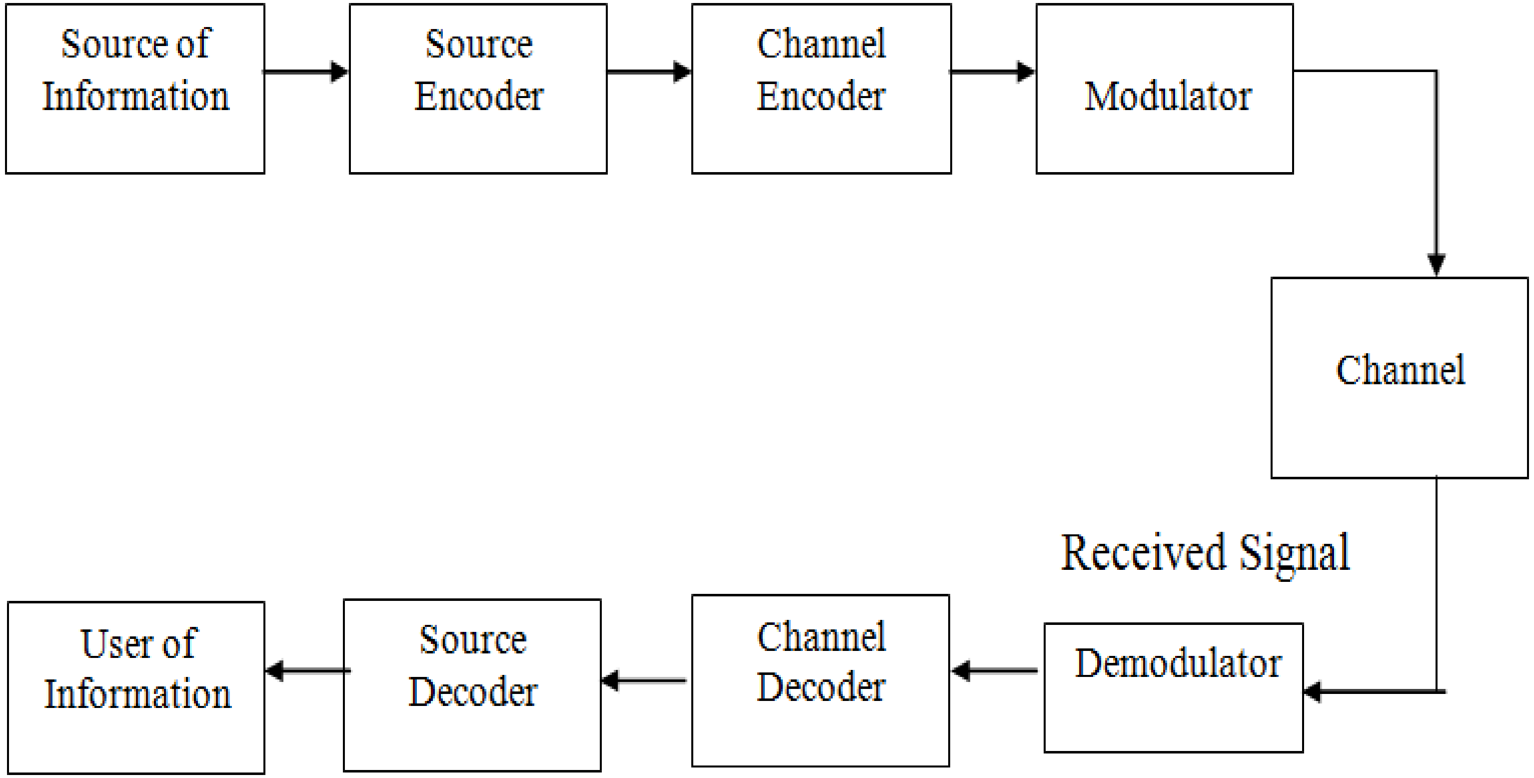
Module-2
Error Control Codes

Digital communication system



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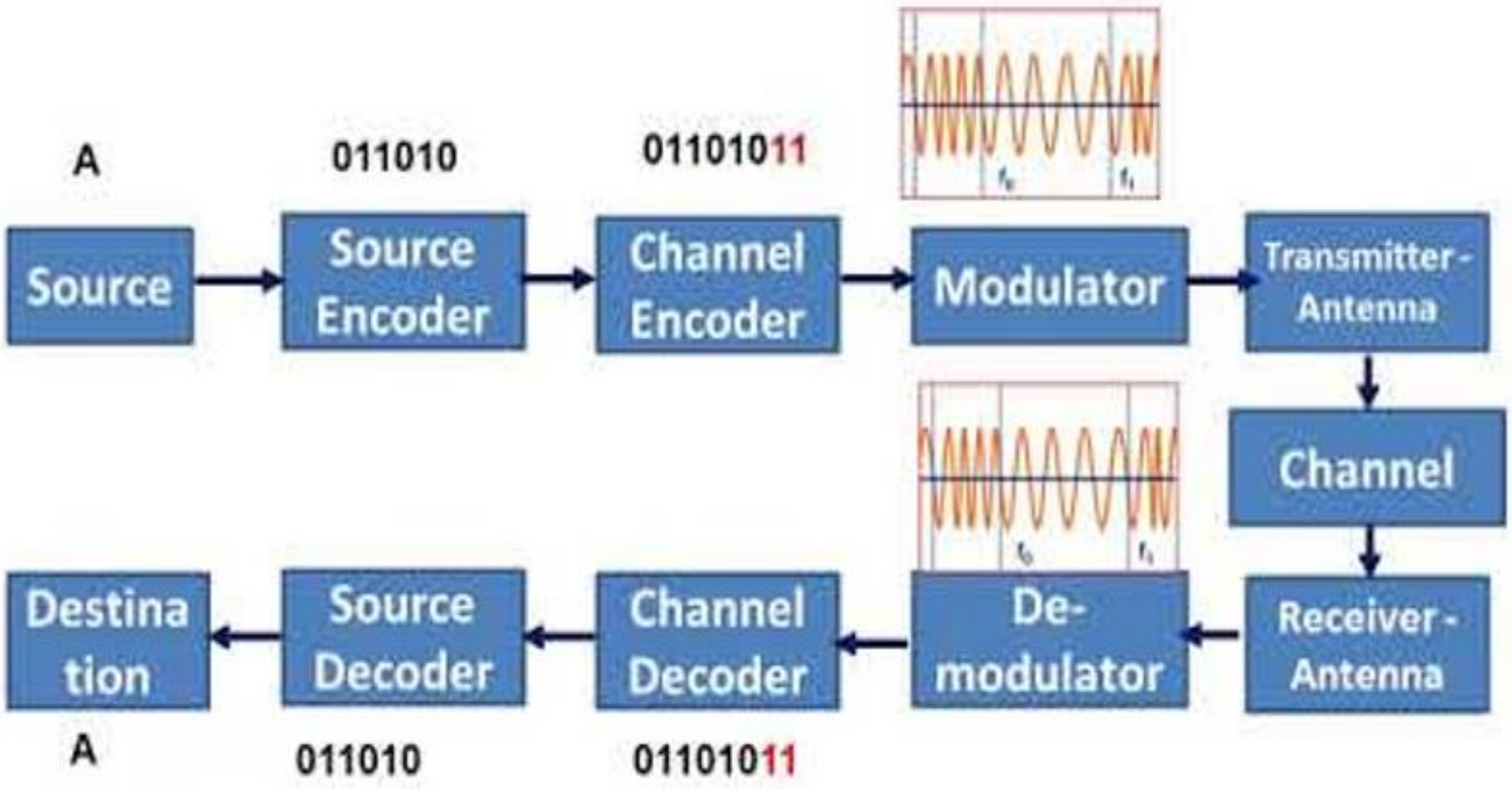
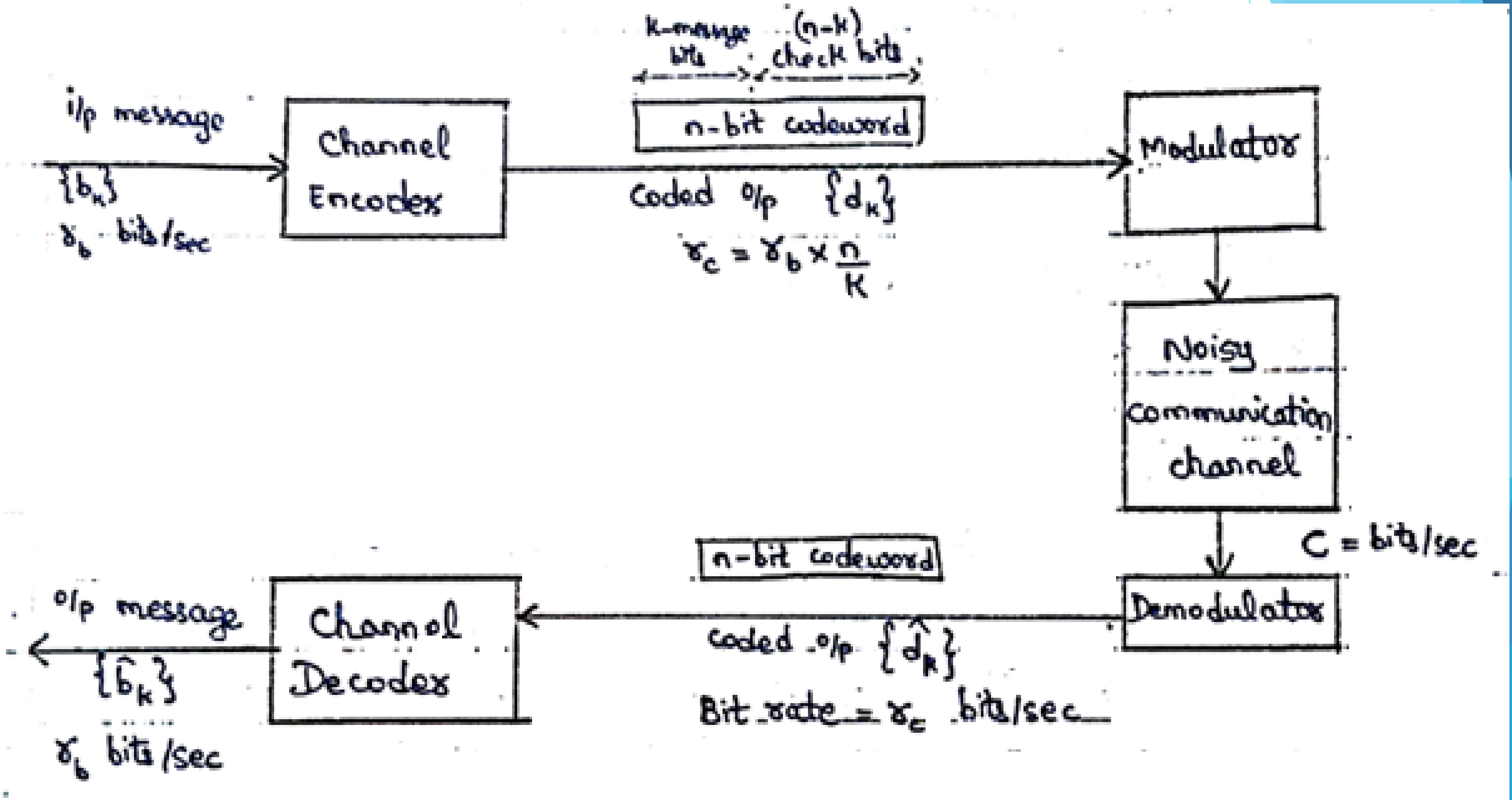


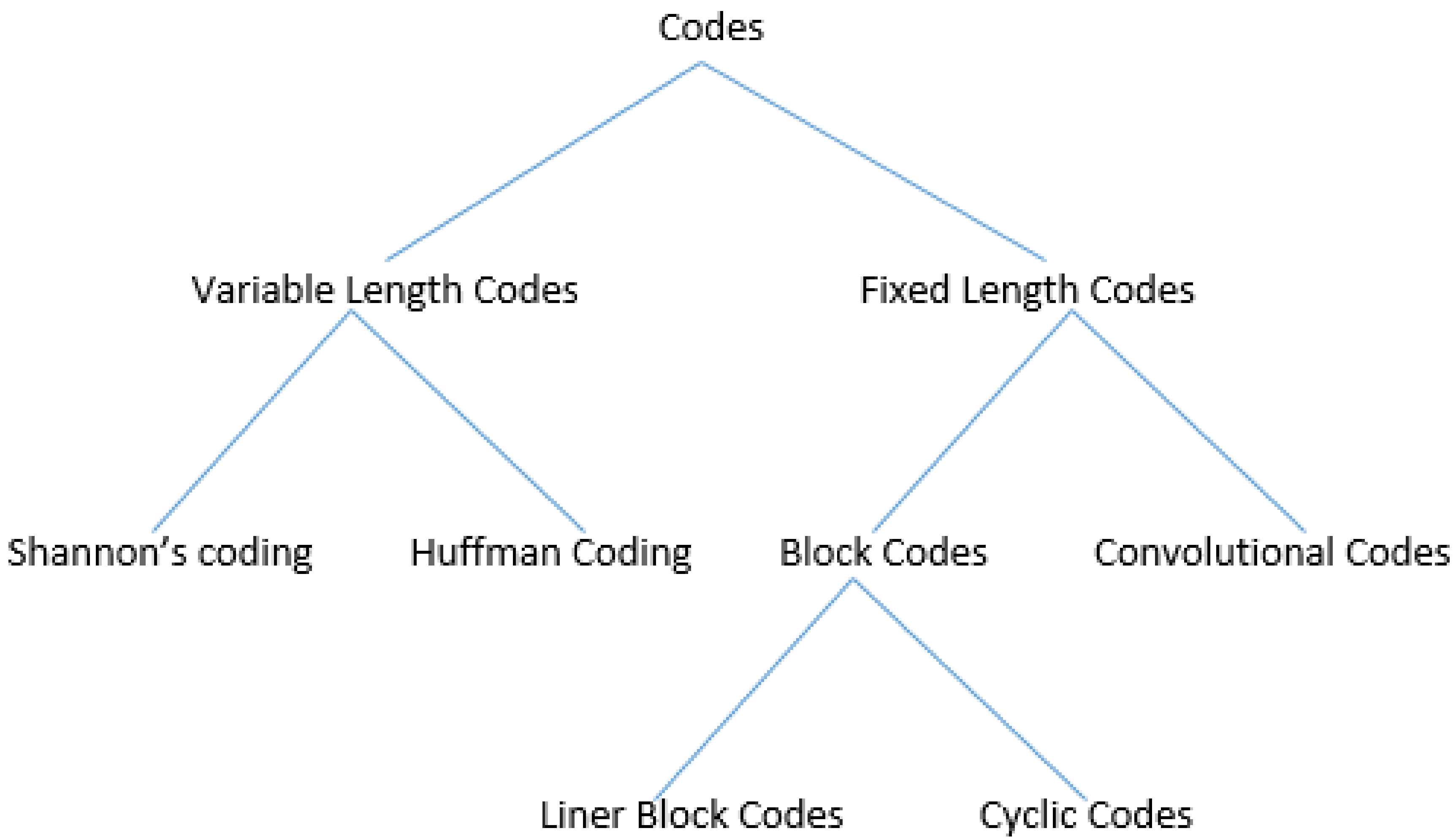
Introduction

- Variable Length Coding
  - Source coding: Minimizing the average word length of the codes so to get higher efficiency
  - Disadvantage: single error effects more than one code word

    Variable length codes is that the time will fluctuate widely.
  - Fixed length codes: A single error will effect only that block
  - To detect and correct the errors, error control coding techniques are used that rely on the systematic addition of redundant symbols
  - ECC is nothing but the calculated use of redundancy. The additional digits carry no information but makes it possible for the channel decoder to detect and correct the error in information bearing digits.
  - The additional digits are redundant digits.

Introduction




Codes
Variable Length Codes
Fixed Length Codes
Shannon’s coding
Huffman Coding
Block Codes
Convolutional Codes
Liner Block Codes
Cyclic Codes

Types of codes

- **Block Codes:**
  - Block code consists of **(n-k)** number of check bits being added to 'K' number of information bits to form '**n**' bit code words. These **(n-k)** number of check bits are derived from '**k**' information bits. At the receiver, the check bits are used to detect and correct the errors which may occur in the entire n-bit code word.
- **Convolutional Codes:**
  - In convolutional codes, the check bits are continuously interleaved with information bits. These check bits verify the information bits not only in the block immediately preceding but in other blocks also

LINEAR BLOCK CODES:-

$\underbrace{c_1 c_2 \ldots c_k}_{\text{msg. bits}}$, $\underbrace{c_{k+1} \ldots c_n}_{\text{check bits}}$

i/p message $\{b_k\}$ → Channel Encoder → $C = c_1, c_2, c_3 \ldots\ldots c_n$

$D = \{d_1, d_2, d_3 \ldots d_k\}$

k msg. bits	(n-k) check bits

(n-k) check bits	k message bits

Systematic code word

A (n, k) block code is said to be (n, k) linear block code if it satisfies the condition given below.

Let C_a & C_b be any 2 code words [n-bits] belonging to a set of (n, k) block code. Then, if $C_a \oplus C_b$ [modulo-2 arithmetic] is also a n-bit code-word belonging to the same set of (n, k) block code, then, such a block code is called (n, k) LINEAR BLOCK CODE.

Matrix Description of Linear Block Codes

- K-bits – Message vector– D={ d1 d2 d3 ......dk}
- Message Vectors-– 2^k
- Each message block is transformed to a code word of length of n-bits
- 2^k code vectors
- C= {C1 C2 C3 ....Cn}
- In a systematic linear block code, the message bits appear at beginning of code vector and remaining (n-k) bits are check bits

$$C = \{\underbrace{c_1\, c_2\, c_2 \cdots\cdots c_k}_{K \text{ message bits}}\; \underbrace{c_{k+1}\, c_{k+2} \cdots\cdots c_n}_{(n-k)\ \text{check bits}}\}$$

These $(n-k)$ check bits are derived from the k message bits using a predefined rule as given below.

$$\left.\begin{aligned}
C_{k+1} &= P_{11}d_1 + P_{21}d_2 + P_{31}d_3 + P_{41}d_4 + \cdots + P_{k1}d_k \\
C_{k+2} &= P_{12}d_1 + P_{22}d_2 + P_{32}d_3 + P_{42}d_4 + \cdots + P_{k2}d_k \\
C_{k+3} &= P_{13}d_1 + P_{23}d_2 + P_{33}d_3 + P_{43}d_4 + \cdots + P_{k3}d_k \\
&\vdots \\
C_n &= P_{1(n-k)}d_1 + P_{2(n-k)}d_2 + P_{3(n-k)}d_3 + \cdots + P_{k(n-k)}d_k
\end{aligned}\right\} \text{(iii)}$$

where $P_{11}, P_{21}, P_{12} \ldots$ are either '0' or '1' & the addition operation is modulo-2 arithmetic.

$$[c_1\; c_2\; c_3 \;\ldots\ldots\; c_n] = [d_1\; d_2\; d_3 \ldots d_k] \left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & \ldots & 0 & P_{11} & P_{12} & \ldots & P_{1(n-k)} \\ 0 & 1 & 0 & 0 & \ldots & 0 & P_{21} & P_{22} & \ldots & P_{2(n-k)} \\ 0 & 0 & 1 & 0 & \ldots & 0 & P_{31} & P_{32} & \ldots & P_{3(n-k)} \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & P_{k1} & P_{k2} & \ldots & P_{k(n-k)} \end{array}\right]$$

$$(1 \times n) = (1 \times k) \; \underbrace{(k \times k) \quad (k \times (n-k))}_{k \times n}$$

$$[C] = [D]\,[G]$$

Code word

[C] = [D] [G]

Where [G]= Generator Matrix or order **(k x n)** which consists of

identity matrix of the order **(k x k)**

Parity matrix of the order k x (n-k)

[G] = [I ¦ P] $_{k \times n}$

In this case, the message bits will be present at the end and the check bits at the beginning of the code.

Problem: The generator for a (6,3) block code is given. Find all the code vectors.

$$G = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

- The message block size for this code is 3 and
- the length of code vector ‘n’ is 6.
- Since k=3, there are $2^k = 2^3 = 8$ message vectors present.

The code vectors are found as follows:

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  The code vectors are found as follows:

$$[C] = [D]\,[G]$$

$$= \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = \left[\begin{array}{ccc:ccc} d_1 & d_2 & d_3 & (d_2+d_3) & (d_1+d_3) & (d_1+d_2) \end{array}\right]$$

Message vector $d_1\ d_2\ d_3$	Code Vector $c_1\ c_2\ c_3\ c_4\ c_5\ c_6$
0 0 0	0 0 0 0 0 0
0 0 1	0 0 1 1 1 0
0 1 0	0 1 0 1 0 1
0 1 1	0 1 1 0 1 1
1 0 0	1 0 0 0 1 1
1 0 1	1 0 1 1 0 1
1 1 0	1 1 0 1 1 0
1 1 1	1 1 1 0 0 0

It can be verified that addition of any 2 code vector is a code vector belonging to the same (6,3) code.

Ex: Consider, $C_4 \oplus C_5$

$$C_4 = 100011$$
$$C_5 = 101101$$
$$\overline{001110} = C_1$$

is a code belonging to same (6,3) code.

$\therefore$ Above code is a linear block code.

Problem: The generator for a (7,4) block code is given. Find all the code vectors.

- n= 7, k=4, (n-k)=3, $2^k = 2^4 = 16$ message vectors present

$$G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Problem: The generator for a (7,4) block code is given. Find all the code vectors.

- n= 7, k=4, (n-k)=3, $2^k = 2^4 = 16$ message vectors present

$$G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$[C] = [D][G]$$

$$= [d_1\ d_2\ d_3\ d_4]\left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$[C] = [d_1, d_2, d_3, d_4, (d_1+d_2+d_3), (d_1+d_2+d_4), (d_1+d_3+d_4)]$$

Message vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	0	1	0	1
0	0	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	1	1	0	1
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	0	1	0
1	0	1	1	1	0	1	1	0	0	1
1	1	0	0	1	1	0	0	0	0	1
1	1	0	1	1	1	0	1	0	1	0
1	1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1

For a systematic (6,3) linear block code, the parity matrix is as given. Find all the possible code vectors.

$n=6$

$k=3$

$2^3 = 8$ message vectors are present.

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For a systematic (6,3) linear block code, the parity matrix is as given. Find all the possible code vectors.

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$n = 6$

$k = 3$

$2^3 = 8$ message vectors are present.

$$[G] = [I_k \vdots P]$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = [D][G]$$

$$= [d_1\ d_2\ d_3] \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= [d_1, d_2, d_3, (d_1+d_3), (d_2+d_3), (d_1+d_2)]$$

Message ~~code~~ vectors	Code vectors
000	000000
001	001110
010	010011
011	011101
100	100101
101	101011
110	110110
111	111000

PARITY CHECK MATRIX:

The generator matrix is given by

$$[G] = [I_K \mid P]$$

$$= \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & \cdots & 0 & P_{11} & P_{12} & \cdots & P_{1(n-K)} \\ 0 & 1 & 0 & \cdots & 0 & P_{21} & P_{22} & \cdots & P_{2(n-K)} \\ \vdots & \vdots & \vdots & \cdots & 0 & P_{31} & P_{32} & \cdots & P_{3(n-K)} \\ & & & & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & P_{K1} & P_{K2} & \cdots & P_{K(n-K)} \end{array}\right]$$

The Parity Check Matrix [H] is given by

$$[H] = [P^T \mid I_{n-K}]$$

$$= \left[\begin{array}{ccccc|ccccc} P_{11} & P_{21} & P_{31} & \cdots & P_{K1} & 1 & 0 & \cdots & \cdots & 0 \\ P_{12} & P_{22} & P_{32} & \cdots & P_{K2} & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1(n-K)} & P_{2(n-K)} & P_{3(n-K)} & \cdots & P_{K(n-K)} & 0 & 0 & \cdots & \cdots & 1 \end{array}\right]$$

The [H] matrix is of order $(n-K) \times n$ and this matrix is used in error correction.

ERROR CORRECTION AND SYNDROME:-

Let $[C] = [c_1, c_2 \ldots c_n]$ be a valid code vector transmitted over a noisy communication channel belonging to a (n, K) linear block code.

Let $[R] = [r_1, r_2 \ldots r_n]$ be a received vector. Due to the noise in the channel, $r_1, r_2 \ldots r_n$ may be different from $c_1, c_2, \ldots c_n$.

The error vector or error pattern E is defined as the difference b/w R & C.

$$\therefore \quad E = R - C$$

Therefore, the error vector can be represented by

$$E = [e_1 \; e_2 \ldots\ldots e_n]$$

From above equation, it is clear that E is also a vector where $e_i = 1$ if $R \neq C$

& $e_i = 0$ if $R = C$

The 1's present in error vector E represent the error's caused by the noise in the channel.

In the equation $E = R - C$, the receiver knows only R & it doesn't know C & E. In order to find E & then C, the receiver does the decoding operation by determining a (n-K) vector S defined as

$$S = RH^T = [S_1 \; S_2 \; S_3 \ldots S_{n-K}]$$

This (n-k) vector is called ~~as~~ ERROR SYNDROME of R.

Consider $S = RH^T$

$$= [C+E][H^T]$$

$$= \cancel{CH^T} + EH^T = EH^T$$

$$\therefore \boxed{S = EH^T}$$

The receiver finds E from the above equation as S & H^T are known. Then, from the equation $R = C + E$, the transmitted code vector C can be found out.

Note that the syndrome S of the received vector will be zero if R is a valid code vector. When $R \neq C$, then $S \neq 0$. The receiver then detects & corrects the error.

For a systematic (6,3) code, find all the transmitted code vector, draw the encoding circuit if received vector $[R] = [110010]$, detect & correct the single error that has occured due to noise.

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad G = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = [D][G]$$
$$= [d_1 \; d_2 \; d_3][\quad]$$

$$[C] = [d_1 \; d_2 \; d_3 \; (d_1+d_3) \; (d_2+d_3) \; (d_1+d_2)]$$

Message vectors $d_1\ d_2\ d_3$	Code vectors
000	000000
001	001110
010	010011
011	011101
100	100101
101	101011
110	110110
111	111000

$$[H] = \left[P^T \,\vdots\, I_{n-k}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[S] = [R][H^T]$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$E = [000100]$

$C = R - E = [110010] - [000100]$

$\therefore C = [110110]$

Comparing the $[S] = [100]$ with the rows of $[H^T]$ matrix. The syndrome vector $[S] = [100]$ is present in the 4^{th} row of $[H^T]$ matrix, & hence the 4^{th} bit in the received vector counting from left is in error.

$\therefore$ The corrected code vector is $[110110]$ which is a valid transmitted code vector.

P) For a systematic (6,3) linear block code, the parity matrix is given by $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

a) Find all possible code vectors

b) Draw the encoding circuit

c) If the received code vector $[R] = [110011]$, find the syndrome, detect and correct the error.

$$[G] = [I_K : P_K]$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$[C] = [D][G]$$

$$= [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$[C] = [d_1 \; d_2 \; d_3 \; (d_1 + d_2 + d_3) \; (d_1 + d_2) \; (d_1 + d_3)]$$

Message vectors	Code vectors
000	000000
001	001101
010	010110
011	011011
100	100111
101	101010
110	110001
111	111100

$$[H] = [P^T \vdots I_{n-k}]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$[S] = [R][H^T]$$

$$= [110011] \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$E = E - R$$
$$= \{000010\} - \{110011\}$$
$$= \{110001\}$$

$$\boxed{[S] = [0\ 1\ 0]} \qquad \therefore [E] = [0.00010]$$

The syndrome vector $[S] = [010]$ is present in 5^{th} row of $[H^T]$ matrix & hence 5^{th} bit in received vector from left is in error.

$\therefore$ The corrected code vector is $[110001]$ which is a valid transmitted code vector.

For a systematic (6,3) code, the received vector $R = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6]$, construct the corresponding syndrome calculation circuit for $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$[H] = [P^T \,\vdots\, I_{n-k}]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[S] = [R][H^T]$$

$$= [r_1\ r_2\ r_3\ r_4\ r_5\ r_6] \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

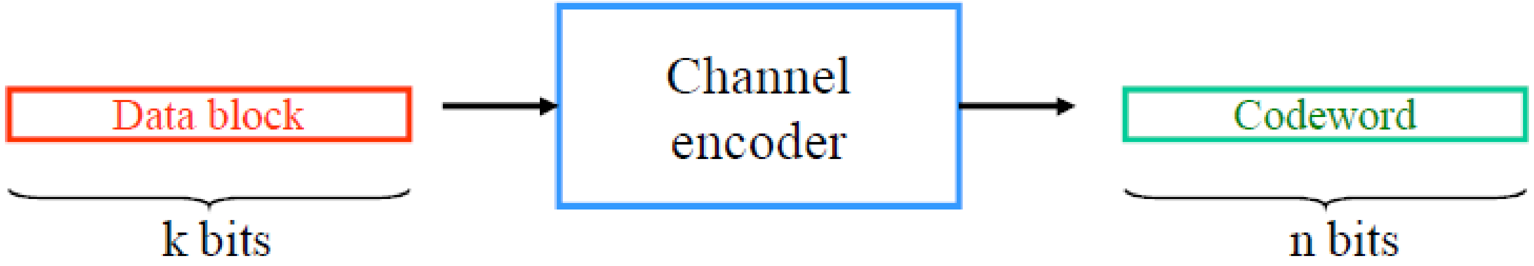
$$= [(r_1 + r_3 + r_4)\ \ (r_2 + r_3 + r_5)\ \ (r_1 + r_2 + r_6)]$$

Binary Cyclic Code

- Binary cyclic codes form the sub class of linear block code

- Advantages
  - Encoding and syndrome calculation circuit can be easily implemented by simple shift register and feedback connection and by some basic gates

  - Cyclic codes have a mathematical structure that makes it possible to design the codes with useful error correcting properly.



$n\text{-}k$ Redundant bits

$R_c = \dfrac{k}{n}$ Code rate

Algebraic structure of cyclic codes

"A (n, k) linear block code is said to be cyclic code if every cyclic shift of the code is also a code vector."

Ex: If $C_1 = 0111110$

$C_2 = 0011111$

$C_3 = 1001111$

$C_4 = 1100111$

⋮ ⋮

If $C_1, C_2, C_3 \ldots$ are also code vectors belonging to the same code, then the code is called CYCLIC CODE.

In general, let the n-bit vector be represented as

$$V = (V_0\ V_1\ V_2 \ldots\ldots V_{n-1})$$

$$V(1) = (V_{n-1}\ V_0\ V_1\ V_2 \ldots\ldots V_{n-2})$$

$$V(2) = (V_{n-2}\ V_{n-1}\ V_0\ V_1 \ldots\ldots V_{n-3})$$

$$\vdots \qquad \vdots$$

$$V(i) = (V_{n-i}\ V_{n-i+1} \ldots\ldots V_0\ V_1\ V_2 \ldots\ldots V_{n-i-1})$$

These equations which are obtained by shifting the 'V' vector cyclically successively are also the code vectors 'C'. This property of cyclic codes also allows to treat the elements of each code vector as a the co-efficients of polynomial of degree ~~(n-1)~~. n

$\therefore$ The equation will be

$$V(x) = V_0 + V_1 x + V_2 x^2 + V_3 x^3 + \ldots\ldots + V_{n-1} x^{n-1}$$

$$V^1(x) = V_{n-1} + V_0 x + V_1 x^2 + V_2 x^3 + \ldots\ldots + V_{n-2} x^{n-1}$$

$$V^2(x) = V_{n-2} + V_{n-1} x + V_0 x^2 + V_1 x^3 + \ldots\ldots + V_{n-3} x^{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$V^i(x) = V_{n-i} + V_{n-i+1} x + V_{n-i+2} x^2 + V_{n-i+3} x^3 + \ldots\ldots + V_{n-i-1} x^{n-1}$$

Cyclic Codes

Binary field :

– The set {0,1}, under modulo 2 binary addition and multiplication forms a field.

Addition	Multiplication
$0 \oplus 0 = 0$	$0 \cdot 0 = 0$
$0 \oplus 1 = 1$	$0 \cdot 1 = 0$
$1 \oplus 0 = 1$	$1 \cdot 0 = 0$
$1 \oplus 1 = 0$	$1 \cdot 1 = 1$

– Binary field is also called Galois field, GF(2).

Modulo-2 Algebra

Find the product of polynomials $f_1(x) = x+1$ & $f_2(x) = x^3+x+1$ using modulo-2 algebra.

$$f_1(x) \cdot f_2(x) = (x+1)(x^3+x+1)$$

$$= x^4 + x^2 + x + x^3 + x + 1$$

$$= x^4 + x^2 + x^3 + x(\overset{0}{\cancel{1+1}}) + 1$$

$$= \underline{x^4 + x^3 + x^2 + 1}$$

Multiply $f_1(x) = 1+x+x^3$ and $f_2(x) = 1+x+x^2+x^4$

$$f_1(x)\cdot f_2(x) = (1+x+x^3)(1+x+x^2+x^4)$$

$$= 1+x+x^2+x^4+x+x^2+x^3+x^5+x^3+x^4+x^5+x^7$$

$$= 1+x\,(\overset{0}{\cancel{1+1}})+x^2(\overset{0}{\cancel{1+1}})+x^3(\overset{0}{\cancel{1+1}})+x^4(\overset{0}{\cancel{1+1}})+x^5(\overset{0}{\cancel{1+1}})+x^7$$

$$= \underline{\underline{1+x^7}}$$

Divide $f_2(x) = x^6+x^5+x^2$ by $f_1(x) = x^3+x+1$

$$
\begin{array}{r|l}
 & x^3+x^2+x \longrightarrow \text{Quotient polynomial} \\
\hline
x^3+x+1 & \cancel{x^6}+x^5+x^2 \\
 & \cancel{x^6}+x^4+x^3 \\
\hline
 & \cancel{x^5}+x^4+\cancel{x^3}+\cancel{x^2} \\
 & \cancel{x^5}+\cancel{x^3}+\cancel{x^2} \\
\hline
 & \cancel{x^4} \\
 & \cancel{x^4}+x^2+x \\
\hline
 & x^2+x \longrightarrow \text{Remainder polynomial}
\end{array}
$$

Properties of cyclic codes

i) For a (n, k) cyclic code, there exists a generator polynomial of degree $(n-k)$ given by $g(x)$

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_{n-k} x^{n-k}$$

ii) The generator polynomial $g(x)$ of a (n, k) cyclic code is a factor of x^n+1

i.e., $x^n + 1 = g(x)\, h(x)$

where, $h(x)$ is another polynomial of degree 'k' called PARITY-CHECK Polynomial.

iii) If $g(x)$ is a polynomial of degree $(n-k)$ & is a factor of x^n+1, then it generates the (n, k) cyclic code.

iv) The code vector polynomial can be found using

$$V(x) = D(x) \cdot g(x)$$

where $D(x) = d_0 + d_1 x + d_2 x^2 + \ldots\ldots + d_{k-1} x^{k-1}$ of degree 'k'. is the message vector polynomial.

This method generates NON-SYSTEMATIC CYCLIC codes.

v) To generate a systematic cyclic code, the remainder polynomial $R(x)$ is obtained from the division of

$$\frac{x^{n-k} D(x)}{g(x)} = R(x)$$

The co-efficients of $R(x)$ are placed in beginning of code vector followed by co-efficients of message polynomial $D(x)$ to get the code vector.

n-bit code vector	
Co-efficients of $R(x)$	Co-efficients of $D(x)$

For (7,4) single error correcting cyclic code, $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$ and $x^n + 1 = x^7 + 1 = (1+x+x^3)(1+x+x^2+x^4)$. Using the generator polynomial $g(x) = (1+x+x^3)$, find all 16 code vectors of cyclic code both in NON-SYSTEMATIC & SYSTEMATIC form.

Non-systematic form:

$V(x) = D(x)g(x)$

Consider the message vector $D = [1011]$

The message vector polynomial $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$

$= 1 + 0 \cdot x + 1 \cdot x^2 + 1 \cdot x^3$

$= 1 + x^2 + x^3$

$V(x) = D(x) \cdot g(x)$

$V(x) = (1 + x^2 + x^3)(1 + x + x^3)$

$= 1 + x + x^3 + x^2 + x^3 + x^5 + x^3 + x^4 + x^6$

$= 1 + x + x^2 + x^3 (1+1+1) + x^4 + x^5 + x^6$

$= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

$\therefore V = [1111111]$ is the code vector

Let $D = 1001$

$\therefore D(x) = 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3$

$= 1 + x^3$

$V(x) = (1 + x^3)(1 + x + x^3)$

$= 1 + x + x^3 + x^3 + x^4 + x^6$

$= 1 + x + x^4 + x^6$

$\therefore V = [1100101]$

Systematic form:

$$\frac{x^{n-k}\, d(x)}{g(x)} = R(x)$$

Let $D = 1011 \Rightarrow D(x) = 1 + x^2 + x^3$

$$\therefore \frac{x^3(1+x^2+x^3)}{1+x+x^3} = \frac{x^3+x^5+x^6}{1+x+x^3}$$

$$\begin{array}{r|l}
 & x^3+x^2+x+1 \\
\hline
x^3+x+1 & \cancel{x^6}+x^5+\cancel{x^3} \\
 & \cancel{x^6}+x^4+\cancel{x^3} \\
\hline
 & \cancel{x^5}+x^4 \\
 & \cancel{x^5}+x^3+x^2 \\
\hline
 & \cancel{x^4}+x^3+x^2 \\
 & \cancel{x^4}+x^2+x \\
\hline
 & \cancel{x^3}+\cancel{x} \\
 & \cancel{x^3}+\cancel{x}+1 \\
\hline
\end{array}$$

$$\therefore R(x) = 1 = R_0 + R_1 x + R_2 x^2$$

$$\therefore R = [100]$$

$$D = [1011]$$

$$C = [R \,\vdots\, D] = [1001011]$$

Let $D = 1001$

$$D(x) = 1 + x^3$$

$$\therefore \frac{x^3(1+x^3)}{1+x+x^3} = \frac{x^3+x^6}{1+x+x^3}$$

$$
\begin{array}{r|l}
 & x^3 + x \\
\hline
x^3+x+1 & \cancel{x^6} + \cancel{x^3} \\
 & \cancel{x^6} + x^4 + \cancel{x^3} \\
\hline
 & \cancel{x^4} \\
 & \cancel{x^4} + x^2 + x \\
\hline
 & x^2 + x
\end{array}
$$

$$\therefore R(x) = x^2 + x = R_0 + R_1 x + R_2 x^2$$

$$R = [0\,1\,1]$$

$$\therefore C = [0\,1\,1\,1\,0\,0\,1]$$

Encoding using (n-k) bit shift register

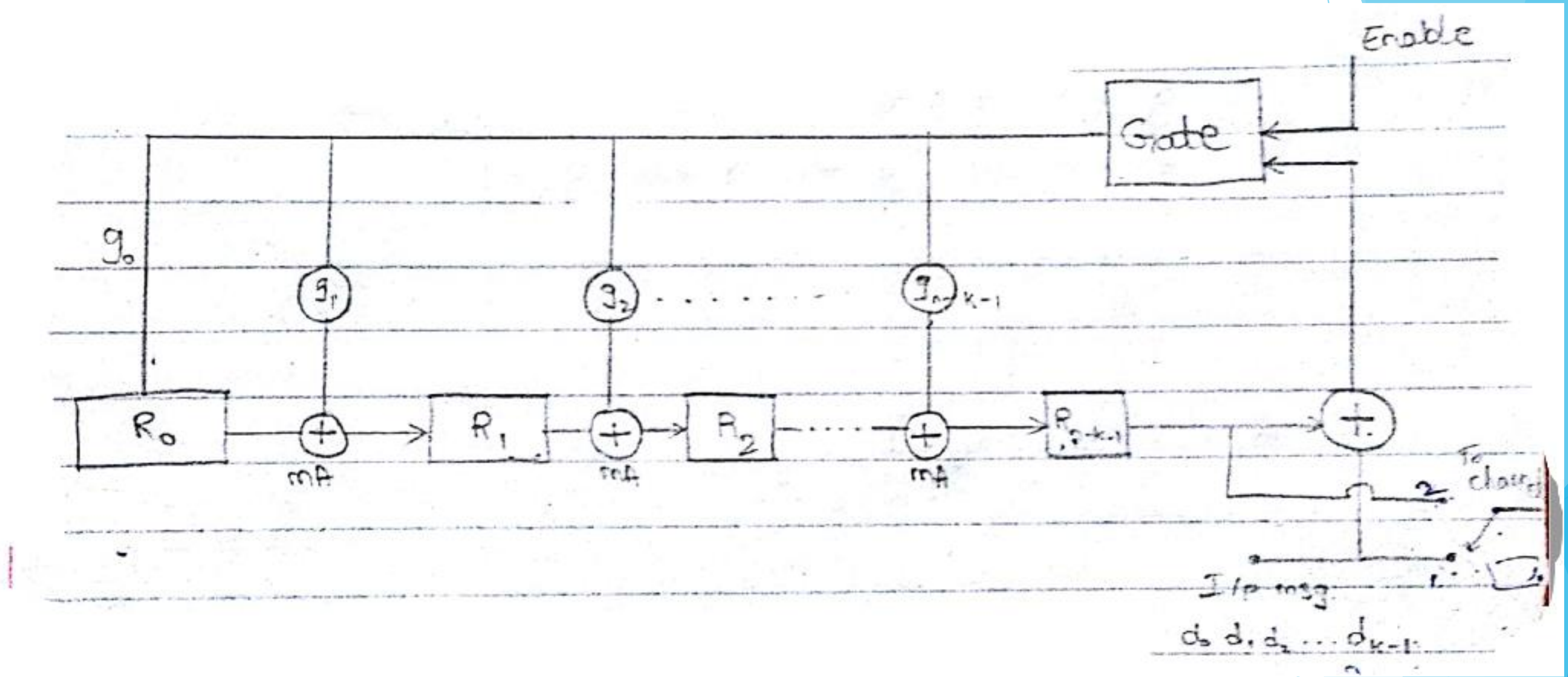
In order to obtain the remainder polynomial $R(x)$, the division of $x^{n-k} D(x)$ by generator polynomial $g(x)$ is done to calculate the parity check polynomial $v(x)$. The hardware required to implement the encoding system consists of

i) $(n-k)$ bit shift register

ii) $(n-k)$ modulo-2 adder

iii) AND gate

iv) Counter to keep track of shifting operation.


Enable
Gate
g_0
g_1
g_2
g_{n-k-1}
R_0
R_1
R_2
R_{n-k-1}
mA
mA
mA
To channel
2
1
I/p msg
$d_0\ d_1\ d_2 \ldots d_{k-1}$

Design an encoder for the (7,4) binary cyclic code generated by g(x)=1+x+x3 and verify its operation using the message vectors (1 0 0 1) and (1 0 1 1)

In general, the generator polynomial is represented as

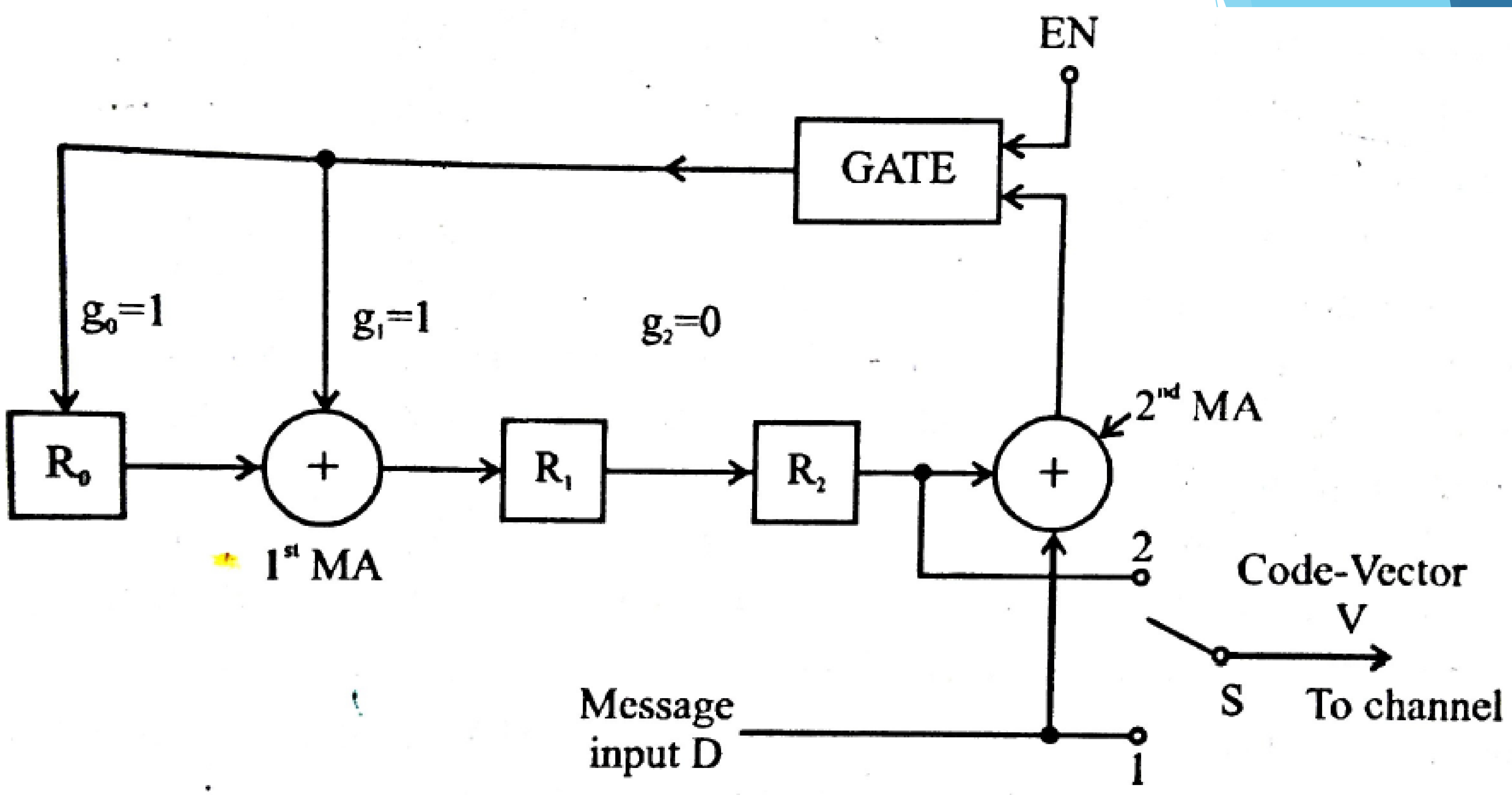
$$g(x) = g_0 + g_1x + g_2x^2 + \dots + g_{n-k}x^{n-k}$$

In the problem given, for (7, 4) cyclic code,

$$g(x) = 1 + x + x^3$$

Comparing the coefficients in equation

$g_0 = 1$, $g_1 = 1$, $g_2 = 0$ and $g_3 = 1$



EN
GATE
$g_0=1$
$g_1=1$
$g_2=0$
R_0
+
R_1
R_2
+
2^{nd} MA
1^{st} MA
2
Code-Vector
V
S
To channel
Message
input D
1

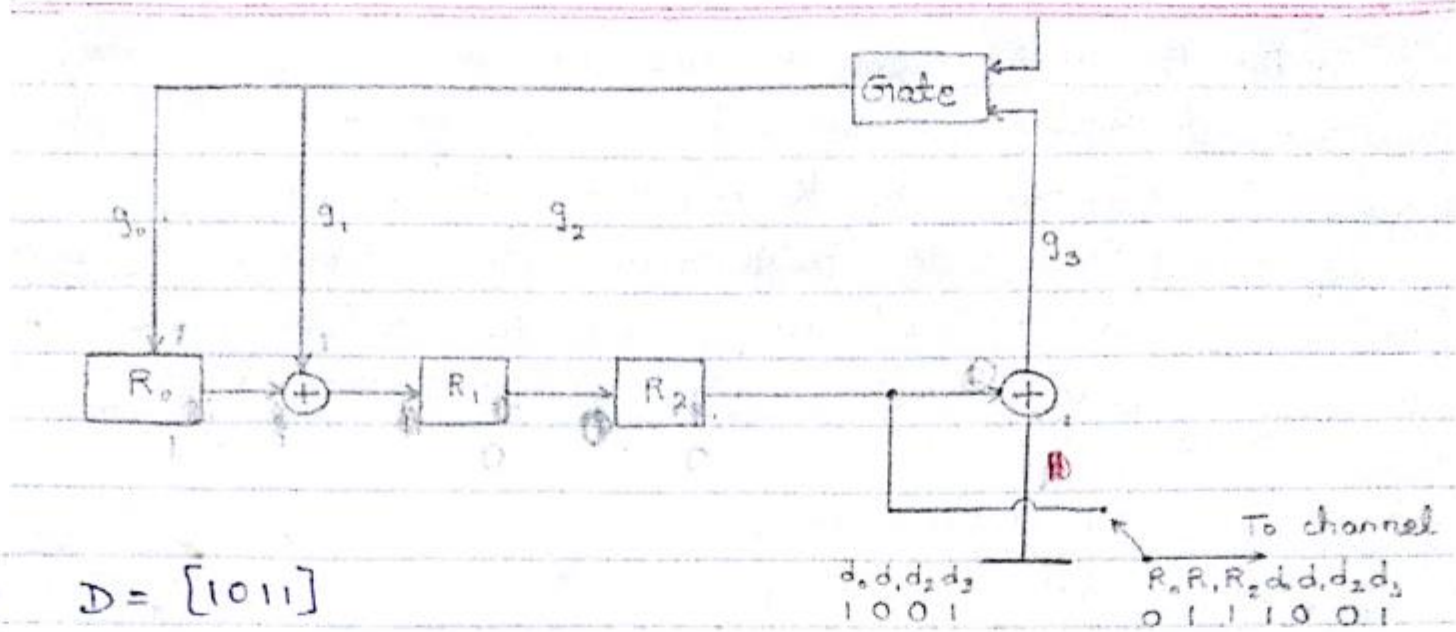
Switch position	No. of Shifts	Input	Shift Register Contents			Remainder
			R_0	R_1	R_2	
			0	0	0	
Switch S in Position 1 Gate is On	1	1	1	1	0	-
	2	0	0	1	1	-
	3	0	1	1	1	-
	4	1	0	1	1	-
Switch S in Position 2 Gate is Off	5	x	0	0	1	1
	6	x	0	0	0	1
	7	x	0	0	0	0

0111001

For the message D = [1 0 0 1], the shift register contents are shown in table

Number of shifts	Input D	Shift Register Contents R_0	R_1	R_2	Remainder bits → R
Initialisation → switch S is in position-1 and gate is turned ON		0	0	0	–
1	1	1	1	0	–
2	0	0	1	1	–
3	0	1	1	1	–
4	1	0	1	1	–
Switch S moves to position-2 and gate is turned OFF					
5	X	0	0	1	1
6	X	0	0	0	1
7	X	0	0	0	0

Contents of shift register in the encoder of figure 6 for message sequence D = 1001

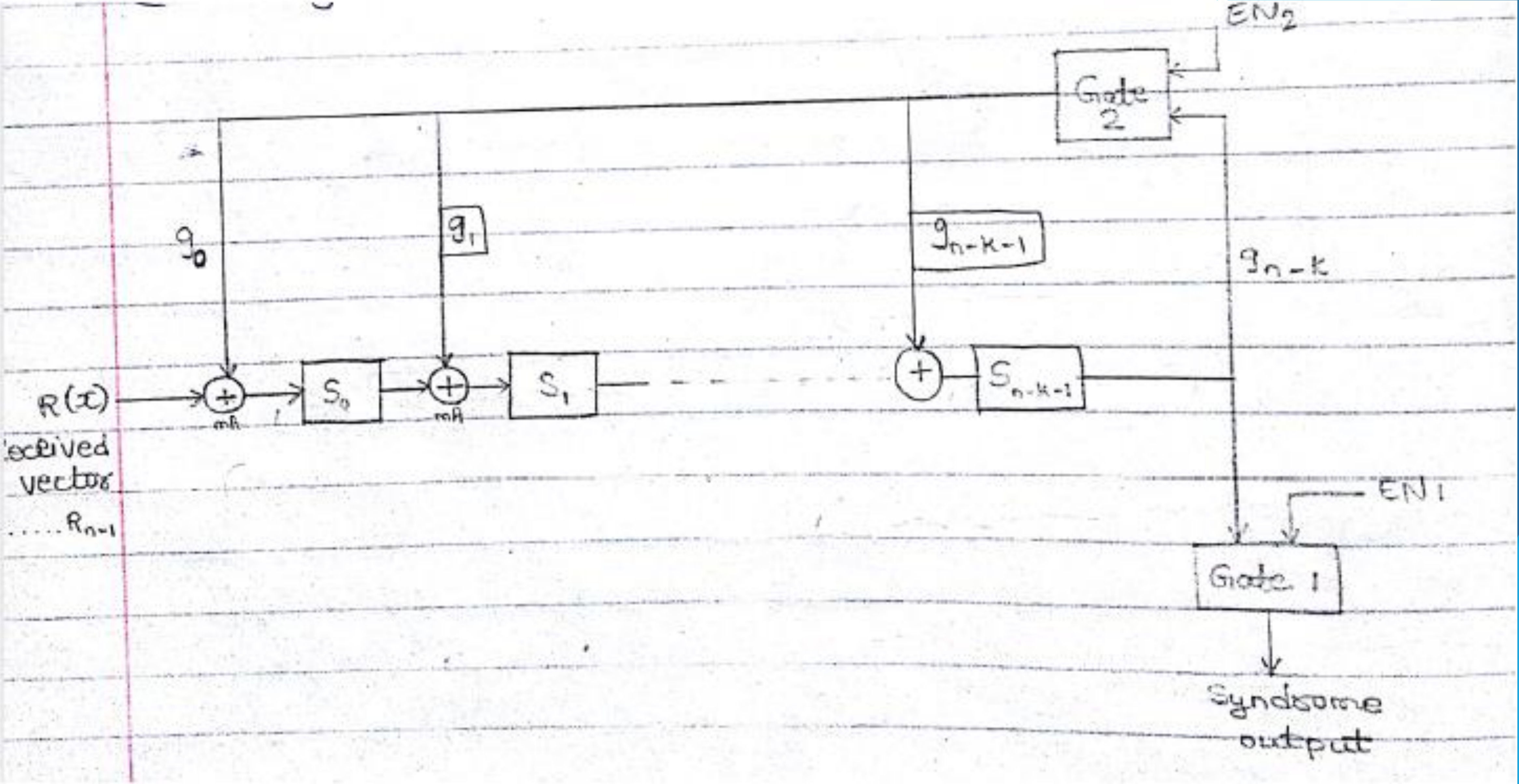


ii) D = [1011]

	No of Shifts	I/P	SR contents R_0	R_1	R_2	Remainder
Switch S in position 1 & gate ON	1	1	0	0	0	—
	2	1	1	1	0	—
	3	0	1	0	0	—
	4	1	1	0	0	—
Switch S in position 2 & gate OFF	5	X	0	1	0	0
	6	X	0	0	1	0
	7	X	0	0	0	1

Syndrome calculation circuit

- If V(x) is the transmitted code vector and R(x) is received code vector and if V(x)=R(x), then the syndrome polynomial S(x)=0
- If V(x) not equal to R(x), then S(x) is not zero
- To calculate syndrome polynomial the received code vector is divided by generator polynomial. If remainder of division is zero then there is no error in received code vector
- To determine the coefficients of syndrome polynomial the dividing circuit for a (n-k) cyclic code is shown


EN2
Gate 2
g_0
g_1
g_{n-k-1}
g_{n-k}
$R(x)$
eceived vector
..... R_{n-1}
S_0
S_1
S_{n-k-1}
EN1
Gate 1
Syndrome output

output

with Gate 1 turned OFF & Gate 2 turned ON, the received code vector is loaded into the shift register with (R_{n-1}) as first digit. At the end of 'n'

clock pulses, the flip-flops will have the co-efficients of syndrome polynomial. After the message is loaded into the shift register, gate 2 is turned OFF & gate 1 is turned ON and the information present in syndrome calculating circuit is shifted to an error detection & correction circuit.

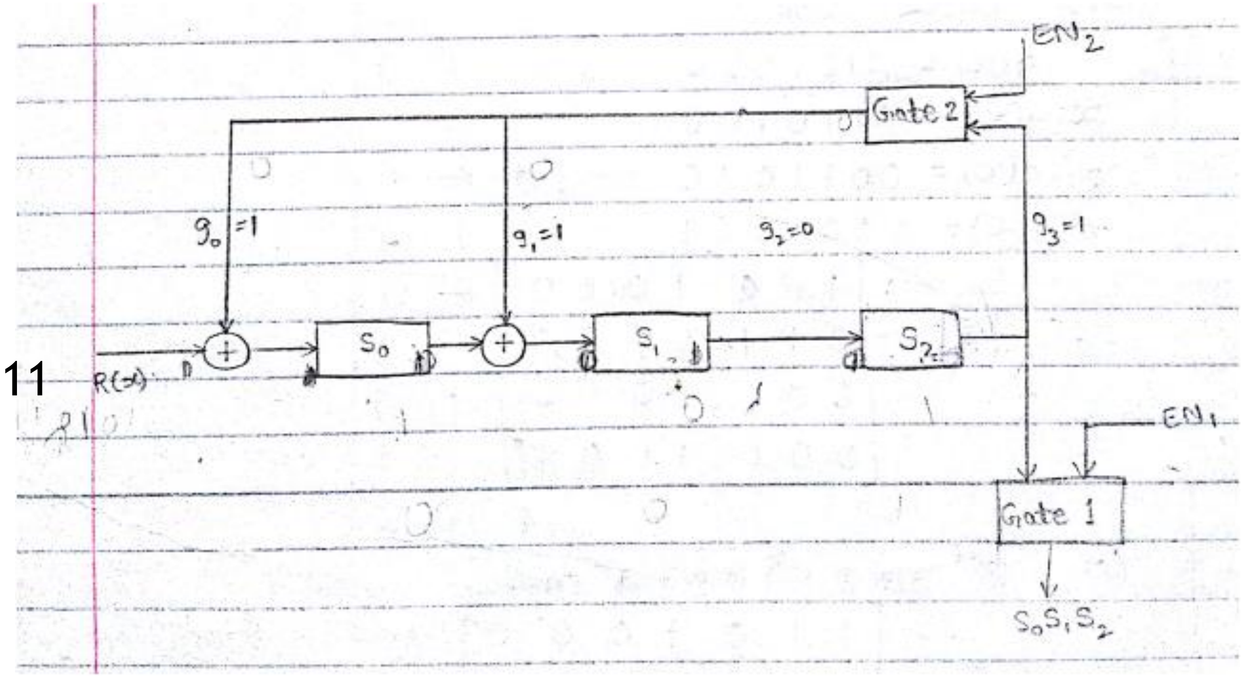
For a (7,4) cyclic code, the received vector is 1110101 and the generator polynomial $g(x) = 1+x+x^3$. Draw the syndrome calculation circuit & correct the single error in the received vector.

$n-k = 7-4 = 3$ bit shift register

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3$$

$$g_0 = 1 \;;\; g_1 = 1 \;;\; g_2 = 0 \;;\; g_3 = 1$$

0111011



	No of shifts	I/p	Shift register S_0	S_1	S_2
Initially		LSB	0	0	0
Gate-1 is OFF	1	1	1	0	0
Gate-2 is ON	2	0	0	1	0
	3	1	1	0	1
	4	0	1	0	0
	5	1	1	1	0
	6	1	1	1	1
	7	1	0	0	1

Indicates error

	No of shifts	I/p (LSB)	Shift register S_0	S_1	S_2
Initially			0	0	0
Gate-1 is OFF	1	1	1	0	0
Gate-2 is ON	2	0	0	1	0
	3	1	1	0	1
	4	0	1	0	0
	5	1	1	1	0
	6	1	1	1	1
	7	1	0	0	1

Indicates error

Consider $g(x) = 1+x+x^3$

It is known that $g(x)$, $x\,g(x)$, $x^2 g(x)$ & $x^3 g(x)$ also represent the code vector polynomial of the same cyclic code.

$$g(x) = 1101000$$

$$x.g(x) = 0110100$$

$$x^2.g(x) = 0011010$$

$$x^3.g(x) = 0001101$$

$$\therefore\ [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Let 3^{rd} row = 1^{st} row + 3^{rd} row

$$[G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Let 4^{th} row = 1^{st} row + 2^{nd} row + 4^{th} row

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Let 4^{th} row = 1^{st} row + 2^{nd} row + 4^{th} row

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ [illegible] & [illegible] & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore [G] = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[H] = [I_{n-k} \; \vdots \; P^T]$$

$$= \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$[H^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

← Syndrome present in 3^{rd} row

$\therefore$ Error pattern = [0 0 1 0 0 0 0]

$\therefore$ Corrected vector = [1 1 0 0 1 0 1]

$R(x) = \underline{011\,|\,1011}$

Generally

$G = [I_k \,\vdots\, P]$

$H = [P^T \,\vdots\, I_{n-k}]$

Now

$G = [P_{n-k} \,\vdots\, I_k]$

$H = [I_{n-k} \,\vdots\, P^T]$

$$H^T = \begin{bmatrix} I_{n-k} \\ \hline P \end{bmatrix}$$

Convolutional codes

- In block codes, a block of ‘n digits generated by the encoder in a particular time unit depends only on one block of ‘k’ input message bits within that time unit.
- A convolutional encoder takes a sequence of message bits and generates a sequence of code bits. In any time unit, a message block consisting of ‘k bits is fed into the encoder and the encoder generates a code block consisting of ‘n code bits.
- The ‘n’ bit code word depends not only on ‘k’ bit message block of the same time unit but also on the previous(m-1) message block.
- The code generated by the above encoder is called **(n, k, m)** convolutional code of rate efficiency ‘k/n’ where

  n – number of outputs=number of modulo-2 adders ,

  k – number of input bits entering at any time

  m – number of stages of the flip-flop
- In convolutional encoder, the message stream continuously runs through the encoder whereas in block coding schemes, the message stream is first divided into long blocks and then encoded.
- In general, there are two methods of generating convolutional codes
  - Time domain approach
  - Transfer domain approach

Encoding of convolutional codes using Time domain approach

- Generator sequence-Impulse response $[g_1^{(j)} \; g_2^{(j)} \; g_3^{(j)} \ldots\ldots g_{m+1}^{(j)}]$

$$C^{(1)} = [d] * g^{(1)}$$
$$C^{(2)} = [d] * g^{(2)}$$

$$C_l^j = \sum_{i=0}^{m} d_{l-i}\, g_{i+1}^j$$

Matrix method

No of rows = No. of digits in in message = L rows

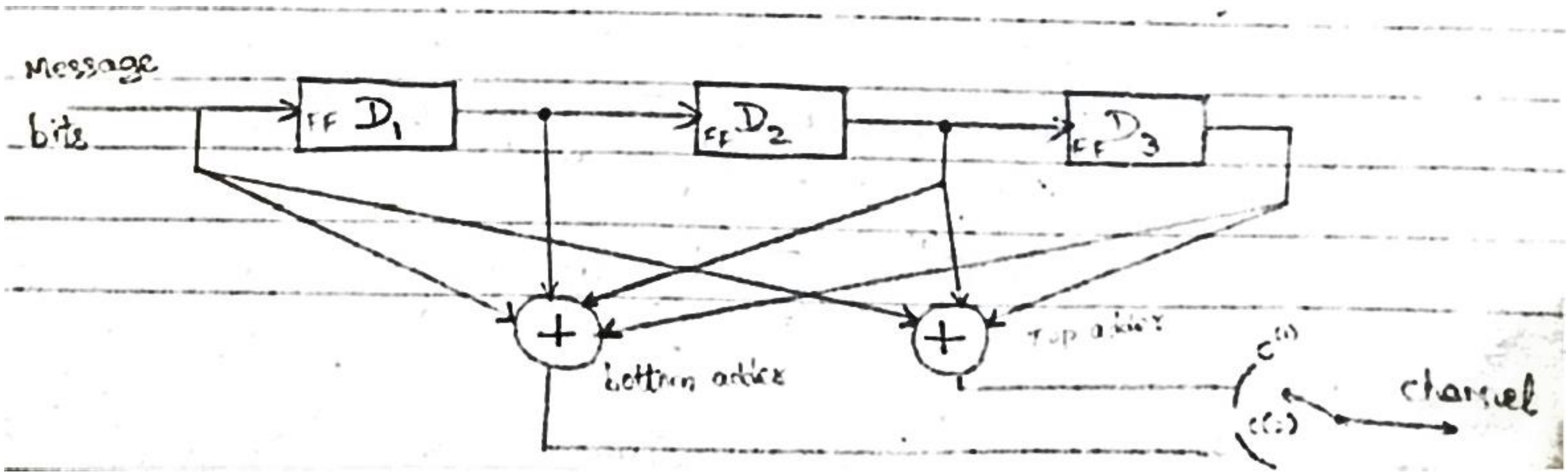
No of columns = n (L + m)

Generator Matrix order = L x n (L + m)

The generator sequence $g_1^{(1)} g_2^{(1)} g_3^{(1)} \ldots g_{m+1}^{(1)}$ for the top adder and $g_1^{(2)} g_2^{(2)} g_3^{(2)} \ldots g_{m+1}^{(2)}$ for the bottom adder can be interlaced & arranged in a matrix form with the no of rows equal to no of digits in the message sequence i.e, L rows & no of columns equal to $n(L+m)$. Such matrix of the order $\{L \times n(L+m)\}$ is called GENERATOR MATRIX of the convolution encoder. In general, for 2 modulo-2 adders convolution encoder, the generator matrix is given by

$$G = \begin{bmatrix} g_1^{(1)} g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & g_3^{(1)} & g_3^{(2)} & \ldots & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0\,0 \ldots & 0\,0 \\ 0 & 0 & g_1^{(1)} & g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & \ldots & g_m^{(1)} & g_m^{(2)} & g_{m+1}^{(1)} g_{m+1}^{(2)} \ldots 0\,0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & & & & & g_{m+1}^{(1)} g_{m+1}^{(2)} \end{bmatrix}$$

Consider a (n,k,m)=(2,1,3) convolutional encoder as shown in fig. Determine the codes using time domain approach for the data (10111) and (11111)



$d = 10111$

$g^{(1)} = 1011$

$g^{(2)} = 1111$

$L = 5 \;;\; C = n(L+m) = 2(5+3) = 16$

$\therefore$ A matrix of (5×16)

$$G = \begin{bmatrix} 11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}$$

$C = [d][G]$

$= [10111][G]$

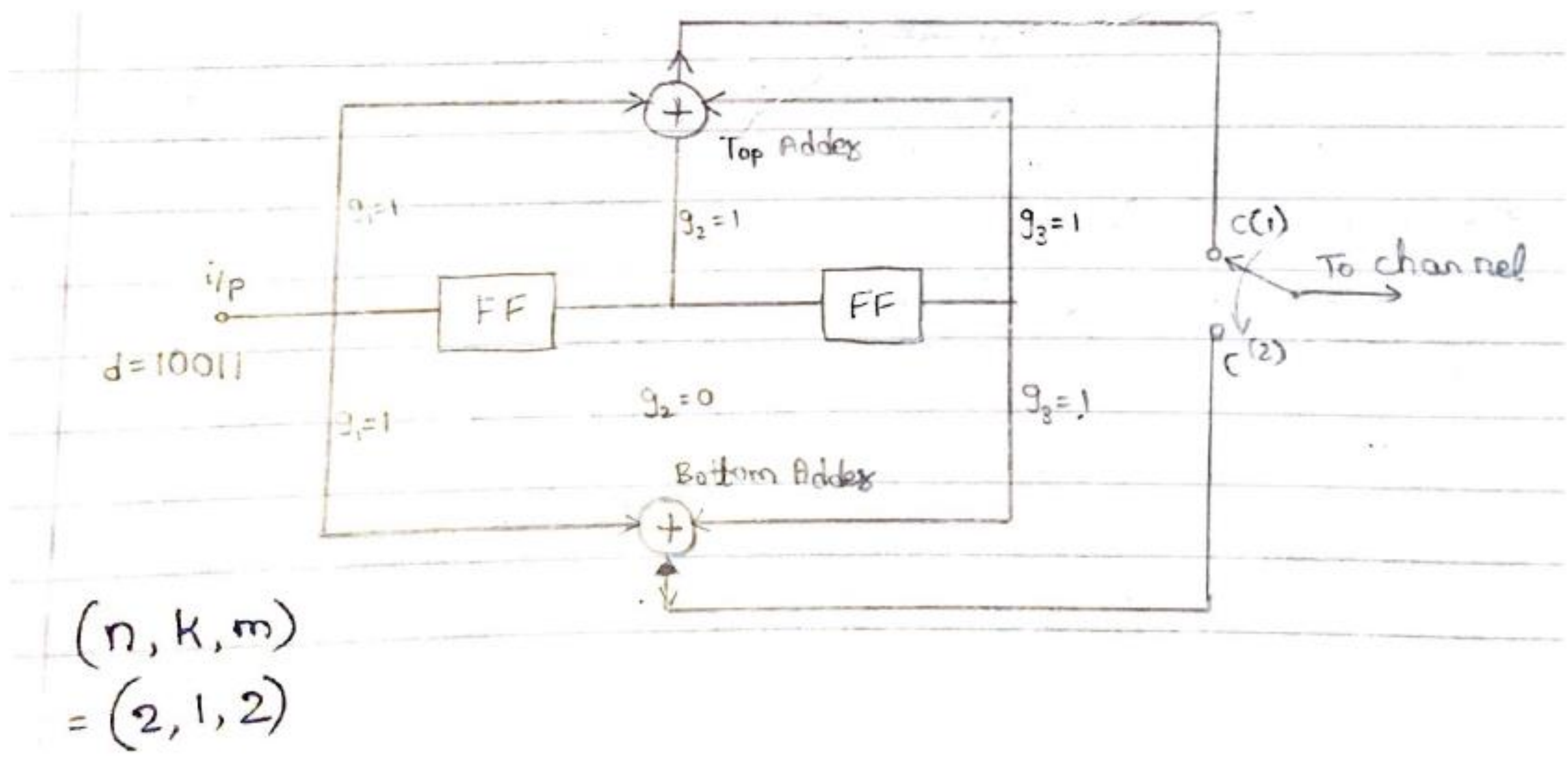
$$\boxed{C = 11, 01, 00, 01, 01, 01, 00, 11}$$

C= [1 0 1 1 1]

1	1	0	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	1	0	1	1	1	1	1	0	0	0	0	0	0
0	0	0	0	1	1	0	1	1	1	1	1	0	0	0	0
0	0	0	0	0	0	1	1	0	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1

C= [1 1 0 1 0 0 0 1 0 1 0 1 0 0 1 1]

For the convolutional encoder shown , d=10011. Find the output sequence using the time domain approaches.



$d = 10011$

$g^{(1)} = 111$

$g^{(2)} = 101$

$R = L = 5$ rows

$C = n(L+m) = 2(5+2) = 14$ columns

$\therefore$ Matrix is of order (5×14)

$$G = \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}$$

$C = [d][G]$

$= [10011][G]$

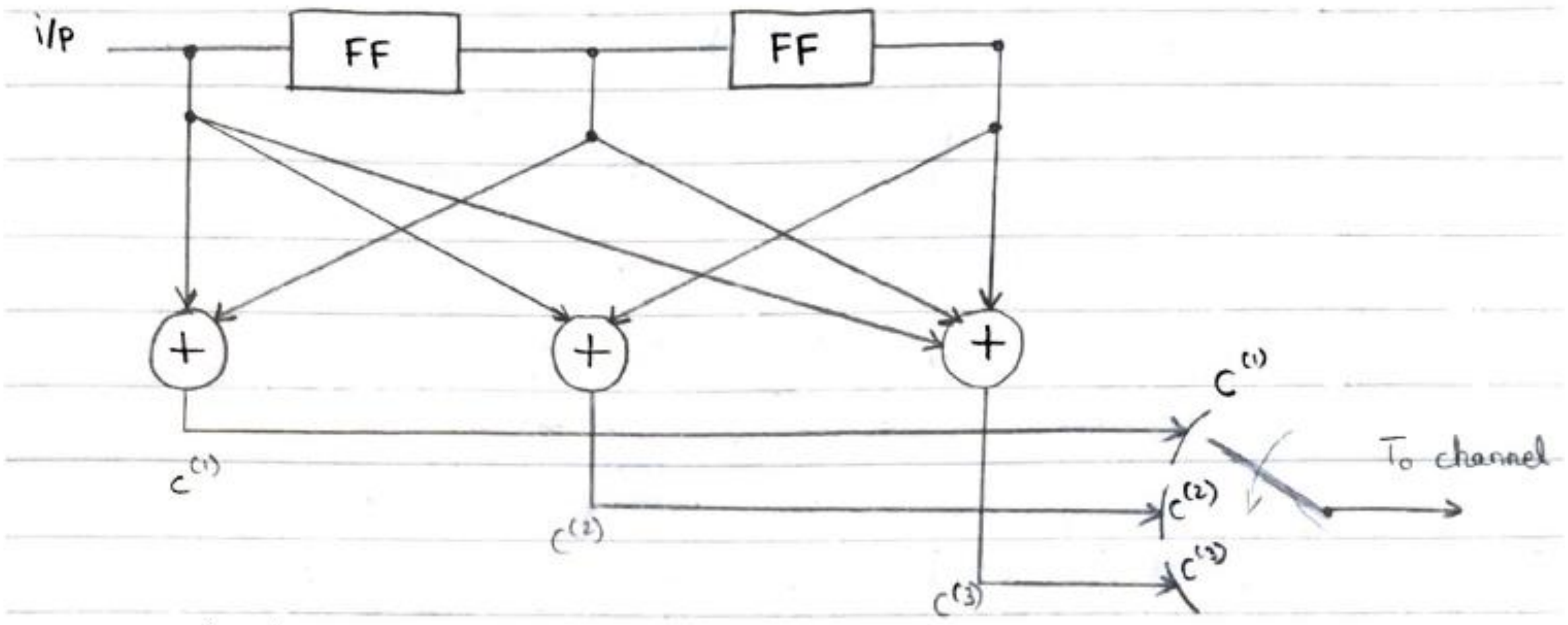
$$\boxed{C = [11, 10, 11, 11, 01, 01, 11]}$$

Consider a (3,1,2) convolution code with $g^{(1)} = 110$, $g^{(2)} = 101$ and $g^{(3)} = 111$.

a) Draw the encoder block diagram

b) Find the generator matrix

c) Find the code word corresponding to $d = 11110$ using time domain approach.



Matrix will have $L = 6$ rows

& $n(L+m) = 3(6+2) = 24$ columns

$g^{(1)} = 110$

$g^{(2)} = 101$

$g^{(3)} = 111$

$$G = \begin{bmatrix} 111 & 101 & 011 & 000 & 000 & 000 & 000 & 000 \\ 000 & 111 & 101 & 011 & 000 & 000 & 000 & 000 \\ 000 & 000 & 111 & 101 & 011 & 000 & 000 & 000 \\ 000 & 000 & 000 & 111 & 101 & 011 & 000 & 000 \\ 000 & 000 & 000 & 000 & 111 & 101 & 011 & 000 \\ 000 & 000 & 000 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}$$

$d = [111101]$

$C = [d][G]$

$= [111, 010, 001; 001; 110; 100; 101; 011]$

Transform domain method

For j no of modulo-2 adders (where j varies from 1 to n), the Generator Polynomial is

$$g^{j}(x) = g_1^{j} + x g_2^{j} + x^2 g_3^{j} + x^3 g_4^{j} + \ldots\ldots + x^m g_{m+1}^{j}$$

where $j \rightarrow 1$ to n

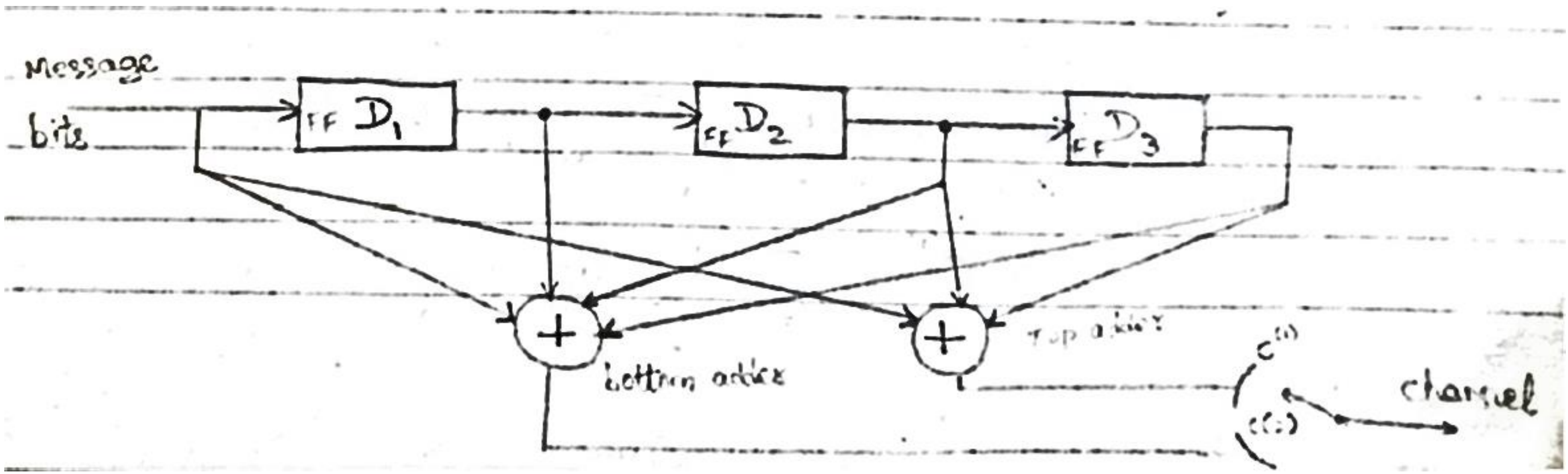
The corresponding o/p of each of the adder is given by $C^{j}(x) = d(x)\, g^{j}(x)$

where $d(x)$ is message vector polynomial.

After getting the polynomials at the o/p of each of the adder, the final encoder o/p polynomial is obtained in the form

$$C(x) = c^{(1)}(x^n) + x\, c^{(2)}(x^n) + x^2 c^{(3)}(x^n) + \ldots + x^{n-1} c^{(n)}(x^n)$$

Consider a (n, k, m)=(2,1,3) convolutional encoder as shown in fig. Determine the codes using transfer domain approach for the data (10111) and (11111)



$g^{(1)} = 1011$

$g^{(2)} = 1111$

$g^{(1)}(x) = 1 + 0.x + 1.x^2 + 1.x^3 = 1 + x^2 + x^3$

$g^{(2)}(x) = 1 + 1.x + 1.x^2 + 1.x^3 = 1 + x + x^2 + x^3$

$d = 10111$

$d(x) = 1 + 0.x + 1.x^2 + 1.x^3 + 1.x^4 = 1 + x^2 + x^3 + x^4$

$$c^{(1)}(x) = d(x)\, g^{(1)}(x)$$

$$= (1+x^2+x^3+x^4)(1+x^2+x^3)$$

$$= 1+\cancel{x^2}+\cancel{x^3}+\cancel{x^2}+\cancel{x^4}+\cancel{x^5}+\cancel{x^3}+\cancel{x^5}+\cancel{x^6}+\cancel{x^4}+\cancel{x^6}+$$

$$= 1+x^7$$

$$c^{(1)}(x) = 1+x^7$$

$$c^{(2)}(x) = d(x)\, g^{(2)}(x)$$

$$= (1+x^2+x^3+x^4)(1+x+x^2+x^3)$$

$$= 1+x+\cancel{x^2}+\cancel{x^3}+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+\cancel{x^5}+x^3+\cancel{x^4}+\cancel{x^5}+\cancel{x^6}$$

$$+x^4+x^5+\cancel{x^6}+x^7$$

$$= 1+x+x^3+x^4+x^5+x^7$$

$$C(x) = c^{(1)} x^n + x\, c^{(2)} x^n$$
$$= c^{(1)}(x)^2 + x\, c^{(2)}(x)^2$$

$$\left[(1+x^7)^2 = 1 + \cancel{x^7} + \cancel{x^7} + x^{14} \right.$$
$$= 1 + x^{14}$$
$$\left. 11\|^{ly} \text{ for } 2^{nd} \text{ term also} \right]$$

$$= (1+x^7)^2 + x\,(1+x+x^3+x^4+x^5+x^7)^2$$
$$= 1+x^{14} + x(1+x^2+x^6+x^8+x^{10}+x^{14})$$
$$= 1+x^{14} + x + x^3 + x^7 + x^9 + x^{11} + x^{15}$$
$$= 1+x+x^3+x^7 + x^9 + x^{11} + x^{14} + x^{15}$$

$$C = [1\,1,0\,1,0\,0,0\,1,0\,1,0\,1,0\,0,1\,1]$$

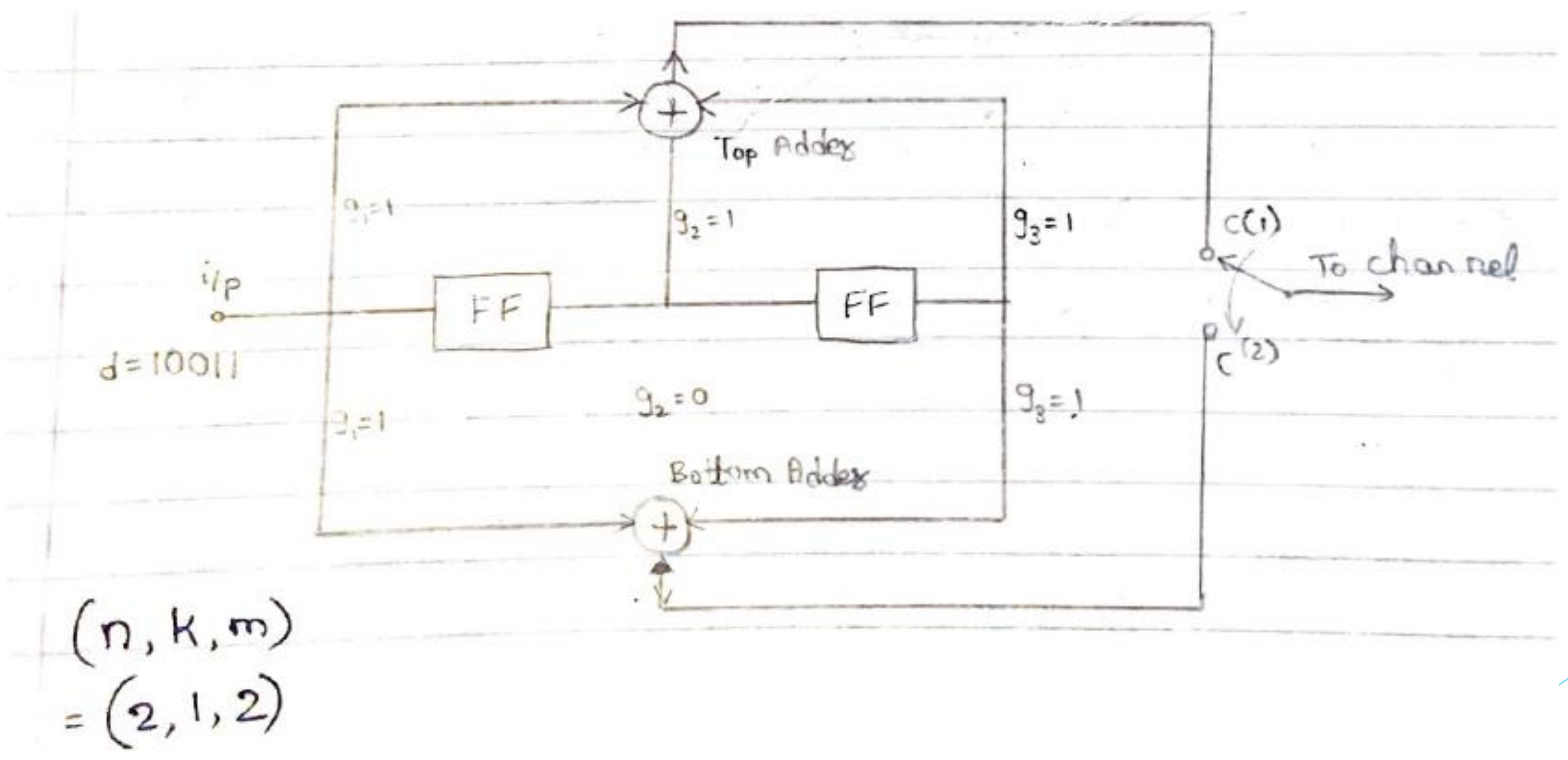
$$
\begin{aligned}
c^{(1)}(x) &= d(x)\, g^{(1)}(x) \\
&= (1+x^2+x^3+x^4)(1+x^2+x^3) \\
&= 1+\cancel{x^2}+\cancel{x^3}+x^2+\cancel{x^4}+x^5+\cancel{x^3}+\cancel{x^5}+\cancel{x^6}+\cancel{x^4}+\cancel{x^6}+ \\
&= 1+x^7 \\
c^{(1)}(x) &= 1+x^7
\end{aligned}
$$

$$
\begin{aligned}
c^{(2)}(x) &= d(x)\, g^{(2)}(x) \\
&= (1+x^2+x^3+x^4)(1+x+x^2+x^3) \\
&= 1+x+\cancel{x^2}+\cancel{x^3}+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+\cancel{x^5}+x^3+\cancel{x^4}+\cancel{x^5}+\cancel{x^6} \\
&\quad +x^4+x^5+\cancel{x^6}+x^7 \\
&= 1+x+x^3+x^4+x^5+x^7
\end{aligned}
$$

$$
\begin{aligned}
C(x) &= c^{(1)}x^n + x\, c^{(2)}x^n \\
&= c^{(1)}(x)^2 + x\, c^{(2)}(x)^2 \\
&= (1+x^7)^2 + x\,(1+x+x^3+x^4+x^5+x^7)^2 \\
&= 1+x^{14} + x(1+x^2+x^6+x^8+x^{10}+x^{14}) \\
&= 1+x^{14}+x+x^3+x^7+x^9+x^{11}+x^{15} \\
&= 1+x+x^3+x^7+x^9+x^{11}+x^{14}+x^{15}
\end{aligned}
$$

$$C = [1\,1, 0\,1, 0\,0, 0\,1, 0\,1, 0\,1, 0\,0, 1\,1]$$

For the convolutional encoder shown , d=10011. Find the output sequence using the transfer domain approach



$g^{(1)} = 111$

$g^{(2)} = 101$

$d = 10011$

$g^{(1)}(x) = 1 + x + x^2$

$g^{(2)}(x) = 1 + x^2$; $d(x) = 1 + x^3 + x^4$

$$
\begin{aligned}
c^{(1)}(x) &= d(x)\, g^{(1)}(x) \\
&= (1 + x^3 + x^4)(1 + x + x^2) \\
&= 1 + x + x^2 + x^3 + \cancel{x^4} + \cancel{x^5} + \cancel{x^4} + \cancel{x^5} + x^6
\end{aligned}
$$

$c^{(1)}(x) = 1 + x + x^2 + x^3 + x^6$

$$
\begin{aligned}
c^{(2)}(x) &= d(x)\, g^{(2)}(x) = (1 + x^3 + x^4)(1 + x^2) \\
&= 1 + x^2 + x^3 + x^5 + x^4 + x^6 \\
&= 1 + x^2 + x^3 + x^4 + x^5 + x^6
\end{aligned}
$$

$$C(x) = C^{(1)}(x)^{n} + x.C^{(2)}(x)^{n}$$

$$= C^{(1)}(x)^{2} + x \cdot C^{(2)}(x)^{2}$$

$$= \{1 + x + x^{2} + x^{3} + x^{6}\}^{2} + x\{1 + x^{2} + x^{3} + x^{4} + x^{5} + x^{6}\}^{2}$$

$$= 1 + x^{2} + x^{4} + x^{6} + x^{12} + x + x^{5} + x^{7} + x^{9} + x^{11} + x^{13}$$

$$= 1 + x^{1} + x^{2} + x^{4} + x^{5} + x^{6} + x^{7} + x^{9} + x^{11} + x^{12} + x^{13}$$

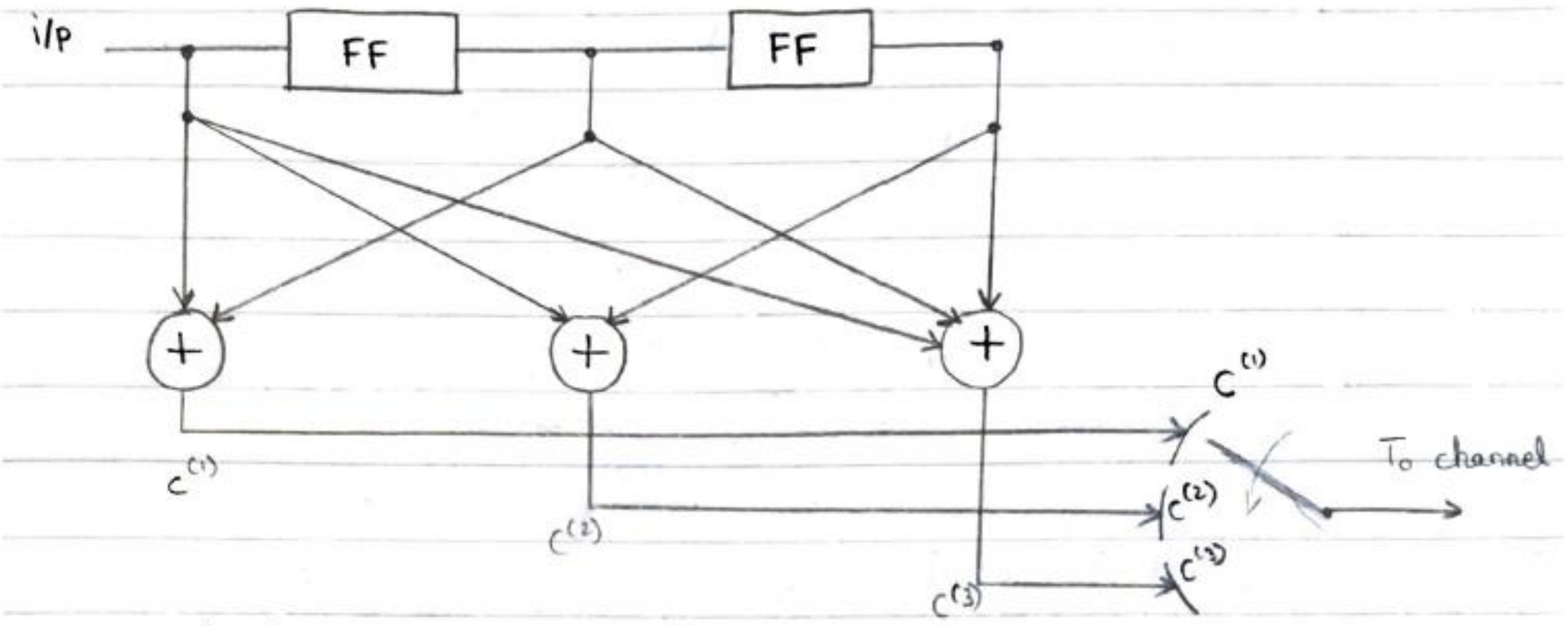
$$C = [1\,1\,1\,0\,1\,1\,1\,1\,0\,1\,0\,1\,1\,1]$$

Consider a (3,1,2) convolution code with $g^{(1)} = 110$, $g^{(2)} = 101$ and $g^{(3)} = 111$.

a) Draw the encoder block diagram

b) Find the generator matrix

c) Find the code word corresponding to $d = 11110$ using time domain approach.



$g^{(1)} = 110$

$g^{(2)} = 101$

$g^{(3)} = 111$

$d = 111101$

$g^{(1)}(x) = 1+x$

$g^{(2)}(x) = 1+x^2$

$g^{(3)}(x) = 1+x+x^2$

$d(x) = 1+x+x^2+x^3+x^5$

$$\begin{aligned}
c^{(1)}(x) &= d(x)\cdot g^{(1)}(x) \\
&= (1+x+x^2+x^3+x^5)(1+x) \\
&= 1+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^5+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^4+x^6 \\
&= 1+x^4+x^5+x^6
\end{aligned}$$

$$\begin{aligned}
c^{(2)}(x) &= d(x)\, g^{(2)}(x) \\
&= (1+x+x^2+x^3+x^5)(1+x^2) \\
&= 1+x+\cancel{x^2}+\cancel{x^3}+\cancel{x^5}+\cancel{x^2}+\cancel{x^3}+x^4+\cancel{x^5}+x^7 \\
&= 1+x+x^4+x^7
\end{aligned}$$

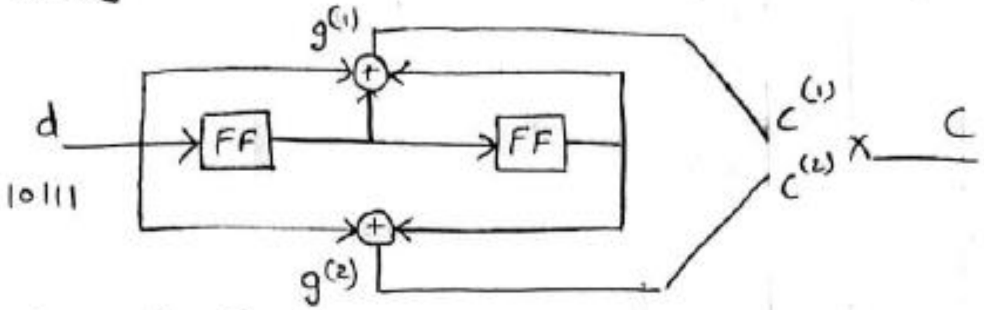
$$\begin{aligned}
c^{(3)}(x) &= d(x)\, g^{(3)}(x) \\
&= (1+x+x^2+x^3+x^5)(1+x+x^2) \\
&= 1+\cancel{x}+x^2+x^3+\cancel{x^5}+\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+x^6 \\
&\qquad +\cancel{x^2}+\cancel{x^3}+\cancel{x^4}+\cancel{x^5}+x^7 \\
&= 1+x^2+x^3+x^6+x^7
\end{aligned}$$

$$\begin{aligned}
c(x) &= c^{(1)}(x)^n + x\,c^{(2)}(x)^n + x^2\,c^{(3)}(x)^n \\
&= c^{(1)}(x^3) + x\,c^{(2)}(x^3) + x^2\,c^{(3)}(x^3) \\
&= 1+x^{12}+x^{15}+x^{18} + x\{1+x^3+x^{12}+x^{21}\} \\
&\qquad + x^2\{1+x^6+x^9+x^{18}+x^{21}\} \\
&= 1+x^{12}+x^{15}+x^{18}+x+x^4+x^{13}+x^{22}+x^2+x^8+x^{11}+x^{20} \\
&\qquad +x^{23} \\
&= 1+x+x^2+x^4+x^8+x^{11}+x^{12}+x^{13}+x^{15}+x^{18}+x^{20}+x^{22}+x^{23}
\end{aligned}$$

$c = [111,010,001,001,110,100,101,011]$

State Diagram and Code Tree

Dec-10 14M Consider the binary Convolutional encoder shown in the figure. Draw the state table, state transition table, state diagram and the corresponding code tree. Find the encoded sequence for the message 10111. Also find Constraint length and rate of efficiency.



Sol:- $(n, k, m) = (2, 1, 2)$

Since $m = 2$, number of States $= 2^m = 2^2 = 4$

State Table :-

$c^{(1)} = d_1 \oplus d_2 \oplus d_3$

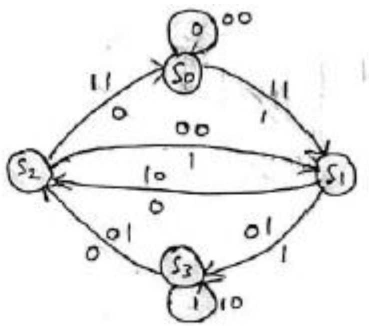
$c^{(2)} = d_1 \oplus d_3$

States	Binary Description
S_0	00
S_1	10
S_2	01
S_3	11

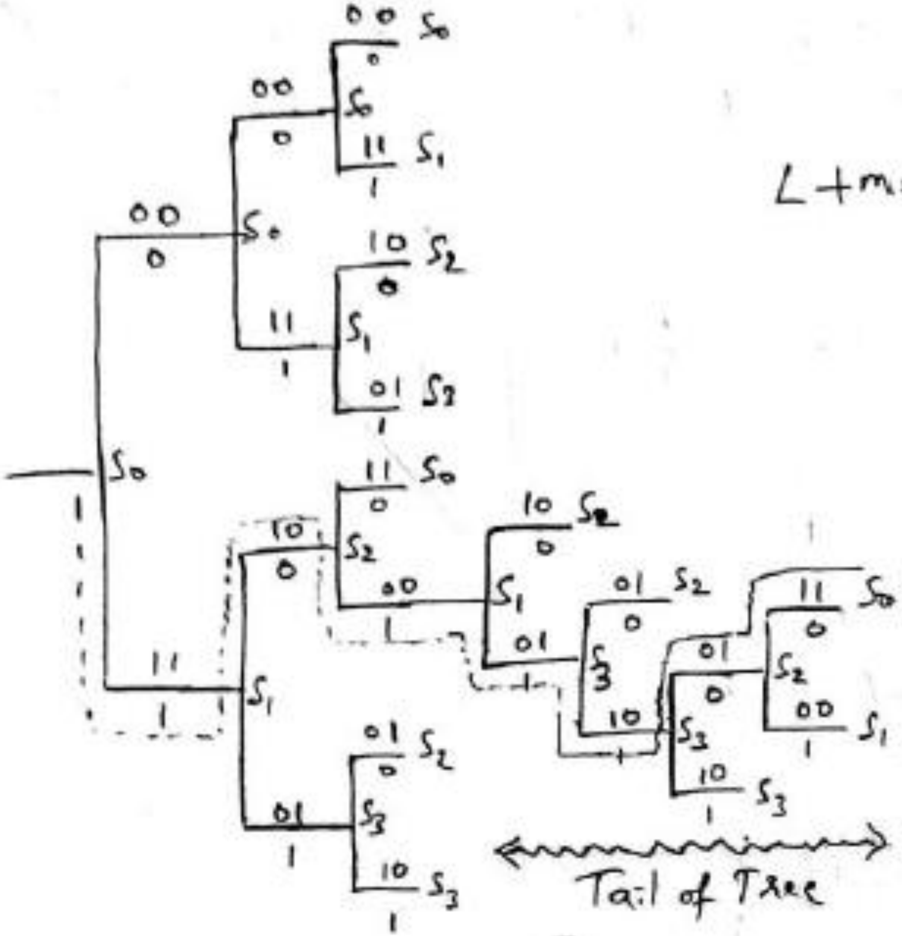
State Transition Table :-

Present State	Binary Description	Input	Next State	Binary Description	Shift Registers d_1 d_2 d_3	Code Vectors $c^{(1)}$ $c^{(2)}$
S_0	00	0	S_0	00	0 0 0	0 0
		1	S_1	10	1 0 0	1 1
S_1	10	0	S_2	01	0 1 0	1 0
		1	S_3	11	1 1 0	0 1
S_2	01	0	S_0	00	0 0 1	1 1
		1	S_1	10	1 0 1	0 0
S_3	11	0	S_2	01	0 1 1	0 1
		1	S_3	11	1 1 1	1 0

State Diagram :-



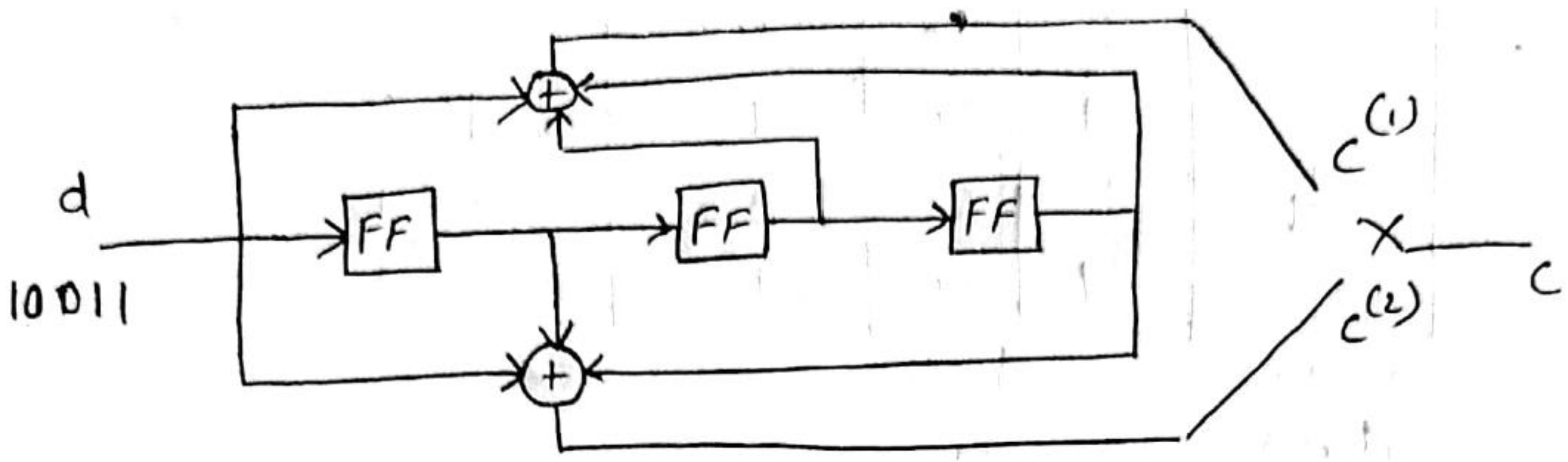
Code Tree :-



$L + m = 5 + 2 = 7$ bits

$C = [11\ 10\ 00\ 01\ 10\ 01\ 11]$

Consider convolutional encoder shown in the figure. Draw state table, state transition table, state diagram. Find the encoder output for the message 10011 by traversing through the state diagram



S_0	000	
S_1	100	
S_2	010	
S_3	110	
S_4	001	
S_5	101	
S_6	011	
S_7	111	