Properties of the DFT

	Name of the Property (NOTATION)	Time domain representation $x(n), y(n)$	Frequency domain Representation $X(K), Y(K)$
(1)	Periodicity	$x(n) = x(n+N)$	$X(K) = X(K+N)$
(2)	Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X(K) + a_2 X_2(K)$
(3)	Symmetry	$x^*(n)$ $x^*(N-n)$	$X^*(N-K)$ $X^*(K)$
(4)	Circular convolution	$x_1(n) \circledast_N x_2(n)$	$X_1(K) \cdot X_2(K)$
(5)	Circular time reversal	$x((-n))_N = x(N-n)$	$X((-K))_N = X(N-K)$
(6)	Circular time shift	$x((n-l))_N$	$X(K) e^{-j\frac{2\pi K l}{N}}$
(7)	Circular freq shift	$x(n) e^{\frac{j2\pi l n}{N}}$	$X((K-l))_N$
(8)	complex conjugate Properties	$x^*(n)$ $x^*((-n))_N = x^*(N-n)$	$X^*((-K))_N = X^*(N-K)$ $X^*(K)$
(9)	Circular correlation	$\tilde{r}_{xy}(l) = x(l) \circledast_N y^*(-l)$	$X(K) Y^*(K)$
(10)	Multiplication of two sequence	$x_1(n) x_2(n)$	$\frac{1}{N} X_1(K) \circledast_N X_2(K)$
(11)	Parsvel's theorem	$\sum_{n=0}^{N-1} \|x(n)\|^2$ $\sum_{n=0}^{N-1} x(n) y^*(n)$	$\frac{1}{N} \sum_{K=0}^{N-1} \|X(K)\|^2$ $\frac{1}{N} \sum_{K=0}^{N-1} X(K) Y^*(K)$

Properties of the DFT :-

* A good understanding of these properties is extremely helpful in the Application of the DFT to practical problems.

> **NOTE :** N - point DFT pair $x(n)$ & $X(k)$ is
>
> $$x(n) \xleftrightarrow[N]{DFT} X(k)$$

(1) **Periodicity :** $x(n)$ & $X(k)$ be DFT Pair.

$$x(n+N) = x(n) \quad \forall n, \text{ then}$$

$$X(k+N) = X(k) \quad \forall k, \text{ then}.$$

Proof : $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(K+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}\, \cancel{W_N^{Nn}} \rightarrow 1$$

$$W_N^{Nn} = e^{-\frac{j2\pi Nn}{N}} = e^{-j2\pi n}$$

$$= \cos 2\pi n - j\sin 2\pi n$$

$$= 1 - j0 = 1$$

$$\therefore X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \cdot 1.$$

$$\underline{\underline{X(K+N) = X(K)}}$$

(2) **Linearity :**

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$

Proof : $$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

where $$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$\therefore X(K) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{Kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{Kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) W_N^{Kn} + a_2 \sum_{n=0}^{N-1} x_2(n) W_N^{Kn}$$

$$X(K) = a_1 X_1(K) + a_2 X_2(K)$$

ie $x(n) \xleftrightarrow[N]{DFT} X(K)$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

(3) <u>Circular symmetries of a sequence</u> :

WKT A periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending $x(n)$ ie

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

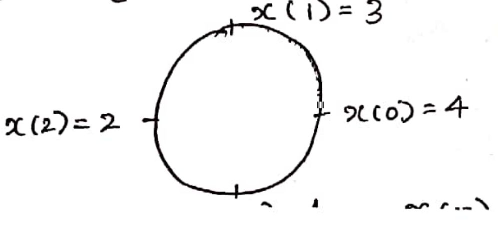
→ Let $x_p(n)$ be shifted by "K" units to the right and this sequence $x'_p(n)$

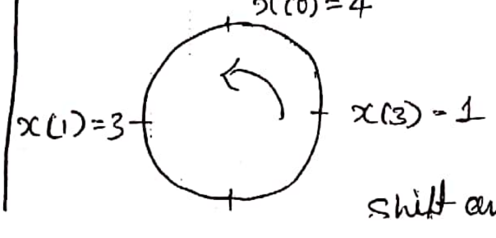
$$x'_p(n) = x_p(n-K) = \sum_{l=-\infty}^{\infty} x(n-K-lN)$$

$$x'(n) = \begin{cases} x'_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$x'(n)$ is related to $x(n)$ by the circular shift.

Ex := $\{4, 3, 2, 1\} = x(n)$ | $x(n-1)$





shift anticlockwise

(3.1) Circular even sequence : sequence that is symmetric about point zero on circle.

$$\text{ie } x(N-n) = x(n)$$

(3.2) Circular odd sequence : Sequence that is antisymmetric about point zero on circle

$$\text{ie } x(N-n) = -x(n)$$

(3.3) Circular folded sequence

$$x((-n))_N = x(N-n)$$

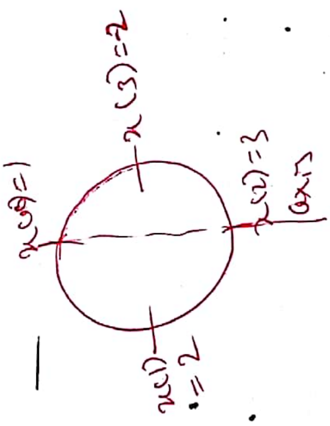
* In general the circular shift of the sequence can be represented as the index modulo N

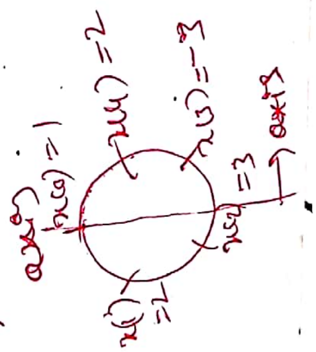
$$\therefore x'(n) = x(n-K, \text{modulo } N)$$
$$= x((n-K))_N.$$

Ex :- If K=2 & N=4 then

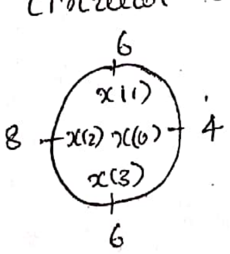
$$x'(n) = x(0-2)_4 = x(-2) = x(2)$$
$$= x(1-2)_4 = x(-1) = x(3)$$
$$= x(2-2)_4 = x(0) = x(0)$$
$$= x(3-2)_4 = x(1) = x(1)$$

Def about the 0th sample → Symmetry



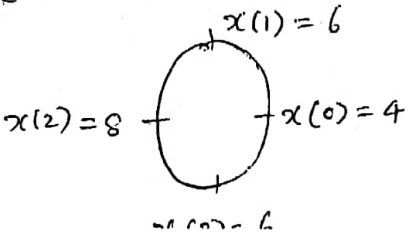


Ex : circular even $x(n) = \{4, 6, 8, 6\}$



$$x(N-n) = x(n)$$
$$x(4-0) = x(4) = x(0)$$
$$x(4-1) = x(3) = x(1)$$
$$x(4-2) = x(2) = x(2)$$
$$x(4-3) = x(1) = x(3)$$

Ex : Circular odd $x(n) = \{4, -6, 8, 6\}$



$$x(N-n) = -x(n)$$
$$x(4-1) = -x(1) = x(3)$$
$$x(4-2) = x(2) = -x(2)$$
$$x(4-3) = x(1) = -x(3).$$

* For periodic sequence $x_p(n)$

even : $x_p(n) = x_p(-n) = x_p(N-n)$

odd : $x_p(n) = -x_p(-n) = -x_p(N-n)$

→ If the periodic sequence is complex valued

Conjugate even : $x_{p_e}(n) = x_p^*(N-n)$

—॥— odd : $x_{p_o}(n) = -x_p^*(N-n)$

→ $x_p(n) = x_{p_e}(n) + x_{p_o}(n)$

$$x_{p_e}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{p_o}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

* In general $x((n-m))_N = x((n+N-M))_N$

* $x((n))_N$ → N-point sequence plotted across the circular anticlockwise, i.e. positive direction

* $x((n-k))_N$ → sequence $x(n)$ shifted anticlockwise by "k" samples, it indicates delay

* $x((n+K))_N$ → Sequence $x(n)$ shifted clockwise by "k" samples which indicates advancing operation

* $x((-n))_N$ → Circular folding, sequence $x(n)$ plotted across circle in clockwise direction i.e. in negative direction.

(4) Symmetry properties :-

Let $x(n)$ be complex valued and expressed as

$$x(n) = x_R(n) + j\, x_I(n) \longrightarrow (1) \quad 0 \le n \le N-1$$

Let DFT of $x(n)$ be expressed as

$$X(k) = X_R(k) + j\, X_I(k) \longrightarrow (2) \quad 0 \le k \le N-1$$

By def^n of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} [x_R(n) + j\, x_I(n)]\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} [x_R(n) + j\, x_I(n)] \cdot \left[\cos\left(\frac{2\pi kn}{N}\right) - j \sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) - j\, x_R(n) \sin\left(\frac{2\pi kn}{N}\right) + j\, x_I(n) \cos\left(\frac{2\pi kn}{N}\right) - j^2\, x_I(n) \sin\left(\frac{2\pi kn}{N}\right)$$

$$= \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi kn}{N}\right) - j\left[x_R(n) \sin\left(\frac{2\pi kn}{N}\right) - x_I(n) \cos\left(\frac{2\pi kn}{N}\right)\right]$$

By comparing above Eq^n with Eq^n (2)

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (3) \quad 0 \le k \le N-1$$

$$X_I(k) = -\sum_{n=0}^{N-1} x_R(n) \sin\left(\frac{2\pi kn}{N}\right) - x_I(n) \cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (4) \quad 0 \le k \le N-1$$

$|||^{ly}$, IDFT is

$$x_R(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_R(K)\cos\left(\frac{2\pi Kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi Kn}{N}\right) \longrightarrow (5)$$

$$x_I(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_R(K)\sin\left(\frac{2\pi Kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi Kn}{N}\right) \longrightarrow (6)$$

(4.1) <u>Symmetry property for real valued $x(n)$</u> :

$$X(N-k) = X^*(K) = X(-K)$$

Proof : $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$X(N-K) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-K)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, \underbrace{W_N^{Nn}}_{1}\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{-Kn}$$

$$= X(-K)$$

$$X(N-K) = \underline{X^*(K)}$$

4.2) <u>If $x(n)$ is real and even</u> :-

$x(n) = x(N-n)$ $\quad 0 \le n \le N-1$

then $X(K) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi Kn}{N}\right)$

Proof : $x(n) = x_R(n) = x_R^e(n)$, $x_I(n) = 0$.

Eq^n (3) & (4) becomes .

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

$$X_I(k) = -\sum_{n=0}^{N-1} x_R^e(n) \sin \frac{2\pi kn}{N}$$

$$X(k) = X_R(k) + j X_I(k)$$

→ $\cos\left(\frac{2\pi kn}{N}\right)$ is an even function & $\sin\left(\frac{2\pi kn}{N}\right)$ is an odd sequence.

→ Product of two even sequence is even and
— " — odd & even sequence is odd
— " — two odd sequence is even.

Sum of all samples of an odd sequence is zero ~~exact~~.

$$\therefore X_I(k) = 0$$

$$\therefore X(k) = X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

* If the time domain sequence is real & even then its DFT $X(k)$ is also real and even.

(4.3) If $x(n)$ is real and odd :-

If $x(n)$ is real and odd, then

$$x(n) = x_R(n) = x_R^0(n) \quad \& \quad x_I(n) = 0$$

Eqn ③ & ④ will be

$$\left.\begin{aligned} X_R(K) &= \sum_{n=0}^{N-1} x_R^0(n) \cos\left(\frac{2\pi Kn}{N}\right) \\ X_I(K) &= \sum_{n=0}^{N-1} x_R^0(n) \sin\left(\frac{2\pi Kn}{N}\right) \end{aligned}\right\} \quad 0 \leq K \leq N-1$$

→ $x_R^0(n) \cos\left(\frac{2\pi}{N} Kn\right)$ is an odd sequence and sum of all samples of an odd sequence is zero.

ie $X_R(K) = 0 \quad \forall K$

→ $x_R^0(n) \sin\left(\frac{2\pi}{N} Kn\right)$ is an even sequence

$$\therefore X(K) = \cancel{X_R(K)}^{0} + j X_I(K)$$

$$= j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi Kn}{N}$$

$$= -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi Kn}{N}$$

→ If the time domain sequence is real and odd then its DFT $X(K)$ is purely imaginary and odd

→ IDFT reduces

$$x(n) = j \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sin \frac{2\pi Kn}{N} \qquad 0 \leq n \leq N-1$$

→ In general; $\boxed{X_I(N-K) = -X_I(K)}$ $\qquad 1 \leq K \leq N-1$

(4.4) Purely imaginary sequence :- & even :-

$$x(n) = j\, x_I(n) = j\, x_I^e(n) \quad \& \quad x_R(n) = 0$$

Eqⁿ ③ & ④

$$\left.\begin{aligned} X_I(K) &= \sum_{n=0}^{N-1} x_I^e(n) \cos\left(\frac{2\pi}{N} Kn\right) \\ X_R(K) &= \sum_{n=0}^{N-1} x_I^e(n) \sin\left(\frac{2\pi}{N} Kn\right) \end{aligned}\right\} \quad 0 \le K \le N-1$$

* $x_I^e(n) \sin\left(\frac{2\pi Kn}{N}\right)$ is an odd sequence, $X_R(K) = 0 \quad \forall K$

$$X_I(N-K) = X_I(K)$$

* $x(n)$ is imaginary and even, then its DFT is also imaginary and even.

(4.5) $x(n)$ is imaginary and odd then,

$$x(n) = j\, x_I(n) = j\, x_I^o(n) \quad \& \quad x_R(n) = 0$$

Eqⁿ ③ & ④ will be

$$\left.\begin{aligned} X_R(K) &= \sum_{n=0}^{N-1} x_I^o(n) \sin\left(\frac{2\pi Kn}{N}\right) \\ X_I(K) &= \sum_{n=0}^{N-1} x_I^o(n) \cos\left(\frac{2\pi}{N} Kn\right) \end{aligned}\right\} \quad 0 \le K \le N-1$$

* $x_I^o(n) \cos\left(\frac{2\pi Kn}{N}\right)$ is an odd sequence i.e $X_I(K) = 0$

$$\boxed{\therefore\ X_R(N-K) = -X_R(K)}$$

* $x(n)$ is imaginary & odd then its DFT is real & odd

(5) Circular Convolution :-

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then $x_1(n) \; Ⓝ \; x_2(n) \xleftrightarrow[N]{DFT} X_1(K) \, X_2(K)$

" Multiplication of two DFT is equivalent to circular convolu-tion of their sequences in time domain "

Proof : By defn DFT

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) \, e^{\frac{-j2\pi Kn}{N}} \qquad 0 \leq K \leq N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) \, e^{\frac{-j2\pi Kn}{N}} \qquad 0 \leq K \leq N-1$$

→ If two DFTS are Multiplied, the result is DFT

$$X_3(K) = X_1(K) \, X_2(K)$$

→ IDFT of $X_3(K)$

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) \, e^{\frac{j2\pi Km}{N}}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) \, X_2(K) \, e^{\frac{j2\pi Km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) \, e^{\frac{-j2\pi Kn}{N}} + \sum_{l=0}^{N-1} x_2(l) \, e^{\frac{-j2\pi Kl}{N}} \right] \cdot e^{\frac{j2\pi Km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{K=0}^{N-1} e^{\frac{j2\pi K(m-n-l)}{N}} \right]$$

* Inner sum in the bracket of above eq^n has the (28)

form of.

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a=1 \\ \frac{1-a^N}{1-a} & a \neq 1 \end{cases} \longrightarrow (i)$$

$$\text{where} \quad a = e^{\frac{j2\pi(m-n-l)}{N}}$$

Condition (i) $a = 1 \Rightarrow \sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}}$

Condition (ii) $a \neq 1 \Rightarrow \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}}$

when $a \neq 1 \Rightarrow (m-n-l)$ is not Multiple of N. only N

from (i)

$$\therefore = \frac{1-a^N}{1-a}$$

$$= \frac{1 - e^{\cancel{\frac{j2\pi k(m-n-l)N}{N}}}}{1 - e^{\frac{j2\pi k(m-n-l)}{N}}} \rightarrow 1$$

$$= \frac{1-1}{1-e^{\frac{j2\pi k(m-n-l)}{N}}} = \underline{\underline{0}}$$

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} = \begin{cases} N & \text{when } (m-n-l) \text{ is of "N" multiple.} \\ 0 & \text{otherwise.} \end{cases}$$

Substitute the above results in $x_3(m)$ eq^n.

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \longrightarrow$$

* Let $(m-n-l)$ Multiple of "N" can be written as "PN" where p is an integer.

→ "p" might be +ve (or) -'ve

$$\therefore \quad m-n-l = -PN$$

$$l = m-n+PN$$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n) \underbrace{x_2(m-n+PN)}$$

↳ Here second summation is lost because "l" doesnot exist.

* $x_2(m-n-PN)$ is a periodic sequence with period "N" this periodic sequence is delayed by "n" samples.

* $x_2(m-n+PN)$ represents sequence $x_2(m)$ shifted circularly by "n" samples.

ie $x_2(m-n+PN) = x_2(m-n, \text{modulo } N) = x_2((m-n))_N$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n) . x_2((m-n))_N , \quad 0 \le m \le N-1$$

* Eq^n has the form of convolution sum; it is not the ordinary linear convolution

* Sequence $x_2(\,)$ is shifted circularly ∴ it is called circular convolution

→ It takes because of circular shift of the sequence

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(6) Time Reversal of a sequence :

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-K))_N = X(N-K)$

" Reversing the N-point sequence in time is equivalent to reversing the DFT values."

Proof : By Definition

$$DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n)\, W_N^{K(n)}$$

$\because$ circular symmetry property
$x(n) = x(N-n)$

let $l = N-n$ (or) $n = N-l$

when $n = 0$, $l = N$
$n = N-1$, $l = 1$

$$DFT[x(N-n)] = \sum_{l=N}^{1} x(l)\, W_N^{K(N-l)}$$

$$= \sum_{l=1}^{N} x(l)\, \cancel{W_N^{KN}}\, W_N^{-Kl} \quad \to 1$$

RHS $\times$led by W_N^{Nl} $\to 1 \;\forall\, l$

$$= \sum_{l=0}^{N-1} x(l)\, (W_N^{Nl})\, W_N^{-Kl}.$$

$$= \sum_{l=0}^{N-1} x(l)\, W_N^{(N-K)l} = X(N-K)$$

$$\boxed{DFT[x(N-n)] = X(N-K) = X((-K))_N}$$

Ex: The 4-pt. DFT of $x(n)$ is $X(K) = \{3, -j, 1, +j\}$. Find the DFT of the sequence $y(n)$, if $y(n)$ is $y(n) = x((-n))_4$.

Sol^n: $y(n) = x((-n))_4 = x(4-n) \xleftrightarrow[N=4]{DFT} Y(K) = X((-K))_4$

$= X(4-K)$

$Y(K) = X(4-K), \quad 0 \leq K \leq 3$

$Y(0) = X(4) = X(0) = 3$

$Y(1) = X(3) = +j$

$Y(2) = X(2) = 1$

$Y(3) = X(1) = -j$

$Y(K) = \{3, j, 1, -j\}$.

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(7) Circular time shift of a sequence :

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-m))_N \xleftrightarrow[N]{DFT} W_N^{km} X(k)$ (or) $X(k)\, e^{-j2\pi km / N}$.

i.e. "shifting the two sequences circularly "m" samples is equivalent to multiplying its DFT by $e^{-j2\pi km / N}$."

Proof : By Def^n

$$DFT\left[x((n-m))_N\right] = \sum_{n=0}^{N-1} x((n-m))_N W_N^{kn} \quad , \quad 0 \le K \le N$$

But $x((n-m))_N = x(n-m+N) \quad , \quad 0 \le n \le m-1$

$$= x(n-m) \quad , \quad m \le n \le N-1$$

Spliting the summation 0 to m-1 & m to N-1

$$\therefore DFT\left[x((n-m))_N\right] = \sum_{n=0}^{m-1} x(n-m+N)\, W_N^{kn} + \sum_{n=m}^{N-1} x(n-m)\, W_N^{kn}$$

let $l = n-m+N$; let $l = n-m$ (or) $n = l+m$

$n = l+m-N$

when $n=0$, $l = N-m$, when $n = M$, $l = 0$

$n = m-1$, $l = N-1$; $n = N-1$, $l = N-1-m$

$$DFT\left[x((n-m))_N\right] = \sum_{l=N-m}^{N-1} x(l)\, W_N^{K(l+m-N)} + \sum_{l=0}^{N-1-m} x(l) \cdot W_N^{K(l+m)}$$

$$= \sum_{l=0}^{N-m-1} x(l) \cdot W_N^{Kl}\, W_N^{Km} + \sum_{l=N-m}^{N-1} x(l)\, W_N^{Kl}\, W_N^{Km}\, \underbrace{W_N^{-KN}}_{1}$$

$$= W_N^{km}\left[\sum_{l=0}^{N-m-1} x(l)\, W_N^{kl} + \sum_{l=N-m}^{N-1} x(l)\cdot W_N^{kl}\right]$$

$$= W_N^{km}\left[\sum_{l=0}^{N-1} x(l)\cdot W_N^{kl}\right]$$

$$\boxed{DFT\left[x((n-m))_N\right] = W_N^{km}\, X(K)}$$

$$\boxed{\therefore\ y(n) = x((n-m))_N \xleftrightarrow[N]{DFT} Y(K) = W_N^{Km}\, X(K) = e^{-j\frac{2\pi Km}{N}} X(K)}$$

Ex: The 4-pt DFT of $x(n)$ is given by $X(K) = \{3, -j, 1, +j\}$ without computing IDFT and DFT find the DFT of the sequence $y(n)$ if (i) $y(n) = x((n-1))_N$
(ii) $y(n) = x((n+1))_N$

Sol^n (i) $y(n) = x((n-1))_N$.

By using circular time-shift property.

$$y(n) = x((n-m))_N \xleftrightarrow[N]{DFT} Y(K) = W_N^{kn} X(K) \qquad 0 \le K \le N-1$$

$$y(n) = x((n-1))_4 \xleftrightarrow[N=4]{DFT} Y(K) = W_4^{kn} X(K), \quad 0 \le K \le 3.$$

$K=0$, $Y(0) = W_4^0 \cdot X(0) = 3$

$K=1$, $Y(1) = W_4^1 \cdot X(1) = -j \times -j = -1$

$K=2$, $Y(2) = W_4^2 \cdot X(2) = -1 \times +1 = -1$

$K=3$, $Y(3) = W_4^3 \cdot X(3) = j \times j = -1$

$\therefore X(K) = \{3, -1, -1, -1\}$.

(ii) $y(n) = x((n+1))_4 \xleftrightarrow[N=4]{DFT} Y(K) = W_4^{-K} X(K)$ (2)

$0 \leq K \leq 3$

$K=0, \quad Y(0) = W_4^{0} X(0) = 3$

$K=1, \quad Y(1) = W_4^{-1} X(1) = j \times j = 1$

$K=2, \quad Y(2) = W_4^{-2} X(2) = -1 \times 1 = -1$

$K=3, \quad Y(3) = W_4^{-3} X(3) = -j \times j = 1$

$\therefore Y(K) = \{3, 1, -1, 1\}$.

Circular Time shift.

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-l))_N \longleftrightarrow e^{-j\frac{2\pi}{N}Kl} X(K) = W_N^{Kl} X(K)$

Proof: $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$

$DFT[x((n-l))_N] = \sum_{n=0}^{N-1} x((n-l))_N e^{-j\frac{2\pi}{N}kn}$

$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j\frac{2\pi}{N}kn}$

$+ \sum_{n=l}^{N-1} x((n-l))_N e^{-j\frac{2\pi}{N}kn}$

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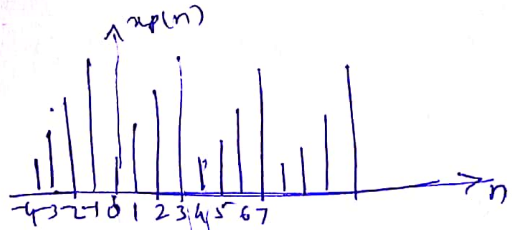
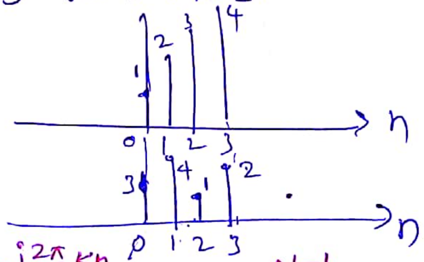


comparing one cycle of $x_p(n)$ & $x_p(n-2)$ from 0 to 3.



$$= \sum_{n=0}^{(l-1)} x(N+n-l)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=l}^{N-1} x(n-l)\, e^{-j\frac{2\pi}{N}kn}$$

Ist term 2nd term.

Performing change of variables in above eqn

In 2nd term: $n-l=m$, $\Rightarrow n=m+l$; In first term: $N+n-l=m \Rightarrow n=m+l-N$

$$DFT[x((n-l))_N] = \sum_{m=N-l}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}k(m+l-N)} + \sum_{m=0}^{N-1-l} x(m)\, e^{-j\frac{2\pi}{N}k(m+l)}$$

$$= \sum_{m=0}^{N-1-l} x(m)\, e^{-j\frac{2\pi}{N}km}\, e^{-j\frac{2\pi}{N}kl} + \sum_{m=N-l}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}km}\, e^{-j\frac{2\pi}{N}kl}$$

Interchanging Ist & 2nd terms.

$$= \left[\sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}km}\right] e^{-j\frac{2\pi}{N}kl}$$

Independent of m ∴ taken out of Σ

$$\text{ie } DFT[x((n-l))_N] = e^{-j\frac{2\pi}{N}kl} \underbrace{\sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}km}}_{\text{DFT of } x(m)}$$

$$\boxed{DFT[x((n-l))_N] = e^{-j\frac{2\pi}{N}kl}\, X(K)}$$

$$* \sum_{m=0}^{N-l-1} + \sum_{m=N-l}^{N-1} = \sum_{m=0}^{N-1}$$

(8) Circular frequency shift :-

" Multiplication of the sequence $x(n)$ with the complex exponential sequence $e^{j2\pi kn/N}$ is equivalent to the circular shift of the DFT by m units in frequency."

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $W_N^{-mn} x(n) \xleftrightarrow[N]{DFT} X((K-m))_N$

Proof : By def^n

$$IDFT[X(K)] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}, \quad 0 \le n \le N-1.$$

put $K = K-m$

$$IDFT[X(K-m)] = \frac{1}{N} \sum_{K=0}^{N-1} X((K-m))_N W_N^{-Kn}, \quad 0 \le n \le N-1$$

but $X((K-m))_N = X((K-m+N)) \quad , 0 \le K \le m-1$

$= X(K-m) \quad , m \le K \le N-1.$

$$\therefore IDFT[x((K-m))_N] = \frac{1}{N}\left[\sum_{K=0}^{m-1} X(K-m+N) W_N^{-Kn} + \sum_{K=m}^{N-1} X(K-m) W_N^{-Kn}\right]$$

let $l = K-m+N$

$K = l+m-N$

when $k=0$, $l = N-m$

$K = m-1$, $l = N-1$

let $l = K-m$

$K = l+m$

when $K=m$, $l=0$

$K = N-1$ $l = N-1-m$

$$\therefore \; IDFT\left[x((K-m))_N\right] = \frac{1}{N}\left[\sum_{l=N-m}^{N-1} x(l)\, W_N^{-(l+m-N)n} + \sum_{l=0}^{N-1-m} x(l)\, W_N^{-(l+m)n}\right]$$

$$= \frac{1}{N}\left[\sum_{l=N-m}^{N-1} x(l)\cdot W_N^{-ln}\, W_N^{-mn}\, \overbrace{W_N^{Nn}}^{1} + \sum_{l=0}^{N-m-1} x(l)\, W_N^{-ln}\, W_N^{-mn}\right]$$

$$= \frac{1}{N}\, W_N^{-mn}\left[\sum_{l=0}^{N-m-1} x(l)\, W_N^{-ln} + \sum_{l=N-m}^{N-1} x(l)\, W_N^{-ln}\right]$$

$$= W_N^{-mn}\left[\frac{1}{N}\sum_{l=0}^{N-1} x(l)\, W_N^{-ln}\right]$$

$$= W_N^{-mn}\, x(n)$$

ie

$$IDFT\; X((K-m))_N = W_N^{-mn}\, x(n)$$

$$\therefore \; y(n) = W_N^{-mn}\, x(n) = e^{\frac{j2\pi mn}{N}}\cdot x(n) \xleftrightarrow[N]{DFT} Y(K) = X((K-m))_N$$

(9) Complex conjugate property :-

If $x(n)$ is complex sequence &

$x(n) \xleftrightarrow[N]{DFT} X(K)$ then

(i) $x^*(n) \xleftrightarrow[N]{DFT} X^*(N-K) = X^*((-K))_N$

(ii) $x^*(N-n) = x^*((-n))_N \xleftrightarrow[N]{DFT} X^*(K)$

Proof : By Definition

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

if $x(n)$ is complex $\Rightarrow x^*(n)$

$$\therefore DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) (W_N^{-Kn})^*$$

$$= \sum_{n=0}^{N-1} (x(n) W_N^{-Kn})^*$$

$$= \sum_{n=0}^{N-1} (x(n) W_N^{Nn} W_N^{-Kn})^* \quad | \because W_N^{Nn} = 1$$

$$= \sum_{n=0}^{N-1} (x(n) \cdot W_N^{(N-K)n})^*$$

$$DFT[x^*(n)] = X^*(N-K)$$

$$\boxed{\therefore DFT[x^*(n)] = [X[N-K]]^* = X^*(N-K) = X^*((-K))_N}$$

(ii) By defⁿ

$$IDFT[X(k)] = x(n)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

$$IDFT[X^*(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) \cdot W_N^{-kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) \cdot (W_N^{kn})^*$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} (X(k) \cdot W_N^{kn})^*$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} [X(k) \; W_N^{-Nn} \; W_n^{kn}]^* \qquad \because W_N^{-Nn} = 1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} [X(k) \; W_N^{-k(N-n)}]^*$$

$$= x^*(N-n)$$

$$\boxed{\therefore \; IDFT[X^*(k)] = [x(N-n)]^* = x^*(N-n) = x^*((-n))_N}$$

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(10) Multiplication of Two sequences :

(or) Modulation property

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X_1(K) \circledast_N X_2(K)\right]$

Proof : $X(K) = DFT[x(n)]$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn} \qquad 0 \leq K \leq N-1$$

let $x(n) = x_1(n)\, x_2(n)$

$$\therefore \quad X(K) = \sum_{n=0}^{N-1} x_1(n)\, x_2(n) \cdot W_N^{Kn} \qquad 0 \leq K \leq N-1$$

$$X(K) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, W_N^{-ln} \cdot \frac{1}{N} \sum_{m=0}^{N-1} X_2(m)\, W_N^{-mn} \right] W_N^{Kn}$$

$$= \frac{1}{N^2} \sum_{l=0}^{N-1} X_1(l) \sum_{m=0}^{N-1} X_2(m) \sum_{n=0}^{N-1} W_N^{-n(L+M-K)}$$

let $a = W_N^{-(L+M-K)} = e^{j\frac{2\pi}{N}[L+M-K]}$ $\quad \therefore \sum_{n=0}^{N-1} a^n$

$$\therefore \quad X(K) = \frac{1}{N^2} \sum_{L=0}^{N-1} X_1[L] \sum_{M=0}^{N-1} X_2[M] \sum_{n=0}^{N-1} a^n$$

where $\displaystyle\sum_{n=0}^{N-1} a^n = \begin{cases} \dfrac{1-a^N}{1-a}, & a \neq 1 \\ N, & a = 1 \end{cases}$

$$a^N = W_N^{-(L+M-K)N} = e^{\frac{j2\pi(L+M-K)N}{N}} = 1.$$

$$\therefore \sum_{n=0}^{N-1} a^n = \begin{cases} 0, & \text{if } a \neq 1 \\ N, & \text{if } a = 1 \end{cases}.$$

$\rightarrow$ $a = 1$ then $L+M-K$ is an integer multiple of N

ie $L+M-K = NP$ where P is an integer

if $L+M-K \neq NP$ then $a \neq 1$.

$$\text{ie } \sum_{n=0}^{N-1} a^n = \begin{cases} N, & \text{if } a=1, \text{ ie } L+M-K=N \ (\text{if } P=1) \text{ (or) } M=((K-L))_N \\ 0, & \text{if } a \neq 1, \text{ ie } L+M-K \neq N \text{ (or) } M \neq ((K-L))_N \end{cases}$$

$$\therefore \sum_{n=0}^{N-1} a^n = N\,\delta(M-((K-L))_N).$$

Substitute the above eqn in $X(K)$ eqn.

$$X(K) = \frac{1}{N^2} \sum_{L=0}^{N-1} x_1[L] \sum_{M=0}^{N-1} x_2[M] \cdot N\,\delta(M-((K-L))_N)$$

$\therefore$ ~~X(K)=~~

$$\boxed{\therefore X(K) = \frac{1}{N} X_1(K) \circledast_N X_2(K)}$$

(12) Parseval's Theorem :- Inner product property

if $x(n) \xleftrightarrow[N]{DFT} X(K)$ and.

$y(n) \xleftrightarrow[N]{DFT} Y(K)$ then.

$$\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K)\, Y(K).$$

Proof : WKT from cirlar correlation.

$$\tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

$$\text{if } l=0 \Rightarrow \tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*(n)$$

$$\text{WKT DFT } \{\tilde{\gamma}_{xy}(l)\} = X(K)\, Y^*(K)$$

$$\tilde{\gamma}_{xy}(l) = \text{IDFT } \{X(K)\, Y^*(K)\}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)\, e^{j\frac{2\pi K l}{N}}$$

with $l=0$, above Eq^n becomes.

$$\tilde{\gamma}_{xy}(0) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)$$

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K).$$

* when $x(n) = y(n)$ then the above Eq^n becomes

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

which represents the energy of the sequence of $x(n)$

Circular Convolution of DFTS (or) multiplication prop

$$\text{DFT}\{x_1(n)\, x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, W_N^{kn}$$

$$\text{IDFT of } x(n) = \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln}$$

$$\text{DFT}\{x_1(n)\, x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n) \left\{ \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln} \right\} W_N^{kn}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \sum_{n=0}^{N-1} x_1(n)\, W_N^{(k-l)n}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, X_1((k-l))_N$$

$$= \frac{1}{N} \underbrace{\sum_{l=0}^{N-1} X_2(l)\, X_1((k-l))_N}_{\text{circular convolution}}$$

$$= \frac{1}{N} X_1(k) \circledast_N X_2(k)$$

This property is also called as modulation property.

(11) Circular correlation :=

If $x(n) \xleftrightarrow[N]{DFT} X(K)$ &

$y(n) \xleftrightarrow[N]{DFT} Y(K)$ then

$$\tilde{\gamma}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(K) = X(K)\, Y^*(K)$$

where $\tilde{\gamma}_{xy}$ is circle correlation given as

$$\tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

" ie Multiplication of DFT of one sequence & conjugate DFT of another sequence is equivalent to circular correlation of these two sequences in time domain "

Proof :
$$\tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N \longrightarrow (1)$$

$y^*((n-l))_N$ can be written as $y^*((-(l-n)))_N$

$$\therefore \tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((-(l-n)))_N \longrightarrow (2)$$

But wkt
$$x_1(m) \circledast_N x_2(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \longrightarrow (3)$$

comparing (2) and (3)

$$\tilde{\gamma}_{xy}(l) = x(l) \circledast_N y^*(-l)$$

From circular convolution property we can write.

$$DFT\{\tilde{\gamma}_{xy}(l)\} = DFT\{x(l)\} \cdot DFT\{y^*(-l)\}$$

$$\tilde{R}_{xy}(K) = X(K)\; DFT\{y^*(-l)\}$$

WKT that by defn of DFT we have

$$DFT\{y^*(-l)\} = \sum_{l=0}^{N-1} y^*(-l)\, e^{-j\frac{2\pi K l}{N}}$$

Let $n = -l$

when $l = 0$, $n = 0$

$l = N-1$, $n = -(N-1)$

$$\therefore DFT\{y^*(-l)\} = \sum_{n=0}^{-(N-1)} y^*(n)\, e^{j\frac{2\pi K n}{N}}$$

But $y^*(n)$ is circular in nature

$\therefore -(N-1) = N-1$

$$\Rightarrow DFT\{y^*(-l)\} = \sum_{n=0}^{N-1} y^*(n)\, e^{j\frac{2\pi K n}{N}}$$

$$= \left[\sum_{n=0}^{N-1} y^*(n)\, e^{-j\frac{2\pi K n}{N}}\right]^* = [Y(K)]^*$$

$$= Y^*(K)$$

$$\therefore \tilde{R}_{xy}(K) = X(K)\cdot Y^*(K)$$

$$\tilde{\gamma}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(K) = X(K)\, X^*(K) = |X(K)|^2$$

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