Chapter 4

Design of IIR Filters

4.1 Introduction

The purpose of a filter is to enhance a wanted signal relative to unwanted signals, interference or noise. This chapter deals with the design techniques in which a desired frequency response is approximated by a transfer function expressed as a ratio of the polynomials. In general, this type of transfer function yields an impulse response of infinite duration. Therefore, the analog filters designed in this chapter are commonly referred to as *infinite impulse response* (IIR) filters. A very important approach to the design of digital filters is to apply transformation to an existing analog filter. Hence, it is utmost important to have a catalog of analog filters that can serve as the prototype or models for the transformation. A typical design procedure of a digital IIR filter involves the following steps:

1. Choosing a method of transformation that maps a stable analog filter into a digital filter having approximately the same frequency response.
2. Converting the specifications of the digital IIR filter to equivalent specifications of an analog IIR filter such that, after the mapping from analog to digital is carried out, the digital IIR filter will meet the given frequency-domain specifications.
3. Designing the analog IIR filter in accordance with the transformed specifications.
4. Transforming the analog filter to an equivalent digital filter.

The main classes of analog filters are Butterworth and Chebyshev filters. These filters differ in the nature of their magnitude responses as well as in their design and implementation. Familiarity with these filters helps one to choose the most suitable filter class for a specific application. In this chapter, we deal with the design of lowpass analog filters mentioned above. The design of analog filters other than lowpass is based on frequency transformation.

Finally, we will deal with different types of transformations that map an analog IIR filter into a digital IIR filter.

4.2 Analog Filter Specifications

An important step in the design of an analog filter is the definition of the frequency response specifications that should be satisfied by the filter frequency response. These specifications describe how the filter reacts in the steady-state to sinusoidal inputs. Fig. 4.1 shows a typical magnitude frequency response (also known as magnitude or amplitude response) of a lowpass filter.

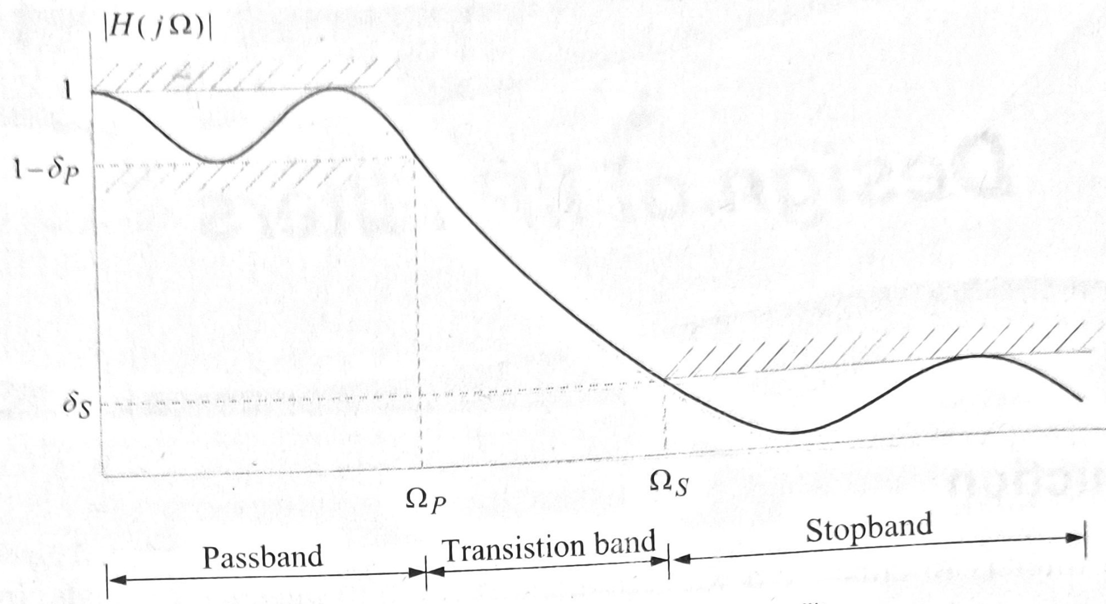


Fig. 4.1 Specifications of a lowpass filter.

In Fig. 4.1, Ω_P and Ω_S denote the passband and stopband edge frequencies respectively. The frequency region between Ω_P and Ω_S is the so called *transistion band* where no specification is provided. The hatched areas in the passband and in the stopband indicate forbidden magnitude values in these bands. There are no forbidden values in the transistion band, but it is usually desired that the magnitude decreases monotonically in this band. The mathematical description of the frequency response shown in Fig. 4.1 is

$$\begin{aligned} 1-\delta_P \leqslant |H(j\Omega)| \leqslant 1, &\quad 0 \leqslant \Omega \leqslant \Omega_P \\ 0 \leqslant |H(j\Omega)| \leqslant \delta_S, &\quad |\Omega| \geqslant \Omega_S \end{aligned} \tag{4.1}$$

The parameter δ_P is the tolerance of the magnitude response in the passband. The desired magnitude response in the passband is 1. The parameter δ_S is the tolerance of the magnitude response in the stopband. The desired magnitude response in the stopband is 0.

We define, $A_P = -20\log(1-\delta_P)$ as the passband ripple in dB. Also, $K_P = -A_P = 20\log(1-\delta_P)$ is defined as the gain at $\Omega = \Omega_P$.

The quantity δ_S is called the *stopband attenuation*. Another useful quantity is $A_S = -20\log\delta_S$. This parameter is known as the stopband [1]attenuation in dB. Finally, $K_S = -A_S$ is known as the *stopband gain* at $\Omega = \Omega_S$.

4.3 Classification of Analog Filters

There are many ways of classifying analog filters:

1. By their magnitue frequency response; lowpass, highpass, bandpass or bandstop. The frequency responses of these filters are shown in Fig. 4.2. Also, shown are the frequency responses for the ideal LP, HP, BP and BS filters which exhibit no transistion.

[1]Gain and attenuation in dB are negative of each other.

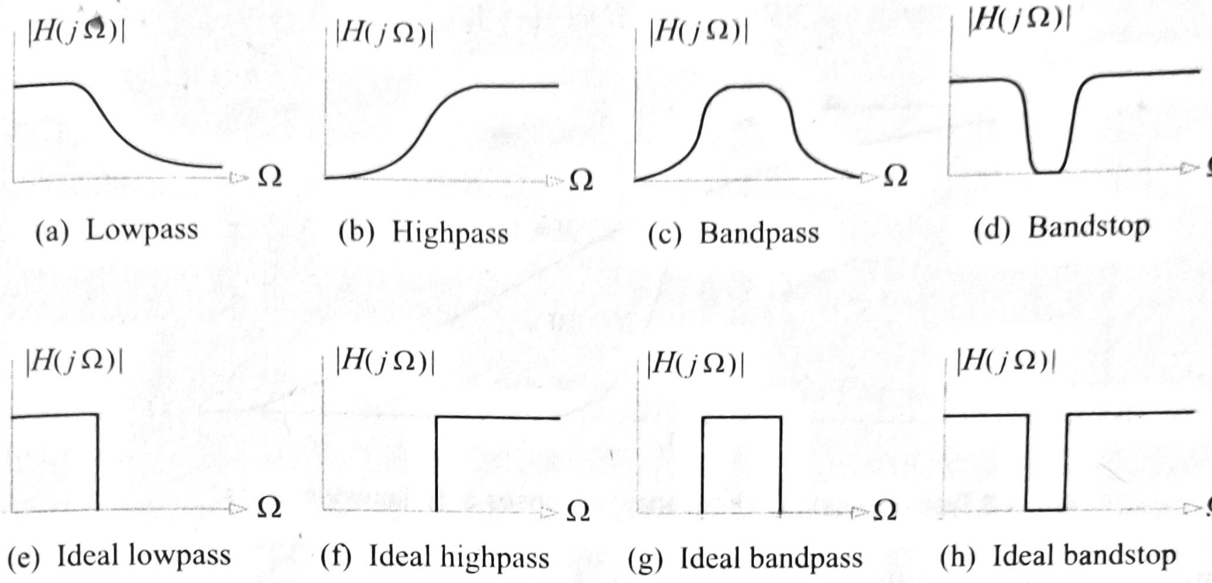


Fig. 4.2 Basic types of frequency responses.

2. By their cutoff frequencies (The cutoff frequency is defined as the frequency where the gain has changed by some specified amount relative to the mean midband gain).
3. By the shape of their amplitude response.
4. By the shape of their phase response.
5. By the nature of devices used; active devices like operational amplifiers or a combination of passive elements like resistors and capacitors. That means the filter may be either active or passive.

A filter is said to be normalized if the cutoff frequency of the filter, Ω_C is 1 rad/sec. It is known that the lowpass, highpass, bandpass and bandstop filters can be designed by applying a specific transformation to a normalized lowpass filter. Therefore, a lot of importance is given to the design of normalized lowpass analog filters. In the sections to follow, the properties and design procedures for the analog Butterworth and Chebyshev lowpass filters are presented along with the procedures and transformations necessary to convert them into other lowpass, highpass, bandpass and bandstop filters.

4.4 Butterworth Filters

Butterworth filters have a very smooth passband, which we *pay* for with a relatively wide transistion region, whereas Chebyshev filters are the opposite, having a sharp transistion region and a not so smooth passband.

A Butterworth filter is characterized by its magnitude frequency response,

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}} \tag{4.2}$$

where N is the order of the filter and Ω_C is defined as the cutoff frequency where the filter magnitude is $\frac{1}{\sqrt{2}}$ times the dc gain ($\Omega = 0$). Typical magnitude frequency responses of Butterworth filters are shown in Fig. 4.3.

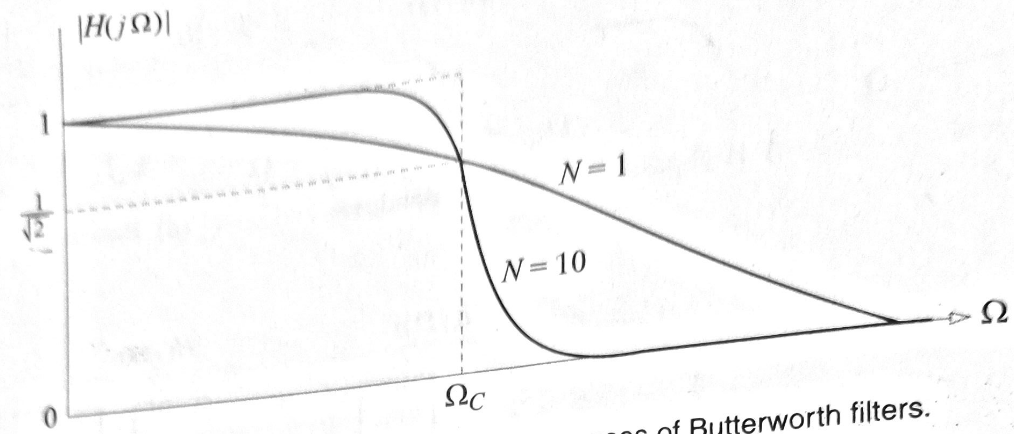


Fig. 4.3 Typical magnitude frequency responses of Butterworth filters.

The following observations are made from Fig. 4.3.

1. $|H(j0)| = 1$ for all N.
2. $|H(j\Omega)| = \frac{1}{\sqrt{2}}$ at $\Omega = \Omega_C$ for all finite N. This means that $20\log_{10}|H(j\Omega_C)| = -3.01$ dB.
3. $|H(j\Omega)| \to 0$ as $\Omega \to \infty$.
4. The magnitude characteristic is said to be maximally flat because $\left.\frac{d^n|H(j\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0$ for $n = 1, 2 \cdots 2N-1$.
5. $|H(j\Omega)|$ is a monotonically decreasing function of frequency, i.e., $|H(j\Omega_2)| < |H(j\Omega_1)|$ for any values of Ω_1 and Ω_2 such that $0 \leqslant \Omega_1 < \Omega_2$.

The magnitude-squared frequency response of the normalized ($\Omega_C = 1$) lowpass Butterworth filter is

$$|H_N(j\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

$$\Rightarrow \quad H_{N(j\Omega)}\, H_N(-j\Omega) = \frac{1}{1+\Omega^{2N}}$$

Replacing $j\Omega$ by s and hence Ω by $\frac{s}{j}$ in the above equation, we get

$$H_N(s)H_N(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} \tag{4.3}$$

The transfer function $H_N(s)H_N(-s)$ has no finite zeros, and therefore $H_N(s)$ itself has no finite zeros. The poles of the product $H_N(s)H_N(-s)$ are determined by equating the denominator to zero, which gives

$$1+\left(\frac{s}{j}\right)^{2N} = 0 \quad \Rightarrow \quad s = (-1)^{\frac{1}{2N}}\, j$$

Since $-1 = e^{j\pi(2k+1)}$, $k = 0, 1 \cdots$ and $j = e^{j\pi/2}$, the poles are given by

$$s_k = e^{j\pi\frac{(2k+1)}{2N}} e^{j\pi/2}, \qquad k = 0, 1, \cdots 2N-1$$

The poles are therefore on a circle with radius unity and are placed at angles,

$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \frac{\pi}{2}, \qquad k = 0, 1 \cdots 2N-1$$

Let us find the poles of $H_N(s)H_N(-s)$ for $N = 2$ and $N = 3$. The same are shown in Fig. 4.4 and Fig. 4.5 respectively.

***Case* (i) $N = 2$:**

$$\theta_k = \frac{\pi}{2}k + \frac{\pi}{4} + \frac{\pi}{2} \qquad k = 0, 1, 2, 3$$

Hence, Left-half poles of $H_N(s)$: $\theta_0 = \frac{3\pi}{4}$, $\theta_1 = \frac{5\pi}{4}$

Right-half poles of $H_N(-s)$: $\theta_2 = \frac{7\pi}{4}$, $\theta_3 = \frac{9\pi}{4}$

$S_k = 1 \angle \theta_k$

***Case* (ii) $N = 3$:**

$$\theta_k = \frac{\pi}{3}k + \frac{\pi}{6} + \frac{\pi}{2}, \qquad k = 0, 1, 2, 3, 4, 5$$

Left-half poles of $H_N(s)$: $\theta_0 = \frac{2\pi}{3}$, $\theta_1 = \pi$, $\theta_2 = \frac{4\pi}{3}$

Right-half poles of $H_N(-s)$: $\theta_3 = \frac{5\pi}{3}$, $\theta_4 = 2\pi$, $\theta_5 = \frac{7\pi}{3}$

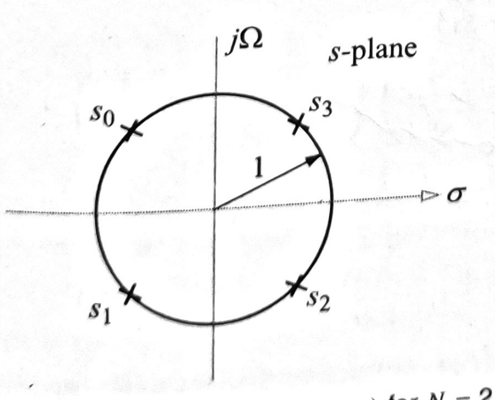


Fig. 4.4 Poles of $H_N(s)H_N(-s)$ for $N = 2$.

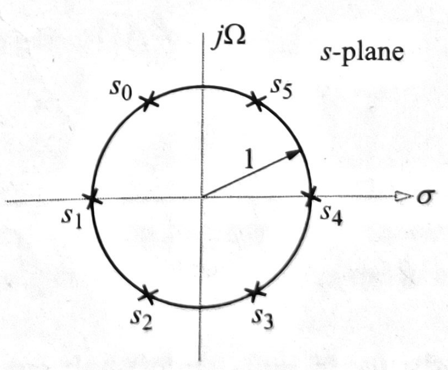


Fig. 4.5 Poles of $H_N(s)H_N(-s)$ for $N = 3$.

From Fig. 4.4 and Fig. 4.5, we find that the poles of $H_N(s)H_N(-s)$ lie on a circle of unit radius and have a relative phase difference of $\frac{\pi}{N}$ radians.

The poles are distributed on the unit circle in the s-plane; half on the left-half plane, and half on the right-half plane. Since, these are the poles of the product $H_N(s)H_N(-s)$ and we want the transfer function $H_N(s)$ to be stable, we assign N poles on the left-half plane to $H_N(s)$, and so the remaining poles on the right-half plane are invariably assigned to the factor $H_N(-s)$.

Thus,
$$H_N(s) = \frac{1}{\prod\limits_{LHP}(s - s_k)} = \frac{1}{B_N(s)}$$

In the above expression, s_k are all the left-half poles of $H_N(s)H_N(-s)$ and $B_N(s)$ is a Butterworth polynomial of order N. Table 4.1 shows the first five Butterworth polynomials in a real factored form.

Table 4.1 Normalized Butterworth polynomials

Order N	***Butterworth polynomial $B_N(s)$***
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$

b_0 $\frac{b_0}{\sqrt{1+\epsilon^2}}$

We shall now demonstrate, how the above table is prepared by synthesizing $B_2(s)$ and $B_3(s)$. For $N = 2$, as is evident from Fig. 4.4, the left-half poles of $H_N(s)H_N(-s)$ are $s_0 = \frac{-1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$ and $s_1 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$. Thus,

$$\begin{aligned}
B_2(s) &= (s - s_0)(s - s_1)\\
&= \left(s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\left(s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)\\
&= \left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\\
&= s^2 + \sqrt{2}s + 1
\end{aligned}$$

Similarly, for $N = 3$, the left-half poles of $H_N(s)H_N(-s)$ are as follows (Refer Fig. 4.5).

$$s_0 = \frac{-1}{2} + j\frac{\sqrt{3}}{2}, \quad s_1 = -1, \quad s_2 = \frac{-1}{2} - j\frac{\sqrt{3}}{2}$$

S | θ_k Rec(1, θ_k)

$\theta_k = \frac{\pi k}{N} + \frac{\pi}{2N}$

Hence,

$$
\begin{aligned}
B_3(s) &= (s - s_0)(s - s_1)(s - s_2) \\
&= \left(s + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(s+1)\left(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \\
&= \left[\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right](s+1) \\
&= (s^2 + s + 1)(s+1)
\end{aligned}
$$

4.5 Frequency Transformations/Spectral Transformations

Let $H(s)$ denote the transfer function of a lowpass analog filter with a passband edge frequency Ω_P equal to 1 rad/sec. This filter is known as *analog lowpass normalised protype*. If Ω_u is the desired passband edge frequency of the new lowpass filter, then the transfer function of this new lowpass filter is obtained by using the transformation:

$$s \longrightarrow \frac{s}{\Omega_u}$$

where $\longrightarrow$ is read as *is replaced by*.

Let $H'(s)$ be the transfer function of the new lowpass filter. Then,

$$
\begin{aligned}
H'(s) &= H(s)\big|_{s \longrightarrow \frac{s}{\Omega_u}} \\
\Rightarrow \quad H'(s) &= H\left(\frac{s}{\Omega_u}\right)
\end{aligned}
$$

The frequency response of the new lowpass filter is obtained by letting $s = j\Omega$ in $H'(s)$.

That is,
$$H'(j\Omega) = H\left(\frac{j\Omega}{\Omega_u}\right)$$

Evaluating the magnitude response at $\Omega = \Omega_u$, we get

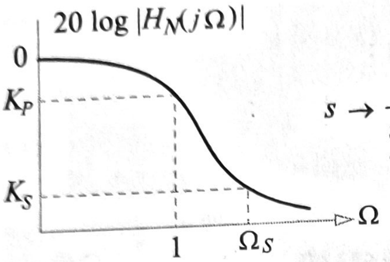
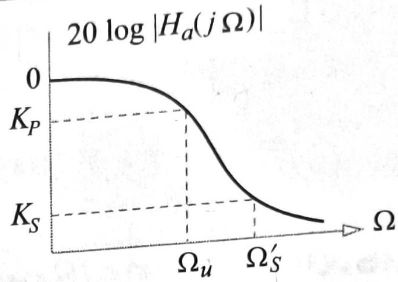
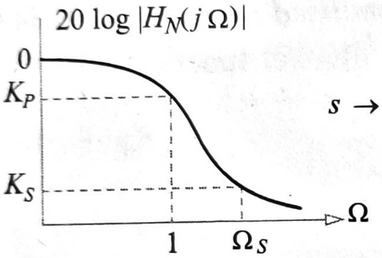
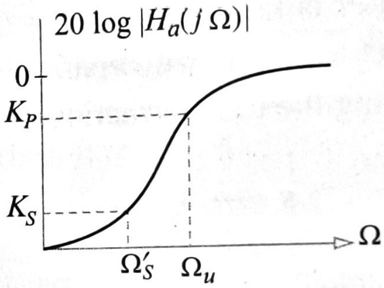
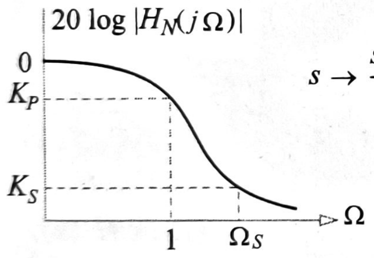
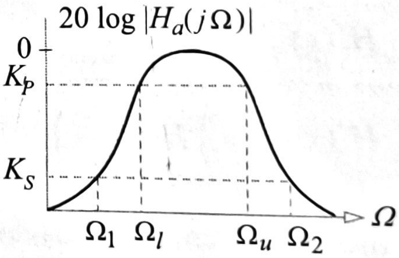
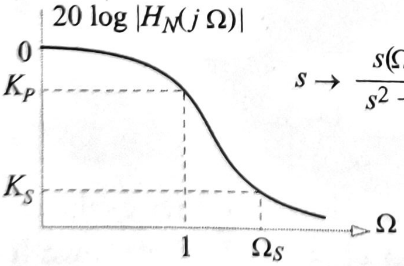
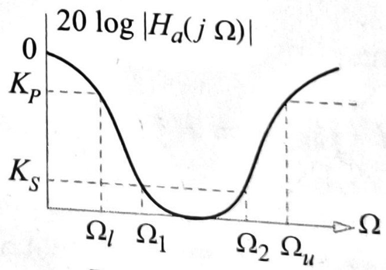
$$|H'(j\Omega_u)| = H(j1)$$

The above equation means that the frequency response of the new lowpass filter evaluated at $\Omega = \Omega_u$ is equal to the value of the prototype transfer function at $\Omega = 1$. In a way, we have translated the cutoff frequency from 1 rad/sec to Ω_u rad/sec. Thus, it justifies the correctness of the lowpass-to-lowpass transformation, $s \longrightarrow \frac{s}{\Omega_u}$.

The lowpass-to-highpass transformation is simply achieved by replacing s by $\frac{1}{s}$. If the desired highpass filter has a passband edge frequency Ω_u, then the transformation is

$$s \longrightarrow \frac{\Omega_u}{s}$$

Table 4.2 Analog-to-Analog transformations

Prototype frequency response	Transformed frequency response	Backward design equations
 $s \to \frac{s}{\Omega_u}$ Ω_c Lowpass, $H_N(s)$	 Lowpass, $H_a(s)$	$\Omega_S = \frac{\Omega'_S}{\Omega_u}$ $S \to \frac{\Omega_P}{\Omega_P}, S$
 $s \to \frac{\Omega_u}{s}$ Ω_c Lowpass, $H_N(s)$	 Highpass, $H_a(s)$	$S \to \frac{\Omega_P \cdot \Omega'_P}{S}$ $\Omega_S = \frac{\Omega_u}{\Omega'_S}$
 $s \to \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}$ Lowpass, $H_N(s)$	 Bandpass, $H_a(s)$	$\Omega_S = \mathrm{Min}\{\lvert A\rvert, \lvert B\rvert\}$ $A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$ $S = -\Omega_P \cdot S^2$
 $s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}$ Lowpass, $H_N(s)$	 Bandstop, $H_a(s)$	$\Omega_S = \mathrm{Min}\{\lvert A\rvert, \lvert B\rvert\}$ $A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u}$ $B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l}$

For a lowpass-to-bandpass transformation, it should be understood that a bandpass filter is essentially a combination of a lowpass filter and a highpass filter. The transformation is given by

$$s \longrightarrow \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}$$

Finally, the lowpass-to-bandstop transformation is

$$s \longrightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

The analog-to-analog transformations are summarized in Table 4.2. The table also supplies the design equations for backward development. For example, if Ω'_S is the desired critical stopband edge frequency of the transformed lowpass filter, the backward design equation gives the value of Ω_S that must be used in the design of the normalized lowpass filter such that going through the transformation, $s \longrightarrow \frac{s}{\Omega_u}$ to the normalized lowpass filter results in the required Ω'_S.

The backward design equations are needed for designing a normalized lowpass filter, which will then be mapped into a desired filter by applying the appropriate transformations.

4.6 Design of Lowpass Butterworth Filters

We are required to design a lowpass Butterworth filter to meet the following frequency-domain specifications.

$$K_P \leqslant 20 \log |H(j\Omega)| \leqslant 0 \qquad \text{for all } \Omega \leqslant \Omega_P$$

and

$$20 \log |H(j\Omega)| \leqslant K_S \qquad \text{for all } \Omega \geqslant \Omega_S$$

It may be noted that Ω_P and Ω_S are the passband and stopband edge frequencies, and K_P and K_S are the corresponding gains at these frequencies. These edge frequencies and the corresponding gains at these frequencies are shown in Fig. 4.6.

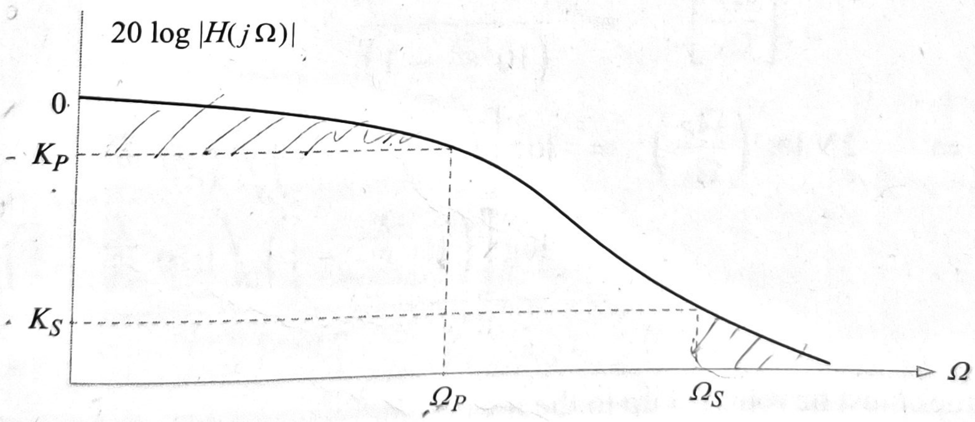


Fig. 4.6 Frequency-domain specifications of a lowpass Butterworth filter.

The design of a lowpass filter amounts to the determination of its transfer function. This necessitates the value of the filter order N and cutoff frequency Ω_C.

The magnitude frequency response of a lowpass Butterworth filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + \left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}} \tag{4.4}$$

Taking 20 log on both the sides of equation (4.4), we get

$$20\log|H(j\Omega)| = K = -20\log\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}$$

$$= -10\log\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right] \tag{4.5}$$

From Fig. 4.6, we find that at $\Omega = \Omega_P$, $K = K_P$. Making use of this fact in equation (4.5), we get

$$K_P = -10\log\left[1+\left(\frac{\Omega_P}{\Omega_C}\right)^{2N}\right]$$

$$\Rightarrow \left[\frac{\Omega_P}{\Omega_C}\right]^{2N} = 10^{\frac{-K_P}{10}} - 1 \tag{4.6}$$

Similarly, at $\Omega = \Omega_S$, we find from Fig. 4.6 that $K = K_S$. Making use of this condition in equation (4.5), we get

$$K_S = -10\log\left[1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}\right]$$

$$\Rightarrow \left[\frac{\Omega_S}{\Omega_C}\right]^{2N} = 10^{\frac{-K_S}{10}} - 1 \tag{4.7}$$

Dividing equation (4.6) by equation (4.7), we get

$$\left[\frac{\Omega_P}{\Omega_S}\right]^{2N} = \frac{\left(10^{\frac{-K_P}{10}} - 1\right)}{\left(10^{\frac{-K_S}{10}} - 1\right)}$$

$$\Rightarrow 2N\log\left(\frac{\Omega_P}{\Omega_S}\right) = \log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]$$

Therefore,

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} \tag{4.8}$$

The filter order must be rounded up to the next larger integer value. For example, if $N = 2.3$, we take $N = 3$.

Once, the order N gets decided, the procedure for finding the cutoff frequency Ω_C is as follows.

1. If we desire to meet the passband requirement exactly and do better than our stopband requirement, the cutoff frequency Ω_C is selected from equation (4.6) as

$$\Omega_C = \frac{\Omega_P}{\left(10^{\frac{-K_P}{10}} - 1\right)^{\frac{1}{2N}}} \tag{4.9}$$

2. If we wish to meet our requirement at Ω_S precisely and do better than our requirement at Ω_P, we find the cutoff frequency Ω_C from equation (4.7) as

$$\Omega_C = \frac{\Omega_S}{\left(10^{\frac{-K_S}{10}} - 1\right)^{\frac{1}{2N}}} \tag{4.10}$$

3. The third option is to take the cutoff frequency as the arithmetic mean of the two cutoff frequencies found above.

Example 4.1 A Butterworth lowpass filter has to meet the following specifications.

a. Passband gain, $K_P = -1$ dB at $\Omega_P = 4$ rad/sec.

b. Stopband attenuation greater than or equal to 20 dB at $\Omega_S = 8$ rad/sec.

Determine the transfer function $H_a(s)$ of the lowest-order Butterworth filter to meet the above specifications.

□ Solution

The specified magnitude frequency response of the lowpass Butterworth filter is shown in Fig. Ex.4.1.

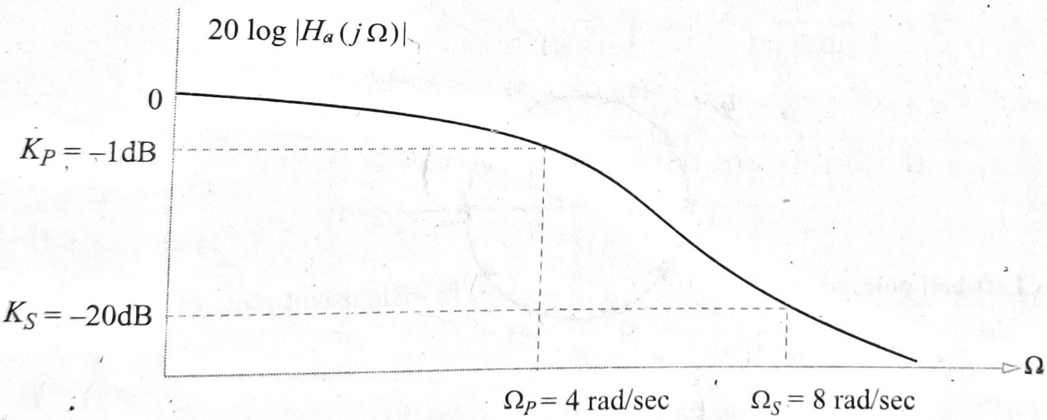


Fig. Ex.4.1 Specified lowpass magnitude frequency response.

Filter order,

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)}$$

$$= 4.289$$

4.05

Rounding off to next larger integer, we get $N = 5$.

Let us now, proceed to find the transfer function of the 5th order normalized lowpass filter. The poles of the normalized lowpass Butterworth filter are located at

$$s_k = 1\,\angle\theta_k\,, \qquad k = 0, 1, \cdots 2N - 1$$

where
$$\theta_k = \frac{\pi k}{N} + \frac{\pi}{2N} + \frac{\pi}{2}$$

Hence,
$$\begin{aligned}
s_0 &= 1 \angle \tfrac{6\pi}{10} = -0.309 + j0.951 \\
s_1 &= 1 \angle \tfrac{8\pi}{10} = -0.809 + j0.588 \\
s_2 &= 1 \angle \pi = -1 \\
s_3 &= 1 \angle \tfrac{12\pi}{10} = -0.809 - j0.588 \\
s_4 &= 1 \angle \tfrac{14\pi}{10} = -0.309 - j0.951 \\
s_5 &= 1 \angle \tfrac{16\pi}{10} = 0.309 - j0.951 \\
s_6 &= 1 \angle \tfrac{18\pi}{10} = 0.809 - j0.588 \\
s_7 &= 1 \angle 2\pi = 1 \\
s_8 &= 1 \angle \tfrac{22\pi}{10} = 0.809 + j0.588 \\
s_9 &= 1 \angle \tfrac{24\pi}{10} = 0.309 + j0.951
\end{aligned}$$

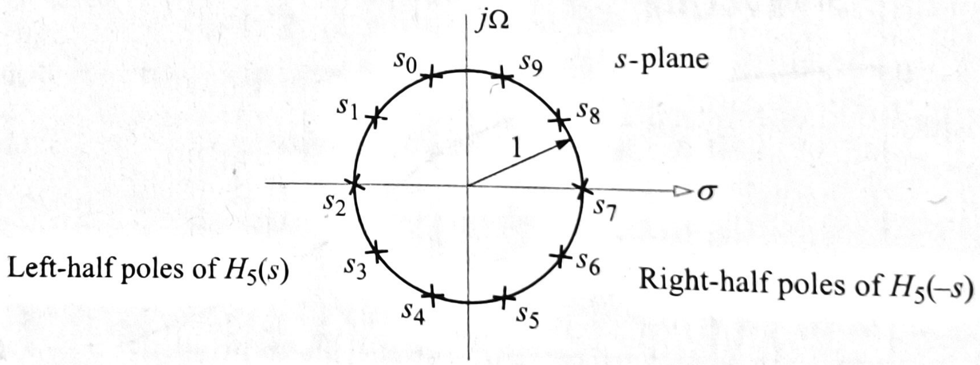


Fig. Ex.4.1 Poles of $H_5(s)H_5(-s)$.

Hence, the transfer function of the 5th order normalized lowpass Butterworth filter is

$$\begin{aligned}
H_5(s) &= \frac{1}{\prod\limits_{\substack{\text{LHP}\\ \text{only}}} (s - s_k)} \\
&= \frac{1}{(s - s_0)(s - s_1)(s - s_2)(s - s_3)(s - s_4)}
\end{aligned}$$

$$\Rightarrow H_5(s) = \frac{1}{(s + 0.309 - j0.951)(s + 0.809 - j0.588)(s + 1)(s + 0.809 + j0.588)(s + 0.309 + j0.951)}$$

$$= \frac{1}{(s + 1)\left[(s + 0.309)^2 + (0.951)^2\right]\left[(s + 0.809)^2 + (0.588)^2\right]}$$

$$= \frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)}$$

$$= \frac{1}{s^5+3.236s^4+5.236s^3+5.236s^2+3.236s+1}$$

Let us now find the cutoff frequency Ω_C to exactly meet the -1 dB requirement.

$$\Omega_C = \frac{\Omega_P}{\left(10^{\frac{-K_P}{10}}-1\right)^{\frac{1}{2N}}}$$

$$= 4.5787 \text{ rad/sec}$$

$s \rightarrow \frac{\Omega_c}{\Omega_c'} \cdot s$

The specified lowpass filter is obtained by applying lowpass-to-lowpass transformation on the normalized lowpass filter.

That is,

$$H_a(s) = H_5(s)\big|_{s\rightarrow\frac{s}{\Omega_C}}$$

$$= H_5(s)\big|_{s\rightarrow\frac{s}{4.5787}}$$

$$= \frac{1}{\left(\frac{s}{4.5787}\right)^5+3.236\left(\frac{s}{4.5787}\right)^4+5.236\left(\frac{s}{4.5787}\right)^3 +5.236\left(\frac{s}{4.5787}\right)^2+3.236\left(\frac{s}{4.5787}\right)+1}$$

$$= \frac{2012.4}{s^5+14.82s^4+109.8s^3+502.6s^2+1422.3s+2012.4}$$

Verification of the design

The frequency response of an analog filter is obtained by letting $s = j\Omega$ in its transfer function.

Hence,

$$H_a(j\Omega) = \frac{2012.4}{(j\Omega)^5+14.82(j\Omega)^4+109.8(j\Omega)^3+502.6(j\Omega)^2 +1422.3(j\Omega)+2012.4}$$

$$= \frac{2012.4}{(14.82\Omega^4-502.6\Omega^2+2012.4)+j\left(\Omega^5-109.8\Omega^3+1422.3\Omega\right)}$$

$$\Rightarrow \quad |H_a(j\Omega)| = \frac{2012.4}{\sqrt{\left(14.82\Omega^4-502.6\Omega^2+2012.4\right)^2+\left(\Omega^5-109.8\Omega^3+1422.3\Omega\right)^2}}$$

Hence,

$$20\log|H_a(j\Omega)|_{\Omega=4} = -1 \text{ dB}$$

and

$$20\log|H_a(j\Omega)|_{\Omega=8} = -24 \text{ dB}$$

Example 4.2 Let $H(s) = \frac{1}{s^2+s+1}$ represent the transfer function of a lowpass filter (not Butterworth) with a passband of 1 rad/sec. Use frequency transformations to find the transfer functions of the following analog filters.

a. A lowpass filter with a passband of 10 rad/sec.

b. A highpass filter with a cutoff frequency of 1 rad/sec.

c. A highpass filter with a cutoff frequency of 10 rad/sec.

d. A bandpass filter with a passband of 10 rad/sec and a center frequency of 100 rad/sec.

e. A bandstop filter with a stopband of 2 rad/sec and a center frequency of 10 rad/sec.

Ω_u = passband edge freq of new lowpass filter. $\Omega_u - \Omega_l =$

$\Omega_0 = \sqrt{\Omega_u \Omega_l}$

□ Solution

Given
$$H(s) = \frac{1}{s^2+s+1}$$

a. The lowpass-to-lowpass transformation is

$$s \longrightarrow \frac{s}{\Omega_u}$$

Hence, the required lowpass filter is

$$\begin{aligned} H_a(s) &= H(s)\big|_{s\longrightarrow \frac{s}{10}} \\ &= \frac{1}{\left(\frac{s}{10}\right)^2 + \left(\frac{s}{10}\right) + 1} = \frac{100}{s^2+10s+100} \end{aligned}$$

b. The lowpass-to-highpass transformation is

$$s \longrightarrow \frac{\Omega_u}{s}$$

Hence, the required highpass filter is

$$\begin{aligned} H_a(s) &= H(s)\big|_{s\longrightarrow \frac{1}{s}} \\ &= \frac{s^2}{s^2+s+1} \end{aligned}$$

c.
$$\begin{aligned} H_a(s) &= \left.\frac{1}{s^2+s+1}\right|_{s\longrightarrow \frac{10}{s}} \\ &= \frac{1}{\left(\frac{10}{s}\right)^2 + \frac{10}{s} + 1} \\ &= \frac{s^2}{s^2+10s+100} \end{aligned}$$

d. The lowpass-to-bandpass transformation is

$$s \longrightarrow \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s^2 + \Omega_o^2}{s B_o}$$

$\Omega_o = 100$ rad/s.

where $\boxed{\Omega_o = \sqrt{\Omega_u \Omega_l}}$ is the center frequency of the bandpass filter and $\boxed{B_o = \Omega_u - \Omega_l}$ is the width of the passband.

$= 10$ rad/s.

Hence, the required bandpass filter is

$$\begin{aligned} H_a(s) &= \left.\frac{1}{s^2 + s + 1}\right|_{s \longrightarrow \frac{s^2 + 10^4}{10s}} \\ &= \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^5 s + 10^8} \end{aligned}$$

e. The lowpass-to-bandstop transformation is

$$s \longrightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} = \frac{s B_o}{s^2 + \Omega_o^2}$$

Hence, the required bandstop filter is

$$\begin{aligned} H_a(s) &= \left.\frac{1}{s^2 + s + 1}\right|_{s \longrightarrow \frac{2s}{s^2 + 100}} \\ &= \frac{(s^2 + 100)^2}{s^4 + 2s^3 + 204s^2 + 200s + 10^4} \end{aligned}$$

Example 4.3 Design an analog bandpass filter to meet the following frequency-domain specifications:

a. a -3.0103 dB upper and lower cutoff frequency of 50 Hz and 20 KHz,

b. a stopband attenuation of atleast 20 dB at 20 Hz and 45 KHz, and

c. a monotonic frequency response.

☐ Solution

The monotonic frequency response can be achieved by using a Butterworth filter. Fig. Ex.4.3(a) shows the frequency response of the required bandpass filter with all the critical requirements.

The recipe is to first design a normalized lowpass filter and then apply the appropriate transformation to get the desired bandpass filter.

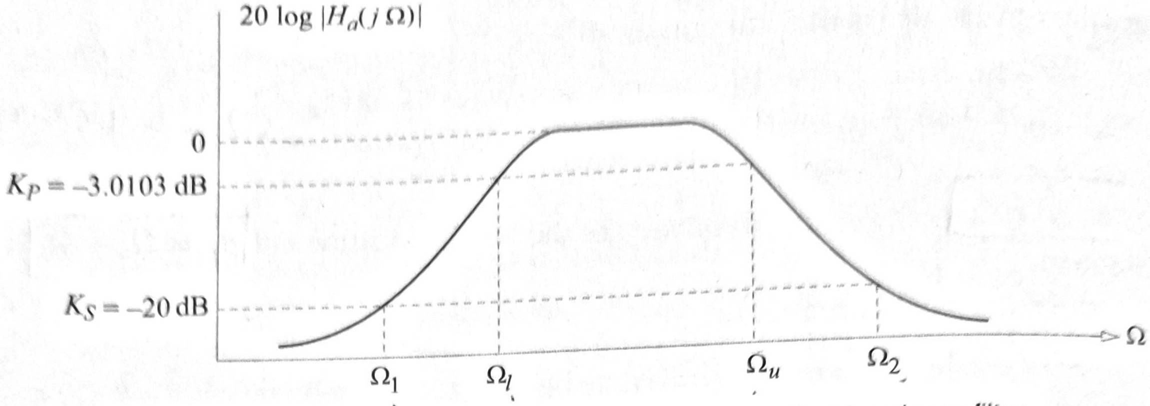


Fig. Ex.4.3(a) Magnitude frequency response of the specified bandpass filter.

We have,

$$\begin{aligned}\Omega_1 &= 2\pi \times 20 = 125.663 \text{ rad/sec}\\ \Omega_2 &= 2\pi \times 45 \times 10^3 = 2.827 \times 10^5 \text{ rad/sec}\\ \Omega_u &= 2\pi \times 20 \times 10^3 = 1.257 \times 10^5 \text{ rad/sec}\\ \Omega_l &= 2\pi \times 50 = 314.159 \text{ rad/sec}\end{aligned}$$

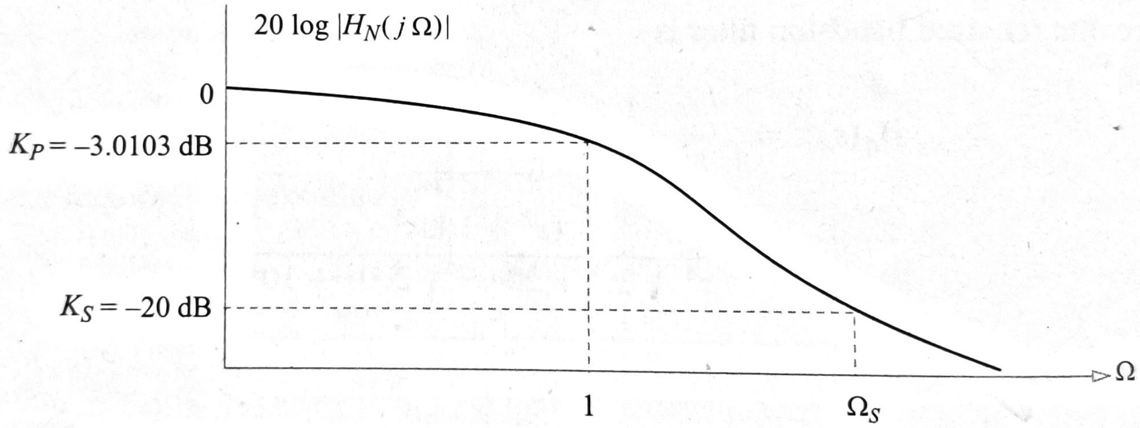


Fig. Ex.4.3(b) Magnitude frequency response of the normalized lowpass filter.

Let us use the backward design equation (Refer Table 4.2) and find the stopband edge frequency Ω_S of the normalized lowpass filter (Refer Fig. Ex.4.3(b)).

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1 (\Omega_u - \Omega_l)} = 2.51$$

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2 (\Omega_u - \Omega_l)} = 2.25$$

Hence,

$$\Omega_S = \text{Min}\{|A|, |B|\} = 2.25$$

The order N of the normalized lowpas Butterworth filter is computed as follows.

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right) \Big/ \left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2 \log\left(\frac{1}{\Omega_S}\right)}$$

$$= \frac{\log\left[\left(10^{0.30103} - 1\right)/\left(10^{2} - 1\right)\right]}{2\log\left(\frac{1}{2.25}\right)} = 2.83$$

Rounding off to the next larger integer, we get, $N = 3$.

Referring to the normalized Butterworth filter table given in Appendix I (or Table 4.1), we have lowpass prototype as

$$H_3(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

Applying the lowpass-to-bandpass transformation to $H_3(s)$, we get the required bandpass filter.

That is,

$$H_a(s) = H_3(s)\Big|_{s \longrightarrow \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}}$$

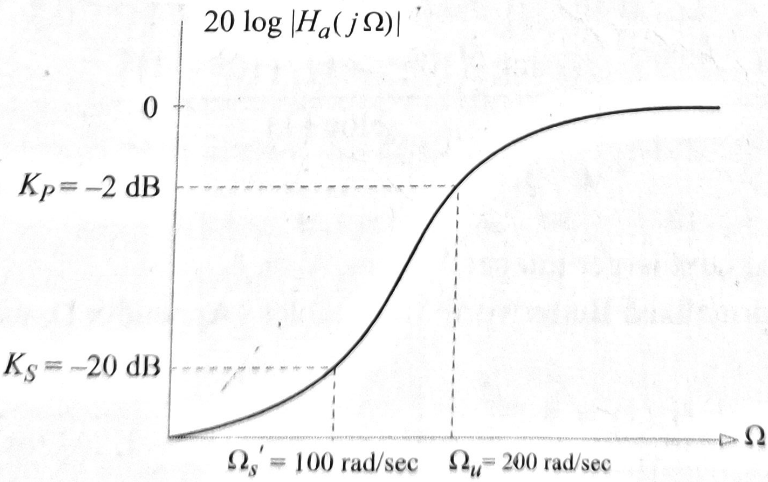
$$= H_3(s)\Big|_{s \longrightarrow \frac{s^2+3.949\times10^7}{s(1.2538\times10^5)}}$$

$$\Rightarrow \quad H_a(s) = \frac{1.9695\times10^{15}s^3}{\left(s^6+2.51\times10^5s^5+3.154\times10^{10}s^4+1.989\times10^{15}s^3 + 1.2453\times10^{18}s^2+3.9073\times10^{20}s+6.1529\times10^{22}\right)}$$

Example 4.4 Design a Butterworth analog highpass filter that will meet the following specifications:

a. Maximum passband attenuation = 2 dB.
b. Passband edge frequency = 200 rad/sec.
c. Minimum stopband attenuation = 20 dB.
d. Stopband edge frequency = 100 rad/sec.

Solution



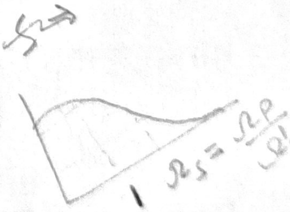


Fig. Ex.4.4(a) Specified highpass magnitude frequency response.

The first step in the analysis is to design a normalized lowpass Butterworth filter having the magnitude frequency response shown in Fig. Ex.4.4(b).

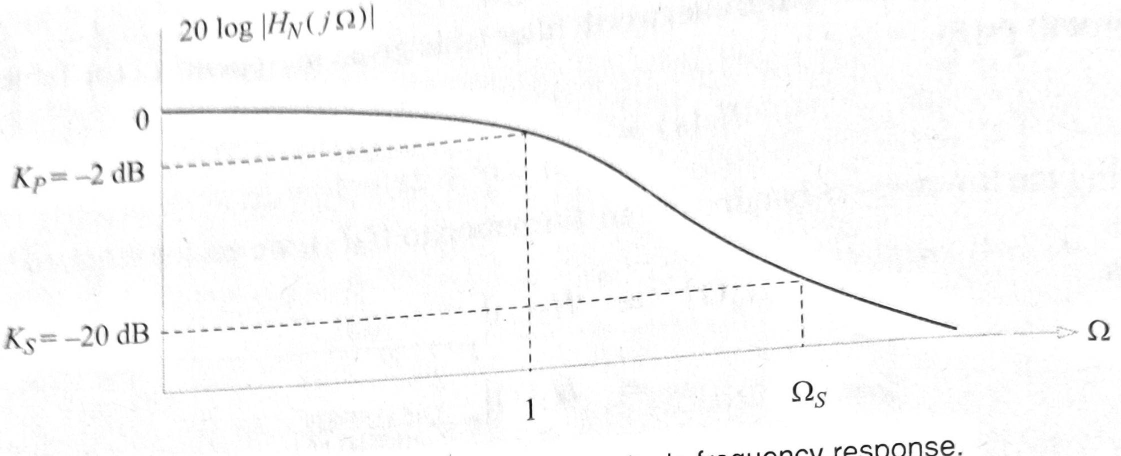


Fig. Ex.4.4(b) Normalized lowpass magnitude frequency response.

Using the backward design equation (Refer Table 4.2), we get

$$\Omega_S = \frac{\Omega_u}{\Omega'_S} = \frac{200}{100} = 2$$

The normalized lowpass Butterworth filter now has the following frequency-domain specifications:

$$\begin{aligned} \Omega_P &= 1, \quad K_P = -2 \text{ dB} \\ \Omega_S &= 2, \quad K_S = -20 \text{ dB} \end{aligned}$$

The order N of the normalized lowpass Butterworth filter is found as follows:

$$\begin{aligned} N &= \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} \\ &= \frac{\log\left[(10^{0.2} - 1)/(10^2 - 1)\right]}{2\log\left(\frac{1}{2}\right)} \\ &= 3.7 \end{aligned}$$

Rounding off to the next larger integer, we get, $N = 4$.
Referring to the normalized Butterworth filter tables (Appendix I), we get

$$H_4(s) = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

Let us now, determine the cutoff frequency Ω_C of the lowpass filter (prototype) so that –2 dB requirement is satisfied exactly.

That is,

$$\Omega_C = \frac{\Omega_P}{\left(10^{\frac{-K_P}{10}} - 1\right)^{\frac{1}{2N}}}$$

$$= \frac{1}{\left(10^{0.2} - 1\right)^{\frac{1}{8}}}$$

$$= 1.0693$$

Hence, the lowpass filter prototype is

$$H_P(s) = H_4(s)\Big|_{s \longrightarrow \frac{s}{\Omega_C}}$$

$$= H_4(s)\Big|_{s \longrightarrow \frac{s}{1.0693}}$$

To get the specified highpass filter $H_a(s)$, let us apply lowpass-to-highpass transformation on $H_P(s)$.

That is,

$$H_a(s) = H_P(s)\Big|_{s \longrightarrow \frac{\Omega_u}{s}}$$

$$= H_P(s)\Big|_{s \longrightarrow \frac{200}{s}}$$

$$= H_4(s)\Big|_{s \longrightarrow \frac{200}{1.0693s}}$$

$$= H_4(s)\Big|_{s \longrightarrow \frac{187.031}{s}}$$

$$= \frac{s^4}{\left(s^2 + 143.1464s + 34980.7521\right)\left(s^2 + 345.5892s + 34980.7521\right)}$$

Verification of the design

The frequency response of the highpass filter is obtained by letting $s = j\Omega$ in $H_a(s)$. Accordingly, we get

$$H_a(j\Omega) = \frac{\Omega^4}{\left[(34980.7521 - \Omega^2) + j143.1464\Omega\right]\left[(34980.7521 - \Omega^2) + j345.5892\Omega\right]}$$

$$\Rightarrow |H_a(j\Omega)| = \frac{\Omega^4}{\sqrt{\left[\left(34980.7521 - \Omega^2\right)^2 + (143.1464\Omega)^2\right]\left[\left(34980.7521 - \Omega^2\right)^2 + (345.5892\Omega)^2\right]}}$$

Therefore,

$$20 \log |H_a(j\Omega)|_{\Omega=200} = -2 \text{ dB}$$

and

$$20 \log |H_a(j\Omega)|_{\Omega=100} = -21.83 \text{ dB}$$

Example 4.5 Design an analog lowpass (maximally flat) filter that will have a −1 dB cutoff frequency at 75 Hz and have greater than 20 dB of attenuation for all frequencies greater than 150 Hz. Verify the design.

☐ Solution

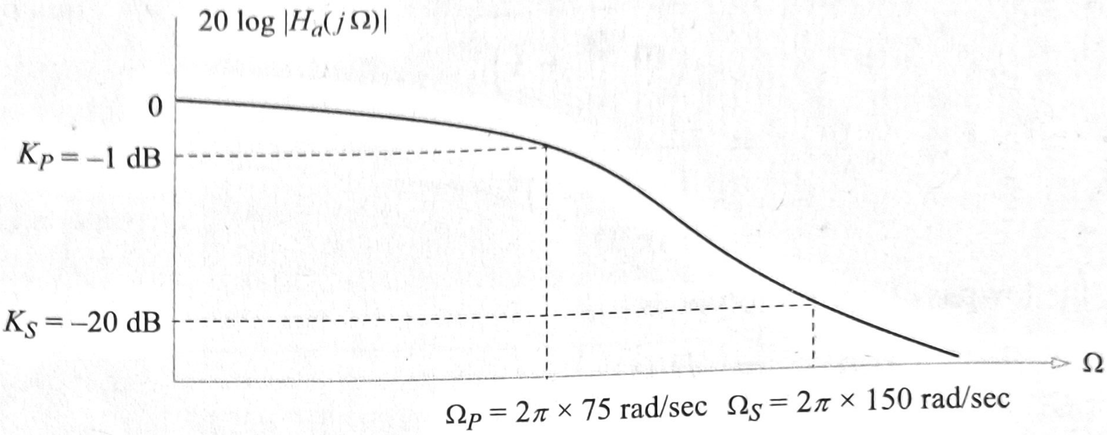


Fig. Ex.4.5 Magnitude frequency response of the specified lowpass filter.

The critical lowpass requirements are:

$$K_P = -1 \text{ dB}, \qquad K_S = -20 \text{ dB}$$
$$\Omega_P = 150\pi \text{ rad/sec}, \qquad \Omega_S = 300\pi \text{ rad/sec}$$

The lowpass Butterworth filter order is

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} = 4.28$$

Rounding off to the next larger integer, we get $N = 5$.

Referring to Table 4.1, the lowpass normalized Butterworth prototype is given by

$$H_5(s) = \frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)} \qquad \text{(Factored form)}$$

$$\Rightarrow \quad H_5(s) = \frac{1}{s^5+3.236s^4+5.236s^3+5.236s^2+3.236s+1} \qquad \text{(Polynomial form)}$$

Let us now, find the cutoff frequency, Ω_C to satisfy -1 dB requirement exactly and to exceed -20 dB requirement.

Accordingly,

$$\Omega_C = \frac{\Omega_P}{\left[10^{\frac{-K_P}{10}}-1\right]^{\frac{1}{2N}}} = 539.4158 \text{ rad/sec}$$

The desired lowpass filter $H_a(s)$ is obtained by applying, lowpass to-lowpass transformation on $H_5(s)$.

That is,

$$H_a(s) = H_5(s)\Big|_{s \longrightarrow \frac{s}{\Omega_C}}$$

$$= H_5(s)\Big|_{s \longrightarrow \frac{s}{539.4158}}$$

$$\Rightarrow \quad H_a(s) = \frac{4.56687 \times 10^{13}}{(s+539.4158)\left(s^2+333.35896s+290969.4053\right)\left(s^2+872.77476s+290969.4053\right)}$$

Verification of the design

The frequency response of the desired lowpass filter is obtained by letting $s = j\Omega$ in $H_a(s)$.

$$H_a(j\Omega) = \frac{4.56687 \times 10^{13}}{(j\Omega+539.4158)\left(-\Omega^2+333.35896j\Omega+290969.4053\right)\left(-\Omega^2+872.77476j\Omega+290969.4053\right)}$$

$$\Rightarrow \quad |H_a(j\Omega)| = \frac{4.56687 \times 10^{13}}{\sqrt{\begin{array}{c}(\Omega^2+539.4158^2)\times \\ \left[(-\Omega^2+290969.4053)^2+(333.35896\Omega)^2\right]\times \\ \left[(-\Omega^2+290969.4053)^2+(872.77476\Omega)^2\right]\end{array}}}$$

Hence,

$$20\log|H_a(j\Omega)|_{\Omega=150\pi} = -1 \text{ dB}$$

and

$$20\log|H_a(j\Omega)|_{\Omega=300\pi} = -24.25 \text{ dB}$$

Example 4.6 Consider a fifth-order lowpass Butterworth filter with a passband of 1 KHz and a maximum passband attenuation of 1 dB. What is the actual attenuation in dB, of the lowpass filter at a frequency of 2 KHz?

Solution

The magnitude frequency response of a lowpass Butterworth filter is

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}}$$

$$\Rightarrow \quad 20\log|H(j\Omega)| = -10\log\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right] \quad (4.12)$$

Given, $20\log|H(j\Omega)| = -1$ at $\Omega = 2\pi \times 1 \times 10^3$ rad/sec and $N = 5$.

Hence,

$$-1 = -10\log\left[1 + \left(\frac{2\pi \times 1 \times 10^3}{\Omega_C}\right)^{10}\right]$$

Solving, we get $\Omega_C = 7192.21$ rad/sec.

We are now required to find $20\log|H(j\Omega)|$ at $\Omega = 2\pi \times 2 \times 10^3 = 4\pi \times 10^3$ rad/sec.

Using equation (4.12), we get

$$\begin{aligned} 20\log|H(j\Omega)|_{\Omega=4\pi\times 10^3} &= -10\log\left[1 + \left(\frac{2\pi \times 2 \times 10^3}{7192.21}\right)^{10}\right] \\ &= -24.25 \text{ dB} \end{aligned}$$

Hence, the stopband attenuation = $A_S = 24.25$ dB.

Example 4.7 Let Ω_P and Ω_S denote the desired passband and stopband edge frequencies of an analog lowpass filter. Let δ_P be the passband ripple and δ_S be the stopband attenuation. Show that the order of the filter required to meet these specifications is

$$N \geqslant \frac{\log\left(\frac{1}{d}\right)}{\log\left(\frac{1}{K}\right)}$$

where $d = \sqrt{\dfrac{(1-\delta_P)^{-2} - 1}{\delta_S^{-2} - 1}}$ = discrimination factor

and $K = \dfrac{\Omega_P}{\Omega_S}$ = Selectivity factor.

Also show that the -3 dB cutoff frequency Ω_C being any value within the range

$$\Omega_P\left[(1-\delta_P)^{-2} - 1\right]^{-\frac{1}{2N}} \leqslant \Omega_C \leqslant \Omega_S\left(\delta_S^{-2} - 1\right)^{-\frac{1}{2N}}.$$

☐ **Solution**

Eventhough δ_P is called the *passband ripple*, we want to emphasize that there are no ripples in the passband. To be more appropriate, δ_P is the tolerance of the magnitude response in the passband.

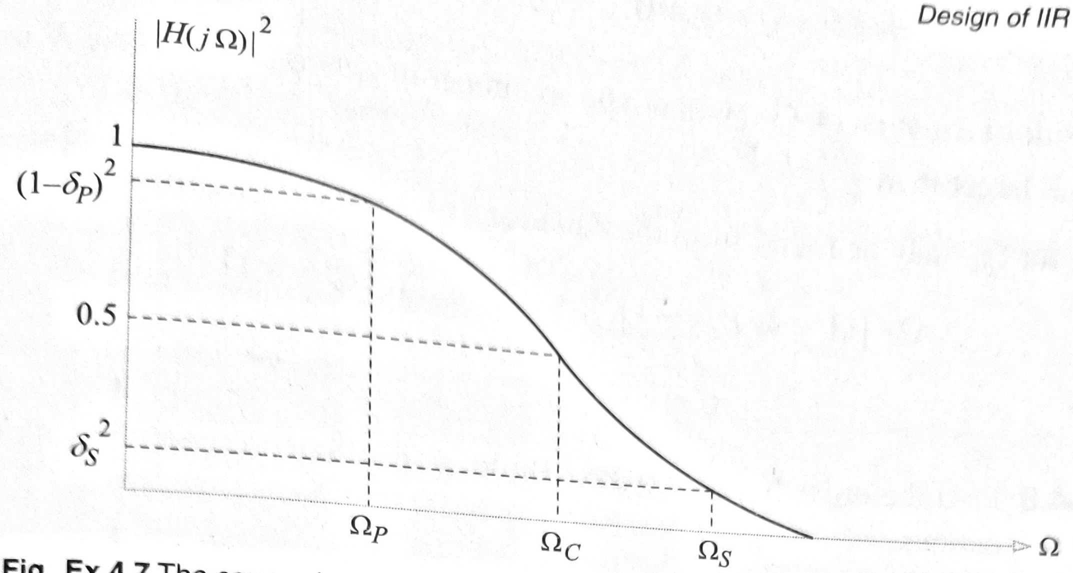


Fig. Ex.4.7 The squared magnitude frequency response of a lowpass Butterworth filter.

We observe from Fig. Ex.4.7 that

$$\begin{aligned}
|H(j\Omega_P)|^2 &\geqslant (1-\delta_P)^2 \\
\Rightarrow \quad \frac{1}{1+\left(\frac{\Omega_P}{\Omega_C}\right)^{2N}} &\geqslant (1-\delta_P)^2 \\
\Rightarrow \quad \left(\frac{\Omega_P}{\Omega_C}\right)^{2N} &\leqslant (1-\delta_P)^{-2}-1 \qquad (4.13a)
\end{aligned}$$

Also, we observe from Fig. Ex.4.7 that

$$\begin{aligned}
|H(j\Omega_S)|^2 &\leqslant \delta_S^2 \\
\Rightarrow \quad \frac{1}{1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}} &\leqslant \delta_S^2 \\
\Rightarrow \quad \left(\frac{\Omega_S}{\Omega_C}\right)^{2N} &\geqslant \delta_S^{-2}-1 \qquad (4.13b)
\end{aligned}$$

Dividing equation (4.13b) by equation (4.13a), we get

$$\left(\frac{\Omega_S}{\Omega_P}\right)^{2N} \geqslant \frac{\delta_S^{-2}-1}{(1-\delta_P)^{-2}-1}$$

Using the definitions of d and K in the above equation, we get

$$\begin{aligned}
\left(\frac{1}{K}\right)^{2N} &\geqslant \left(\frac{1}{d}\right)^2 \\
\Rightarrow \quad 2N\log\left(\frac{1}{K}\right) &\geqslant 2\log\left(\frac{1}{d}\right) \\
\Rightarrow \quad N &\geqslant \frac{\log\left(\frac{1}{d}\right)}{\log\left(\frac{1}{K}\right)} \qquad (4.13c)
\end{aligned}$$

The right side of equation (4.13c) will not be an integer in general, so we take N as the smallest integer which is larger than $\frac{\log\left(\frac{1}{d}\right)}{\log\left(\frac{1}{K}\right)}$.

The range for Ω_C may be found from the equations (4.13a) and (4.13b) as

$$\Omega_P\left[(1-\delta_P)^{-2}-1\right]^{\frac{-1}{2N}} \leqslant \Omega_C \leqslant \Omega_S\left[\delta_S^{-2}-1\right]^{\frac{-1}{2N}}$$

Example 4.8 Find the order N of a lowpass Butterworth filter to meet the following specifications.

$$\begin{aligned} \delta_P &= 0.001, & \delta_S &= 0.001 \\ \Omega_P &= 1\ \text{rad/sec}, & \Omega_S &= 2\ \text{rad/sec} \end{aligned}$$

☐ **Solution**

Discrimination factor,
$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 4.4755 \times 10^{-5}$$

Selectivity factor,
$$K = \frac{\Omega_P}{\Omega_S} = 0.5$$

We know that,
$$N \geqslant \frac{\log\left(\frac{1}{d}\right)}{\log\left(\frac{1}{K}\right)}$$
$$\Rightarrow \quad N \geqslant 14.45$$
$$\Rightarrow \quad N = 15$$

Alternate method

$$K_P = 20\log(1-\delta_P) = -8.69 \times 10^{-3}$$
$$K_S = 20\log\delta_S = -60$$
$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)}$$
$$= 14.45 = 15 \quad \text{(rounded off value).}$$

4.7 Chebyshev Filters

In order to understand the frequency-domain behaviour of Chebyshev filters, it is utmost important to define a Chebyshev polynomial and then its properties. A Chebyshev polynomial of degree N is defined as

Chebyshev filters.

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & |x| \leqslant 1 \\ \cosh(N \cosh^{-1} x), & |x| > 1 \end{cases} \tag{4.14}$$

Also, it is possible to generate Chebyshev polynomials using the following recursive formula.

$$\begin{aligned} T_N(x) &= 2xT_{N-1}(x) - T_{N-2}(x), \qquad N \geqslant 2 \\ \text{with} \quad T_0(x) &= 1 \quad \text{and} \quad T_1(x) = x \end{aligned} \tag{4.15}$$

The above relations stated by equation (4.15) can be derived using equation (4.14) as follows:

$$\begin{aligned} \text{Let} \quad x &= \cos\theta \\ \Rightarrow \quad \theta &= \cos^{-1} x \end{aligned}$$

Case(i) $|x| \leqslant 1$:

$$\begin{aligned} T_N(x) &= \cos\left(N \cos^{-1} x\right) \\ &= \cos N\theta \\ \Rightarrow \quad T_{N+1}(x) &= \cos((N+1)\theta) \\ &= \cos N\theta \cos\theta - \sin N\theta \sin\theta \\ \text{and} \quad T_{N-1}(x) &= \cos((N-1)\theta) \\ &= \cos N\theta \cos\theta + \sin N\theta \sin\theta \\ \text{Then} \quad T_{N+1}(x) + T_{N-1}(x) &= 2\cos N\theta \cos\theta \\ &= 2\cos\theta\, T_N(x) \\ \Rightarrow \quad T_{N+1}(x) &= 2\cos\theta\, T_N(x) - T_{N-1}(x) \\ &= 2xT_N(x) - T_{N-1}(x) \\ \text{Thus,} \quad T_N(x) &= 2xT_{N-1}(x) - T_{N-2}(x) \\ \text{Also,} \quad T_0(x) &= \cos 0 = 1 \\ T_1(x) &= \cos\left(\cos^{-1} x\right) = x \end{aligned} \tag{4.16}$$

Case (ii) $|x| > 1$:

$$\text{Let} \quad \theta = \cosh^{-1}(x) \quad \Rightarrow \quad T_N(x) = \cosh(N\theta) \tag{4.17}$$

$$\begin{aligned} \text{Then,} \quad T_{N+1}(x) &= \cosh((N+1)\theta) \\ &= \cosh(N\theta)\cosh\theta + \sinh(N\theta)\sinh\theta \\ \text{and} \quad T_{N-1}(x) &= \cosh((N-1)\theta) \\ &= \cosh(N\theta)\cosh\theta - \sinh(N\theta)\sinh\theta \\ \text{Hence,} \quad T_{N+1}(x) + T_{N-1}(x) &= 2\cosh(N\theta)\cosh\theta \\ \Rightarrow \quad T_{N+1}(x) + T_{N-1}(x) &= 2T_N(x)x \\ \text{Thus} \quad T_N(x) &= 2xT_{N-1}(x) - T_{N-2}(x) \\ \text{Also,} \quad T_0(x) = 1 \quad &\text{and} \quad T_1(x) = x \end{aligned}$$

We know that,
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$\Rightarrow \quad T_5\left(\frac{\Omega}{2\pi \times 10^3}\right) = 16\left(\frac{\Omega}{2\pi \times 10^3}\right)^5 - 20\left(\frac{\Omega}{2\pi \times 10^3}\right)^3 + 5\left(\frac{\Omega}{2\pi \times 10^3}\right)$$

Hence,
$$|H(j\Omega)| = \frac{1}{\left[1 + (0.50885)^2\left(16\left(\frac{\Omega}{2\pi\times10^3}\right)^5 - 20\left(\frac{\Omega}{2\pi\times10^3}\right)^3 + 5\left(\frac{\Omega}{2\pi\times10^3}\right)\right)^2\right]^{\frac{1}{2}}}$$

Hence,
$$A_1 = -20\log|H(j\Omega)|_{\Omega=2\pi\times10^3} = 1\text{ dB}$$
and
$$A_2 = -20\log|H(j\Omega)|_{\Omega=2\pi\times2\times10^3} = 45.31\text{ dB}$$

4.8 Digital Filters

In the previous sections, we have perfected the procedure for designing analog filters that is straightforward and almost mechanical. However, the real goal of this chapter is to design *recursive* filters (IIR filters). The design of digital IIR filter is largely based on analog filter design techniques. To simulate an analog filter, the digital filter is used in A/D-Digital filter-D/A structure as shown in Fig. 4.9.

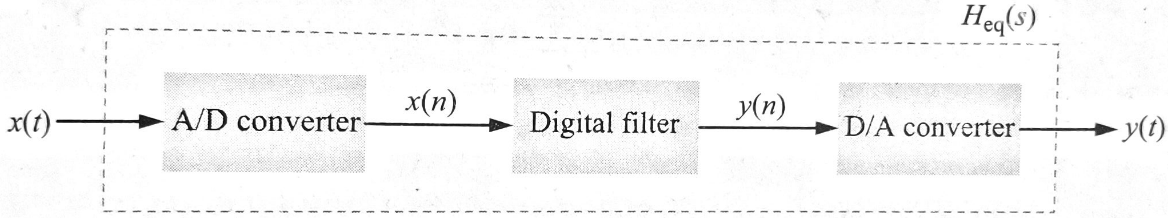


Fig. 4.9 Simulation of an analog filter.

The A/D-Digital filter-D/A structure behaves like an equivalent analog filter having a transfer function, $H_{eq}(s)$. A digital filter takes a discrete-time input sequence $x(n)$ and produces a discrete time output sequence $y(n)$. A special class of digital filters can be characterised by a unit sample response $h(n)$, a system function $H(z)$, or a difference equation realization. A typical design procedure of a digital IIR filter involves the following steps.

1. Selecting a method of transformation of a given analog filter to a digital filter having roughly the same frequency response.
2. Mapping the specifications of the digital IIR filter to equivalent specifications of an analog IIR filter such that, after the mapping from analog to digital is carried out, the digital IIR filter will meet the given specifications.
3. Designing the analog IIR filter according to the mapped specifications.
4. Transforming the analog filter to an equivalent digital filter.

The next few sections will look at different transformations that map a given analog filter to a digital filter having roughly the same frequency response. They are:

i. the backward difference method,
ii. the bilinear transformation,
iii. the impulse invariant transformation, and
iv. the matched-$\mathcal{Z}$ transformation.

Bilinear Transformation

4.10 Bilinear Transformation

The bilinear transformation is used for transforming an analog filter to a digital filter. The bilinear transformation can be regarded as a correction of the backward difference method. Bilinear transformation uses trapezoidal rule for integrating a continuous-time function. It is defined by the substitution:

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

To derive this transformation, let us consider the derivative,

$$\frac{dy(t)}{dt} = x(t) \tag{4.35}$$

Taking Laplace transform on both the sides of the above equation with all initial conditions neglected, we get

$$sY(s) = X(s) \tag{4.36}$$

To find $y(t)$ within the limits $(n-1)T$ and nT, where T is the sampling period, we integrate both the sides of equation (4.35) with respect to t. Accordingly, we get

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt}dt = \int_{(n-1)T}^{nT} x(t)\,dt \tag{4.37}$$

$$\Rightarrow \quad y(nT) - y[(n-1)T] = \int_{(n-T)T}^{nT} x(t)dt \tag{4.38}$$

The integral on the right-hand side of equation (4.38) is approximated by the trapezoidal rule. This rule states that if T is small, the area (integral) can be approximated by the mean height of $x(t)$ between the two limits and then multiplying by the width. That is,

$$\int_{(n-1)T}^{nT} x(t)dt = \left[\frac{x(nT) + x[(n-1)T]}{2}\right]T \tag{4.39}$$

Equating equations (4.38) and (4.39), we get

$$y(nT) - y[(n-1)T] = \left[\frac{x(nT) + x[(n-1)T]}{2}\right]T$$

Since, $x(n) \triangleq x(nT)$ and $y(n) \triangleq y(nT)$, the above equation may be written as

$$y(n) - y(n-1) = \left[\frac{x(n) + x(n-1)}{2}\right]T$$

Taking $\mathcal{Z}$-transform of the above equation gives

$$Y(z) - z^{-1}Y(z) = \left[\frac{X(z) + z^{-1}X(z)}{2}\right]T$$

$$\Rightarrow \quad Y(z)[1 - z^{-1}] = \frac{[1 + z^{-1}]T\,X(z)}{2}$$

$$\Rightarrow \quad X(z) = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]Y(z) \tag{4.40}$$

Comparing equations (4.36) and (4.40), we get the transformation from s to z as

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right] \tag{4.41}$$

Similarly, the transformation from z to s is

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

Unlike the backward difference method, the left-half s-plane is now mapped entirely inside the unit circle, $|z| = 1$, rather than to a part of it. Also, the imaginary axis is mapped to the unit circle, refer Fig. 4.11. Therefore, equation (4.41) is a true frequency-to-frequency transformation. This is proved as follows.

We know that

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right] = \frac{2}{T}\left[\frac{z - 1}{z + 1}\right]$$

Putting $s = \sigma + j\Omega$ and $z = re^{j\omega}$ in the above equation, we get

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right] = \frac{2}{T}\frac{[(r\cos\omega - 1) + jr\sin\omega]}{[(r\cos\omega + 1)] + jr\sin\omega]}$$

Rationalizing the right-hand side of the above equation, we get

$$\sigma + j\Omega = \frac{2}{T}\frac{[(r\cos\omega - 1) + jr\sin\omega]}{[(r\cos\omega + 1) + jr\sin\omega]} \times \frac{[(r\cos\omega + 1) - jr\sin\omega]}{[(r\cos\omega + 1) - jr\sin\omega]}$$

$$\Rightarrow \quad \sigma + j\Omega = \frac{2}{T}\left[\frac{r^2 - 1}{r^2 + 1 + 2r\cos\omega} + j\frac{2r\sin\omega}{r^2 + 1 + 2r\cos\omega}\right] \tag{4.42}$$

Equating real and imaginary parts on both the sides of equation (4.42), we get

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{r^2 + 1 + 2r\cos\omega}\right] \tag{4.43}$$

and

$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{r^2 + 1 + 2r\cos\omega}\right] \tag{4.44}$$

From the above discussion, following conclusions are made.

1. If $r < 1$, we get $\sigma < 0$. This means that the left-hand side of the s-plane is mapped inside the circle, $|z| = 1$.
2. If $r > 1$, we get $\sigma > 0$. Thus, the right-hand side of the s-plane is mapped outside the circle $|z| = 1$.
3. If $r = 1$, we get $\sigma = 0$. This shows that the imaginary axis in the s-domain is mapped to the circle of unit radius centered at $z = 0$ in the z-domain.

Fig. 4.11 shows the mapping of the s-plane into the z-plane with the bilinear transformation.

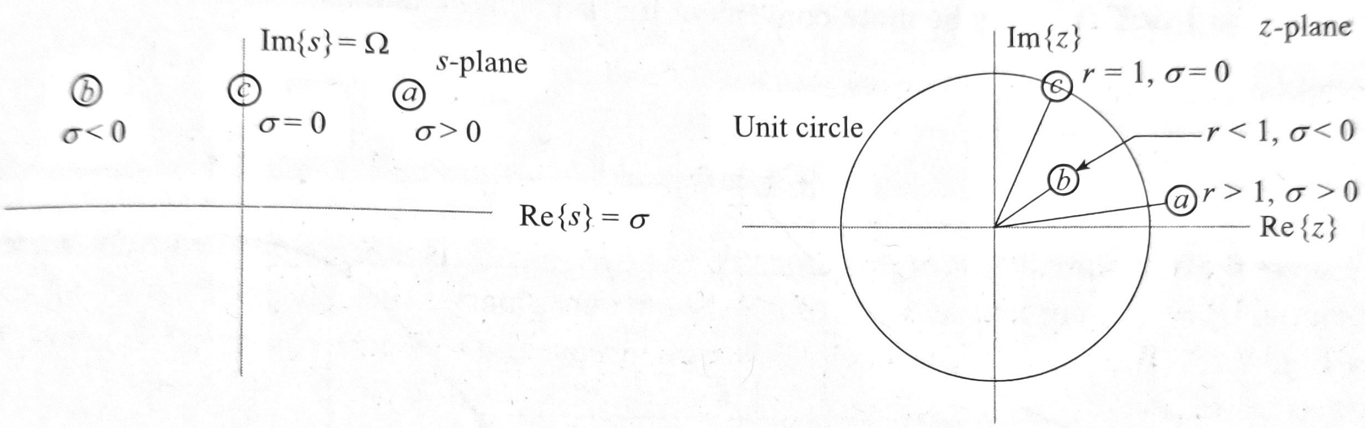


Fig. 4.11 Effect of bilinear transformation: $s = \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}}$.

The above analysis means that the bilinear transformation preserves the stability of the transformed filter.

When $r = 1$, we get $\sigma = 0$ and

$$\Omega = \frac{2}{T}\left[\frac{2\sin\omega}{1^2 + 1 + 2\cos\omega}\right]$$

$$\Rightarrow \quad \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \tag{4.45}$$

$$\Rightarrow \quad \omega = \frac{2}{T}\tan^{-1}\left(\frac{\Omega T}{2}\right) \tag{4.46}$$

The digital-domain frequency ω is therefore warped with respect to the analog frequency Ω, the warping function being $\frac{2}{T}\tan^{-1}\left(\frac{\Omega T}{2}\right)$. The analog frequencies, $\Omega = \pm\infty$ are mapped to digital frequencies, $\omega = \pm\pi$. The frequency mapping is not aliasied; that is, the relationship between Ω and ω is one-to-one. As a consequence of this, there are no major restrictions on the use of bilinear transformation; it is adequate for all filter types.

Fig. 4.12 shows the frequency warping introduced by the bilinear transformation. The filter taken for illustrating frequency response is a lowpass filter having ripples in both the bands. To circumvent the frequency warping introduced by the bilinear transformation, it is necessary to prewarp the specifications of the analog filter, so that after warping they will be located at the desired frequencies. Suppose, we wish to design a lowpass digital filter with passband and stopband edge frequencies ω_P and ω_S. We first convert these frequencies to corresponding analog-domain band-edge frequencie Ω'_p and Ω'_s using the following equations:

$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)$$
$$\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right).$$

Using Ω'_P and Ω'_S, we design the analog filter. Finally, we obtain the required digital lowpass filter by using the bilinear transformation on the analog filter. Since, prewarping is done in the beginning of the design procedure, and bilinear transformation is performed in the end, the value of T used is immaterial, as long as it is the same in both. Eventhough T is the sampling interval, taking $T = 1$ or $T = 2$ may be more convenient for hand computations.

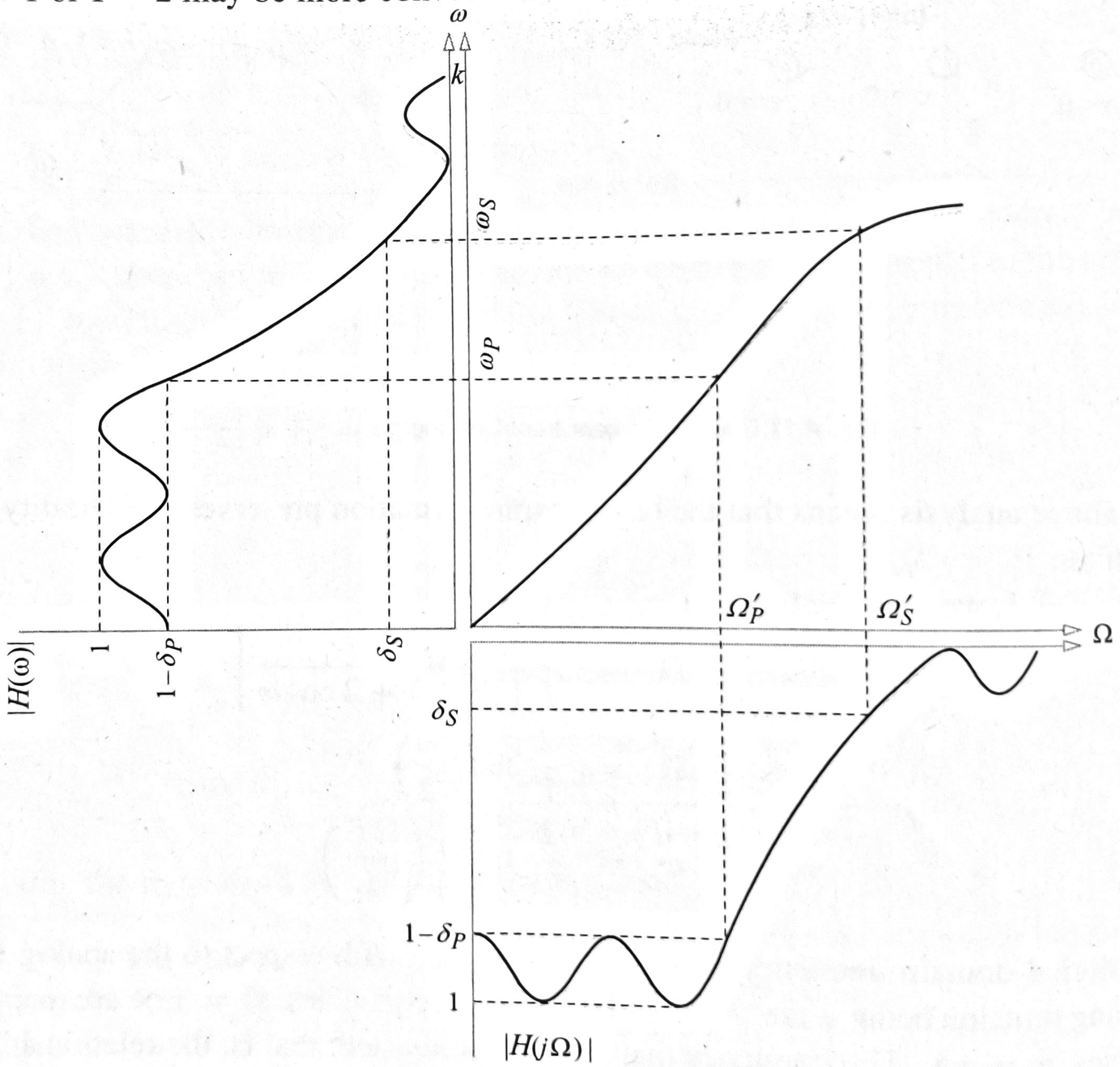


Fig. 4.12 Mapping of an analog lowpass filter to digital lowpass filter via the bilinear transformation.

The summary of design procedure for a digital IIR filter is as follows:

1. Prewarp each specified band-edge frequency of the digital filter to a corresponding band-edge frequency of an analog filter using the equations given below:

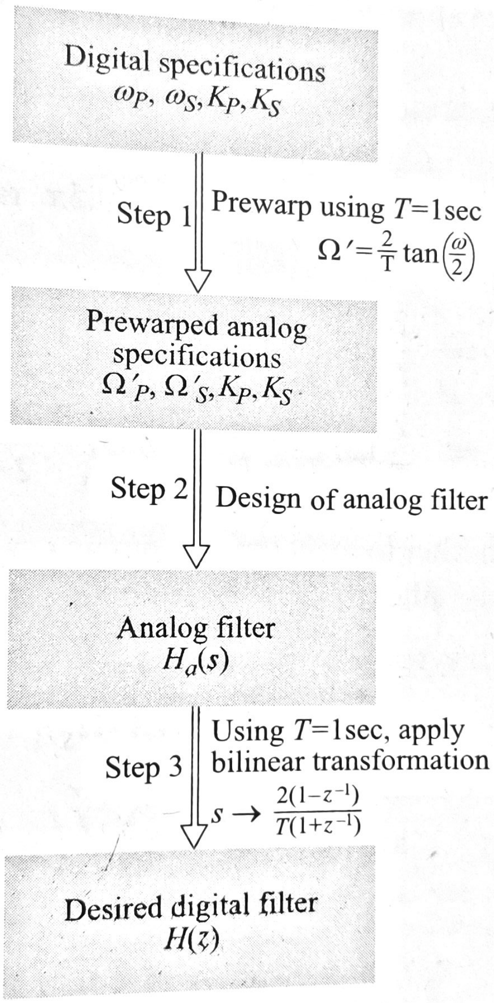
$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)$$
$$\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right)$$

Take $T = 1$, since it is more convenient for hand computations and has no effect on the design if it is maintained at the same value when the bilinear transformation is performed in the end.

During prewarping, passband gain $K_P = 20\log(1-\delta_P)$ and stopband gain $K_S = 20\log\delta_S$ remain unchanged.

2. Design an analog filter $H_a(s)$ of the desired type, according to prewarped specifications.
3. Transform $H_a(s)$ to a digital filter $H(z)$ using $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ with $T = 1$ sec.

The above mentioned steps are represented by a flowchart, which is much easier to remember.



Example 4.16 A digital lowpass filter is required to meet the following specifications:

a. Monotonic passband and stopband.

b. -3.01 dB cutoff frequency of 0.5π rad.

c. Stopband attenuation of atleast 15 dB at 0.75π rad.

Find the system function $H(z)$ and the difference equation realization. Verify the design by checking for the passband and stopband specifications.

N = 2
fc = 0.5083
b = 0 0 0.2583
a = 1 0.7188 0.2583

☐ **Solution**

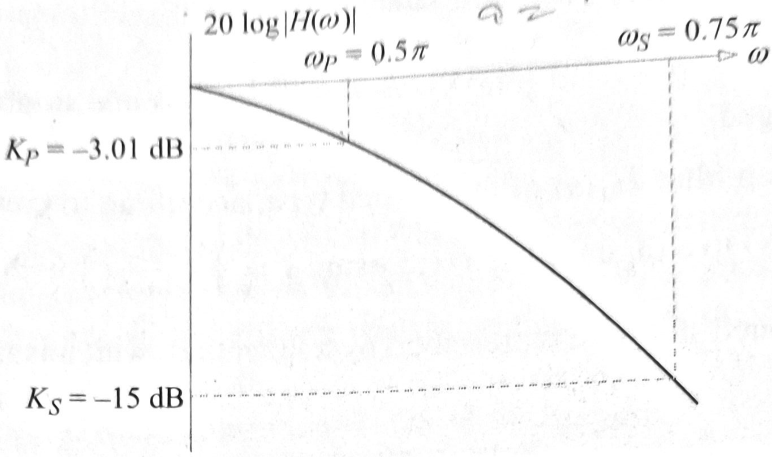


Fig. Ex.4.16 Magnitude frequency response of the specified lowpass digital IIR filter.

Step 1: Prewarp the band-edge frequencies $\omega_P = 0.5\pi$ rad and $\omega_S = 0.75\pi$ rad using $T = 1$ sec to get Ω'_P and Ω'_S.

$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) = \frac{2}{1}\tan\left(\frac{0.5\pi}{2}\right) = 2$$

$$\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) = \frac{2}{1}\tan\left(\frac{0.75\pi}{2}\right) = 4.8282$$

Step 2: Choose a Butterworth filter to meet the monotonic passband and stopband requirements. Let us design a lowpass filter to meet the following specifications:

$$K_P = -3.01 \text{ dB} \leqslant 20\log\left|H_a(j\Omega'_P)\right| \leqslant 0$$

and

$$20\log\left|H_a(j\Omega'_S)\right| \leqslant K_S = -15 \text{ dB}$$

Order,

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega'_P}{\Omega'_S}\right)} = 1.94$$

Rounding off to the next larger integer, we get $N = 2$. Referring to normalized lowpass Butterworth filter tables, we get

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Let us determine the cutoff frequency Ω_C to meet the passband requirement precisely.

$$\Omega_C = \frac{\Omega'_P}{\left[10^{\frac{-K_P}{10}} - 1\right]^{\frac{1}{2N}}} = 2 \text{ rad/sec}$$

The prewarped analog lowpass filter $H_a(s)$ is obtained by using lowpass-to-lowpass transformation to $H_2(s)$

That is,

$$\begin{aligned} H_a(s) &= H_2(s)\big|_{s \to \frac{s}{\Omega_C}} \\ &= \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s\to\frac{s}{2}} \\ &= \frac{4}{s^2+2\sqrt{2}s+4} \end{aligned}$$

Step 3: Obtain $H(z)$ by applying bilinear transformation ($T = 1$) to $H_a(s)$.

That is,

$$\begin{aligned} H(z) &= H_a(s)\big|_{s\to 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{4}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 2\sqrt{2}\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+4} \\ &= \frac{(1+z^{-1})^2}{3.4142+0.5858z^{-2}} \end{aligned}$$

$1+2z^{-1}+z^{-2}$
$3.4142+0.5858z^{-2}$

Verification of the design

The frequency response of the digital lowpass filter is obtained by letting $z = e^{j\omega}$ in $H(z)$.

That is,

$$\begin{aligned} H(e^{j\omega}) = H(\omega) &= \frac{\left(1+e^{-j\omega}\right)^2}{3.4142+0.5858e^{-j2\omega}} \\ &= \frac{[(1+\cos\omega)-j\sin\omega]^2}{(3.4142+0.5858\cos 2\omega)-j0.5858\sin 2\omega} \\ \Rightarrow \quad |H(\omega)| &= \frac{(1+\cos\omega)^2+\sin^2\omega}{\sqrt{(0.5858\cos 2\omega+3.4142)^2+(0.5858\sin 2\omega)^2}} \end{aligned}$$

Hence,

$$20\log|H(\omega)|_{\omega=0.5\pi} = -3.01 \text{ dB}$$

and

$$20\log|H(\omega)|_{\omega=0.75\pi} = -15.44 \text{ dB}$$

Difference equation realization

Let

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+z^{-2}}{0.5858z^{-2}+3.4142}$$

Cross-multiplying, we get

$$0.5858z^{-2}\,Y(z)+3.4142\,Y(z) = X(z)+2z^{-1}\,X(z)+z^{-2}\,X(z)$$

Taking inverse $\mathcal{Z}$-transform on both the sides and rearranging, we get

$$y(n) = -0.1715y(n-2) + 0.2928x(n) + 0.5857x(n-1) + 0.2928x(n-2)$$

Example 4.17 An ideal analog integrator has the system function $H_a(s) = \frac{1}{s}$. A digital integrator is designed by applying bilinear transformation to $H_a(s)$.

That is, $$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

a. Obtain the difference equation for the digital integrator relating the input $x(n)$ to the output $y(n)$.

b. Roughly sketch the magnitude $|H_a(j\Omega)|$ and $\theta(\Omega)$ for the analog integrator.

c. Roughly sketch $|H(\omega)|$ and $\theta(\omega)$ for the digital integrator.

d. Compare the magnitude and phase characteristics obtained in parts (b) and (c).

☐ Solution

a. Let $T = 2$ secs.

We know that,

$$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$\Rightarrow \quad H(z) = \frac{1}{s}\Big|_{s \to \left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}}$$

$$\Rightarrow \quad Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

Taking inverse $\mathcal{Z}$-transform on both the sides, we get

$$y(n) - y(n-1) = x(n) + x(n-1)$$

$$\Rightarrow \quad y(n) = y(n-1) + x(n) + x(n-1)$$

b. The frequency response of the analog integrator is obtained by letting $s = j\Omega$ in $H_a(s)$.

Hence,

$$H_a(j\Omega) = \frac{1}{j\Omega}$$

$$\Rightarrow \quad |H_a(j\Omega)| = \frac{1}{|\Omega|}$$

and

$$\theta(\Omega) = \measuredangle H_a(j\Omega) = \begin{cases} -\frac{\pi}{2}, & \Omega > 0 \\ \frac{\pi}{2}, & \Omega < 0 \end{cases}$$

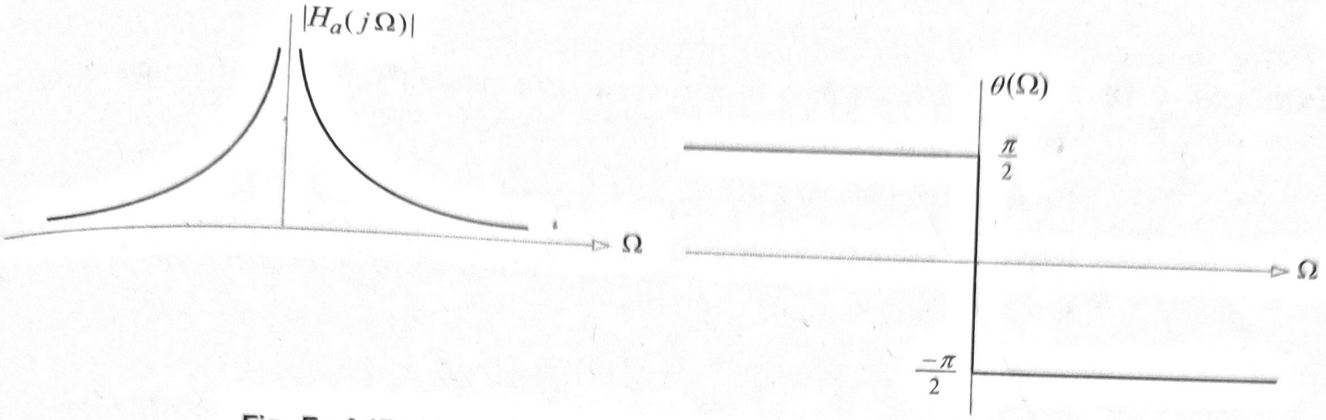


Fig. Ex.4.17(a) Magnitude and phase responses of analog integrator.

c. We know that

$$H(z) = H_a(s)\Big|_{s \longrightarrow \frac{1-z^{-1}}{1+z^{-1}}}$$

$$\Rightarrow \quad H(z) = \frac{1+z^{-1}}{1-z^{-1}} \qquad (T = 2 \text{ secs})$$

The frequency respone of the digital integrator is obtained by letting $z = e^{j\omega}$ in $H(z)$.

That is, $$H\left(e^{j\omega}\right) = H(\omega) = \frac{1+e^{-j\omega}}{1-e^{-j\omega}}$$

$$\Rightarrow \quad H(\omega) = \frac{e^{-j\omega/2}\left[e^{j\omega/2}+e^{-j\omega/2}\right]}{e^{-j\omega/2}\left[e^{j\omega/2}-e^{-j\omega/2}\right]} = -j\cot\frac{\omega}{2}$$

Hence, $$|H(\omega)| = \left|\cot\left(\frac{\omega}{2}\right)\right|$$

and $$\theta(\omega) = \measuredangle H(\omega) = \begin{cases} -\frac{\pi}{2}, & \omega > 0 \\ \frac{\pi}{2}, & \omega < 0 \end{cases}$$

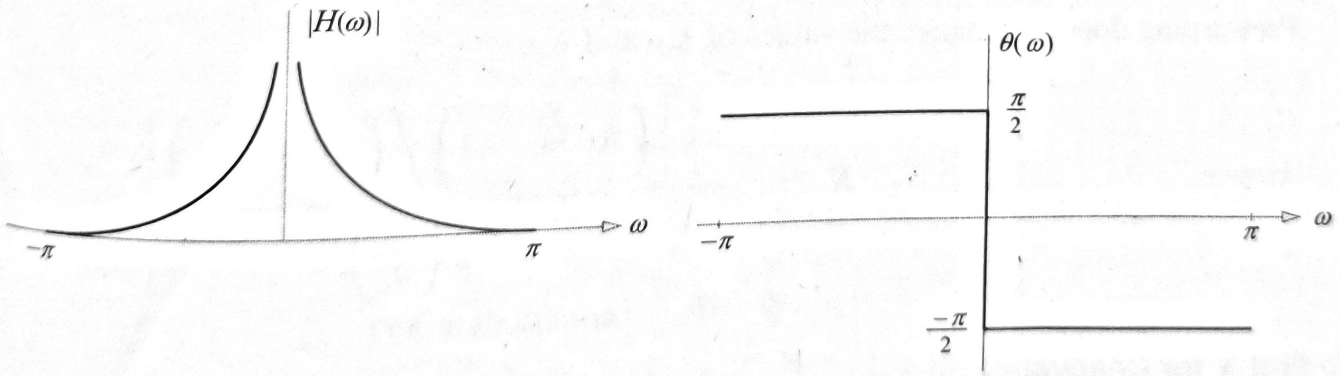


Fig. Ex.4.17(b) Magnitude and phase responses of digital integrator.

d. From the figures in parts b and c we see that, the magnitude characteristic of the digital integrator closely matches with that of the analog integrator. The two phase characteristics are identical. The above comparisons are made for Ω varying from $-\pi$ to $+\pi$.

Example 4.18 A digital IIR lowpass filter is required to meet the following frequency-domain specifications.

Passband ripple: $\leqslant 1$ dB.

Passband edge frequency: 0.33π rad.

Stopband attenuation: $\geqslant 40$ dB.

Stopband edge frequency: 0.5π rad.

The digital filter is to be designed by applying bilinear transformation on an analog system function. Determine the order, N of Butterworth and Chebyshev filters needed to meet the specifications in the digital implementation.

☐ Solution

We are given,

$$K_P = -1\text{ dB}, \quad \omega_P = 0.33\pi \text{ rad}$$
$$K_S = -40\text{ dB}, \quad \omega_S = 0.5\pi \text{ rad}$$

To find N for Butterworth filter

Prewarping the band-edge frequencies $\omega_P = 0.33\pi$ rad and $\omega_S = 0.5\pi$ rad using $T = 1$ sec, we get

$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) = \frac{2}{1}\tan\left(\frac{0.33\pi}{2}\right) = 1.1408 \text{ rad/sec}$$

$$\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) = \frac{2}{1}\tan\left(\frac{0.5\pi}{2}\right) = 2 \text{ rad/sec}$$

Prewarping does not change the values of K_P and K_S.

Hence,

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega'_P}{\Omega'_S}\right)} = 10 \quad \text{(minimum order)}$$

To find N for Chebyshev filter

Discrimination factor,

$$d = \sqrt{\frac{(1-\delta_P)^{-2} - 1}{\delta_S^{-2} - 1}} = 5.0887 \times 10^{-3}$$

Selectivity factor,

$$K = \frac{\Omega'_P}{\Omega'_S} = 0.5704$$

Filter order,

$$N \geq \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)}$$

Minimum filter order,

$$N = \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)} = 6$$

Example 4.19 Determine the system function $H(z)$ of the lowest-order Chebyshev filter that meets the following specifications:

a. 3 dB ripple in the passband $0 \leqslant |\omega| \leqslant 0.3\pi$.

b. Atleast 20 dB attenuation in the stopband $0.6\pi \leqslant |\omega| \leqslant \pi$.

Use the bilinear transformation.

Solution

The specified magnitude frequency response is shown in Fig. Ex.4.19(a).

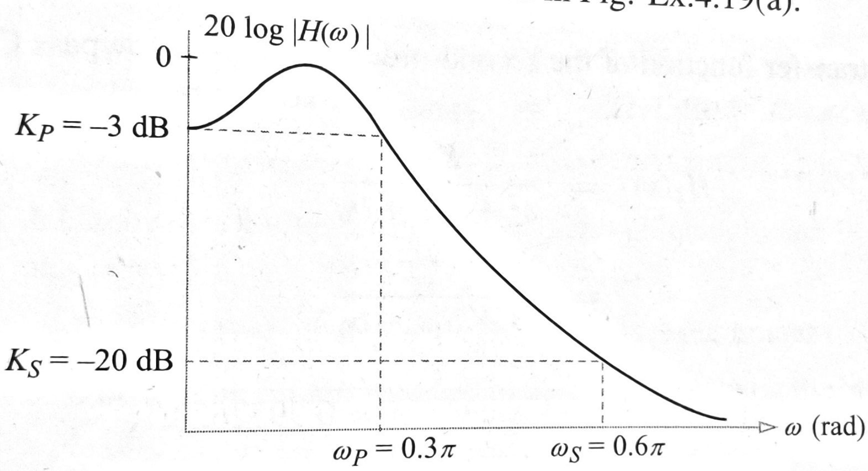


Fig. Ex.4.19(a) Specified lowpass magnitude response.

Step 1: Prewarping the band-edge frequencies ω_P and ω_S using $T = 1$ sec, we get

$$\begin{aligned}
\Omega'_P &= \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.3\pi}{2}\right) \\
&= 1.019, \quad K_P = -3 \text{ dB} \\
\Omega'_S &= \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.6\pi}{2}\right) \\
&= 2.75, \quad K_S = -20 \text{ dB}
\end{aligned}$$

Step 2: Let us design a prewarped analog lowpass Chebyshev I filter having a transfer function $H_a(s)$ to meet the specifications of step 1.

$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.1$$

$$K = \frac{\Omega'_P}{\Omega'_S} = 0.3705$$

Minimum filter order, $$N = \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)} = 1.8$$

Rounding off to the next larger integer, we get $N = 2$.

Referring to the normalized 3 dB ripple Chebyshev I filter tables (provided in Appendix II), we get for $N = 2$, the following filter coefficients.

$$b_0 = 0.7079478$$
$$b_1 = 0.6448996$$

Hence, the transfer function of the second-order normalized lowpass Chebyshev I filter is

$$H_2(s) = \frac{K_N}{s^2 + b_1 s + b_0} = \frac{\frac{b_0}{\sqrt{1+\epsilon^2}}}{s^2 + b_1 s + b_0}$$

Since, $K_P = 20\log\left(\frac{1}{\sqrt{1+\epsilon^2}}\right) = -3$, we get $\epsilon^2 = 0.9952623$.

Hence, $$H_2(s) = \frac{\frac{0.7079478}{\sqrt{1+0.995263}}}{s^2 + 0.6448996s + 0.7079478} = \frac{0.50119}{s^2 + 0.6448996s + 0.7079478}$$

Since, we want the cutoff at Ω'_P, the required prewarped lowpass Chebyshev I filter $H_a(s)$ is obtained by applying lowpass-to-lowpass transformation to $H_2(s)$.

That is, $$H_a(s) = H_2(s)\Big|_{s \longrightarrow \frac{s}{\Omega'_P}} = \frac{0.50119}{s^2 + 0.6448996s + 0.7079478}\Bigg|_{s \longrightarrow \frac{s}{1.019}} = \frac{0.52}{s^2 + 0.6571526924s + 0.7351053856}$$

Step 3: Finally, the transfer function $H(z)$ of the digital filter is obtained by applying bilinear tranformation to $H_a(s)$ with $T = 1$ sec.

That is,

$$\begin{aligned} H(z) &= H_a(s)\Big|_{s\longrightarrow\frac{2}{1}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \\ &= \frac{0.52}{4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 0.6571526924\times 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 0.7351053856} \\ &= \frac{0.52\left(1+z^{-1}\right)^2}{6.0494 - 6.53z^{-1} + 3.420805z^{-2}} \end{aligned}$$

Verification of the design

The frequency response of the digital filter is obtained by letting $z = e^{j\omega}$ in $H(z)$.

That is,

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{0.52\left(1+e^{-j\omega}\right)^2}{6.0494 - 6.53e^{-j\omega} + 3.420805e^{-2j\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.52\left[(1+\cos\omega)^2 + \sin^2\omega\right]}{\sqrt{\begin{array}{l}(6.0494 - 6.53\cos\omega + 3.420805\cos 2\omega)^2\\ +(6.53\sin\omega - 3.420805\sin 2\omega)^2\end{array}}}$$

Hence,

$$20\log|H(\omega)|_{\omega=0.3\pi} = -3 \text{ dB}$$

and

$$20\log|H(\omega)|_{\omega=0.6\pi} = -22.7 \text{ dB}$$

Example 4.20 A Chebyshev I filter of order $N = 3$ and unit bandwidth is known to have a pole at $s = -1$.

a. Find the two other poles of the filter and parameter ϵ.

b. The analog filter is mapped to the z-domain using the bilinear transformation with $T = 2$. Find the transfer function $H(z)$ of the digital filter.

☐ Solution

The poles of Chebyshev I filter are located on an ellipse as shown in Fig. Ex.4.20.

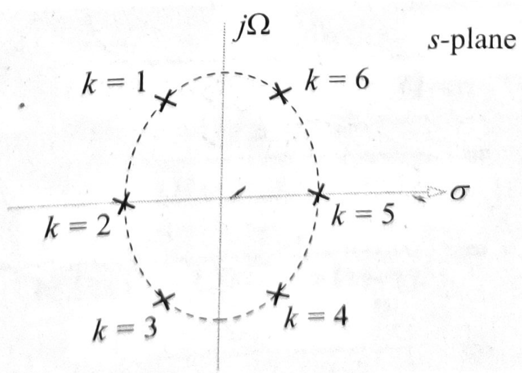


Fig. Ex.4.20 Poles of Chebyshev I filter for N=3.

The locations of these poles are computed using the following formulae:

$$\begin{aligned} \sigma_k &= -a \sin\left[(2k-1)\frac{\pi}{2N}\right] \\ \Omega_k &= b \cos\left[(2k-1)\frac{\pi}{2N}\right], \qquad k = 1, 2 \cdots 2N \end{aligned} \tag{4.47a}$$

where
$$a = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} - \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}} \tag{4.47b}$$

and
$$b = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} + \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}}$$

a. In the present context, $k = 2$ and $\sigma_k = -1$. Hence, from the equation (4.47a), we find that

$$\begin{aligned} -1 &= -a \sin\left[3 \times \frac{\pi}{6}\right] \\ \Rightarrow \quad a &= 1 \end{aligned}$$

Now, using $a = 1$ and $N = 3$ in equation (4.47b) and solving for ϵ, we get

$$\epsilon = \frac{1}{7}$$

The locations of the remaining poles are as follows:

k	σ_k	Ω_k
1	−0.5	$+j1.2247$
3	−0.5	$-j1.2247$

b. The transfer function of third-order normalized Chebyshev I filter is

$$\begin{aligned} H_3(s) &= \frac{K_N}{(s-s_1)(s-s_2)(s-s_3)} \\ &= \frac{K_N}{(s-s_2)(s-s_1)(s-s_3)} \\ &= \frac{K_N}{(s+1)(s+0.5-j1.2247)(s+0.5+j1.2247)} \\ &= \frac{K_N}{(s+1)(s^2+s+1.75)} \end{aligned}$$

Since, N is odd, $K_N = b_0$. In the present context, $b_0 = 1.75$. Hence, $K_N = 1.75$.

Thus,
$$H_3(s) = \frac{1.75}{(s+1)(s^2+s+1.75)}$$

Applying bilinear transformation to $H_3(s)$ with $T = 2$ sec, we get the transfer function $H(z)$ of the digital filter.

That is,

$$H(z) = H_3(s)\Big|_{s \to \frac{2}{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{0.2333\left(1+z^{-1}\right)^3}{1+0.4z^{-1}+0.4666z^{-2}}$$

Example 4.21 Using the bilinear transformation, $z = \frac{2+s}{2-s}$;

a. Find the image of $s = -1 + j$ in the z-plane.

b. Plot the image of $s = -1 + j\Omega$ as Ω varies from $-\infty$ to ∞.

c. If $H(s)$ is a stable analog filter, will the bilinear transformatin used in this context lead to a stable digital filter?

□ Solution

a. The image of $s = -1 + j$ is

$$z_0 = \frac{2+(-1+j)}{2-(-1+j)} = \frac{1+j}{3-j} = 0.2 + j0.4$$

b. For $s = -1 + j\Omega$, we have

$$z = \frac{2-1+j\Omega}{2+1-j\Omega} = \frac{1+j\Omega}{3-j\Omega}$$

$$= \frac{(1+j\Omega)(3+j\Omega)}{9+\Omega^2} = \frac{3-\Omega^2}{\Omega^2+9} + j\frac{4\Omega}{\Omega^2+9}$$

As Ω varies from $-\infty$ to ∞, a circle is traced as shown in Fig. Ex.4.21.

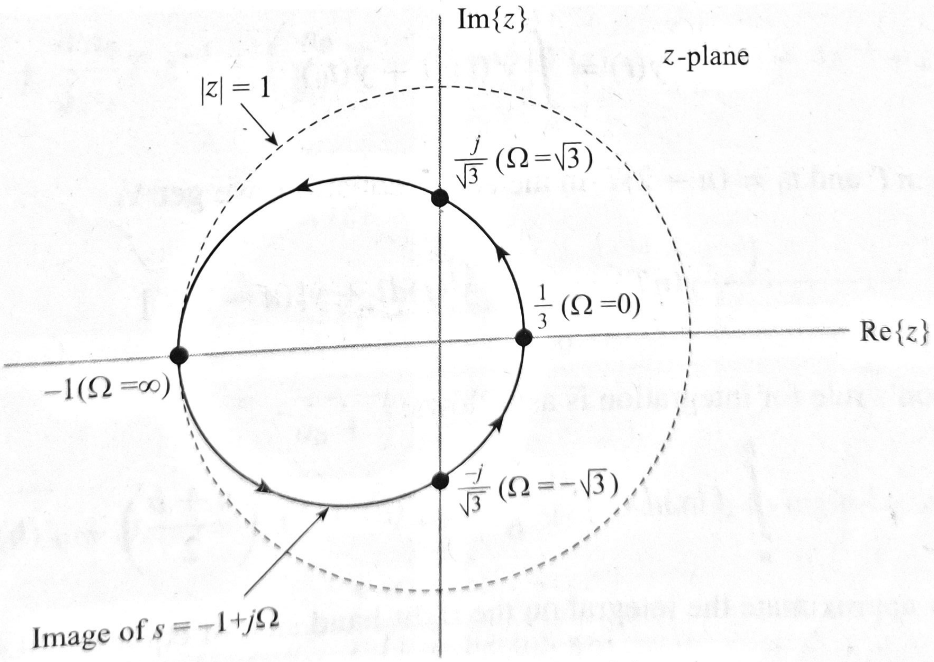


Fig. Ex.4.21 Image of $s = -1 + j\Omega$ in z-plane.

when,
$$\begin{aligned} \Omega &= -\sqrt{3}, & z &= -j\tfrac{1}{\sqrt{3}} \\ \Omega &= 0, & z &= \tfrac{1}{3} \\ \Omega &= \sqrt{3}, & z &= j\tfrac{1}{\sqrt{3}} \\ \Omega &= \infty, & z &= -1 \end{aligned}$$

c. The bilinear transformation results into a stable digital filter since the entire left-half of the s-plane is mapped inside the unit circle. This aspect is proved in the section 4.9 and also for $s = -1 + j\Omega$ in part (b) of this problem.

Example 4.22 Obtain a transformation for the solution of a first-order linear constant coefficient differential equation by using Simpson's rule for integral approximation instead of trapezoidal approximation.

☐ Solution

Let the first-order differential equation be defined as follows:

$$a_1\, y'(t) + a_0\, y(t) = b_0\, x(t) \tag{4.48a}$$

Taking Laplace transform on both the sides of the above equation with all initial conditions ignored gives

$$\begin{aligned} a_1\, sY(s) + a_0\, Y(s) &= b_0\, X(s) \\ \Rightarrow \qquad H(s) \triangleq \frac{Y(s)}{X(s)} &= \frac{b_0}{a_0 + a_1 s} \end{aligned} \tag{4.48b}$$

The fundamental theorem of integral calculus allows us to write

$$y(t) = \int_{t_0}^{t} y'(t)dt + y(t_0) \tag{4.48c}$$

Letting $t = nT$ and $t_0 = (n-2)T$ in the above equation, we get

$$y(nT) = \int_{(n-2)T}^{nT} y'(t)dt + y[(n-2)T] \tag{4.48d}$$

The Simpson's rule for integration is as follows:

$$\int_a^b f(x)dx = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Let us now approximate the integral on the right-hand side of equation (4.48d) by Simpson's rule. Then,

$$y(nT) = y[(n-2)T] + \left[\frac{nT-(n-2)T}{6}\right]\left(y'(nT) + 4y'\left[\frac{(nT+nT-2T)}{2}\right] + y'[(n-2)T]\right.$$

$$\Rightarrow \quad y(nT) = y[(n-2)T] + \frac{T}{3}\left(y'(nT) + 4y'[(n-1)T] + y'[(n-2)T]\right) \tag{4.48e}$$

From equation (4.48a), we have

$$y'(t) = -\frac{a_0}{a_1}y(t) + \frac{b_0}{a_1}x(t) \tag{4.48f}$$

Replacing t by nT, $(n-1)T$ and $(n-2)T$ in the above equation and replacing the corresponding $y'(t)$ in equation (4.48e), we get

$$\begin{aligned}
y(nT) &= y[(n-2)T] + \frac{T}{3}\left(-\frac{a_0}{a_1}y(nT) + \frac{b_0}{a_1}x(nT) - 4\frac{a_0}{a_1}y[(n-1)T] + 4\frac{b_0}{a_1}x[(n-1)T]\right. \\
&\qquad \left. -\frac{a_0}{a_1}y[(n-2)T] + \frac{b_0}{a_1}x[(n-2)T]\right) \\
&= -\frac{a_0}{a_1}\frac{T}{3}y(nT) - \frac{4\,a_0}{3\,a_1}T\,y[(n-1)T] + \left(1 - \frac{a_0}{3a_1}T\right)y[(n-2)T] \\
&\qquad + \frac{T\,b_0}{3\,a_1}\left(x(nT) + 4x[(n-1)T] + x[(n-2)T]\right)
\end{aligned}$$

Taking $\mathcal{Z}$-transform on both the sides and ignoring all the initial conditions, we get

$$\begin{aligned}
Y(z) &= -\frac{a_0}{a_1}\frac{T}{3}Y(z) - \frac{4\,a_0}{3\,a_1}T\,z^{-1}Y(z) + \left(1 - \frac{a_0}{3a_1}T\right)z^{-2}\,Y(z) \\
&\qquad + \frac{T\,b_0}{3\,a_1}\left[X(z) + 4z^{-1}X(z) + z^{-2}X(z)\right]
\end{aligned}$$

$$\Rightarrow \quad \left[\left(1 + \frac{a_0}{a_1}\frac{T}{3}\right) + \frac{4\,a_0}{3\,a_1}T\,z^{-1} + \left(\frac{a_0}{3a_1}T - 1\right)z^{-2}\right]Y(z) = \frac{T\,b_0}{3\,a_1}\left[1 + 4z^{-1} + z^{-2}\right]X(z)$$

$$\begin{aligned}
\Rightarrow \quad H(z) &\triangleq \frac{Y(z)}{X(z)} \\
&= \frac{b_0T\left(1 + 4z^{-1} + z^{-2}\right)}{a_0\left(T + 4T\,z^{-1} + T\,z^{-2}\right) + \left(3 - 3z^{-2}\right)a_1} \\
&= \frac{b_0}{a_0 + \frac{3}{T}\frac{(1-z^{-2})a_1}{(1+4z^{-1}+z^{-2})}}
\end{aligned}$$

Comparing $H(z)$ with $H_a(s) = \frac{b_0}{a_0 + a_1 s}$, we get the required analog-to-digital transformation as

$$s \longrightarrow \frac{3}{T}\frac{\left(1 - z^{-2}\right)}{\left(1 + 4z^{-1} + z^{-2}\right)}$$

Example 4.23 We are given the digital filter $H(z)$ having two zeros at $z = -1$ and poles at $z = \pm ja$, where a is real and is bounded by $0.6 < a < 1$. This filter $H(z)$ was obtained from an analog counterpart by applying bilinear transformation to it.

a. Sketch an approximate plot of $|H(\omega)|$ versus ω.

b. Evaluate $H(s)$ and express it as a ratio of two polynomials, with a and T as parameters.

c. If $a = \frac{1}{\sqrt{2}}$ and $T = 1$ sec, is $H(s)$ Butterworth or Chebyshev I filter?

☐ Solution

The transfer function of the digital filter is

$$\begin{aligned} H(z) &= \frac{K(z+1)^2}{(z+ja)(z-ja)} \\ &= \frac{K(z+1)^2}{z^2+a^2} \end{aligned}$$

The frequency response of the stable digital filter is obtained by letting $z = e^{j\omega}$ in $H(z)$.

Then,

$$\begin{aligned} H\left(e^{j\omega}\right) = H(\omega) &= \frac{K\left(e^{j\omega}+1\right)^2}{e^{j2\omega}+a^2} \\ &= \frac{K[(1+\cos\omega)+j\sin\omega]^2}{(\cos 2\omega + j\sin 2\omega)+a^2} \\ &= \frac{K[(1+\cos\omega)+j\sin\omega]^2}{\left(\cos 2\omega + a^2\right)+j\sin 2\omega} \end{aligned}$$

$$\Rightarrow \qquad |H(\omega)| = \frac{K[(1+\cos\omega)^2+\sin^2\omega]}{\sqrt{\left(a^2+\cos 2\omega\right)^2+\sin^2 2\omega}}$$

Hence,

$$\begin{aligned} |H(0)| &= \frac{4K}{1+a^2}, \\ \left|H\left(\frac{\pi}{2}\right)\right| &= \frac{2K}{1-a^2} \quad \text{and} \\ |H(\pi)| &= 0 \end{aligned}$$

a. For $a > \frac{1}{3}$, we find that $\left|H\left(\frac{\pi}{2}\right)\right| > |H(0)|$. Thus, the magnitude frequency response of the digital filter is as shown in Fig. Ex.4.23.

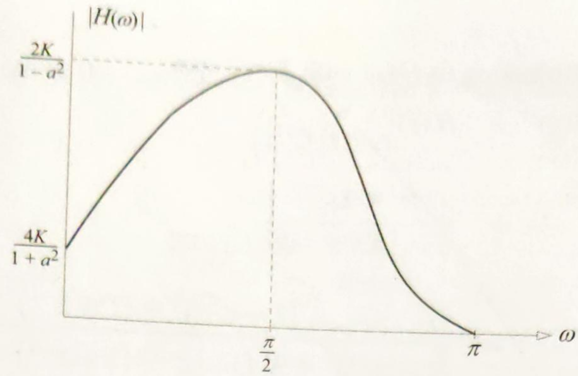


Fig. Ex.4.23 Magnitude frequency response of the digital filter.

b. The bilinear transformation is

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\Rightarrow \qquad z = \frac{1+0.5Ts}{1-0.5Ts}$$

Hence,

$$\begin{aligned} H(s) &= H(z)\Big|_{z=\frac{1+0.5Ts}{1-0.5Ts}} \\ &= \frac{K(z+1)^2}{(z^2+a^2)}\Bigg|_{z=\frac{1+0.5Ts}{1-0.5Ts}} \\ &= \frac{4K}{0.25T^2(1+a^2)s^2+(1-a^2)Ts+1+a^2} \end{aligned}$$

c. If $a = \frac{1}{\sqrt{2}}$ and $T = 1$ sec, we get

$$H(s) = \frac{\frac{32}{3}K}{s^2+\frac{4}{3}s+4}$$

The poles of $H(s)$ are at $s = -\frac{2}{3} \pm j\frac{4\sqrt{2}}{3}$. These poles do not lie on the unit circle. Hence, $H(s)$ is not a Butterworth filter but a Chebyshev I filter, the only other option.

Example 4.24 A second-order analog notch-filter has the transfer function

$$H(s) = \frac{s^2+\Omega_0^2}{s^2+Ks+\Omega_0^2}$$

Using bilinear transformation, show that the transfer function $H(z)$ of the digital notch filter is

$$H(z) = \frac{1}{2}\left[\frac{(1+\alpha)-2\beta(1+\alpha)z^{-1}+(1+\alpha)z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right]$$

where

$$\alpha = \frac{1+\Omega_0^2-K}{1+\Omega_0^2+K} \quad \text{and} \quad \beta = \frac{1-\Omega_0^2}{1+\Omega_0^2}$$

Solution

Let us apply bilinear transformation to $H(s)$ with $T = 2$ secs.

Then,

$$
\begin{aligned}
H(z) &= H(s)\Big|_{s \rightarrow \frac{2}{2}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \\
&= \left.\frac{s^2+\Omega_0^2}{s^2+Ks+\Omega_0^2}\right|_{s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{\left(1-z^{-1}\right)^2+\left(1+z^{-1}\right)^2\Omega_0^2}{\left(1-z^{-1}\right)^2+K\left(1-z^{-1}\right)\left(1+z^{-1}\right)+\left(1+z^{-1}\right)^2\Omega_0^2} \\
&= \frac{\left(1-2z^{-1}+z^{-2}\right)+\Omega_0^2\left(1+2z^{-1}+z^{-2}\right)}{1+K+\Omega_0^2-\left(2-2\Omega_0^2\right)z^{-1}+\left(1-K+\Omega_0^2\right)z^{-2}} \\
&= \frac{\left(1+\Omega_0^2\right)-\left(2-2\Omega_0^2\right)z^{-1}+\left(1+\Omega_0^2\right)z^{-2}}{1+K+\Omega_0^2-\left(2-2\Omega_0^2\right)z^{-1}+\left(1-K+\Omega_0^2\right)z^{-2}} \\
&= \frac{\frac{1+\Omega_0^2}{1+K+\Omega_0^2}-\frac{2\left(1-\Omega_0^2\right)z^{-1}}{1+K+\Omega_0^2}+\frac{\left(1+\Omega_0^2\right)z^{-2}}{1+K+\Omega_0^2}}{1-\frac{2\left(1-\Omega_0^2\right)z^{-1}}{1+K+\Omega_0^2}+\frac{\left(1-K+\Omega_0^2\right)z^{-2}}{1+K+\Omega_0^2}}
\end{aligned}
$$

Given

$$
\alpha = \frac{1+\Omega_0^2-K}{1+\Omega_0^2+K}, \quad \Rightarrow \quad \frac{1}{2}(1+\alpha) = \frac{1+\Omega_0^2}{1+\Omega_0^2+K}
$$

$$
\beta = \frac{1-\Omega_0^2}{1+\Omega_0^2}, \quad \Rightarrow \quad \beta(1+\alpha) = \frac{2\left(1-\Omega_0^2\right)}{1+\Omega_0^2+K}
$$

Thus,

$$
\begin{aligned}
H(z) &= \frac{\frac{1}{2}(1+\alpha)-\beta(1+\alpha)z^{-1}+\frac{1}{2}(1+\alpha)z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \\
&= \frac{1}{2}\left[\frac{(1+\alpha)-2\beta(1+\alpha)z^{-1}+(1+\alpha)z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right]
\end{aligned}
$$

Example 4.25 A second-order Butterworth lowpass analog filter with a half-power frequency of 1 rad/second is converted to a digital filter $H(z)$, using the bilinear transformation at a sampling rate, $\frac{1}{T} = 1$ Hz.

a. What is the transfer function $H(s)$ of the analog filter?

b. What is the transfer function $H(z)$ of the digital filter?

c. Are the dc gains of $H(z)$ and $H(s)$ identical? Explain.

d. Are the gains $H(z)$ and $H(s)$ at their respective half-power frequencies identical? Explain.

☐ Solution

a. The transfer function of the second-order normalized lowpass Butterworth filter is

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

b.

$$\begin{aligned} H(z) &= H_2(s)\big|_{s \longrightarrow \frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{(z+1)^2}{7.8284z^2 - 6z + 2.1716} \\ &= \frac{\left(1+z^{-1}\right)^2}{2.1716z^{-2} - 6z^{-1} + 7.8284} \end{aligned}$$

c. The dc gain of $H(s)$ is obtained by letting $s = 0$ in $H(s)$, while the dc gain of $H(z)$ is obtained by letting $z = 1$ in $H(z)$.

$$\begin{aligned} \text{Dc gain of} \quad H(s) &= \left.\frac{1}{s^2 + \sqrt{2}s + 1}\right|_{s=0} = 1 \\ \text{Dc gain of} \quad H(z) &= \left.\frac{(1+z^{-1})^2}{2.1716z^{-2} - 6z^{-1} + 7.8284}\right|_{z=1} = 1 \end{aligned}$$

That means the dc gains match.

d. Gain of $H(s)$ at $\Omega = 1$ rad/sec is obtained by letting $s = j$ in $H(s)$ and then finding the magnitude.

$$\begin{aligned} H(j) &= \left.\frac{1}{s^2 + \sqrt{2}s + 1}\right|_{s=j} \\ &= \frac{1}{-1 + \sqrt{2}j + 1} \\ &= \frac{1}{j\sqrt{2}} \\ \Rightarrow \quad |H(j)| &= \frac{1}{\sqrt{2}} \end{aligned}$$

Gain of $H(z)$ at $\omega = 1$ rad is obtained by letting $z = e^j$ in $H(z)$ and then finding the magnitude.

$$\begin{aligned} H\left(e^j\right) &= \frac{\left(1+e^{-j}\right)^2}{2.1716e^{-j2} - 6e^{-j} + 7.8284} \\ \Rightarrow \quad \left|H\left(e^j\right)\right| &= 0.6421 \end{aligned}$$

Therefore, in this case the gains do not match.

4.11 Impulse Invariant Transformation (IIT)

If $h_a(t)$ represents the impulse response of an analog filter, then the unit sample response of a discrete-time filter $H(z)$ used in an A/D-$H(z)$-D/A structure is selected to be the sampled version of $h(t)$.

That is,

$$\begin{aligned} h(n) &= h_a(nT) \\ &= h_a(t)\big|_{t=nT} \end{aligned}$$

and the discrete (digital) transfer function is

$$H(z) = \mathcal{Z}\{h(n)\}$$

Notice that the digital transfer function $H(z)$ is the $\mathcal{Z}$-transform of the unit sample response $h(n)$, while the analog transfer function $H_a(s)$ is the Laplace transform of the unit impulse response $h_a(t)$. Don't be tempted to write $H(z) = H(s)\big|_{s=z}$, because it is incorrect.

Let us now generalize this procedure and at the same time show that $H(z)$ can be obtained directly from $H_a(s)$ without intervening steps of finding $h_a(t)$ and then $h_a(nT)$.

Consider an analog transfer function with N different poles that has the s-domain transfer function written in partial fraction expansion form as

$$H_a(s) = \sum_{i=1}^{N} \frac{C_i}{s - s_i} \tag{4.49}$$

with the corresponding unit impulse response

$$h_a(t) = \sum_{i=1}^{N} C_i\, e^{s_i t} \tag{4.50}$$

In the above expression, C_i is the constant associated with the partial fraction expansion of $H_a(s)$. If this response is sampled every T seconds ($t = nT$), we have the sampled response

$$h_a(nT) = h(n) = \sum_{i=1}^{N} C_i\, e^{s_i nT} \tag{4.51}$$

Finally, we take the $\mathcal{Z}$-transform of equation (4.51) to obtain the discrete transfer function of the digital IIR filter.

$$\begin{aligned} H(z) &= \mathcal{Z}\{h(n)\} \\ &= \sum_{n=0}^{\infty} h(n) z^{-n} \end{aligned} \tag{4.52}$$

The lower limit of summation in the above equation is zero because the filter is assumed to be causal.

Impulse

Substituting equation (4.51) in equation (4.52), we get

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty}\sum_{i=1}^{N} C_i\, e^{s_i nT} z^{-n} \\ &= \sum_{i=1}^{N} C_i \sum_{n=0}^{\infty} [e^{s_i T} z^{-1}]^n \\ &= \sum_{i=1}^{N} C_i \frac{1}{1 - e^{s_i T} z^{-1}} \end{aligned} \tag{4.53}$$

Comparing equations (4.49) and (4.53), we find that

$$\frac{1}{s - s_i} \stackrel{\text{IIT}}{\Longrightarrow} \frac{1}{1 - e^{s_i T} z^{-1}} = \frac{z}{z - e^{s_i T}} \tag{4.54}$$

Equation (4.54) shows that the analog pole at $s = s_i$ is mapped to a digital pole at $z_i = e^{s_i T}$. The transformed digital filter $H(z)$ has the following properties:

1. Its order is same as that of $H_a(s)$ because the common denominator on the right-hand side has degree N.
2. Its poles are mapped according to

$$s_i \stackrel{\text{IIT}}{\Longrightarrow} z_i = e^{s_i T}, \qquad 1 \leqslant i \leqslant N$$

That is, the analog and digital poles are related as per the equation

$$z = e^{sT}$$

Letting $z = re^{j\omega}$ and $s = \sigma + j\Omega$ in the above equation, we get

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Hence,

$$r = e^{\sigma T}$$

and

$$\omega = \Omega T$$

Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$. When $\sigma = 0$, we have $r = 1$. Hence, the left-half pole is mapped inside the unit circle in the z-plane and the right half pole in s is mapped outside the unit circle in the z-plane. Thus, a stable analog filter $H_a(s)$ is transformed to a stable digital filter $H(z)$. Also, the image of the $j\Omega$ axis in the z-plane is the unit circle as indicated above. However, the mapping of the $j\Omega$ axis is not one-to-one. The mapping, $\omega = \Omega T$ implies that the interval $-\frac{\pi}{T} \leqslant \Omega \leqslant \frac{\pi}{T}$ maps into the interval $-\pi \leqslant \omega \leqslant \pi$. In general, the interval $(2q-1)\frac{\pi}{T} \leqslant \Omega \leqslant (2q+1)\frac{\pi}{T}$ also maps into the interval $-\pi \leqslant \omega \leqslant \pi$, where q is an integer. Thus, the mapping of analog frequency, Ω in to digital frequency ω is many-to-one, which simply reflects the effects of aliasing due to sampling. Fig. 4.13 shows the mapping from s-plane to z-plane using impulse invariant transformation.

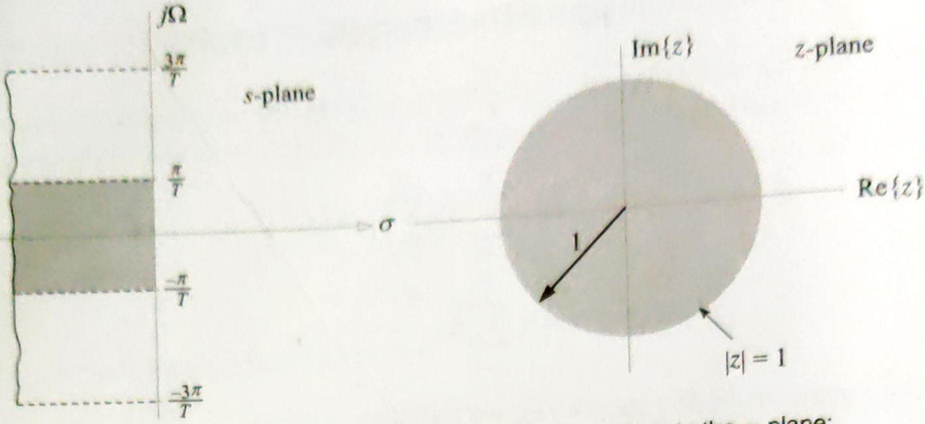


Fig. 4.13 The impulse invariant transformation from the s-plane to the z- plane: the imaginary axis maps to the unit circle, a strip of $\frac{2\pi}{T}$ maps to the disk.

3. The frequency response $H(\omega)$ of the digital filter is related to the frequency response of the analog filter [3] $H_a(\Omega)$ by the sampling theorem,

$$H(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H\left(\frac{\omega - 2\pi k}{T}\right) \tag{4.55}$$

Since, for a rational analog filter, $H(\Omega)$ is never band limited, the frequency response of the digital filter is always aliased. If $H_a(s)$ is lowpass or bandpass, aliasing can be made very minimal by choosing the sampling frequency $\frac{1}{T}$ high enough such that the fraction of the energy in the range $|\omega| > \frac{\pi}{T}$ will be negligible. On the otherhand, if $H_a(s)$ is highpass or bandstop, the impulse invariant method cannot be used at all. This is because the frequency response of these two filter classes does not decay to zero, so the right-hand side of equation (4.55) does not converge.

4. The zeros of $H(z)$ and $H_a(s)$ do not share a simple relationship. When the right-hand side of equation (4.53) is brought to a common denominator, the numerator will be a polynomial of degree $(N-1)$ in z^{-1}. This is regardless of the degree (q) of the numerator polynomial of $H_a(s)$.

Example 4.26 A third-order Butterworth lowpass filter has the transfer function

$$H(s) = \frac{1}{(s+1)\left(s^2+s+1\right)}$$

Design $H(z)$ using impulse invariant technique.

[3] $H_a(\Omega)$ was denoted earlier by $H_a(j\Omega)$.

☐ **Solution**

$$H(s) = \frac{1}{(s+1)\left(s^2+s+1\right)}$$

$$= \frac{1}{(s+1)(s+0.5-j0.866)(s+0.5+j0.866)}$$

$$= \frac{C_1}{s+1} + \frac{C_2}{s+0.5-j0.866} + \frac{C_2^*}{s+0.5+j0.866}$$

Using partial fraction expansion, we find $\cos\theta - j\sin\theta$.

$$C_1 = 1, \quad C_2 = 0.577\, e^{-j2.62} \quad \text{and} \quad C_2^* = 0.577\, e^{j2.62}$$

Hence,
$$H(s) = \frac{1}{s+1} + \frac{0.577\, e^{-j2.62}}{s+0.5-j0.866} + \frac{0.577\, e^{j2.62}}{s+0.5+j0.866}$$

The three poles are

$$s_1 = -1, \quad s_2 = -0.5 + j0.866 \quad \text{and} \quad s_3 = -0.5 - j0.866$$

We know that
$$H(z) = \sum_{i=1}^{3} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$= \frac{C_1}{1-e^{s_1 T}z^{-1}} + \frac{C_2}{1-e^{s_2 T}z^{-1}} + \frac{C_3}{1-e^{s_3 T}z^{-1}}$$

Here,
$$C_3 = C_2^*$$

Hence,
$$H(z) = \frac{C_1}{1-e^{s_1 T}z^{-1}} + \frac{C_2}{1-e^{s_2 T}z^{-1}} + \frac{C_2^*}{1-e^{s_3 T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} + \frac{0.577\, e^{-j2.62}}{1-e^{(-0.5+j0.866)T}z^{-1}} + \frac{0.577\, e^{j2.62}}{1-e^{(-0.5-j0.866)T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} + \frac{0.577\, e^{-j2.62}}{1-e^{-0.5T}e^{j0.866T}z^{-1}} + \frac{0.577\, e^{j2.62}}{1-e^{-0.5T}e^{-j0.866T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} + \frac{2(0.577)\cos(-2.62) - 2(0.577)e^{-0.5T}z^{-1}\cos(-2.62-0.866T)}{1-2\,e^{-0.5T}\cos(0.866T)z^{-1} + e^{-T}z^{-2}}$$

Multiplying the numerator and the denominator of first term on the right-hand side by z and by z^2 for the second term on the right- hand side, the above equation becomes

$$H(z) = \frac{z}{z-e^{-T}} + \frac{-z^2 - 1.154e^{-0.5T}\cos\left(\frac{5\pi}{6} + 0.866T\right)z}{z^2 - 2\,e^{-0.5T}\cos(0.866T)z + e^{-T}}$$

In terms of the sampling interval T, the filter transfer function is

$$H(z) = \frac{b_0 z^2 + b_1 z}{z^3 - a_1 z^2 - a_2 z - a_3}$$

$$= \frac{b_0 z^{-1} + b_1 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

where
$$b_0 = -2\, e^{-0.5T} \cos(0.866T) + e^{-T} + 1.154\, e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right)$$

$$b_1 = e^{-T} + 1.154\, e^{-1.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right)$$

$$a_1 = e^{-T} + 2\, e^{-0.5T} \cos(0.866T)$$

$$a_2 = -e^{-T} - 2\, e^{-1.5T} \cos(0.866T)$$

$$a_3 = e^{-2T}$$

Example 4.27 Let $H_a(s) = \frac{b}{(s+a)^2+b^2}$ be a causal second-order analog transfer function. Show that the causal second-order digital transfer function $H(z)$ obtained from $H_a(s)$ through impulse invariance method is given by

$$H(z) = \frac{e^{-aT} \sin bT\ z^{-1}}{1 - 2e^{-aT} \cos bT\ z^{-1} + e^{-2aT} z^{-2}}$$

Also, find $H(z)$ when $H_a(s) = \frac{1}{s^2+2s+2}$.

☐ Solution

Poles of $H_a(s)$ are obtained from

$$(s+a)^2 + b^2 = 0.$$

Solving, we get

$$s = -a \pm jb$$

Let $s_1 = -a + jb$ and $s_2 = -a - jb$

We may now write the analog transfer function in factored form as

$$H_a(s) = \frac{b}{(s + a - jb)(s + a + jb)}$$

$$= \frac{C_1}{s + a - jb} + \frac{C_2}{s + a + jb}$$

where
$$C_1 = \left.\frac{b}{s + a + jb}\right|_{s=-a+jb} = \frac{1}{j2}$$

and
$$C_2 = C_1^* = -\frac{1}{j2}$$

We know that,
$$H(z) = \sum_{i=1}^{N} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence,
$$H(z) = \frac{1}{j2}\left(\frac{z}{z - e^{(-a+jb)T}} - \frac{z}{z - e^{(-a-jb)T}}\right)$$

$$= \frac{1}{j2}\left(\frac{z^2 - ze^{-aT}e^{-jbT} - z^2 + ze^{-aT}e^{jbT}}{z^2 - ze^{-aT}e^{-jbT} - ze^{-aT}e^{jbT} + e^{-2aT}}\right)$$

$$= \frac{ze^{-aT}\sin bT}{z^2 - 2e^{-aT}\cos bT z + e^{-2aT}}$$

$$= \frac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Given
$$H_a(s) = \frac{1}{s^2 + 2s + 2}$$

$$\Rightarrow \quad H_a(s) = \frac{1}{(s+1)^2 + 1^2}$$

Hence,
$$H(z) = \frac{e^{-T}\sin T z^{-1}}{1 - 2e^{-T}\cos T z^{-1} + e^{-2T} z^{-2}}$$

Example 4.28 Let $H_a(s) = \frac{s+a}{(s+a)^2+b^2}$ be a causal second-order analog transfer function. Show that the causal second-order digital transfer function $H(z)$ is obtained from $H_a(s)$ through impulse invariance method is given by

$$H(z) = \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2\cos bT\, e^{-aT} z^{-1} + e^{-2aT}\, z^{-2}}$$

☐ Solution

The poles of $H_a(s)$ are obtained from

$$(s+a)^2 + b^2 = 0$$

$$\Rightarrow \quad s = -a \pm jb$$

Let $s_1 = -a + jb$ and $s_2 = -a - jb$

The analog transfer function $H_a(s)$ is written in the factored form as

$$H_a(s) = \frac{s+a}{(s+a-jb)(s+a+jb)}$$

$$= \frac{C_1}{s+a-jb} + \frac{C_2}{s+a+jb}$$

where $$C_1 = \left.\frac{s+a}{s+a+jb}\right|_{s=-a+jb} = \frac{1}{2}$$

and $$C_2 = C_1^* = \frac{1}{2}$$

We know that $$H(z) = \sum_{i=1}^{N} \frac{C_i\, z}{z - e^{s_i T}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence, $$H(z) = C_1 \frac{z}{z - e^{s_1 T}} + C_2 \frac{z}{z - e^{s_2 T}}$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{(-a+jb)T}} + \frac{z}{z - e^{(-a-jb)T}}\right)$$

$$= \frac{1}{2}\left(\frac{z^2 - ze^{-aT}e^{-jbT} + z^2 - ze^{-aT}e^{jbT}}{z^2 - ze^{-aT}e^{-jbT} - ze^{-aT}e^{jbT} + e^{-2aT}}\right)$$

$$\Rightarrow \quad H(z) = \frac{z^2 - ze^{-aT}\cos bT}{z^2 - 2z\cos bT\; e^{-aT} + e^{-2aT}}$$

$$= \frac{1 - e^{-aT}\cos bT\; z^{-1}}{1 - 2\cos bT\; e^{-aT} z^{-1} + e^{-2aT} z^{-2}}$$

Example 4.29 Transform the analog filter

$$H_a(s) = \frac{(s+1)}{s^2 + 5s + 6}$$

into $H(z)$ using impulse invariant transformation. Take $T = 0.1$ sec.

☐ Solution

Given, $$H_a(s) = \frac{(s+1)}{s^2 + 5s + 6}$$

$$= \frac{s+1}{(s+2)(s+3)}$$

$$= \frac{C_1}{s+2} + \frac{C_2}{s+3}$$

where $$C_1 = \left.\frac{s+1}{s+3}\right|_{s=-2} = -1$$

and $$C_2 = \left.\frac{s+1}{s+2}\right|_{s=-3} = 2$$

Also, the poles of $H_a(s)$ are $s_1 = -2$ and $s_2 = -3$.

We know that, $$H(z) = \sum_{i=1}^{N} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$= \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence, $$H(z) = \frac{C_1\, z}{z - e^{s_1 T}} + \frac{C_2\, z}{z - e^{s_2 T}}$$

$$= \frac{-z}{z - e^{-0.2}} + \frac{2z}{z - e^{-0.3}}$$

$$= \frac{-z}{z - 0.8186} + \frac{2z}{z - 0.7408}$$

$$= \frac{z^2 - 0.8964z}{z^2 - 1.559z + 0.6065}$$

$$= \frac{1 - 0.8964\, z^{-1}}{1 - 1.559\, z^{-1} + 0.6065\, z^{-2}}$$

Example 4.30 Consider the analog filter having the transfer function

$$H_a(s) = \frac{1}{s+2}$$

a. Transform $H_a(s)$ to a digital filter $H(z)$, using impulse invariance technique. Assume that the sampling rate, $S = 2$ Hz.

b. Will the impluse response $h(n)$ match the impulse response $h(t)$ of the analog filter at the sampling instants? should it? Explain.

c. Will the step response $S(n)$ match the step response $S(t)$ of the analog filter at the sampling instants? Explain.

Solution

a. We know that

$$H_a(s) = \frac{1}{s - s_i} \xrightarrow{\text{IIT}} H(z) = \frac{1}{1 - e^{s_i T} z^{-1}}$$

where
$$T = \frac{1}{S}$$

Here,
$$T = \frac{1}{2} \text{ sec.}$$

Hence,
$$H_a(s) = \frac{1}{s+2} \xrightarrow{\text{IIT}} H(z) = \frac{1}{1 - e^{-2\times\frac{1}{2}} z^{-1}} = \frac{1}{1 - e^{-1} z^{-1}}$$

b.
$$H(z) = \frac{z}{z - e^{-1}}$$

Assuming the system to be causal, we get

$$h(n) = \left(e^{-1}\right)^n u(n) = e^{-n} u(n) \tag{4.56a}$$

Given
$$H_a(s) = \frac{1}{s+2}$$

$$\Rightarrow \quad h_a(t) = e^{-2t}, \quad t \geqslant 0$$

Letting $t = nT$, we get

$$h_a(nT) = h(n) = e^{-2nT}, \qquad n \geqslant 0$$

Since $T = \frac{1}{2}$ sec, we get

$$h(n) = e^{-n} u(n) \tag{4.56b}$$

From equations (4.56a) and (4.56b), we find that the impulse responses of analog and digital filters will match at the sampling instants.

c. Let
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - e^{-1}}$$

$$\Rightarrow \quad Y(z) = \frac{z}{z - e^{-1}} X(z)$$

If $x(n) = u(n)$, we get

$$Y(z) = \frac{z}{\left(z - e^{-1}\right)} \frac{z}{z-1}; \qquad \text{ROC} : |z| > 1$$

$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{z}{(z-1)(z-e^{-1})} = \frac{K_1}{z-1} + \frac{K_2}{z-e^{-1}}$$

where

$$K_1 = \left.\frac{z}{z-e^{-1}}\right|_{z=1} = 1.582$$

$$K_2 = \left.\frac{z}{z-1}\right|_{z=e^{-1}} = -0.582$$

Hence,

$$Y(z) = 1.582\frac{z}{z-1} - 0.582\frac{z}{z-e^{-1}}$$

Taking inverse $\mathcal{Z}$-transform of $Y(z)$ we get

$$y(n) = S(n) = 1.582\,u(n) - 0.582\,e^{-n}u(n) \tag{4.56c}$$

Let

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s+2}X(s)$$

Since $x(t) = u(t)$, the above equation becomes

$$Y(s) = \frac{1}{s(s+2)} = \frac{K_3}{s} + \frac{K_4}{s+2}$$

where

$$K_3 = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2}$$

$$K_4 = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$$

Hence,

$$Y(s) = \frac{1}{2s} - \frac{1}{2(s+2)}$$

$$\Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}e^{-t}, \quad t \geqslant 0$$

Letting $t = nT$, we get

$$y(nT) = S(n) = \frac{1}{2} - \frac{1}{2}e^{-nT}, \quad n \geqslant 0$$

$$= \frac{1}{2}u(n) - \frac{1}{2}e^{-\frac{n}{2}}u(n) \tag{4.56d}$$

From equations (4.56c) and (4.56d), we find that the step responses of analog and digital filters will not match at the sampling instants.

Example 4.31 The following causal IIR digital filters were designed using the impulse invariant transformation.

Find their respective causal analog counter parts. Take $T = 0.3$ sec.

a. $H(z) = \dfrac{2z}{z - e^{-0.9}} + \dfrac{3z}{z - e^{-1.2}}$

b. $H(z) = \dfrac{z^2 - ze^{-0.6}\cos(0.9)}{z^2 - 2ze^{-0.6}\cos(0.9) + e^{-1.2}}$

☐ Solution

a. Given, $$H(z) = \frac{2z}{z - e^{-0.9}} + \frac{3z}{z - e^{-1.2}}$$

Comparing the above transfer function with

$$\begin{aligned} H(z) &= \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}} \\ &= \frac{C_1\, z}{z - e^{s_1 T}} + \frac{C_2\, z}{z - e^{s_2 T}} \end{aligned}$$

we get

$$\begin{aligned} s_1 T &= -0.9 \Rightarrow s_1 = \frac{-0.9}{0.3} = -3, \\ s_2 T &= -1.2 \Rightarrow s_2 = \frac{-1.2}{0.3} = -4, \\ C_1 &= 2 \qquad \text{and} \qquad C_2 = 3 \end{aligned}$$

Hence,

$$\begin{aligned} H_a(s) &= \frac{C_1}{s - s_1} + \frac{C_2}{s - s_2} \\ &= \frac{2}{s + 3} + \frac{3}{s + 4} \end{aligned}$$

b.

$$\begin{aligned} H(z) &= \frac{z^2 - ze^{-0.6}\cos(0.9)}{z^2 - 2ze^{-0.6}\cos(0.9) + e^{-1.2}} \\ &= \frac{1 - e^{-0.6}\cos(0.9)z^{-1}}{1 - 2e^{-0.6}\cos(0.9)z^{-1} + e^{-1.2}z^{-2}} \end{aligned}$$

Comparing the above equation with

$$H_a(s) = \frac{s + a}{(s + a)^2 + b^2} \stackrel{\text{IIT}}{\Longrightarrow} H(z) = \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}\, z^{-2}}$$

we get,
$$aT = 0.6 \Rightarrow a = \frac{0.6}{0.3} = 2$$

and
$$bT = 0.9 \Rightarrow b = \frac{0.9}{0.3} = 3$$

Hence,
$$H_a(s) = \frac{s+2}{(s+2)^2+9}$$

Example 4.32 Transform the causal analog filter represented by

$$H_a(s) = \frac{4s^2+10s+8}{(s^2+2s+3)(s+1)}$$

into a causal IIR digital filter. Assume, $T = 0.2$ secs.

☐ Solution

Given,
$$H_a(s) = \frac{4s^2+10s+8}{(s^2+2s+3)(s+1)}$$

Making use of partial fraction expansion, we can write

$$\begin{aligned} H_a(s) &= \frac{1}{s+1} + \frac{3s+5}{s^2+2s+3} \\ &= \frac{1}{s+1} + \frac{3(s+1)}{(s+1)^2+\left(\sqrt{2}\right)^2} + \frac{\sqrt{2}\left(\sqrt{2}\right)}{(s+1)^2+\left(\sqrt{2}\right)^2} \end{aligned}$$

We know that

1. $$\frac{1}{s-s_i} \xrightarrow{\text{IIT}} \frac{1}{1-e^{s_iT}z^{-1}}$$
2. $$\frac{s+a}{(s+a)^2+b^2} \xrightarrow{\text{IIT}} \frac{1-e^{-aT}\cos bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$
3. $$\frac{b}{(s+a)^2+b^2} \xrightarrow{\text{IIT}} \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$

Hence,
$$\begin{aligned} H(z) &= \frac{1}{1-e^{-T}z^{-1}} + 3\left[\frac{1-e^{-T}\cos\left(\sqrt{2}T\right)z^{-1}}{1-2e^{-T}\cos\left(\sqrt{2}T\right)z^{-1}+e^{-2T}z^{-2}}\right] \\ &\quad +\sqrt{2}\left[\frac{e^{-T}\sin\left(\sqrt{2}T\right)z^{-1}}{1-2e^{-T}\cos\left(\sqrt{2}T\right)z^{-1}+e^{-2T}z^{-2}}\right]. \end{aligned}$$

Letting $T = 0.2$ secs in the above equation, we get

$$H(z) = \frac{1}{1-0.81873z^{-1}} + \frac{3-2.3585z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}} + \frac{0.32314z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}}$$

$$= \frac{1}{1-0.81873z^{-1}} + \frac{3-2.03536z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}}$$

Example 4.33 A digital lowpass filter is required to meet the following specifications:

$$20\log|H(\omega)|_{\omega=0.2\pi} \geqslant -1.9328 \text{ dB}$$
$$20\log|H(\omega)|_{\omega=0.6\pi} \leqslant -13.9794 \text{ dB}$$

The filter must have a maximally flat frequency response. Find $H(z)$ to meet the above specifications using impulse invariant transformation.

❑ **Solution**

We are given the following specifications for the digital filter:

$$K_P = -1.9328 \text{ dB}, \qquad \omega_P = 0.2\,\pi$$
$$K_S = -13.9794 \text{ dB}, \qquad \omega_S = 0.6\,\pi$$

Step 1: Convert the edge-band digital frequencies into analog frequencies using the formula, $\Omega = \frac{\omega}{T}$ with $T = 1$ sec.

Hence,
$$\Omega_P = 0.2\pi \text{ rad/sec}, \qquad K_P = -1.9328 \text{ dB}$$
$$\Omega_S = 0.6\pi \text{ rad/sec}, \qquad K_S = -13.9794 \text{ dB}$$

The effect of T gets cancelled out in the design. Hence, $T = 1$ is taken as a matter of convenience.

Step 2: A Butterworth analog filter is chosen as the analog prototype, to meet the maximally flat condition for the frequency response. Using the analog specifications determined in step 1, let us design an analog lowpass filter, $H_a(s)$.

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} = 1.7$$

Rounding off to the next larger integer, we get $N = 2$.

The poles of the second-order normalized lowpass Butterworth filter are as found as follows:

$$s_k = 1\underline{/\theta_k}$$

where
$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \frac{\pi}{2}, \quad k = 0, 1, \cdots 2N-1$$

k	s_k
0	$-0.707 + j0.707$
1	$-0.707 - j0.707$

Hence,
$$H_2(s) = \frac{1}{\prod\limits_{\substack{\text{LHP}\\ \text{only}}} (s - s_k)} = \frac{1}{(s - s_0)(s - s_1)}$$

$$= \frac{1}{(s + 0.707 - j0.707)(s + 0.707 + j0.707)}$$

$$= \frac{1}{(s + 0.707)^2 + (0.707)^2}$$

$$= \frac{1}{s^2 + 1.414s + 1}$$

Let us determine the cutoff frequency Ω_C to meet the passband requirement precisely.

$$\Omega_C = \frac{\Omega_P}{\left[10^{\frac{-K_P}{10}} - 1\right]^{\frac{1}{2N}}} = 0.7255$$

The required lowpass analog protype is obtained by applying lowpass-to-lowpass analog frequency transformation to $H_2(s)$.

That is,
$$H_a(s) = H_2(s)\big|_{s \rightarrow \frac{s}{\Omega_C}}$$

$$\Rightarrow \quad H_a(s) = \frac{1}{s^2 + 1.414s + 1}\bigg|_{s \rightarrow \frac{s}{0.7255}}$$

$$= \frac{1}{\left(\frac{s}{0.7255}\right)^2 + 1.414\left(\frac{s}{0.7255}\right) + 1}$$

$$= \frac{0.52635}{s^2 + 1.02586s + 0.52635}$$

$$= \frac{0.52635}{s^2 + 1.02586s + 0.263097 - 0.263097 + 0.52635}$$

$$= \frac{1.0259 \times 0.513082}{(s + 0.5129298)^2 + (0.513082)^2}$$

Step 3: Let us design $H(z)$ using IIT with $T = 1$ sec.

We know that,

$$\frac{b}{(s+a)^2+b^2} \xrightarrow{\text{IIT}} \frac{e^{-aT}\sin bT z^{-1}}{1-2e^{-aT}\cos bT z^{-1}+e^{-2aT}z^{-2}}$$

Hence, $$H(z) = 1.0259\left[\frac{e^{-0.5129298}\sin(0.513082)z^{-1}}{1-2e^{-0.5129298}\cos(0.513082)z^{-1}+e^{-2\times 0.5129298}z^{-2}}\right]$$

$$= \frac{0.301512z^{-1}}{1-1.0433z^{-1}+0.3585z^{-2}}$$

Verification

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{0.301512e^{-j\omega}}{1-1.0433e^{-j\omega}+0.3585e^{-j2\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.301512\sqrt{\cos^2\omega+\sin^2\omega}}{\sqrt{(1-1.0433\cos\omega+0.3585\cos 2\omega)^2+(1.0433\sin\omega-0.3585\sin 2\omega)^2}}$$

Therefore $$20\log|H(\omega)|_{\omega=0.2\pi} = -2\text{ dB}$$

and $$20\log|H(\omega)|_{\omega=0.6\pi} = -14.4\text{ dB}$$

It may be noted that the passband specification is slightly exceeded and this is due to aliasing. This will not be the case when $H(z)$ is designed using bilinear transformation. If the resulting $H(z)$ designed using IIT fails to meet the given specifications because of aliasing, there is no alternative with impulse invariance but to try again with a higher-order filter or with a different adjustment of the filter parameter, holding N fixed.

Example 4.34 Apply impulse invariant technique to the analog transfer function given by

$$H_a(s) = \frac{s^2+4.525}{s^2+0.692s+0.504}$$

with $T = 1$ sec.

☐ **Solution**

We can write the given analog transfer function as

$$H_a(s) = \frac{s^2+4.525+0.692s-0.692s+0.504-0.504}{s^2+0.692s+0.504}$$

$$= \frac{(s^2+0.692s+0.504)-0.692s+4.021}{s^2+0.692s+0.504}$$

$$= 1 + \left[\frac{-0.692(s+0.346) + 4.021 + 0.692 \times 0.346}{(s+0.346)^2 + 0.(6199)^2}\right]$$

$$= 1 - \frac{0.692(s+0.346)}{(s+0.346)^2 + (0.6199)^2} + \frac{4.26}{(s+0.346)^2 + (0.6199)^2}$$

$$= 1 - \frac{0.692(s+0.346)}{(s+0.346)^2 + (0.6199)^2} + 6.872\frac{0.6199}{(s+0.346)^2 + (0.6199)^2}$$

We know that

1. $\dfrac{s+a}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1}}$

2. $\dfrac{b}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT z^{-1} + e^{-2aT} z^{-2}}$

3. $1 \overset{\text{IIT}}{\Longrightarrow} 1$

   (please note that $\mathcal{L}\{\delta(t)\} = 1$ and $\mathcal{Z}\{\delta(n)\} = 1$)

$$\text{Hence,}\quad H(z) = 1 - \frac{0.692\left[1 - e^{-0.346}\cos(0.6199)z^{-1}\right]}{1 - 2\,e^{-0.346}\cos(0.6199)z^{-1} + e^{-0.692}z^{-2}}$$

$$+ \frac{6.872\,e^{-0.346}\sin(0.6199)z^{-1}}{1 - 2\,e^{-0.346}\cos(0.6199)z^{-1} + e^{-0.692}z^{-2}}$$

$$= 1 - \left[\frac{0.692 - 3.2234z^{-1}}{1 - 1.1517z^{-1} + 0.5006z^{-2}}\right]$$

Example 4.35 Design a digital Chebyshev I filter that satisfies the following constraints.

$$0.8 \leqslant |H(\omega)| \leqslant 1, \qquad 0 \leqslant \omega \leqslant 0.2\pi$$
$$|H(\omega)| \leqslant 0.2, \qquad 0.6\pi \leqslant \omega \leqslant \pi$$

Use impluse invariant transformation.

☐ Solution

We are given the following digital specifications:

Passband ripple: $\delta_P = 1 - 0.8 = 0.2$.

Passband-edge frequency: $\omega_P = 0.2\pi$.

Stopband tolerance: $\delta_S = 0.2$.

Stopband-edge frequency: $\omega_S = 0.6\pi$.

Step 1: Convert the above edge-band digital frequencies into analog frquencies using the formula $\Omega = \frac{\omega}{T}$ with $T = 1$ sec.

Hence,

$$\Omega_P = 0.2\pi, \quad \delta_P = 1 - 0.8 = 0.2$$
$$\Rightarrow \quad K_P = 20\log(1-\delta_P) = -1.94 \text{ dB}$$
$$\Omega_S = 0.6\,\pi, \quad \delta_S = 0.2$$
$$\Rightarrow \quad K_S = 20\log\delta_S = -14 \text{ dB}$$

Step 2: Design a chebyshev I lowpass analog prototype filter to meet the specifications listed in step 1.

$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.153$$

$$K = \frac{\Omega_P}{\Omega_S} = 0.33$$

Filter order:
$$N \geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)}$$
$$\Rightarrow \quad N \geqslant 1.446$$

Hence, the minimum filter order is $N = 2$.

We know that
$$1 - \delta_P = \frac{1}{\sqrt{1+\epsilon^2}}$$
$$\Rightarrow \quad \epsilon = 0.75$$

$$a = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} - \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}}$$
$$= 0.57735$$

$$b = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} + \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}}$$
$$= 1.1547$$

$$\sigma_k = -a\sin\left[(2k-1)\frac{\pi}{2N}\right], \quad k = 1, 2, 3, 4$$

$$\Omega_k = b\cos\left[(2k-1)\frac{\pi}{2N}\right], \quad k = 1, 2, 3, 4$$

k	σ_k	Ω_k	s_k
1	−0.4082481	0.8164962	$-0.4082481 + j0.8164962$
2	−0.4082481	−0.8164962	$-0.4082481 - j0.8164962$
3	0.4082481	−0.8164962	$0.4082481 - j0.8164962$
4	0.4082481	0.8164962	$0.4082481 + j0.8164962$

$$\text{Hence,}\qquad H_2(s) = \frac{K_N}{\prod\limits_{\substack{\text{LHP}\\ \text{only}}} (s - s_k)} = \frac{K_N}{(s - s_1)(s - s_2)}$$

$$= \frac{K_N}{(s + 0.4082481 - j0.8164962)\,(s + 0.4082481 + j0.8164962)}$$

$$= \frac{K_N}{(s + 0.4082481)^2 + (0.8164962)^2}$$

$$= \frac{K_N}{s^2 + 0.8164962s + 0.833333}$$

$$\text{where}\qquad K_N = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.833333}{\sqrt{1 + (0.75)^2}} = 0.667$$

$$\text{Thus,}\qquad H_2(s) = \frac{0.667}{s^2 + 0.8164962s + 0.833333}$$

Since, we want the cutoff at $\Omega_P = 0.2\pi$, let us apply lowpass-to-lowpass transformation on $H_2(s)$ and get $H_a(s)$.

$$\text{That is,}\qquad H_a(s) = H_2(s)\big|_{s \longrightarrow \frac{s}{0.2\pi}}$$

$$= \frac{0.667}{\left(\frac{s}{0.2\pi}\right)^2 + 0.8164962\left(\frac{s}{0.2\pi}\right) + 0.833333}$$

$$= \frac{0.263321}{s^2 + 0.51302s + 0.32899} = \frac{0.263321}{(s+0.25651)^2 - (0.25651)^2 + 0.32899}$$

$$= \frac{0.263321}{(s + 0.25651)^2 + (0.51302)^2}$$

$$= \frac{0.263321}{0.51302} \times \frac{0.51302}{(s + 0.25651)^2 + (0.51302)^2}$$

$$= 0.513276 \times \frac{0.51302}{(s + 0.25651)^2 + (0.51302)^2}$$

Step 3: Design $H(z)$ using IIT with $T = 1$ sec.

$$\text{Recall :}\qquad \frac{b}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} \frac{e^{-aT} \sin bT z^{-1}}{1 - 2e^{-aT} \cos bT z^{-1} + e^{-2aT} z^{-2}}$$

$$\text{Hence,}\qquad H(z) = \frac{0.513276 \times e^{-0.25651} \sin(0.51302) z^{-1}}{1 - 2e^{-0.25651} \cos(0.51302) z^{-1} + e^{-2\times 0.25651} z^{-2}}$$

$$= \frac{0.19492 z^{-1}}{1 - 1.34828 z^{-1} + 0.598685 z^{-2}}$$

Verification

$$H(e^{j\omega}) = H(\omega) = \frac{0.19492e^{-j\omega}}{1 - 1.34828e^{-j\omega} + 0.598685e^{-j2\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.19492}{\sqrt{\begin{array}{l}(1 - 1.34828\cos\omega + 0.598685\cos 2\omega)^2 \\ +(1.34828\sin\omega - 0.598685\sin 2\omega)^2\end{array}}}$$

Therefore $\quad 20\log|H(\omega)|_{\omega=0.2\pi} = -1.886$ dB

and $\quad 20\log|H(\omega)|_{\omega=0.6\pi} = -19.69$ dB

4.12 Matched $\mathcal{Z}$-transform Design

This method of design is conceptually straightforward. Poles and zeros of the analog transfer function $H_a(s)$ are transformed directly to poles and zeros of the digital transfer function $H(z)$ by making a simple substitution. Consider a term representing a simple pole such as $\frac{1}{s+s_i}$. The development leading to the appropriate substitution is as follows.

$$\mathcal{L}^{-1}\left\{\frac{1}{s+s_i}\right\} = e^{-s_i t} \xrightarrow[\text{yields}]{\text{sampled}} e^{-s_i nT} \xrightarrow[\text{yields}]{\mathcal{Z}\text{-transformed}} \frac{1}{1 - e^{-s_i T}z^{-1}}$$

and we say that a pole or zero factor such as

$$s + s_i \longrightarrow \left(1 - e^{-s_i T}z^{-1}\right)$$

or the analog pole or zero at $s = -s_i$ maps to a digital pole or zero at $z = e^{-s_i T}$.

Following the same line of reasoning as above, we find that

$$s + a - jb \longrightarrow 1 - e^{-(a-jb)T}z^{-1}$$

$$\Rightarrow \quad (s + a - jb)(s + a + jb) \longrightarrow \left(1 - e^{-aT}e^{jbT}z^{-1}\right)\left(1 - e^{-aT}e^{-jbT}z^{-1}\right)$$

$$= \left(1 - 2z^{-1}e^{-aT}\cos bT + e^{-2aT}z^{-2}\right)$$

The matched $\mathcal{Z}$-transform technique is very simple to apply but it suffers from the fact that unlike BLT, IIT, and backward difference methods, an all-pole analog filter will be transformed as an all-pole digital filter. As a result of this, there are no zeros to help shape the frequency response. Even if $H(s)$ does have zeros, if their imaginary parts are greater than $\frac{\omega_s}{2}$, the resulting zeros in $H(z)$ will produce serious aliasing errors.

Example 4.36 Given the analog transfer function,

$$H(s) = \frac{s+2}{(s+1)(s+3)}$$

Find $H(z)$ using matched $\mathcal{Z}$-transform design. The system uses a sampling rate of 10 Hz ($T = 0.1$ sec).

☐ Solution

Recall the fact:

$$
\begin{aligned}
& s + s_i &\longrightarrow\quad & 1 - e^{-s_i T} z^{-1} \\
\Rightarrow\quad & s + 1 &\longrightarrow\quad & \left(1 - e^{-T} z^{-1}\right) = 1 - 0.905 z^{-1} \\
& s + 2 &\longrightarrow\quad & \left(1 - e^{-2T} z^{-1}\right) = 1 - 0.819 z^{-1} \\
& s + 3 &\longrightarrow\quad & \left(1 - e^{-3T} z^{-1}\right) = 1 - 0.741 z^{-1}
\end{aligned}
$$

Hence, the discrete-time transfer function $H(z)$ becomes

$$
H(z) = \frac{1 - 0.819 z^{-1}}{\left(1 - 0.905 z^{-1}\right)\left(1 - 0.741 z^{-1}\right)}
$$

Observations

- The stable poles of $H(s)$ at $s = -1$ and $s = -3$ become stable poles of $H(z)$ at $z = e^{-T} = 0.905$ and $z = e^{-3T} = 0.741$.
- It is common practice to have $H(j\Omega)\big|_{\Omega=0}$ and $H(\omega)\big|_{\omega=0}$ to have unit values. In this case,

$$
H(j\Omega)\big|_{\Omega=0} = H(s)\big|_{s=0} = \frac{2}{1 \times 3} = 0.66
$$

and

$$
H(\omega)\big|_{\omega=0} = H(z)\big|_{z=1} = \frac{1 - 0.819}{(1 - 0.905)(1 - 0.741)} = 7.3562
$$

Hence, the analog frequency plot should be scaled by $\frac{1}{0.66} = 1.515$ and the digital frequency plot should be scaled by $\frac{1}{7.3562} = 0.1359$ to make them equal at zero frequency.

4.13 Analog Design Using Digital Filters

Often we are interested in simulating an analog filter $H_{\text{eq}}(s)$ using the A/D-$H(z)$-D/A structure shown in Fig. 4.14. We are usually given the analog requirements in terms of edge-band frequencies Ω_P and Ω_S and the corresponding decibel gains K_P and K_S. The sampling rate $f_S = \frac{1}{T}$ of the A/D converter will be specified or can be determined from the baseband spectrum of the input signal using sampling theorem. The recipe for the analog design using digital filters is as follows:

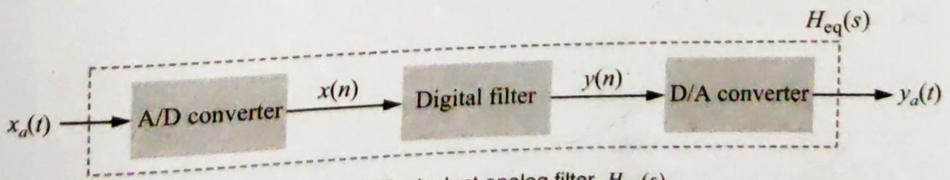


Fig. 4.14 Equivalent analog filter, $H_{\text{eq}}(s)$.

Step 1: Convert each specified band-edge frequency of the analog filter to a corresponding band-edge frequency of digital filter using the equation, $\omega = \Omega T$. Leave the pass-band gain K_P and stopband gain K_S unchanged. The relation $\omega = \Omega T$ can be proved as follows.

Let the input to the equivalent analog filter be

$$x_a(t) = \sin(\Omega t)$$

The output of the A/D converter can be obtained by letting $t = nT$ in the above expression. Accordingly, we get

$$x_a(nT) = x(n) = \sin(\Omega\, nT) = \sin(\omega\, n)$$

Thus, the magnitude of $x(n)$ is same as that of $x_a(t)$. However, the digital frequency ω is given interms of the analog frequency Ω by

$$\omega = \Omega T$$

Hence, the specifications for the digital filter become ω_P and ω_S with corresponding decibel gains K_P and K_S. We again wish to remind the reader that $K_P = 20\log(1-\delta_P)$ and $K_S = 20\log\delta_S$, where δ_P is the passband tolerance and δ_S is the stopband tolerance.

Step 2: Prewarp each specified band-edge frequency of the digital filter to a corresponding band-edge frequency of an analog filter using the relation, $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$. Leave K_P and K_S unchanged. The effect of T gets cancelled out in the bilinear transformation and hence T may be taken as 1 to simplify the calculations.

Step 3: Design an analog filter $H_a(s)$ of the specified type, according to prewarped specifications of Step 2.

Step 4: Transform $H_a(s)$ to a digital filter $H(z)$ using bilinear transformation in which $s \longrightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ with $T = 1$ sec.

Example 4.37 Design a digital filter $H(z)$ that when used in an A/D-H(z)-D/A structure gives an equivalent analog filter with the following specifications:

Passband ripple : $\leqslant 3.01$ dB.

Passband edge : 500 Hz.

Stopband attenuation : $\geqslant 15$ dB.

Stopband edge : 750 Hz.

Sample rate : 2 KHz.

The filter is to be designed by performing a bilinear transformation on an analog system function, use Butterworth prototype. Also, plot the complete magnitude frequency response and obtain the difference equation realization.

□ Solution

The analog specifications are

$$\Omega_P = 2\pi \times 500 = \pi \times 10^3 \text{ rad/sec}, \qquad K_P = -3.01 \text{ dB}$$
$$\Omega_S = 2\pi \times 750 = 1.5\pi \times 10^3 \text{ rad/sec}, \qquad K_S = -15 \text{ dB}$$

Also, $T = \dfrac{1}{f_S} = \dfrac{1}{2000}$ secs.

Step 1: The corresponding digital specifications are obtained as follows.

$$\omega_P = \Omega_P T = \pi \times 10^3 \times \frac{1}{2000} = 0.5\pi \text{ rad}, \qquad K_P = -3.01 \text{ dB}$$
$$\omega_S = \Omega_S T = 1.5\pi \times 10^3 \times \frac{1}{2000} = 0.75\pi \text{ rad}, \qquad K_S = -15 \text{ dB}$$

Step 2: Prewarp the band-edge digital frequencies using $T = 1$ sec. Leave K_P and K_S unchanged.

$$\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) = 2\tan\left(\frac{0.5\pi}{2}\right) = 2, \qquad K_P = -3.01 \text{ dB}$$
$$\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) = 2\tan\left(\frac{0.75\pi}{2}\right) = 4.8282, \qquad K_S = -15 \text{ dB}$$

Step 3: Design an analog lowpass filter having the transfer function $H_a(s)$ to meet the prewarped specifications of Step 2.

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right) \Big/ \left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega'_P}{\Omega'_S}\right)} = 1.944$$

Rounding off N to the next larger integer, we get $N = 2$.

The cutoff frequency Ω_C is found to satisfy the passband requirement exactly.

$$\Omega_C = \frac{\Omega'_P}{\left[10^{\frac{-K_P}{10}} - 1\right]^{\frac{1}{2N}}} = 2$$

Referring the normalized lowpass Butterworth filter tables, we get

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Therefore, the required prewarped analog filter is obtained by applying lowpass-to-lowpass transformation to $H_2(s)$.

That is,

$$\begin{aligned} H_a(s) &= H_2(s)\big|_{s\to\frac{s}{2}} \\ &= \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s\to\frac{s}{2}} \\ &= \frac{4}{s^2+2\sqrt{2}s+4} \end{aligned}$$

Step 4: Apply bilinear transformation to $H_a(s)$ with $T = 1$ sec and get $H(z)$.

That is,

$$\begin{aligned} H(z) &= H_a(s)\big|_{s\to\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{4}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2+2\sqrt{2}\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+4} \\ &= \frac{1+2z^{-1}+z^{-2}}{3.4142+0.5858z^{-2}} \\ &= \frac{(1+z^{-1})^2}{3.4142+0.5858z^{-2}} \end{aligned}$$

Difference equation realization

Let

$$H(z) = \frac{Y(z)}{X(z)}$$

Then,

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+z^{-2}}{3.4142+0.5858z^{-2}}$$

Cross multiplying and taking inverse $\mathcal{Z}$-transform yield

$$3.4142y(n)+0.5858y(n-2) = x(n)+2x(n-1)+x(n-2)$$

$$\Rightarrow \quad y(n) = -0.1715y(n-2)+0.2928x(n)+0.5857x(n-1)+0.2928x(n-2)$$

Verification of the design

Letting $z = e^{j\omega}$ in $H(z)$, we get

$$\begin{aligned} H\left(e^{j\omega}\right) = H(\omega) &= \frac{(1+e^{-j\omega})^2}{3.4142+0.5858e^{-j2\omega}} \\ &= \frac{[(1+\cos\omega)-j\sin\omega]^2}{(3.4142+0.5858\cos 2\omega)-j\,0.5858\sin 2\omega} \\ \Rightarrow \quad |H(\omega)| &= \frac{[(1+\cos\omega)^2+\sin^2\omega]}{\sqrt{(3.4142+0.5858\cos 2\omega)^2+(0.5858\sin 2\omega)^2}} \end{aligned}$$

Therefore,

$$\begin{aligned} 20\log|H(\omega)|_{\omega=0.5\pi} &= -3.01\text{ dB} \\ 20\log|H(\omega)|_{\omega=0.75\pi} &= -15.44\text{ dB} \end{aligned}$$

The complete magnitude frequency response is drawn making use of the table given below and is shown in Fig. Ex.4.37.

Table: Ex.4.37

ω	$20 \log \lvert H(\omega) \rvert$
0	0
0.2π	-0.048
0.3π	-0.28
0.4π	-1.067
0.5π	-3.01
0.6π	-6.617
0.8π	-19.57
π	$-\infty$

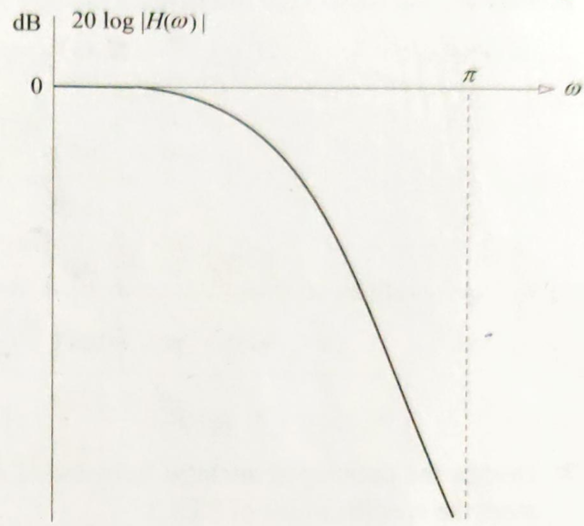


Fig. Ex.4.37 Magnitude frequency response for $H(z) = \frac{(1+z^{-1})^2}{3.4142+0.5858z^{-2}}$.

Example 4.38 Design an IIR digital filter that when used in the prefilter A/D-H(z)-D/A structure will satisfy the following specifications (use Chebyshev protype):

a. lowpass filter with -2 dB cutoff at 100 Hz,

b. stopband attenuation of 20 dB or greater at 500 Hz, and

c. sampling rate of 4000 samples/sec.

Verify the design.

☐ Solution

We are given the following analog requirements:

$$\Omega_P = 2\pi \times 100 = 200\pi \text{ rad/sec}, \qquad K_P = -2 \text{ dB}$$
$$\Omega_S = 2\pi \times 500 = 1000\pi \text{ rad/sec}, \qquad K_S = -20 \text{ dB}$$

Also,
$$T = \frac{1}{4000} \text{ secs}$$

In the present problem, the value of T is provided. In practice, the value of T is found using lowpass sampling theorem: $T \leqslant \frac{1}{2f_x}$, where f_x is the highest frequency present in the input signal, $x_a(t)$.

Step 1: Convert the above analog frequencies into equivalent digital frequencies using the relation $\omega = \Omega T$ with $T = \frac{1}{4000}$ secs. The values of K_P and K_S remain unchanged.

$$\omega_P = \Omega_P T = 0.05\pi, \qquad K_P = -2 \text{ dB}$$
$$\omega_S = \Omega_S T = 0.25\pi, \qquad K_S = -20 \text{ dB}$$

Step 2: Prewarp the band-edge frequencies ω_P and ω_S using $T = 1$ sec.

$$\begin{aligned}
\Omega'_P &= \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.05\pi}{2}\right) \\
&= 0.1574, \qquad K_P = -2 \text{ dB} \\
\Omega'_S &= \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.25\pi}{2}\right) \\
&= 0.8284, \qquad K_S = -20 \text{ dB}
\end{aligned}$$

Step 3: Design the prewarped analog Chebyshev I filter having the transfer function $H_a(s)$ to meet the specifications of Step 2.

Given

$$\begin{aligned}
K_P = -2 &= 20\log(1-\delta_P) \\
\Rightarrow \qquad \delta_P &= 0.20567
\end{aligned}$$

and

$$\begin{aligned}
K_S = -20 &= 20\log\delta_S \\
\Rightarrow \qquad \delta_S &= 0.1
\end{aligned}$$

$$\begin{aligned}
d &= \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.077 \\
K &= \frac{\Omega'_P}{\Omega'_S} = 0.19 \\
N &\geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)} = 1.39 \\
\Rightarrow \qquad N &\geqslant 1.39
\end{aligned}$$

Minimum filter order, $N = 2$.

To find $H_2(s)$:

$$\begin{aligned}
K_P &= 20\log\left(\frac{1}{\sqrt{1+\epsilon^2}}\right) = -2 \\
\Rightarrow \qquad \epsilon &= 0.76478 \\
a &= \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{1}{N}} - \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{-1}{N}} \\
&= 0.56839 \\
b &= \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{1}{N}} + \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{-1}{N}} \\
&= 1.15024
\end{aligned}$$

We know that
$$\sigma_k = -a\sin\left[(2k-1)\frac{\pi}{2N}\right], \qquad k = 1, 2, \cdots 2N$$
$$\Rightarrow \quad \sigma_k = -0.56839\sin\left[(2k-1)\frac{\pi}{4}\right], \qquad k = 1, 2, 3, 4$$
and
$$\Omega_k = b\cos\left[(2k-1)\frac{\pi}{2N}\right], \qquad k = 1, 2, \cdots 2N$$
$$\Rightarrow \quad \Omega_k = 1.15024\cos\left[(2k-1)\frac{\pi}{4}\right], \qquad k = 1, 2, 3, 4$$

k	σ_k	Ω_k	$s_k = \sigma_k + j\Omega_k$
1	-0.40191	0.81334	$-0.40191 + j0.81334$
2	-0.40191	-0.81334	$-0.40191 - j0.81334$

The values of $k = 3$ and 4 give the poles of $H_2(s)H_2(-s)$ on the right-half of the s-plane and hence are not considered. In fact, $k = 3$ and 4 give the poles of $H_2(-s)$.

Thus,
$$H_2(s) = \frac{K_N}{\prod\limits_{\substack{\text{LHP}\\ \text{only}}}(s - s_k)} = \frac{K_N}{(s - s_1)(s - s_2)}$$
$$= \frac{K_N}{(s + 0.40191 - j0.81334)(s + 0.40191 + j0.81334)}$$
$$= \frac{K_N}{(s + 0.40191)^2 + (0.81334)^2}$$
$$= \frac{K_N}{s^2 + \underbrace{0.80382}_{b_1}\, s + \underbrace{0.82305}_{b_0}}$$

Since N is even, the normalizing factor $K_N = \frac{b_0}{\sqrt{1+\epsilon^2}}$.

Hence,
$$K_N = \frac{0.82305}{\sqrt{1 + (0.76478)^2}} = 0.65377$$

Therefore,
$$H_2(s) = \frac{0.65377}{s^2 + 0.80382s + 0.82305}$$

Since, we want the cutoff at $\Omega'_p = 0.1574$, we apply lowpass-to-lowpss transformation to $H_2(s)$ and get the required lowpass analog filter $H_a(s)$.

That is,
$$H_a(s) = H_2(s)\big|_{s \to \frac{s}{0.1574}}$$
$$= \frac{0.65377}{\left(\frac{s}{0.1574}\right)^2 + 0.80382\left(\frac{s}{0.1574}\right) + 0.82305}$$
$$= \frac{0.0162}{s^2 + 0.12652s + 0.02039}$$

Step 4: Applying the bilinear transformation with $T = 1$ sec to $H_a(s)$ will take the prewarped analog filter to a digital filter with transfer function $H(z)$ that will satisfy the digital specifications mentioned in step 1. The $H(z)$ thus designed meets the specifications of $H_{eq}(s)$ of the A/D-$H(z)$-D/A structure.

That is,

$$\begin{aligned} H(z) &= H_a(s)\Big|_{s \to \frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{0.0162}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.12652 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.02039} \\ &= \frac{0.0162(1+z^{-1})^2}{4.27343 - 7.95922z^{-1} + 3.76735z^{-2}} \end{aligned}$$

Verification of the design

The frequency response of the digital filter is obtained by letting $z = e^{j\omega}$ in $H(z)$. Accordingly, we get

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{0.0162\left(1+e^{-j\omega}\right)^2}{4.27343 - 7.95922e^{-j\omega} + 3.76735e^{-j2\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.0162\left[(1+\cos\omega)^2 + \sin^2\omega\right]}{\sqrt{\begin{array}{c}(4.27343 - 7.95922\cos\omega + 3.76735\cos 2\omega)^2 \\ +(7.9592\sin\omega - 3.76735\sin 2\omega)^2\end{array}}}$$

Hence,

$$20\log|H(\omega)|_{\omega=0.05\pi} = -2 \text{ dB}$$

and

$$20\log|H(\omega)|_{\omega=0.25\pi} = -32.3 \text{ dB}$$

Example 4.39 A digital lowpass filter is required to meet the following specifications:

Passband ripple : $\leqslant 1$ dB.

Passband edge : 4 KHz.

Stopband attenuation : $\geqslant 40$ dB.

Stopband edge : 6 KHz.

Sample rate : 24 KHz.

The filter is to be designed by performing bilinear transformation on an analog system function. Determine the order of Butterworth and Chebyshev I analog designs that must be used to meet the specifications in the digital implementation.

☐ **Solution**

We are given the following analog requirements:

$$K_P = -1 \text{ dB}, \qquad \Omega_P = 2\pi \times 4 \times 10^3 \text{ rad/sec}$$

$$K_S = -40 \text{ dB}, \qquad \Omega_S = 2\pi \times 6 \times 10^3 \text{ rad/sec}$$

Also,

$$T = \frac{1}{f_S} = \frac{1}{24 \times 10^3} \text{ sec}$$

Step 1: Let us convert the above analog specifications into digital specifications.

$$\omega_P = \Omega_P T = 2\pi \times 4 \times 10^3 \times \frac{1}{24 \times 10^3} = \frac{\pi}{3} \text{ rad}, \qquad K_P = -1 \text{ dB}$$
$$\omega_S = \Omega_S T = 2\pi \times 6 \times 10^3 \times \frac{1}{24 \times 10^3} = \frac{\pi}{2} \text{ rad}, \qquad K_S = -40 \text{ dB}$$

Step 2: Prewarp the band-edge digital frequencies using $T = 1$ sec. The values of K_P and K_S remain unchanged.

$$\Omega'_P = \frac{2}{1}\tan\left(\frac{\omega_P}{2}\right) = 1.155, \qquad K_P = -1 \text{ dB}$$
$$\Omega'_S = \frac{2}{1}\tan\left(\frac{\omega_S}{2}\right) = 2, \qquad K_S = -40 \text{ dB}$$

Butterworth filter:

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}} - 1\right)\Big/\left(10^{\frac{-K_S}{10}} - 1\right)\right]}{2\log\left(\frac{\Omega'_P}{\Omega'_S}\right)} = 9.618$$

Rounding off to next larger integer, we get $N = 10$.

Chebyshev I filter:

$$K_P = 20\log(1 - \delta_P) = -1$$
$$\Rightarrow \quad \delta_P = 0.11$$
$$K_S = 20\log\delta_S = -40$$
$$\Rightarrow \quad \delta_S = 0.01$$
$$d = \sqrt{\frac{(1-\delta_P)^{-2} - 1}{\delta_S^{-2} - 1}} = 5.1234 \times 10^{-3}$$
$$K = \frac{\Omega'_P}{\Omega'_S} = 0.5775$$
$$N \geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)}$$
$$\Rightarrow \quad N \geqslant 5.21$$

Hence, minimum filter order $N = 6$.

***Conclusion*:** For the same set of frequency-domain specifications, the filter order N for a Chebyshev filter is less than or equal to that of a Butterworth filter.

Example 4.40 Design a lowpass filter that will operate on the sampled analog data such that the cutoff frequency is 200 Hz (1 dB acceptable ripple) and at 400 Hz, the attenuation is atleast 20 dB with a monotonic shape past 200 Hz. Take $T = \frac{1}{2000}$ secs and use normalized lowpass filter tables.

Solution

We are given the following analog specifications:

$$\begin{aligned}
\Omega_P &= 2\pi \times 200 = 400\pi \text{ rad/sec}, \qquad K_P = -1 \text{ dB} \\
\Omega_S &= 2\pi \times 400 = 800\pi \text{ rad/sec}, \qquad K_S = -20 \text{ dB}
\end{aligned}$$

Also,

$$T = \frac{1}{2000} \text{ secs}$$

Step 1: Convert the above analog frequencies into equivalent digital specifications using the formula, $\omega = \Omega T$ with $T = \frac{1}{2000}$ secs. The values of K_P and K_S remain unchanged.

$$\begin{aligned}
\omega_P &= \Omega_P T = 400\pi \times \frac{1}{2000} = 0.2\pi \text{ rad}, \qquad K_P = -1 \text{ dB} \\
\omega_S &= \Omega_S T = 800\pi \times \frac{1}{2000} = 0.4\pi \text{ rad}, \qquad K_S = -20 \text{ dB}
\end{aligned}$$

Step 2: Prewarp the critical digital frequencies using $T = 1$ sec.

$$\begin{aligned}
\Omega'_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) &= \frac{2}{1}\tan\left(\frac{0.2\pi}{2}\right) \\
&= 0.6498, \qquad K_P = -1 \text{ dB} \\
\Omega'_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) &= \frac{2}{1}\tan\left(\frac{0.4\pi}{2}\right) \\
&= 1.4531, \qquad K_S = -20 \text{ dB}
\end{aligned}$$

Step 3: A Chebyshev I filter is taken to meet the specification of a monotonic magnitude response beyond passband. Let us design an analog lowpass filter having the transfer function $H_a(s)$ to meet the prewarped specifications of Step 2.

We know that

$$K_P = 20\log(1-\delta_P)$$

Hence,

$$\begin{aligned}
-1 &= 20\log(1-\delta_P) \\
\Rightarrow \quad \delta_P &= 0.11
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_S &= 20\log\delta_S \\
\Rightarrow \quad -20 &= 20\log\delta_S \\
\Rightarrow \quad \delta_S &= 0.1
\end{aligned}$$

$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.05$$

$$K = \frac{\Omega'_P}{\Omega'_S} = 0.45$$

$$N \geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)} = 2.567$$

$$\Rightarrow \quad N \geqslant 2.567$$

Hence, the minimum filter order is $N = 3$.

Referring to the normalized filter tables for 1 dB ripple with $N = 3$, we get the following normalized filter coefficients.

$$b_0 = 0.4913067 \approx 0.49131$$
$$b_1 = 1.2384092 \approx 1.23841$$
$$b_2 = 0.9883412 \approx 0.98834$$

Since, N is odd, the normalizing factor $K_N = b_0 = 0.49131$.

Therefore,
$$H_3(s) = \frac{b_0}{s^3 + b_2\, s^2 + b_1\, s + b_0} = \frac{0.49131}{s^3 + 0.98834s^2 + 1.23841s + 0.49131}$$

Since, we want the cutoff at $\Omega'_P = 0.6498$, apply lowpass-to-lowpass transformation to $H_3(s)$ and get the required lowpass analog filter, $H_a(s)$.

That is,
$$H_a(s) = H_3(s)\Big|_{s \longrightarrow \frac{s}{\Omega'_P}}$$
$$= \frac{0.49131}{s^3 + 0.98834s^2 + 1.23841s + 0.49131}\Bigg|_{s \longrightarrow \frac{s}{0.6498}}$$
$$= \frac{0.1348}{s^3 + 0.64222s^2 + 0.5229s + 0.1348}$$

Step 4: Transform $H_a(s)$ into $H(z)$ by applying bilinear transformation to $H_a(s)$ with $T = 1$ sec.

That is,
$$H(z) = H_a(s)\Big|_{s \longrightarrow \frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$
$$= \frac{0.1348}{8\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 0.64222 \times 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.5229 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1348}$$

Simplifying, we get
$$H(z) = \frac{0.1348(1+z^{-1})^3}{11.74982 - 25.11864z^{-1} + 20.7895z^{-2} - 6.34206z^{-3}}$$

Verification of the design

Letting $z = e^{j\omega}$ in $H(z)$, we get

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{0.1348\left(1+e^{-j\omega}\right)^3}{11.74982 - 25.11864e^{-j\omega} + 20.7895e^{-j2\omega} - 6.34206e^{-j3\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.1348\left((1+\cos\omega)^2 + \sin^2\omega\right)^{\frac{3}{2}}}{\sqrt{\begin{array}{l}(11.74982 - 25.11864\cos\omega + 20.7895\cos 2\omega - 6.34206\cos 3\omega)^2 \\ +(25.11864\sin\omega - 20.7895\sin 2\omega + 6.34206\sin 3\omega)^2\end{array}}}$$

$$\Rightarrow \quad |H(\omega)|_{\omega=0.2\pi} = 0.8912$$

$$\text{and} \quad |H(\omega)|_{\omega=0.4\pi} = 0.051642$$

Therefore, $20\log|H(\omega)|_{\omega=0.2\pi} = -1$ dB

and $20\log|H(\omega)|_{\omega=0.4\pi} = -25.74$ dB.

Example 4.41 Design a highpass filter $H(z)$ to be used in Fig. Ex.4.41(a) to meet the specifications shown in Fig. Ex.4.40(b). The sampling rate is fixed at 1000 samples per sec.

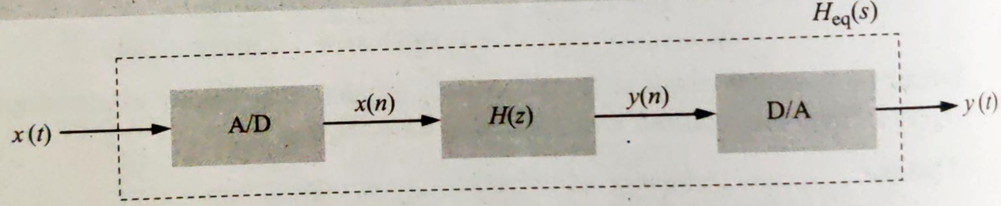


Fig. Ex.4.41(a) Equivalent analog filter having the transfer function, $H_{eq}(s)$.

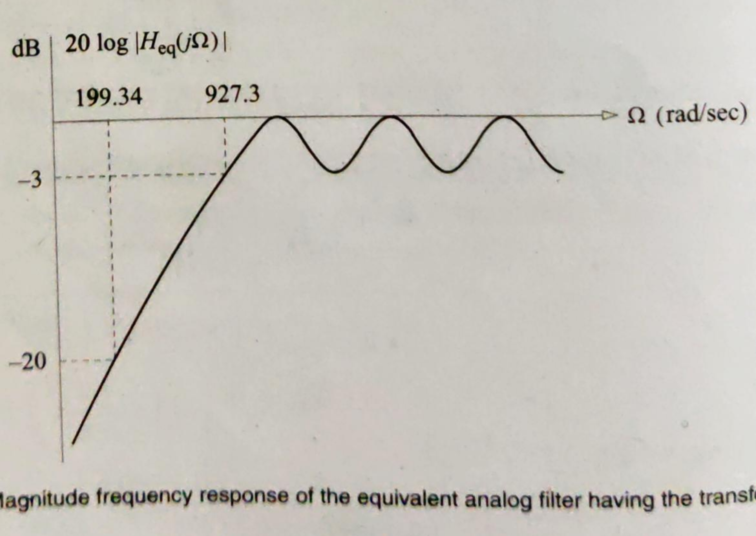


Fig. Ex.4.41(b) Magnitude frequency response of the equivalent analog filter having the transfer function, $H_{eq}(s)$.

☐ **Solution**

We are given the following analog requirements:

$$\Omega_P = 927.3 \text{ rad/sec}, \qquad K_P = -3 \text{ dB}$$
$$\Omega_S = 199.34 \text{ rad/sec}, \qquad K_S = -20 \text{ dB}$$

Also,

$$T = \frac{1}{1000} \text{ sec.}$$

Step 1: Convert the band-edge analog frequencies into digital frequencies.

$$\omega_P = \Omega_P T = 927.3 \times \frac{1}{1000} = 0.9273 \text{ rad}, \qquad K_P = -3 \text{ dB}$$
$$\omega_S = \Omega_S T = 199.34 \times \frac{1}{1000} = 0.19934 \text{ rad}, \qquad K_S = -20 \text{ dB}$$

Step 2: Prewarp the band-edge digital frequencies using $T = 1$ sec.

$$\begin{aligned}
\Omega'_P &= \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.9273}{2}\right) \\
&= 1, \qquad K_P = -3 \text{ dB} \\
\Omega'_S &= \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) \\
&= \frac{2}{1}\tan\left(\frac{0.19934}{2}\right) \\
&= 0.2, \qquad K_S = -20 \text{ dB}
\end{aligned}$$

Step 3: Design a prototype analog highpass filter $H_a(s)$ to meet the specifications of Step 2.

A normalized lowpass filter is converted into a highpass filter using the relation $s \longrightarrow \frac{1}{s}$. Hence, the specifications of the normalized lowpass filter become

$$\Omega'_P = \frac{1}{1} = 1, \qquad K_P = -3 \text{ dB}$$
$$\Omega'_S = \frac{1}{0.2} = 5, \qquad K_S = -20 \text{ dB}$$

We know that

$$\begin{aligned}
K_P &= 20\log(1-\delta_P) \\
\Rightarrow \quad -3 &= 20\log(1-\delta_P) \\
\Rightarrow \quad \delta_P &= 0.2920
\end{aligned}$$

and

$$\begin{aligned}
K_S &= 20\log\delta_S \\
\Rightarrow \quad -20 &= 20\log\delta_S
\end{aligned}$$

$$\Rightarrow \quad \delta_S = 0.1$$

$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.01$$

$$K = \frac{\Omega'_P}{\Omega'_S} = 0.2$$

Filter order:

$$N \geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)}$$

$$\Rightarrow \quad N \geqslant 1.3$$

Hence, the minimum filter order, $N = 2$. Referring to normalized lowpass Chebyshev filter tables, we have

$$b_0 = 0.7079478 \approx 0.70795$$

$$b_1 = 0.6448996 \approx 0.6449$$

Hence,

$$H_2(s) = \frac{K_N}{s^2 + b_1 s + b_0}$$

$$= \frac{\frac{b_0}{\sqrt{1+\epsilon^2}}}{s^2 + b_1 s + b_0}$$

$$= \frac{0.5012}{s^2 + 0.6449s + 0.70795}$$

The required highpass analog Chebyshev prototype is obtained by applying lowpass-t highpass transformation to $H_2(s)$.

That is,

$$H_a(s) = H_2(s)\big|_{s\longrightarrow\frac{1}{s}}$$

$$= \frac{0.5012}{\frac{1}{s^2} + \frac{0.6449}{s} + 0.70795}$$

$$= \frac{0.5012s^2}{0.70795s^2 + 0.6449s + 1}$$

Step 4: Finally, the digital filter h[illegible] the system function $H(z)$ is obtained by applying bilin transform to $H_a(s)$ wit[illegible] sec.

That is,

$$H = H_a(s)\big|_{s\longrightarrow\frac{2}{1}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{0.5012 \times 4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2}{0.70795 \times 4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 0.6449 \times 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 1}$$

$$= \frac{0.391432\left[1-z^{-1}\right]^2}{1 - 0.7153z^{-1} + 0.49328z^{-2}}$$

Example 4.42 A lowpass analog filter is defined by the following transfer function and the corresponding impulse response.

$$H_a(s) = \frac{\alpha}{s+\alpha} \Longleftrightarrow h_a(t) = \alpha e^{-\alpha t}$$

a. What is the gain at dc? At what radian frequency is the analog frequency response 3 dB down from its dc value? At what frequency is the analog frequency response zero? At what time has the analog frequency response decayed to $\frac{1}{e}$ of the initial value?

b. Prewarp the parameter α and perform bilinear transformation to obtain the digital transfer function $H(z)$ from the analog design. What is the gain at dc? At what frequency is the response zero? Give an expression for the 3 dB radian frequency. Also find $h(n)$.

☐ Solution

a. The dc gain is obtained by letting $s = 0$ in $H_a(s)$.

That is, dc gain: $H_a(0) = 1$.

3 dB radian frequency Ω_C is obtained by using the condition:

$$\begin{aligned}
\left|H_a(j\Omega)\right|^2_{\Omega=\Omega_C} &= \frac{1}{2} \\
\Rightarrow \quad \left[\frac{\alpha}{\sqrt{\alpha^2+\Omega_C^2}}\right]^2 &= \frac{1}{2} \\
\Rightarrow \quad \frac{\alpha^2}{\alpha^2+\Omega_C^2} &= \frac{1}{2} \\
\Rightarrow \quad \Omega_C &= \alpha \text{ rad/sec}
\end{aligned}$$

The analog frequency response, $H_a(j\Omega)$ is zero only for $\Omega = \infty$.
Taking inverse Laplace transform of $H_a(s)$, we get

$$h_a(t) = \alpha e^{-\alpha t}$$

The initial value of $h_a(t)$ is obtained by letting $t = 0$ in the above expression.
That is, initial value: $h(0) = \alpha$.
Let at $t = \tau$, $h_a(t)$ be equal to $\frac{1}{e} \times h(0)$.

Then,

$$\begin{aligned}
h_a(\tau) &= \frac{1}{e} \times \alpha = \alpha e^{-\alpha\tau} \\
\Rightarrow \quad \tau &= \frac{1}{\alpha}
\end{aligned}$$

b. Applying BLT to $H_a(s)$, we get

$$H(z) = H_a(s)\Big|_{s\to\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{\alpha}{s+\alpha}\Big|_{s\to\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Rightarrow \quad H(z) = \frac{\alpha}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\alpha} = \frac{\alpha T\left(1+z^{-1}\right)}{2+\alpha T+(\alpha T-2)z^{-1}}$$

The frequency response of the digital filter is obtained by letting $z = e^{j\omega}$. The frequency $\omega = 0$ corresponds to $z = 1$. Hence, dc gain of the digital filter is obtained by letting $z = 1$.

$$\text{Dc gain:} \quad H\left(e^{j\omega}\right)\Big|_{\omega=0} = H(\omega)\Big|_{\omega=0} = H(z)\Big|_{z=1} = 1$$

Also, from the expression $z = e^{j\omega}$, we find that $\omega = \pi$ corresponds to $z = -1$. Putting $z = -1$ gives $H(z) = 0$. Hence, the digital frequency response is zero at $\omega = \pi$.

An analog frequency Ω_C is mapped to a digital frequency ω_C by

$$\omega_C = 2\tan^{-1}\left(\frac{\Omega_C T}{2}\right)$$

Hence,

$$\omega_C = 2\tan^{-1}\left(\frac{\alpha T}{2}\right)$$

We know that

$$H(z) = \frac{\alpha T(1+z^{-1})}{2+\alpha T+(\alpha T-2)z^{-1}}$$

Let

$$a = \frac{2-\alpha T}{2+\alpha T}$$

Then,

$$H(z) = \frac{1-a}{2}\left[1+\frac{(1+a)z^{-1}}{1-az^{-1}}\right]$$

Recall for causal sequences:

$$\mathcal{Z}\left\{a^n u(n)\right\} = \frac{z}{z-a}$$

$$\mathcal{Z}\left\{a^{n-1}u(n-1)\right\} = \frac{1}{z-a}$$

Hence,

$$h(n) = \frac{1-a}{2}\left[\delta(n)+(1+a)a^{n-1}u(n-1)\right]$$

4.14 Digital Frequency Transformations

Digital frequency transformations are available for converting a digital lowpass filter into another digital lowpass filter, a highpass filter, a bandpass filter, or a bandstop filter. These transformations are shown in Table 4.4 along with the formulae for design parameters. In each case, ω_C' represents the cutoff frequency of the original lowpass filter and ω_C is the cutoff frequency of the resulting lowpass or highpass filter. The lower and upper cutoff frequencies of the derived bandpass or bandstop filters are denoted by ω_1 and ω_2 respectively, and the center frequency by ω_0.

Table 4.4 Spectral transformations of a lowpass filter with a cutoff frequency ω'_C.

Filter type	*Transformation*	*Design parameters*
Lowpass	$\dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\omega'_C-\omega_C}{2}\right)}{\sin\left(\frac{\omega'_C+\omega_C}{2}\right)}$
Highpass	$\dfrac{-\left(z^{-1}+\alpha\right)}{1+\alpha z^{-1}}$	$\alpha = \dfrac{-\cos\left(\frac{\omega'_C+\omega_C}{2}\right)}{\cos\left(\frac{\omega'_C-\omega_C}{2}\right)}$
Bandpass	$\dfrac{-\left(z^{-2}-\frac{2\alpha K z^{-1}}{K+1}+\frac{K-1}{K+1}\right)}{\frac{K-1}{K+1}z^{-2}-\frac{2\alpha K}{K+1}z^{-1}+1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_2+\omega_1}{2}\right)}{\cos\left(\frac{\omega_2-\omega_1}{2}\right)} = \cos\omega_0$ $K = \cot\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega'_C}{2}\right)$
Bandstop	$\dfrac{z^{-2}-\frac{2\alpha z^{-1}}{K+1}+\frac{1-K}{1+K}}{\frac{1-K}{1+K}z^{-2}-\frac{2\alpha}{1+K}z^{-1}+1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_2+\omega_1}{2}\right)}{\cos\left(\frac{\omega_2-\omega_1}{2}\right)} = \cos\omega_0$ $K = \tan\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega'_C}{2}\right)$

Example 4.43 The system function of a lowpass digital filter is given by

$$H(z) = \frac{1}{2}\left(\frac{1+z^{-1}}{2-z^{-1}}\right)$$

Show that the cutoff frequency $\omega'_C = 0.6435$. Use a lowpass transformation to obtain another lowpass filter with $\omega_C = 1$ rad.

☐ Solution

Given
$$H(z) = \frac{1}{2}\left(\frac{1+z^{-1}}{2-z^{-1}}\right)$$

Letting $z = e^{j\omega}$ in $H(z)$, we get

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{1}{2}\left(\frac{1+e^{-j\omega}}{2-e^{-j\omega}}\right)$$

$$\Rightarrow \quad H(0) = \frac{1}{2}\left(\frac{1+1}{2-1}\right) = 1$$

Also,

$$|H(\omega)|^2 = \frac{1}{4}\frac{\left[(1+\cos\omega)^2+\sin^2\omega\right]}{\left[(2-\cos\omega)^2+\sin^2\omega\right]}$$

$$= \frac{1+\cos\omega}{2(5-4\cos\omega)}$$

$$\Rightarrow \quad |H(\omega_C')|^2 = \frac{1+\cos 0.6435}{2(5-4\cos 0.6435)} = \frac{1}{2}$$

Hence, ω_C' is the cutoff frequency.
Design parameter α is computed as

$$\alpha = \frac{\sin\left[\frac{(\omega_C'-\omega_C)}{2}\right]}{\sin\left[\frac{(\omega_C'+\omega_C)}{2}\right]}$$

$$= \frac{\sin\left[\frac{(0.6435-1)}{2}\right]}{\sin\left[\frac{(0.6435+1)}{2}\right]} = -0.2421$$

The lowpass-to-lowpass digital transformation is

$$z^{-1} \longrightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$$

Hence,

$$H_{LP}(z) = \frac{1}{2}\left[\frac{1+\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}}{2-\frac{(z^{-1}-\alpha)}{1-\alpha z^{-1}}}\right]$$

$$= \frac{(1-\alpha)z^{-1}+(1-\alpha)}{2[2+\alpha-(1+2\alpha)z^{-1}]}$$

$$\Rightarrow \quad H_{LP}(z) = \frac{1}{2}\times\frac{1-\alpha}{2+\alpha}\left[\frac{1+z^{-1}}{1-\frac{(1+2\alpha)z^{-1}}{2+\alpha}}\right]$$

$$= 0.3533\left[\frac{1+z^{-1}}{1-0.2934z^{-1}}\right]$$

Example 4.44 A digital lowpass filter is described by the following system function:

$$H(z) = \frac{1+2z^{-1}}{4-z^{-1}}$$

Find the following:

a. cutoff frequency ω_C',
b. another lowpass filter with $\omega_C = 2$ rad,
c. a highpass filter with $\omega_C = 2$ rad,
d. a bandpass filter with $\omega_1 = \frac{\pi}{4}$ rad and $\omega_2 = \frac{3\pi}{4}$ rad, and
e. a bandstop filter with $\omega_1 = \frac{\pi}{4}$ rad and $\omega_2 = \frac{3\pi}{4}$ rad.

☐ Solution

a.

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{1+2e^{-j\omega}}{4-e^{-j\omega}}$$

$$\Rightarrow \quad \left|H\left(\omega_C'\right)\right|^2 = \frac{5+4\cos\omega_C'}{17-8\cos\omega_C'} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\omega_C' = \frac{7}{16}$$

Hence,

$$\omega_C' = 1.118 \text{ rad}$$

b.

$$\omega_C = 2 \text{ rad}$$

$$\alpha = \frac{\sin\left(\frac{\omega_C'-\omega_C}{2}\right)}{\sin\left[\frac{(\omega_C'+\omega_C)}{2}\right]} = -0.4269$$

The lowpass-to-lowpass transformation is

$$z^{-1} \longrightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$$

Hence,

$$\begin{aligned} H_{LP}(z) &= H(z)\Big|_{z^1 \longrightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}} \\ &= \frac{1-2\alpha+(2-\alpha)z^{-1}}{4+\alpha-(4\alpha+1)z^{-1}} \\ &= \frac{0.5188\left(1+1.3091z^{-1}\right)}{1+0.1980z^{-1}} \end{aligned}$$

c.

$$\omega_C = 2 \text{ rad}$$

$$\alpha = \frac{-\cos\left(\frac{\omega_C'+\omega_C}{2}\right)}{\cos\left(\frac{\omega_C'-\omega_C}{2}\right)} = -0.0130$$

The lowpass-to-highpass transformation is

$$z^{-1} \longrightarrow \frac{-\left(z^{-1}+\alpha\right)}{1+\alpha z^{-1}}$$

Hence,

$$\begin{aligned} H_{HP}(z) &= \frac{1-2\frac{(z^{-1}+\alpha)}{1+\alpha z^{-1}}}{4+\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}} \\ &= \frac{1-2\alpha+(\alpha-2)z^{-1}}{4+\alpha+(1+4\alpha)z^{-1}} \end{aligned}$$

$$\Rightarrow \quad H_{HP}(z) = \frac{0.2573\left(1-1.9620z^{-1}\right)}{1+0.2378z^{-1}}$$

d.

$$\omega_1 = \frac{\pi}{4}, \qquad \omega_2 = \frac{3\pi}{4}$$

$$\frac{1}{2}(\omega_1 + \omega_2) = \frac{\pi}{2}, \qquad \frac{1}{2}(\omega_2 - \omega_1) = \frac{\pi}{4}$$

Design parameters:

$$\begin{aligned}
\alpha &= \frac{\cos\left[\frac{1}{2}(\omega_2 + \omega_1)\right]}{\cos\left[\frac{1}{2}(\omega_2 - \omega_1)\right]} = 0 \\
K &= \cot\left[\frac{1}{2}(\omega_2 - \omega_1)\right]\tan\left(\frac{\omega'_C}{2}\right) = 0.6256 \\
\frac{K-1}{K+1} &= \frac{0.6256 - 1}{0.6256 + 1} = -0.2303
\end{aligned}$$

The lowpass-to-bandpass transformation is

$$z^{-1} \longrightarrow \frac{-\left[z^{-2} - \frac{2\alpha K}{K+1}z^{-1} + \frac{K-1}{K+1}\right]}{\frac{K-1}{K+1}z^{-2} - \frac{2\alpha K}{K+1}z^{-1} + 1} = \frac{z^{-2} - 0.2303}{0.2303z^{-2} - 1}$$

$$\begin{aligned}
H_{BP}(z) &= \frac{1 + 2\frac{(z^{-2} - 0.2303)}{0.2303z^{-2} - 1}}{4 - \frac{(z^{-2} - 0.2303)}{(0.2303z^{-2} - 1)}} \\
&= \frac{0.3875\left(1 - 1.527z^{-2}\right)}{1 + 0.0209z^{-2}}
\end{aligned}$$

e.

$$\omega_1 = \frac{\pi}{4}, \qquad \omega_2 = \frac{3\pi}{4}$$

Design parameters:

$$\begin{aligned}
\alpha &= \frac{\cos\left(\frac{\omega_2 + \omega_1}{2}\right)}{\cos\left(\frac{\omega_2 - \omega_1}{2}\right)} = 0 \\
K &= \tan\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\left(\frac{\omega'_C}{2}\right) \\
&= \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{1.118}{2}\right) = 0.6256 \\
\frac{1-K}{1+K} &= 0.2303
\end{aligned}$$

The lowpass-to-bandstop transformation is

$$z^{-1} \longrightarrow \frac{z^{-2} - \frac{2\alpha}{1+K}z^{-1} + \frac{1-K}{1+K}}{\frac{1-K}{1+K}z^{-2} - \frac{2\alpha}{1+K}z^{-1} + 1} = \frac{z^{-2} + 0.2303}{0.2303z^{-2} + 1}$$

Hence,

$$H_{BS}(z) = \frac{1+2\left(\frac{z^{-2}+0.2303}{0.2303z^{-2}+1}\right)}{4-\left(\frac{z^{-2}+0.2303}{0.2303z^{-2}+1}\right)}$$

$$= \frac{0.3875\left(1+1.577z^{-2}\right)}{1-0.029z^{-2}}$$

Example 4.45 A second-order lowpass butterworth filter with a 3-dB frequency of 0.4π rad is given by the system function:

$$H(z) = \frac{0.207+0.413z^{-1}+0.207z^{-2}}{1-0.369z^{-1}+0.196z^{-2}}$$

Design a second-order highpass filter with 3-dB frequency of 0.3π by transforming the above system function using spectral transformation.

☐ Solution

Design parameter:

$$\alpha = \frac{-\cos\left(\frac{\omega_C'+\omega_C}{2}\right)}{\cos\left(\frac{\omega_C'-\omega_C}{2}\right)}$$

$$= \frac{-\cos\left(\frac{0.4\pi+0.3\pi}{2}\right)}{\cos\left(\frac{0.4\pi-0.3\pi}{2}\right)} = -0.45965$$

The lowpass-to-highpass digital transformation is

$$z^{-1} \longrightarrow \frac{-(z^{-1}+\alpha)}{1+\alpha z^{-1}} = -\left(\frac{z^{-1}-0.45965}{1-0.45965z^{-1}}\right)$$

Hence,

$$H_{HP}(z) = H(z)\Big|_{z^{-1}\longrightarrow-\left(\frac{z^{-1}-0.45965}{1-0.45965z^{-1}}\right)}$$

$$\Rightarrow \quad H_{HP}(z) = \frac{0.44057-0.88065z^{-1}+0.44057z^{-2}}{0.8718-0.6525z^{-1}+0.2377z^{-2}}$$

4.15 Advantages and Disadvantages of IIR Filters

Advantages

- The design of IIR filters is very straight forward due to the existence of standard analog filter design techniques and simple transformation procedures.
- The practical implementation of IIR filters is less complicated when compared to FIR filters.
- Relatively short delays, since IIR filters are usually of minimum phase.

Disadvantages

- Causal IIR filters do not have linear-phase.
- IIR filters are not flexible in achieving non-standard magnitude frequency responses.
- The design techniques of obtaining a digital filter from a prototype analog filter are only readily available, other techniques are complex to develop and implement.
- A digital filter $H(z)$ is stable if and only if, all the poles of $H(z)$ are confined inside the unit circle, $|z| = 1$. It is quite possible that although an IIR filter is theoretically stable, it may become unstable when their coefficients are truncated to a finite word length. Hence, stability must be carefully checked after the coefficients are rounded off.

Reinforcement Problems

RP 4.1 Assume that $H_a(s)$ has an m^{th} order pole at $s = -s_i$ so that

$$H_a(s) = \frac{1}{(s+s_i)^m}$$

Find $H(z)$ by using impulse invariance technique.

☐ Solution

Given

$$H_a(s) = \frac{1}{(s+s_i)^m}$$

$$\Rightarrow \quad h_a(t) = e^{-s_i t}\frac{t^{m-1}}{(m-1)!}, \quad t \geqslant 0$$

Letting $t = nT$ in the above expression, we get

$$\begin{aligned} h(n) &= h_a(nT) \\ &= e^{-s_i nT}\frac{(nT)^{m-1}}{(m-1)!}u(n) \end{aligned}$$

The system function is given by

$$H(z) = \frac{1}{(m-1)!}\sum_{n=0}^{\infty} e^{-s_i nT}(nT)^{m-1}z^{-n}$$

We know that

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-s_i nT}z^{-n} &= \sum_{n=0}^{\infty}\left[e^{-s_i T}z^{-1}\right]^n \\ &= \left[1 - e^{-s_i T}z^{-1}\right]^{-1} \end{aligned}$$

$$\Rightarrow \quad \frac{d^{m-1}}{ds_i^{m-1}} \sum_{n=0}^{\infty} e^{-s_i nT} z^{-n} = \sum_{n=0}^{\infty} (-nT)^{m-1} e^{-s_i nT} z^{-n}$$

$$= \frac{d^{m-1}}{ds_i^{m-1}} \left[1 - e^{-s_i T} z^{-1}\right]^{-1}$$

$$\Rightarrow \quad (-1)^{m-1} \sum_{n=0}^{\infty} (nT)^{m-1} e^{-s_i nT} z^{-n} = \frac{d^{m-1}}{ds_i^{m-1}} \left[1 - e^{-s_i T} z^{-1}\right]^{-1}$$

Hence,

$$H(z) = \frac{1}{(m-1)!} \sum_{n=0}^{\infty} e^{-s_i nT} (nT)^{m-1} z^{-n}$$

$$= \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds_i^{m-1}} \left[1 - e^{-s_i T} z^{-1}\right]^{-1} \quad \left[\because (-1)^{-(m-1)} = (-1)^{m-1}\right]$$

Thus,

$$H_a(s) = \frac{1}{(s+s_i)^m} \xRightarrow{\text{IIT}} H(z) = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds_i^{m-1}} \left[1 - e^{-s_i T} z^{-1}\right]^{-1}$$

RP-4.2 Suppose that we are given a continuous-time lowpass filter with frequency response $H_a(j\Omega)$ such that

$$1 - \delta_1 \leqslant |H_a(j\Omega)| \leqslant 1 + \delta_1, \quad |\Omega| \leqslant \Omega_P$$

$$|H_a(j\Omega)| \leqslant \delta_2, \quad |\Omega| \geqslant \Omega_S$$

A set of discrete-time lowpass filters are obtained from $H_a(s)$ by using the bilinear transformation, i.e.,

$$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

with T variable.

a. Assuming that passband-edge frequency Ω_P is fixed, find the value of T such that the corresponding passband cutoff frequency for the discrete-time system is $\omega_P = \frac{\pi}{2}$.

b. With the passband-edge frequency Ω_P fixed, sketch ω_P as a function of $0 < T < \infty$.

c. With both passband and stopband-edge frequencies fixed, sketch the transistion region $\Delta\omega = \omega_S - \omega_P$ as a function of $0 < T < \infty$.

Solution

a. The bilinear transform is

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

The frequency response of the analog filter is obtained by letting $s = j\Omega$ in $H_a(s)$, while the frequency response of the digital filter is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

Hence, letting $s = j\Omega$ and $z = e^{j\omega}$ in the above equation, we get

$$\begin{aligned} j\Omega &= \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] \\ &= \frac{2}{T}\left[\frac{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}\right)}\right]. \\ &= \frac{2}{T}j\tan\left(\frac{\omega}{2}\right) \end{aligned}$$

Hence,
$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$
$$\Rightarrow \quad T = \frac{2}{\Omega_P}\tan\left(\frac{\omega_P}{2}\right)$$
$$\Rightarrow \quad T = \frac{2}{\Omega_P}\tan\left(\frac{\pi}{4}\right) = \frac{2}{\Omega_P}$$

b.
$$\omega_P = 2\tan^{-1}\left(\frac{T\Omega_P}{2}\right)$$

Using the above equation, a sketch of ω_P versus T is made. The same is shown in Fig. RP.4.2(a) below.

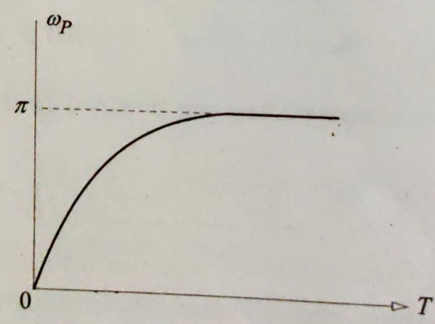


Fig. RP.4.2(a) Plot of ω_P versus T.

c. We have
$$\omega_S = 2\tan^{-1}\left(\frac{T\Omega_S}{2}\right)$$
and
$$\omega_P = 2\tan^{-1}\left(\frac{T\Omega_P}{2}\right)$$
Hence,
$$\Delta\omega = \omega_S - \omega_P = 2\left[\tan^{-1}\left(\frac{T\Omega_S}{2}\right) - \tan^{-1}\left(\frac{T\Omega_P}{2}\right)\right]$$

Using the above equations, sketches of ω_P versus T, ω_S versus T, and $\Delta\omega = \omega_S - \omega_P$ versus T are made and are as shown in Fig. RP.4.2(b), RP.4.2(c) and RP.4.2(d) respectively.

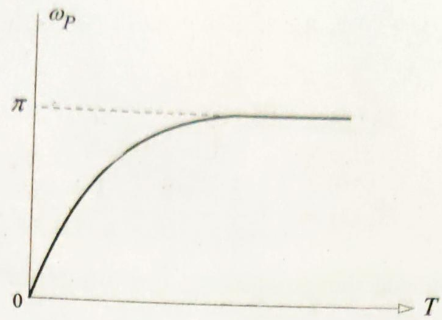


Fig. RP.4.2(b) Plot of ω_P versus T.

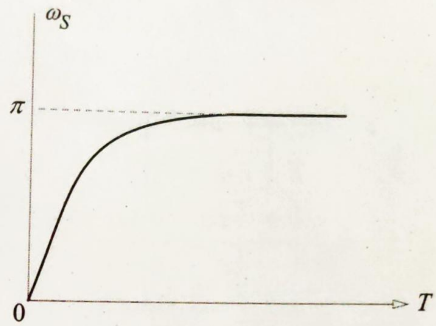


Fig. RP.4.2(c) Plot of ω_S versus T.

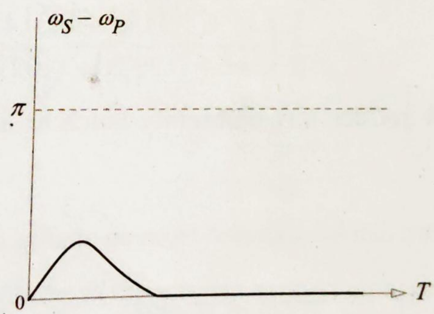


Fig. RP.4.2(d) Plot of $\Delta\omega = \omega_S - \omega_P$ versus T.

RP- 4.3 The transfer function of a discrete-time system is

$$H(z) = \frac{2}{1 - e^{-0.1}z^{-1}} - \frac{1}{1 - e^{-0.4}z^{-1}}$$

a. Assume that the transfer function $H(z)$ was designed using the impulse invariance method with $T = 2$ secs, i.e., $h(n) = h_a(2n)$, where $h_a(t)$ is real. Find the system function $H_a(s)$ of a continuous-time filter that could have been the protype for the design. Is your answer unique? If not find another system function $H_a(s)$.

b. Assume that $H(z)$ was obtained by BLT with $T = 2$ secs. Find the system function $H_a(s)$ that could have been the protype for the design. Is your answer unique? If not, find another $H_a(s)$.

☐ Solution

Recall the relationship:

$$\frac{1}{s - s_i} \stackrel{IIT}{\Longrightarrow} \frac{1}{1 - e^{s_i T} z^{-1}}$$

Hence, we get $$H_a(s) = \frac{2}{s + 0.1} - \frac{1}{s + 0.4}$$

The above solution is not unique due to aliasing, a more general answer is

$$H_a(s) = \frac{2}{s + \left(0.1 + \frac{2\pi k}{T}\right)} - \frac{1}{s + \left(0.4 + \frac{2\pi l}{T}\right)}$$

where k and l are integers.

b. We know that

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

$$\Rightarrow \quad z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} = \frac{1 + s}{1 - s}$$

Hence,
$$\begin{aligned} H_a(s) &= \frac{2}{1 - e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1 - e^{-0.4}\left(\frac{1-s}{1+s}\right)} \\ &= \frac{1}{2}\left(\frac{s^2 + 1.314s + 0.314}{s^2 + 0.297s + 0.02}\right) \end{aligned}$$

Since, the BLT does not introduce any ambiguity (such as aliasing), the transfer function $H_a(s)$ is unique.

RP- 4.4 A digital highpass filter can be obtained from an analog filter using the following transformation:

$$H(z) = H_a(s)\big|_{s = \frac{1+z^{-1}}{1-z^{-1}}}$$

a. Show that the above transformation maps the $j\Omega$-axis of the s-plane onto the circle $|z| = 1$.

b. Prove that if $H_a(s)$ is a rational function with all its poles inside the left-half s-plane, then $H(z)$ will be a rational function with all its poles inside the unit circle of the z-plane.

c. Consider a highpass discrete-time filter having the following specifications.

$$|H(\omega)| \leqslant 0.01, \qquad |\omega| \leqslant \frac{\pi}{3}$$
$$0.95 \leqslant |H(\omega)| \leqslant 1.05, \quad \frac{\pi}{2} < |\omega| \leqslant \pi$$

Find the specifications of the continuous-time lowpass filter, so that the discrete-time highpass filter defined above results from the transformation given in the problem.

□ Solution

a. Given

$$s = \frac{1+z^{-1}}{1-z^{-1}}$$

$$\Rightarrow \quad z = \frac{s+1}{s-1}$$

Letting $s = j\Omega$ in the above expression, we get

$$z = \frac{j\Omega+1}{j\Omega-1} \quad \Rightarrow \quad |z| = \frac{\sqrt{\Omega^2+1}}{\sqrt{\Omega^2+1}} = 1$$

b. Our next objective is to show that $|z| < 1$ if $\mathrm{Re}\{s\} < 0$.

We know that,

$$z = \frac{s+1}{s-1}$$

Letting $s = \sigma + j\Omega$ in the above expression, we get

$$z = \frac{\sigma+j\Omega+1}{\sigma+j\Omega-1}$$

$$\Rightarrow \quad |z| = \frac{\sqrt{(\sigma+1)^2+\Omega^2}}{\sqrt{(\sigma-1)^2+\Omega^2}}$$

Hence, if $|z| < 1$, we get

$$\frac{(\sigma+1)^2+\Omega^2}{(\sigma-1)^2+\Omega^2} < 1$$

$$\Rightarrow \quad (\sigma+1)^2+\Omega^2 < (\sigma-1)^2+\Omega^2$$

$$\Rightarrow \quad \sigma < -\sigma$$

$$\Rightarrow \quad 2\sigma < 0$$

$$\Rightarrow \quad \sigma < 0$$

Hence, the left half s-plane maps to the interior of the unit circle on the z-plane.

c. We know that

$$s = \frac{1+z^{-1}}{1-z^{-1}}$$

Letting $s = j\Omega$ and $z = e^{j\omega}$, we get

$$j\Omega = \frac{1+e^{-j\omega}}{1-e^{-j\omega}}$$

$$\Rightarrow \quad \Omega = -\frac{1}{\tan\left(\frac{\omega}{2}\right)}$$

$$\Rightarrow \quad \Omega_P = -\frac{1}{\tan\left(\frac{\pi}{4}\right)}$$

$$\Rightarrow \quad |\Omega_P| = 1$$

Also,

$$\Omega_S = -\frac{1}{\tan\left(\frac{\pi}{6}\right)}$$

$$\Rightarrow \quad |\Omega_S| = \sqrt{3}$$

RP- 4.5 Suppose we are given the following differential equation:

$$y_a'(t) + C y_a(t) = A x(t)$$

where A and C are constants or equivalently by the transfer function

$$H_a(s) = \frac{A}{s + C}$$

a. Show that $y_a(nT)$ can be expressed in terms of $y_a'(t)$ as

$$y_a(nT) = \int_{(nT-T)}^{nT} y_a'(\tau)d\tau + y_a(nT - T)$$

Using trapezoidal approximation, find an expression for $y_a(nT)$ in terms of $y_a(nT - T)$, $y_a'(nT)$, and $y_a'(nT - T)$.

b. Use the differential equation to obtain an expression for $y_a'(nT)$ and use this expression in the expression obtained in part (a).

c. Define $x(n) = x_a(nT)$ and $y(n) = y_a(nT)$. Using this notation and the difference equation obtained in part (b), find

$$H(z) = \frac{Y(z)}{X(z)}$$

d. Prove that

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

☐ Solution

a.

$$\begin{aligned} \text{RHS} &= \int_{(nT-T)}^{nT} y_a'(\tau)d\tau + y_a(nT - T) \\ &= y_a(\tau)\Big|_{nT-T}^{nT} + y_a(nT - T) \\ &= y_a(nT) = \text{LHS} \end{aligned}$$

Using the trapezodial approximation for the integral, we get

$$y_a(nT) = \frac{T}{2}\left[y_a'(nT) + y_a'(nT - T)\right] + y_a(nT - T) \tag{4.57}$$

b. Given

$$\begin{aligned} & y_a'(t) + C y_a(t) = A\, x_a(t) \\ \Rightarrow\quad & y_a'(nT) + C y_a(nT) = A\, x_a(nT) \\ \Rightarrow\quad & y_a'(nT) = A x_a(nT) - C y_a(nT) \end{aligned} \tag{4.58}$$

Making use of equation (4.58) in equation (4.57), we get

$$y_a(nT) = [Ax_a(nT) - Cy_a(nT) + Ax_a(nT - T) - Cy_a(nT - T)]\frac{T}{2} + y_a(nT - T)$$

c.

$$y(n) = [Ax(n) - Cy(n) + A\,x(n-1) - Cy(n-1)]\frac{T}{2} + y(n-1)$$

$$\Rightarrow \quad y(n)\left[1 + C\frac{T}{2}\right] - y(n-1)\left[1 - C\frac{T}{2}\right] = A\frac{T}{2}\,[x(n) + x(n-1)]$$

Taking $\mathcal{Z}$-transform on both the sides of the above equation, we get

$$Y(z)\left[1 + C\frac{T}{2}\right] - Y(z)z^{-1}\left[1 - C\frac{T}{2}\right] = A\frac{T}{2}X(z)\left[1 + z^{-1}\right]$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{A\frac{T}{2}\left(1 + z^{-1}\right)}{1 + C\frac{T}{2} - z^{-1} + z^{-1}C\frac{T}{2}}$$

d.

$$\begin{aligned} H(z) &= \left.\frac{A}{s + C}\right|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{A\frac{T}{2}\left(1 + z^{-1}\right)}{1 - z^{-1} + C\frac{T}{2}\left(1 + z^{-1}\right)} \end{aligned}$$

The above result is in agreement with part (c).

RP- 4.6 Use the matched $\mathcal{Z}$-transform,

$$s + \alpha \Rightarrow \frac{z - e^{\alpha T}}{z}$$

with $T = 0.5$ sec and gain matching at dc, transform each analog filter $H(s)$ to a digital filter $H(z)$.

a. $H(s) = \dfrac{1}{s+1} + \dfrac{1}{s+2}$

b. $H(s) = \dfrac{s+1}{(s+1)^2 + \pi^2}$

Solution

a.

$$\begin{aligned} H(s) &= \frac{1}{s+1} + \frac{1}{s+2} \\ &= \frac{2(s+1.5)}{(s+1)(s+2)} \end{aligned}$$

$$H(z) = \frac{K\left(\frac{z-e^{-1.5\times0.5}}{z}\right)}{\left(\frac{z-e^{-1\times0.5}}{z}\right)\left(\frac{z-e^{-2\times0.5}}{z}\right)}$$

$$= \frac{Kz\left(z-e^{-0.75}\right)}{\left(z-e^{-0.5}\right)\left(z-e^{-1}\right)}$$

$$= \frac{K\left(z^2-0.4724z\right)}{z^2-0.9744z+0.2231}$$

As per the problem,

$$\Rightarrow \quad H_{dc}(z) = H_{dc}(s)$$

$$\Rightarrow \quad H(z)\big|_{z=1} = H(s)\big|_{s=0}$$

$$\Rightarrow \quad \frac{0.5726K}{0.2487} = 1.5$$

$$\Rightarrow \quad K = 0.6515$$

Hence,

$$H(z) = \frac{0.6515\left(z^2-0.4724z\right)}{z^2-0.9744z+0.2231}$$

$$= \frac{0.6515\left(1-0.4274z^{-1}\right)}{1-0.9744z^{-1}+0.2231z^{-2}}$$

b. Given

$$H(s) = \frac{s+1}{(s+1)^2+\pi^2}$$

$$\Rightarrow \quad H(s) = \frac{(s+1)}{(s+1+j\pi)(s+1-j\pi)}$$

We know that,

$$(s+a+jb)(s+a-jb) \longrightarrow \frac{z-e^{-(a+jb)T}}{z} \times \frac{z-e^{-(a-jb)T}}{z}$$

$$= \frac{z^2-2z^{-aT}\cos bT+e^{-2aT}}{z^2}$$

Here, $a = 1$ and $b = \pi$.

Hence,

$$H(z) = \frac{K\left(\frac{z-e^{-0.5}}{z}\right)}{\frac{z^2-2ze^{-1\times0.5}\cos(\pi\times0.5)+e^{-2\times1\times0.5}}{z^2}}$$

$$= \frac{K\left(z^2-0.6065z\right)}{z^2+0.3679}$$

$$H_{dc}(z) = H(z)\big|_{z=1} = \frac{0.3935K}{1.3679}$$

and

$$H_{dc}(s) = H(s)\big|_{s=0} = \frac{1}{1+\pi^2}$$

As per the problem,

$$
\begin{aligned}
H_{dc}(z) &= H_{dc}(s)\\
\Rightarrow \quad \frac{0.3935K}{1.3679} &= \frac{1}{1+\pi^2}\\
\Rightarrow \quad K &= 0.3198
\end{aligned}
$$

Hence,

$$H(z) = \frac{0.3198(1-0.6065z^{-1})}{1+0.3679z^{-2}}$$

RP- 4.7 For historical reasons, the design formulae for analog filters are given assuming a peak gain of 1 in the passband. In terms of ϵ and A_S, the filter specifications have the form

$$\frac{1}{\sqrt{1+\epsilon^2}} \leqslant |H(j\Omega)| \leqslant 1$$

$$|H(j\Omega)| \leqslant \frac{1}{A_S}$$

Suppose that we wish to use BLT to design a digital lowpass filter $H(z)$ that satisfies the following constraints:

$$1-\delta_P \leqslant |H(\omega)| \leqslant 1+\delta_P, \quad 0 \leqslant \omega \leqslant \omega_P$$

$$|H(\omega)| \leqslant \delta_S, \qquad \omega_S \leqslant \omega \leqslant \pi$$

Find the relationship between the parameters δ_P and δ_S for the digital filter and between the parameters ϵ and A_S for the analog filter.

☐ Solution

Given:

$$1-\delta_P \leqslant |H(\omega)| \leqslant 1+\delta_P$$

Dividing both the sides by $1+\delta_P$, we get

$$\frac{1-\delta_P}{1+\delta_P} \leqslant |H(\omega)| \leqslant 1$$

Setting

$$\frac{1-\delta_P}{1+\delta_P} = \frac{1}{\sqrt{1+\epsilon^2}}$$

we get

$$1+\epsilon^2 = \left(\frac{1+\delta_P}{1-\delta_P}\right)^2$$

$$\Rightarrow \quad \epsilon^2 = \left(\frac{1+\delta_P}{1-\delta_P}\right)^2 - 1 = \frac{4\delta_P}{(1-\delta_P)^2}$$

With a stopband ripple of δ_S, the normalization of peak passband gain to 1 produces a peak stopband ripple of $\frac{\delta_S}{1+\delta_P}$.

Hence,

$$\frac{1}{A_S} = \frac{\delta_S}{1+\delta_P}$$

$$\Rightarrow \quad A_S = (1+\delta_P)\,\delta_S^{-1}$$

Summary of Important Points and Inferences

- The classical analog filters having different ripple characteristics: Butterworth is monotone at all frequencies, Chebyshev I is monotone in the stopband and equiripple in the passband and Chebyshev II is monotone in the passband and equiripple in the stopband.
- When an analog filter other than lowpass needs to be designed, we apply a suitable frequency transformation to a normalized lowpass filter.
- A Chebyshev filter implementation normally gives the lowest filter order compared to Butterworth filter assuming equal cutoff attenuation and stopband attenuation.
- An analog filter is mapped to a digital filter by using backward difference method, bilinear transformation and impulse invariant transformation. The preferred technique is bilinear transformation. The bilinear transformation preserves the order and stability of the analog filter.
- In the impulse invariance IIR design method, the impulse response of an analog filter with the desired behaviour is sampled to obtain the impulse response for the equivalent digital filter.
- IIR filters are well suited to applications requiring frequency-selective filters with sharp cutoffs or where linear-phase is relatively unimportant.

  The main advantages of an IIR filter are standardized easy design, and low filter order.
- IIR filters cannot give linear-phase in the passband and are quite susceptible to the effects of coefficients quantization.
- IIR filters are used in graphic equalizers for digital audio and filters for digital filters where linear-phase is relatively unimportant. If linear-phase is of paramount importance as in biomedical signal processing, it is best to use FIR filters.

Exercise Problems

EP-4.1 The transfer function $H_5(s)$ represents a normalized fifth-order Butterworth filter.

a. Give $H_5(s)$ in quadratic form by finding the poles of $H_5(s)$.

b. Find the gain in dB at $\Omega = 1$ rad/sec.

Ans:

a. $$H_5(s) = \frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)}$$

b. -3.01 dB.

EP-4.2 Design an analog Butterworth filter to meet the following specifications:

passband attenuation: 2 dB,

passband edge frequency: 20 rad/sec,

stopband attenuation: 10 dB,

stopband edge frequency: 30 rad/sec.

Ans:

$$H_a(s) = \frac{0.20921 \times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

EP-4.3 Design a lowpass Chebyshev filter to meet the following frequency-domain specifications:

acceptable passband ripple: 2 dB,

passband edge-frequency: 40 rad/sec,

stopband attenuation: $\geqslant 20$ dB,

stopband edge-frequency: 52 rad/sec.

Ans:

$$H_a(s) = \frac{8.366 \times 10^6}{(s+8.73212)(s^2+5.3969s+1523.44)(s^2+14.1292s+628.984)}$$

EP-4.4 Design a digital Chebyshev lowpass filter to meet the following specifications:
acceptable passband ripple: 1 dB,
passband edge-frequency: 400 π rad/sec,
minimum stopband attenuation: 20 dB,
stopband edge-frequency: 800 π rad/sec,
sampling rate: 1000 samples/sec.

Ans:

$$H(z) = \frac{2.07478(1+z^{-1})^2}{9.51817 - 3.3441z^{-1} + 3.13773z^{-2}}$$

EP-4.5 Given $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$. Find $H(z)$ using impulse invariant transformation. Take $T = 1$ sec.

Ans:

$$H(z) = \frac{1+0.8975z^{-1}}{1+1.791z^{-1}+0.8187z^{-2}}$$

EP-4.6 Given $H_a(s) = \frac{16(s+2)}{(s+3)(s^2+2s+5)}$. Find $H(z)$ using impulse invariant transformation. Assume $T = 0.2$ sec.

Ans:

$$H(z) = \frac{-2}{1-0.5848z^{-1}} + \frac{2+0.3314z^{-1}}{1-1.2348z^{-1}+0.4493z^{-2}}$$

EP-4.7 The transfer function $H(z)$ given below was designed using bilinear transformation method with $T = 2$. Determine the parent causal transfer function $H_a(s)$.

$$H(z) = \frac{5z^2+4z-1}{8z^2+4z}$$

Ans:

$$H_a(s) = \frac{12s}{7s^2+16s+12}$$

EP-4.8 We wish to implement the analog filter $H_a(s) = \frac{2s+1}{s^2+s+1}$ by a discrete-time approximation, using the bilinear transformation. The sampling rate is 10 Hz.

a. Determine zeros and poles of both $H(s)$ and $H(z)$.

b. Determine the linear difference equation for the digital filter.

Ans:

a. Zeros :

$$s = \frac{-1}{2} \xrightarrow{\text{BLT}} z = 0.9512$$

$$s = \infty \xrightarrow{\text{BLT}} z = -1$$

Poles :

$$s = e^{\pm j2\pi/3} \xrightarrow{\text{BLT}} z = 0.948 \pm j0.822$$

b.

$$y(n) - 1.895y(n-1) + 0.905y(n-2)$$
$$= 0.0976[x(n) + 0.048x(n-1) - 0.9512x(n-2)]$$

EP-4.9 A lowpass Butterworth filter has a gain of -3 dB at the edge of its passband, at 1 KHz, and a stopband attenuation of 30 dB at 12 KHz . Find $H(z)$, the difference equation, and the frequency response of the filter, if the sampling rate is 25 KHz.

Ans:

$$\begin{aligned} H(z) &= \frac{0.112(1+z^{-1})}{1-0.7757z^{-1}} \\ y(n) &= 0.7757y(n-1) + 0.1122x(n) + 0.1122x(n-1) \\ H(e^{j\omega}) &= H(\omega) = \frac{0.1122(1+e^{-j\omega})}{1-0.7757e^{-j\omega}} \end{aligned}$$

EP-4.10 Design $H(z)$ by applying impulse invariance to an appropriate Butterworth continuous-time filter. The specifications for the discrete-time filter are given below.

$$\begin{aligned} 0.89125 \leqslant |H(\omega)| \leqslant 1, \quad & 0 \leqslant |\omega| \leqslant 0.2\pi \\ |H(\omega)| \leqslant 0.17783, \quad & 0.3\pi \leqslant |\omega| \leqslant \pi \end{aligned}$$

Ans:

$$H(z) = \frac{0.287 - 0.447z^{-1}}{1 - 1.297z^{-1} + 0.695z^{-2}} + \frac{-2.143 + 1.145z^{-1}}{1 - 1.069z^{-1} + 0.370z^{-2}} + \frac{1.856 - 0.63z^{-1}}{1 - 0.997z^{-1} + 0.257z^{-2}}$$

EP-4.11 Given the transfer function,

$$H_a(s) = \frac{1}{(s+1)(s+2)}$$

determine $H(z)$ with the design based on step-invariant method.

Ans:

$$H(z) = \frac{b_0 z^{-1} + b_1 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

where
$$\begin{aligned} b_0 &= 0.5 - e^{-T} + 0.5e^{-2T} \\ b_1 &= 0.5e^{-T} - e^{-2T} + 0.5e^{-3T} \\ a_1 &= e^{-T} + e^{-2T} \\ a_2 &= -e^{-3T} \end{aligned}$$

EP-4.12 Design a lowpass digital filter for a 20 KHz sampling rate that is maximally flat in the passband of 0 to the $-$ 3 dB cutoff frequency, 2 KHz and has the attenuation of atleast 10 dB for frequences exceeding 4 KHz.

Ans:

$$H(z) = \frac{0.068(1 + z^{-1})^2}{1 - 1.142z^{-1} + 0.413z^{-2}}$$

Difference equation:

$$y(n) = 1.142y(n-1) - 0.413y(n-2) + 0.068x(n)$$
$$+0.136x(n-1) + 0.068x(n-2)$$

EP-4.13 As the order of an analog Butterworth filter is increased, the slope of $|H(j\Omega)|^2$ at the 3-dB cutoff frequency, Ω_C, increases. Derive an expression for the slope of $|H(j\Omega)|^2$ at Ω_C as a function of the filter order, N.

Ans:

$$\left.\frac{d}{d\Omega}|H(j\Omega)|^2\right|_{\Omega=1} = -\frac{N}{2}$$

EP-4.14 Show that BLT maps the $j\Omega$ axis in the s-plane onto the unit circle, $|z| = 1$, and maps the left-half s-plane, $\text{Re}\{s\} = \sigma < 0$ inside the unit circel, $|z| < 1$.

Chapter 5

Design of FIR Filters

5.1 Introduction

In chapter 4, IIR digital filters were designed to give a desired magnitude frequency response without caring for the phase response. In many applications, a linear-phase is required throughout the passband of the filter to preserve the shape of a given signal in the passband. To understand the significance of linear-phase, let us consider a lowpass filter having the frequency response given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases} \tag{5.1}$$

Notice that $H_d(\omega)$ is periodic with a period 2π.

That is, $$H_d(\omega + 2\pi) = H_d(\omega)$$

The magnitude frequency response and phase frequency response of an ideal lowpass filter are shown in Fig. 5.1.

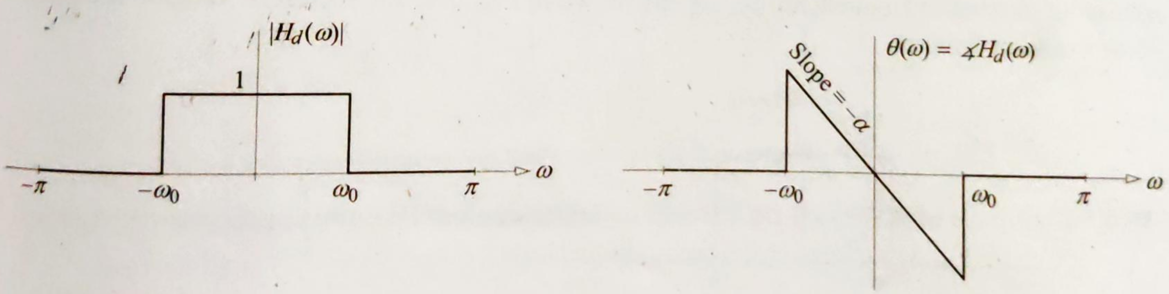


Fig. 5.1 **(a)** Magnitude frequency response. **(b)** Phase frequency response.

Let $x(n)$ be the input and $y(n)$ be the output of an ideal lowpass filter having an impulse response $h_d(n)$. Since, lowpass filter is an LTI system, we may write

$$y(n) = x(n) * h_d(n) \tag{5.2}$$

Taking DTFT on both the sides of equation (5.2), we get

$$Y(\omega) = X(\omega)\, H_d(\omega) \tag{5.3}$$

Substituting equation (5.1) in equation (5.3), we get

$$Y(\omega) = \begin{cases} X(\omega)e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases} \tag{5.4}$$

Taking inverse DTFT on both the sides of equation (5.4), we get

$$y(n) = x(n-\alpha) \tag{5.5}$$

Thus, the output of the lowpass filter is nothing but the input delayed by an amount α. Hence, the linear-phase has not altered the shape of $x(n)$, simply translated it by an amount α. Conversely, if the phase had not been linear, the output signal would have been a distorted version of $x(n)$.

We wish to inform the readers that a causal IIR filter cannot give a linear-phase characteristic and only special types of FIR filter that exhibit center symmetry in its impulse response give the linear-phase.

5.2 Paley-Wiener Theorem

If $h_c(n)$ is causal sequence with $h_c(n) = 0$ for $n < 0$ and it has finite energy, then

$$\int_{-\pi}^{\pi} |\log|H_c(\omega)|| \, d\omega < +\infty \tag{5.6}$$

where $$H_c(\omega) = \text{DTFT}\{h_c(n)\}$$

The immediate implication of this result is that an ideal filter cannot be causal, because it is zero within an interval of frequencies (ω_C, π) as shown in Fig. 5.2, and therefore integral in equation (5.6) would be infinite.

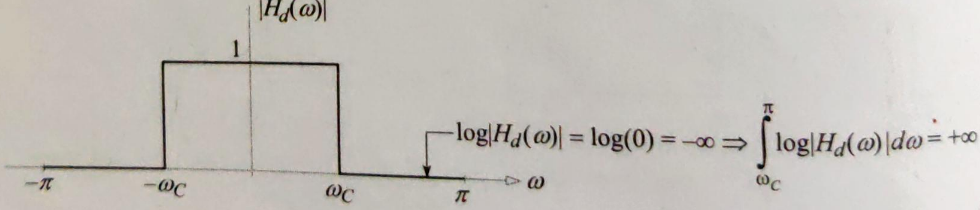


Fig. 5.2 Ideal lowpass filter and the Paley-Wiener condition.

One important inference we make from Paley-Wiener theorem is that $|H_c(\omega)|$ can be zero at some frequencies, but it cannot be zero over any finite band of frequencies, because the integral then becomes unbounded.

5.3 Symmetric and Antisymmetric FIR Filters

If $h(n)$ represents the impulse response of a discrete-time filter, a necessary and sufficient condition for linear-phase is that $h(n)$ must have a finite duration N and is either symmetric or antisymmetric about its midpoint.

Mathematically,
$$h(n) = h(N-1-n) \quad \text{[Symmetric]}, \quad n = 0, 1, \cdots N-1$$
or
$$h(n) = -h(N-1-n) \quad \text{[Antisymmetric]}, \quad n = 0, 1, \cdots N-1$$

gives a linear-phase FIR filter.

Let us now prove that a linear-phase filter results if symmetry condition for $h(n)$ is met with. Fig. 5.3 and 5.4 illustrate the general shapes of $h(n)$ that give a linear-phase.

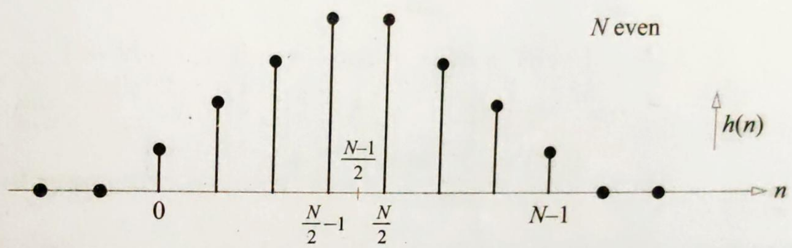


Fig. 5.3 An impulse response $h(n)$ that is symmetric with respect to midpoint for N even.

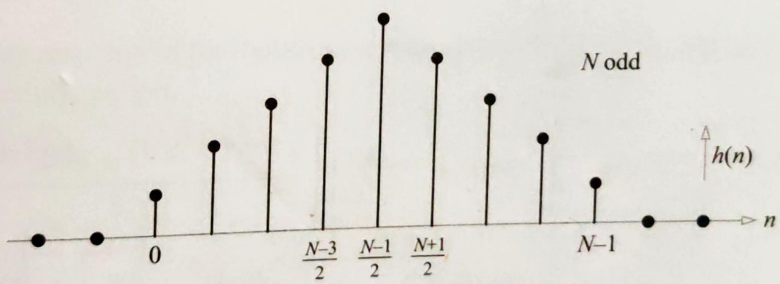


Fig. 5.4 An impulse response $h(n)$ that is symmetric with respect to midpoint and for N odd.

The $\mathcal{Z}$-transform of a causal finite duration sequence $h(n)$ that begins at $n = 0$ and ends at $N-1$ is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \tag{5.7}$$

Referring Fig. 5.3, the above summation can be broken into two summations as given below.

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n) z^{-n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) z^{-n} \tag{5.8}$$

Letting $m = N - 1 - n$ in the second summation on the right-hand side of the above equation, we get

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n)z^{-n} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m)z^{-(N-1-m)}$$

for N even

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)z^{-n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)z^{-(N-1-m)} \tag{5.9}$$

Since, m is a dummy variable, it can be replaced by n. Accordingly equation (5.9) becomes,

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n)z^{-n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)z^{-(N-1-n)} \tag{5.10}$$

Since, $\quad h(N-1-n) = h(n) \quad$ for $\quad n = 0, 1, \cdots N-1$

we get

$$H(z) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[z^{-n} + z^{-(N-1-n)}\right]$$

The frequency response of the digital filter is obtained by letting $z = e^{j\omega}$ in $H(z)$. Accordingly, we get

$$H\left(e^{j\omega}\right) = H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} 2h(n)e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\frac{e^{-j\omega\left[n-\left(\frac{N-1}{2}\right)\right]} + e^{j\omega\left[n-\left(\frac{N-1}{2}\right)\right]}}{2}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\left(\omega\left[n-\left(\frac{N-1}{2}\right)\right]\right) \tag{5.11}$$

Comparing equation (5.11) with

$$H(\omega) = H_r(\omega)e^{j\theta(\omega)}$$

where $H_r(\omega)$ is a real function of ω, we get

$$H_r(\omega) = \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\left(\omega\left[n-\left(\frac{N-1}{2}\right)\right]\right), \quad N \text{ even}$$

and

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases} \tag{5.12}$$

Referring Fig. 5.4, the summation in equation (5.7) can be broken into the following form.

$$H(z) = \sum_{n=0}^{\frac{(N-3)}{2}} h(n)z^{-n} + h\left(\frac{N-1}{2}\right)z^{-\frac{(N-1)}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)z^{-n} \qquad \text{for } N \text{ odd} \tag{5.13}$$

Letting $m = N - 1 - n$ in the third term on the right-hand side of equation (5.13), we get

$$H(z) = \sum_{n=0}^{\frac{(N-3)}{2}} h(n)z^{-n} + h\left(\frac{N-1}{2}\right)z^{-\frac{(N-1)}{2}} + \sum_{m=\frac{N-3}{2}}^{0} h(N-1-m)z^{-(N-1-m)}$$

Since, m is a dummy variable, it can be replaced by n and also the order of summation could be reversed. Accordingly, the above equation becomes,

$$H(z) = \sum_{n=0}^{\frac{(N-3)}{2}} h(n)z^{-n} + h\left(\frac{N-1}{2}\right)z^{-\frac{(N-1)}{2}} + \sum_{n=0}^{\frac{(N-3)}{2}} h(N-1-n)z^{-(N-1-n)} \tag{5.14}$$

Making use of symmetry condition: $h(N-1-n) = h(n)$, we get

$$H(z) = \sum_{n=0}^{\frac{(N-3)}{2}} h(n)\left[z^{-n} + z^{-(N-1-n)}\right] + h\left(\frac{N-1}{2}\right)z^{-\frac{(N-1)}{2}} \tag{5.15}$$

The frequency response of the digital filter represented by $H(z)$ is obtained by letting $z = e^{j\omega}$ in $H(z)$. Accordingly, we get,

$$\begin{aligned} H\left(e^{j\omega}\right) &= H(\omega) \\ &= h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{(N-3)}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right] \\ &= h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{\frac{(N-3)}{2}} 2h(n)\left(\frac{e^{-j\omega\left[n-\frac{(N-1)}{2}\right]} + e^{j\omega\left[n-\frac{(N-1)}{2}\right]}}{2}\right) \\ &= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{(N-3)}{2}} 2h(n)\cos\left(\omega\left[n - \left(\frac{N-1}{2}\right)\right]\right)\right] \end{aligned} \tag{5.16}$$

Hence, $$H_r(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{(N-3)}{2}} 2h(n)\cos\left(\omega\left[n - \left(\frac{N-1}{2}\right)\right]\right)$$

and $$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases} \tag{5.17}$$

Inspection of equations (5.12) and (5.17) reveals that $\theta(\omega)$ is same irrespective of whether N is even or odd. Also, these equations reveal that the phase $\theta(\omega)$ is linear when the symmetry condition, $h(n) = h(N-1-n)$ for $n = 0, 1, \cdots N-1$ is met with.

When $h(n) = -h(N-1-n)$, the unit impulse response is said to be antisymmetric. For N odd, the midpoint of the antisymmetric $h(n)$ is $n = \frac{(N-1)}{2}$.

Consequently,

$$h\left(\frac{N-1}{2}\right) = 0$$

But, if N is even, each term in $h(n)$ has a matching term of opposite sign.

Proceeding in a similar manner as we did for a symmetric $h(n)$, the frequency response of an antisymmetric $h(n)$ can be expressed as

$$H\left(e^{j\omega}\right) = H(\omega) = H_r(\omega)\, e^{j\theta(\omega)}$$

where

$$H_r(\omega) = 2\sum_{n=0}^{\frac{(N-3)}{2}} h(n)\sin\left(\omega\left[\left(\frac{N-1}{2}\right) - n\right]\right), \quad N \text{ odd}$$

and

$$H_r(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin\left(\omega\left[\left(\frac{N-1}{2}\right) - n\right]\right), \quad N \text{ even}$$

Also, for any N

$$\theta(\omega) = \begin{cases} \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ \frac{3\pi}{2} - \omega\left(\frac{N-1}{2}\right), & \text{if } H_r(\omega) < 0 \end{cases}$$

For N odd, equation (5.17) indicates the slope of $-\alpha = -\frac{(N-1)}{2}$ causes a delay in the output of α, which is an integer number of samples, whereas for N even, the slope causes a noninteger delay. The noninteger delay will cause the values of the sequence to be changed, which, in some cases, may be undesirable.

5.4 Locations of Zeros of Linear-Phase FIR Filters

The system function $H(z)$ of a causal FIR filter is given by

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n) z^{-n} \\ &= h(0) + h(1)z^{-1} + \cdots + h(N-1)z^{-(N-1)} \end{aligned} \tag{5.18}$$

The frequency response of an FIR filter will exhibit linear-phase if $h(n)$ is symmetric about its midpoint.

That is,

$$h(n) = h(N-1-n), \qquad n = 0, 1, \cdots N-1 \tag{5.19}$$

Making use of equation (5.19) in equation (5.18), we get

$$\begin{aligned}
H(z) &= h(N-1) + h(N-2)z^{-1} + \cdots + h(0)z^{-(N-1)} \\
&= z^{-(N-1)}\left[h(0) + h(1)z^{1} + \cdots + h(N-1)z^{(N-1)}\right] \\
&= z^{-(N-1)} \sum_{n=0}^{N-1} h(n) z^{n} \\
&= z^{-(N-1)} \sum_{n=0}^{N-1} h(n) \left(z^{-1}\right)^{-n} \\
\Rightarrow \quad H(z) &= z^{-(N-1)} H\left(z^{-1}\right) \qquad (5.20)
\end{aligned}$$

The above equation implies that the roots of the polynomial $H(z)$ are identical to the roots of the polynomial $H\left(z^{-1}\right)$. Also, the roots of $H(z) = 0$ are nothing but the zeros of $H(z)$. Consequently, zeros of $H(z)$ must occur in reciprocal pairs. That is, if z_1 is a zero of $H(z)$, then $\frac{1}{z_1}$ is also a zero. We consider the following typical cases:

1. If $z_1 = -1$, then $z_1^{-1} = z_1 = -1$. This means that the zero z_1 lies at -1.
2. If z_2 is a real zero with $|z_2| < 1$, then z_2^{-1} is also a zero. Thus, there are two zeros as shown in Fig. 5.5.
3. If z_3 is a complex zero with $|z_3| = 1$, then $z_3^{-1} = z_3^*$. Thus, there are two zeros in this group.
4. If z_4 is a complex zero with $|z_4| \neq 1$, then we have four zeros: z_4, z_4^{-1}, z_4^* and $\left(z_4^*\right)^{-1}$.

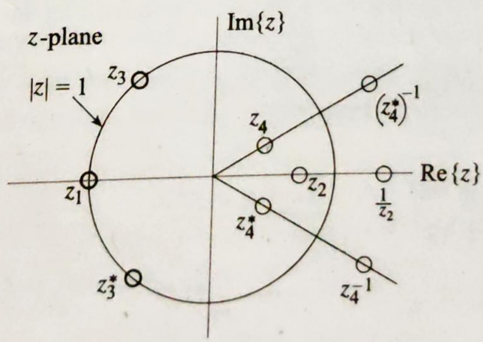


Fig. 5.5 Symmetry of zero locations for a linear-phase filter.

5.5 Design of Linear-Phase FIR Filters Using Windows

Ideal frequency responses $h_d(n)$ have infinite impulse responses. However, by Parseval's theorem, such impulse responses have finite energy. [1]Truncating the impulse response of the ideal filter on both the right and the left sides thus yields a finite impulse response whose associated frequency response approximates that of the ideal filter. This is the basic idea of the impulse response

[1]Read Appendix-III for additional information.

window. Only those coefficients that can be "seen" are used in the filter. That is, if $h_d(n)$ represents the impulse response of a desired IIR filter, then an FIR filter with impulse response $h(n)$ can be obtained as follows:

$$h(n) = \begin{cases} h_d(n), & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.21}$$

The impulse response $h_d(n)$ is truncated at $n = 0$, since we are interested in a causal FIR filter. It is possible to write equation (5.21) alternatively as

$$h(n) = h_d(n)\, w(n) \tag{5.22}$$

Where $w(n)$ in equation (5.22) is said to be a rectangular window defined by

$$w(n) = \begin{cases} 1, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.23}$$

Taking DTFT on both the sides of equation (5.22), we get

$$H(\omega) = H_d(\omega) * W(\omega) \tag{5.24}$$

Now, we would, of course like $H_d(\omega)$ to be equal to $H(\omega)$. This means that $W(\omega)$ must be equal to $\delta(\omega)$, which in turn means that $w(n) = 1$ for all n. This gives us the infinite-length ideal impulse response that cannot be realized in practice. The point here is that we desire a finite-length window $w(n)$ whose DTFT $W(\omega)$ is close to an impulse.

The frequency response of the rectangular window is obtained by taking the DTFT of $w(n)$.

That is,

$$\begin{aligned} W(\omega) &\triangleq \sum_{n=-\infty}^{\infty} w(n) e^{-j\omega n} \\ \Rightarrow \quad W(\omega) &= \sum_{n=0}^{N-1} 1 \times e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\ &= e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \end{aligned}$$

The magnitude and phase of $W(\omega)$ are

$$|W(\omega)| = \left| \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right| \tag{5.25}$$

and

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right), & \text{when } \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi, & \text{when } \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} < 0 \end{cases}$$

The frequency response of the *rectangular window*, therefore has a linear-phase. This is essential because we want the designed FIR filter to have linear-phase. The magnitude frequency response of the rectangular window is plotted using equation (5.25) and it appears as shown in Fig. 5.6.

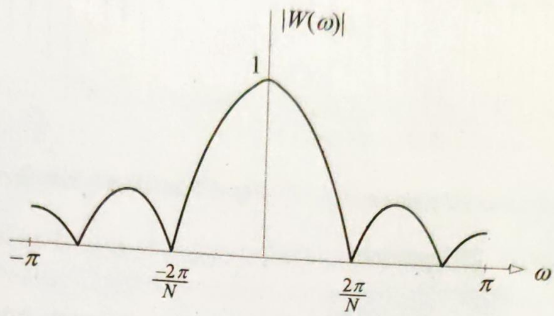


Fig. 5.6 Magnitude response of the rectangular window.

From Fig. 5.6, we find that the magnitude response is *impulse like*; that is, $W(\omega)$ is a narrow pulse, and it falls off sharply (sinc function). It has a *main lobe* and few *sidelobes* (ripples). When $W(\omega)$ is convolved with $H_d(\omega)$, the sidelobes cause ripples in $H(\omega)$. Hence, it is advantageous to use windows having low sidelobe amplitudes. In addition, narrower the main lobe, more the $W(\omega)$ will look like $\delta(\omega)$. Also, the width of the main lobe of $W(\omega)$ is approximately equal to the transistion width of the desired FIR filter. The width of the main lobe is found as $\frac{4\pi}{N}$ as shown in Fig. 5.6. If one desires to have a narrow transistion band for the FIR filter, the main lobe width of $|W(\omega)|$ should be narrow and this enforces the window length N should be as large as possible. In general, we are left with a trade-off of making N large enough so that $H(\omega)$ looks like $H_d(\omega)$, yet small enough to allow reasonable implementation. A lot of work has been done on adjusting $w(n)$ to satisfy certain main lobe and sidelobe requirements. Some of the most commonly used windows are the *Rectangular, Bartlett, Hanning, Hamming, Blackman*, and *Kaiser windows*. These windows are discussed below:

Rectangular window

$K\frac{2\pi}{N} = \Delta\omega$

$N \geq \frac{K2\pi}{\Delta\omega}$

The N-term rectangular window is defined by

$$w_R(n) = \begin{cases} 1, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Fig. 5.7(a), 5.7(b) and 5.7(c) illustrate the impulse response of the rectangular window, magnitude response of the rectangular window and the magnitude response for the filter made with the rectangular window.

From Fig. 5.7(b), it is found that the biggest sidelobe is 13 dB below the dc magnitude. A lowpass filter produced using the rectangular window should have an approximate difference of 21 dB between its passband and stopband gains as seen in Fig. 5.7(c).

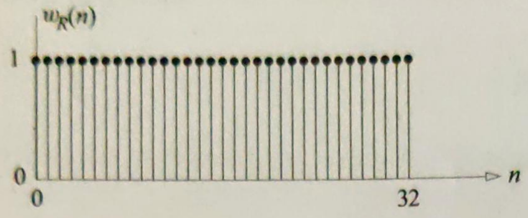


Fig. 5.7(a) Impulse response for rectangular window.

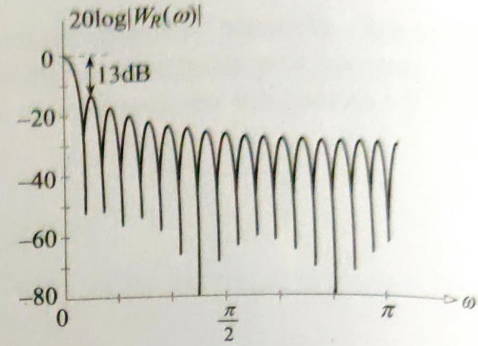


Fig. 5.7(b) Magnitude response for rectangular window.

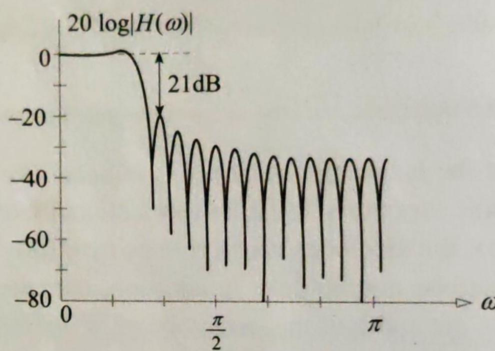


Fig. 5.7(c) Filter shape (magnitude frequency response) for filter made with rectangular window.

Bartlett window

The N-term Bartlett window is defined by

$$w_B(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{(N-1)}{2}\right|}{N-1}, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

Fig. 5.8(a), 5.8(b) and 5.8(c) illustrate the impulse response of the Bartlett window, magnitude response of the Bartlett window and the filter shape for the filter made with Bartlett window respectively.

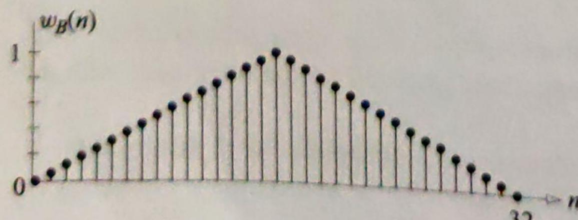


Fig. 5.8 (a) Impulse response for Bartlett window.

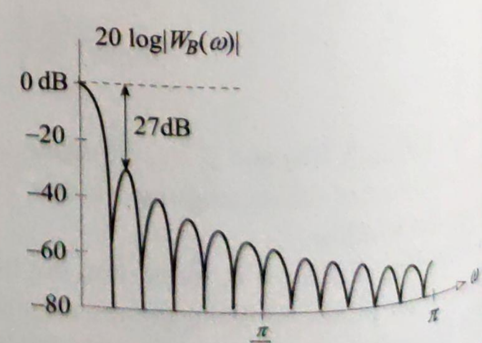


Fig. 5.8 (b) Magnitude response for Bartlett window.

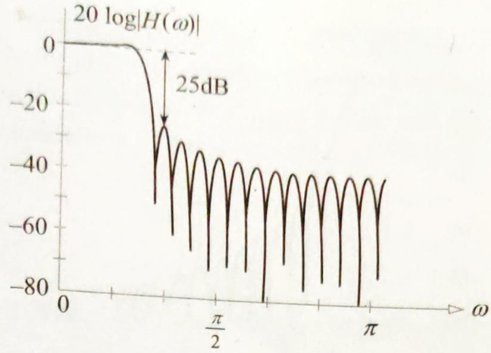


Fig. 5.8(c) Filter shape for filter made with Bartlett window.

From Fig. 5.8(b), it is found that the biggest sidelobe is 27 dB below the dc magnitude. A lowpass filter produced using Bartlett window shows an approximate difference of 25 dB between its passband and stopband gains as seen in Fig. 5.8(c).

Hanning window

The N-term Hanning window is defined by the equation

$$w_{\text{Han}}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

Fig. 5.9(a) shows the window function for $N = 33$, and Fig. 5.9(b) shows its magnitude response. It is evident that the smoother contours in the window function lead to smaller sidelobes than for the rectangular window. The sidelobes for the Hanning window are 32 dB below its dc magnitude. Fig. 5.9(c) shows the magnitude response of a lowpass FIR filter designed with a Hanning window. It is found that the stopband gain for the biggest sidelobe is 44 dB below the passband gain.

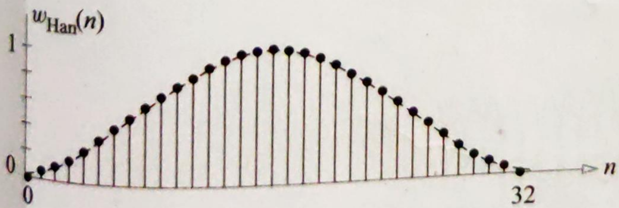


Fig. 5.9(a) Impulse response for Hanning window.

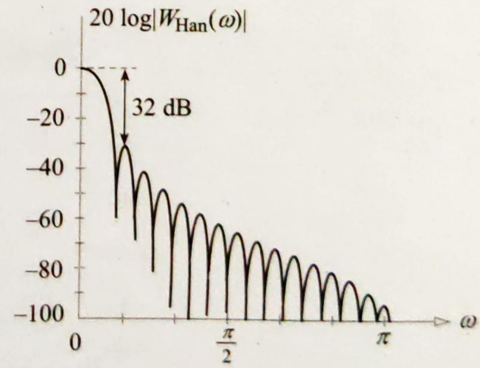


Fig. 5.9(b) Magnitude response for Hanning window.

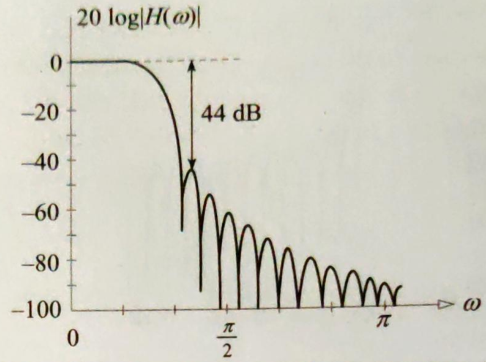


Fig. 5.9(c) Filter shape for filter made with Hanning window.

Hamming window

The impulse response of an N-term Hamming window is defined as follows:

$$w_{\text{Ham}}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

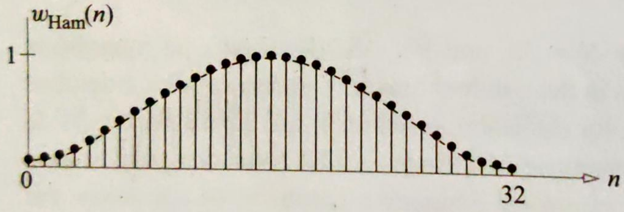


Fig. 5.10(a) Impulse response for Hamming window.

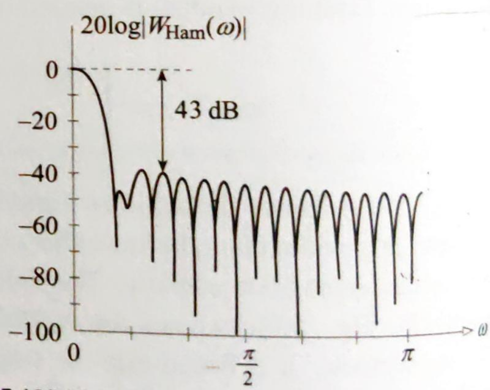


Fig. 5.10(b) Magnitude response for Hamming window.

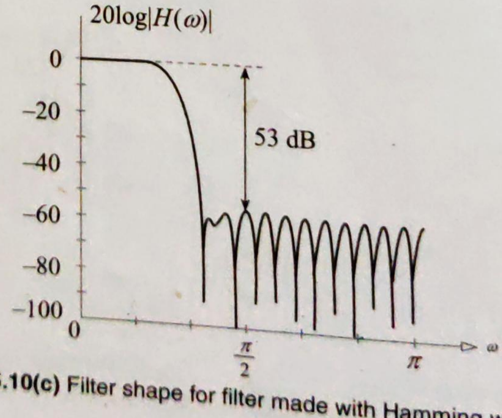


Fig. 5.10(c) Filter shape for filter made with Hamming window.

Fig. 5.10(a) shows the impulse response for N = 33 and Fig. 5.10(b) shows its magnitude response. Again, the smooth contours in the window function lead to small sidelobes. The sidelobes for the Hamming window are 43 dB below its dc magnitude and the lowpass filter designed with Hamming window, the largest sidelobe in the stopband is 53 dB below the passband gain as shown in Fig. 5.10(c).

Blackman window

The impulse response of an N-term Blackman window is defined as follows:

$$w_{\mathrm{Bl}}(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

Fig. 5.11(a) shows the impulse response for N = 33 and Fig. 5.11(b) shows its magnitude response. Fig. 5.11(b) indicates a 58 dB difference between the dc magnitude and the largest sidelobe. A lowpass filter designed with a Blackman window gives sidelobes that are 74 dB below the passband gain and the same is shown in Fig. 5.11(c).

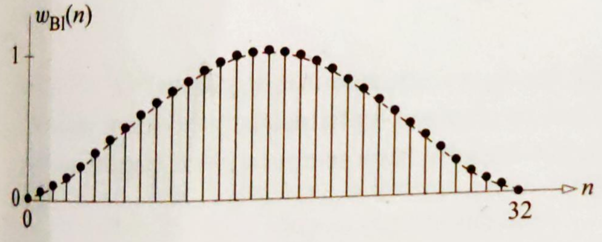


Fig. 5.11(a) Impulse response for Blackman window.

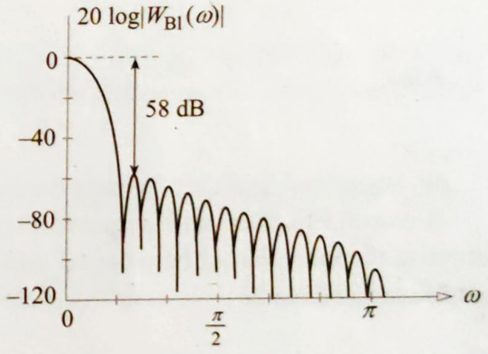


Fig. 5.11(b) Magnitude response for Blackman window.

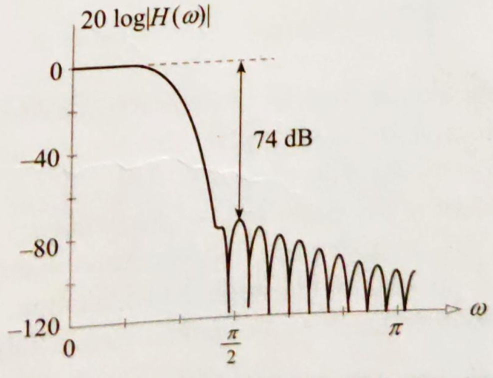


Fig. 5.11(c) Filter shape for filter made with Blackman window.

Cross multiplying and taking inverse $\mathcal{Z}$-transform on both the sides of the above equation, we get

$$
\begin{aligned}
y(n) &= -0.1273[x(n) - x(n-10)] - 0.2122[x(n-2) - x(n-8)] \\
&\quad -0.6366[x(n-4) - x(n-6)]
\end{aligned}
$$

c. Since, N is odd and $h(n) = -h(N-1-n)$, the magnitude frequency response is

$$
\begin{aligned}
|H(\omega)| &= |H_r(\omega)| \\
&= \left| 2\sum_{n=0}^{\frac{(N-3)}{2}} h(n) \sin\left[\omega\left(\left(\frac{N-1}{2}\right) - n\right)\right]\right| \\
\Rightarrow \quad |H(\omega)| &= \left| 2\sum_{n=0}^{4} h(n) \sin[\omega(5-n)]\right| \\
\Rightarrow \quad |H(\omega)| &= \left| \begin{array}{c} 2h(0)\sin 5\omega + 2h(1)\sin 4\omega + 2h(2)\sin 3\omega \\ +2h(3)\sin 2\omega + 2h(4)\sin\omega \end{array} \right| \\
&= \left| \begin{array}{c} 2\times -0.1273\sin 5\omega + 0 + 2\times -0.2122\sin 3\omega + 0 \\ +2\times -0.6366\sin\omega \end{array} \right| \\
&= |-0.2546\sin 5\omega - 0.4244\sin 3\omega - 1.2732\sin\omega|
\end{aligned}
$$

5.9 Frequency Sampling Design of FIR Filters

In the frequency sampling method, we sample the desired frequency response $H_d(\omega)$ as shown in Fig. 5.15 at N equally spaced points in the interval $(0, 2\pi)$ by replacing ω by $\omega_k = \frac{2\pi k}{N}$ for $k = 0, 1 \cdots N-1$. This is clearly a sampling of the discrete time Fourier transform (DTFT), which is actually the process of getting discrete Fourier transform (DFT) from DTFT. That is,

$$
\begin{aligned}
H(k) &= H_d(\omega)\big|_{\omega=\omega_k}, \quad k = 0, 1, \cdots N-1 \\
&= H_d\left(\frac{2\pi k}{N}\right), \quad k = 0, 1, \cdots N-1 \qquad (5.27)
\end{aligned}
$$

Note that in equation (5.27), $H(k)$ does not have conjugate symmetry about the folding index, $\frac{N}{2}$ for $k = 0, 1, 2, \cdots N-1$. This symmetry is compulsory for the inverse DFT to yield a filter having real coefficients for its impulse response $h(n)$. Hence, we force the conjugate symmetry as follows: for N odd, we must have $H(0)$ real and

$$H(N-k) = H^*(k), \qquad k = 1, 2, \cdots \frac{(N-1)}{2}$$

and for N even, we must have $H(0)$ real and

$$H(N-k) = H^*(k), \qquad k = 1, \cdots \frac{N}{2} - 1 \quad \text{and} \quad H\left(\frac{N}{2}\right) = 0$$

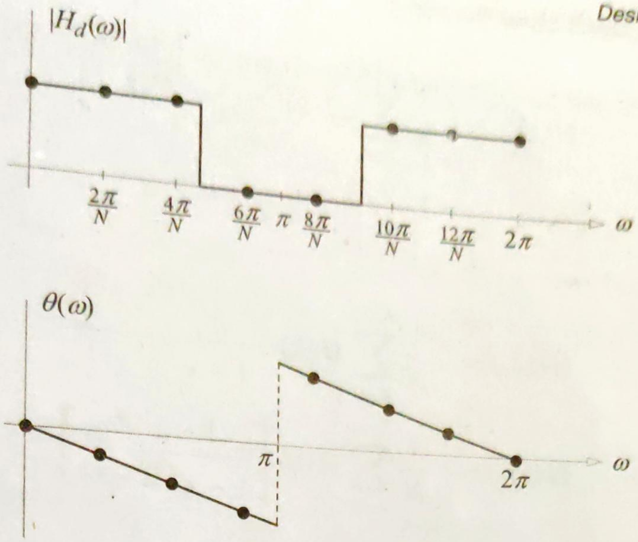


Fig. 5.15 Sampling of a desired frequency response with samples taken at intervals of $\frac{2\pi}{N}$.

The FIR filter coefficients, $h(n)$, are obtained simply by taking the inverse DFT of $H(k)$.

That is,
$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j\frac{2\pi}{N}nk}, \quad n = 0, 1, \cdots N-1 \tag{5.28}$$

If, we let $H(N-k) = \mathrm{H}^*(k)$, $k = 1 \cdots \frac{(N-1)}{2}$ with N being odd, equation (5.28) may be written as

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \mathrm{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right] \tag{5.29}$$

Similary with $H(N-k) = H^*(k)$, $k = 1, 2, \cdots \frac{N}{2}-1$ with N being even and with $H\left(\frac{N}{2}\right) = 0$, equation (5.28) may be written as

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \mathrm{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right] \tag{5.30}$$

Taking $\mathcal{Z}$-transform on both the sides of equation (5.28), we get

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n) z^{-n} \\ &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j\frac{2\pi nk}{N}}\right] z^{-n} \end{aligned}$$

Interchanging the order of summations, we get

$$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk}\, z^{-n} \tag{5.31}$$

$$\Rightarrow \quad H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^n$$

We know that $$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \quad a \neq 1$$

Hence, $$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \frac{1-\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^N}{1-e^{j\frac{2\pi k}{N}} z^{-1}}$$

$$\Rightarrow \quad H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi K}{N}} z^{-1}}\right]$$

The term $1-z^{-N}$ has a factor $1-e^{j\frac{2\pi k}{N}}z^{-1}$. Hence, pole-zero cancellation takes place and $H(z)$ contains only zeros, a necessary and sufficient condition for an FIR filter.

The realization of an FIR filter based on frequency sampling design is given in Fig. 5.16.

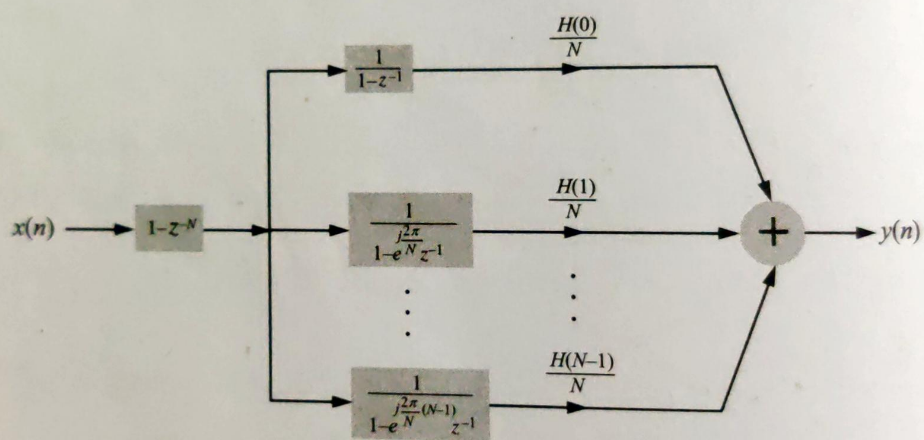


Fig. 5.16 Realization of an FIR filter based on frequency sampling design.

To find the frequency response

The frequency response of the FIR filter is obtained by letting $z = e^{j\omega}$ in $H(z)$.

That is, $$H(e^{j\omega}) = H(\omega)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} H(k) \left(\frac{1-e^{-j\omega N}}{1-e^{j\frac{2\pi k}{N}} e^{-j\omega}}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} H(k) \frac{e^{-j\omega\frac{N}{2}}\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{e^{j\frac{2\pi}{N}\frac{k}{2}} e^{-\frac{j\omega}{2}}\left(e^{-j\frac{2\pi}{N}\frac{k}{2}} e^{\frac{j\omega}{2}} - e^{j\frac{2\pi}{N}\frac{k}{2}} e^{-\frac{j\omega}{2}}\right)}$$

$$= e^{-j\omega\frac{(N-1)}{2}} \sum_{k=0}^{N-1} \frac{H(k)}{N} e^{-j\frac{\pi k}{N}} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}$$

Example 5.19 A lowpass filter has the desired frequency response

$$H_d(\omega) = H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

Determine $h(n)$ based on frequency-sampling technique. Take $N = 7$.

Solution

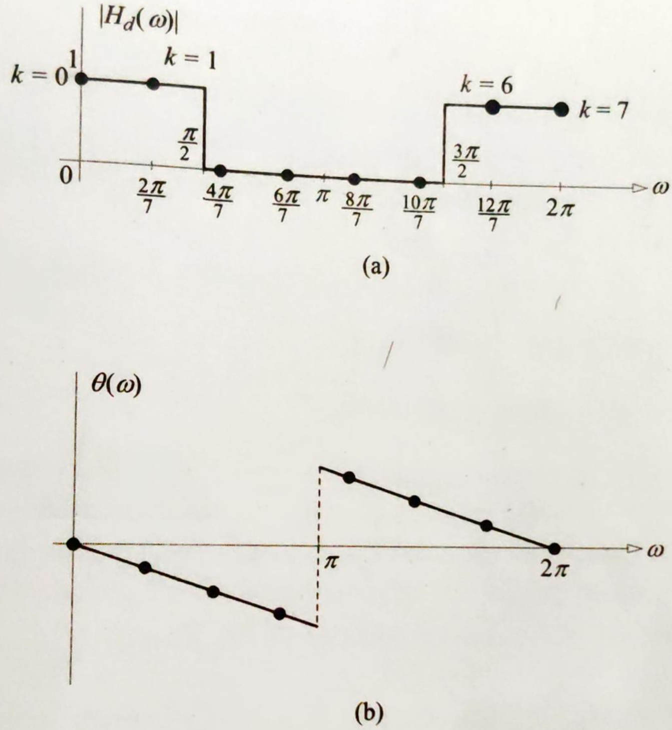


Fig. Ex.5.19 Magnitude and phase responses of the desired lowpass filter.

Let the ideal response of a linear-phase lowpass filter be

$$H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude frequency response is taken to be symmetric about π while the ideal phase response is taken to be antisymmetric about π. The ideal magnitude and phase responses with samples taken for $N = 7$ are shown in Fig. Ex.5.19.

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0, \cdots N-1$. The range of k is found as follows:

i. for $0 \leqslant \omega < \frac{\pi}{2}$, the values of k are 0, 1

ii. for $\frac{\pi}{2} < \omega < \frac{3\pi}{2}$, the values of k are 2, 3, 4, 5

iii. for $\frac{3\pi}{2} < \omega < 2\pi$, the value of k is 6.

From Fig Ex.5.19(a), we find that

$$|H(k)| = \begin{cases} 1, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ 1, & k = 6 \end{cases}$$

Also, from Fig. Ex.5.19(b), we find that

$$\theta_k = -3\omega_k = -3 \times \frac{2\pi}{N}k = \frac{-6\pi}{7}k \quad \text{for } k = 0, 1, 2, 3$$

and

$$\theta_k = \frac{-6\pi}{7}(k-7) \qquad \text{for } k = 4, 5, 6$$

Since $H(k)$ is complex, we may write

$$H(k) = |H(k)|e^{j\theta_k}$$

Hence,

$$H(k) = \begin{cases} e^{-j\frac{6\pi k}{7}}, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ e^{-j\frac{6\pi}{7}(k-7)}, & k = 6 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29). That is,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \operatorname{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right]$$

Hence,

$$\begin{aligned} h(n) &= \frac{1}{7}\left[H(0) + 2\sum_{k=1}^{3} \operatorname{Re}\left\{H(k)e^{j\frac{2\pi nk}{7}}\right\}\right] \\ &= \frac{1}{7}\left[H(0) + 2\operatorname{Re}\left\{H(1)e^{j\frac{2\pi n}{7}}\right\}\right] \\ &= \frac{1}{7}\left[1 + 2\operatorname{Re}\left\{e^{-j\frac{6\pi}{7}}e^{j\frac{2\pi n}{7}}\right\}\right] \\ &= \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right], \quad 0 \leqslant n \leqslant 6 \end{aligned}$$

0.623 1 0.28 ✓
0.3447 0.6084

The filter coefficients are tabulator below:

n	$h(n)$	n	$h(n)$
0	−0.11456	4	0.320997
1	0.07928	5	0.07928
2	0.320997	6	−0.11456
3	0.42857		

Example 5.20 Design a 17-tap linear-phase FIR filter with a cutoff frequency $\omega_C = \frac{\pi}{2}$. The design is to be done based on frequency sampling technique.

☐ Solution

The ideal lowpass frequency response with a linear-phase is

$$H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude response is taken to be even symmetric about π while the phase response is taken to be odd symmetric about π .

The ideal magnitude and phase responses with samples for $N = 17$ are shown in Fig. Ex.5.20. The heavy dots denote the frequency samples of $H_d(\omega)$ taken at $\omega = \omega_k = \frac{2\pi k}{N}$ for $k = 0, \cdots N-1$.

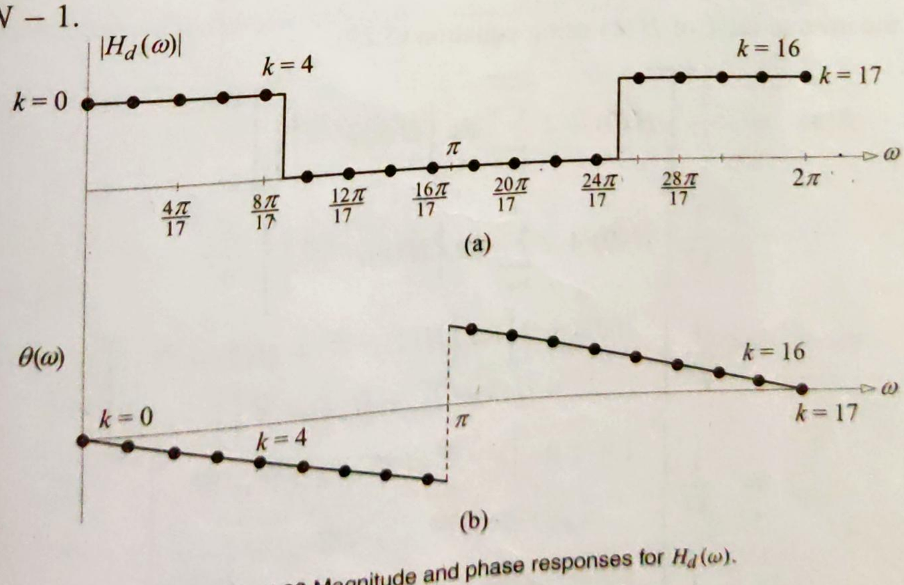


Fig. Ex.5.20 Magnitude and phase responses for $H_d(\omega)$.

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j8\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0, 1, \cdots N-1$. From Fig. Ex.5.20(a), we find that

$$|H(k)| = \begin{cases} 1, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ 1, & 13 \leqslant k \leqslant 16 \end{cases}$$

Also, from Fig. Ex.5.20(b), we find that

$$\begin{aligned} \theta_k = -8\omega_k &= -8 \times \frac{2\pi k}{N} \\ &= \frac{-16\pi k}{17}, \quad 0 \leqslant k \leqslant 8 \end{aligned}$$

and

$$\theta_k = \frac{-16\pi}{17}(k-17), \quad 9 \leqslant k \leqslant 16.$$

Since, $H(k)$ is complex, we may write

$$H(k) = |H(k)|e^{j\theta_k}$$

$$\Rightarrow \quad H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}}, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ e^{-j\frac{16\pi}{17}(k-17)}, & 13 \leqslant k \leqslant 16 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29).

That is,

$$\begin{aligned} h(n) &= \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right] \\ &= \frac{1}{17}\left[H(0) + 2\sum_{k=1}^{8} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{17}}\right\}\right] \\ &= \frac{1}{17}\left[H(0) + 2\left[\text{Re}\left\{H(1)e^{j\frac{2\pi n}{17}} + H(2)e^{j\frac{4\pi n}{17}} + H(3)e^{j\frac{6\pi n}{17}} + H(4)e^{j\frac{8\pi n}{17}}\right\}\right]\right] \\ &= \frac{1}{17}\left[1 + 2\,\text{Re}\left\{e^{-j\frac{16\pi}{17}}e^{j\frac{2\pi n}{17}} + e^{-j\frac{32\pi}{17}}e^{j\frac{4\pi n}{17}} + e^{-j\frac{48\pi}{17}}e^{j\frac{6\pi n}{17}} + e^{-j\frac{64\pi}{17}}e^{j\frac{8\pi n}{17}}\right\}\right] \\ &= \frac{1}{17}\left[1 + 2\cos\left[\frac{2\pi}{17}(n-8)\right] + 2\cos\left[\frac{4\pi}{17}(n-8)\right] + 2\cos\left[\frac{6\pi}{17}(n-8)\right] + 2\cos\left[\frac{8\pi}{17}(n-8)\right]\right], \quad 0 \leqslant n \leqslant 16 \end{aligned}$$

The FIR filter coefficients are tabulated below:

n	$h(n)$	n	$h(n)$	n	$h(n)$
0	0.0398	6	−0.0299	12	0.03154
1	−0.0488	7	0.31876	13	0.06598
2	−0.03459	8	0.5294	14	−0.03459
3	0.06598	9	0.31876	15	−0.0488
4	0.03154	10	−0.0299	16	0.0398
5	−0.10747	11	−0.10747		

Example 5.21 Determine the filter coefficients $h(n)$ obtained by sampling $H_d(\omega)$ given by

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

Also, obtain the frequency response $H(\omega)$. Take $N = 7$.

□ Solution

The computation of $h(n)$ is done in Example 5.19. The result is given below:

$$h(n) = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right]$$

Let us now, proceed to find the frequency response $H(\omega)$.

We have,
$$h(n) = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right]$$

$$\Rightarrow \quad h(n) = \frac{1}{7}\left[1 + e^{j\frac{2\pi}{7}(n-3)} + e^{-j\frac{2\pi}{7}(n-3)}\right]$$

We know that
$$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

$$\Rightarrow \quad H(\omega) = \frac{1}{7}\sum_{n=0}^{6}\left[1 + e^{j\frac{2\pi}{7}(n-3)} + e^{-j\frac{2\pi}{7}(n-3)}\right]e^{-j\omega n}$$

We know that
$$\sum_{n=0}^{N-1} e^{-j\omega n} = e^{-j\frac{(N-1)\omega}{2}}\frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\Rightarrow \quad \sum_{n=0}^{6} e^{-j\omega n} = e^{-j3\omega}\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Hence,
$$7\,H(\omega) = \sum_{n=0}^{6} e^{-jn\omega} + e^{-j\frac{6\pi}{7}} \sum_{n=0}^{6} e^{-jn(\omega-\frac{2\pi}{7})} + e^{j\frac{6\pi}{7}} \sum_{n=0}^{6} e^{-jn(\omega+\frac{2\pi}{7})}$$

$$= e^{-j3\omega}\left[\frac{\sin(\frac{7\omega}{2})}{\sin(\frac{\omega}{2})} + e^{-j\frac{6\pi}{7}} e^{-j3(\frac{-2\pi}{7})} \frac{\sin\left[\frac{7}{2}(\omega-\frac{2\pi}{7})\right]}{\sin\left[\frac{1}{2}(\omega-\frac{2\pi}{7})\right]} + e^{j\frac{6\pi}{7}} e^{-j3(\frac{2\pi}{7})} \frac{\sin\left[\frac{7}{2}(\omega+\frac{2\pi}{7})\right]}{\sin\left[\frac{1}{2}(\omega+\frac{2\pi}{7})\right]}\right]$$

$$\Rightarrow \qquad H(\omega) = \frac{1}{7} e^{-j3\omega}\left[\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} + \frac{\sin\left[\frac{7}{2}\left(\omega-\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega-\frac{2\pi}{7}\right)\right]} + \frac{\sin\left[\frac{7}{2}\left(\omega+\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega+\frac{2\pi}{7}\right)\right]}\right]$$

5.9.1 Advantages and Disadvantages of frequency sampling design

- The realization of FIR filter given in Fig. 5.16 is suitable for implementation in parallel processors, where each processor performs a maximum of two complex multiplications per sample.
- The filter realization does not require an inverse DFT, which may be computationally tedious for long filters.
- For single-processor implementation, the FIR realization given in Fig. 5.16 requires $(2N-1)$ complex multiplications and 1 real multiplication per sample, whereas a Direct form-I FIR realization needs only N real multiplications.
- The pole-zero cancellation may not actually happen when the filter is implemented in a finite wordlength processor. This may lead to instability.

5.10 Equiripple Filters

The design of an FIR lowpass filter using the window design technique is simple and generally results in a filter with relatively good performance. However, the window design does not yield optimal filters. The reasons for not being optimal are:

1. The passband and stopband deviations, δ_P and δ_S, are approximately equal. Eventhough, it is common to require δ_S to be much smaller than δ_P, these parameters cannot be independently controlled in the window design method. Hence, with the window design method, it is necessary to over design the filter in the passband in order to meet the stricter requirements in the stopband.
2. For most windows, the ripple is not uniform either in the passband or in the stopband and generally decreases when moving away from the transistion band. Allowing the ripple to be uniformly spread over the entire band would result in a smaller peak ripple.

5.11 Advantages and Disadvantages of FIR Filters

Advantages

- Linear-phase.
- Since no poles are present in $H(z)$, it is inherently stable.
- Design procedures are highly flexible in achieving almost any desired amplitude response.
- Low sensitivity to finite word length effects.
- Presence of convenient design techniques and sophisticated tools.

Disadvantages

- Practical realization is complex, since large orders are needed to achieve strict tolerances and narrow transistion bands.
- Unduly large delays, which may not be acceptable in certain applications.

Reinforcement Problems

RP-5.1 The ideal analog differentiator is given by

$$y(t) = \frac{dx(t)}{dt}$$

where $x(t)$ and $y(t)$ are the input and output respectively.

a. Find the frequency response of the differentiator. Plot the magnitude and phase responses.

b. The digital differentiator is described by $y(n) = x(n) - x(n-1)$. Plot the magnitude and phase responses.

Solution

a. The ideal analog differentiatior is given as

$$y(t) = \frac{dx(t)}{dt}$$

Taking FT on both the sides of the above equation, we get

$$\begin{aligned} Y(j\Omega) &= j\Omega\, X(j\Omega) \\ \Rightarrow \quad H(j\Omega) &\triangleq \frac{Y(j\Omega)}{X(\Omega)} = j\Omega \end{aligned}$$