{-2n} + X(3) W_4^{-3n} \}$$

At $n=p$: $x(n) = \frac{1}{4}\{10 + (-2+2j)\, W_4^{-1} + (-2) W_4^{-2} + (-2-2j) W_4^{-3}\}$

$$x(n) = \frac{1}{4}\{10 + (-2+2j)\, j + (-2)(-1) + (-2-2j)(-j)\}$$

$$x(n) = \frac{1}{4}\{10 - 2j - 2 + 2 + 2j - 2\}$$

$$x(n) = \frac{1}{4}\{8\} = \underline{2}$$

At $n=0$: $x(0) = \frac{1}{4}\{10 + (-2+2j) + (-2) + (-2-2j)\}$

$$x(0) = \frac{1}{4}\{10 - 2 + 2j - 2 - 2 - 2j\}$$

$$x(0) = \frac{1}{4}\{4\} = 1.$$

At $n=2$: $x(2) = \frac{1}{4}\{10 + (-2+2j)\, W_4^{-2} + (-2) W_4^{-4} + (-2-2j) W_4^{-6}\}$

$$x(2) = \frac{1}{4}\{10 + (-2+2j)(-1) + (-2)(1) + (-2-2j)(-1)\}$$

$$x(2) = \frac{1}{4}\{10 + 2 - 2j - 2 + 2 + 2j\}$$

$$x(2) = \frac{1}{4}\{12\} = \underline{3}$$

At $n=3$: $x(3) = \frac{1}{4}\{10 + (-2+2j)\, W_4^{-3} + (-2)\, W_4^{-6} + (-2-2j) W_4^{-9}\}$

$$x(3) = \frac{1}{4}\{10 + (-2+2j)(-j) + (-2)(-1) + (-2-2j)(j)\}$$

$$= \frac{1}{4}\{10 + \cancel{2j} + 2 + 2 - \cancel{2j} + 2\}$$

$$x(3) = \frac{1}{4}\{16\} = 4$$

$\therefore \quad x(n) = \{1, 2, 3, 4\}$

12 II Method :-

w.k.T $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$

Above IDFT eq^n can also be written as

$$x_N = \frac{1}{N} \cdot X_K \cdot W_N^*$$

given $N=4$; $x_4 = \frac{1}{4} \cdot X_4 \cdot W_4^*$

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 & -2+2j & -2 & -2-2j \end{bmatrix}$$

$$x(n) = \frac{1}{4}\begin{bmatrix} 10-2+2j-2-2-2j = 4 \\ 10-2j-2+2+2j-2 = 8 \\ 10+2-2j-2+2+2j = 12 \\ 10+2j+2+2-2j+2 = 16 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$

$\therefore x(n) = \{1, 2, 3, 4\}$.

7. find the DFT of $x(n) = \delta(n)$

Solution :- From the defination of DFT, w.k.T

$$D.F.T\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \;; \quad k = 0, 1, 2, \ldots (N-1).$$

$$D.F.T\{\delta(n)\} = \sum_{n=0}^{N-1} \delta(n) \cdot W_N^{kn}.$$

w.k.T

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$ Above eq^n becomes

$$DFT\{\delta(n)\} = \delta(0) \cdot W_N^0 = 1 \cdot 1 = 1$$

$$\therefore \boxed{\delta(n) \underset{IDFT}{\overset{DFT}{\longleftrightarrow}} 1}$$

8. Find the DFT of $x(n) = \delta(n-n_0)$.

Solution:- From the definition of DFT, we have

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}, \quad K = 0, 1, \ldots, (N-1)$$

$$DFT\{\delta(n-n_0)\} = \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{kn}$$

W.K.T $\delta(n-n_0) = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$

$$X(K) = \delta(0) \cdot W_N^{kn_0}$$

$\therefore \boxed{X(K) = W_N^{kn_0}}$ $\qquad \therefore \delta(n-n_0) \underset{IDFT}{\overset{DFT}{\rightleftharpoons}} W_N^{kn_0}$

ex:- $\delta(n-5) \underset{IDFT}{\overset{DFT}{\rightleftharpoons}} W_N^{5k}$

$\delta(n-3) \rightleftharpoons W_N^{3k}$

$W_N^{10k} \rightleftharpoons \delta(n-10)$

$W_N^{2k} \rightleftharpoons \delta(n-2)$

9. Find N-pt DFT of $x(n) = e^{j\frac{2\pi}{N}k_0 n}$

Solution:- From the definition of DFT, W.K.T

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$DFT\{e^{j\frac{2\pi}{N}k_0 n}\} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n} \cdot W_N^{kn}$$

$$X(K) = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(K-K_0)n} = \frac{1 - \left[e^{-j\frac{2\pi}{N}(K-K_0)}\right]^N}{1 - e^{-j\frac{2\pi}{N}(K-K_0)}}$$

At $K = K_0$, $X(K) = \frac{0}{0}$

$\therefore$ using L-Hospital's method,

(14)

$$x(k) = \frac{0 - \left(-j\,2\pi \cdot e^{-j2\pi(k-k_0)}\right)}{0 - \left(-j\frac{2\pi}{N}\, e^{-j\frac{2\pi}{N}(k-k_0)}\right)}\Bigg|_{k=k_0}$$

At $k = k_0$:

$$x(k) = \frac{j2\pi \cdot e^0}{j\frac{2\pi}{N}\, e^0} = N$$

$\therefore\ x(k) = N$ at $k = k_0$

(or) $\boxed{x(k) = N \cdot \delta(k-k_0)}$

at $k \neq k_0$, $x(k) = 0$

$$\boxed{x(k) = \begin{cases} N, & \text{at } k = k_0 \\ 0, & \text{at } k \neq k_0 \end{cases}}$$

10. Find the DFT of $x(n) = e^{-j\frac{2\pi}{N}k_0 n}$.

Solution :- From the definition of DFT :-

$$DFT\{x(n)\} = x(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$DFT\left\{e^{-j\frac{2\pi}{N}k_0 n}\right\} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k_0 n} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$x(k) = \sum_{n=0}^{N-1} \left[e^{-j\frac{2\pi}{N}(k+k_0)}\right]^n$$

$$= \frac{1 - \left[e^{-j\frac{2\pi}{N}(k+k_0)}\right]^N}{1 - e^{-j\frac{2\pi}{N}(k+k_0)}}\Bigg|_{k=k_0}$$

At $k = -k_0$: $x(k) = \frac{0}{0}$

using L-Hospital's method;

$$x(k) = \frac{0 - \left[-j2\pi \cdot e^{j2\pi(k+k_0)}\right]}{0 - \left(-j\frac{2\pi}{N}\, e^{-j\frac{2\pi}{\pi}(k+k_0)}\right)}$$

At $k = -k_0$

$$x(k) = \frac{j2\pi \cdot e^0}{j\frac{2\pi}{N} \cdot e^0} = N$$

$x(K) = N \cdot \delta(K+K_0)$ or $N \cdot \delta(K-N+K_0)$.

At $K \neq -K_0$; $x(K) = 0$

$$\therefore \boxed{x(K) = \begin{cases} N & , \text{ at } K = -K_0 \\ 0 & , \text{ at } K \neq -K_0 \end{cases}}$$

(or)

$$\boxed{X(K) = N \cdot \delta(K+K_0)}$$

11. Find N-pt DFT of $x(n) = \cos \frac{2\pi}{N} K_0 n$.

From the definition of DFT, w.k.T

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad K = 0, 1, \ldots (N-1).$$

$$DFT\left\{\cos \frac{2\pi}{N} K_0 n\right\} = \sum_{n=0}^{N-1} \cos \frac{2\pi}{N} K_0 n \cdot W_N^{Kn}$$

$$X(K) = \sum_{n=0}^{N-1} \left\{ \frac{e^{j\frac{2\pi}{N}K_0 n} + e^{-j\frac{2\pi}{N}K_0 n}}{2} \right\} e^{-j\frac{2\pi}{N} \cdot Kn}$$

$$X(K) = \frac{1}{2} \sum_{n=0}^{N-1} \left\{ e^{-j\frac{2\pi}{N}(K-K_0)n} + e^{-j\frac{2\pi}{N}(K+K_0)n} \right\}$$

$$X(K) = \frac{1}{2}\left[N\,\delta(K-K_0) + N\,\delta(K+K_0)\right]$$

$$X(K) = \frac{N}{2}\left[\delta(K-K_0) + \delta(K+K_0)\right]$$

12. Find 5-pt DFT of $x(n) = \{1, 1, 1\}$

From the definition of DFT

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad K = 0, 1, 2, \ldots (N-1)$$

$\underline{N=5}$

$$X(K) = \sum_{n=0}^{4} x(n)\, W_5^{Kn} \quad \text{where } K = 0, 1, 2, 3, 4$$

$$X(K) = x(0)\,W_5^0 + x(1)\,W_5^K + x(2)\,W_5^{2K} + x(3)\,W_5^{3K} + x(4)\,W_5^{4K}$$

At $K=0$; $X(0) = 1+1+1 = 3$

(16) At $k=1$; $x(1) = 1 + W_5^1 + W_5^2 = 0.5 - j1.5388$.

At $k=2$; $x(2) = 1 + W_5^2 + W_5^4 = 0.5 - j0.3633$.

At $k=3$; $x(3) = 1 + W_5^3 + W_5^6 = 0.5 - j0.3633$

At $k=4$; $x(4) = 1 + W_5^4 + W_5^8 = 0.5 + j0.5388$

W.K.T $\quad W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$

$$W_5^0 = e^{-j\frac{2\pi}{5}\cdot 0} = e^0 = 1$$

$$W_5^1 = e^{-j\frac{2\pi}{5}\cdot 1} = \cos\left(\frac{2\pi}{5}\right) - j\sin\left(\frac{2\pi}{5}\right) = 0.3090 - j0.9510.$$

$$W_5^2 = e^{-j\frac{2\pi}{5}\cdot 2} = \cos\left(\frac{4\pi}{5}\right) - j\sin\left(\frac{4\pi}{5}\right) = -0.809 - j0.5878.$$

$$W_5^3 = e^{-j\frac{2\pi}{5}\cdot 3} = \cos\left(\frac{6\pi}{5}\right) - j\sin\left(\frac{6\pi}{5}\right) = -0.809 + j0.587$$

$$W_5^4 = e^{-j\frac{2\pi}{5}\cdot 4} = \cos\left(\frac{8\pi}{5}\right) - j\sin\left(\frac{8\pi}{5}\right) = 0.309 + j0.951.$$

$\therefore\ x(k) = \{3,\ 0.5 - j1.5388,\ 0.5 + j0.3633,\ 0.5 - j0.3633,\ 0.5 + j0.5388\}$

ex :- for problem (11)

1) $\cos\frac{4\pi}{N}\cdot n = \cos\left(\frac{2\pi}{N}\right)2n = \frac{N}{2}\left[\delta(k-2) + \delta(k+2)\right]$

2) $\cos\frac{6\pi}{N}\cdot n = \cos\left(\frac{2\pi}{N}\right)3n = \frac{N}{2}\left[\delta(k-3) + \delta(k+3)\right]$.

13. Find IDFT for the sequence.

$x(k) = \{4,\ 0,\ (1-j),\ 0,\ 1,\ 0,\ (1+j),\ 0\}$

Solution :- $\quad N = 8$

By the eqn of IDFT

$$\text{IDFT}\{x(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, W_N^{-kn} \qquad n = 0, 1, \ldots (N-1)$$

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} x(k)\, W_8^{-kn}$$

$$x(n) = \frac{1}{8}\left\{x(0) + x(1)W_8^{-n} + x(2)W_8^{-2n} + x(3)W_8^{-3n} + x(4)W_8^{-4n} + x(5)W_8^{-5n} + x(6)W_8^{-6n} + x(7)W_8^{-7n}\right\}$$

$$x(n) = \frac{1}{8}\{5 + (1-j)W_8^{-2n} + W_8^{-4n} + (1+j)W_8^{-6n}\}$$

At $n=0$; $x(0) = \frac{1}{8}\{5 + 1 - \cancel{j} + 1 + 1 + \cancel{j}\} = \frac{1}{8}\{8\} = 1$

At $n=1$; $x(1) = \frac{1}{8}\{5 + (1-j)W_8^{-2} + (1)W_8^{-4} + (1+j)W_8^{-6}\}$

$$x(1) = \frac{1}{8}\{5 + (1-j)(j) + (-1) + (1+j)(-j)\}$$

$$x(1) = \frac{1}{8}\{5 + \cancel{j} + j^2 - 1 - \cancel{j} - j^2\} = \frac{1}{8}\{5 + 1 - \cancel{1} + \cancel{1}\} = \frac{1}{8}\{6\}$$

$$x(1) = \frac{3}{4}$$

At $n=2$; $x(2) = \frac{1}{8}\{5 + (1-j)W_8^{-4} + W_8^{-8} + (1+j)W_8^{-12}\}$

$$= \frac{1}{8}\{5 + (1-j)(-1) + 1 + (1+j)(-1)\}$$

$$= \frac{1}{8}\{5 - 1 + j + 1 - 1 - j\} = \frac{1}{8}\{4\} = \frac{1}{2}$$

At $n=3$; $x(3) = \frac{1}{8}\{5 + (1-j)W_8^{-6} + W_8^{-12} + (1+j)W_8^{-18}\}$

$$= \frac{1}{8}\{5 + (1-j)(-j) + (-1) + (1+j)j\}$$

$$= \frac{1}{8}\{5 - \cancel{j} + j^2 - 1 + \cancel{j} + j^2\}$$

$$= \frac{1}{8}\{5 - 1 - 1 - 1\} = \frac{2}{8} = \frac{1}{4}$$

At $n=4$; $x(4) = \frac{1}{8}\{5 + (1-j)W_8^{-8} + W_8^{-16} + (1+j)W_8^{-24}\}$

$$= \frac{1}{8}\{5 + (1-j)1 + 1 + (1+j)1\}$$

$$= \frac{1}{8}\{5 + 1 - j + 1 + 1 + j\}$$

$$= \frac{1}{8}\{8\} = 1$$

At $n=5$; $x(5) = \frac{1}{8}\{5 + (1-j)W_8^{-10} + W_8^{-20} + (1+j)W_8^{-30}\}$

$$= \frac{1}{8}\{5 + (1-j)j + (-1) + (1+j)(-j)\}$$

$$= \frac{1}{8}\{5 + \cancel{j} - j^2 - 1 - \cancel{j} - j^2\} = \frac{1}{8}\{5 + 1 - \cancel{1} + \cancel{1}\}$$

$$= \frac{1}{8}\{6\} = \frac{3}{4}$$

(18) At $n=6$; $x(6) = \frac{1}{8}\{5+(1-j)W_8^{-12} + W_8^{-24} + (1+j)W_8^{-36}\}$

$$= \frac{1}{8}\{5+(1-j)(-1)+1+(1+j)(-1)\}$$

$$= \frac{1}{8}\{5-1+\cancel{j}+\cancel{1}-1-\cancel{j}\} = \frac{1}{8}\{4\} = \frac{1}{2}$$

At $n=7$: $x(7) = \frac{1}{8}\{5+(1-j)W_8^{-14} + W_8^{-28} + (1+j)W_8^{-42}\}$

$$= \frac{1}{8}\{5+(1-j)(-j)+(-1)+(1+j)j\}$$

$$= \frac{1}{8}\{5-j+j^2-1+j+j^2\}$$

$$= \frac{1}{8}\{5-1-1-1\} = \frac{1}{8}\{2\} = -\frac{1}{4}$$

$\therefore$ $x(n) = \{1, 3/4, 1/2, 1/4, 1, 3/4, 1/2, -1/4\}$.

14. Find DFT of the sequence $x(n) = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$

for $N=8$. plot $|X(K)|$ & $\underline{|X(K)}$

Solution :- From the definition of DFT,

$$\text{DFT}\{x(n)\} = \{X(K)\} = \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn}$$

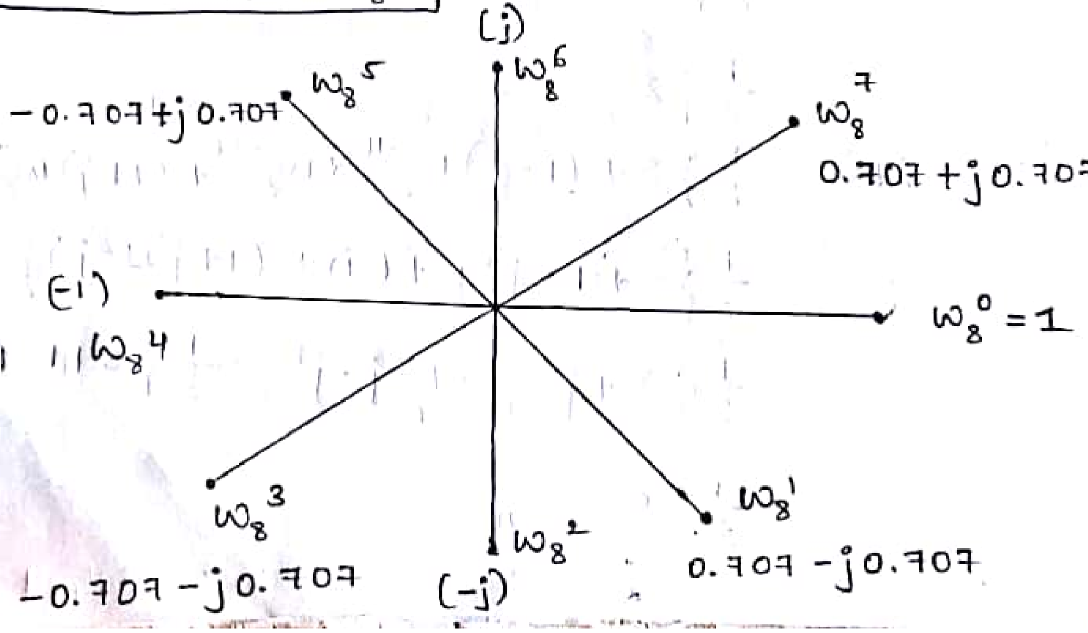
$\underline{N=8}$

$$X(K) = \sum_{n=0}^{7} x(n)\, W_5^{Kn}$$

given $x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$X(K) = 1 + W_8^K + W_8^{2K} + 0 + 0$ where $K = 0, 1, 2, 3, 4$

$$\boxed{X(K) = 1 + W_8^K + W_8^{2K}}$$



At $k=0$; $x(0) = 1+1+1 = 3$

At $k=1$; $x(1) = 1+\omega_8^1+\omega_8^2 = 1+0.707-j0.707-j$

$x(1) = 1.707 - j1.707$

At $k=2$; $x(2) = 1+\omega_8^2+\omega_8^4$

$x(2) = 1+(-j)+(-1) = -j$

At $k=3$; $x(3) = 1+\omega_8^3+\omega_8^6$

$= 1+(-0.707-j0.707)+j = 0.293+j0.293$

At $k=4$; $x(4) = 1+\omega_8^4+\omega_8^8$

$= 1+(-1)+1 = 1$

At $k=5$; $x(5) = 1+\omega_8^5+\omega_8^{10} = 1+\omega_8^5+\omega_8^2$

$= 1+(-0.707+j0.707)+(-j)$

$= 0.293-j0.293$

At $k=6$; $x(6) = 1+\omega_8^6+\omega_8^{12} = 1+\omega_8^6+\omega_8^4$

$= 1+(+j)+(-1) = +j$

At $k=7$; $x(7) = 1+\omega_8^7+\omega_8^{14} = 1+\omega_8^7+\omega_8^6$

$x(7) = 1+(0.707+j0.707)+j = 1.707+j1.707$

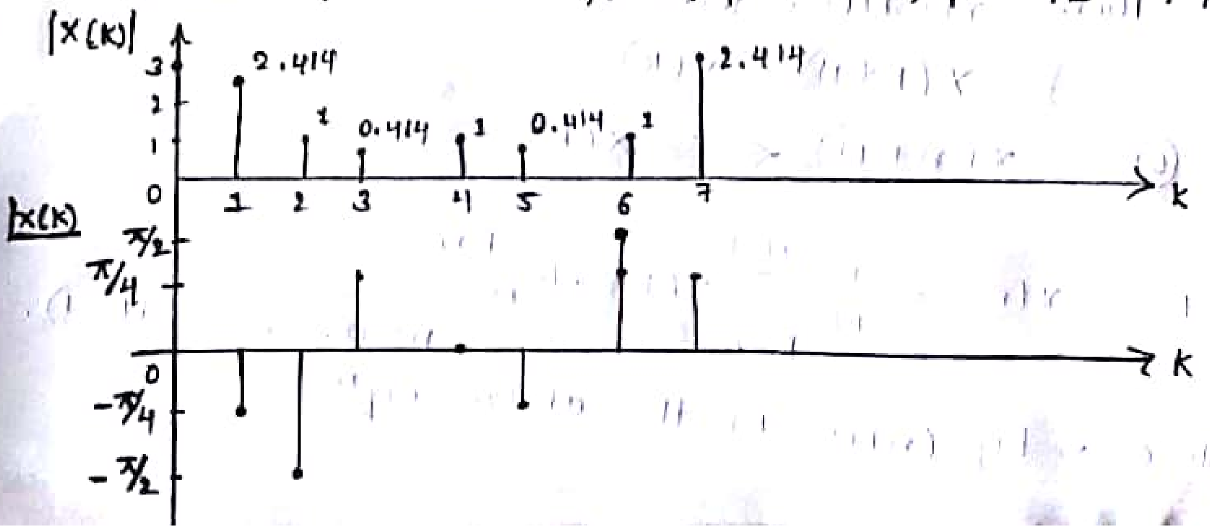
$X(k) = \{3, 1.707-j1.707, -j, 0.293+j0.293, 1, 0.293-j0.293, j, 1.707+j1.707\}$

Magnitude plot :-

$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

phase plot :-

$\angle X(k) = \{0, -\pi/4, -\pi/2, \pi/4, 0, -\pi/4, \pi/2, \pi/4\}$



PROPERTIES OF DFT

1. Linearity :-

Statement : If $x_1(n) \xleftrightarrow[DFT]{N-pt} X_1(K)$

$x_2(n) \longleftrightarrow X_2(K)$

then

$$y(n) = a x_1(n) + b x_2(n) \longleftrightarrow a X_1(K) + b X_2(K) = Y(K)$$

proof :- W.K.T

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

Let $x(n) = a\, x_1(n)$

then $$X_1(K) = \sum_{n=0}^{N-1} a \cdot x_1(n) \cdot W_N^{kn} \quad \text{——— (1)}$$

Let $x(n) = b\, x_2(n)$

then $$X_2(K) = \sum_{n=0}^{N-1} b \cdot x_2(n) \cdot W_N^{kn} \quad \text{——— (2)}$$

Let $y(n) = a\, x_1(n) + b x_2(n)$

then $$Y(K) = \sum_{n=0}^{N-1} a \cdot x_1(n) \cdot W_N^{kn} + b\, x_2(n) \cdot W_N^{kn} \quad \text{——— (3)}$$

Since $eq^n(1) + eq^n(2) = eq^n(3)$

hence the proof.

2. periodicity :-

Statement : If $x(n) \longleftrightarrow X(K)$

then $x(n+N) = x(n)$

& $X(K+N) = X(K)$

(or) $x(n+N) \longleftrightarrow X(K)$

proof :-

a) W.K.T $$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot W_N^{-kn}$$

$n = 0, 1, 2, \ldots (N-1)$.

Replace n by $(n+N)$ in the above eq^n.

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).\, W_N^{-k(n+N)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k).\, W_N^{-kn}.\, W_N^{-kN} \qquad \text{W.K.T } [W_N^{-kN}=1]$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).\, W_N^{-kn}.\, 1 = x(n)$$

Hence proved $\boxed{x(n+N) = x(n)}$

b] W.K.T $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$ where $k = 0, 1, 2 \dots (N-1)$

Replace k by (N+k)

$$X(N+k) = \sum_{n=0}^{N-1} x(n).\, W_N^{(N+k)n}$$

$$X(N+k) = \sum_{n=0}^{N-1} x(n).\, W_N^{kn}.\, W_N^{Nn} \qquad \text{W.K.T } [W_N^{Nn}=1]$$

$$= \sum_{n=0}^{N-1} x(n).\, W_N^{kn}.\, 1 = X(k)$$

$\boxed{X(k+N) = X(k)}$ Hence proof

c] W.K.T $X(k) = \sum_{n=0}^{N-1} x(n).\, W_N^{kn}$ where $k = 0, 1 \dots (N-1)$

Replace $x(n)$ by $x(n+N)$

then

$$DFT\{x(n+N)\} = \sum_{n=0}^{N-1} x(n+N).\, W_N^{kn}$$

put $l = n+N$ $\quad (n = -N + l)$

at $n=0,\ l=N$

$n = N-1,\ l = 2N-1$

$$D.F.T\{x(n+N)\} = \sum_{l=N}^{2N-1} x(l)\ W_N^{k(-N+l)}$$

$$= \sum_{l=N}^{2N-1} x(l).\, W_N^{kl}\ W_N^{-kN} \qquad [W_N^{-Nk} = 1]$$

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$$= \sum_{J=N}^{2N-1} x(J) \cdot W_N^{+KJ} \quad W_N^{-KN}$$

$$= \sum_{J=N}^{2N-1} x(J) \cdot W_N^{+KJ} = \sum_{J=0}^{N-1} x(J) \cdot W_N^{KJ} = X(K)$$

$\therefore$ $\boxed{x(n+N) \longleftrightarrow X(K)}$ hence proved

[since $W_N^{-KN} = 1$ & limits N to 2N-1 can be replaced by 0 to N-1 using periodicity]

3. CIRCULAR TIME SHIFT :-

Circular time shift operation on an N-point sequence $x(n)$ is given by $x(n-m) \bmod N$ or $x((n-m))_N$ or $x(n-m \,.\bmod N)$

statement :- If $x(n) \longleftrightarrow X(k)$

then $\quad x((n-m))_N \longleftrightarrow X(k)\, e^{-j\frac{2\pi}{N}km}$

or $\quad x((n-m))_N \longleftrightarrow X(k)\, W_N^{km}$

IIIly $\quad x((n+m))_N \longleftrightarrow X(k)\, e^{j\frac{2\pi}{N}km}$ or $X(k)\, W_N^{-km}$

Proof :- W.K.T

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

Replace 'n' by (n-m)

$$x((n-m))_N = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-k(n-m)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \{X(k) \cdot W_N^{km}\}\, W_N^{-kn}$$

i.e

$$x((n-m))_N = \text{IDFT}\{X(k)\, W_N^{km}\}$$

or

$$\text{DFT}\{x[(n-m)]_N\} = X(k) \cdot W_N^{km}$$

II Method :-

W.K.T $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

let $x(n) = x((n-m))_N$

then $DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N \cdot W_N^{kn}$

put $n-m = l \rightarrow n = l+m$

at $n=0$; $l = -m$

$n = N-1$; $l = N-1-m$

$$D.F.T\{x((n-m))_N\} = \sum_{l=-m}^{N-1-m} x(l) \cdot W_N^{k(l+m)}$$

$$= \sum_{l=-m}^{N-1-m} x(l)\, W_N^{kl} \cdot W_N^{km}$$

$$= X(k) \cdot W_N^{km}$$

$$\boxed{D.FT\{x((n-m))_N\} = X(k) \cdot W_N^{km}}$$

PROBLEMS :-

1) If $x(n) = \{1, 2, 3, 4\}$ find $y(n)$. given $y(n) = x((n-3))_4$

Solution :- consider $y(n) = x((n-3))_4$

at $n=0$; $y(0) = x((-3))_4 = x(4-3) = x(1) = 2$

at $n=1$; $y(1) = x((-2))_4 = x(4-2) = x(2) = 3$

at $n=2$; $y(2) = x((-1))_4 = x(4-1) = x(3) = 4$

at $n=3$; $y(3) = x(0) = 1$

$\therefore y(n) = \{2, 3, 4, 1\}$.

2) If $x(n) = \{1, 2, 2, 1\}$ find the DFT of

$y(n) = x((n-2))_4$

Solution :- WKT $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

By Linear Transformation Method.

$$X(K) = [1\ 2\ 2\ 1] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 1+2+2+1 = 6 \\ 1-2j-2+j = -1-j \\ 1-2+2-1 = 0 \\ 1+2j-2-j = -1+j \end{bmatrix}$$

$$X(K) = \{6, (-1-j), 0, (-1+j)\}$$

. Given $y(n) = x((n-2))_4$

using circular time shift property

$$y(K) = X(K)\, W_4^{2K} \quad \text{where } K = 0,1,2,3$$

At $K=0$, $y(0) = X(0) \cdot W_4^0 = 6 \cdot 1 = 6$

At $K=1$, $y(1) = X(1)\, W_4^2 = (-1-j)(-1) = 1+j$

At $K=2$, $y(2) = X(2)\, W_4^4 = 0(1) = 0$

At $K=3$, $y(3) = X(3)\, W_4^6 = (-1+j)(-1) = 1-j$

$$\therefore \boxed{y(K) = \{6, 1+j, 0, 1-j\}}$$

4. Time Reversal :-

Statement :- If $x(n) \longleftrightarrow X(K)$

then $x((-n))_N = x(N-n) \longleftrightarrow X((-K))_N$ or $X(N-K)$

proof :- From the Definition of DFT, we have

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

Replace $x(n)$ by $x(N-n)$

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) \cdot W_N^{kn}$$

put $m = N-n$ then $n = N-m$

at $n=0$, $m=N$

$n=N-1$, $m=N-(N-1)=1$

$$DFT\{x(N-n)\} = \sum_{m=N}^{1} x(m) \cdot W_N^{k(N-m)}$$

$$= \sum_{m=0}^{(N-1)} x(m) \cdot W_N^{-km} \; W_N^{kN}$$

W.K.T $W_N^{kN} = 1$

$$= \sum_{m=0}^{(N-1)} x(m) \cdot W_N^{-km} = X((-k))_N$$

(Or)

$$= \sum_{m=0}^{(N-1)} x(m) \; W_N^{-km} \cdot W_N^{Nm}$$

$$= \sum_{m=0}^{(N-1)} x(m) \cdot W_N^{(N-k)m}$$

$$= X(N-k).$$

-: problems :-

1) Given $x(n)=\{5,2,-3,7\}$ find $y(n)=x((-n))_4$

Solution :- consider $y(n)=x((-n))_4 = x(4-n)$.

at $n=0$; $y(0)=x(4)=x(0)=5$

at $n=1$; $y(1)=x(4-1)=x(3)=7$

at $n=2$; $y(2)=x(4-2)=x(2)=-3$

at $n=3$; $y(3)=x(4-3)=x(1)=2$

$y(n)=\{5,7,-3,2\}$

2) Given $x(n)=\{1,2,3,4\}$ with 4-pt DFT $X(k)=\{10, -2+2j, -2, -2-2j\}$. find the 4pt DFT of the sequence $y(n)=\{1,4,3,2\}$.

Solution :- By observation ; $y(n)=x((-n))_4$

$y(n) = x(4-n)$

(26) Taking DFT on both sides

$$y(k) = X(4-k)$$

at $k=0$; $y(0) = X(4-0) = X(4) = X(0) = 10$

at $k=1$; $y(1) = X(4-1) = X(3) = -2-2j$

at $k=2$; $y(2) = X(4-2) = X(2) = -2$

at $k=3$; $y(3) = X(4-3) = X(1) = -2+2j$

$\therefore$ $y(k) = \{10, -2-2j, -2, -2+2j\}$

5. Circular frequency shift :

Statement :- If $x(n) \longleftrightarrow X(k)$

then $x(n) \cdot e^{j\frac{2\pi}{N}mn} \longleftrightarrow X((k-m))_N$

or $x(n) \cdot (W_N)^{-mn} \longleftrightarrow X((k-m))_N$

Similarly, $x(n) \cdot e^{-j\frac{2\pi}{N}mn} \longleftrightarrow X((k+m))_N$

or $x(n) \cdot W_N^{mn} \longleftrightarrow X((k+m))_N$

proof :- I Method

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$DFT\{x(n) \cdot e^{j\frac{2\pi}{N}mn}\} = \sum_{n=0}^{N-1} x(n) \cdot e^{j\frac{2\pi}{N}mn} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}(k-m)n}$$

$$= X((k-m))_N$$

$$\therefore \boxed{DFT\{x(n) \cdot e^{j\frac{2\pi}{N}mn}\} = X((k-m))_N}$$

Hence proof

Multiplication of exponential in time domain is shifting in frequency domain.

II Method :- consider IDFT eq^n ;

$$IDFT\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$ where $n = 0, 1, \ldots (N-1)$.

put $l = k - m$ then $k = l + m$

at $k = 0$; $l = -m$

$k = N-1$; $l = N-1-m$

$$x(n) = \frac{1}{N}\sum_{l=-m}^{N-1-m} X(l+m) \cdot W_N^{-(l+m)n}$$

$$x(n) = \frac{1}{N}\sum_{l=0}^{N-1}\left\{X(l) \cdot W_N^{-ln}\right\} \cdot W_N^{-mn}$$

Replacing 'l' by k , using periodicity property

$$x(n) = \left\{\frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}\right\} \cdot W_N^{-mn}$$

$$\therefore \boxed{IDFT\{X((k-m))_N\} = x(n) \cdot W_N^{-mn}}$$

hence proof

problems :-

1) Find N-point DFT of $x_1(n) = x(n) \cdot \cos\left(\frac{2\pi}{N}k_0 n\right)$

Solution :- given $x_1(n) = x(n) \cdot \cos\left(\frac{2\pi}{N}k_0 n\right)$

$$x_1(n) = x(n) \cdot \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right]$$

W.K.T

$$x(n) \longleftrightarrow X(k)$$

using circular frequency shift property

$$x(n) \cdot e^{j\frac{2\pi}{N}k_0 n} \longleftrightarrow X((k-k_0))_N$$

$$\& \; x(n) \cdot e^{-j\frac{2\pi}{N}k_0 n} \longleftrightarrow X((k+k_0))_N$$

$$\boxed{X_1(k) = \frac{1}{2}\left[X((k-k_0))_N + X((k+k_0))_N\right]}$$

2. Find the N-pt DFT of $x_2(n) = x(n) \cdot \sin\left(\frac{2\pi}{N} k_0 n\right)$

Solution :- $x_2(n) = x(n) \cdot \sin\left(\frac{2\pi}{N} k_0 n\right)$

$$x_2(n) = x(n) \left[\frac{e^{j\frac{2\pi}{N}k_0 n} - e^{-j\frac{2\pi}{N}k_0 n}}{2j} \right]$$

By circular frequency shift property & taking DFT:

$$\boxed{X_2(k) = \frac{1}{2j}\left[X((k-k_0))_N - X((k+k_0))_N \right]}$$

Examples :-

1) $x(n) \cdot \cos\left(\frac{4\pi}{N} \cdot n\right) \longleftrightarrow \frac{1}{2}\left[X((k-2))_N + X((k+2))_N \right]_N$

2) $x(n) \cdot \cos\left(\frac{10\pi}{N} n\right) \longleftrightarrow \frac{1}{2}\left[X((k-5))_N + X((k+5))_N \right] = x(n) \cdot \cos\left(\frac{2\pi}{N} 5n\right)$

3) $x(n) \cdot \sin\left(\frac{8\pi}{N} \cdot n\right) = x(n) \sin\left(\frac{2\pi}{N} \cdot 4n\right) \longleftrightarrow \frac{1}{2j}\left[X((k-4))_N - X((k+4))_N \right]$

6. Circular convolution ~~of~~ or Time domine convolution or Frequency domain Multiplication.

Statement :- if $x(n) \longleftrightarrow X(k)$ & $h(n) \longleftrightarrow H(k)$

then $y(n) = x(n) \, \textcircled{N} \, h(n) \longleftrightarrow X(k) \cdot H(k) = Y(k)$

proof :-

<u>I Method</u> :-

Consider two sequences $x(n)$ & $h(n)$ of length N, then circular convolution is defined as :-

$$y(n) = x(n) \, \textcircled{N} \, h(n)$$

$$\boxed{y(n) = \sum_{m=0}^{N-1} x(m) \cdot h((n-m))_N}$$

$$\text{DFT}\{y(n)\} = Y(k) = \sum_{n=0}^{N-1} y(n) \cdot W_N^{kn}$$

$$Y(k) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(m) \cdot h((n-m))_N \cdot W_N^{kn}$$

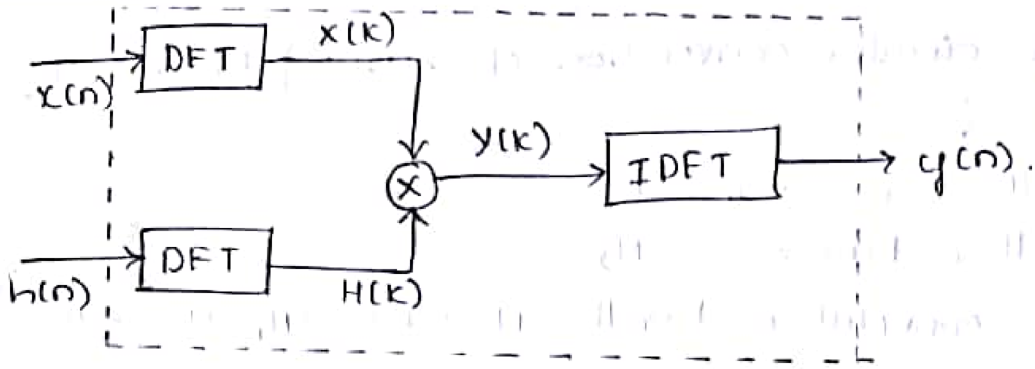
$$Y(k) = \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} h((n-m))_N \cdot W_N^{kn} \Rightarrow H(k) \cdot W_N^{km}$$

By using time shifting property

$$y(k) = \sum_{m=0}^{N-1} x(m) \cdot W_N^{km} \cdot H(k)$$

$$\boxed{y(k) = X(k) \cdot H(k)}$$ Hence proof

II Method :-



consider two sequence, $x(n)$ & $h(n)$

then

$$DFT\{x(n)\} = X(k) = \sum_{m=0}^{N-1} x(m) \cdot W_N^{km}$$

$$DFT\{h(n)\} = H(k) = \sum_{l=0}^{N-1} h(l)\, W_N^{kl}$$

consider $X(k) \cdot H(k) = Y(k)$

$$IDFT\{y(k)\} = y(n) = \frac{1}{N} \sum_{k=0}^{N-1} y(k)\, W_N^{-kn}$$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot H(k) \cdot W_N^{-kn}$$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x(m) \cdot W_N^{km} \right\} \cdot \left\{ \sum_{l=0}^{N-1} h(l) \cdot W_N^{kl} \right\} W_N^{-kn}$$

consider.

$$\sum_{k=0}^{N-1} W_N^{k(l+m-n)} = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-l-m)}$$

put $n-l-m = PN$

then.

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kPN} = \sum_{k=0}^{N-1} 1 = \underline{\underline{N}}$$

& $l = n-m = PN$

or $l = ((n-m))_N$

$$\therefore y(n) = \frac{1}{N}\sum_{m=0}^{N-1} x(m).\, h((n-m))_N \cdot N$$

$$\boxed{y(n) = \sum_{m=0}^{N-1} x(m)\, .\, h((n-m))_N = x(n) \, N \, h(n)}$$

problems :-

1) compute the circular convolution of $x(n) = \{1, 2, 3, 4\}$ & $h(n) = \{1, 2, 2\}$.

Solution :- length of $x(n) = 4 = N_1$

length of $h(n) = 3 = N_2$

Circular convolution length, $N = \max(N_1, N_2) = 4$

Since $N_2 = 3$, we add one zero, $h(n) = \{1, 2, 2, 0\}$

I Method :- Time Domain approach or concentric circle method.

$$y(n) = x(m) \, N \, h(n) = \sum_{m=0}^{N-1} x(m)\, .\, h((n-m))_N$$

$n = 0, 1, 2 \dots (N-1)$.

$$y(n) = x(0)\, h((n))_4 + x(1)\, h((n-1))_4 + x(2)\, h((n-2))_4 + x(3).\, h((n-3))_4$$

at $n=0$

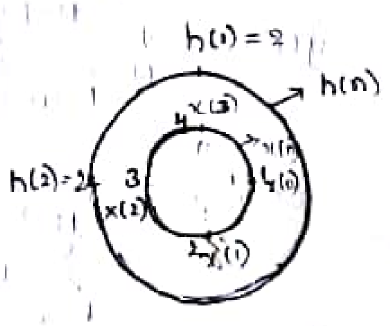
$y(0) = x(0)\, h(0) + x(1)\, h((-1))_4 + x(2)\, h((-2))_4 + x(3)\, h((-3))_4$

$y(0) = x(0)\, h(0) + x(1)\, h(3) + x(2)\, h(2) + x(3)\, h(1)$

$y(0) = (1)(1) + (2)(0) + (3)(2) + (4)(2)$

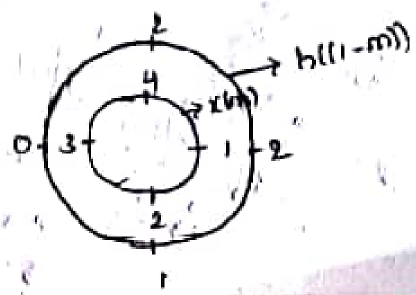
$y(0) = 1 + 0 + 6 + 8$

$\boxed{y(0) = 15}$



At $n = 1$:

$$y(1) = \sum_{m=0}^{3} x(m)\, .\, h((1-m))_4$$



$y(1) = 2 + 2 + 0 + 8$

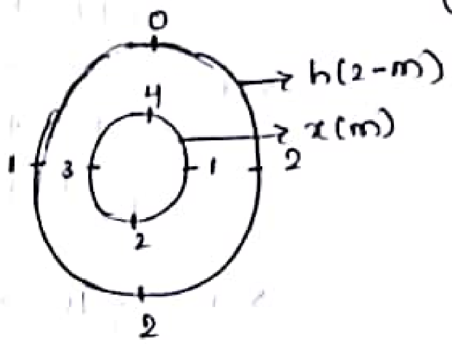
$\boxed{y(1) = 12}$

at $n=2$;

$$y(2) = \sum_{m=0}^{3} x(m) \cdot h((2-m))_4$$

$$= 2+4+3+0$$

$$\boxed{y(2) = 9}$$

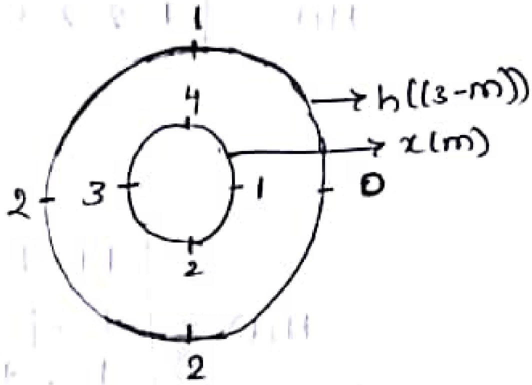


at $n=3$;

$$y(3) = \sum_{m=0}^{3} x(m) \cdot h((3-m))_4$$

$$= 0+4+6+4$$

$$\boxed{y(3) = 14}$$



$$\boxed{y(n) = \{15, 12, 9, 14\}}$$

Verification:-

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+8+6+0 = 15 \\ 2+2+8+0 = 12 \\ 3+4+2+0 = 9 \\ 4+6+4+0 = 14 \end{bmatrix}$$

$$\therefore \; y(n) = [15 \;\; 12 \;\; 9 \;\; 14]$$

II Method :- using DFT & IDFT eq^n or Transformed domain approach or Stockhalm's method.

w.k.T $\quad y(n) = x(n) \; \textcircled{N} \; h(n)$

applying DFT ; $\quad Y(K) = X(K) \cdot H(K)$

$$X(K) = [1 \;\; 2 \;\; 3 \;\; 4] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

(32)

$$X(K) = \begin{bmatrix} 1+2+3+4 = 10 \\ 1-2j-3+4j = -2+2j \\ 1-2+3-4 = -2 \\ 1+2j-3-4j = -2-2j \end{bmatrix}$$

$$X(K) = \{10, -2+2j, -2, -2-2j\}$$

$$H(K) = [1\ 2\ 2\ 0] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$H(K) = \begin{bmatrix} 1+2+2+0 = 5 \\ 1-2j-2+0 = -1-2j \\ 1-2+2+0 = 1 \\ 1+2j-2-0 = -1+2j \end{bmatrix}$$

$$H(K) = \{5, -1-2j, 1, -1+2j\}$$

W.K.T $\quad y(K) = X(K) \cdot H(K)$

$$y(K) = \{10, -2+2j, -2, -2-2j\}\{5, -1-2j, 1, -1+2j\}$$

$$y(K) = \{50, 6+2j, -2, 6-2j\}$$

$$y(n) = IDFT\{y(K)\} = \frac{1}{N}\sum_{K=0}^{N-1} x(K)\, W_N^{-Kn}$$

$$y(n) = IDFT\{y(K)\} = \frac{1}{4}[50\ \ 6+2j\ \ -2\ \ 6-2j] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 50+6+\cancel{2j}-2+6-\cancel{2j} \\ 50+\cancel{6j}-\cancel{2}+\cancel{2}-\cancel{6j}-2 \\ 50-6-\cancel{2j}-2-6+\cancel{2j} \\ 50-\cancel{6j}+2+2+\cancel{6j}+2 \end{bmatrix}$$

$$(-2+2j)(-1-2j)$$
$$= 2+4j-2j+4$$
$$= 6+2j$$

$$(-2-2j)(-1+2j)$$
$$= 2+2j-4j+4$$
$$= 6-2j$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 60 \\ 48 \\ 36 \\ 56 \end{bmatrix}$$

$$\boxed{y(n) = \{15 \quad 12 \quad 9 \quad 14\}}$$

III Method :- using DFT & IDFT eqn (or) Transform domine approach (or) Stockholm's method.

$$y(n) = x(n) \, (N) \, h(n)$$

$$Y(K) = X(K) \, H(K) \longrightarrow \text{By Taking DFT}$$

N=4 since $x(n) = \{1, 2, 3, 4\}$

Taking DFT ; $X(K) = 1 + 2W_4^k + 3W_4^{2k} + 4W_4^{3k}$

Illy. $h(n) = \{1, 2, 2\}$

Taking DFT ; $H(K) = 1 + 2W_4^k + 2W_4^{2k}$

$$Y(K) = X(K) \cdot H(K)$$

$$Y(K) = \{1 + 2W_4^k + 3W_4^{2k} + 4W_4^{3k}\}\{1 + 2W_4^k + 2W_4^{2k}\}$$

$$Y(K) = 1 + 2W_4^k + 3W_4^{2k} + 4W_4^{3k}$$
$$+ 2W_4^k + 4W_4^{2k} + 6W_4^{3k} + 8W_4^{4k}$$
$$+ 2W_4^{2k} + 4W_4^{3k} + 6W_4^{4k} + 8W_4^{5k}$$

$$Y(K) = 1 + 4W_4^k + 9W_4^{2k} + 14W_4^{3k} + 14W_4^{4k} + 8W_4^{5k}$$

By periodicity property : $W_4^{4k} = W_4^0 = 1$

$$W_4^{5k} = W_4^{(4+1)k} = W_4^k$$

$$\therefore \quad Y(K) = 15 + 12W_4^k + 9W_4^{2k} + 14W_4^{3k}$$

Taking IDFT :

$$y(n) = 15\delta(n) + 12\delta(n-1) + 9\delta(n-2) + 14\delta(n-3)$$

$$\therefore \boxed{y(n) = \{15, 12, 9, 14\}}$$

(34) 7. Multiplication of two sequences (or) Modulation property (or) Convolution in frequency domain. 6.

Statement :- If $x_1(n) \longleftrightarrow X_1(K)$ & $x_2(n) \longleftrightarrow X_2(K)$

then $$x_1(n) \cdot x_2(n) \longleftrightarrow \frac{1}{N}\left[X_1(k) \; (N) \; X_2(k)\right]$$

Proof :- consider two sequences $x_1(n)$ & $x_2(n)$ of length N with DFT $X_1(K)$ & $X_2(k)$

$$IDFT\{X_1(K)\} = x_1(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_1(l) \cdot W_N^{-ln}$$

$$IDFT\{X_2(k)\} = x_2(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_2(m) \cdot W_N^{-mn}$$

let $y(n) = x_1(n) \cdot x_2(n)$

applying DFT :

$$y(K) = \sum_{n=0}^{N-1} y(n) \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x_1(n) \cdot x_2(n) \; W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N}\left\{\sum_{l=0}^{N-1} X_1(l) \cdot W_N^{-ln}\right\} \cdot \left\{\frac{1}{N}\sum_{m=0}^{N-1} X_2(m) \cdot W_N^{-mn}\right\} W_N^{kn}.$$

$$y(K) = \frac{1}{N^2}\sum_{n=0}^{N-1}\left[\left\{\sum_{l=0}^{N-1} X_1(l) \cdot W_N^{-ln}\right\}\right]\left\{\sum_{m=0}^{N-1} X_2(m) \cdot W_N^{-mn}\right\} W_N^{kn}$$

consider

$$y(K) = \frac{1}{N^2}\sum_{l=0}^{N-1} X_1(l) \sum_{m=0}^{N-1} X_2(m) \cdot \sum_{n=0}^{N-1} W_N^{(k-l-m)n}$$

consider

$$\sum_{n=0}^{N-1} W_N^{(k-l-m)n} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l-m)n}$$

if $k-l-m = PN$

$m = k-l-PN = ((k-l))_N$

or $l = k-m-PN = ((k-m))_N$

$= \;\; \cdots$

then $\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}\cdot pN} = \sum_{n=0}^{N-1} 1 = \underline{N}$

$\therefore \; y(k) = \frac{1}{N^2} \cdot N \cdot \sum_{l=0}^{N-1} x_1(l) \cdot x_2((k-l))_N$

$y(k) = \frac{1}{N} \sum_{l=0}^{N-1} x_1(l) \cdot x_2((k-l))_N$

or $y(k) = \frac{1}{N}\left[x_1(k) \; (N) \; x_2(k)\right]$

8. Symmetry property of a complex valued sequence.

Statement :- if $x(n) \longleftrightarrow x(k)$

then $x^*(n) \longleftrightarrow x^*(N-k) = x^*((-k))_N$

or

$x^*((-n))_N = x^*(N-n) \longleftrightarrow x^*(k)$

proof :- (a) w.k.T $DFT\{x(n)\} = x(k) = \sum_{n=0}^{N-1} x(n) \cdot w_N^{kn}$

Taking complex conjugate on both sides.

$x^*(k) = \sum_{n=0}^{N-1} x^*(n) \cdot w_N^{-kn}$

Replace k by N-k

$x^*(N-k) = \sum_{n=0}^{N-1} x^*(n) \cdot w_N^{-(N-k)n}$

$x^*(N-k) = \sum_{n=0}^{N-1} \left\{ x^*(n) \cdot w_N^{kn} \right\} w_N^{-Nn}$ w.k.T $w_N^{-Nn} = 1$

$x^*(N-k) = DFT\{x^*(n)\}$

or $\boxed{x^*(n) \longleftrightarrow x^*(N-k)}$

(b) To proove that : $x^*((-n))_N = x^*(N-n) \longleftrightarrow x^*(k)$

proof :- w.k.T $x(k) = \sum_{n=0}^{N-1} x(n) \cdot w_N^{kn}$

Taking complex conjugate on both sides.

$$x^*(k) = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{-kn}$$

Replace 'n' by (N-n).

$$x^*(k) = \sum_{n=0}^{N-n} x^*(N-n) \cdot W_N^{-k(N-n)}$$

$$x^*(k) = \sum_{n=0}^{N-1} x^*(N-n) \cdot W_N^{kn} \cdot W_N^{-Nn}$$

$$x^*(k) = \sum_{n=0}^{N-1} x^*(N-n) \cdot W_N^{kn} \qquad \left[\because W_N^{-Nn} = 1\right]$$

$$x^*(k) = DFT\{x^*(N-n)\} = DFT\{x^*((-n))_N\}$$

$$\therefore \boxed{x^*((-n))_N \longleftrightarrow x^*(k)}$$

9. Symmetry property of Real valued Sequence :-

Statement :- if $x(n)$ is real valued sequence.

then $x(k) = x^*(N-k)$

proof :-

W.K.T

$$x(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

Taking complex conjugate on both sides.

$$x^*(k) = \sum_{n=0}^{N-1} x^*(n) \cdot W_N^{-kn}$$

Since $x(n)$ is real, $x^*(n) = x(n)$

$$x^*(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{-kn}$$

(or)

$$x^*(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{-kn} \cdot W_N^{Nn}$$

$$x^*(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{(N-k)n}$$

$$x^*(k) = x(N-k)$$

(or) $\boxed{X(K) = X^*(N-K)}$

problems :-

1) The first five points of the 8-point DFT of a real valued sequence are {0.25, (0.125-j0.3018), 0, (0.125-j0.0518), 0}. Determine the remaining 3 points.

Solution :- Given $N=8$

$X(0) = 0.25$

$X(1) = 0.125-j0.3018$

$X(2) = 0$

$X(3) = 0.125-j0.0518$

$X(4) = 0$

Since $x(n)$ is real valued,

$$X^*(K) = X(N-K)$$

or

$$X(K) = X^*(N-K)$$

At $k=5$; $X(5) = X^*(8-5) = X^*(3) = 0.125+j0.0518$

At $k=6$; $X(6) = X^*(8-6) = X^*(2) = 0$

At $k=7$; $X(7) = X^*(8-7) = X^*(1) = 0.125+j0.3018$

$\therefore$ $X(K) = \{0.25, (0.125-j0.3018), 0, (0.125-j0.0518), 0, (0.125+j0.0518), 0, (0.125+j0.3018)\}$

10. Circular correlation :-

Statement: For a complex valued sequences, $x(n)$ & $y(n)$.

if $x(n) \longleftrightarrow X(K)$ & $y(n) \longleftrightarrow Y(K)$

then $\tilde{r}_{xy}(l) \longleftrightarrow \tilde{R}_{xy}(K) = X(K) \cdot Y^*(K)$.

where $\tilde{r}_{xy}(l)$ is the circular cross-correlation sequence defined by

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N$$

38 proof : we can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$

i.e. $\tilde{r}_{xy}(l) = x(l) \, (N) \, y^*(-l)$

Taking DFT on both sides

$$\boxed{\tilde{R}_{xy}(K) = X(K) \cdot Y^*(-K)}$$

using circular convolution & complex conjugate property. if $x(n) = y(n)$,

$$\boxed{\tilde{r}_{xx}(l) \longleftrightarrow \tilde{R}_{xx}(K) = |X(K)|^2}$$

11. **Parseval's Theorem :-**

Statement : for a complex valued sequences $x(n)$ & $y(n)$

if $x(n) \longleftrightarrow X(K)$ & $y(n) \longleftrightarrow Y(K)$

then $$\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot Y^*(K)$$

proof :- cross - correlation of a sequence is defined as

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N$$

At $l = 0$:

$$\tilde{r}_{xy}(0) = \sum_{n=0}^{N-1} x(n) \cdot y^*(n) \quad \text{——— (a)}$$

W.K.T

$$\text{IDFT}\{\tilde{R}_{xy}(K)\} = \tilde{r}_{xy}(l)$$

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{K=0}^{N-1} \tilde{R}_{xy}(K) \cdot e^{j\frac{2\pi}{N}Kl}$$

(or) $$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot Y^*(K) \cdot e^{j\frac{2\pi}{N}Kl}$$

At $l=0$

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot Y^*(k) \quad \text{——— (b)}$$

Equating the Eqn (a) & (b);

$$\boxed{\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot Y^*(k)}$$

if $x(n) = y(n)$. then

$$\boxed{E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \text{ Joules}}$$

which is the Energy in the finite duration sequence $x(n)$ interms of the frequency components $X(k)$.

problems :-

1. Determine N-pt circular correlation of $x(n) = \cos \frac{2\pi}{N} n$ & $y(n) = \sin \frac{2\pi}{N} n$.

Solution :- given: $x(n) = \cos \frac{2\pi}{N} n$

using circular frequency shift & Taking DFT ;

$$X(k) = \frac{N}{2} \{ \delta(k-1) + \delta(k+1) \}$$

$$y(n) = \sin\left(\frac{2\pi}{N}\right) n$$

Taking N-pt-DFT ; $Y(k) = \frac{N}{2j} [\delta(k-1) - \delta(k+1)]$

From the definition of circular correlation;

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N$$

Or

$$\tilde{r}_{xy}(l) = x(l) \text{ (N) } y^*(-l)$$

Taking DFT on both sides.

$$\tilde{R}_{xy}(k) = X(k) \cdot Y^*(k)$$

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$$\tilde{R}_{xy}(k) = \left\{\frac{N}{2}\left[\delta(k-1)+\delta(k+1)\right]\right\}\left\{\frac{N}{2j}\left[\delta(k-1)-\delta(k+1)\right]\right\}^*$$

$$\tilde{R}_{xy}(k) = \frac{-N^2}{4j}\left[\delta(k-1)-\delta(k+1)\right]$$

$$\tilde{R}_{xy}(k) = \frac{-N}{2}\cdot\frac{N}{2j}\left[\delta(k-1)-\delta(k+1)\right]$$

Taking IDFT :

$$\tilde{r}_{xy}(l) = -\frac{N}{2}\sin\left(\frac{2\pi}{N}\right)n$$

2. find the circular auto correlation of $x(n) = \cos\left(\frac{2\pi}{N}\right)n$.

Solution :- given $x(n) = \cos\left(\frac{2\pi}{N}n\right)$

Taking DFT : $X(k) = \frac{N}{2}\left\{\delta(k-1)+\delta(k+1)\right\}$

w.k.T $\tilde{R}_{xy}(k) = X(k) \cdot Y^*(k)$

Since $x(n) = y(n)$

$$\tilde{R}_{xx}(k) = X(k)\cdot X^*(k)$$

$$\tilde{R}_{xx}(k) = \left[\frac{N}{2}\left\{\delta(k-1)+\delta(k+1)\right\}\right]\left[\frac{N}{2}\left\{\delta(k-1)+\delta(k+1)\right\}\right]$$

$$\tilde{R}_{xx}(k) = \frac{N^2}{4}\left[\delta(k-1)+\delta(k+1)\right]$$

$$\tilde{R}_{xx}(k) = \frac{N}{2}\cdot\frac{N}{2}\left[\delta(k-1)+\delta(k+1)\right]$$

Taking IDFT

$$\boxed{\tilde{r}_{xx}(l) = \frac{N}{2}\cos\left(\frac{2\pi}{N}\right)n}$$

3. Find the circular auto correlation of the sequence $x(n) = \{1, 2, 3, 4\}$

Solution :

$x(n) = \{1 . 2 . 3 . 4\}$ Here $N=4$

$$X(K) = [1\ 2\ 3\ 4]\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \{10, -2+2j, -2, -2-2j\}$$

Auto correlation of the sequence $x(n)$ is

$$\tilde{r}_{xx}(l) \longleftrightarrow \tilde{R}_{xx}(K) = X(K).X^*(K) = |X(K)|^2$$

$$X^*(K) = \{10, -2-2j, -2, -2+2j\}$$

$$\tilde{R}_{xx}(K) = \{100, 8, 4, 8\}$$

$$\tilde{r}_{xx}(l) = \text{IDFT}\{\tilde{R}_{xx}(K)\}$$

$$\tilde{r}_{xx}(l) = \frac{1}{4}[100\ 8\ 4\ 8]\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\tilde{r}_{xx}(l) = [25\ 2\ 1\ 2]\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\tilde{r}_{xx}(l) = \begin{bmatrix} 25+2+1+2 \\ 25+2j-1-2j \\ 25-2+1-2 \\ 25-2j-1+2j \end{bmatrix} = \{30, 24, 22, 24\}$$

4) Find the Circular correlation given

$$x(n) = \{1, 2, 3, 4\} \ \& \ y(n) = \{1, 2, 2, 0\}$$

Solution: $x(n) = \{1, 2, 3, 4\}$

Taking DFT; $X(K) = [1\ 2\ 3\ 4]\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

(Q2) $x(k) = \begin{bmatrix} 1+2+3+4 = 10 \\ 1-2j-3+4j = -2+2j \\ 1-2+3-4 = -2 \\ 1+2j-3-4j = -2-2j \end{bmatrix}$

$\therefore \; x(k) = \{10, -2+2j, -2, -2-2j\}$

$y(n) = \{1, 2, 2, 0\}$

Taking DFT: $y(k) = [1 \; 2 \; 2 \; 0] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

$y(k) = \begin{bmatrix} 1+2+2+0 = 5 \\ 1-2j-2+0 = -1-2j \\ 1-2+2-0 = 1 \\ 1+2j-2-0 = -1+2j \end{bmatrix}$

$y(k) = \{5, -1-2j, 1, -1+2j\}$

Circular correlation: $\tilde{R}_{xy}(k) = x(k) \cdot y^{*}(k)$

$y^{*}(k) = \{5, -1+2j, 1, -1-2j\}$

$\tilde{R}_{xy}(k) = \{10, -2+2j \;\; -2 \;\; -2-2j\}\{5 \;\; -1+2j \;\; 1 \;\; -1-2j\}$

$\tilde{R}_{xy}(k) = \{50 \;\; -2-6j \;\; -2 \;\; -2+6j\}$

$\tilde{r}_{xy}(l) = \text{IDFT}\{\tilde{R}_{xy}(k)\}$

$\tilde{r}_{xy}(l) = \frac{1}{4}[50 \;\; -2-6j \;\; -2 \;\; -2+6j] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$

$\tilde{r}_{xy}(l) = \begin{bmatrix} 50-2-\cancel{6j}-2-2+\cancel{6j} = 44 \\ 50-\cancel{2j}+6+2+\cancel{2j}+6 = 64 \\ 50+2+\cancel{6j}-2+2-\cancel{6j} = 52 \\ 50+\cancel{2j}-6+2-\cancel{2j}-6 = 40 \end{bmatrix}$

$$\tilde{r}_{xy}(l) = \{11, 16, 13, 10\}$$

6) Given $x(n) = \{1, 2, 3, 4\}$ find the energy and hence verify parseval's theorem.

Solution:

$$x(n) = \{1, 2, 3, 4\}$$

apply DFT:

$$X(K) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 1+2+3+4 = 10 \\ 1-2j-3+4j = -2+2j \\ 1-2+3-4 = -2 \\ 1+2j-3-4j = -2-2j \end{bmatrix}$$

$$X(K) = \{10, -2+2j, -2, -2-2j\}$$

From parseval's theorem WKT

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

$$\text{Energy}, E = \sum_{n=0}^{N-1} |x(n)|^2$$

$$E = \sum_{n=0}^{3} |x(n)|^2 = \{x(0)\}^2 + \{x(1)\}^2 + \{x(2)\}^2 + \{x(3)\}^2$$

$$E = 1+4+9+16$$

$$\boxed{E = 30\,J}$$

$$\sum_{K=0}^{N-1} |X(K)|^2 = N \cdot \sum_{n=0}^{N-1} |x(n)|^2$$

or

$$E = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

$$E = \frac{1}{4}\left[(10)^2 + (\sqrt{8})^2 + 2^2 + (\sqrt{8})^2\right] = \frac{1}{4}\left[100+8+4+8\right]$$

$$E = \frac{1}{4}[120] = 30\,J$$

$$\therefore \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Hence parseval's Theorem is verified.

6) Find the energy of N- point sequence

$$x(n) = \cos \frac{2\pi}{N} k_0 n$$

Solution :-

$$x(n) = \cos \frac{2\pi}{N} k_0 n = \frac{1}{2}\left[e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}\right]$$

w.k.T

$$DFT\{x(n)\} = \frac{N}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]$$

From parseval's theorem w.k.T

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left| DFT\left\{\cos \frac{2\pi}{N} k_0 n\right\}\right|^2$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left\{\frac{N}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]\right\}^2$$

$$= \frac{1}{\cancel{N}} \cdot \frac{N^{\cancel{2}}}{4} \sum_{k=0}^{N-1} \left[\delta(k-k_0) + \delta(k+k_0)\right]$$

$$E = \frac{N}{4} \sum_{k=0}^{N-1} \left[\delta(k-k_0) + \delta(k+k_0)\right] \text{ Joules}$$

II Method :-

$$E = \sum_{n=0}^{N-1} |x(n)|^2$$

$$E = \sum_{n=0}^{N-1} \left|\cos \frac{2\pi}{N} k_0 n\right|^2$$

$$= \sum_{n=0}^{N-1} \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right]^2$$

$$= \sum_{n=0}^{N-1} \frac{1}{4}\left[e^{j\frac{4\pi}{N}k_0 n} + e^{-j\frac{4\pi}{N}k_0 n} + 2\right]$$

$$E = 1 + \frac{1}{4} \sum_{n=1}^{N-1} \left[e^{j\frac{4\pi}{N}k_0 n} + e^{-j\frac{4\pi}{N}k_0 n} + 2\right] \text{ Joules.}$$

7) Let $x(n) = \{1, 2, 3, 4\}$ with 4-pt DFT of $x(K)$ find $\sum_{K=0}^{3} |x(K)|^2$.

Solution :- From parseval's theorm:

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

$$\sum_{K=0}^{N-1} |X(K)|^2 = N \cdot \sum_{n=0}^{N-1} |x(n)|^2 . \quad \text{here } \underline{N=4}$$

$$\sum_{K=0}^{3} |X(K)|^2 = 4 \{ |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2 \}$$

$$\sum_{K=0}^{3} |X(K)|^2 = 4 \{ 1^2 + 2^2 + 3^2 + 4^2 \} = 4 \{ 1 + 4 + 9 + 16 \}$$

$$\sum_{K=0}^{3} |X(K)|^2 = 4(30) = 120 \text{ joules.}$$

12. Circular Symmetries of a sequence :-

We know that N-pt DFT of a finite duration sequence $x(n)$ of length $L \leq N$ is equivalent to N-pt DFT of a periodic sequence $x_p(n)$ of period N, which is obtained by periodically extending $x(n)$.

$$\text{i.e., } x_p(n) = \sum_{l=-\infty}^{+\infty} x(n - l_N)$$

If we shift $x_p(n)$ by 'm' units to the right then we obtain periodic sequence

$$x_p'(n) = x_p(n-m) = \sum_{l=-\infty}^{+\infty} x(n-m-l_N)$$

then the finite duration sequence

$$x'(n) = \begin{cases} x_p'(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

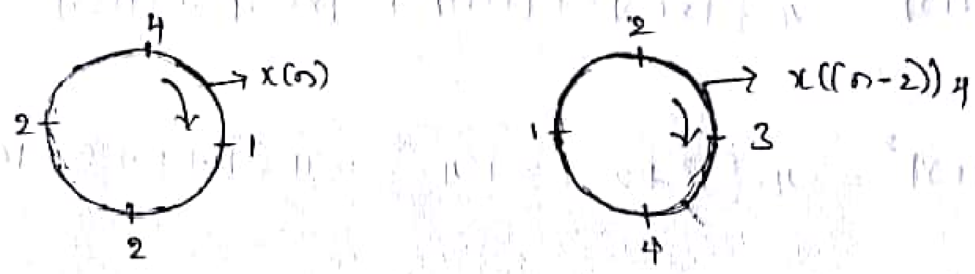
46 where $x'(n)$ is related to $x(n)$ by a circular shift circular shift of the sequence is represented as the index modulo N

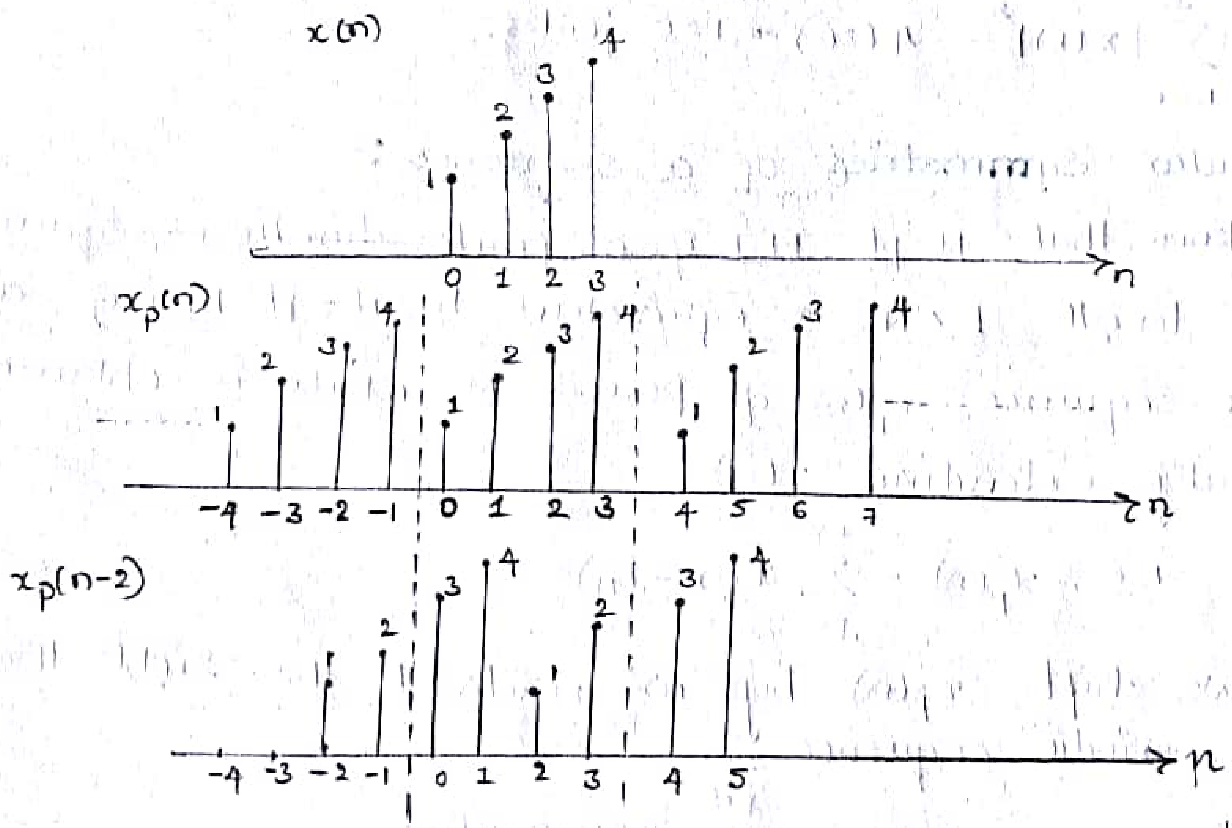
$$x'(n) = x(n-k, \text{ modulo } N)$$

$$x'(n) = x((n-k))_N$$

Ex :- let $x(n) = \{1, 2, 3, 4\}$

then $x((n-2))_4 = \{3, 4, 1, 2\}$





ODD & EVEN SYMMETRY :-

* An N-point sequence is called circularly even if it is symmetric about the point on the circle.

i.e, $x(N-n) = x(n)$ $\quad 1 \le n \le N-1$

* An N-point sequence is called circularly odd if it is antisymmetric about the point zero on the circle.

i.e, $x(N-n) = -x(n)$ $\quad 1 \le n \le N-1$

* The time reversal of an N-point sequence is obtained by reversing its samples about the point - zero on the circle.

$$x((-n))_N = x(N-n) \quad 0 \le n \le N-1$$

for even signals ; $x_p(n) = x_p(-n) = x_p(N-n)$

for odd signals ; $x_p(n) = -x_p(-n) = -x_p(N-n)$

if $x_p(n)$ is complex valued, then

even conjugate $= x_p(n) = x_p^*(N-n)$

odd conjugate $= x_p(n) = -x_p^*(N-n)$

Decomposition of even and odd components :-

W.K.T $\quad x_p(n) = x_{p_e}(n) + x_{p_o}(n)$ ——— (1)

put $n = -n$ in eqn (1)

$$x_p(-n) = x_{p_e}(-n) + x_{p_o}(-n)$$

$$x_p(-n) = x_{p_e}(n) - x_{p_o}(n) \text{ ——— } (2)$$

adding eqn (1) & (2)

$$x_p(n) + x_p(-n) = 2x_{p_e}(n)$$

$$x_{p_e}(n) = \frac{x_p(n) + x_p(-n)}{2}$$

(or) $\quad \boxed{x_{p_e}(n) = \frac{x_p(n) + x_p^*(N-n)}{2}}$

Subtracting eqn (1) & (2)

$$x_p(n) - x_p(-n) = 2x_{p_o}(n)$$

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$$x_{p_o}(n) = \frac{x_p(n) - x_p(-n)}{2}$$

(or)

$$x_{p_o}(n) = \frac{x_p(n) - x_p(-n)}{2}$$

$$\boxed{x_{p_o}(n) = \frac{x_p(n) - x_p^*(N-n)}{2}}$$

Symmetry properties of DFT

let us assume that $x(n)$ & $X(k)$ are complex valued then

$$x(n) = x_R(n) + j x_I(n) \text{ ——— (1)}$$

& $$X(k) = X_R(k) + j X_I(k) \text{ ——— (2)}$$

where $n = 0, 1, 2, \ldots (N-1)$

& $k = 0, 1, 2, \ldots (N-1)$

from the definition of DFT, WKT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \text{ ——— (3)}$$

Substituting eq (1) in eq (3)

$$X(k) = \sum_{n=0}^{N-1} \{ x_R(n) + j x_I(n) \} W_N^{kn}$$

W.K.T $W_N^{kn} = e^{-j\frac{2\pi}{N}kn} = \cos\frac{2\pi}{N}kn - j\sin\frac{2\pi}{N}kn$

$$X(k) = \sum_{n=0}^{N-1} \{ x_R(n) + j x_I(n) \} \{ \cos\frac{2\pi}{N}kn - j\sin\frac{2\pi}{N}kn \}$$

$$X(k) = \sum_{n=0}^{N-1} \{ x_R(n) \cdot \cos\frac{2\pi}{N}kn - j x_R(n) \sin\frac{2\pi}{N}kn + j x_I(n) \cos\frac{2\pi}{N}kn + x_I(n) \sin\frac{2\pi}{N}kn \} \text{ —(4)}$$

comparing eq (2) & (4)

$$X_R(k) = \sum_{n=0}^{N-1} \{ x_R(n) \cos\frac{2\pi}{N}kn + x_I(n) \sin\frac{2\pi}{N}kn \} \text{ —— (5)}$$

$$X_I(k) = \sum_{n=0}^{N-1} \{ -x_R(n) \sin\frac{2\pi}{N}kn + x_I(n) \cos\frac{2\pi}{N}kn \} \text{ —— (6)}$$

Similarly, w.k.t $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot W_N^{-kn}$ ——(7)

Substituting eqn (2) in eqn (7)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\{X_R(k) + j X_I(k)\} W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\{X_R(k) + j X_I(k)\}\left\{\cos\frac{2\pi}{N}kn + j\sin\frac{2\pi}{N}kn\right\}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\Big\{X_R(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn + j X_I(k)\cos\frac{2\pi}{N}kn + j X_R(k)\sin\frac{2\pi}{N}kn\Big\} \quad (8)$$

comparing eqns (1) & (8) :-

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left\{X_R(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn\right\} \quad (9)$$

$$x_I(n) = \frac{1}{N}\sum\left\{X_R(k)\sin\frac{2\pi}{N}kn + X_I(k)\cos\frac{2\pi}{N}kn\right\} \quad (10)$$

Real valued sequence :-

Case - 1 . If $x(n)$ is real then $X(N-k) = X^*(k) = X((-k))$

& $|X(N-k)| = |X(k)|$

$\angle X(N-k) = -\angle X(k)$ for $x_I(n) = 0$

hence $x(n)$ can be calculated using eqn (9)

Case - 2 : Real and even sequence

If $x(n) = x(N-n)$, $0 \le n \le N-1$ i.e. $x(n)$ is real & even

then $X_I(k) = 0$

hence DFT reduces to

$$X(k) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi}{N}kn$$

where $X(k)$ is real valued & even

since $X_I(k) = 0$, IDFT reduces to

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\frac{2\pi}{N}kn$$

case - 3 : Real and odd sequences

If $x(n)$ is real and odd i.e, $x(n) = x(N-n)$

then $X_R(K) = 0$

$$X(K) = -j \sum_{n=0}^{N-1} x(n) \cdot \sin \frac{2\pi}{N} kn$$

which is purely imaginary & odd

Since $X_R(K) = 0$, IDFT reduces to

$$x(n) = j \cdot \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot \sin \frac{2\pi}{N} kn$$

case - 4 :- purely imaginary sequence

i.e $x(n) = j\, x_I(n)$

then $X_R(K) = \sum_{n=0}^{N-1} x_I(n) \cdot \sin \frac{2\pi}{N} kn \longrightarrow (a)$

$X_I(K) = \sum_{n=0}^{N-1} x_I(n) \cdot \cos \frac{2\pi}{N} kn \longrightarrow (b)$

i.e, $X_R(K)$ is odd if $X_I(K)$ is even

if $x_I(n)$ is odd, then $X_I(k) = 0$, hence $X(K)$ is purely real.

If $x_I(n)$ is even, then $X_R(K) = 0$, hence $X(K)$ is purely imaginary.

Relationship of the DFT To other Transforms

1. Relationship to the fourier series of a periodic sequence

A periodic sequence $x_p(n)$. with fundamental period N can be Represented in Fourier Series as:

$$x_p(n) = \sum_{k=0}^{(N-1)} c_k \cdot e^{j\frac{2\pi}{N}kn} \quad \text{——(1)} \qquad -\infty < n < +\infty$$

with Fourier Series co-efficients

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi}{N}kn} \quad \text{——(2)}$$

$$k = 0, 1, \ldots \ldots (N-1)$$

w.k.T

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn} \text{——(3)}, \quad 0 \le k \le N-1$$

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \quad \text{——(4)}$$

where $0 \le n \le N-1$

comparing eq^n (2) & (3)

$$c_k = \frac{1}{N} X(k)$$

(or) $\boxed{X(k) = N \cdot c_k}$

2. Relationship to the fourier Transform of a non-periodic sequence.

Consider a non-periodic finite energy sequence $x(n)$ with F.T $X(e^{j\omega})$

i.e; $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$$

if we sample $X(e^{j\omega})$ at 'N' equally spaced frequencies, $\omega_k = \frac{2\pi}{N}k$, $0 \le k \le (N-1)$

we get;

$$X(k) = X(e^{j\omega})\Big|_{\omega_k = \frac{2\pi}{N}k} \qquad 0 \le k \le (N-1)$$

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$$\boxed{X(K) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\frac{2\pi}{N}kn}}$$

in above eqn., $X(K)$ is the DFT co-efficients of periodic sequence $x_p(n)$ of period N given by,

$$x_p(n) = \sum_{l=-\infty}^{+\infty} x(n-lN)$$

i.e, $x_p(n)$ is obtained by shifting $x(n)$ by N-sample

3. Relationship to the Z-Transform :-

Z.Transform of a sequence $x(n)$ is given by

$$Z.T\{x(n)\} = X(Z) = \sum_{n=-\infty}^{+\infty} x(n) \cdot z^{-n}$$

with ROC that includes unit circle [stability]

if $X(Z)$ is sampled at the N-equally spaced points on the unit circle, $Z_k = e^{+j\frac{2\pi}{N}k}$

i.e, $X(K) = X(Z)\Big|_{z = e^{j\frac{2\pi}{N}k}}$

$$X(K) = \sum_{n=-\infty}^{+\infty} x(n) \cdot \left(e^{j\frac{2\pi}{N}k}\right)^{-n}$$

$$X(K) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\frac{2\pi}{N}kn}$$ $\longrightarrow$ In this eqn is identical to F.T $\{X(e^{j\omega})\}$ evaluated at N-equally spaced frequencies $\omega_k = \frac{2\pi}{N}k$

if $x(n)$ is a finite duration sequence of length $L \leq N$, then sequence can be recovered from its N-point DFT.

The relationship b/w $X(Z)$ & $X(K)$ is

$$X(Z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n}$$

using IDFT eqn:

i.e., $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) . e^{j\frac{2\pi}{N}kn}$

$$X(z) = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} X(k) . e^{j\frac{2\pi}{N}kn} \right\} z^{-n}$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) . \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k} . z^{-1} \right)^n$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\frac{1 - \left(e^{j\frac{2\pi}{N}k} . z^{-1} \right)^{N-1+1}}{1 - e^{j\frac{2\pi}{N}k} . z^{-1}} \right]$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} z^{-1}} \right]$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} z^{-1}} \right]$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

if above eqn is evaluated on the unit circle at N equally spaced points $\left(z_k = e^{j\frac{2\pi}{N}k} \right)$ we get, the F.T of the finite duration sequence $x(n)$, interms of its DFT

if $z = e^{j\frac{2\pi}{N}k}$, we get

$$X(e^{jw}) = \frac{1 - e^{j}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi}{N}k} . e^{-j\frac{2\pi}{N}k}}$$

4. **Relationship to the fourier series of a periodic continuous time signal.**

consider a periodic continuous - time signal $x_a(t)$ with fundamental frequency $w = 2\pi f$ (or) $w = \frac{2\pi}{T}$, then $x_a(t)$ can be expressed in Fourier series as:

$$x_a(t) = \sum_{k=-\infty}^{+\infty} c_k . e^{jwkt}$$

where $w = 2\pi f_0$

& c_k is the fourier co-efficients

(53) If we sample $x_a(t)$ at a uniform rate.

$F_s = \frac{N}{T_P} = \frac{1}{T}$, we obtain discrete time sequence.

i.e : $x(n) = x_a(t)\Big|_{t=nT}$

$$= x_a(nT) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{j2\pi f_0 k \cdot nT}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cdot e^{j\frac{2\pi}{N}kn}$$

$\frac{N}{T_P} = \frac{1}{T}$

$N \cdot f = \frac{1}{T}$

$N = \frac{T_P}{T}$

∴ Above eqn can also be written as

$$x(n) = \sum_{k=0}^{N-1} \left\{ \sum_{l=-\infty}^{\infty} c_{k-lN} \right\} e^{j\frac{2\pi}{N}kn}$$

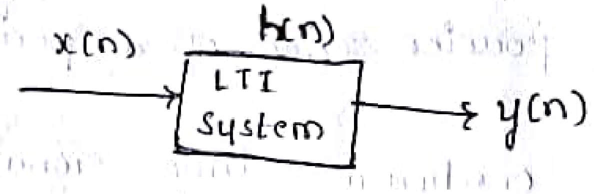
Above eqn. is of the form of IDFT

where $X(k) = N \cdot \sum_{l=-\infty}^{+\infty} c_{k-lN} = N \cdot \tilde{c}_k$

& $\tilde{c}_k = \sum_{l=-\infty}^{+\infty} c_{k-lN}$

Thus $\tilde{c}_k$ sequence is an aliased version of the sequence c_k.

∴ Liner filtering using DFT :-



W.K.T output of LTI system is governed by convolution sum

i.e ; $y(n) = x(n) * h(n)$

where $* \Rightarrow$ Linear convolution

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k) \quad . \qquad -\infty \le n \le \infty$$

- if L_1 is the length of $x(n)$
  & L_2 is the length of $h(n)$
  then convolved o/p $y(n)$ will have the length

$$\boxed{L = L_1 + L_2 - 1}$$

W.K.T DFT based frequency is more efficient compared to time domain processing.

In circular convolution both $x(n)$ & $h(n)$ should be of same length.

Differences B/w Linear & Circular Convolution :-

Linear	Circular
1. $y(n) = x(n) * h(n)$ $y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$	1. $y(n) = x(n) \textcircled{N} h(n)$ $y(n) = \sum_{m=0}^{N-1} x(m) \cdot h((n-m))_N$
2. length of $x(n)$ & $h(n)$ need not be same	2. length of $x(n)$ & $h(n)$ must be same
3. Zero padding is not Required	3. Zero padding is ~~not Required~~. necessary to make both sequence of same length.
4. If L_1 is the length of $x(n)$ & L_2 is length of $h(n)$, then convolved o/p will have length $L = L_1 + L_2 - 1$	4. if N_1 is length of $x(n)$ & N_2 is length of $h(n)$ then convolution length. $N = Max(N_1, N_2)$.
5. No aliasing effect	5. Aliasing occurs
6. applicable to aperiodic signals.	6. applied for periodic signals.

Linear convolution using circular convolution.

We can use circular convolution to get linear convol-ution o/p by making both sequences of length

$L = L_1 + L_2 - 1$ (or) $N = N_1 + N_2 - 1$ using zero padding. Thus aliasing effect can be avoided.

problems :-

1) Let $x(n) = \{1, 2, 3, 4\}$ & $h(n) = \{1, 2, 2\}$ compute
   (a) circular convolution.
   (b) Linear convolution.
   (c) Linear convolution using circular convolution. comment on the result.

Solution:- a) Circular convolution

$x(n) = \{1, 2, 3, 4\}$ & $h(n) = \{1, 2, 2\}$

Zero padding is done to h(n) to make its length equal to length of x(n).

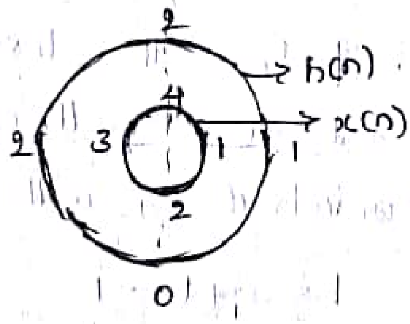
$$y(n) = x(n) \, ⓝ \, h(n)$$

Taking DFT : $Y(K) = X(K) \cdot H(K)$

$$y(n) = \sum_{m=0}^{N-1} x(m) \cdot h((n-m))_N$$

$$y(0) = \sum_{m=0}^{3} x(m) \cdot h((-m))_N = \sum_{m=0}^{3} x(m) \cdot h((-m))_4$$

$$y(0) = 1+0+6+8 = 15$$

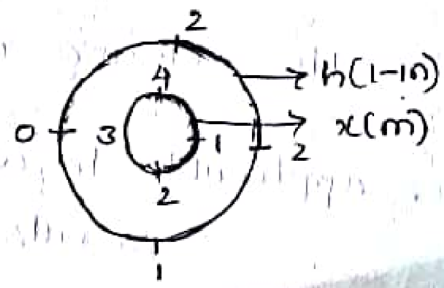


$$y(1) = \sum_{m=0}^{3} x(m) \cdot h((1-m))_4$$

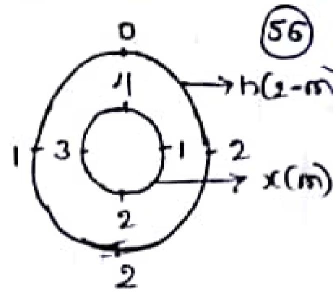
$$y(1) = 2+2+0+8 = 12$$

$$y(2) = \sum_{m=0}^{3} x(m) \; h[(2-m)]_4$$

$$y(2) = 2+4+3+0 = 9$$

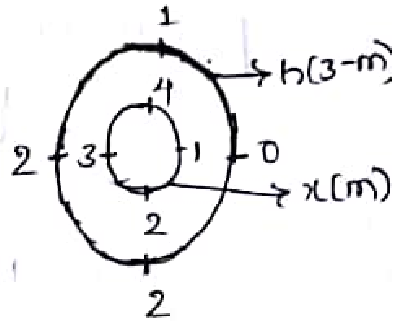


$$y(3) = \sum_{m=0}^{3} x(m) \cdot h((3-m))_4$$



$$y(3) = 0+4+6+4 = 14$$

$$\therefore \boxed{y(n) = \{15, 12, 9, 14\}}$$



b) Linear convolution :-

W.K.T $\quad y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

Let $x(n) = \{1, 2, 3, 4\}$ be represented in terms of shifted impulses.

$$x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$$

$$h(n) = \delta(n) + 2\delta(n-1) + 2\delta(n-2).$$

$$y(n) = \{\delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)\} * \{\delta(n) + 2\delta(n-1) + 2\delta(n-2)\}$$

$$\begin{aligned} y(n) = \{ & \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3) \\ & + 2\delta(n-1) + 4\delta(n-2) + 6\delta(n-3) + 8\delta(n-4) \\ & + 2\delta(n-2) + 4\delta(n-3) + 6\delta(n-4) + 8\delta(n-5) \end{aligned}$$

$$y(n) = \delta(n) + 4\delta(n-1) + 9\delta(n-2) + 14\delta(n-3) + 14\delta(n-4) + 8\delta(n-5)$$

(or) $\boxed{y(n) = \{\underset{\uparrow}{1}, 4, 9, 14, 14, 8\}}$

c) Linear convolution using circular convolution :-

$x(n) = \{1, 2, 3, 4\}$, $N_1 = 4$

$h(n) = \{1, 2, 2\}$, $N_2 = 3$

To avoid aliasing $\quad N = N_1 + N_2 - 1$

$$= 4 + 3 - 1$$

$$\underline{N = 6}$$

Zero padding is done to make length of both $x(n)$ & $h(n)$ (57)

length $= \underline{6}$

$x(n) = \{1, 2, 3, 4, 0, 0\}$

$h(n) = \{1, 2, 2, 0, 0, 0\}$

performing 6-point circular convolution

$$y(n) = x(n) \; ⑥ \; h(n)$$

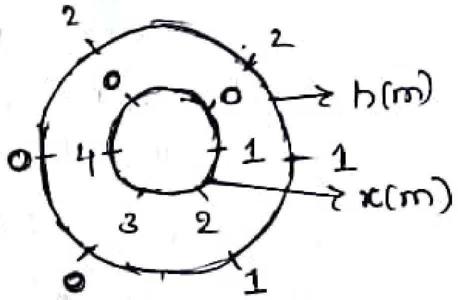
$$y(n) = \sum_{m=0}^{N-1} x(m) \cdot h((n-m))_N$$

since $N=6$: $\quad y(n) = \sum_{m=0}^{5} x(m) \cdot h((n-m))_6$

where $n = 0, 1, 2, 3, 4, 5$

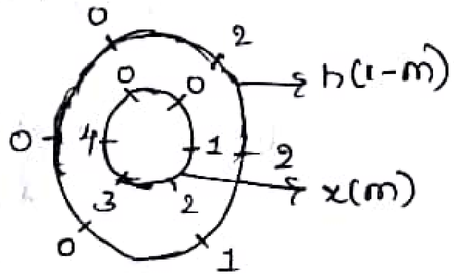
At $n=0$; $y(0) = \sum_{m=0}^{5} x(m) \cdot h((-m))_6$

$y(0) = 1 + 0 + 0 + 0 + 0 + 0 = 1$



at $n=1$; $y(1) = \sum_{m=0}^{5} x(m) \cdot h((1-m))_6$

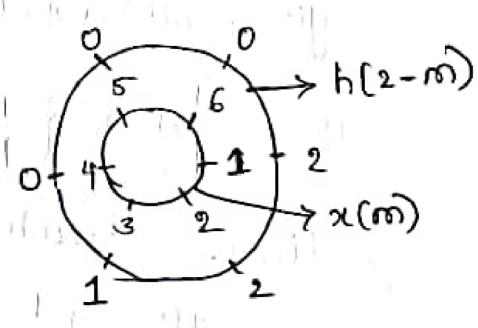
$y(1) = 2 + 2 + 0 + 0 + 0 + 0 = 4$



at $n=2$; $y(2) = \sum_{m=0}^{5} x(m) \cdot h((2-m))_6$

$y(2) = 2 + 4 + 3 + 0 + 0 + 0$

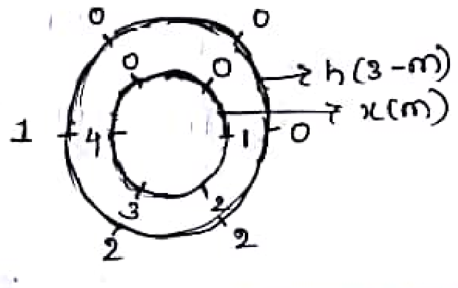
$= \underline{\underline{9}}$



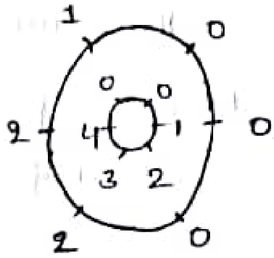
at $n=3$; $y(3) = \sum_{m=0}^{5} x(m) \cdot h((3-m))_6$

$y(3) = 0 + 4 + 6 + 4 + 0 + 0$

$= \underline{\underline{14}}$



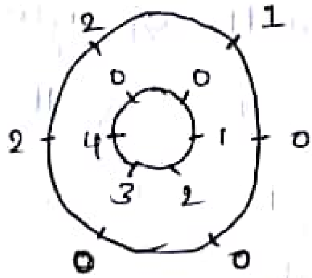
at $n=4$; $y(4) = \sum_{m=0}^{5} x(m) \cdot h((4-m))_6$



$y(4) = 0+0+6+8+0+0$

$= 14$

at $n=5$; $y(5) = \sum_{m=0}^{5} x(m) \cdot h((5-m))_6$



$= 0+0+0+8+0+0$

$= 8$

$\therefore \boxed{y(n) = \{1, 4, 9, 14, 14, 8\}}$

Comments :-

$y(n)_{Linear} = \{1 \quad 4 \quad 9 \quad 14 \quad \boxed{14 \quad 8}\}$

$y(n)_{Circular} = \{\overset{+14}{15} \quad \overset{+8}{12} \quad 9 \quad 14\}$

In circular convolution, length is limited to N=4 Thus Last 2 samples will overlap with first 2 samples causing aliasing effect. Thus circular convolution gives aliasing.

This can be solved by making the sequences of length $N = N_1 + N_2 - 1$ i.e., Instead of performing 4-point circular convolution, if we do 6-point circular convolution, aliasing can be avoided.

Filtering of Long Data Sequence

In practical applications involving linear filtering of signals, the i/p sequence x(n) is very large sequence.

To perform linear filtering long data sequence is segmented & fast convolution is performed on each section, hence it is also known as Sectional convolution (or) Fast convolution.

There are two techniques :

1. Overlap Save Method.
2. Overlap Add Method

problems :-

1. Find the o/p $y(n)$ of a filter whose impulse response is $h(n) = \{1, 2\}$ & the i/p signal to the filter is $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$ using overlap save Method.

Sol^n: $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$

& $h(n) = \{1, 2\}$

W.K.T $N = L + M - 1$

where L = length of each block of sub-sequence

M = length of impulse response $h(n)$

N = size of DFT & IDFT or convolution length.

$M = 2$, $N = 2^M = 2^2 = 4$ $\boxed{N=4}$

$\therefore$ $L = 4 + 1 - 2 = 3$

$\boxed{L=3}$

* Given i/p data sequence is segmented into blocks of length $L = 3$

$x_1(n) = \{1, 2, -1\}$

$x_2(n) = \{2, 3, -2\}$

$x_3(n) = \{-3, -1, 1\}$

$x_4(n) = \{1, 2, -1\}$

* Since we need to perform 4-pt circular convolution as $N=4$, we have to do zero padding.

$h(n) = \{1, 2, 0, 0\}$

in overlap - save method, first sub-sequence will have first (M-1) zeroes & next 'L' i/p samples.

* The modified sub-sequences are

$x_1(n) = \{0, 1, 2, -1\}$

$x_2(n) = \{\underbrace{-1}, 2, 3, -2\}$

↳ (M-1) sample from previous block.

$x_3(n) = \{-2, -3, -1, 1\}$

$x_4(n) = \{1, 1, 2, -1\}$

$x_5(n) = \{-1, 0, 0, 0\}$

* To find the o/p of each sub sequence

$y_1(n) = x_1(n) \, (4) \, h(n) = \{-2, 1, 4, 3\}$

$y_2(n) = x_2(n) \, (4) \, h(n) = \{-5, 0, 7, 4\}$

$y_3(n) = x_3(n) \, (4) \, h(n) = \{0, -7, -7, -1\}$

$y_4(n) = x_4(n) \, (4) \, h(n) = \{-1, 3, 4, 3\}$

$y_5(n) = x_5(n) \, (4) \, h(n) = \{-1, -2, 0, 0\}$

* $y_1(n) = x_1(n) \, (4) \, h(n)$

Taking DFT on both sides

$y_1(k) = x_1(k) \cdot H(k)$

$y_1(k) = \{0 + W_4^k + 2W_4^{2k} - W_4^{3k}\} \cdot \{1 + 2W_4^k\}$

$y_1(k) = W_4^k + 2W_4^{2k} - W_4^{3k}$

$+ 2W_4^{2k} + 4W_4^{3k} - 2W_4^{4k}$

$y_1(k) = W_4^k + 4W_4^{2k} + 3W_4^{3k} - 2W_4^{4k}$

w.k.T $W_4^{4k} = W_4^0 = 1$ by periodicity property

$y_1(k) = -2 + W_4^k + 4W_4^{2k} + 3W_4^{3k}$

Taking IDFT on both sides

$$y_1(n) = \{-2 \quad 1 \quad 4 \quad 3\}$$

* $y_2(n) = x_2(n) \circledast_4 h(n)$

Taking DFT on both sides

$$Y_2(k) = X_2(k) \cdot H(k)$$

$$Y_2(k) = \{-1 + 2W_4^{k} + 3W_4^{2k} - 2W_4^{3k}\}\{1 + 2W_4^{k}\}$$

$$Y_2(k) = -1 + 2W_4^{k} + 3W_4^{2k} - 2W_4^{3k}$$
$$\quad - 2W_4^{k} + 4W_4^{2k} + 6W_4^{3k} - 4W_4^{4k}$$

$$Y_2(k) = -1 + 0W_4^{k} + 7W_4^{2k} + 4W_4^{3k} - 4W_4^{4k}$$

$$Y_2(k) = -5 + 0W_4^{k} + 7W_4^{2k} + 4W_4^{3k} \quad [\because W_4^{4k} = 1]$$

Taking IDFT on both sides

$$y_2(n) = \{-5 \quad 0 \quad 7 \quad 4\}$$

* $y_3(n) = x_3(n) \circledast_4 h(n)$

Taking DFT on both sides

$$Y_3(k) = X_3(k) \cdot H(k)$$

$$Y_3(k) = \{-2 - 3W_4^{k} - W_4^{2k} + W_4^{3k}\}\{1 + 2W_4^{k}\}$$

$$Y_3(k) = -2 - 3W_4^{k} - W_4^{2k} + W_4^{3k}$$
$$\quad -4W_4^{k} - 6W_4^{2k} - 2W_4^{3k} + 2W_4^{4k}$$

$$Y_3(k) = -2 - 7W_4^{k} - 7W_4^{2k} - W_4^{3k} + 2W_4^{4k}$$

$$Y_3(k) = 0 - 7W_4^{k} - 7W_4^{2k} - W_4^{3k} \quad [\because W_4^{4k} = 1]$$

Taking IDFT on both sides

$$y_3(n) = \{0 \quad -7 \quad -7 \quad -1\}$$

* $y_4(n) = x_4(n) \circledast_4 h(n)$

Taking DFT on both sides

$$Y_4(k) = X_4(k) \cdot H(k)$$

$$Y_4(k) = \{1 + W_4^{k} + 2W_4^{2k} - W_4^{3k}\}\{1 + 2W_4^{k}\}$$

$$y_4(k) = 1 + \omega_4^{k} + 2\omega_4^{2k} - \omega_4^{3k}$$
$$+2\omega_4^{k} + 2\omega_4^{2k} + 4\omega_4^{3k} - 2\omega_4^{4k}$$

$$y_4(k) = 1 + 3\omega_4^{k} + 4\omega_4^{2k} + 3\omega_4^{3k} - 2\omega_4^{4k}$$

$$y_4(k) = -1 + 3\omega_4^{k} + 4\omega_4^{2k} + 3\omega_4^{3k} \quad [\because \omega_4^{4k} = 1]$$

Taking IDFT on both sides

$$y_4(n) = \{-1 \quad 3 \quad 4 \quad 3\}$$

* $y_5(n) = x_5(n)$ (4) $h(n)$

Taking DFT on both sides

$$y_5(k) = x_5(k) \cdot H(k)$$

$$y_5(k) = \{-1\}\{1 + 2\omega_4^{k}\} = -1 - 2\omega_4^{k}$$

Taking IDFT on both sides

$$y_5(n) = \{-1, -2, 0, 0\}$$

combining all the o/p's. we get the final o/p.

$y_1(n) = \boxed{-2} \quad 1 \quad 4 \quad 3$

$y_2(n) =$ Discard $\leftarrow \boxed{-5} \quad 0 \quad 7 \quad 4$

$y_3(n) = \quad \boxed{0} \quad -7 \quad -7 \quad -1$

$y_4(n) = \quad$ Discard $\leftarrow \boxed{-1} \quad 3 \quad 4 \quad 3$

$y_5(n) = \quad$ Discard $\rightarrow \boxed{-1} \quad -2 \quad 0 \quad 0$

$$y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2\}$$

NOTE :- To find o/p's $y_1(n)$, $y_2(n)$ . . . . DFT & IDFT Eqns can be used or concentric circles can also be used.

2) The unit impulse response of a certain FIR digital filter is $h(n) = \{\underset{\uparrow}{1}, 2, 1\}$. Determine its step response using overlap save method. sketch the step response. also verify from the response whether the filter is stable & causal.

Soln: $h(n) = \{\underset{\uparrow}{1}, 2, 1\}$

Step response means response of an LTI system to (, step i/p. i.e $x(n) = u(n)$

$u(n) = \{1, 1, 1, 1, 1, 1, 1, 1 \ldots\}$

Here $m = 3$, $N = 2^m = 2^3 = 8$, $\boxed{N=8}$

$N = L + m - 1 \Rightarrow L = N + 1 - m = 8 + 1 - 3$ $\therefore \boxed{L=6}$

* Dividing the given i/p sequence into blocks of length $L = 6$

$x_1(n) = \{1, 1, 1, 1, 1, 1\}$

$x_2(n) = \{1, 1, 1, 1, 1, 1\}$

⋮

upto ∞

* Since $N = 8$, Rewrite the above eqn as:

$x_1(n) = \{0, 0, 1, 1, 1, 1, 1, 1\}$

$x_2(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

⋮

& $h(n) = \{1, 2, 1, 0, 0, 0, 0, 0\}$

* To find the o/p of each sub-sequences:

$y_1(n) = x_1(n) \, ⑧ \, h(n)$

$y_2(n) = x_2(n) \, ⑧ \, h(n)$

$y_3(n) = x_3(n) \, ⑧ \, h(n)$

$y_4(n) = x_4(n) \, ⑧ \, h(n) \ldots\ldots$ etc

* $y_1(n) = x_1(n) \, ⑧ \, h(n)$

Taking DFT on both sides

$Y_1(K) = X_1(K) \cdot H(K)$

$Y_1(K) = \{0 + 0W_8^K + W_8^{2K} + W_8^{3K} + W_8^{4K} + W_8^{5K} + W_8^{6K} + W_8^{7K}\}.$

$\{1 + 2W_8^K + W_8^{2K}\}$

$$y_1(k) = W_8^{2k} + W_8^{3k} + W_8^{4k} + W_8^{5k} + W_8^{6k} + W_8^{7k}$$
$$+ 2W_8^{3k} + 2W_8^{4k} + 2W_8^{5k} + 2W_8^{6k} + 2W_8^{7k} + 2W_8^{8k}$$
$$+ W_8^{4k} + W_8^{5k} + W_8^{6k} + W_8^{7k} + W_8^{8k} + W_8^{9k}$$

$$y_1(k) = W_8^{2k} + 3W_8^{3k} + 4W_8^{4k} + 4W_8^{5k} + 4W_8^{6k} + 4W_8^{7k} + 3W_8^{8k} + W_8^{9k}$$

$$y_1(k) = 3 + W_8^{k} + W_8^{2k} + 3W_8^{3k} + 4W_8^{4k} + 4W_8^{5k} + 4W_8^{6k} + 4W_8^{7k}$$

Taking IDFT on both sides $\left[\because W_8^{8k} = 1,\; W_8^{9k} = W_8^{k}\right]$

$$y_1(n) = \{3, 1, 1, 3, 4, 4, 4, 4\}$$

* $y_2(n) = x_2(n) \; ⑧ \; h(n)$

Taking DFT on both sides

$$y_2(k) = X_2(k) \cdot H(k)$$

$$y_2(k) = \{1 + W_8^{k} + W_8^{2k} + W_8^{3k} + W_8^{4k} + W_8^{5k} + W_8^{6k} + W_8^{7k}\}\{1 + 2W_8^{k} + W_8^{2k}\}$$

$$y_2(k) = 1 + W_8^{k} + W_8^{2k} + W_8^{3k} + W_8^{4k} + W_8^{5k} + W_8^{6k} + W_8^{7k}$$
$$+ 2W_8^{k} + 2W_8^{2k} + 2W_8^{3k} + 2W_8^{4k} + 2W_8^{5k} + 2W_8^{6k} + 2W_8^{7k} + 2W_8^{8k}$$
$$+ W_8^{2k} + W_8^{3k} + W_8^{4k} + W_8^{5k} + W_8^{6k} + W_8^{7k} + W_8^{8k} + W_8^{9k}$$

$$y_2(k) = 1 + 3W_8^{k} + 4W_8^{2k} + 4W_8^{3k} + 4W_8^{4k} + 4W_8^{5k} + 4W_8^{6k} + 4W_8^{7k} + 3W_8^{8k} + W_8^{9k}.$$

$$y_2(k) = 4 + 4W_8^{k} + 4W_8^{2k} + 4W_8^{3k} + 4W_8^{4k} + 4W_8^{5k} + 4W_8^{6k} + 4W_8^{7k}$$

Taking IDFT on both sides

$$y_2(n) = \{4, 4, 4, 4, 4, 4, 4, 4\}$$

* combining all the o/p's to get the final o/p.

$y_1(n) = \boxed{3 \;\; 1} \;\; 1 \;\; 3 \;\; 4 \;\; 4 \;\; 4 \;\; 4$

Discard

$y_2(n) = \qquad\qquad\qquad \boxed{4 \;\; 4} \;\; 4 \;\; 4 \;\; 4 \;\; 4 \;\; 4 \;\; 4$

$$y(n) = \{1, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4 \ldots\ldots\}$$

③ A long sequence $x(n)$ is filtered through a filter with impulse response $h(n)$ to yield the o/p $y(n)$ if $h(n) = \{1, -1\}$ & $x(n) = \{1, 1, 1, 1, 1, 3, 1, 1, 4, 2, 1, 1, 3, 1\}$. compute $y(n)$ using overlap - save techniques [use only 5- point circular convolution].

Soln :- $h(n) = \{1, -1\}$; $m = 2$

$N = 5$, $L = N+1-m = 5+1-2 = 4$ $\boxed{L = 4}$

* given i/p sequence is divided into sub-sequences of length $L = 4$

$x_1(n) = \{1, 1, 1, 1\}$

$x_2(n) = \{1, 3, 1, 1\}$

$x_3(n) = \{4, 2, 1, 1\}$

$x_4(n) = \{3, 1, 0, 0\}$

* Since $N=5$, we want to perform 5-pt circular convolution using overlap - save method. hence above eqns can be re-written as

$h(n) = \{1, -1, 0, 0, 0\}$

$x_1(n) = \{0, 1, 1, 1, 1\}$

$x_2(n) = \{1, 1, 3, 1, 1\}$

$x_3(n) = \{1, 4, 2, 1, 1\}$

$x_4(n) = \{1, 3, 1, 0, 0\}$

* To find the o/p of each sub-sequence

$y_1(n) = x_1(n)$ ⑤ $h(n)$

$y_2(n) = x_2(n)$ ⑤ $h(n)$

$y_3(n) = x_3(n)$ ⑤ $h(n)$

$y_4(n) = x_4(n)$ ⑤ $h(n)$

* $y_1(n) = x_1(n)$ ⑤ $h(n)$

Taking DFT : $y_1(K) = x_1(K) \cdot H(K)$

$y_1(K) = \{0 + \omega_5^{K} + \omega_5^{2K} + \omega_5^{3K} + \omega_5^{4K}\}\{1 - \omega_5^{K}\}$

$$y_1(k) = W_5^{k} + W_5^{2k} + W_5^{3k} + W_5^{4k}$$
$$\quad - W_5^{2k} - W_5^{3k} - W_5^{4k} - W_5^{5k}$$

$$y_1(k) = -1 + W_5^{k}$$

Taking IDFT : $y_1(n) = \{-1, 1, 0, 0, 0\}$

* $y_2(n) = x_2(n) \; ⑤ \; h(n)$

Taking DFT : $Y_2(k) = X_2(k) \cdot H(k)$

$$Y_2(k) = \{1 + W_5^{k} + 3W_5^{2k} + W_5^{3k} + W_5^{4k}\}\{1 - W_5^{k}\}$$

$$Y_2(k) = 1 + W_5^{k} + 3W_5^{2k} + W_5^{3k} + W_5^{4k}$$
$$\quad - W_5^{k} - W_5^{2k} - 3W_5^{3k} - W_5^{4k} - W_5^{5k}$$

$$Y_2(k) = 2W_5^{2k} - 2W_5^{3k}$$

Taking IDFT : $y_2(n) = \{0, 0, 2, -2, 0\}$

* $y_3(n) = x_3(n) \; ⑤ \; h(n)$

Taking DFT : $Y_3(k) = X_3(k) \cdot H(k)$

$$Y_3(k) = \{1 + 4W_5^{k} + 2W_5^{2k} + W_5^{3k} + W_5^{4k}\}\{1 - W_5^{k}\}$$

$$Y_3(k) = 1 + 4W_5^{k} + 2W_5^{2k} + W_5^{3k} + W_5^{4k}$$
$$\quad - W_5^{k} - 4W_5^{2k} - 2W_5^{3k} - W_5^{4k} - W_5^{5k}$$

$$Y_3(k) = 3W_5^{k} - 2W_5^{2k} - W_5^{3k}$$

Taking IDFT : $y_3(n) = \{0, 3, -2, -1, 0\}$

* $y_4(n) = x_4(n) \; ⑤ \; h(n)$

Taking DFT : $Y_4(k) = X_4(k) \cdot H(k)$

$$Y_4(k) = \{1 + 3W_5^{k} + W_5^{2k}\}\{1 - W_5^{k}\}$$

$$Y_4(k) = 1 + 3W_5^{k} + W_5^{2k}$$
$$\quad - W_5^{k} - 3W_5^{2k} - W_5^{3k}$$

$$Y_4(k) = 1 + 2W_5^{k} - 2W_5^{2k} - W_5^{3k}$$

Taking IDFT : $y_4(n) = \{1, 2, -2, -1, 0\}$

* combining all above o/p's to obtain final o/p.

$y_1(n) = \boxed{-1}\ 1\ 0\ 0\ 0$

$y_2(n) = \qquad\qquad\qquad \boxed{0}\ 0\ 2\ -2\ 0$

$y_3(n) = \qquad\qquad\qquad\qquad\qquad\qquad \boxed{0}\ 3\ -2\ -1\ 0$

$y_4(n) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boxed{1}\ 2\ -2\ -1$

$y(n) = \{1, 0, 0, 0, 0, 2, -2, 0, 3, -2, -1, 0, 2, -2, -1\}$

4) Find the o/p $y(n)$ of a filter whose impulse response is $h(n) = \{1, -2\}$ & i/p is given by $x(n) = \{3, -2, 4, 1, 5, 7, 2, -9\}$ using overlap-add method. use only 5-point circular convolution in your approach.

Sol^n:- $x(n) = \{3, -2, 4, 1, 5, 7, 2, -9\}$

$h(n) = \{1, -2\}$

$m = 2 \ , \ N = 5$

$L = N + 1 - m = 5 + 1 - 2 = 4$

$\boxed{L = 4}$

* Divide the given i/p sequence into sub-sequence of block length $\boxed{L = 4}$

$x_1(n) = \{3, -2, 4, 1\}$

$x_2(n) = \{5, 7, 2, -9\}$

* using overlap-add method, we pad $(m-1)$ zeros at the end of each sub-sequence.

$x_1(n) = \{3, -2, 4, 1, 0\}$

$x_2(n) = \{5, 7, 2, -9, 0\}$

$h(n) = \{1, -2, 0, 0, 0\}$

* To find the o/p of each sub-sequence

$y_1(n) = x_1(n) \ ⑤ \ h(n)$

$y_2(n) = x_2(n) \ ⑤ \ h(n)$

* $y_1(n) = x_1(n) \circledS h(n)$

Taking DFT ; $Y_1(K) = X_1(K) \cdot H(K)$

$Y_1(K) = \{3 - 2\omega_5^k + 4\omega_5^{2k} + \omega_5^{3k}\}\{1 - 2\omega_5^k\}$

$$Y_1(K) = 3 - 2\omega_5^k + 4\omega_5^{2k} + \omega_5^{3k}$$
$$-6\omega_5^k + 4\omega_5^{2k} - 8\omega_5^{3k} - 2\omega_5^{4k}$$

$Y_1(K) = 3 - 8\omega_5^k + 8\omega_5^{2k} - 7\omega_5^{3k} - 2\omega_5^{4k}$

Taking IDFT ; $y_1(n) = \{3, -8, 8, -7, -2\}$

* $y_2(n) = x_2(n) \circledS h(n)$

Taking DFT : $Y_2(K) = X_2(K) \cdot H(K)$

$Y_2(K) = \{5 + 7\omega_5^k + 2\omega_5^{2k} - 9\omega_5^{3k}\}\{1 - 2\omega_5^k\}$

$$Y_2(K) = 5 + 7\omega_5^k + 2\omega_5^{2k} - 9\omega_5^{3k}$$
$$-10\omega_5^k - 14\omega_5^{2k} - 4\omega_5^{3k} + 18\omega_5^{4k}$$

$Y_2(K) = 5 - 3\omega_5^k - 12\omega_5^{2k} - 13\omega_5^{3k} + 18\omega_5^{4k}$

Taking IDFT ; $y_2(n) = \{5, -3, -12, -13, 18\}$

* Adding last (M-1) samples of $y_1(n)$ with first (M-1) samples of $y_2(n)$ to obtain $y(n)$

$y_1(n) =$	3	-8	8	-7	-2				
$y_2(n) =$					+5	-3	-12	-13	18

$y(n) = \{3, -8, 8, -7, 3, -3, -12, -13, 18\}$

5) Determine the Response of an LTI system with $h(n) = \{1, -1, 2\}$ for an i/p $x(n) = \{1, 0, +1, -2, 1, 2, 3, -1, 0, 2\}$ Employ overlap-add method. with block length $L = 4$

Soln :- $L = 4$, $M = 3$

$N = L + M - 1 = 4 + 3 - 1$

$\boxed{N = 6}$

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* Divide the given i/p sequence $x(n)$ into sub-sequence of length, $L=4$

$x_1(n) = \{1, 0, 1, -2\}$

$x_2(n) = \{1, 2, 3, -1\}$

$x_3(n) = \{0, 2, 0, 0\}$

* Since $N=6$, above eq^{n}s are rewritten using overlapped-add method as:

$x_1(n) = \{1, 0, 1, -2, 0, 0\}$ padding $(M-1)$ zeroes

$x_2(n) = \{1, 2, 3, -1, 0, 0\}$ at end of each sub-sequence.

$x_3(n) = \{0, 2, 0, 0, 0, 0\}$

$h(n) = \{1, -1, 2, 0, 0, 0\}$

* To find the o/p of each sub-sequence

$y_1(n) = x_1(n)$ ⑥ $h(n)$

$y_2(n) = x_2(n)$ ⑥ $h(n)$

$y_3(n) = x_3(n)$ ⑥ $h(n)$

* $y_1(n) = x_1(n)$ ⑥ $h(n)$

Taking DFT: $Y_1(K) = X_1(K) \cdot H(K)$

$Y_1(K) = \{1 + W_6^{2K} - 2W_6^{3K}\}\{1 - W_6^{K} + 2W_6^{2K}\}$

$$
\begin{aligned}
Y_1(K) = {} & 1 + 0\,W_6^{K} + W_6^{2K} - 2W_6^{3K} \\
& - W_6^{K} - 0\,W_6^{2K} - W_6^{3K} + 2W_6^{4K} \\
& + 2W_6^{2K} + 0\,W_6^{3K} + 2W_6^{4K} - 4W_6^{5K}
\end{aligned}
$$

$Y_1(K) = 1 - W_6^{K} + 3W_6^{2K} - 3W_6^{3K} + 4W_6^{4K} - 4W_6^{5K}$

Taking IDFT: $y_1(n) = \{1, -1, 3, -3, 4, -4\}$

* $y_2(n) = x_2(n)$ ⑥ $h(n)$

Taking DFT: $Y_2(K) = X_2(K) \cdot H(K)$

$Y_2(K) = \{1 + 2W_6^{K} + 3W_6^{2K} - W_6^{3K}\}\{1 - W_6^{K} + 2W_6^{2K}\}$

$$Y_2(k) = 1 + 2W_6^{k} + 3W_6^{2k} - W_6^{3k}$$
$$- W_6^{k} - 2W_6^{2k} - 3W_6^{3k} + W_6^{4k}$$
$$+ 2W_6^{2k} + 6W_6^{4k} - 2W_6^{5k} + 4W_6^{3k}$$

$$Y_2(k) = 1 + W_6^{k} + 3W_6^{2k} - 2W_6^{5k} + 7W_6^{3k}$$

Taking IDFT : $y_2(n) = \{1, 1, 3, 0, 7, -2\}$

* $y_3(n) = x_3(n)$ ⑥ $h(n)$

Taking DFT on both sides : $Y_3(k) = X_3(k) \cdot H(k)$

$$Y_3(k) = \{2W_6^{k}\}\{1 - W_6^{k} + 2W_6^{2k}\}$$

$$Y_3(k) = 2W_6^{k} - 2W_6^{2k} + 4W_6^{3k}$$

Taking IDFT on both sides

$$y_3(n) = \{0, 2, -2, 4, 0, 0\}$$

* combining & Adding all the o/p's of subsequences to get the final o/p.

$y_1(n) =$	1	-1	3	-3	4	-4								
$y_2(n) =$					+1	+1	3	0	7	-2				
$y_3(n) =$								+0	+2	-2	-4	0	0	

$y(n) = \{1, -1, 3, -3, 5, -3, 3, 0, 7, 0, +2, -4, 0, 0\}$

$y(n) = \{1, -1, 3, -3, 5, -3, 3, 0, 7, 0, -2, -4\}$

$\therefore y(n) = \{1, -1, 3, -3, 5, -3, 3, 0, 7, 0, -2, -4\}$

6) perform $x(n) * h(n)$ using overlap - add technique given $h(n) = \{1, 1, 1\}$ & $x(n) = \{1, 2, 3, 4, -2, 1, 0, 6, 2, 8, 9, 6, 2, 2, 1\}$.

Sol^n :- $x(n) = \{1, 2, 3, 4, -2, 1, 0, 6, 2, 8, 9, 6, 2, 2, 1\}$

& $h(n) = \{1, 1, 1\}$

$m = 3$. $N = 2^m = 2^3 = 8$ $\therefore$ $\underline{N = 8}$

$L = N + 1 - m = 8 + 1 - 3 = 6$ $\boxed{L = 6}$

* Divide the given input sequence $x(n)$ to sub-sequences of length $L=6$

(71)

$x_1(n) = \{1, 2, 3, 4, -2, 1\}$

$x_2(n) = \{0, 6, 2, 8, 9, 6\}$

$x_3(n) = \{2, 2, 1, 0, 0, 0\}$

* using overlap - add technique, padding of zeroes is done to make the length of all subsequences equal to $N=8$

$x_1(n) = \{1, 2, 3, 4, -2, 1, 0, 0\}$

$x_2(n) = \{0, 6, 2, 8, 9, 6, 0, 0\}$

$x_3(n) = \{2, 2, 1, 0, 0, 0, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

* To find the o/p of each sub-sequence :

$y_1(n) = x_1(n) \circledast_8 h(n)$

$y_2(n) = x_2(n) \circledast_8 h(n)$

$y_3(n) = x_3(n) \circledast_8 h(n)$

* $y_1(n) = x_1(n) \circledast_8 h(n)$

Taking DFT : $Y_1(K) = X_1(K) . H(K)$

$Y_1(K) = \{1 + 2W_8^{k} + 3W_8^{2K} + 4W_8^{3K} - 2W_8^{4K} + W_8^{5K}\}\{1 + W_8^{k} + W_8^{2K}\}$

$$
\begin{aligned}
Y_1(K) = & \ 1 + 2W_8^{K} + 3W_8^{2K} + 4W_8^{3K} - 2W_8^{4K} + W_8^{5K} \\
& + W_8^{k} + 2W_8^{2K} + 3W_8^{3K} + 4W_8^{4K} - 2W_8^{5K} + W_8^{6K} \\
& + W_8^{2K} + 2W_8^{3K} + 3W_8^{4K} + 4W_8^{5K} - 2W_8^{6K} + W_8^{7K}
\end{aligned}
$$

$Y_1(K) = 1 + 3W_8^{k} + 6W_8^{2K} + 9W_8^{3K} + 5W_8^{4K} + 3W_8^{5K} - W_8^{6K} + W_8^{7K}$

Taking IDFT ; $y_1(n) = \{1, 3, 6, 12, 5, 3, -1, 1\}$

* $y_2(n) = x_2(n) \circledast_8 h(n)$

Taking DFT : $Y_2(K) = X_2(K) . H(K)$

$Y_2(k) = \{6W_8^{k} + 2W_8^{2K} + 8W_8^{3K} + 9W_8^{4K} + 6W_8^{5K}\}\{1 + W_8^{K} + W_8^{2K}\}$

$$y_2(k) = 6w_8^k + 2w_8^{2k} + 8w_8^{3k} + 9w_8^{4k} + 6w_8^{5k}$$
$$+ 6w_8^{2k} + 2w_8^{3k} + 8w_8^{4k} + 9w_8^{5k} + 6w_8^{6k}$$
$$+ 6w_8^{3k} + 2w_8^{4k} + 8w_8^{5k} + 9w_8^{6k} + 6w_8^{7k}$$

$$y_2(k) = 6w_8^k + 8w_8^{2k} + 16w_8^{3k} + 19w_8^{4k} + 23w_8^{5k} + 15w_8^{6k} + 6w_8^{7k}$$

Taking IDFT : $y_2(n) = \{0, 6, 8, 16, 19, 23, 15, 6\}$

* $y_3(n) = x_3(n) ⑧ h(n)$

Taking DFT : $Y_3(k) = X_3(k) \cdot H(k)$

$$Y_3(k) = \{2 + 2w_8^k + w_8^{2k}\}\{1 + w_8^k + w_8^{2k}\}$$

$$Y_3(k) = 2 + 2w_8^k + w_8^{2k}$$
$$+ 2w_8^k + 2w_8^{2k} + w_8^{3k}$$
$$+ 2w_8^{2k} + 2w_8^{3k} + w_8^{4k}$$

$$Y_3(k) = 2 + 4w_8^k + 5w_8^{2k} + 3w_8^{3k} + w_8^{4k}$$

Taking IDFT : $y_3(n) = \{2, 4, 5, 3, 1, 0, 0, 0\}$

* combining all o/p's to get final o/p

y(n) =	1	3	6	9	5	3	-1	1									
							+0	+6	8	16	19	23	15	6			
													+2	+4	5 3 1 0 0 0		

$$y(n) = \{1, 3, 6, 9, 5, 3, -1, 7, 8, 16, 19, 23, 17, 10, 5, 3, 1\}$$

N-point DFT's OF TWO REAL SEQUENCES USING A SINGLE N-POINT DFT

Let $g(n)$ & $h(n)$ be two real sequences of length N, with N point DFT's $G(k)$ & $H(k)$. These two DFT's can be computed using a single N-point DFT of a complex sequence defined as:

$$x(n) = g(n) + j\,h(n) \quad \text{——(1)}$$

Taking DFT on both sides

$$X(k) = G(k) + j\,H(k)$$

& $$x^*(n) = g(n) - j\,h(n) \quad \text{——(2)}$$

Taking DFT on both sides

$$X^*(k) = G(k) - j\,H(k)$$

Adding eqns (1) & (2)

$$x(n) = g(n) + \cancel{j\,h(n)}$$
$$x^*(n) = g(n) - \cancel{j\,h(n)}$$

$$x(n) + x^*(n) = 2g(n)$$

$$\therefore \quad g(n) = \frac{x(n) + x^*(n)}{2}$$

$$\& \quad G(K) = \frac{X(K) + X^*(-K)}{2}$$

Substracting eqn (1) & (2)

$$x(n) = \cancel{g(n)} + j\,h(n)$$
$$\underset{(-)}{x^*(n)} = \underset{(-)}{\cancel{g(n)}} \underset{(+)}{-\, j\,h(n)}$$

$$\therefore \quad x(n) - x^*(n) = 2j\,h(n)$$

$$\therefore \quad h(n) = \frac{x(n) - x^*(n)}{2j}$$

$$\& \quad H(K) = \frac{X(K) - X^*(-K)}{2j}$$

$$\Rightarrow \boxed{G(K) = \frac{X(K) + X^*(N-K)}{2}}$$

$$\boxed{H(K) = \frac{X(K) - X^*(N-K)}{2j}}$$

$\because \; x^*(n) \xrightarrow{DFT} X^*(-K)$ or $X^*(N-K)$ by periodicity property.

problems:-

1. find the 4-pt DFT's of two sequences $g(n)$ & $h(n)$ defined below using a single 4-pt DFT given $g(n) = \{1, 2, 0, 1\}$ & $h(n) = \{2, 2, 1, 1\}$

Soln:- Let $x(n) = g(n) + j\,h(n)$

$$x(n) = \{1+2j,\; 2+2j,\; j,\; 1+j\}$$

From the definition of DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \qquad \text{where } k = 0, 1, 2, 3$$

$$X(K) = [1+2j,\; 2+2j,\; j,\; 1+j] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 1+2j+2+2j+j+1+j = 4+6j \\ 1+2j+2-2j-j+j-1 = 2 \\ 1+2j-2-2j+j-1-j = -2 \\ 1+2j+2j-2-j-j+1 = 2j \end{bmatrix}$$

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$x(k) = \{4+6j,\ 2,\ -2,\ 2j\}$

$x^*(k) = \{4-6j,\ 2,\ -2,\ -2j\}$

$$G(k) = \frac{x(k) + x^*(-k)}{2} = \frac{x(k) + x^*(N-k)}{2}$$

at $k=0$; $G(0) = \frac{x(0)+x^*(0)}{2} = \frac{4+\cancel{6j}+4-\cancel{6j}}{2} = 4$

at $k=1$; $G(1) = \frac{x(1)+x^*(4-1)}{2} = \frac{x(1)+x^*(3)}{2} = \frac{2-2j}{2} = 1-j$

at $k=2$; $G(2) = \frac{x(2)+x^*(4-2)}{2} = \frac{x(2)+x^*(2)}{2} = \frac{-2-2}{2} = -2$

at $k=3$; $G(3) = \frac{x(3)+x^*(4-3)}{2} = \frac{x(3)+x^*(1)}{2} = \frac{2j+2}{2} = 1+j$

$$\boxed{\therefore G(k) = \{4,\ 1-j,\ -2,\ 1+j\}}$$

$$H(k) = \frac{x(k) - x^*(N-k)}{2j}$$

where $k = 0, 1, 2, 3$

at $k=0$; $H(0) = \frac{x(0)-x^*(0)}{2j} = \frac{\cancel{4}+6j-\cancel{4}+6j}{2j} = 6$

at $k=1$; $H(1) = \frac{x(1)-x^*(4-1)}{2j} = \frac{x(1)-x^*(3)}{2j} = \frac{2+2j}{2j} = \frac{1+j}{j}$

$H(1) = 1-j$

at $k=2$; $H(2) = \frac{x(2)-x^*(4-2)}{2j} = \frac{x(2)-x^*(2)}{2j} = \frac{2-2}{2j} = 0$

at $k=3$; $H(3) = \frac{x(3)-x^*(4-3)}{2j} = \frac{x(3)-x^*(1)}{2j} = \frac{2j-2}{2j}$

$H(3) = \frac{j-1}{j} = (1+j)$

$\therefore H(k) = \{6,\ (1-j),\ 0,\ (1+j)\}$

III) EXAM QUESTIONS SOLVED

1) let $x(n)$ be a finite length sequence with DFT $X(K) = \{10, -2+2j, -2, -2-2j\}$ using the properties of DFT, find the DFT's of the following sequences.

i) $x_1(n) = x((n+2))_4$

ii) $x_2(n) = x(4-n) = x((-n+4))_4$

Soln :- i) $x_1(n) = x((n+2))_4$

Taking DFT on both sides & using time shifting property

$X_1(K) = X(K) \cdot W_4^{-2k}$ where $k = 0, 1, 2, 3$

at $k=0$; $X_1(0) = X(0) \cdot W_4^0 = 10 \cdot (1) = 10$

$k=1$; $X_1(1) = X(1) \cdot W_4^{-2} = (-2+2j)(-1) = 2-2j$

$k=2$; $X_1(2) = X(2) \cdot W_4^{-4} = (-2)(1) = -2$

$k=3$; $X_1(3) = X(3) \cdot W_4^{-6} = (-2-2j)(-1) = 2+2j$

$\therefore \boxed{X_1(K) = \{10, 2-2j, -2, 2+2j\}}$

ii) $x_2(n) = x(4-n) = x((-n))_4$

Taking DFT on both sides & using time shifting property

$X_2(K) = X((-K))_4 = X(N-K)$ where $k = 0, 1, 2, 3$

at $k=0$; $X_2(0) = X((0))_4 = X(0) = 10$

$k=1$; $X_2(1) = X((-1))_4 = X(4-1) = X(3) = -2-2j$

$k=2$; $X_2(2) = X((-2))_4 = X(4-2) = X(2) = -2$

$k=3$; $X_2(3) = X((-3))_4 = X(4-3) = X(1) = -2+2j$

$\boxed{X_2(K) = \{10, -2-2j, -2, -2+2j\}}$

2) let $x(n)$ be a real sequence defined by $x(n) = \{1, 2, 3, -4\}$ without evaluating its DFT $X(K)$, find i) $\sum_{k=0}^{3} X(K)$ ii) $X(0)$

Soln:- i) $\sum_{k=0}^{3} X(K)$

By IDFT Eqn: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \cdot W_N^{-kn}$

Here $N=4$ & $n=0$

$$N \cdot x(n) = \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

$$4 \cdot x(0) = \sum_{k=0}^{3} X(k) \cdot W_4^{0}$$

$$4 \cdot (1) = \sum_{k=0}^{3} X(k)$$

$$\therefore \boxed{\sum_{k=0}^{3} X(k) = 4}$$

ii) $\cdot X(0)$

By DFT Eqn : $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

at $k=0$: $X(0) = \sum_{n=0}^{3} x(n) \cdot W_4^{0} = \sum_{n=0}^{3} x(n)$

$$X(0) = x(0) + x(1) + x(2) + x(3)$$

$$X(0) = 1 + 2 + 3 - 4$$

$$\boxed{X(0) = 2}$$

3) PT $\sum_{n=0}^{N-1} x^*(n) \cdot y(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot Y(k)$.

Soln : By IDFT Eqn ; $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$

Taking conjugate on both sides :

$$x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot W_N^{kn}$$

consider, $\sum_{n=0}^{N-1} x^*(n) \cdot y(n) = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot W_N^{kn} \cdot y(n)$.

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \boxed{\sum_{n=0}^{N-1} y(n) \cdot W_N^{kn}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot Y(k)$$ Hence proof

4) If $x(n) = \{1, 2, 0, 3, -2, 4, 7, 5\}$. Evaluate the following : i) $X(0)$ ii) $X(4)$ iii) $\sum_{k=0}^{7} X(k)$ iv) $\sum_{k=0}^{7} |X(k)|^2$

Soln :- $N = 8$

i) $X(0)$

From the definition of DFT ; $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

at $k = 0$;

$$X(0) = \sum_{n=0}^{7} x(n) \cdot W_8^0 = \sum_{n=0}^{7} x(n) \cdot (1)$$

$$X(0) = x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$$

$$X(0) = 1 + 2 + 0 + 3 + (-2) + 4 + 7 + 5$$

$$\boxed{X(0) = 20}$$

ii) $X(4)$

From DFT Eqn ; $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

at $k = 4$; $X(4) = \sum_{n=0}^{7} x(n) \; W_8^{4n}$

$$X(4) = x(0) W_8^0 + x(1) W_8^4 + x(2) W_8^8 + x(3) W_8^{12} + x(4) W_8^{16} + x(5) W_8^{20} + x(6) W_8^{24} + x(7) W_8^{28}$$

$$X(4) = (1)(1) + (2)(-1) + (0)(1) + (3)(-1) + (-2)(1) + (4)(-1) + (7)(1) + (5)(-1).$$

$$X(4) = 1 - 2 - 3 - 2 - 4 + 7 - 5$$

$$\boxed{X(4) = -8}$$

iii) $\sum_{k=0}^{7} X(k)$

By IDFT Eqn ; $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$

at $n = 0$; $8 \cdot x(0) = \sum_{k=0}^{7} X(k) \cdot W_8^0$

$$\sum_{k=0}^{7} X(k) = 8(1) = \underline{\underline{8}}$$

iv) $\sum_{k=0}^{7} |X(k)|^2$

From parseval's Theorem:

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

at $N=8$:

$$\sum_{k=0}^{7} |X(k)|^2 = 8\sum_{n=0}^{7} |x(n)|^2$$

$$\sum_{k=0}^{7} |X(k)|^2 = 8\{|x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2 + |x(4)|^2 + |x(5)|^2 + |x(6)|^2 + |x(7)|^2\}$$

$$\sum_{k=0}^{7} |X(k)|^2 = 8\{1+4+0+9+4+16+49+25\}$$

$$= 8\{108\} = \underline{864}$$

$$\therefore \boxed{\sum_{k=0}^{7} |X(k)|^2 = 864}$$

5) Let $x(n)$ be a finite length sequence with $X(k)=\{0, 1+j, 1, 1-j\}$ using the properties of DFT. find DFT's of the following sequen -ces.

i) $x_1(n) = e^{j\frac{\pi}{2}n}.\, x(n)$

ii) $x_2(n) = \cos\frac{\pi}{2}n.\, x(n)$

iii) $x_3(n) = x((n-1))_4$

Soln :- i) $x_1(n) = e^{j\frac{\pi}{2}n}.\, x(n) = e^{j\frac{2\pi}{4}n}.\, x(n)$

applying DFT & using circular frequency shifting property

$$X_1(k) = X((k-1))_4$$

at $k=0$: $X_1(0) = X((-1))_4 = X(4-1) = X(3)$

$X_1(0) = (1-j)$

at $k=1$: $X_1(1) = X((0))_4 = X(0) = 0$

at $k=2$: $X_1(2) = X((1))_4 = X(1) = 1+j$

at $k=3$: $X_1(3) = X((2))_4 = X(2) = 1$

$$\therefore X_1(k) = \{(1-j),\ 0,\ 1+j,\ 1\}$$

ii) $x_2(n) = \cos\frac{\pi}{2}n \cdot x(n) = \cos\frac{2\pi}{4}n \cdot x(n)$

Taking DFT & using circular frequency shifting property

$$X_2(K) = \frac{1}{2}\left[X((K-1))_4 + X((K+1))_4\right]$$

Since $\cos\frac{2\pi}{4}\cdot n = \dfrac{e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}}{2}$

at $k=0$; $X((0+1))_4 = X((1))_4 = X(1) = (1+j)$

$k=1$; $X((1+1))_4 = X((2))_4 = X(2) = 1$

$k=2$; $X((2+1))_4 = X((3))_4 = X(3) = (1-j)$

$k=3$; $X((3+1))_4 = X((4))_4 = X(0) = 0$

$$X_2(K) = \frac{1}{2}\left[X((K-1))_4 + X((K+1))_4\right]$$

at $k=0$; $X_2(0) = \frac{1}{2}\left[1-\cancel{j} + 1+\cancel{j}\right] = 1$

$k=1$; $X_2(1) = \frac{1}{2}\left[0+1\right] = \frac{1}{2}$

$k=2$; $X_2(2) = \frac{1}{2}\left[1+\cancel{j} + 1-\cancel{j}\right] = 1$

$k=3$; $X_2(3) = \frac{1}{2}\left[1+0\right] = 1/2$

$\therefore \boxed{X_2(K) = \{1, 1/2, 1, 1/2\}}$

iii) $x_3(n) = x((n-1))_4$

Taking DFT & using time shifting property

$$X_3(K) = X(K) \cdot W_4^{K}$$

at $k=0$; $X_3(0) = X(0)\cdot W_4^0 = 0\,(1) = 0$

at $k=1$; $X_3(1) = X(1)\cdot W_4^1 = (1+j)(-j) = -j-j^2 = 1-j$

at $k=2$; $X_3(2) = X(2)\cdot W_4^2 = (1)(-1) = -1$

at $k=3$; $X_3(3) = X(3)\cdot W_4^3 = (1-j)(j) = j-j^2 = 1+j$

$\boxed{X_3(K) = \{0, 1-j, -1, 1+j\}}$

(80)

6) Given $x(n)=\{1, 3/4, 1/2, 1/4\}$ with 4-point DFT $X(K)$, plot the sequence $y(n)$ whose DFT is $Y(K)=X(K).W_4^{3k}$

Sol^n:- $Y(K) = X(K). W_4^{3K}$

Taking IDFT & using circular time shift property

$y(n) = x((n-3))_4$

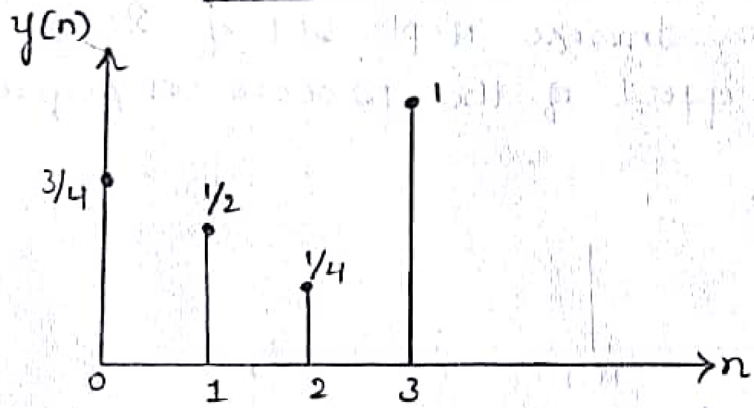
at $n=0$; $y(0) = x((-3))_4 = x(4-3) = x(1) = 3/4$

at $n=1$; $y(1) = x((-2))_4 = x(4-2) = x(2) = 1/2$

at $n=2$; $y(2) = x((-1))_4 = x(4-1) = x(3) = 1/4$

at $n=3$; $y(3) = x((0))_4 = x(0) = 1$

$$\boxed{y(n) = \{3/4, 1/2, 1/4, 1\}}$$



7) Let $X(K)$ denote 6pt DFT of real sequence $x(n)=\{1, -1, 2, 3, 0, 0\}$ without computing the IDFT, determine $y(n)$, whose 6pt DFT is given by $Y(K)=X(K). W_3^{2k}$

Sol^n:- $Y(K) = X(K). W_3^{2k} = Y(K) = X(K). W_6^{4k}$

Taking IDFT & using circular time shift property

$y(n) = x((n-4))_6$

at $n=0$; $y(0) = x((-4))_6 = x(6-4) = x(2)$

$y(0) = \underline{2}$

$\therefore x((-n))_N = x(N-n)$

at $n=1$; $y(1) = x((1-4))_6 = x((-3))_6 = x(6-3) = x(3)$

$y(1) = x(3) = \underline{3}$

at $n=2$; $y(2) = x((2-4))_6 = x((-2))_6 = x(6-2) = x(4)$

$y(2) = 0$

at $n=3$; $y(3) = x((3-4))_6 = x((-1))_6 = x(6-1) = x(5)$

$y(3) = 0$

at $n=4$; $y(4) = x((4-4))_6 = x(0) = 1$

at $n=5$; $y(5) = x((5-4))_6 = x(1) = -1$

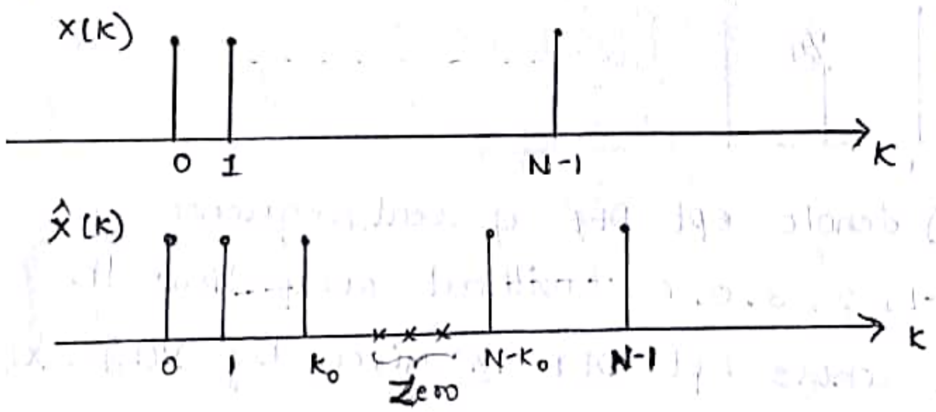
$\therefore y(n) = \{2, 3, 0, 0, 1, -1\}$

8) Let $X(K)$, $0 \le k \le N-1$ be the N-pt DFT of the sequence $x(n)$, $0 \le n \le N-1$, we define.

$$\hat{X}(K) = \begin{cases} X(K), & 0 \le k \le k_0 \;,\; N-k_0 \le k \le N-1 \\ 0 & k_0 < k < N-k_0 \end{cases}$$

and we compute the Inverse N-pt DFT of $\hat{X}(K)$. $0 \le K \le N-1$ what is the effect of the process on sequence $x(n)$. Explain.

Soln :-



W.K.T $\quad X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$ where $k = 0, 1, \ldots (N-1)$

$\hat{X}(K)$ is a DFT sequence, having sequence from 0 to k_0 & $(N-K_0)$ to $(N-1)$

$$\text{IDFT}\{\hat{X}(K)\} = \frac{1}{N}\left[\sum_{k=0}^{K_0} \hat{X}(K) \cdot W_N^{kn} + \sum_{k=N-K_0}^{N-1} \hat{X}(K) \cdot W_N^{kn}\right]$$

Since $X(K)$ is symmetric, & the samples are retained 0 to K_0 & $(N-K_0)$ to $(N-1)$, it is equivalent to low pass filtering of $x(n)$.

Q) Find the N-point DFT of the following signals also find its energy using parseval's theorem. (82)

$$x(n) = \begin{cases} 1 , & n = 0, 2, 4 \ldots\ldots (N-2) \\ 2 , & n = 1, 3, 5, \ldots\ldots (N-1) \end{cases}$$

Sol^n :- From the definition of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$X(k) = \sum_{n=0}^{N-1} x(2n)\, W_N^{k2n} + \sum_{n=0}^{N-1} x(2n+1)\, W_N^{k(2n+1)}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} 1 \cdot W_N^{2kn} + \sum_{n=0}^{\frac{N}{2}-1} 2 \cdot W_N^{k(2n+1)}$$

$$W_N^{k(2n+1)} = W_N^{k \cdot 2n} \cdot W_N^{k}$$

$$\therefore \quad X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left(W_N^{2kn} + 2 W_N^{2kn} \cdot W_N^{k} \right)$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} W_N^{2kn} \left[1 + 2W_N^{k}\right] = \sum_{n=0}^{(N/2-1)} e^{-j\frac{2\pi}{N} \cdot 2kn} \left[1 + 2W_N^{k}\right]$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} e^{-j\frac{2\pi}{(N/2)} \cdot kn} \left[1 + 2W_N^{k}\right]$$

$$X(k) = \frac{1 - \left(e^{j\frac{2\pi}{N/2} k}\right)^{N/2}}{1 - e^{-j\frac{2\pi}{N/2} k}} \left(1 + 2W_N^{k}\right)$$

$$X(k) = \frac{1 - e^{-j\frac{2\pi k}{N/2} N/2}}{1 - e^{-j\frac{2\pi k}{N/2}}} \left[1 + 2W_N^{k}\right] \quad \forall\ k = 0, 1, \ldots (N-1)$$

at $k = 0$; solving

$$X(0) = \frac{N}{2}[1+2] = N + \frac{N}{2} = \frac{3N}{2}$$

at $k = \frac{N}{2}$, $X\left(\frac{N}{2}\right) = -N/2$

$$X(k) = \begin{cases} N + \frac{N}{2} & , \text{for } k = 0 \\ \frac{-N}{2} & , \text{for } k = \frac{N}{2} \\ 0 & , \text{otherwise} \end{cases}$$

W.K.T

$$\sum_{n=0}^{N/2-1} a^n = \frac{1 - a^{N/2-1+1}}{1 - a} = \frac{N}{2} \quad \forall |a| < 1$$

To find energy :- I Method :-

From parseval's theorem , w.k.T

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \left[\left(\frac{3N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 \right]$$

$$= \frac{1}{N} \left[\frac{9N^2}{4} + \frac{N^2}{4} \right] = \frac{1}{N} \cdot \frac{N^2}{4} \cdot 10$$

$$= \frac{5}{2} N = 2.5N \text{ joules}$$

II Method :-

x(n) can be expressed as

$$x(n) = 1.5 - (-1)^n . (0.5)$$

(or) $x(n) = x_1(n) - x_2(n)$

applying DFT ; $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

$$X(k) = \sum_{n=0}^{N-1} \{1.5 - (-1)^n 0.5\} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} 1.5\, W_N^{kn} - \sum_{n=0}^{N-1} (-1)^n\, 0.5 . W_N^{kn}$$

Solving. we get ;

$$X(k) = 1.5N . \delta(k) - 0.5N . \delta(k - N/2)$$

10) If X(k) is the N-pt DFT of the signal x(n). Indicate the DFT of

i) $x((n-1))_N + x((1-n))_N$

ii) $x(n) . x^*(-n)$.

Sol^n:- i) w.k.T $x(n) \underset{IDFT}{\overset{DFT}{\longleftrightarrow}} X(k)$

$$x((n-1))_N \longrightarrow X(k) . W_N^{-k}$$

$$x((-n))_N \longrightarrow X((-k))_N$$

$$x((1-n))_N \longrightarrow X((-k))_N . W_N^{+k}$$

$\because\ x((1-n))_N = x((-n+1))_N = x((\{-n-(-1)\}))_N$

$\therefore\ x((n-1))_N + x((1-n))_N \longrightarrow X(k)\cdot W_N^{-k} + X((-k))_N \cdot W_N^{-k}$

$$\boxed{x((n-1))_N + x((1-n))_N \longrightarrow W_N^{-k}\left(X(k) + X((-k))_N\right)}$$

ii) From the property of frequency domine convolution (multiplication in time domine).

W.K.T $\quad x_1(n)\cdot x_2(n) \longrightarrow \frac{1}{N}\left[X_1(k) \; Ⓝ \; X_2(k)\right]$

$x(n)\cdot x^*(-n) \longrightarrow \frac{1}{N}\left[X(k) \; Ⓝ \; X^*(k)\right]$

11) x(n) is 8-length signal with circular odd symmetry if its four samples are $\{0, 2, 2, -2\}$. find x(n).

Sol^n:- From the circular odd symmetry property

W.K.T $\quad x((N-n)) = -x(n)$

given $\quad N=8;\ x((8-n)) = -x(n)$

$x(0)=0$ But 'n' takes values from 0 to (N-1)

$x(1)=2$ i.e, n=0 to 7

$x(2)=2$

$x(3)=-2$

at $n=4$; $\quad x(8-4) = -x(4) = 0$

at $n=5$; $\quad x(8-5) = -x(3) = -(-2) = 2$

at $n=6$; $\quad x(8-6) = -x(2) = -(2) = -2$

at $n=7$; $\quad x(8-7) = -x(1) = -(2) = -2$

$\therefore\ x(n) = \{0, 2, 2, -2, 0, 2, -2, -2\}$

12) Let x(n) be a given sequence with N-pt DFT of X(k). Denote the operation of finding DFT as follows; $X(k) = F\{x(n)\}$ what is the resulting sequence x(n) operated upon four times. i.e, Determine Y(k) where

$$Y(k) = F\{F\{F\{F\{x(n)\}\}\}\}$$

Sol^n:- $X(K) = F\{x(n)\}$

then, $x(n) \xrightarrow{DFT} X(K)$

$X(K) \xrightarrow{IDFT} x(n) \cdot \frac{1}{N}$

$\therefore F^{-1} = \frac{1}{N} \cdot (F)$

$\therefore F = N F^{-1}$

$y(K) = F\{F\{F\{F\{x(n)\}\}\}\}$

$y(K) = NF^{-1}\{F\{NF^{-1}\{F\{x(n)\}\}\}\}$

$$\boxed{y(K) = N^2 \cdot x(n)}$$

13) consider the Sequence $x_1(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4\}$
$x_2(n) = \{\underset{\uparrow}{0}, 1, 0, 0, 0\}$, $\delta(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 0\}$ & their 5 pt DFT

i) Determine a sequence $y(n)$ so that $Y(K) = X_1(K) \cdot X_2(K)$

ii) Is there a Sequence $x_3(n)$ such that $S(K) = X_1(K) \cdot X_3(K)$

Sol^n:- i) $Y(K) = X_1(K) \cdot X_2(K)$

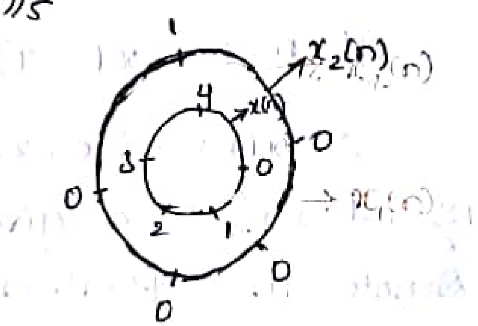
Taking IDFT; $y(n) = x_1(n) \, Ⓝ \, x_2(n)$

Here $N=5$; From the definition of circular convolution

$$y(n) = \sum_{n=0}^{N-1} x_1(m) \cdot x_2((n-m))_N$$

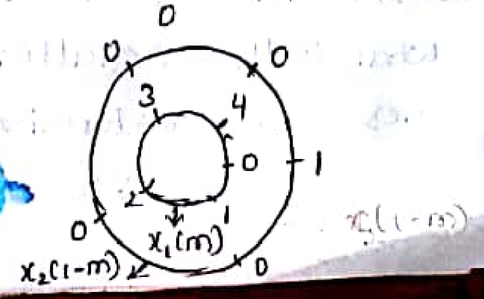
at $n=0$; $y(0) = \sum_{n=0}^{4} x_1(m) \cdot x_2((-m))_5$

$\underline{\underline{y(0) = 4}}$



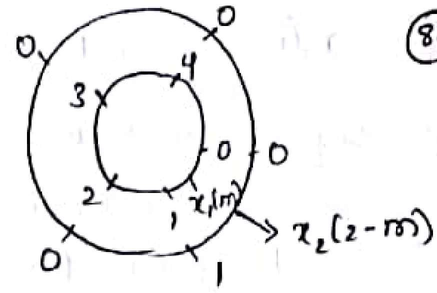
at $n=1$; $y(1) = \sum_{m=0}^{4} x_1(m) \cdot x_2((1-m))_5$

$y(1) = 0$



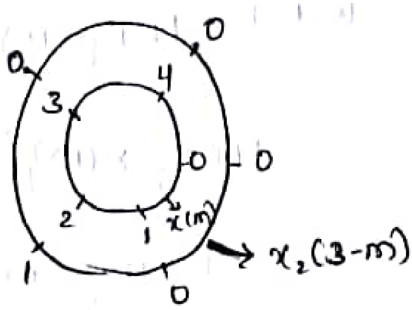
at $n=2$; $y(2) = \sum_{m=0}^{4} x_1(m) \cdot x_2((2-m))_5$

$y(2) = 1$



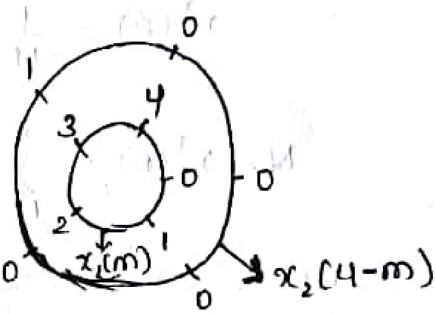
at $n=3$; $y(3) = \sum_{m=0}^{4} x_1(m) \cdot x_2((3-m))_5$

$y(3) = 2$



at $n=4$; $y(4) = \sum_{m=0}^{4} x_1(m) \cdot x_2((4-m))_5$

$y(4) = 3$



$\therefore \boxed{y(n) = \{4, 0, 1, 2, 3\}}$

ii) $S(k) = X_1(k) \cdot X_3(k)$

To find $x_3(n)$, taking I.D.F.T

$s(n) = x_1(n) \; (5) \; x_3(n)$

let $x_3(n) = \{x_0, x_1, x_2, x_3, x_4\}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$4x_1 + 3x_2 + 2x_3 + x_4 = 1$

$x_0 + 4x_2 + 3x_3 + 2x_4 = 0$

$2x_0 + x_1 + 4x_3 + 3x_4 = 0$

$3x_0 + 2x_1 + x_3 + 4x_4 = 0$

$4x_0 + 3x_1 + 2x_3 + x_4 = 0$

By solving we get

$x_3(n) = \{-0.18, 0.22, 0.02, 0.02, 0.02\}$

14) Suppose that we are given a program to find the DFT of a complex valued sequence $x(n)$. How can we use this program to find the inverse DFT of $X(K)$.

Sol^n: w.r.t

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad \text{——— (1)}$$

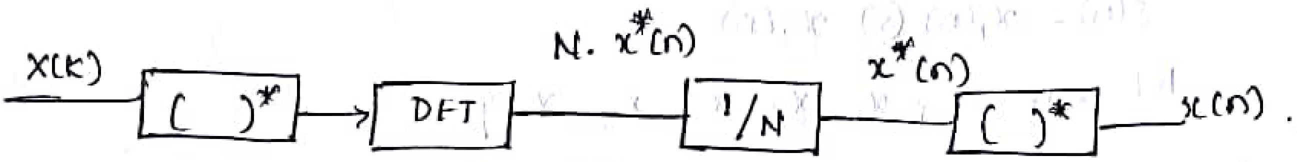
$$IDFT\{X(K)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \cdot W_N^{-kn} \quad \text{——— (2)}$$

Taking conjugate of eqn (2)

$$x^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K) \cdot W_N^{kn}$$

$$N \cdot x^*(n) = \sum_{K=0}^{N-1} X^*(K) \cdot W_N^{kn}$$

i.e., we can use DFT program to compute IDFT by calculating $X^*(K)$ & passing through DFT program. we get o/p $N \cdot x^*(n)$ and scale by $(\frac{1}{N})$ & again take conjugate.



15) A design is having a number of 8-point FFT chips. Show explicitly how we should inter connect three chips in order to compute a 24-pt DFT.

Sol^n:- Create 3 sub-sequences of length 8 each

$x(0), x(3), x(6) \ldots\ldots x(21)$

$x(1), x(4), x(7) \ldots\ldots x(22)$

$x(2), x(5), x(8) \ldots\ldots x(23)$

$$X(K) = \sum_{n=0,3,6}^{21} x(n) \cdot W_N^{kn} + \sum_{n=1,4,7}^{22} x(n) \cdot W_N^{kn} + \sum_{n=2,5,8}^{23} x(n) \cdot W_N^{kn}$$

(or)

$$X(K) = \sum_{m=0}^{7} x(3m)\, W_N^{3km} + \sum_{m=0}^{7} x(3m+1)\, W_N^{(3m+1)k} + \sum_{m=0}^{7} x(3m+2) W_N^{(3m+2)k}$$

$$\underset{\text{24-pt}}{X(K)} = \underset{\text{8-pt}}{X_1(K)} + \underset{\text{8-pt}}{X_2(K)} + \underset{\text{8-pt}}{X_3(K)}$$

16> Evaluate the Sum $S = \sum_{n=0}^{15} x_1(n) . x_2^*(n)$.

when $x_1(n) = \cos\left(\frac{3\pi n}{8}\right)$, $X_2(K) = 3$, $0 \le K \le 15$

Soln:- $X_2(K) = 3$

Taking IDFT ; $x_2(n) = 3 . \delta(n)$

$$x_2^*(n) = 3 . \delta(n)$$

$$S = \sum_{n=0}^{15} \left\{ \cos \frac{3\pi n}{8} \right\} . 3\, \delta(n)$$

$$\boxed{S = 3}$$

17> Let $N = 4M$ where M is an Integer . &

$$X(K) = \begin{cases} 0.5 , & K = M \\ 0.5 , & K = 3M \\ 0 & \text{otherwise} \end{cases}$$

compute $x(n)$. let $y(n) = (-1)^n . x(2n)$, $0 \le n \le 2M-1$

compute $Y(K)$.

Soln:- W.K.T $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) . W_N^{-kn}$

$$x(n) = \frac{1}{N} \sum_{k=0}^{4M-1} X(K) . W_{4M}^{-kn}$$

$$x(n) = \frac{1}{N} \left(0.5\, W_{4N}^{-Nn} + 0.5\, W_{4N}^{-2Mn} \right)$$

$$W_{4M}^{-Mn} = e^{j \frac{2\pi}{4M} . Mn} = e^{j \frac{2\pi}{4} n}$$

$$W_4^{-2mn} = e^{j \frac{2\pi}{4M} 3mn} = e^{j \frac{6\pi}{4} n}$$

$$x(n) = \frac{1}{N}\left[e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}\right]$$

$$x(n) = \frac{1}{N}\cos\frac{2\pi}{4}n$$

To find $x(2n)$

$$x(2n) = \frac{1}{N}\cos\left(\frac{2\pi \cdot 2n}{4}\right) = \frac{1}{N}(-1)^n$$

given; $y(n) = (-1)^n \cdot x(2n)$

$$y(n) = (-1)^n \cdot \frac{1}{N}(-1)^n$$

$$y(n) = \frac{1}{N}(-1)^{2n}$$

$$\therefore \boxed{y(n) = \frac{1}{N}}$$

<u>To find $y(k)$</u>

W.K.T $\text{DFT}\{y(n)\} = y(k) = \sum_{n=0}^{N-1} y(n) \cdot W_N^{kn}$

given; $y(n) = \frac{1}{N}$, $0 \le n \le (2M-1)$

$$y(k) = \sum_{n=0}^{2m-1} \frac{1}{N} \cdot W_{2M}^{kn} = \frac{1}{N}\left[\frac{1-(W_{2m}^{k})^{2m-1+1}}{1-W_{2m}^{k}}\right]$$

$$y(k) = \begin{cases} \frac{1}{N} \cdot 2m, & \text{at } k=0 \\ 0, & k \neq 0 \end{cases}$$

given $N = 4M$;

$$y(k) = \frac{1}{4M} \cdot 2M = \frac{1}{2}\delta(k)$$

$$\therefore \boxed{y(k) = \frac{1}{2} \cdot \delta(k)}$$

18) ST the product of two complex numbers $(a+jb)$ & $(c+jd)$ can be performed with 3 real multiplications and five additions.

<u>Soln</u>:

$$(a+jb)(c+jd) = ac + jbc + jad - bd$$

$$= (ac - bd) + j(bc + ad)$$

$$= \{ac - bd + ad - ad\} + j\{bc + ad + bd - bd\}$$

$$= d(a-b) + a(c-d) + j\{d(a-b) + b(c+d)\}$$

$\therefore$ we have 3 multiplications

$(a-b)d$

$(c-d)a$

$(c+d)b$

We have 5 additions

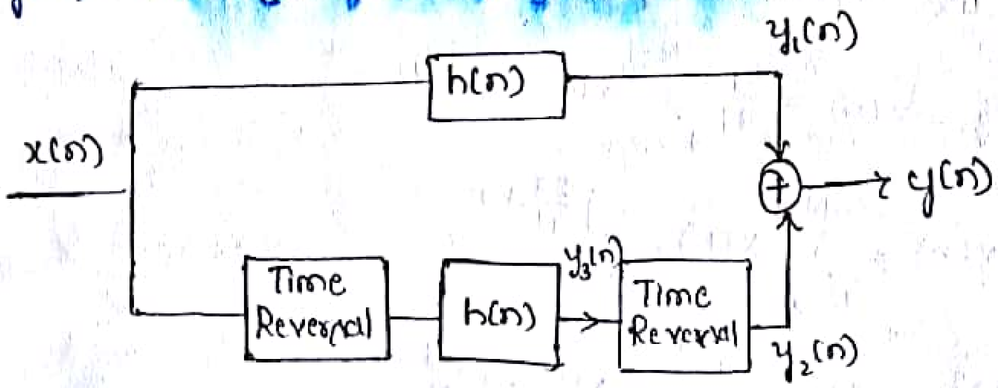
$$[(a-b)d + a(c-d)] + j[(a-b)d + (c+d)b]$$

$(a-b)$

$(c-d)$ + 2 additions.

$(c+d)$

21) consider the system shown in fig where $x(n)$, $h(n)$ & $y(n)$ are finite length real sequences use DFT & IDFT's to compute $y(n)$ interms of $x(n)$ & $h(n)$.



Soln:- From the above block diagram

$$y_1(n) = x(n) \, Ⓝ \, h(n)$$

Taking DFT ; $Y_1(K) = X(K) \cdot H(K)$

$$y_3(n) = x((-n))_N \, Ⓝ \, h(n)$$

Taking DFT ; $Y_3(K) = X((-K))_N \cdot H(K)$

$$y_2(n) = y_3((-n))_N$$

Taking DFT : $Y_2(k) = Y_3((-k))_N$

$$y(n) = y_1(n) + y_2(n)$$

Taking DFT : $Y(k) = Y_1(k) + Y_2(k)$

$$Y(k) = X(k) \cdot H(k) + Y_3((-k))_N$$

$$Y(k) = X(k) \cdot H(k) + X(k) \cdot H((-k))_N$$

$$Y(k) = X(k) \left[H(k) + H((-k))_N \right]$$

Taking IDFT : $\boxed{y(n) = x(n) \, (N) \, [h(n) + h((-n))_N]}$

22) Evaluate the following function without computing the DFT $\sum_{k=0}^{11} e^{-j\frac{4\pi k}{6}} \cdot X(k)$ for a given 12-point sequence x(n).

$$x(n) = \{8, 4, 7, -1, 2, 0, -2, -4, -5, 1, 4, 3\}$$

Soln :- given $\sum_{k=0}^{11} X(k) \cdot e^{-j\frac{4\pi}{6}k} = \sum_{k=0}^{11} X(k) \cdot e^{-j\frac{2\pi}{12} \cdot 4k}$

From the definition of IDFT :

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N} \cdot nk}$$

$$N \cdot x(n) = \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N} \cdot kn}$$

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

$$\text{IDFT}\{X(k) \cdot e^{-j\frac{2\pi}{12} \cdot 4k}\} = \frac{1}{12} \sum_{k=0}^{11} X(k) \cdot e^{j\frac{2\pi}{12} kn} \cdot e^{-j\frac{2\pi}{12} 4k}$$

using time shifting property

$$x((n-4))_{12} = \frac{1}{12} \sum_{k=0}^{11} X(k) \cdot e^{-j\frac{2\pi}{12} \cdot 4k} \cdot e^{+j\frac{2\pi}{12} nk}$$

at $n=0$;

$$x((-4))_{12} = \frac{1}{12} \sum_{k=0}^{11} X(k) \cdot e^{-j\frac{2\pi}{12}4k} \cdot e^{0}$$

But $x((-4))_{12} = x(12-4) = x(8) = -5$

$$\therefore \sum_{k=0}^{11} X(k) \cdot e^{-j\frac{2\pi}{12}4k} = -5 \times 12 = -60$$

23) let $x(n)$ be a length N real sequence with N-pt DFT $X(K)$. PT $X(0)$ is real.

Soln:- W.K.T $X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$

at $k=0$; $X(0) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{0}$

$$X(0) = \sum_{n=0}^{N-1} x(n)$$

Since $x(n)$ is real, $X(0)$ is real.

24) A 498 point DFT of a real valued sequence $x(n)$ has the following DFT samples.

$X(0) = 2$	$X(K_2) = -4.7 + j1.9$
$X(11) = 7 + j3.1$	$X(249) = 2.9$
$X(K_1) = -2.2 - j1.5$	$X(309) = -4.7 - j1.9$
$X(112) = 3 + j0.7$	$X(K_3) = 3 - j0.7$
	$X(412) = -2.2 + j1.5$
	$X(K_4) = 7 - j3.1$

The other samples have a value zero. find the values of K_1, K_2, K_3, K_4.

Soln:- for a real valued sequence

WKT $X(K) = X^*(N-K)$

$N = 498$

a) for $K = 412$, $X(412) = X^*(498 - 412) = X^*(86)$

$X(412) = X^*(86)$

$X^*(86) = -2.2 + j1.5$

$\therefore X(86) = -2.2 - j1.5 = X(K_1)$

$\therefore \boxed{k_1 = 86}$

b) for $k = 309$, $X(309) = X^*(498-309) = X^*(189)$

$$-4.7 - j1.9 = X^*(189)$$

$$\therefore X(189) = -4.7 + j1.9 = X(k_2)$$

$$\therefore \boxed{k_2 = 189}$$

c) for $k = 112$, $X(112) = X^*(498-112) = X^*(386)$

$$3 + j0.7 = X^*(386)$$

$$\therefore X(386) = 3 - j0.7 = X(k_3)$$

$$\therefore \boxed{k_3 = 386}$$

d) for $k = 11$, $X(11) = X^*(498-11) = X^*(487)$

$$7 + j3.1 = X^*(487)$$

$$X(487) = 7 - j3.1 = X(k_4)$$

$$\therefore \boxed{k_4 = 487}$$

$\therefore$ $k_1 = 86$
$k_2 = 189$
$k_3 = 386$
$k_4 = 487$

25) for the two sequences $x_1(n) = \{2, 1, 1, 2\}$ & $x_2(n) = \{1, -1, -1, 1\}$ compute the circular convolution using DFT & IDFT.

Soln :-

$$X_1(k) = DFT\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n)\, W_N^{kn}$$

$$X_1(k) = [2 \quad 1 \quad 1 \quad 2] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} = \begin{bmatrix} 2+1+1+2 = 6 \\ 2-j-1+2j = 1+j \\ 2-1+1-2 = 0 \\ 2+j-1-2j = 1-j \end{bmatrix}$$

$\therefore\ X_1(K) = \{6,\ 1+j,\ 0,\ 1-j\}$

$$X_2(K) = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} 1-1-1+1 = 0 \\ 1+j+1+j = 2+2j \\ 1+1-1-1 = 0 \\ 1-j+1-j = 2-2j \end{bmatrix}$$

$X_2(K) = \{0,\ 2+2j,\ 0,\ 2-2j\}$

$y(n) = x_1(n) \text{ Ⓝ } x_2(n)$

Taking DFT : $Y(K) = X_1(K) \cdot X_2(K)$

$Y(K) = \{6,\ 1+j,\ 0,\ 1-j\}\{0,\ 2+2j,\ 0,\ 2-2j\}$

$Y(K) = \{0,\ 4j,\ 0,\ -4j\}$

Taking IDFT : $y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(K) \cdot W_N^{-kn}$

$$y(n) = \frac{1}{4}\begin{bmatrix} 0 & 4j & 0 & -4j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} = \begin{bmatrix} 0+4j+0-4j = 0 \\ 0-4-0-4 = -8 \\ 0-4j+0+4j = 0 \\ 0+4-0+4 = 8 \end{bmatrix} \cdot \frac{1}{4}$$

$y(n) = \frac{1}{4}[0\ \ -8\ \ 0\ \ 8]$

$\therefore\ \boxed{y(n) = \{0,\ -2,\ 0,\ 2\}}$

26) Let $x_p(n)$ be a periodic sequence with fundamental period N. If the N-point DFT$\{x_p(n)\} = X_1(K)$ & 3N point DFT$\{x_p(n)\} = X_3(K)$

i) Find the relationship b/w $X_1(K)$ & $X_3(K)$.

ii) Verify the above result for $\{2, 1\}$ & $\{2, 1, 2, 1, 2, 1\}$

Soln:- i) W.K.T N-pt DFT$\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x_p(n) \cdot W_N^{kn}$

N-pt DFT$\{x_p(n)\} = X_1(K) = \sum_{n=0}^{N-1} x_p(n) \cdot W_N^{kn}$

$$3N\text{-pt DFT}\{x_p(n)\} = X_3(k) = \sum_{n=0}^{3N-1} x_p(n) \cdot W_{3N}^{kn}$$

$$= \sum_{n=0}^{N-1} x_p(n) \cdot W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n+N)\, W_{3N}^{(n+N)k}$$

$$+ \sum_{n=0}^{N-1} x_p(n+2N)\, W_{3N}^{(n+2N)k}$$

$$= \sum_{n=0}^{N-1} x_p(n) \cdot W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n) \cdot W_{3N}^{kn} \cdot W_{3N}^{2Nk} + \sum_{n=0}^{N-1} x_p(n) \cdot W_{3N}^{nk} \cdot W_{3N}^{2N}$$

i.e, from periodicity property : $x_p(n+N) = x_p(n)$

& $x_p(n+2N) = x_p(n)$

& $W_{3N}^{Nk} = W_3^{k}$

$W_{3N}^{2Nk} = W_3^{2k}$

$W_{3N}^{nk} = W_N^{n \cdot \frac{k}{3}}$

$$\therefore \quad X_3(k) = \sum_{n=0}^{N-1} x_p(n) \left[1 + W_3^{k} + W_3^{2k}\right] W_N^{n \cdot \frac{k}{3}}$$

$$X_3(k) = \left[1 + W_3^{k} + W_3^{2k}\right] \sum_{n=0}^{N-1} x_p(n) \cdot W_N^{n \cdot k/3}$$

$$\boxed{X_3(k) = \left[1 + W_3^{k} + W_3^{2k}\right] X_1\left(\frac{k}{3}\right)}$$

ii) given $x_1(n) = \{2, 1\}$

$x_3(n) = \{2, 1, 2, 1, 2, 1\}$

Taking DFT : $X_1(k) = 2 + W_2^{k}$

$$X_3(k) = 2 + W_6^{k} + 2W_6^{2k} + W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

$$W_6^{k} = W_{2\times 3}^{k} = W_2^{k/3}$$

$$X_3(k) = \left[2 + W_2^{k/3}\right] + W_6^{2k}\left[2 + W_6^{k}\right] + W_6^{4k}\left[2 + W_6^{k}\right]$$

$$X_3(k) = \left[2 + W_2^{k/3}\right] + W_6^{2k}\left[2 + W_2^{k/3}\right] + W_6^{4k}\left[2 + W_2^{k/3}\right]$$

$$X_3(K) = [2 + W_2^{K/3}][1 + W_6^{2K} + W_6^{4K}]$$

$$X_3(K) = X_1\left(\frac{K}{3}\right)[1 + W_3^{K} + W_3^{2K}]$$ hence verified.

27) Let $x(n) = \{1, 2, 3, 4\}$ with $X(K) = \{10, -2+2j, -2, -2-2j\}$ find the DFT of $x_1(n) = \{1, 0, 2, 0, 3, 0, 4, 0\}$ without actually calculating the DFT.

Soln :- $$DFT\{x_1(n)\} = X_1(K) = \sum_{n=0}^{N-1} x_1(n) \cdot W_N^{kn}$$

Dividing the above summation into even & odd parts

$$X_1(K) = \sum_{n=0}^{3} x_1(2n) \cdot W_N^{2nk} + \sum_{n=0}^{3} x_1(2n+1) W_N^{(2n+1)K}$$

Here $\underline{N=8}$

$$X_1(K) = \sum_{n=0}^{3} x_1(2n) W_8^{2nk} + 0$$ because odd samples are zero.

$$X_1(K) = \sum_{n=0}^{3} x(n) \cdot W_{8/2}^{nk} = \sum_{n=0}^{3} x(n) W_4^{nk}$$

or $$X_1(K) = \begin{cases} X(K), & n = 0, 1, 2, 3 \\ X(K), & n = 4, 5, 6, 7 \end{cases}$$

$$\therefore X_1(K) = X(K)$$

$$X_1(K) = \{10, -2+2j, -2, -2-2j, 10, -2+2j, -2, -2-2j\}$$

using periodicity property.

$$X_1(4) = X(4) = X(0)$$

28) Determine N-pt DFT of $x(n)$

$$x(n) = \frac{1}{2} + \cos^2\left(\frac{2\pi n}{N}\right)$$ where $n = 0, 1 \ldots\ldots (N-1)$

Soln :-
$$x(n) = \frac{1}{2} + \cos^2\left(\frac{2\pi n}{N}\right)$$

W.K.T $\quad \cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$x(n) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\cos\frac{2\pi \cdot 2n}{N}$$

$$x(n) = 1 + \frac{1}{2}\cos\frac{2\pi}{N}\cdot 2n$$

W.K.T . $\quad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$$x(n) = 1 + \frac{1}{4}\left[e^{j\frac{2\pi}{N}2n} + e^{-j\frac{2\pi}{N}2n}\right]$$

Taking DFT on both sides & using sampling property

$$DFT\{1\} = N\cdot\delta(k).$$

$$\frac{1}{4}\cdot W_N^{-2n} = e^{j\frac{2\pi}{N}\cdot 2n} \longleftrightarrow \frac{1}{4}N\,\delta(k-2)$$

$$\frac{1}{4}\cdot W_N^{2n} = e^{-j\frac{2\pi}{N}\cdot 2n} \longleftrightarrow \frac{1}{4}N\,\delta(k+2)$$

$$\therefore\ X(k) = N\cdot\delta(k) + \frac{1}{4}\cdot N\cdot\delta(k-2) + \frac{1}{4}\cdot N\cdot\delta(k+2).$$

29) Let $X(k)$ denote the N-point DFT of an N-pt sequence $x(n)$. If the DFT of $X(k)$ is computed to obtained a sequence $x_1(n)$. determine $x_1(n)$ interms of $x(n)$.

W.K.T $\quad DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi}{N}kn}$

Let

$$x_1(m) = DFT\{X(k)\}$$
$$= \sum_{m=0}^{N-1} X(k)\, e^{-j\frac{2\pi}{N}km}$$
$$= \sum_{m=0}^{N-1}\left\{\sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi}{N}kn}\right\}\cdot e^{-j\frac{2\pi}{N}km}$$
$$= \sum_{n=0}^{N-1} x(n)\cdot \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(m+n)k}$$

W.K.T $e^{-j\frac{2\pi}{N}k(m+n)} = \begin{cases} 1, & \text{if } m+n=0 \text{ (or) } m+n=PN \\ 0, & \text{otherwise} \end{cases}$

consider $m+n=0$ (or) $m=-n$

$$x_1(m) = x(-m).N$$

(or) $x_1(n) = N \cdot x(-n)$.

30) let $x(n)$ be a N-pt sequence with an N-pt DFT $X(K)$

i) if $x(n)$ is symmetric, satisfying the condition.

$x(n) = x(N-1-n)$. ST $X(\frac{N}{2})=0$ for N even.

ii) if $x(n)$ is antisymmetric, satisfying the condition

$x(n) = -x(N-1-n)$, ST $X(0)=0$ for N even

iii) if N is even & $x(n) = -x(n+\frac{N}{2})$ then

P.T $X(K)=0$ for k=even.

Soln:-

i) W.K.T $\quad X(K) = \sum_{n=0}^{N-1} x(n) . W_N^{kn}$

Since N is even

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) . W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) . W_N^{kn}$$

put $m=N-1-n$ in 2nd summation

$$X(K) = \sum_{n=0}^{N/2-1} x(n) . W_N^{kn} + \sum_{m=0}^{N/2-1} x(N-1-m) . W_N^{k(N-1-m)} \quad \text{—(1)}$$

Since $x(n)$ is symmetric:

$$x(N-1-n) = x(n)$$

$$X(K) = \sum_{n=0}^{N/2-1} x(n) \left\{ W_N^{kn} + W_N^{k(N-1-m)} \right\}$$

at $k = N/2$:

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N/2-1} x(n)\left[e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{\cancel{2}}n} + e^{-j\frac{\cancel{2}\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{\cancel{2}}(N-1-n)}\right]$$

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N/2-1} x(n)\left\{e^{-j\pi n} + e^{-j\pi(N-1-n)}\right\}$$

$$= \sum_{n=0}^{N/2-1} x(n)\cdot e^{-j\pi\frac{(N-1)}{2}}\left\{e^{-j\pi\left(\frac{N-1}{2}-n\right)} + e^{-j\pi\left(\frac{N-1}{2}-n\right)}\right\}$$

$$X\left(\frac{N}{2}\right) = e^{-j\pi\left(\frac{N-1}{2}\right)}\cdot\sum_{n=0}^{N/2-1} 2\cdot x(n)\cdot\cos\pi\left(\frac{N-1}{2}-n\right)$$

$$\therefore \boxed{X\left(\frac{N}{2}\right) = 0}$$

ii) consider eqn (1)

$$X(k) = \sum_{n=0}^{N/2-1} x(n)\cdot e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N/2-1} x(N-1-n)\cdot e^{-j\frac{2\pi}{N}k(N-1-n)}$$

if $x(n)$ is antisymmetric

$$x(N-1-n) = -x(n)$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n)\cdot e^{-j\frac{2\pi}{N}kn} - \sum_{n=0}^{N/2-1} x(n)\cdot e^{-j\frac{2\pi}{N}\pi(N-1-n)k}$$

at $k=0$;

$$X(0) = \sum_{n=0}^{N/2-1} x(n) - \sum_{n=0}^{N/2-1} x(n) = 0$$

$$\therefore \boxed{X(0) = 0}$$

iii) W.K.T $\quad X(k) = \sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi}{N}kn}$

since $N = $ even

$$X(K) = \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn} + \sum_{n=N/2}^{N-1} x(n) . e^{-j\frac{2\pi}{N}kn}$$

put $m = n - \frac{N}{2}$ in second summation, simplify Replace m by n in second summation:

$$X(K) = \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2})\, e^{-j\frac{2\pi}{N}k(n+\frac{N}{2})}$$

given $x(n) = -x(n+\frac{N}{2})$

or $x(n+\frac{N}{2}) = -x(n)$

$$X(K) = \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn} - \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn} . e^{-j\frac{2\pi}{\cancel{N}}.k.\frac{\cancel{N}}{2}}$$

$e^{-j\pi k} = (e^{-j\pi})^k = 1$ because k is even

$$X(K) = \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn} - \sum_{n=0}^{N/2-1} x(n) . e^{-j\frac{2\pi}{N}kn}$$

$\boxed{X(K) = 0}$ for $k = \text{even}$.

July 2015 (9M)

31) compute the DFT of the sequence $x(n) = \cos\left(\frac{n\pi}{4}\right)$ for N=4, plot $|X(K)|$ & $\angle X(K)$

Sol^n: given N=4, $n = 0,1,2,3.$

$\therefore x(n) = \{1, 0.707, 0, -0.707\}$

From the def^n of DFT W.K.T

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}.$$

$$X(K) = \sum_{n=0}^{3} x(n) W_4^{kn}.$$

$$X(K) = x(0) + x(1) W_4^{K} + x(2) W_4^{2K} + x(3) W_4^{3K}$$

$X(K) = 1 + 0.707\, W_4^{K} + 0 - 0.707\, W_4^{3K}$, where $0 \le K \le 3$.

at $K=0$
$X(0) = 1 + 0.707 - 0.707 = 1$

at $K=1$ $\quad X(1) = 1 + 0.707 W_4^{1} - 0.707 W_4^{3} = 1 - j1.414$

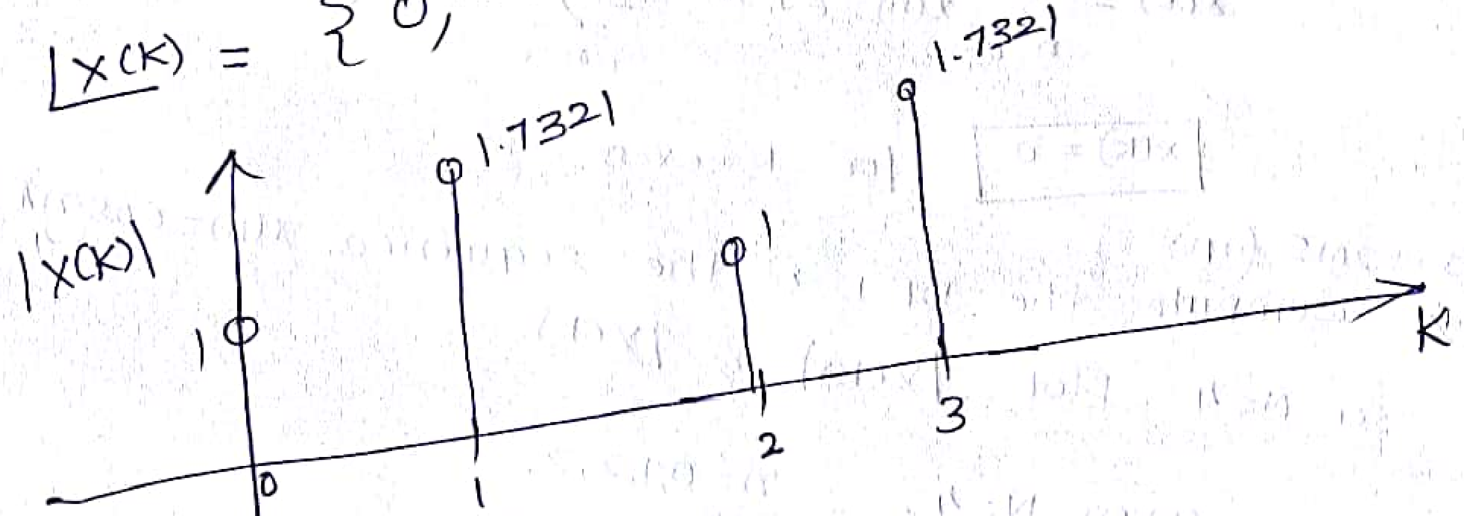
at $K=2$ $\quad X(2) = 1 + 0.707 W_4^{2} - 0.707 W_4^{6} = 1.$

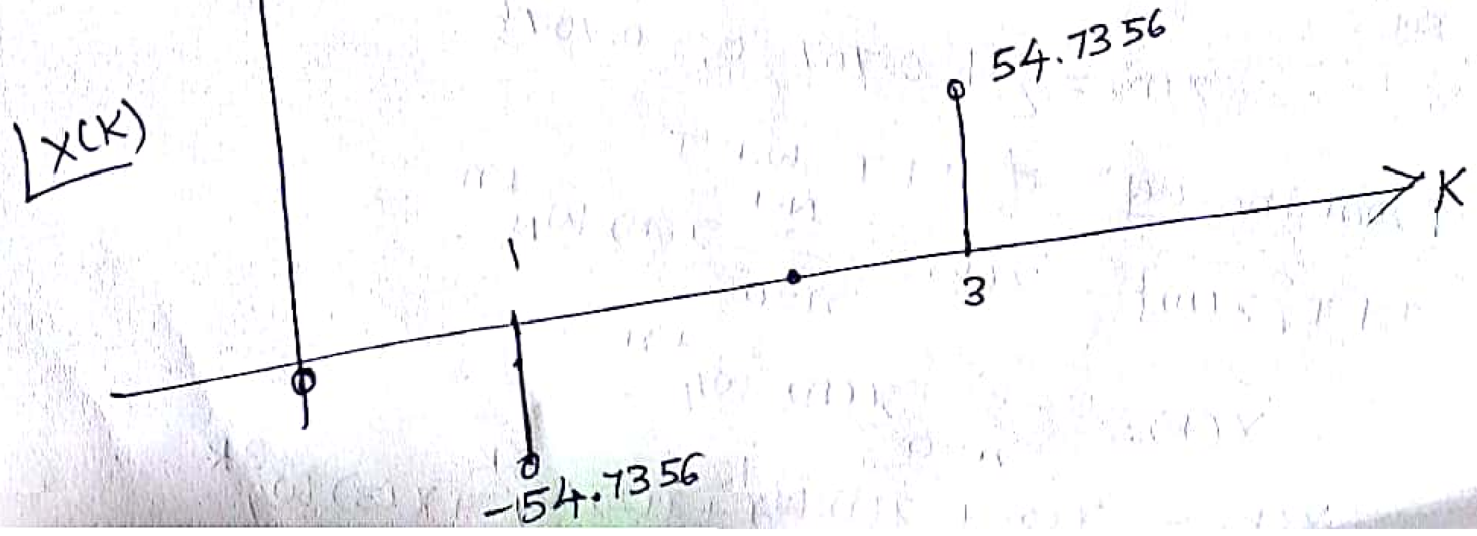
at $K=3$ $\quad X(3) = 1 + 0.707 W_4^{3} - 0.707 W_4^{9} = 1 + j1.414$

$\therefore \boxed{X(K) = \{1, (1 - j1.414), 1, (1 + j1.414)\}}$

$|X(K)| = \{1, 1.7321, 1, 1.7321\}$.

$\angle X(K) = \{0, -54.7356, 0, 54.7356\}$





(B) compute N-point DFT of $x(n) = a^n$

W.K.T

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$DFT\{a^n\} = X(k) = \sum_{n=0}^{N-1} a^n \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (aW_N^k)^n$$

$$= \frac{1 - (aW_N^k)^{N-\cancel{1}+\cancel{1}}}{1 - aW_N^k} \qquad |aW_N^k| < 1$$

$$= \frac{1 - a^N \cdot W_N^{kN}}{1 - aW_N^k} \qquad \text{W.K.T } \boxed{W_N^{kN} = 1}$$

$$= \frac{1 - a^N}{1 - aW_N^k}$$

$$\therefore \quad a^n \underset{DFT}{\overset{N\text{-pt}}{\longleftrightarrow}} \frac{1 - a^N}{1 - aW_N^k}$$

(or)

$$a^n \longleftrightarrow \frac{a^N - 1}{aW_N^k - 1}$$

(P) compute N-point DFT of $x(n) = an$.

$$DFT\{an\} = X(k) = \sum_{k=0}^{N-1} an \cdot W_N^{kn}$$

$$= a \sum_{k=0}^{N-1} n \cdot W_N^{kn} \quad \text{——} (1)$$

$$\text{W.K.T} \quad \sum_{n=0}^{N-1} b^n = \frac{1 - b^N}{1 - b} = \frac{b^N - 1}{b - 1} \qquad b \neq 1$$

Diff B.S, wrt b

$$\sum_{n=0}^{N-1} n \cdot b^{n-1} = \frac{(b-1)\,N b^{N-1} - (b^N - 1)(1-0)}{(b-1)^2}$$

$$\frac{\sum_{0}^{N-1} n b^n}{b} = \frac{b \cdot N b^{N-1} - N b^{N-1} - b^N + 1}{(b-1)^2}$$

$$\sum_{0}^{N-1} n b^n = \frac{b\left[N b^N - N b^{N-1} - b^N + 1\right]}{(b-1)^2}$$

$$\sum_{0}^{N-1} n b^n = \frac{b\left[b^N (N-1) - N b^{N-1} + 1\right]}{(b-1)^2}$$

using above relationship eq^n 1 becomes

$$x(k) = a\left\{\frac{w_N^k\left(w_N^{kN}(N-1) - N (w_N^k)^{N-1} + 1\right)}{(w_N^k - 1)^2}\right\}$$

w.k.T $w_N^N = 1$

$$= a\left[\frac{w_N^k\left(N-1 - N\left(w_N^{kN} \cdot w_N^{-k}\right) + 1\right)}{(w_N^k - 1)^2}\right]$$

$$= a\left[\frac{w_N^k\left(N-1 - N w_N^{-k} + 1\right)}{(w_N^k - 1)^2}\right]$$

$$= a\left[\frac{N w_N^k - N w_N^k w_N^{-k}}{(w_N^k - 1)^2}\right]$$

$$= a\left[\frac{N[w_N^k - 1]}{(w_N^k - 1)^2}\right]$$

$$\boxed{x(k) = a\left[\frac{N}{w_N^k - 1}\right]} \quad 0 \le k \le N-1$$

case (i) :- at $k=0$

$X(0) = a \cdot \frac{N}{0} \longrightarrow$ using L Hospital's method Sol^n.

(or) eq^n (1) becomes

at $k=0$

$$X(0) = a \cdot \sum_{k=0}^{N-1} n \cdot W_N^0$$

$$= a \cdot \sum_{k=0}^{N-1} n.$$

W.K.T

$$\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2}$$

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\boxed{X(0) = \frac{aN(N-1)}{2}}$$

$$\therefore \quad X(k) = \begin{cases} a \cdot \frac{N(N-1)}{2} & , \text{ at } k=0 \\ \frac{aN}{W_N^k - 1} & , \text{ at } k \neq 0 \end{cases}$$

July-15 (07M)

33) Find the DFT of the sequence $x(n) = 0.5^n u(n)$ for $0 \le n \le 3$, by evaluating $x(n) = a^n$ for $0 \le n \le N-1$.

Sol^n: given $x(n) = a^n \quad , \quad 0 \le n \le N-1$

$$X(K) = \sum_{n=0}^{N-1} a^n W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a W_N^K)^n = \frac{1-(aW_N^K)^{N-1+1}}{1-aW_N^K}$$

$$X(K) = \frac{1-a^N W_N^{KN}}{1-aW_N^K}.$$

WKT $W_N^{KN} = 1$

$$\therefore \boxed{X(K) = \frac{1 - a^N}{1 - aW_N^K}}$$

given $x(n) = 0.5^n u(n)$, $0 \le n \le 3$.

$a = 0.5$, & $N = 4$.

$$X(K) = \frac{1 - (0.5)^4}{1 - 0.5W_4^K} \qquad 0 \le K \le 3$$

at $K=0$ $\quad X(0) = 1.875$.

at $K=1$ $\quad X(1) = 0.75 - j0.375$

at $K=2$ $\quad X(2) = 0.625$.

at $K=3$ $\quad X(3) = 0.75 + j0.735$

$$\boxed{X(K) = \{1.875, (0.75 - j0.375), 0.625, (0.75 + j0.375)\}}$$