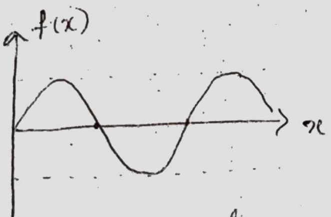
DISCRETE FOURIER TRANSFORMS (DFT)

Discrete Fourier Transforms (DFT) : DFT as a linear transformation, its relationship with other transforms. Properties of DFT (multiplication of two DFTs - the circular convolution, additional DFT Properties)

Introduction to Digital signal processing :-

Signal : Signal is a function, function is a dependent variable of some variables which are Independent variables.
(or) Signal is a function of Independent variables.
Ex: $f(x_1, x_2 \cdots)$

x_i = Time, Distance, Temperature



1-D signal
(one-Dimension signal)

where x = time axis.

$f(x)$ V/S x.

* Examples for signals : speech, Music, picture and video signals.

Types of signals :

(i) Based on Dimensions
- 1D
- 2D
- Multydimension (MD)

(ii) signals
- Natural
- Synthetic (generated by computer simulations)

(iii) signals
- continuous time (CT)
- Discrete time (DT) → A/D → Digital signal

(iv) Signals
- Analog
  - continuous time
  - Discrete time → A/D → Digital signal
- Digital

* All continuous signals are Analog signals but all analog signals are not continuous.

* Analog signal : If time is discrite & not amplitude (or) If Independent variable is discrete and not the dependent variable then it is analog signal.

* Digital signal : Time & Amplitude both are descrete (or) Both Independent & the dependent variables are descrete.

* 1-D (one-dimensional) : Is a function of a single Independent variable (Ex : speech signal variable is time)

* 2-D (Two-dimensional) : Is a function of a two Independent Variable.

  Ex : Image (or) photograph, two Independent variables are spatial Variables.

* M-D (Multidimensional) : Is a function of more than one variable.

  Ex : 3-D signals.
  Black & white video signals.

* Few signals Representations

* A signal carries information
* Objective of signal processing is to extract useful Information carried by the signal.
* Information extraction depends on the type of signal & the nature of the information being carried by the signal.
  - Signal processing is concerned with the mathematical representation of the signal and the algorithmic operation carried out on it to extract the information present. (Ex: MUX, DMUX, Noise filtering)
  - Representation of the signal can be in terms of basis functions in the domain of the original independent variable(s) or it can be in terms of basis functions in a transformed domain.
  - The information extraction process may be carried out in the original domain of the signal or in a transformed domain.

* Few Signal processing operations :

(1) Elementary Time - Domain Operations
(2) Filtering
(3) Generation of complex signal.
(4) Modulation and Demodulation
(5) Multiplexing and Demultiplexing.
(6) Quadrature Amplitude Modulation
(7) Signal Generation.

Few Typical signals

(1) Electrocardiography (ECG) Signal
(2) Electroencephalogram (EEG) Signal
(3) Seismic signals
(4) Diesel Engine Signal
(5) Speech Signals
(6) Musical signals
(7) Time series
(8) Images

Few typical signal processing Applications

(1) Sound Recording Applications
- compressors & limiters
- Expanders and noise gates
- Equalizers and filters
- Noise reduction system
- Delay & reverberation systems
- Special effects

(2) Telephone Dialing Applications
(3) FM stereo Applications
(4) Electronic music synthesis
- Subtractive Synthesis
- Additive synthesis

(5) Echo cancellation in telephone Networks

Signal processing

2a

- → Analog Signal processing
  Adv (?) Disadv (?)
- → Digital Signal processing
  Adv (?) Disadv (?)
- → Mixed (Analog + Digital) signal processing
  Adv (?) Disadv (?)

* Analog Dominated prior to 70's
* Digital Boosted up Due.. to IC's
* Mixed is the Current practice with dominating DSP
* Analog signal processing (ASP) Advanced with Greandau Jumps with the inventions in Devices And circuits

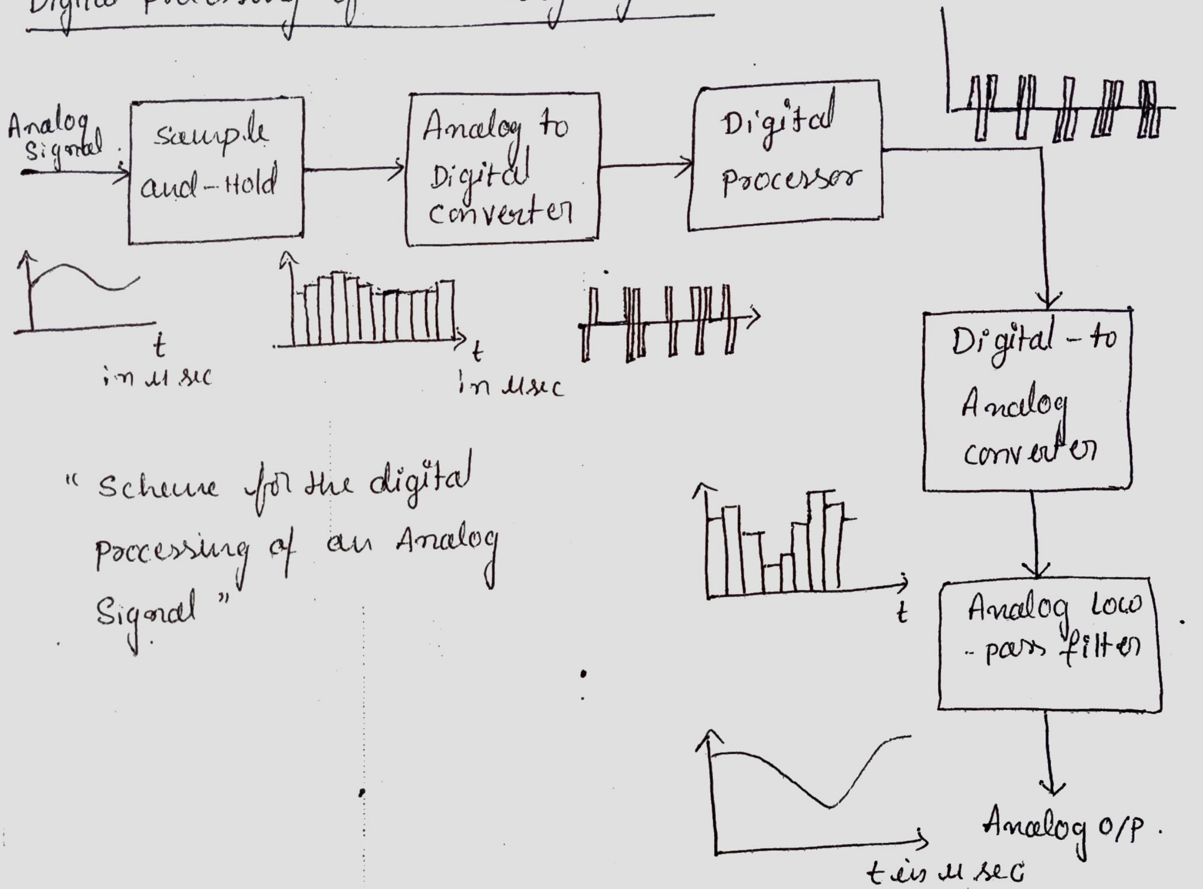
  Ex: Coherer & Spark gaps
  Vacuum tubes, Triode ~~&c~~ ep.
  Transistor, OP Amps & IC's
  SSI (Small Scale Integrated Ckt's)
  MSI (Medium —— " ——)
  LSI (Large —— " ——)
  VLSI (Very large —— " ——)

* Analog signal still in progress why (?)
  Advances in ASP are going in full swing
  ∵ most signals of interest are analog & the ultimate desired o/p in most cases is analog

  Ex: Speech.

* Need for processing
  — Choic of processing is Analog (or) Digital
  ⇒ Natural choice of Digital signal is DSP
  ⇒ —— " —— Analog signal is DSP (or) ASP.
* Processings are Add^n, Multiplication, Delay
* DSP IC's low cost compared to ASP IC's.

Digital processing of an Analog signals



" Scheme for the digital processing of an Analog Signal "

* Digital processing of an analog signal consists basically of three steps.
    (1) Conversion of the analog signal into a digital form
    (2) processing of the digital version
    (3) Conversion of the processed digital signal back into an Analog signal form.

Sample and-Hold : The Amplitude of the analog input signal varies with time, a sample-&-hold (S/H) circuit is used first to sample the analog input at periodic intervals & hold the sampled value constant at the input of the A/D converter to permit accurate digital conversion.

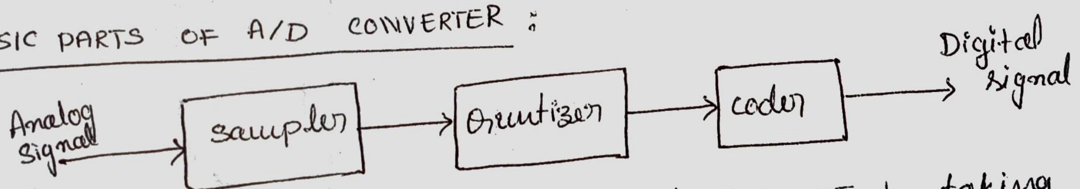
A/D converter : Input is a staircase-type analog signal. The O/P of the A/D converter is a binary data stream that is next processed by Digital Processor. (consist of switch, Requires time for conversion)

Digital processor : I/P is a binary data stream from A/D converter. (coded Binary data)
Digital processor implementing the desired signal processing algorithm (Addⁿ, Mulᵗ, Subⁿ)
O/P is a Again binary data stream

D/A converter : I/P is a binary data stream from Digital processor
Converted into a staircase-type analog signal

Analog Low pass filter : To remove all undesired high-frequency components
Delivers at its O/P the desired processed analog signal. ..

BASIC PARTS OF A/D CONVERTER :



Sampler : Conversion of CT signal into DT by taking samples at discrete time instances

$x_a(t) \rightarrow x_a(nT)$

$F_s = 1/T$

where
$T \rightarrow$ sampling Period
$F_s \rightarrow$ sampling frequency

CT - continuous time

DT - Discrete time.

Quantization : conversions of DT signal into digital signal
$x_a(nT) \rightarrow x_q(n)$
- Value of each sample is represented by a value selected from set of possible values.

Coding : Each Discrete value $x_q(t)$ is represented by a n-bit binary sequence.

Advantages of DSP :

- Less sensitivity to component tolerance & Environmental changes.
- Volume production without the need for Adjustment During Construction (or) use.
- Amenable to Full integration
- Accuracy increases as wordlength increases which lead in increase of cost.
- Dynamic Range increases with floating point Arithmetic.
- Time sharing of a given processor
- Time Multiplexing (even & odd time signal generation).
  ∴ 2 signals can transmit by multiplexing.
- Easy adjustment of processor characteristics By changing co-efficient (Ex: In adaptive filters variable cutoff filters)
- Exact linear phase (No delay distortion)
- Multirate processing (different frequency of same signal Step up & stepdown frequency) *
- No loading problem due to cascading
- Easy storage in tapes, Disks (Including optical)
- Very low frequency processing with Easy. ie No Inductance problems.

Disadvantages of DSP :

- Complexity increased because of S/H, A/D, D/A
- More Devises.
- Speed limited because of A/D converter
- limited rate samples ie frequency range limited.
  ∴ S/H, A/D.
- Frequency required for No aliasing.
  - $\Rightarrow f_s = 2f_m$
  - $\Rightarrow$ If f_s increases, A/D resolution decreases.
  - $\Rightarrow$ For 1 GHz f_s, 6 bits A/D resolution.
  - $\Rightarrow$ In general. 12 to 16 bits resolution needed in most Application limits f_s to 10 MHz.
- Power | Dissipation
  - $\Rightarrow$ AT &T DSP 32C containing 405000 transistors dissipates 1W ie No such problems with LC Ckts But Resistor dissipates power.
  - $\Rightarrow$ Less power use (or) No power use for digital ckts
    TI is working on that
    ie For Implanting inside the human body Should not use power (or) battery ∴ Such devises should work on the body temperature of the Live body.

Digital Signal :

$x_a(t) \longrightarrow x_a(nT) \longrightarrow \boxed{A/D} \longrightarrow x(n)$

Analog signal — Sample & hold — Analog Discrete signal : — Digital signal.

(T is normalized to 1 sec)

* $x_a(nT)$ & $x(n)$ can be interchangably used because differential, Z-transform etc are same for both the signal notation.

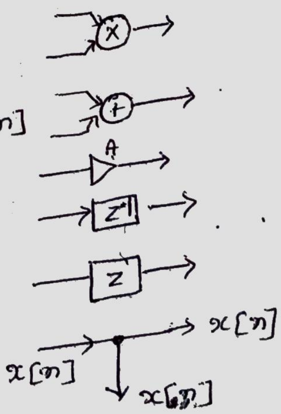
  $\therefore$ $x_a(nT)$ & $x(n)$ can use interchangably.

* $T = 1/f_s$ where f_s = sampling frequency
  T = sampling rate.
* sampling → up-sample / down-sample
* Sequence $x(n)$ or $\{x(n)\}$ means the same.
* sampling rate $= R = \dfrac{F_T'}{F_T}$

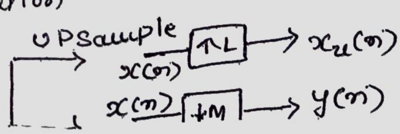
  where F_T' = Desired sampling rate
  F_T = sampling rate

  ⇒ if $R > 1$ ⇒ Processes is interpolation
  $R < 1$ ⇒ —— " —— Decimation.

* operation of sequences.
  — Basic operations.
  ⇒ Modulator $w_1[n] = x[n]y[n]$
  ⇒ adder $w_2[n] = x[n] + y[n]$
  ⇒ Multiplier $w_3[n] = A\,x[n]$
  ⇒ unit delay $w_4[n] = x[n-1]$
  ⇒ unit advance $w_5[n] = x[n+1]$
  ⇒ pick-off node $w_5[n] = x[n]$



  — Combination of Basic operation
  — Sampling Rate Alteration



$\Rightarrow x_{e}(n) = \begin{cases} x(n/L), & n = 0, \pm L, \pm 2L \dots \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow y[n] = x[nM]$

* Few sequences :

$\Rightarrow$ Finite length sequences

$\Rightarrow$ Infinite —— " ——

$\Rightarrow$ Right sided signal (RSS)

$\Rightarrow$ Left —— " —— (LSS)

$\Rightarrow$ Conjugate - symmetric sequence $x[n] = x^{*}[-n]$

$\Rightarrow$ Conjugate - antisymmetric sequence $x[n] = -x^{*}[-n]$

$\Rightarrow$ complex sequence $x[n] = \underbrace{x_{cs}[n]}_{\text{conjugate-symmetric part}} + \underbrace{x_{ca}[n]}_{\text{conjugate-Antisymmetric part}}$

$\Rightarrow x_{cs}[n] = \frac{1}{2}(x[n] + x^{*}[-n])$

$\Rightarrow x_{ca}[n] = \frac{1}{2}(x[n] - x^{*}[-n])$

$\Rightarrow$ general sequence $x[n] = x_e[n] + x_o[n]$

even sequence

$x_e[n] = x_e[-n]$

$$\boxed{x_e[n] = \frac{x[n] + x[-n]}{2}}$$

odd sequence

$x_o[n] = -x_o[-n]$

$$\boxed{x_o(n) = \frac{x(n) - x(-n)}{2}}$$

$\Rightarrow$ Periodic & Aperiodic signals .

* $\tilde{x}[n] = \tilde{x}[n + kN]$ for all n is called a periodic sequence with a period N where N is a positive integer.

* A sequence is called an aperiodic sequence if it is not periodic.

⑥

⇒ Energy & power signals

Total energy of a sequence $x[n]$ is defined by

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

⇒ Average power

energy over the length $2K+1$

$$\sum_{n=-K}^{K} |x(n)|^2$$

Average power length will be

$$\frac{1}{2K+1} \sum_{n=-K}^{K} |x(n)|^2$$

→ we need to cover total power Average power.

$$\therefore \; P_{av} \triangleq \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x(n)|^2$$

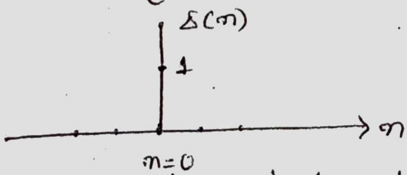
If signal is periodic signal.

$$P_{av} \triangleq \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

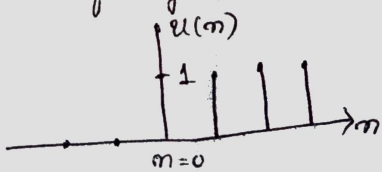
Elementary Digital signal :

(1) Digital Impulse :

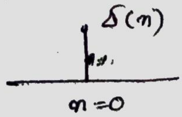
$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



(2) Digital step :- It's combination of digital Impulse & unit step.



$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u(n) = \sum_{K=0}^{\infty} \delta(n-K)$$ or $$\delta(n) = u(n) - u(n-1)$$

evaluate FT for K at $\omega = 2\pi K/N$

where K is the index for the samples.

$\therefore$ Eq^n ① will be

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N} \quad , \quad k = 0, 1 \ldots N-1$$

Subdivide the above Eq^n into an infinite number of summations. where each sum contains 'N' terms

$$X\left(\frac{2\pi}{N}k\right) = \cdots\cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$+ \sum_{n=N}^{2N-1} x(n)\, e^{-j2\pi kn/N} + \cdots\cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi}{N}k.n}$$

$\cdot$Change the index of inner sum from n to $n - lN$

| ie $n = n - lN$ ie $lN = 0$

$$\therefore X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{-j\frac{2\pi}{N}k(n-lN)}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{-j\frac{2\pi}{N}kn} + \cancel{e^{\frac{j2\pi k lN}{N}}} \nearrow 1$$

by interchanging the order of summation.

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\underbrace{\sum_{l=-\infty}^{\infty} x(n-lN)}_{x_p(n)}\; e^{-j\frac{2\pi k}{N}.n} \right]$$

for $k = 1, 2, 3 \ldots N-1$

But $x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$

$= \cdots + x(n+2N) + x(n+N) + x(n) + x(n-N) + x(n-2N) + \cdots$

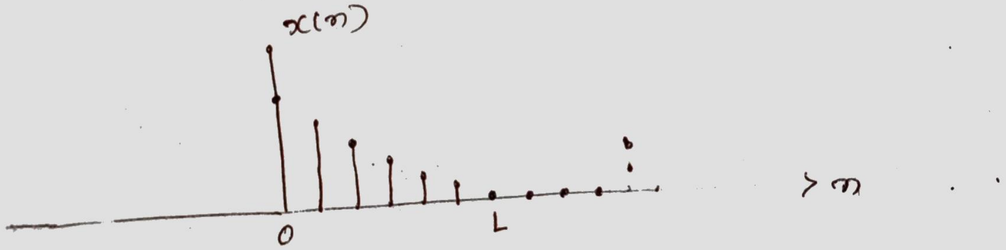
where $x_p(n) \rightarrow$ is periodic repetation of $x(n)$ with the period of N-samples.

$$\boxed{\therefore X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi k n}{N}}}$$

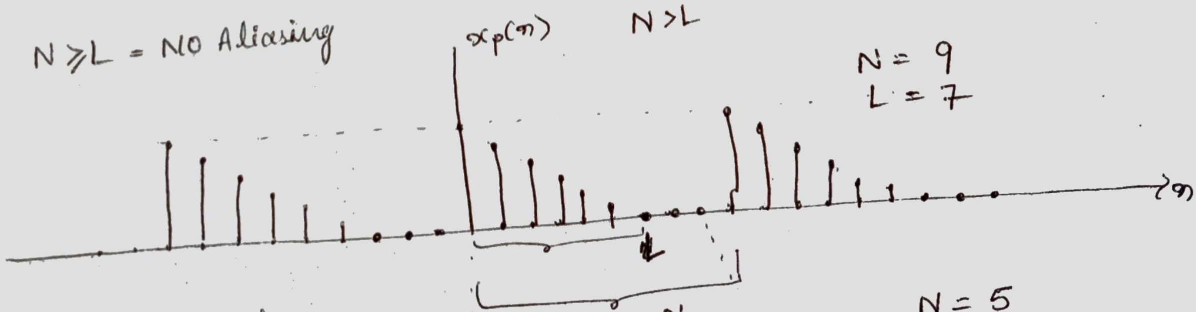
Case I : N > L ie NO Aliasing.

L → block length of signal $x(n)$

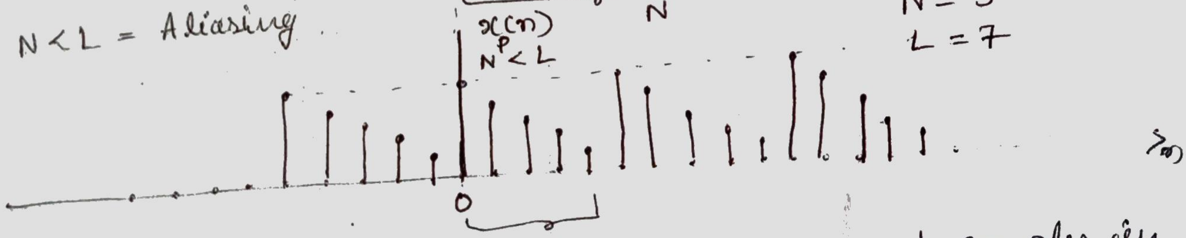
N → NO of. samples per period



N ≥ L = NO Aliasing



N < L = Aliasing.



* To avoid aliasing in time domain, the NO of samples in the frequency spectrum must be greater than NO of samples in time domain sequence ie N ≥ L

* if $N < L$, not possible to obtain $x(n)$ from $x_p(n)$

Reconstruction :

$x_p(n)$ is periodic with period N, It can be expressed by discrete Fourier series as

$$x_p(n) = \sum_{K=0}^{N-1} C(K)\, e^{j\frac{2\pi K}{N} n} \qquad n = 0,1,\dots N-1$$

where $C(K)$ = fourier co-efficients

$$C(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi K}{N} n} = X\left(\tfrac{2\pi}{N} K\right)$$

$K = 0,1 \dots N-1$

WKT $$X\left(\frac{2\pi}{N} K\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N} K n}$$

$K = 0,1,2 \dots N-1$.

$$\therefore C(K) = \frac{1}{N} X\left(\frac{2\pi}{N} \cdot K\right) \qquad K = 0,1,2 \dots N-1.$$

$$\boxed{\therefore x_p(n) = \sum_{K=0}^{N-1} \frac{1}{N} X\left(\frac{2\pi}{N} K\right) e^{j\frac{2\pi K}{N} n}} \qquad n = 0,1,2 \dots n-1$$

↳ This Eq^n gives the expression for time domain sequence $x_p(n)$ from frequency domain Samples $X\left(\frac{2\pi}{N} K\right)$

ie Time domain sequence can be reconstructed from frequency domain simply.

Def^n of DFT and IDFT :

Consider the Eq^n

$$X\left(\frac{2\pi}{N}k\right) = \sum_{k=0}^{N-1} x_p(n)\, e^{-\frac{j2\pi kn}{N}}$$

wkt no of samples in $x(n)$ are less than "N"

$\therefore$ No aliasing. , So that the above Eq^n can be written as

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{L-1} x[n]\, e^{-j\frac{2\pi kn}{N}}$$

$N > L$ to avoid aliasing in time domain $\therefore$ upper limit of summation can be made equal to $N-1$

$$\boxed{\cancel{X}(k) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n}}$$

$\rightarrow$ DFT

where $k \rightarrow$ indicates the index of freq.

by considering the periodic Eq^n

$$x_p(n) = \sum_{k=0}^{N-1} \frac{1}{N} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi kn}{N}}$$

, If we evaluate the above Eq^n for $n = 0, 1, \dots N-1$

$$\boxed{x_p(n) = x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}}$$

$n = 0, 1, 2 \dots N-1$

$\rightarrow$ IDFT

NOTE : $x(n)$ & $X(K)$ both contain "N" samples if $x(n)$ has "L" samples and $N > L$ under such condition $N-L$ zeroes are appended at the end of $x(n)$ to make it N

(OR)

If $x(n)$ is a discrete time non-periodic signal defined interval $0 \le n \le N-1$ having "N" samples then DFT is defined as

$$\boxed{X(K) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi}{N}K.n}}$$

where $X(K)$ is the frequency spectrum of $x(n)$ and it is discrete and periodic with period "N"

The IDFT is defined as

$$\boxed{x(n) = IDFT[X(K)] = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\cdot e^{\frac{j2\pi Kn}{N}}}$$

$$\left.\begin{array}{l}\text{Twiddle factor}\\ \text{or phase factor}\end{array}\right\} \Rightarrow \begin{array}{l} w = e^{-j2\pi} = \cos 2\pi - j \sin 2\pi \\ w_N = e^{-j2\pi/N} = \cos 2\pi/N - j \sin 2\pi/N \end{array}$$

① Find the 4 pt DFT of signal $x(n) = \{1, 1, 1, 1\}$

Soln

wkt $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$ where $N = 4$

(9a)

$= \sum_{n=0}^{3} x(n)\, W_4^{kn}$

$= \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi kn}{4}}$

$= x(0) W_4^0 + x(1) W_4^K + x(2) W_4^{2K} + x(3) W_4^{3K}$

$X(0) = 1+1+1+1 = 4$

$X(1) = 0+0+0+0 = 0$

$X(2) = 0+0+0+0 = 0$

$X(3) = 0+0+0+0 = 0$

$W_N = e^{-j2\pi/N} = \cos\frac{2\pi}{N} - j\sin\frac{2\pi}{N}$

$W_N^K = e^{-j\frac{2\pi K}{N}} = \cos\frac{2\pi K}{N} - j\sin\frac{2\pi}{N}k$

$W_4^0 = \cos(0) - j\sin(0) = 1 - j(0)$
$= 1$

$\Rightarrow W_4^0 = W_4^4 = W_4^8 = 1$

$\Rightarrow W_4^1 = W_4^5 = W_4^7 = -j$

$\Rightarrow W_4^2 = W_4^6 = W_4^{10} = -1$

$\Rightarrow W_4^3 = W_4^7 = W_4^{11} = +j$

$\{4, 0, 0, 0\}$

② Find the 4 pt DFT of $x(n) = \cos\left(\frac{\pi n}{N}\right)$, $x(n) = \sin\left(\frac{\pi n}{4}\right)$

$x(0) = 1$

$x(1) = 1/\sqrt{2}$

$x(2) = 0$

$x(3) = -1/\sqrt{2}$

$x(n) =$

$X_1(K) = \{1,\ 1-j1.414,\ 1,\ 1+j1.414\}$

$X_2(K) = \{2.414,\ -1,\ -0.414,\ -1\}$

(3) compute DFT of following sequences

(i) $x(n) = \{0, 1, 2, 3\}$ $\quad X(K) = \{6, -2+2j, -2, -2-2j\}$

(ii) $x(n) = \{1, 2, 3, 1\}$ $\quad X(K) = \{7, -2-j, 1, -2+j\}$

(iii) $x(n) = \{1, 0, 1, 0\}$ $\quad X(K) = \{2, 0, 2, 0\}$ $\quad$ 9b

(iv) $x(n) = \{1, -1, +1, -1\}$ $\quad X(K) = \{0, 0, 4, 0\}$

(v) $x(n) = \{1, 2, 1, 0\}$. $\quad X(K) = \{3, -2j, 0, +2j\}$

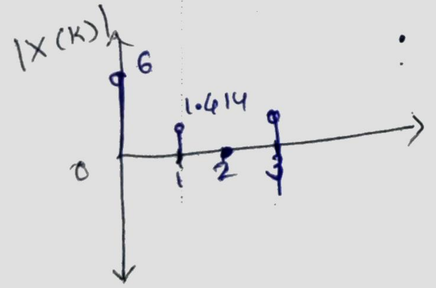
(4) $x(n) = \{1, 2, 2, 1\}$ magnitude & Phase response.

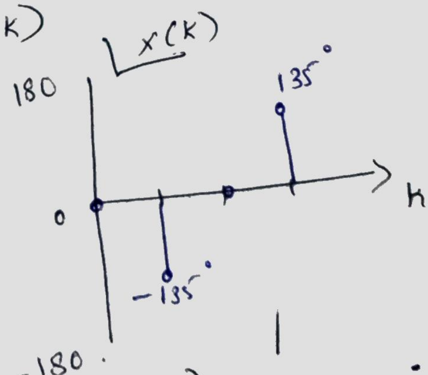
$X(K) = \{6, -1-j, 0, -1+j)$

K	$\lvert X(K) \rvert$	$\angle X(K)$
0	6	0
1	1.4142	-135° (0.7853)
2	0	0
3	1.4142	135 (−0.7853)

$$|X(K)| = \sqrt{X_R(K)^2 + X_I(K)^2}$$

$$\angle X(K) = \tan^{-1} \frac{X_I(K)}{X_R(K)}$$





$|X(K)|$ has circular even symmetry

$-\angle X(K)$

$\angle X(K)$ has circular odd symmetry.

1> Find DFT of $x(n) = \{(1+2j), (3+4j), (2-j), (3+j)\}$

$X(k) = \{(0.0001 + 0.0001j), (2.2916 + 2.2915j), (-4.583 - 0.0000j), (2.2915 - 2.2915j)\} \times 10^5$

2> Find DFT of $x(n) = \{(1+j), (1+2j), (2-j), (2+3j)\}$

$X(k) = \{(0.0001 + 0.0001j), (4.583 + 0.0000j), (0 - 0.0001j), (4.583 - 0j)\} \times 10^5$

ie $X(k) = \{(10 + 10j), 4.583 \times 10^5, -10j, 4.583 \times 10^5\}$

3> DFT of $x(n) = \{1, -2, 3, -4\}$

$X(k) = \{-2, -2-2j, 10, -2+2j\}$

5pt DFT of $x(n) = ?$

$X(k) = \{-2, (1.191 - 2.2124j), (2.309 + 7.833j), (2.309 - 7.833j), (1.191 + 2.2124j)\}$

$|X(k)| = \{2.5126, 8.1662, 8.1662, 2.5126\}$

$\angle X(k) = \{3.1416^c, -1.077^c, 1.2841^c, -1.2841^c, +1.077^c\}$

for 4 pt DFT

$|X(k)| = \{2 \quad 2.8284 \quad 10 \quad 2.8284\}$

$\angle X(k) = \{3.1416, -2.3562, 0, 2.3562\}$.

9d

find N-DFT of $x(n) = [1+j,\ 2+2j,\ -1+3j,\ 3+4j]$

$$X(K) = (1+j) + (2+2j)\,\omega_4^{K} + (-1+3j)(\omega_4^{2K}) + (3+4j)\,\omega_4^{3K}$$

$$X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j \\ 2+2j \\ -1+3j \\ 3+4j \end{bmatrix} = \begin{bmatrix} 5+10j \\ -1j \\ -5-2j \\ 4-3j \end{bmatrix}$$

$$X_1(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2+j \\ -5 \\ 2-j \end{bmatrix}$$

$$X_2(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} j \\ 2j \\ 3j \\ 4j \end{bmatrix} = \begin{bmatrix} 10j \\ -2-2j \\ -2j \\ +2-2j \end{bmatrix}$$

$$X_1(K) + X_2(K) = \begin{bmatrix} 5+10j \\ -j \\ -5-2j \\ 4-3j \end{bmatrix}$$

DFT as a Linear Transformation :-

Let $\omega_N = e^{-j2\pi/N}$ → N^{th} root of unity.

∴ DFT & IDFT will be

$$\text{DFT} \Rightarrow X(k) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn}, \quad k = 0, 1, \dots N-1$$

$$\text{IDFT} \Rightarrow x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, \omega_N^{-kn}, \quad n = 0, 1, \dots N-1$$

NOTE : $W_N^{k+N} = \omega_N^{k}$.

$$\begin{aligned} \text{LHS} &= W_N^{k+N} \\ &= e^{-j\frac{2\pi}{N}(k+N)} \\ &= e^{-j\frac{2\pi}{N}k}\, \cancel{e^{-j\frac{2\pi}{N}N}}^{\,1} \\ &= e^{-j\frac{2\pi}{N}k} \\ &= W^{k}. \end{aligned}$$

$$\left[\begin{array}{c} \cos 2\pi - j\sin 2\pi \\ 1 - 0 = 1. \end{array}\right]$$

LHS = RHS.

$W_N^{k+N} = \omega^{k}$ is periodicity Property of twiddle factor.

* Each point of the DFT is calculated using N complex multiplications and $(N-1)$ complex additions.
 ie N-point DFT can be calculated in a total of N^2 complex multiplication and $N(N-1)$ complex addition.

$$\omega_N^{k} = e^{-j\frac{2\pi}{N}k} = 1 \angle{-\frac{2\pi}{N}k}$$

$$\left.\begin{aligned} \text{DFT} &\longrightarrow X_N = [W_N]\, x_N \\ \text{IDFT} &\longrightarrow x_N = \frac{1}{N}[W_N^{*}]\, X_N \end{aligned}\right\}$$ Expressed in matrix form.

* The N-point DFT of $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \leq k \leq N-1$$

$$X(k) = x(0) W_N^0 + x(1) W_N^k + x(2) W_N^{2k} + \cdots + x(N-1) W_N^{k(N-1)}$$

If $k=0$

$$X(0) = x(0) + x(1) + x(2) + \cdots + x(N-1)$$

$k=1$

$$X(1) = x(0) + x(1) W_N^1 + x(2) W_N^2 + \cdots + x(N-1) W_N^{(N-1)}$$

$k=2$

$$X(2) = x(0) + x(1) W_N^2 + x(2) W_N^4 + \cdots + x(N-1) W_N^{2(N-1)}$$

$\vdots$

$k=N-1$

$$X(N-1) = x(0) + x(1) W_N^{N-1} + \cdots + x(N-1) W_N^{(N-1)(N-1)}$$

Above Eqⁿs in the matrix form are

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}}_{N\times 1} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-2} & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-2)} & W_N^{2(N-1)} \\ \vdots & & & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)(N-1)} & \end{bmatrix}}_{N\times N} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}}_{N\times 1}$$

$$X_N = W_N x_N \Rightarrow \text{N-point DFT}$$

where $W_N \rightarrow$ is the matrix of the linear transformation.

$\Rightarrow$ we can observe that W_N is symmetric matrix

$\Rightarrow$ By assuming inverse of W_N exists ie W_N^{-1}.

$\therefore$ × ly W_N^{-1} both the side

$$W_N^{-1} X_N = \underbrace{W_N^{-1} W_N}_{I} x_N \Rightarrow \boxed{x_N = W_N^{-1} X_N}$$

Expression for IDFT

But IDFT is expressed as

$$\boxed{x_N = \frac{1}{N} W_N^* X_N}$$

where $W_N^* \rightarrow$ is complex conjugate of the matrix W_N

$\Rightarrow$ By comparing above two Eqn.

$$\boxed{W_N^{-1} = \frac{1}{N} W_N^*} \qquad \boxed{[W_N^*] = W_N^{-kn}}$$

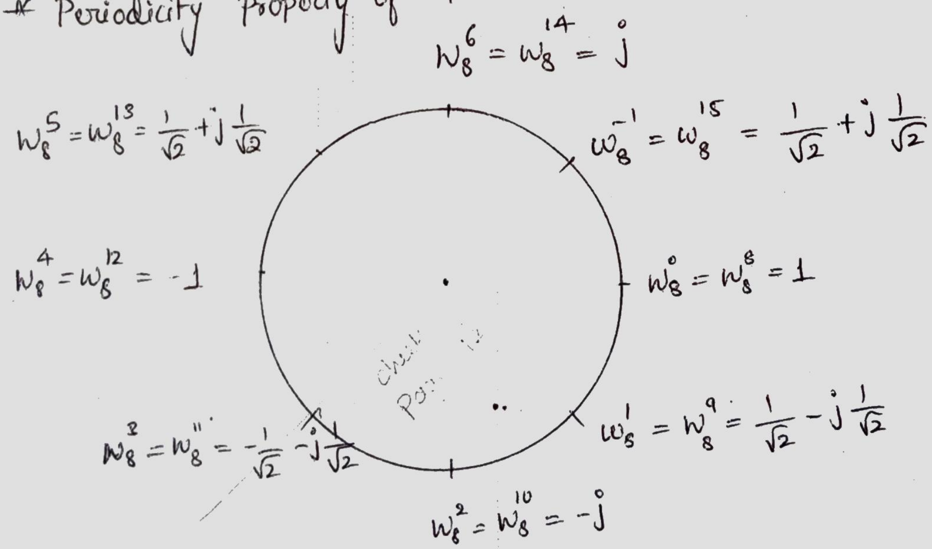
ie $\boxed{W_N W_N^* = I_N N}$

where $I_N \Rightarrow$ NXN identity matrix.

$\Rightarrow$ Matrix W_N is the transformation is an orthogonal (unitary) matrix

$\Rightarrow$ It's Inverse exists & is given as $\frac{W_N^*}{N}$

* Periodicity Property of twiddle factor (N=8)



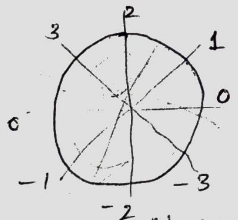
* For N=4 matrix will be

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

* Circular Representation

$$x(n) = \{-3, -2, -1, \underset{\uparrow}{0}, 1, 2, 3\}.$$

- → Right shift ⇒ rotate anticlockwise
- → Left shift ⇒ rotate clockwise
- → on circular representation, anticlockwise direction is considered as a positive and clockwise direction is negative



- → The N-point of W_N^k will lie on a unit radius Circular in the complex plane
- → The angle b/w two consegaitive points on the Circle is $2\pi/N$
- → $X(K) = X_R(K) + X_I(K)$

$$|X(K)| = \sqrt{[X_R(K)]^2 + [X_I(K)]^2}$$

$$\angle X(K) = \tan^{-1}\frac{X_I(K)}{X_R(K)}$$

(1) when $N=4$, $W_N^k = e^{-j\frac{2\pi}{N}k} = 1\angle -\frac{2\pi}{N}k$, $0 \le K \le N-1$

$$W_4^k = e^{-j\frac{2\pi}{4}k} = 1\angle -\frac{2\pi}{4}k$$

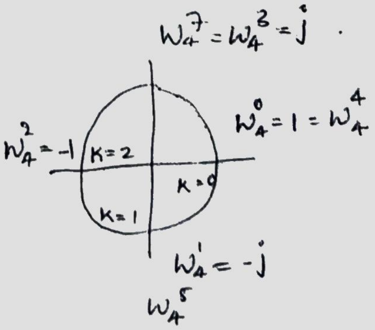
$$= 1\angle -\frac{\pi}{2}k \qquad 0 \le K \le 3$$

if $K=0$, $W_4^0 = 1\angle 0 = 1$

$K=1$, $W_4^1 = 1\angle -\pi/2 = -j$

$K=2$, $W_4^2 = 1\angle -\pi = -1$

$K=3$, $W_4^3 = 1\angle -3\pi/2 = +j$



(2) when $N = 8$

$$W_N^K = e^{-j\frac{2\pi}{N}K} = 1 \underline{\left|-\frac{2\pi}{N}K\right.}$$

$$= 1 \underline{\left|-\frac{2\pi}{8}K\right.} = 1 \underline{\left|-\pi/4 \, K\right.} \qquad 0 \le K \le 7$$

if $K=0$, $W_8^0 = 1 \underline{|0} = 1$

$K=1$, $W_8^1 = 1 \underline{|-\pi/4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$K=2$, $W_8^2 = 1 \underline{|-\pi/2} = -j$

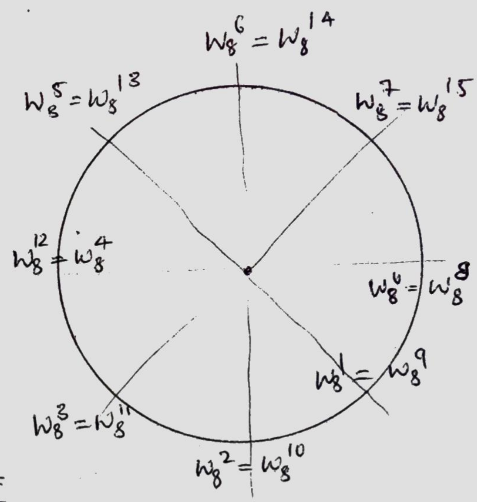
$K=3$, $W_8^3 = 1 \underline{|(-\pi/4)3} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$K=4$, $W_8^4 = 1 \underline{|-\pi} = -1$

$K=5$, $W_8^5 = 1 \underline{|-5\pi/4} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$K=6$, $W_8^6 = 1 \underline{|-3\pi/2} = j$

$K=7$, $W_8^7 = 1 \underline{|-7\pi/4} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$



*) Compute the DFT of the four-point sequence $x(n) = (0, 1, 2, 3)$ using linear transformation

$$W_N^{K+N/2} = -W_N^K$$

Periodicity property & symmetry property.

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4 x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Relationship of DFT with other transforms :-

* The DFT is an important computational tool for performing frequency analysis of signals on digital signal processors.
  ⇒ ie, It is Important to establish the relationships of the DFT to the other frequency analysis tools & transforms that have developed.

1) Relationship betweet DFT and Z-transform

Consider a sequence $x(n)$ having the Z-Transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \longrightarrow ①$$

→ with the ROC that includes the unit circle

→ If $X(z)$ is sampled at the "N" equally spaced points on the unit circle ie

$$Z_K = e^{j\frac{2\pi K}{N}}, \quad 0 \leq K \leq N-1$$

$Z = r \cdot e^{j\omega}$

where $\omega = \frac{2\pi K}{N}$

$$\therefore \quad X(K) \equiv X(z)\Big|_{z = e^{j\frac{2\pi K}{N}}} \qquad K = 0, 1 \cdots N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi nK/N} \longrightarrow ②$$

→ The discrete time fourier transform of $x(n)$ is defined as

$$X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow ③$$

is continuous and periodic with period 2π

→ Eq^n ② can be obtained from Eq^n ③ if $X(\omega)$ is uniformly sampled to get N-points at frequency $\omega = \frac{2\pi}{N} K$, $0 \leq K \leq N-1$

→ If $x(n)$ is a finite length sequence defined in the interval $0 \leq n \leq N-1$

Eqⁿ ② will be

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi K}{N}n} \longrightarrow ④$$

→ $x(n)$ can be obtained by taking IDFT

$$x(n) = \text{IDFT}[X(K)] = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad 0 \le n \le N-1$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{+j\frac{2\pi K}{N}n} \longrightarrow ⑤ \qquad 0 \le n \le N-1$$

→ By Defⁿ of z-transform

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

Substitute $x(n)$ in above Eqⁿ from Eqⁿ ⑤

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{+j\frac{2\pi Kn}{N}}\right] z^{-n}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} e^{j\frac{2\pi Kn}{N}}\, z^{-n}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1}\left(e^{j\frac{2\pi K}{N}}\, z^{-1}\right)^n$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, \frac{1-\left(e^{j\frac{2\pi K}{N}}\, z^{-1}\right)^N}{1-e^{j\frac{2\pi K}{N}}\, z^{-1}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\left[\frac{1-e^{+j\frac{2\pi KN}{N}}\, z^{-N}}{1-e^{j\frac{2\pi K}{N}}\, z^{-1}}\right]$$

$$X(z) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\left[\frac{1-z^{-N}}{1-e^{-j\frac{2\pi K}{N}}\, z^{-1}}\right]$$

$$X(z) = \frac{1-z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{[1 - e^{-j\frac{2\pi K}{N}} \cdot z^{-1}]}$$

(ii) Relation b/w DFT and DTFS :

If $x_1(n)$ is a DT periodic signal with fundamental period "N" then DTFS co-efficient representation of the periodic signal is given by

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi Kn}{N}} \longrightarrow (1)$$

where $\frac{2\pi}{N} = \Omega_0$ (or) w_0 is the angular frequency in radians.

$$\therefore x_1(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi Kn}{N}} \longrightarrow (2)$$

$x_1(n) \Rightarrow$ DTFS of periodic signal

where C_K is the fourier coefficient of $x_1(n)$ & it is discrete and periodic with period "N"

If $x(n)$ is DT finite length sequence having N-samples than DFT pair of eq^n are given by.

CTP → NP DF → DTFS
CTNP → NP CF → DTFT
DTP → P DF → DTFS
DTNP → P CF → DTFT

$$X(K) = \text{DFT}[x(n)]$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \longrightarrow (3) \quad 0 \leq K \leq N-1$$

Illy $x(n) = \text{IDFT}[X(K)]$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn} = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}Kn} \quad 0 \leq n \leq N-1$$

$$= \sum_{K=0}^{N-1} \frac{X(K)}{N} e^{j\frac{2\pi Kn}{N}} \longrightarrow (4)$$

If $x_1(n) = x(n) \quad 0 \leq n \leq N-1$

then Eq^n 1 = 3 & 2 = 4
(or) 3 in 1

$\therefore N\, C_K = X(K)$

$$\boxed{C_K = \frac{X(K)}{N}}$$

Ex: If $x(n)$ is DT periodic sequence with fundamental period $N=4$ Find the fourier series co-efficients if $x(n) = \{1, 2, -1, 2\}$

$$\text{DFT}, \quad X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \leq n \leq N-1$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -4 \\ 2 \end{bmatrix}$$

$$X[k] = [\underset{\uparrow}{4}, 2, -4, 2]$$

Fourier series co-efficient are $C_K = \dfrac{X(K)}{N} = \dfrac{X(K)}{4}$

$$C_K = \{1, 0.5, -1, 0.5\}$$

iii) Relation b/w DFT and DTFT :-

* we have seen that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$,

$X(\omega) \Rightarrow$ sampled at "N" equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$

where $k = 0, 1, \dots N-1$

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi n k}{N}} \qquad K = 0, 1, \dots N-1$$

→ DFT co-efficients of the periodic sequence of period "N" given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

→ $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \leq n \leq N-1$, The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

→ if $x(n)$ is of finite duration and length $L \leq N$

it will be

$$x(n) = \hat{x}(n) \qquad 0 \leq n \leq N-1$$

ie only in this case the IDFT of $\{X(K)\}$ yield the original sequence $\{x(n)\}$

Properties of DFT :=

(1) Periodicity :-

$$x(n+N) = x(n) \quad \forall n$$
$$X(K+N) = X(K) \quad \forall K$$

(2) Linearity :- $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$

(3) circular symmetries of a sequence :=

(i) $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$

(ii) $x(N-n) = x(n)$ [for circular even sequence]

(iii) $x(N-n) = -x(n)$ [for circular odd sequence]

(iiii) $x((-n))_N = x(N-n)$ for circular folded sequence

(4) Symmetry properties :=

(i) Symmetry property for real valued $x(n)$

$$X(N-K) = X^*(K) = X(-K)$$

(ii) $x(n)$ is real and even

$$x(n) = x(N-n)$$

then $X(K) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi Kn}{N}\right)$

(iii) $x(n)$ is real and odd

$$X(K) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi Kn}{N}$$

$$X_I(N-K) = -X_I(K)$$

(iv) If $x(n)$ is imaginary & even

$$X_I(N-K) = X_I(K)$$

(v) If $x(n)$ is imaginary & odd

$$X_R(N-K) = -X_R(K)$$

(5) Circular convolution :-

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then $x_1(n) \, Ⓝ \, x_2(n) \xleftrightarrow[N]{DFT} X_1(K)X_2(K)$

(6) Time reversal of a sequence :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-K))_N = X(N-K)$

(7) Circular time shift of sequence :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-m))_N \xleftrightarrow[N]{DFT} W_N^{km} X(K)$ (or) $X(K) e^{-j\frac{2\pi km}{N}}$

(8) Circular frequency shift :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $W_N^{-mn} x(n) \xleftrightarrow[N]{DFT} X((K-m))_N$

(or) $W_N^{-mn} x(n) = e^{\frac{j2\pi mn}{N}} \cdot x(n) \xleftrightarrow[N]{DFT} Y(K) = X((K-m))_N$

(9) Complex conjugate property :-

If $x(n)$ is complex sequence

$$x(n) \xleftrightarrow[N]{DFT} X(K)$$

(i) $$x^*(n) \xleftrightarrow[N]{DFT} X^*(N-K) = X^*((-K))_N$$

(ii) $$x^*(N-n) = x^*((-n))_N \xleftrightarrow[N]{DFT} X^*(K)$$

(10) Multiplication of two sequences. (or) Modulation property

If $$x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$$

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$$

then $$x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X_1(K) \circledast_N X_2(K)\right]$$

(11) Circular co-relation :-

$$x(n) \xleftrightarrow[N]{DFT} X(K) \ \&$$

$$y(n) \xleftrightarrow[N]{DFT} Y(K) \text{ then}$$

$$\tilde{\gamma}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(K) = X(K)\,Y^*(K)$$

where $$\tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Illy $$\tilde{\gamma}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(K) = X(K)X^*(K) = |X(K)|^2$$

12) Parseval's theorem :-

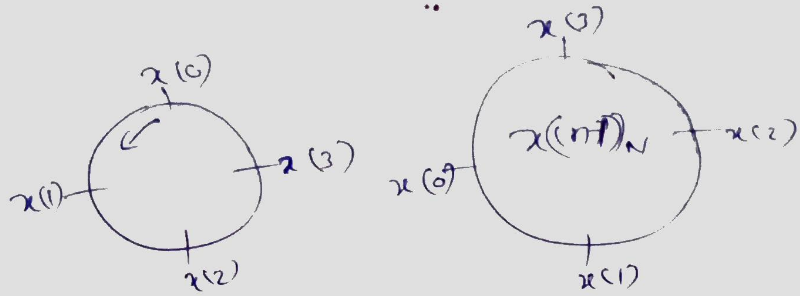
$$x(n) \xleftrightarrow[N]{DFT} X(K)$$

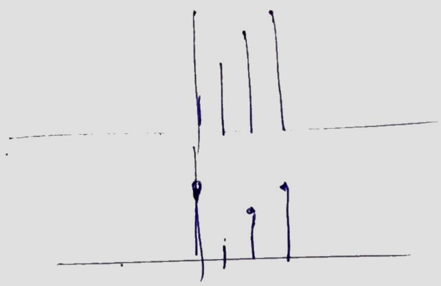
$$y(n) \xleftrightarrow[N]{DFT} Y(K)$$

$$\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K)\, Y(K)$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

(12)





$$x((n-k))_N = \sum_{0}^{N-1}$$

$$= \sum_{n=0}^{l} x((n-k))_N \, e$$

$$+ \sum_{n=l}^{N-1} x(n-k) \, e$$

* Portions → Max 2 can be up to $1\frac{1}{2}$ units
* Mention CO & Blooms levels in QP by adding extra column
* Class teacher collect student list with critical attendance status. & give to KM.
* 2 QP to be prepared for each test.
* Eligibility for make up >45 CIE.

Q. $x(n)$ be an N-point sequence & let $w(n) = \frac{1}{2}+\frac{1}{2}\cos\left[\frac{2\pi}{N}(n-N/2)\right]$ find the DFT of the windowed sequence, $x(n)\,w(n)$, from the DFT of the unwindowed sequence?

Sol^n

$$w(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\pi}{N}\left(n-\frac{N}{2}\right)\right]$$

$$= \frac{1}{2}+\frac{1}{2}\left[\frac{e^{j\frac{2\pi}{N}(n-N/2)} + e^{-j\frac{2\pi}{N}(n-N/2)}}{2}\right]$$

$$= \frac{1}{2}+\frac{1}{4}e^{j\frac{2\pi}{N}(n-N/2)} + \frac{1}{4}e^{-j\frac{2\pi}{N}(n-N/2)}$$

$$= \frac{1}{2}+\frac{1}{4}e^{j\frac{2\pi n}{N}}\cdot \underbrace{\cancel{e^{-j\frac{2\pi}{N}\times\frac{N}{2}}}}_{-1} + \frac{1}{4}e^{-j\frac{2\pi}{N}n}\,\underbrace{\cancel{e^{+j\frac{2\pi}{N}\frac{N}{2}}}}_{-1}$$

$$= \frac{1}{2}+\frac{1}{4}e^{j\frac{2\pi n}{N}}(-1) + \frac{1}{4}e^{-j\frac{2\pi}{N}n}(-1)$$

$$e^{-j\pi} = \cos\pi + j\sin\pi$$

$$e^{+j\pi} =$$

$$= \frac{1}{2}-\frac{1}{4}e^{j\frac{2\pi n}{N}} - \frac{1}{4}e^{-j\frac{2\pi n}{N}}$$

$$\therefore x(n)\,w(n) = \frac{1}{2}x(n) - \underbrace{\frac{1}{4}e^{j\frac{2\pi n}{N}}x(n)} - \underbrace{\frac{1}{4}e^{-j\frac{2\pi n}{N}}x(n)}$$

circular frequency shift

$$e^{j\frac{2\pi n}{N}}x(n) \xleftrightarrow{DFT} X((K-1))_N.$$

$$e^{-j\frac{2\pi n}{N}}x(n) \xleftrightarrow{DFT} X((K+1))_N.$$

$$\therefore e^{j\frac{2\pi mn}{N}}x(n) \xleftrightarrow{DFT} X((K-m))_N$$

$\therefore$ circular frequency shift

$$\therefore x(n)\,w(n) \xLeftrightarrow{DFT} \frac{1}{2}X(K) - \frac{1}{4}X((K-1))_N - \frac{1}{4}X((K+1))_N$$

*) The 4-point DFT of $x(n)$ is $X(K)$ find DFT of

i) $x_1(n) = x(n)(-1)^n$ ii) $x_2(n) = x(4-n)$

iii) $x_3(n) = x((n-1))_4 + x((n-2))_4$

i) $X_1(K) = DFT[x(n)[e^{jn\pi}]] = DFT[x(n)\, e^{\frac{jn\pi \cdot 4}{4}}]$

$= X(K-4)$

ii) $X_2(K) = DFT[x((-n))_N] = X(N-K) = X((-K))_N$

iii) $X_3(K) = DFT[x((n-1))_4] + DFT[x((n-2))_4]$

$= e^{-j\frac{2\pi}{4}K1} X(K) + e^{-j\frac{2\pi}{4}K2} X(K)$

$= e^{-j\frac{K\pi}{2}} X(K) + e^{-jK\pi} X(K).$

Properties of the DFT :-

* A good understanding of these properties is extremely helpful in the Application of the DFT to practical problems.

> **NOTE :** N - point DFT pair $x(n)$ & $X(k)$ is
>
> $$x(n) \xleftrightarrow[N]{DFT} X(k)$$

(1) **Periodicity** : $x(n)$ & $X(K)$ be DFT Pair.

$$x(n+N) = x(n) \quad \forall n, \text{ then}$$

$$X(k+N) = X(k) \quad \forall k, \text{ then}.$$

Proof : $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(K+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}\, W_N^{Nn} \nearrow 1$$

$$W_N^{Nn} = e^{-\frac{j2\pi Nn}{N}} = e^{-j2\pi n}$$

$$= \cos 2\pi n - j\sin 2\pi n$$

$$= 1 - j0 = 1$$

$$\therefore X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \cdot 1.$$

$$\underline{\underline{X(K+N) = X(K)}}$$

(2) **Linearity** :

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$

Proof : $$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

where $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore X(k) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + a_2 \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$$

$$X(k) = a_1 X_1(k) + a_2 X_2(k)$$

ie $x(n) \xleftrightarrow[N]{DFT} X(k)$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$$

(3) Circular symmetries of a sequence :

WKT A periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending $x(n)$ ie

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

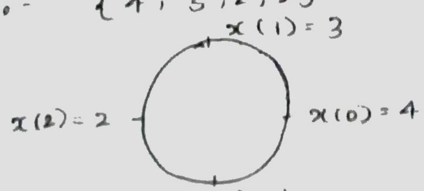
→ Let $x_p(n)$ be shifted by "k" units to the right and this sequence $x'_p(n)$

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

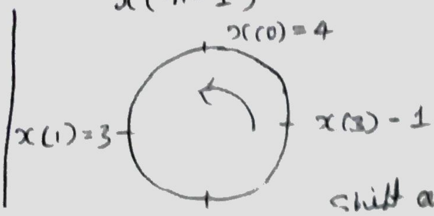
$$x'(n) = \begin{cases} x'_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$x'(n)$ is related to $x(n)$ by the circular shift.

Ex :- $\{4, 3, 2, 1\} = x(n)$



$x(n-1)$



shift anticlockwise

(3.1) Circular even sequence : sequence that is symmetric about point zero on circle

$$\text{ie } x(N-n) = x(n)$$

(3.2) Circular odd sequence : Sequence that is antisymmetric about point zero on circle

$$\text{ie } x(N-n) = -x(n)$$

(3.3) Circular folded sequence

$$x((-n))_N = x(N-n)$$

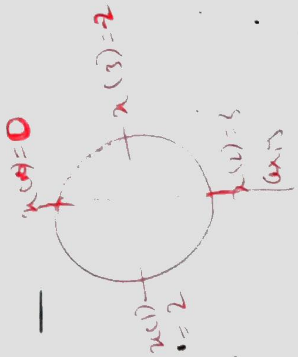
* In general the circular shift of the sequence can be represen-ted as the index modulo N

$$\therefore x'(n) = x(n-K, \text{modulo } N)$$
$$= x((n-K))_N.$$

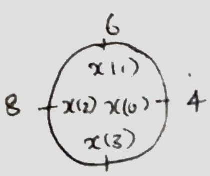
Ex :- If K=2 & N=4 then

$$x'(n) = x(0-2)_4 = x(-2) = x(2)$$
$$= x(1-2)_4 = x(-1) = x(3)$$
$$= x(2-2)_4 = x(0) = x(0)$$
$$= x(3-2)_4 = x(1) = x(1)$$

step the 0th sample about

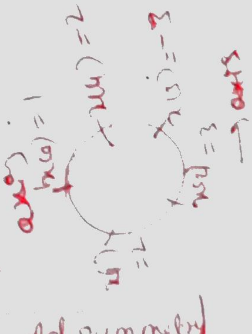


Ex : circular even $x(n) = \{4, 6, 8, 6\}$

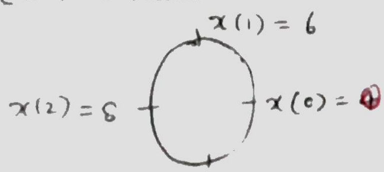


$$x(N-n) = x(n)$$
$$x(4-0) = x(4) = x(0)$$
$$x(4-1) = x(3) = x(1)$$
$$x(4-2) = x(2) = x(2)$$
$$x(4-3) = x(1) = x(3)$$

Symmetry



Ex : Circular odd $x(n) = \{0, -6, 8, 6\}$



$$x(N-n) = -x(n)$$
$$x(4-1) = -x(1) = x(3)$$
$$x(4-2) = x(2) = -x(2)$$
$$x(4-3) = x(1) = -x(3)$$

* for odd symmetry $x(0) = 0$

* For periodic sequence $x_p(n)$

even : $x_p(n) = x_p(-n) = x_p(N-n)$

odd : $x_p(n) = -x_p(-n) = -x_p(N-n)$

→ If the periodic sequence is complex valued

Conjugate even : $x_{p_e}(n) = x_p^*(N-n)$

—— " —— odd : $x_{p_o}(n) = -x_p^*(N-n)$

→ $x_p(n) = x_{p_e}(n) + x_{p_o}(n)$

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

* In general $x((n-m))_N = x((n+N-M))_N$

* $x((n))_N$ → N-point sequence plotted across the circular anticlockwise, ~~the~~ positive direction

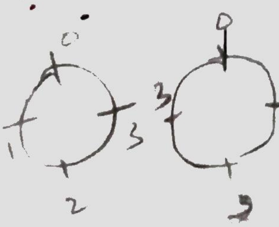
* $x((n-k))_N$ → sequence $x(n)$ shifted anticlockwise by "k" samples, it indicates delay

* $x((n+k))_N$ → Sequence $x(n)$ shifted clockwise by "k" samples which indicates advancing operation

* $x((-n))_N$ → circular folding, sequence $x(n)$ plotted across circle in clockwise direction ie in negative direction.

$(x(-k))_N = +x(k)$ → even

$(x(-k))_N = -x(k)$ → odd



$x(N-k) = x(k)$ K = 0 to N-1

$x(-k) = x(N-k)_N$

$x(4-0) = x(4) = x(0)$

$x(4-1) = x(3) = -x(1)$

$x(4-2) = x(2) = -x(2)$

$x(1) = -x(4)$

$x(1) = x(3)$

(4) Symmetry properties :-

Let $x(n)$ be complex valued and expressed as

$$x(n) = x_R(n) + j\, x_I(n) \longrightarrow (1) \qquad 0 \le n \le N-1$$

Let DFT of $x(n)$ be expressed as

$$X(k) = X_R(k) + j\, X_I(k) \longrightarrow (2) \qquad 0 \le K \le N-1$$

By defn of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}}$$

$$= \sum_{n=0}^{N-1} [x_R(n) + j\, x_I(n)]\, e^{-j\frac{2\pi Kn}{N}}$$

$$= \sum_{n=0}^{N-1} [x_R(n) + j\, x_I(n)] \left[\cos\left(\frac{2\pi Kn}{N}\right) - j \sin\left(\frac{2\pi Kn}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi Kn}{N}\right) - j\, x_R(n) \sin\left(\frac{2\pi Kn}{N}\right) + j\, x_I(n) \cos\left(\frac{2\pi Kn}{N}\right) - j^2\, x_I(n) \sin\left(\frac{2\pi Kn}{N}\right)$$

$$= \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi Kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi Kn}{N}\right) - j\left[x_R(n) \sin\left(\frac{2\pi Kn}{N}\right) - x_I(n) \cos\left(\frac{2\pi Kn}{N}\right)\right]$$

By comparing above Eqn with Eqn (2)

$$X_R(K) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi Kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi Kn}{N}\right) \longrightarrow (4) \qquad 0 \le K \le N-1$$

$$X_I(K) = -\sum_{n=0}^{N-1} x_R(n) \sin\left(\frac{2\pi Kn}{N}\right) - x_I(n) \cos\left(\frac{2\pi Kn}{N}\right) \longrightarrow (4) \qquad 0 \le K \le N-1$$

$|||^{ly}$ IDFT is

$$x_R(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_R(K)\cos\left(\frac{2\pi Kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right) \longrightarrow (5)$$

$$x_I(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_R(K)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right) \longrightarrow (6)$$

(4.1) Symmetry property for real valued $x(n)$:

$$X(N-k) = X^*(K) = X(-K)$$

Proof: $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-K)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, \cancel{W_N^{Nn}}^{\,1}\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{-Kn}$$

$$= X(-K)$$

$$X(N-K) = X^*(K)$$

4.2) If $x(n)$ is real and even :-

$$x(n) = x(N-n) \qquad 0 \le n \le N-1$$

then $X(K) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right)$

Proof: $x(n) = x_R(n) = x_R^e(n)$, $x_I(n) = 0$

Eq^n (3) & (4) becomes

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

$$X_I(k) = -\sum_{n=0}^{N-1} x_R^e(n) \sin \frac{2\pi kn}{N}$$

$$X(k) = X_R(k) + j X_I(k)$$

→ $\cos\left(\frac{2\pi kn}{N}\right)$ is an even function & $\sin\left(\frac{2\pi kn}{N}\right)$ is an odd sequence.

→ Product of two even sequence is even and
—"— odd & even sequence is odd
—"— two odd sequence is even.

Sum of all samples of an odd sequence is zero ~~product~~.

$$\therefore X_I(k) = 0$$

$$\therefore X(k) = X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

* If the time domain sequence is real & even then its DFT X(k) is also real and even.

$$\boxed{\begin{aligned} x(n) &= x_R^e(n) + x_R^o(n) + j x_I^e(n) + j x_I^o(n) \\ X(k) &= X_R^e(k) + X_R^o(k) + j X_I^e(k) + j X_I^o(k) \end{aligned}}$$

(4.3) <u>If $x(n)$ is real and odd</u> :-

If $x(n)$ is real and odd, then

$$x(n) = x_R(n) = x_R^0(n) \quad \& \quad x_I(n) = 0$$

Eq^n (3) & (4) will be

$$\left.\begin{aligned} X_R(K) &= \sum_{n=0}^{N-1} x_R^0(n) \cos\left(\frac{2\pi Kn}{N}\right) \\ X_I(K) &= \sum_{n=0}^{N-1} x_R^0(n) \sin\left(\frac{2\pi Kn}{N}\right) \end{aligned}\right\} \quad 0 \leq K \leq N-1$$

→ $x_R^0(n) \cos\left(\frac{2\pi Kn}{N}\right)$ is an odd sequence and sum of all samples of an odd sequence is zero.

i.e. $X_R(K) = 0 \quad \forall k$

→ $x_R^0(n) \sin\left(\frac{2\pi Kn}{N}\right)$ is an even sequence

$$\therefore X(K) = \cancel{X_R(K)}^{\,0} + j X_I(K)$$

$$= j \sum_{n=0}^{N-1} x(n) \sin\frac{2\pi Kn}{N}$$

$$= -j \sum_{n=0}^{N-1} x(n) \sin\frac{2\pi Kn}{N}$$

→ If the time domain sequence is real and odd then its DFT $X(K)$ is <u>purely imaginary</u> and <u>odd</u>

→ IDFT reduces

$$x(n) = j\frac{1}{N} \sum_{k=0}^{N-1} X(K) \sin\frac{2\pi kn}{N} \qquad 0 \leq n \leq N-1$$

→ In general ;, $\boxed{X_I(N-K) = -X_I(K)}$ $\qquad 1 \leq K \leq N-1$

(4.4) Purely imaginary sequence :- & even :-

$$x(n) = j\, x_I(n) = j\, x_I^e(n) \quad \& \quad X_R(n) = 0$$

Eqⁿ ③ & ④

$$\left.\begin{aligned} X_I(K) &= \sum_{n=0}^{N-1} X_I^e(n) \cos\left(\frac{2\pi}{N} kn\right) \\ X_R(K) &= \sum_{n=0}^{N-1} X_I^e(n) \sin\left(\frac{2\pi}{N} kn\right) \end{aligned}\right\} \quad 0 \le K \le N-1$$

* $X_I^e(n) \sin\left(\frac{2\pi kn}{N}\right)$ is an odd sequence, $X_R(K) = 0 \quad \forall k$

$$X_I(N-k) = X_I(k)$$

* $x(n)$ is imaginary and even, then its DFT is also imaginary and even.

(4.5) $x(n)$ is imaginary and odd then,

$$x(n) = j\, x_I(n) = j\, x_I^o(n) \quad \& \quad x_R(n) = 0$$

Eqⁿ ③ & ④ will be

$$\left.\begin{aligned} X_R(K) &= \sum_{n=0}^{N-1} x_I^o(n) \sin\left(\frac{2\pi kn}{N}\right) \\ X_I(K) &= \sum_{n=0}^{N-1} x_I^o(n) \cos\left(\frac{2\pi}{N} kn\right) \end{aligned}\right\} \quad 0 \le K \le N-1$$

* $x_I^o(n) \cos\left(\frac{2\pi kn}{N}\right)$ is an odd sequence i.e $X_I(K) = 0$

$$\boxed{\therefore \quad X_R(N-K) = -X_R(K)}$$

* $x(n)$ is imaginary & odd then its DFT is real & odd.

(5) Circular Convolution :-

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

$x_2(n) \xleftrightarrow{DFT} X_2(K)$

then $x_1(n) \circledN x_2(n) \xleftrightarrow[N]{DFT} X_1(K)\, X_2(K)$

" Multiplication of two DFT is equivalent to circular convolution of their sequences in time domain "

Proof : By defn DFT

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-\frac{j2\pi Kn}{N}} \qquad 0 \leq K \leq N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-\frac{j2\pi Kn}{N}} \qquad 0 \leq K \leq N-1$$

→ If two DFTS are Multiplied , the result is DFT

$$X_3(K) = X_1(K)\, X_2(K)$$

→ IDFT of $X_3(K)$

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K)\, e^{\frac{j2\pi Km}{N}}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X_1(K)\, X_2(K)\, e^{\frac{j2\pi Km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n)\, e^{-\frac{j2\pi Kn}{N}} \sum_{l=0}^{N-1} x_2(l)\, e^{-\frac{j2\pi Kl}{N}} \right] \cdot e^{\frac{j2\pi Km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{K=0}^{N-1} e^{\frac{j2\pi K(m-n-l)}{N}} \right]$$

* Inner sum in the bracket of above Eq^n has the form of. (28)

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a=1 \\ \dfrac{1-a^N}{1-a} & a \neq 1 \end{cases} \longrightarrow (i)$$

where $a = e^{\frac{j2\pi(m-n-l)}{N}}$

Condition (i) $a = 1 \Rightarrow \sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}}$

Condition (ii) $a \neq 1 \Rightarrow \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}}$

when $a \neq 1 \Rightarrow (m-n-l)$ is not Multiple of N. only N

from (i)

$$\therefore = \frac{1-a^N}{1-a}$$

$$= \frac{1 - e^{\frac{j2\pi k(m-n-l)\cancel{N}}{\cancel{N}}}}{1 - e^{\frac{j2\pi k(m-n-l)}{N}}}$$

$$= \frac{1-1}{1-e^{\frac{j2\pi k(m-n-l)}{N}}} = 0$$

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} = \begin{cases} N & \text{when } (m-n-l) \text{ is of "N" multiple} \\ 0 & \text{otherwise} \end{cases}$$

Substitute the above results in $x_3(m)$ Eq^n

$$x_3(m) = \frac{1}{\cancel{N}} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot \cancel{N}$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \longrightarrow$$

* Let $(m-n-l)$ Multiple of "N" can be written as "PN" where P is an integer.

→ "P" might be +ve (or) -ve

$$\therefore \quad m-n-l = -PN$$

$$l = m-n+PN$$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, \underbrace{x_2(m-n+PN)}$$

↳ Here second summation is lost because "l" doesnot exist.

* $x_2(m-n-PN)$ is a periodic sequence with period "N" this periodic sequence is delayed by "n" samples.

* $x_2(m-n+PN)$ represents sequence $x_2(m)$ shifted circularly by "n" samples.

ie $x_2(m-n+PN) = x_2(m-n, \text{modulo } N) = x_2((m-n))_N$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N, \quad 0 \le m \le N-1$$

* Eq.^{n} has the form of convolution sum, it is not the ordinary linear convolution

* Sequence $x_2(\,)$ is shifted circularly $\therefore$ it is called circular convolution

→ It takes because of circular shift of the sequence

(6) Time Reversal of a sequence :

If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

" Reversing the N-point sequence in time is equivalent to reversing the DFT values."

Proof : By Definition

$$DFT[x(n)] = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) W_N^{k(n)}$$

$\because$ circular symmetry property
$x(n) = x(N-n)$

Let $l = N-n$ (or) $n = N-l$

when $n=0$, $l=N$

$n=N-1$, $l=1$

$$DFT[x(N-n)] = \sum_{l=N}^{1} x(l) W_N^{k(N-l)}$$

$$= \sum_{l=1}^{N} x(l) \cancel{W_N^{kN}} W_N^{-kl}$$

$\rightarrow 1$

$\rightarrow 1 \ \forall l$

RHS $\times$ led by W_N^{Nl}

$$= \sum_{l=0}^{N-1} x(l) \left(W_N^{Nl}\right) W_N^{-kl}$$

$$= \sum_{l=0}^{N-1} x(l) W_N^{(N-k)l} = X(N-k)$$

$$\boxed{DFT[x(N-n)] = X(N-k) = X((-k))_N}$$

Ex: The 4-pt DFT of $x(n)$ is $X(K) = \{3, -j, 1, +j\}$
Find the DFT of the sequence $y(n)$, if $y(n)$ is
$y(n) = x((-n))_4$.

Soln : $y(n) = x((-n))_4 = x(4-n) \xleftrightarrow[N=4]{DFT} Y(K) = X((-K))_4$
$= X(4-K)$

$Y(K) = X(4-K), \quad 0 \leq K \leq 3$

$Y(0) = X(4) = X(0) = 3$

$Y(1) = X(3) = +j$

$Y(2) = X(2) = 1$

$Y(3) = X(1) = -j$

$Y(K) = \{3, j, 1, -j\}$.

(7) Circular time shift of a sequence :

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-m))_N \xleftrightarrow[N]{DFT} W_N^{km} X(k)$ (or) $X(k)\, e^{\frac{-j2\pi km}{N}}$.

ie " shifting the two sequences circularly "m" samples is equivalent to multiplying its DFT by $e^{-\frac{j2\pi km}{N}}$. "

Proof : By Def^n

$$DFT\left[x((n-m))_N\right] = \sum_{n=0}^{N-1} x((n-m))_N\, W_N^{kn} \quad , \quad 0 \le K \le N$$

But $x((n-m))_N = x(n-m+N) \quad , \quad 0 \le n \le m-1$

$= x(n-m) \quad , \quad m \le n \le N-1$

Spliting the Summation 0 to m-1 & m to N-1

$$\therefore DFT\left[x((n-m))_N\right] = \sum_{n=0}^{m-1} x(n-m+N)\, W_N^{kn} + \sum_{n=m}^{N-1} x(n-m)\, W_N^{kn}$$

let $l = n-m+N$ in first Σ, let $l = n-m$ (or) $n = l+m$ in 2nd Σ .

$n = l+m-N$

when $n=0$, $l = N-m$, when $n = m$, $l = 0$

$n = m-1$, $l = N-1$ $\quad$ $n = N-1$, $l = N-1-m$

$$DFT\left[x((n-m))_N\right] = \sum_{l=N-m}^{N-1} x(l)\, W_N^{k(l+m-N)} + \sum_{l=0}^{N-1-m} x(l) \cdot W_N^{k(l+m)}$$

$$= \sum_{l=0}^{N-m-1} x(l) \cdot W_N^{kl}\, W_N^{km} + \sum_{l=N-m}^{N-1} x(l)\, W_N^{kl}\, W_N^{km}\, \underbrace{\cancel{W_N^{-kN}}}_{} \rightarrow 1$$

$$= W_N^{km}\left[\sum_{l=0}^{N-m-1} x(l)\, W_N^{kl} + \sum_{l=N-m}^{N-1} x(l)\cdot W_N^{kl}\right]$$

$$= W_N^{km}\left[\sum_{l=0}^{N-1} x(l)\cdot W_N^{kl}\right]$$

$$\boxed{DFT\left[x((n-m))_N\right] = W_N^{km}\, X(k)}$$

$$\boxed{\therefore\ y(n) = x((n-m))_N \xleftrightarrow[N]{DFT} Y(k) = W_N^{km}\, X(k) = e^{-j\frac{2\pi km}{N}} X(k)}$$

$$|||^{ly}\quad y(n) = x((n+m))_N \xleftrightarrow[N]{DFT} Y(k) = W_N^{-km}\, X(k) = e^{j\frac{2\pi}{N}km} X(k)$$

Ex: The 4-pt DFT of $x(n)$ is given by $X(k) = \{3, -j, 1, +j\}$ without computing IDFT and DFT find the DFT of the sequence $y(n)$ if (i) $y(n) = x((n-1))_N$
(ii) $y(n) = x((n+1))_N$

Sol^n (i) $y(n) = x((n-1))_N$

By using circular time-shift property.

$$y(n) = x((n-m))_N \xleftrightarrow[N]{DFT} Y(k) = W_N^{kn} X(k) \qquad 0 \leq k \leq N-1$$

$$y(n) = x((n-1))_4 \xleftrightarrow[N=4]{DFT} Y(k) = W_4^{kn} X(k), \quad 0 \leq k \leq 3$$

$k=0$, $Y(0) = W_4^0 \cdot X(0) = 3$

$k=1$, $Y(1) = W_4^1 \cdot X(1) = -j \times -j = -1$

$k=2$, $Y(2) = W_4^2 \cdot X(2) = -1 \times 1 = -1$

$k=3$, $Y(3) = W_4^3 \cdot X(3) = j \times j = -1$

$\therefore X(k) = \{3, -1, -1, -1\}$.

(ii) $y(n) = x((n+1))_4 \xleftrightarrow[N=4]{DFT} Y(K) = W_4^{-K} X(K)$

$0 \le K \le 3$

$K=0$, $Y(0) = W_4^{0} X(0) = 3$

$K=1$, $Y(1) = W_4^{-1} X(1) = j \times j = 1$

$K=2$, $Y(2) = W_4^{-2} X(2) = -1 \times 1 = -1$

$K=3$, $Y(3) = W_4^{-3} X(3) = -j \times j = 1$

$\therefore Y(K) = \{3, 1, -1, 1\}$.

(8) Circular frequency shift :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $W_N^{-mn} x(n) \xleftrightarrow[N]{DFT} X((K-m))_N$

" Multiplication of the sequence $x(n)$ with the complex exponential sequence $e^{j2\pi Kn/N}$ is equivalent to the circular shift of the DFT by m units in frequency."

Proof : By defn

$$IDFT[X(K)] = \frac{1}{N}\sum_{K=0}^{N-1} X(K) W_N^{-Kn}, \quad 0 \le n \le N-1$$

put $K = K-m$

$$IDFT[X(K-m)] = \frac{1}{N}\sum_{K=0}^{N-1} X((K-m))_N W_N^{-Kn}, \quad 0 \le n \le N-1$$

but $X((K-m))_N = X((K-m+N))$, $0 \le K \le m-1$

$= X(K-m)$, $m \le K \le N-1$

$$\therefore IDFT[X((K-m))_N] = \frac{1}{N}\left[\sum_{K=0}^{m-1} X(K-m+N) W_N^{-Kn} + \sum_{K=m}^{N-1} X(K-m) W_N^{-Kn}\right]$$

let $l = K-m+N$

$K = l+m-N$

when $K=0$, $l = N-m$

$K = m-1$, $l = N-1$

let $l = K-m$

$K = l+m$

when $K=m$, $l=0$

$K = N-1$ $l = N-1-m$

$$\therefore \text{IDFT}\left[x((k-m))_N\right] = \frac{1}{N}\left[\sum_{l=N-m}^{N-1} x(l)\, W_N^{-(l+m-N)n} + \sum_{l=0}^{N-1-m} x(l)\, W_N^{-(l+m)n}\right]$$

$$= \frac{1}{N}\left[\sum_{l=N-m}^{N-1} x(l)\cdot W_N^{-ln}\, W_N^{-mn}\, W_N^{\overset{1}{\cancel{Nn}}} + \sum_{l=0}^{N-m-1} x(l)\, W_N^{-ln}\, W_N^{-mn}\right]$$

$$= \frac{1}{N}\, W_N^{-mn}\left[\sum_{l=0}^{N-m-1} x(l)\, W_N^{-ln} + \sum_{l=N-m}^{N-1} x(l)\, W_N^{-ln}\right]$$

$$= W_N^{-mn}\left[\frac{1}{N}\sum_{l=0}^{N-1} x(l)\, W_N^{-ln}\right]$$

$$= W_N^{-mn}\; x(n)$$

ie

$$\boxed{\begin{array}{l} \text{IDFT}\; X((k-m))_N = W_N^{-mn}\, x(n) \\ \therefore\; y(n) = W_N^{-mn}\, x(n) = e^{\frac{j2\pi mn}{N}}\cdot x(n) \xleftrightarrow[N]{\text{DFT}} Y(k) = X((k-m))_N \end{array}}$$

$|||^{ly}$ $\quad y(n) = W_N^{mn}\, x(n) = e^{-j\frac{2\pi}{N} mn}\, x(n) \xleftrightarrow[N]{\text{DFT}} Y(k) = X((k+m))_N$

(9) Complex conjugate property :-

If $x(n)$ is complex sequence &

$$x(n) \xleftrightarrow[N]{DFT} X(K) \text{ then}$$

(i) $x^*(n) \xleftrightarrow[N]{DFT} X^*(N-K) = X^*((-K))_N$

(ii) $x^*(N-n) = x^*((-n))_N \xleftrightarrow[N]{DFT} X^*(K)$

Proof : By Definition

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

if $x(n)$ is complex $\Rightarrow x^*(n)$

$$\therefore DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) \left(W_N^{-Kn}\right)^*$$

$$= \sum_{n=0}^{N-1} \left(x(n) W_N^{-Kn}\right)^*$$

$$= \sum_{n=0}^{N-1} \left(x(n) W_N^{Nn} W_N^{-Kn}\right)^* \quad \Big| \because W_N^{Nn} = 1$$

$$= \sum_{n=0}^{N-1} \left(x(n) \cdot W_N^{(N-K)n}\right)^*$$

$$DFT[x^*(n)] = X^*(N-K)$$

$$\boxed{\therefore DFT[x^*(n)] = [X[N-K]]^* = X^*(N-K) = X^*((-K))_N}$$

(ii) By defⁿ

$$\mathrm{IDFT}[X(k)] = x(n)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

$$\mathrm{IDFT}[X^*(k)] = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot (W_N^{kn})^*$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} (X(k)\, W_N^{kn})^*$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} [X(k)\, W_N^{-Nn}\, W_n^{kn}]^* \qquad \because W_N^{-Nn} = 1$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} [X(k)\, W_N^{-k(N-n)}]^*$$

$$= x^*(N-n)$$

$$\boxed{\therefore\ \mathrm{IDFT}[X^*(k)] = [x(N-n)]^* = x^*(N-n) = x^*((-n))_N}$$

(10) Multiplication of Two sequences :

(or) Modulation property

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X_1(K) \circledast_N X_2(K)\right]$

Proof: $X(K) = DFT[x(n)]$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn} \qquad 0 \leq K \leq N-1$$

let $x(n) = x_1(n)\, x_2(n)$

$$\therefore X(K) = \sum_{n=0}^{N-1} x_1(n)\, x_2(n) \cdot W_N^{Kn} \qquad 0 \leq K \leq N-1$$

$$X(K) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{L=0}^{N-1} X_1(L)\, W_N^{-Ln} \cdot \frac{1}{N} \sum_{M=0}^{N-1} X_2(M)\, W_N^{-Mn}\right] W_N^{Kn}$$

$$= \frac{1}{N^2} \sum_{L=0}^{N-1} X_1(L) \sum_{M=0}^{N-1} X_2(M) \sum_{n=0}^{N-1} W_N^{-n(L+M-K)}$$

let $a = W_N^{-(L+M-K)} = e^{j\frac{2\pi}{N}[L+M-K]}$ $\quad \therefore \sum_{n=0}^{N-1} a^n$

$$\therefore X(K) = \frac{1}{N^2} \sum_{L=0}^{N-1} X_1[L] \sum_{M=0}^{N-1} X_2[M] \sum_{n=0}^{N-1} a^n$$

$$\text{where } \sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a}, & a \neq 1 \\ N, & a = 1 \end{cases}$$

$$a^N = W_N^{-(L+M-K)N} = e^{\frac{j2\pi(L+M-K)N}{N}} = 1$$

$$\therefore \sum_{n=0}^{N-1} a^n = \begin{cases} 0, & \text{if } a \neq 1 \\ N, & \text{if } a = 1 \end{cases}$$

$\longrightarrow$ $a = 1$ then $L+M-K$ is an integer multiple of N

ie $L+M-K = NP$ where P is an integer

if $L+M-K \neq NP$ then $a \neq 1$.

$$\text{ie } \sum_{n=0}^{N-1} a^n = \begin{cases} N, & \text{if } a=1, \text{ ie } L+M-K=N \ (\text{if } P=1) \text{ or } M=((K-L))_N \\ 0, & \text{if } a\neq 1, \text{ ie } L+M-K\neq N \text{ or } M\neq((K-L))_N \end{cases}$$

$$\therefore \sum_{n=0}^{N-1} a^n = N\,\delta(M-((K-L))_N).$$

Substitue the above eqn in $X(K)$ eqn

$$X(K) = \frac{1}{N^2}\sum_{L=0}^{N-1} x_1[L] \sum_{M=0}^{N-1} x_2[M] \cdot N\,\delta(M-((K-L))_N)$$

$\therefore$ ~~$X(K)$~~ =

$$\boxed{\therefore X(K) = \frac{1}{N} X_1(K) \circledast_N X_2(K)}$$

(12) Parseval's Theorem :- Inner product property

if $x(n) \xleftrightarrow[N]{DFT} X(K)$ and .

$y(n) \xleftrightarrow[N]{DFT} Y(K)$ then.

$$\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K)\, Y(K)$$

Proof : wkt from circular correlation.

$$\tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

if $l=0 \Rightarrow \tilde{\gamma}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*(n)$

wkt $DFT\{\tilde{\gamma}_{xy}(l)\} = X(K)\, Y^*(K)$

$$\tilde{\gamma}_{xy}(l) = IDFT\{X(K)\, Y^*(K)\}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)\, e^{\frac{j2\pi K l}{N}}$$

with $l=0$, above Eqn becomes.

$$\tilde{\gamma}_{xy}(0) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)$$

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)$$

* when $x(n) = y(n)$ then the above Eqn becomes

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

which represents the energy of the sequence of $x(n)$

Circular Convolution of DFTS @ multiplication prop

$$DFT\{x_1(n)\, x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, W_N^{kn}$$

$$IDFT \text{ of } x(n) = \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln}$$

$$DFT\{x_1(n)\, x_2(n)\} \triangleq \sum_{n=0}^{N-1} x_1(n) \left\{ \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln} \right\} W_N^{kn}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \sum_{n=0}^{N-1} x_1(n)\, W_N^{(K-l)n}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, X_1((K-l))_N$$

$$= \frac{1}{N} \underbrace{\sum_{l=0}^{N-1} X_2(l)\, X_1((K-l))_N}_{\text{circular convolution}}$$

$$= \frac{1}{N} X_1(K) \circledast_N X_2(K)$$

This property is also called as modulation property.

(11) Circular correlation :-

If $x(n) \xleftrightarrow[N]{DFT} X(K)$ &

$y(n) \xleftrightarrow[N]{DFT} Y(K)$ then

$$\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(K) = X(K)\, Y^*(K)$$

where $\tilde{r}_{xy}$ is circular correlation given as

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

" ie Multiplication of DFT of one sequence & conjugate DFT of another sequence is equivalent to circular correlation of these two sequences in time domain "

Proof : $$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N \longrightarrow (1)$$

$y^*((n-l))_N$ can be written as $y^*((-(l-n)))_N$

$$\therefore \tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((-(l-n)))_N \longrightarrow (2)$$

But wkt $$x_1(m) \circledast_N x_2(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \longrightarrow (3)$$

comparing (2) and (3)

$$\tilde{r}_{xy}(l) = x(l) \circledast_N y^*(-l)$$

From circular convolution property we can write

$$DFT\{\tilde{r}_{xy}(l)\} = DFT\{x(l)\}\cdot DFT\{y^*(-l)\}$$

$$\tilde{R}_{xy}(K) = X(K)\; DFT\{y^*(-l)\}$$

wkt that by def^n of DFT we have

$$DFT\{y^*(-l)\} = \sum_{l=0}^{N-1} y^*(-l)\, e^{-\frac{j2\pi k l}{N}}$$

let $n = -l$

when $l=0$, $n=0$

$l = N-1$, $n = -(N-1)$

$$\therefore\ DFT\{y^*(-l)\} = \sum_{n=0}^{-(N-1)} y^*(n)\, e^{\frac{j2\pi k n}{N}}$$

But $y^*(n)$ is circular in nature.

$\therefore\ -(N-1) = N-1$.

$$\Rightarrow DFT\{y^*(-l)\} = \sum_{n=0}^{N-1} y^*(n)\, e^{\frac{j2\pi k n}{N}}$$

$$= \left[\sum_{n=0}^{N-1} y^*(n)\, e^{-\frac{j2\pi k n}{N}}\right]^* = [y(k)]^*$$

$$= y^*(k)$$

$$\therefore\ \tilde{R}_{xy}(k) = X(k) \cdot y^*(k)$$

$$\tilde{\gamma}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(k) = \underline{\underline{X(k)\, X^*(k)}} = |X(k)|^2$$

Properties of the DFT

Name of the Property (NOTATION)	Time domain representation $x(n), y(n)$	Frequency domain Representation $X(K), Y(K)$
(1) Periodicity	$x(n) = x(n+N)$	$X(K) = X(K+N)$
(2) Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X(K) + a_2 X_2(K)$
(3) Symmetry	$x^*(n)$ $x^*(N-n)$	$X^*(N-K)$ $X^*(K)$
(4) $\odot$lar convolution	$x_1(n) \circledast_N x_2(n)$	$X_1(K) \cdot X_2(K)$
(5) $\odot$lar time reversal	$x((-n))_N = x(N-n)$	$X((-K))_N = X(N-K)$
(6) $\odot$lar time shift	$x((n-l))_N$	$X(K) e^{-j\frac{2\pi K l}{N}}$
(7) $\odot$lar freq shift	$x(n) e^{\frac{j2\pi l n}{N}}$	$X((K-l))_N$
(8) complex conjugates Properties	$x^*(n)$ $x^*((-n))_N = x^*(N-n)$	$X^*((-K))_N = X^*(N-K)$ $X^*(K)$
(9) $\odot$lar correlation	$\tilde{r}_{xy}(l) = x(l) \circledast_N y^*(-l)$	$X(K) Y^*(K)$
(10) Multiplication of two sequence	$x_1(n) x_2(n)$	$\frac{1}{N} X_1(K) \circledast_N X_2(K)$
(11) Parsvel's theorem	$\sum_{n=0}^{N-1} \lvert x(n) \rvert^2$ $\sum_{n=0}^{N-1} x(n) y^*(n)$	$\frac{1}{N} \sum_{K=0}^{N-1} \lvert X(K) \rvert^2$ $\frac{1}{N} \sum_{K=0}^{N-1} X(K) Y^*(K)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n\ \text{even}} x(n)\, W_N^{kn} + \sum_{n\ \text{odd}} x(n)\, W_N^{kn}$$

$$= \sum_{r=0}^{N/2-1} x(2r)\, W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1)\, W_N^{k(2r+1)}$$

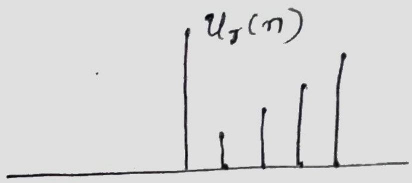
$$= \sum g_1(r)\, W_{N/2}^{kr} + W_N^{k} \sum_{r=0}^{N/2-1} g_2(r)\, W_{N/2}^{kr}$$

$$X(k) = G_1(k) + W_N^{k}\, G_2(k)$$

$$X\left(k+\frac{N}{2}\right) = G_1(k) - W_N^{k}\, G_2(k)$$

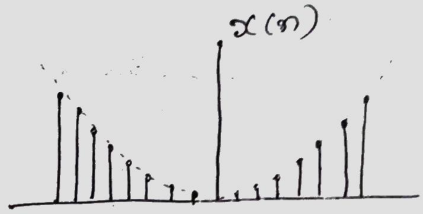
(3) <u>Unit ramp signal $u_r(n)$</u> :

$$u_r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



(4) <u>Exponential signal</u> :

$$x(n) = a^n \quad \forall n$$



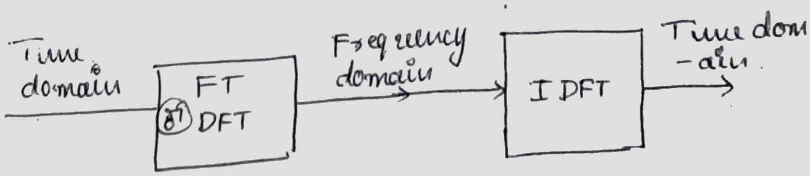
<u>Digital process</u> : Processor can be a computer (or) a mathematical tool (or) any processor that can compute (or) Perform operations on the digital signals.

- For many applications the processing of signals will be <u>appropriate</u> in frequency domain than in time domain.

  ⇒ Hence transformation of signals from time to frequency domain is required

  ⇒ Transformation can be obtained with the help of Fourier Transform (FT) & Discrete fourier transform (DFT)

- For Discrete signals DFT is used
- once the computations are done the signals are transformed back to time domain from frequency domain using Inverse Discrete fourier transform (IDFT)



Discrete Fourier Transform (DFT) :

* The frequency domain representation of a DT signal obtained using DTFT will be a continuous and periodic function of "ω" with periodicity of 2π

* To obtain discrete function of "ω", the DTFT can be sampled at sufficient number of frequency Intervals.

Frequency - Domain Sampling and Reconstruction of Discrete-Time signals :

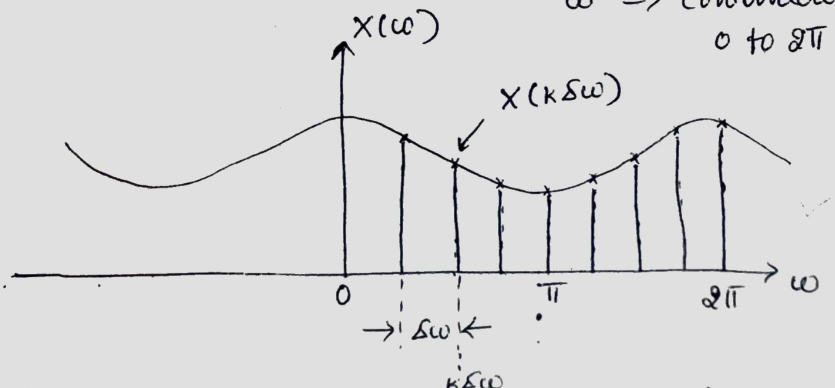
Let us consider aperiodic discrete-time signal $x(n)$ with its Fourier transformation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \longrightarrow (1)$$

where $x(n) \Rightarrow$ Aperiodic Discrete-time signal

$X(\omega) \Rightarrow$ Fourier transform of $x(n)$

$\omega \Rightarrow$ continuous time-frequency 0 to 2π



$\omega = e^{j\omega}$

$\delta\omega \rightarrow$ Spacing in radians b/w successive samples

$X(\omega) \rightarrow$ is periodic with period 2π

Let us take "N" equidistant samples in the interval $0 \le \omega < 2\pi$

$$\therefore \quad \delta\omega = \frac{2\pi}{N}$$

where $N \rightarrow$ number of samples in the frequency domain.