4 IIR Filter Design and Applications

Example 1 : *Use impulse invariance method to design a digital filter from an analog prototype that has a system function,*

$$H_a(s) = \frac{s+a}{(s+a)^2+b^2}$$

$$\text{and } H_a(s) = \frac{s}{(s+a)^2+b^2}$$

Use T = 1 sec

Solution : i) Consider

$$H_a(s) = \frac{s+a}{(s+a)^2+b^2}$$

$$= \frac{\frac{1}{2}}{s-(-a+jb)} + \frac{\frac{1}{2}}{s-(-a-jb)}$$

We know that the impulse invariant transformation is given as,

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}}$$

Applying this transformation to $H_a(s)$,

$$\text{H(z)} = \frac{1/2}{1-e^{(-a+jb)T} z^{-1}} + \frac{1/2}{1-e^{(-a-jb)T} z^{-1}}$$

$$= \frac{1/2}{1-e^{-aT} \cdot e^{jbT} z^{-1}} + \frac{1/2}{1-e^{-aT} e^{-jbT} z^{-1}}$$

$$= \frac{\frac{1}{2}-\frac{1}{2} e^{-aT} e^{-jbT} z^{-1} + \frac{1}{2}-\frac{1}{2} e^{-aT} e^{jbT} z^{-1}}{1-e^{-aT} e^{-jbT} z^{-1} - e^{-aT} e^{jbT} z^{-1} + e^{-2aT} z^{-2}}$$

$$= \frac{1-\frac{1}{2} e^{-aT} z^{-1} [e^{-jbT} + e^{jbT}]}{1-e^{-aT} z^{-1} [e^{-jbT} + e^{jbT}] + e^{-2aT} z^{-2}}$$

Here $e^{-jbT} + e^{jbT}$ = 2cos bT. Hence above equation becomes,

$$H(z) = \frac{1-e^{-aT}z^{-1}\cos bT}{1-2e^{-aT}z^{-1}\cos bT + e^{-2aT}z^{-2}}$$

Observe that this equation is same as eq. 4.3.8 in the book.

ii) Consider, $H_a(s) = \dfrac{s}{(s+a)^2+b^2}$

Rearranging above equation,

$$H_a(s) = \frac{s+a-a}{(s+a)^2+b^2}$$

$$= \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$$

Rearranging the second term,

$$H_a(s) = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{b}\cdot\frac{b}{(s+a)^2+b^2}$$

apply eq. 4.3.8 to first term and eq. 4.3.9 to second term we get,

$$H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}} - \frac{a}{b}\,\frac{e^{-aT}\sin(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$= \frac{1-e^{-aT}\cos(bT)z^{-1}-\frac{a}{b}e^{-aT}\sin(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$= \frac{1-\left[\cos(bT)+\frac{a}{b}\sin(bT)\right]e^{-aT}z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

This is the required system function of digital filter.

Example 2 : *Given that* $|H_a(j\Omega)|^2 = \dfrac{1}{1+64\Omega^6}$, *determine the analog filter system function* $H_a(s)$.

Solution : Consider the given magnitude squared function,

$$|H_a(j\Omega)|^2 = \frac{1}{1+64\Omega^6}$$

$$= \frac{1}{1+\left(\frac{\Omega}{1/2}\right)^6}$$

$$= \frac{1}{1+\left(\frac{\Omega}{1/2}\right)^{2\times 3}}$$

On comparing this equation with,

$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

we obtain order of the filter as $N = 3$ and cutoff frequency $\Omega_c = \frac{1}{2}$.

The poles of the butterworth filter are given by equation 4.5.6 in the book as,

$$p_k = \pm \Omega_c \, e^{j\,(N+2k+1)\pi/2N}, \quad k = 0, 1, \ldots.. N-1$$

Putting for N and Ω_c in above equation we get,

$$p_k = \pm \frac{1}{2} \, e^{j(3+2k+1)\pi/(2\times 3)}$$

$$= \pm \frac{1}{2} \, e^{j(4+2k)\pi/6}, \quad k = 0, 1, 2.$$

For different values of k in above equation we get following poles :

With $k = 0$, $\quad p_0 = \frac{1}{2} e^{j\,4\pi/6} = \frac{1}{2} e^{j\,2\pi/3}$

$$= -0.25 + j\,0.433 \text{ and } 0.25 - j\,0.433$$

With $k = 1$, $\quad p_1 = \pm \frac{1}{2} e^{j\pi}$

$$= -0.5 \text{ and } 0.5$$

With $k = 2$, $\quad p_2 = \pm \frac{1}{2} e^{j\,8\pi/6} = \frac{1}{2} e^{j\,4\pi/3}$

$$= -0.25 - j\,0.433 \text{ and } 0.25 + j\,0.433$$

Thus there are total 6 poles as calculated above. These are the poles of $H_a(s) \cdot H_a(-s)$. These poles are plotted in the s-plane as shown in Fig. 1. In this figure observe that all the poles lie on the circle of radius $\Omega_c = \frac{1}{2} = 0.5$. Also observe that the angular separation between the poles is $\frac{\pi}{N}$ i.e. $\frac{\pi}{3}$. If we want stable filter, then $H(s)$ should have all the poles in left half of the s-plane. Hence we have to consider the poles lying in left half of the s-plane. These poles are as shown in Fig. 1 :

Pair 1 : $\quad p_0$ and $p_2 \Rightarrow -0.25 \pm j\,0.433$

$\qquad p_1 \Rightarrow -0.5$

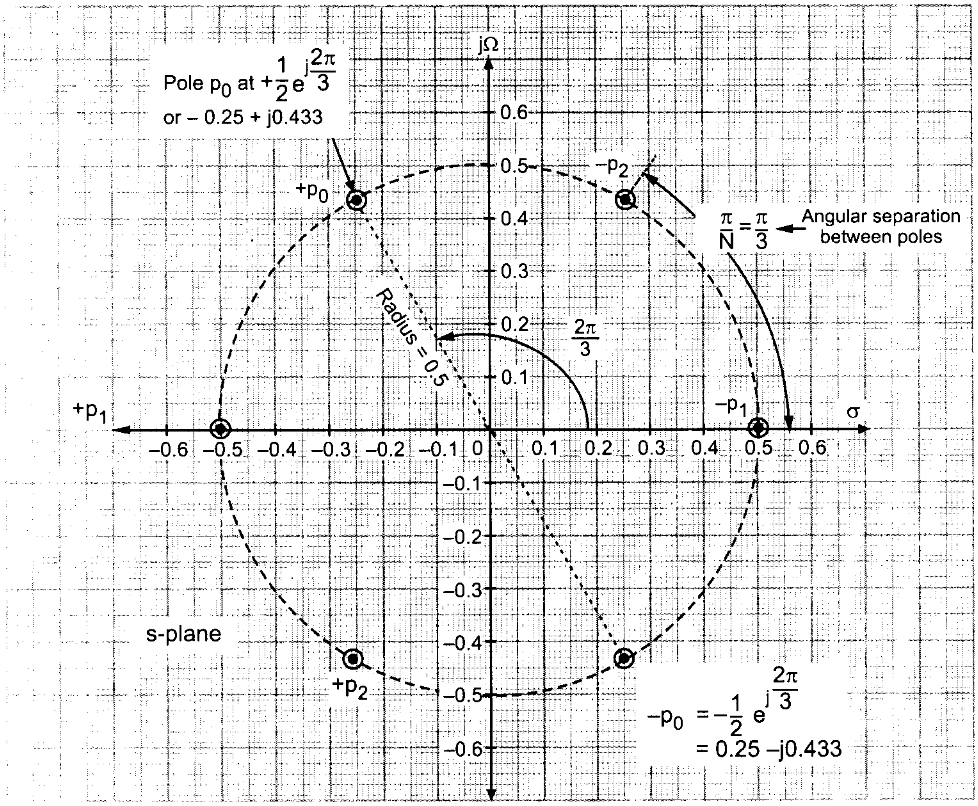


Fig. 1 Poles of $H_a(s) \cdot H_a(-s)$ of example 2

Thus p_0 and p_2 are complex conjugates of each other. Let us name these poles as follows for convenience :

$$s_1 = -0.25 + j\,0.433 \quad \text{i.e. } p_0$$

$$s_1^* = -0.25 - j\,0.433 \quad \text{i.e. } p_2$$

$$s_2 = -0.5 \quad \text{i.e. } p_1$$

To get $H_a(s)$, we have to combine complex conjugate poles so that all the coefficients will be real. For $N = 3$ we can write $H_a(s)$ as,

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)\left(s-s_1^*\right)(s-s_2)}$$

Putting values in above equation,

$$H_a(s) = \frac{(0.5)^3}{(s+0.25-j\,0.433)\,(s+0.25+j\,0.433)\,(s+0.5)}$$

$$= \frac{(0.5)^3}{\left[(s+0.25)^2+(0.433)^2\right](s+0.5)}$$

$$= \frac{0.125}{(s+0.5)\left(s^2+0.5\,s+0.25\right)}$$

This is the required system function of the analog filter.

Example 3 *Design an analog filter with maximally flat response in the passband and an acceptable attenuation of −2 dB at 20 radians/second. The attenuation in the stopband should be more than 10 dB beyond 30 radians/second.*

Solution : The maximally flat response is required, means we have to use butterworth approximation. The given data is,

Attenuation in the passband, A_p = 2 dB

Passband edge frequency, Ω_p = 20 radians/sec

Attenuation in the stopband, A_s = 10 dB

Stopband edge frequency, Ω_s = 30 radians/sec.

To determine order N and cutoff frequency Ω_c :

When the specifications are given in dB, order of the filter is given by equation 4.5.19 in the book as,

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\,A_s}-1}{10^{0.1\,A_p}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

Putting values in above equation,

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1\times 10}-1}{10^{0.1\times 2}-1}\right]}{\log\left(\dfrac{30}{20}\right)}$$

Solving the above equation we get,

$$\text{N} = 3.37$$

Since the order is between 3 and 4, we will select the order of the filter be 4. Hence,

$$\text{N} = 4$$

Cutoff frequency Ω_c can be obtained with the help of equation 4.5.15 in the book i.e.,

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right]$$

With the help of equation 4.5.17 and equation 4.5.18 in the book we can write above equation as,

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left(10^{0.1\,A_p\,dB}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(10^{0.1\,A_s\,dB}-1\right)^{\frac{1}{2N}}}\right] \qquad \text{...(1)}$$

Putting the values in above equation,

$$\Omega_c = \frac{1}{2}\left\{\frac{20}{\left(10^{0.1\times 2}-1\right)^{\frac{1}{2\times 4}}} + \frac{30}{\left(10^{0.1\times 10}-1\right)^{\frac{1}{2\times 4}}}\right\}$$

On solving the above equation we get,

$$\Omega_c = 22 \text{ rad/sec}$$

To obtain poles of $H_a(s)$:

The poles of $H_a(s)\cdot H_a(-s)$ are given by equation 4.5.6 in the book. i.e.,

$$p_k = \pm\,\Omega_c\, e^{j(N+2k+1)\pi/2N}, \quad k = 0, 1, 2, \ldots\ldots N-1$$

Putting for Ω_c and N in above equation,

$$p_k = \pm\, 22\, e^{j(4+2k+1)\pi/(2\times 4)}, \qquad k = 0, 1, 2, 3.$$

$$= \pm\, 22\, e^{j(5+2k)\pi/8}$$

$k = 0 \Rightarrow$ $\quad p_0 = \pm\, 22\, e^{j5\pi/8}$

$$= -8.419 + j\,20.325 \text{ and } 8.419 - j\,20.325$$

$k = 1 \Rightarrow$ $\quad p_1 = \pm\, 22\, e^{j7\pi/8}$

$$= -20.325 + j\,8.419 \text{ and } 20.325 - j\,8.419$$

$k = 2 \Rightarrow$ $\quad p_2 = \pm\, 22\, e^{j9\pi/8}$

$$= -20.325 - j\,8.419 \text{ and } 20.325 + j\,8.419$$

$k = 3 \Rightarrow$ $\quad p_3 = \pm\, 22\, e^{j11\pi/8}$

$$= -8.419 - j\,20.325 \text{ and } 8.419 + j\,20.325$$

It can be shown easily that all the above poles lie on the circle of radius $\Omega_c = 22\ rad/\sec$. We want stable analog filter. Hence we should consider the poles lying in left half of the s-plane. These poles are as follows :

$$\left.\begin{aligned} p_0 &= -8.419 + j\,20.325 \\ p_3 &= -8.419 - j\,20.325 \end{aligned}\right\} \text{Pair 1}$$

$$\left.\begin{aligned} p_1 &= -20.325 + j\,8.419 \\ p_2 &= -20.325 - j\,8.419 \end{aligned}\right\} \text{Pair 2}$$

The above poles are combined as complex conjugate pairs. These are the poles of $H_a(s)$, since they lie in left half of the s-plane. Let us name these poles as follows for convenience :

$$\begin{aligned} s_1 &= -8.419 + j\,20.325 \quad \text{i.e. } p_1 \\ s_1^* &= -8.419 - j\,20.325 \quad \text{i.e. } p_3 \\ s_2 &= -20.325 + j\,8.419 \quad \text{i.e. } p_1 \\ s_2^* &= -20.325 - j\,8.419 \quad \text{i.e. } p_2 \end{aligned}$$

To obtain the system function $H_a(s)$:

The system function $H_a(s)$ can be obtained from the above poles as follows :

$$H_a(s) = \frac{\Omega_c^N}{(s - s_1)(s - s_1^*)(s - s_2)(s - s_2^*)}$$

Here note that we have combined complex conjugate poles so that $H_a(s)$ will have real coefficients. Putting the values in above equation,

$$H_a(s) = \frac{(22)^4}{(s + 8.419 - j\,20.325)(s + 8.419 + j\,20.325)(s + 20.325 - j\,8.419)(s + 20.325 + j\,8.419)}$$

$$= \frac{(22)^4}{\left[(s + 8.419)^2 + (20.325)^2\right]\left[(s + 20.325)^2 + (8.419)^2\right]}$$

$$= \frac{(22)^4}{(s^2 + 16.838\,s + 484)(s^2 + 40.65\,s + 484)}$$

This is the system function of the required analog filter with butterworth approximation.

Example 4: *Design a lowpass 1 rad/sec bandwidth chebyshev filter with the following characteristics :*

(i) Acceptable passband ripple of 2 dB

(ii) Cutoff radian frequency of 1 rad/sec

(iii) Stopband attenuation of 20 dB or greater beyond 1.3 rad/sec.

Solution : (i) Given data

Here the cutoff radian frequency is given as 1 rad/sec. This means we have to design a normalized lowpass chebyshev filter.

Passband ripple A_p = 2 dB

Consider equation 4.5.22 in the book,

$$\varepsilon = \left(10^{0.1\, A_p\, dB} - 1\right)^{\frac{1}{2}} = \left(10^{0.1 \times 2} - 1\right)^{\frac{1}{2}}$$

$$= 0.764$$

And $\quad |H_a(j\Omega)| = -20$ dB for $\Omega = \Omega_s = 1.3$ rad/sec.

(ii) To determine order 'N' :

The order of the filter can be obtained from equation 4.6.12 in the book. i.e.,

$$|H_a(j\Omega)| = -20\, log_{10}\, \varepsilon - 6\,(N-1) - 20\, N\, log_{10}\, \Omega$$

Putting values in above equation,

$$-20 = -20\, log_{10}\, (0.764) - 6\,(N-1) - 20\, N\, log_{10}\, (1.3)$$

Solving the above equation for 'N' we get,

$$N = 3.42$$

Since 'N' must be an integer, we will select next higher integer. i.e.,

$$N = 4$$

(iii) To obtain system function $H_a(s)$:

The poles of the system function can be obtained using equation 4.6.16 and equation 4.6.17 in the book. The polynomial for the system function can also be obtained directly from standard tables. The generalized form of such polynomial is given by equation 4.6.8 in the book. i.e.,

$$H_a(s) = \frac{K}{s^N + b_{N-1}\, s^{N-1} + \ldots + b_1\, s + b_0}$$

For $N = 4$,

$$H_a(s) = \frac{K}{s^4 + b_3\, s^3 + b_2\, s^2 + b_1\, s + b_0}$$

For $N = 4$ and ripple of 2 dB the denominator polynomial of above equation can be obtained from table 4.6.2 (c) in the book. i.e.,

$$H_a(s) = \frac{K}{s^4 + 0.716\, s^3 + 1.25\, s^2 + 0.516\, s + 0.205}$$

Here $N = 4$ i.e. even. Hence K can be obtained using equation 4.6.19 in the book as,

$$K = \frac{b_0}{\sqrt{1+\varepsilon^2}} = \frac{0.205}{\sqrt{1+(0.764)^2}}$$

$$= 0.1628$$

Hence system function $H_a(s)$ becomes,

$$H_a(s) = \frac{0.1628}{s^4 + 0.716\, s^3 + 1.25\, s^2 + 0.516\, s + 0.205}$$

This is the required system function of the normalized chebyshev filter.

Example 5 : *Determine H(z) for a lowest order butterworth filter satisfying following constraints.*

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \qquad \frac{3\pi}{4} \le \omega \le \pi$$

with T=1s. Apply impulse invariant transformation.

Solution : Given data

$$A_P = \sqrt{0.5}, \quad \omega_p = \frac{\pi}{2}$$

$$A_s = 0.2 \quad \omega_s = \frac{3\pi}{4}$$

$$T = 1s$$

Specifications of equivalent analog filter

For impulse invariant transformation,

$$\omega = \Omega T$$

$$\therefore \quad \Omega = \frac{\omega}{T} = \omega \quad \text{since } T = 1$$

Hence specification of equivalent analog filter will be,

$$A_p = \sqrt{0.5} = 0.707 \quad \Omega_p = \frac{\pi}{2} = 0.5\,\pi$$

$$A_s = 0.2 \qquad \Omega_s = \frac{3\pi}{4} = 0.75\,\pi$$

To determine order of butterworth filter

Order of butterworth filter is given as,

$$N = \frac{1}{2}\,\frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{\log\left(\frac{\Omega s}{\Omega p}\right)}$$

$$= \frac{1}{2}\frac{\log\left[\left(\frac{1}{(0.2)^2}-1\right)\Big/\left(\frac{1}{(0.70712)^2}-1\right)\right]}{\log\left(\frac{0.75\pi}{0.5\pi}\right)} = 3.918 \approx 4$$

To determine cutoff frequency Ω_c

Ω_c is given by eq. 4.5.15 in the book as,

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}}+\frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.5\pi}{\left(\frac{1}{0.707)^2}-1\right)^{\frac{1}{8}}}+\frac{0.75\pi}{\left(\frac{1}{(0.2)^2}-1\right)^{\frac{1}{8}}}\right\} = 1.577 \text{ rad/sec}$$

To determine poles of $H_a(s)$

Poles of $H_a(s)$ are given as,

$$p_k = \pm\Omega_c\, e^{j(N+2k+1)\pi/2N},\ k = 0, 1, 2, \ldots N-1$$

$$\therefore \quad p_k = \pm 1.577\, e^{j(5+2k)\pi/8},\ k = 0, 1, 2, 3$$

$$\therefore \quad p_o = \pm\, 1.577 e^{j5\pi/8} = -\,0.6 + j\,1.457 \text{ and } 0.6 - j\,1.457$$

$$p_1 = \pm\, 1.577 e^{j7\pi/8} = -\,1.457 + j\,0.6 \text{ and } 1.457 - j\,0.6$$

$$p_2 = \pm\, 1.577 e^{j9\pi/8} = -\,1.457 + j\,0.6 \text{ and } 1.457 + j\,0.6$$

$$p_3 = \pm\, 1.577 e^{j11\pi/8} = -\,0.6 - j\,1.457 \text{ and } 0.6 + j\,1.457$$

The poles in LHS of s-plane in complex conjugate pairs must be selected for stable filter,

$$\therefore \quad s_1 = -\,0.6 + j\,1.457 \text{ and } s_1^* = -\,0.6 - j\,1.457$$

$$s_2 = -\,1.457 + j\,0.6 \text{ and } s_2^* = -\,1.457 - j\,0.6$$

To determine system function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^4}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

$$= \frac{(1.577)^4}{(s+0.6-j1.457)\,(s+0.6+j1.457)\,(s+1.457-j0.6)\,(s+1.457+j0.6)}$$

$$= \frac{6.185}{[(s+0.6)^2+(1.457)^2]\,[(s+1.457)^2+(0.6)^2]}$$

Let us rearrange above equation as,

$$H_a(s) = 7.075\,\frac{1.457}{(s+0.6)^2+(1.457)^2}\times\frac{0.6}{(s+1.457)^2+(0.6)^2}$$

To determine H(z) by impulse invariance

We will apply equation 4.3.9 in the book to the above $H_a(s)$ to get H(z). Thus,

$$H(z) = 7.075\times\frac{e^{-0.6}(\sin 1.457)z^{-1}}{1-2e^{-0.6}(\cos 1.457)^{z^{-1}}+e^{-1.2}z^{-2}}\times\frac{e^{-1.457}(\sin 0.6)z^{-1}}{1-2e^{-1.457}(\cos 0.6)z^{-1}+e^{-2.914}z^{-2}}$$

$$= 7.075\times\frac{0.549z^{-1}}{1-0.125z^{-1}+0.3z^{-2}}\times\frac{0.132\,z^{-1}}{1-0.384z^{-1}+0.054z^{-2}}$$

This is the required system function of digital filter.

Example 6 : *Design a digital LPF to satisfy the following passband ripple $1 \le |H(j\Omega)| \le 0$ for $0 \le \Omega \le 1404\pi$ rad/sec and stopband attenuation $|H(j\Omega)|$ dB > 60 for $\Omega \ge 8268\,\pi$ rad/sec. The sampling interval $T_s = 10^{-4}$ sec. Use bilinear transformation technique for designing.*

Solution : Given data

$$A_p = 1\text{dB}, \qquad \Omega_p = 1404\,\pi$$

$$A_s = 60\text{dB}, \qquad \Omega_s = 8268\,\pi$$

$$F_{SF} = \frac{1}{T_s} = \frac{1}{10^{-4}} = 10^4 \text{ Hz}$$

Specifications of digital filter

$$A_p = 1\text{dB}, \qquad \omega_p = \frac{\Omega_p}{F_{SF}} = 0.441$$

$$A_s = 60\text{dB}, \qquad \omega_s = \frac{\Omega_s}{F_{SF}} = 2.597$$

Specifications of equivalent analog filter for bilinear transformation

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

$$= \tan\frac{\omega_p}{2} \qquad \text{Assuming } \frac{2}{T} = 1$$

$$= \tan\frac{0.441}{2} = 0.224$$

$$\Omega_s = \tan \frac{\omega_s}{2}$$

$$= \tan \frac{2.597}{2}$$

$$= 3.581$$

Order of the butterworth filter

$$N = \frac{1}{2} \frac{\log\left[\dfrac{10^{0.1 A_{sdB}} - 1}{10^{0.1 A_{pdB}} - 1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{1}{2} \frac{\log\left[\dfrac{10^{0.1 \times 60} - 1}{10^{0.1 \times 1} - 1}\right]}{\log\left(\dfrac{3.581}{0.224}\right)}$$

$$= 2.7359 \approx 3$$

Cutoff frequency

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(10^{0.1 A_{pdB}} - 1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(10^{0.1 A_{sdB}} - 1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.224}{\left(10^{0.1 \times 1} - 1\right)^{\frac{1}{6}}} + \frac{3.581}{\left(10^{0.1 \times 60} - 1\right)^{\frac{1}{6}}}\right\}$$

$$= 0.3193 \text{ rad/sec}$$

Poles of $H_a(s)$

Poles of $H_a(s)$ are given as,

$$p_k = \pm \Omega_c \, e^{j(N+2k+1)\pi/2N}$$

$$= \pm 0.3193 \, e^{j(4+2k)\pi/6}, \; k = 0, 1, 2$$

$$p_o = \pm 0.3193 \, e^{j4\pi/6}$$

$$= -0.159 + j\,0.276 \quad \text{and} \quad 0.159 - j\,0.276$$

$$P_1 = \pm\; 0.3193 \, e^{j\pi} = -0.3193 \quad \text{and} \quad +j\,0.3193$$

$$P_2 = \pm 0.3193 \, e^{j8\pi/6}$$

$$= -0.159 - j\,0.276 \qquad \text{and} \qquad 0.159 + j\,0.276$$

The complex conjugate pairs are,

$$s_o = -0.3193$$

$$s_1 = -0.159 + j\,0.276 \quad \text{and} \quad s_1^* = -0.154 - j\,0.276$$

System function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)(s-s_1^*)}$$

$$= \frac{(0.3193)^3}{(s+0.3193)(s+0.159-j0.276)(s+0.159+j0.276)}$$

$$= \frac{(0.3193)^3}{(s+0.3193)(s^2+0.318s+0.101)}$$

Applying bilinear transformaton to $H_a(s)$

Putting $s = \dfrac{1-z^{-1}}{1+z^{-1}}\left(Assuming\,\dfrac{2}{T}=1\right)$ in above equation,

$$H(z) = \frac{(0.3193)^3}{\left(\dfrac{1-z^{-1}}{1+z^{-1}}+0.3193\right)\left[\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2+0.318\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.101\right]}$$

$$= \frac{[0.3193(1+z^{-1})]^3}{(1.3193-0.68z^{-1})(1.419-1.798z^{-1}+0.783z^{-2})}$$

$$= \frac{0.0173(1+z^{-1})^3}{(1-0.515z^{-1})(1-1.267z^{-1}+0.552z^{-2})}$$

This is the system function of the digital filter

Example 7 : *The digital lowpass filter is to be designed that has a passband cutoff frequency $\omega_p = 0.375\pi$ with $\delta_p = 0.01$ and a stopband cutoff frequency $\omega_s = 0.5\pi$ with $\delta_s = 0.01$. The filter is to be designed using bilinear transformation. What orders of the butterworth, chebyshev filters are necessary to meet design specifications.*

Solution : Given data

$$\omega_p = 0.375\,\pi, \qquad \delta_p = 0.01$$

$$\omega_s = 0.5\,\pi, \qquad \delta_s = 0.01$$

For bilinear transformation, the equivalent analog filter specifications

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \tan\frac{0.375\pi}{2} \left(Assuming\frac{2}{T}=1\right) = 0.668$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.5\pi}{2} = 1$$

To determine order of butterworth filter

Here observe that ripple/decay of passband edge is $\delta_p = 0.01$. Hence passband edge attenuation is,

$$A_p = 1 - 0.1 = 0.99$$

And $$A_s = \delta_s = 0.01$$

$$N = \frac{1}{2}\frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{\log\left(\frac{\Omega s}{\Omega p}\right)}$$

$\therefore$ $$N = \frac{1}{2}\frac{\log\left[\left(\frac{1}{(0.01)^2}-1\right)\Big/\left(\frac{1}{(0.99)^2}-1\right)\right]}{\log\left(\frac{1}{0.668}\right)}$$

$$= 16.243 \approx 17$$

Thus the butterworth filter of order 17 is required.

To determine order of Chebyshev filter

We know that,

$$A_p^2 = \frac{1}{1+\varepsilon^2}$$

$\because$ $$(0.99)^2 = \frac{1}{1+\varepsilon^2}$$

$\therefore$ $$\varepsilon = 0.1425$$

Order of chebyshev filter is given as,

$$|H_a(j\Omega)|\,dB = -20\log_{10}\varepsilon - 6(N-1) - 20N\log_{10}\Omega$$

$$A_s dB = -20\log A_s = -20\log 0.01 = 40 \text{ dB}$$

$\therefore$ $$-40 = -20\log(0.1425) - 6(N-1) - 20N\log 1$$

$\therefore$ $$N = 10.487 \approx 11$$

Thus chebyshev filter of order 11 is required. The order of chebyshev filter is lower than that of butterworth filter.

Example 8 : *Use bilinear transformation to design a first order lowpass butterworth filter that has a 3dB cutoff frequency* $\omega_c = 0.2\pi$.

Solution : To determine cutoff frequency of equivalent analog filter

The cutoff frequency of equivalent analog filter for bilinear transformation is given as

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \tan\frac{\omega_c}{2} \quad \text{for } \frac{2}{T} = 1$$

$$= \tan\frac{0.2\pi}{2} = 0.325$$

Normalized lowpass butterworth filter

The system function of first order normalized butterworth filter is given as,

$$H_{an}(s) = \frac{1}{s+1}$$

To obtain H(s) using frequency transformation

We have the low pass filter of $\Omega_c = 1$. Now we want the lowpass filter of $\Omega_c = 0.325$. The low-pass to low-pass frequency transformation is given as,

$$s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s$$

Here $\Omega_p = 1$, $\Omega_{LP} = 0.325$. Hence $s \rightarrow \frac{s}{0.325}$

$$\therefore \quad H(s) = \frac{1}{\frac{s}{0.325}+1} = \frac{0.325}{s+0.325}$$

To determine H(z) using bilinear transformation

Bilinear transformation is given as $\left(\textit{assuming } \frac{2}{T} = 1 \right)$

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

$$\therefore \quad H(z) = \frac{0.325}{\frac{1-z^{-1}}{1+z^{-2}}+0.325}$$

$$= \frac{0.325\,(1+z^{-1})}{1.325-0.675z^{-1}} = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$

This is the system function of the required digital filter.

Example 9 : *Design an IIR digital filter that when used in the prefilter $A / D - H(z) - D / A$ structure will satisfy the following equivalent analog specifications :*
(i) LPF with −1 dB cutoff at 100 π rad/sec
(ii) Stopband attenuation of 35 dB or greater at 1000 π rad/sec
(iii) Monotonic stopband and passband
(iv) Sampling rate of 2000 samples/sec

Solution : (i) Given data :

The passband and stopbands are monotonic. This characteristic is obtained by butterworth approximation. Hence we have to design butterworth filter.

Sampling frequency $F_{SF} = 2000\ Hz$

Passband attenuation, $A_p = 1$ dB for $\Omega_p = 100\,\pi$ rad/sec or $F_p = 50$ Hz.

Stopband attenuation, $A_s = 35$ dB for $\Omega_s = 1000\,\pi$ rad/sec or $F_s = 500$ Hz.

Here digital filter transformation is not given. Hence we will assume bilinear transformation.

(ii) To define specifications of digital filter :

Let us convert the analog frequencies to their discrete time values i.e. $\left(\text{with } f = \frac{F}{F_{SF}}\right)$,

$$f_p = \frac{F_p}{F_{SF}} = \frac{50}{2000} = 0.025, \text{ Hence } \omega_p = 2\pi f_p = 2\pi \times 0.025 = 0.157$$

$$f_s = \frac{F_s}{F_{SF}} = \frac{500}{2000} = 0.25, \text{ Hence } \omega_s = 2\pi f_s = 2\pi \times 0.25 = 1.57$$

Thus the specifications of equivalent digital filter will be,

$$\left.\begin{array}{ll} A_p = 1\ dB\ , & \omega_p = 0.157 \\ A_s = 35\ dB, & \omega_s = 1.57 \end{array}\right\} \quad \ldots (1)$$

(iii) To obtain specifications of equivalent analog filter for bilinear transformation (prewarping) :

Here we are using bilinear transformation. Hence frequency specifications of equation 1 should be converted to their equivalent analog values according to bilinear transformation. This is called prewarping. The bilinear transformation frequency relationship is given as,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

We have seen that $\frac{2}{T}$ cancels out during calculations. Hence $\frac{2}{T}$ can be considered 1. Hence above equation will be,

$$\Omega = tan\frac{\omega}{2}$$

$$\therefore \quad \Omega_p = tan\frac{\omega_p}{2}$$

$$= tan\frac{0.157}{2} = 0.078 \text{ rad/sec}$$

$$\Omega_s = tan\,\frac{\omega_s}{2}$$

$$= tan\,\frac{1.57}{2} = 0.999 \text{ rad/sec}$$

Thus the specifications of equivalent analog filter according to bilinear transformation are

$$\left.\begin{array}{ll} A_p = 1\ dB, & \Omega_p = 0.078\ rad\ /\ \text{sec} \\ A_s = 35\ dB, & \Omega_s = 0.999\ rad\ /\ \text{sec} \end{array}\right\} \quad \text{... (2)}$$

(iv) To obtain order of the butterworth filter :

Here attenuations are given in dB hence let us use equation 4.5.19 in the book to obtain order of the filter. i.e.,

$$N = \frac{1}{2}\frac{log\left[\frac{10^{0.1\,A_s\,dB}-1}{10^{0.1\,A_p\,dB}-1}\right]}{log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Putting the values in above equation from equation (2),

$$N = \frac{1}{2}\frac{log\left[\frac{10^{0.1\times 35}-1}{10^{0.1\times 1}-1}\right]}{log\left(\frac{0.999}{0.078}\right)}$$

$$= 1.844$$

Hence select $N = 2$.

(v) To determine cutoff frequency Ω_c :

The 3 dB cutoff frequency can be obtained either by equation 4.5.11, equation 4.5.12 or equation 4.5.15 in the book. Here let us use equation 4.5.14 i.e.,

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}\right]$$

From equation 4.5.18 in the book we know that $\frac{1}{A_p^2} - 1 = \left(10^{0.1\, A_p\, dB} - 1\right)$ hence above equation can be written as,

$$\Omega_c = \frac{1}{2} \left[\frac{\Omega_p}{\left(10^{0.1\, A_p\, dB} - 1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(10^{0.1\, A_s\, dB} - 1\right)^{\frac{1}{2N}}} \right]$$

Putting values in above equation,

$$= \frac{1}{2} \left[\frac{0.078}{\left(10^{0.1 \times 1} - 1\right)^{\frac{1}{2 \times 2}}} + \frac{0.999}{\left(10^{0.1 \times 35} - 1\right)^{\frac{1}{2 \times 2}}} \right]$$

$$= 0.121 \quad \text{rad/sec} \qquad \ldots (3)$$

(v) To determine system function of the normalized (prototype) lowpass butterworth filter :

The normalized butterworth system function is given by equation 4.5.31 in the book as,

$$H_{an}(s) = \frac{1}{s^N + b_{N-1}\, s^{N-1} + \ldots + b_1\, s + 1}$$

For $N = 2$, $\quad H_{an}(s) = \frac{1}{s^2 + b_1\, s + 1}$

The value of b_1 can be obtained from table 4.5.1 in the book for $N = 2$. It is $b_1 = \sqrt{2}$. Hence above equation becomes,

$$H_{an}(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1} \qquad \ldots (4)$$

(vii) To obtain the system function of required analog filter by frequency transformation :

We obtained the system function of second order normalized lowpass filter (equation 4). We want the lowpass filter having cutoff frequency $\Omega_c = 0.121$ rad/sec. Hence let us use lowpass to lowpass frequency transformation. It is given by equation 4.7.1 as,

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}}\, s$$

Here Ω_p is passband edge frequency of the normalized filter, i.e. $\Omega_p = 1$ rad/sec and Ω_{LP} is passband edge frequency of the desired filter, i.e. $\Omega_c = \Omega_{LP} = 0.121$ rad/sec. Hence above transformation will be,

$$s \rightarrow \frac{1}{0.121}\, s$$

Applying this transformation to equation (4),

$$H_a(s) = H_{an}(s)\Big|_{s \to \frac{s}{0.121}}$$

$$= \frac{1}{\left(\frac{s}{0.121}\right)^2 + \sqrt{2}\left(\frac{s}{0.121}\right) + 1}$$

$$= \frac{0.0146}{s^2 + 0.171\, s + 0.0146} \qquad \text{... (5)}$$

This is the system function of equivalent analog filter.

(viii) To determine $H(z)$ using bilinear transformation :

Next step is to apply bilinear transformation to $H_a(s)$ to get $H(z)$. Bilinear transformation is given as,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

We have considered $\frac{2}{T} = 1$, since it cancels out. Hence above equation will be,

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

Applying this transformation to equation (5) we get,

$$H_a(s) = H_{an}(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.0146}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.171\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.0146}$$

$$= \frac{0.0123\left(1+z^{-1}\right)^2}{1-1.66\, z^{-1} + 0.711\, z^{-2}}$$

This is the system function of the required butterworth digital filter.

Example 10 : *A system is represented by a transfer function $H(z)$ is given by,*

$$H(z) = 3 + \frac{4z}{z-\frac{1}{2}} - \frac{2}{z-\frac{1}{4}}$$

(i) Does this $H(z)$ represent a FIR or IIR filter. Why ?

(ii) Give a difference equation realization of this system using direct form-I.

(iii) Draw the block diagram for the direct form-II canonic realization, and give the governing equations for implementation.

Solution : (i) To check whether FIR or IIR filter :

Consider the given system function. It can be written as,

$$H(z) = 3+\frac{4z}{z-0.5}-\frac{2}{z-0.25}$$

$$= \frac{7z^2-5.25z+1.375}{z^2-0.75z+0.125} \qquad \dots (1)$$

Observe that the system function has numerator polynomial of order 2 as well as denominator polynomial of order 2.

The system function has poles as well as zeros. Hence it represents IIR filter. The FIR filter has all zero system function.

(ii) Direct form-I realization :

We can write equation (1) as follows :

$$H(z) = \frac{7-5.25z^{-1}+1.375z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} \qquad \dots (2)$$

Let us expand summations of equation 4.10.1 in the book for $M=N=2$. i.e.,

$$H(z) = \frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

The direct form-I realization of above equation for generalized M and N is shown in Fig. 4.10.4 in the book. On comparing above equation with equation (2) we get,

$b_0=7$, $b_1 = -5.25$ $b_2=1.375$ and

$a_1 = -0.75$, $a_2=0.125$

Based on the above values and Fig. 4.10.4 in the book, the direct form-I realization is given in Fig. 2.

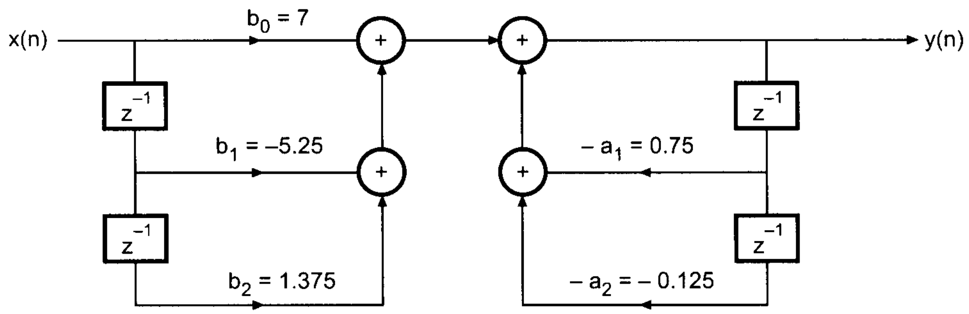


Fig. 2 Direct form-I realization of $H(z)$ of example 10

(iii) Direct form-II canonic realization :

We can write equation 1 as follows :

$$H(z) = \frac{7 - 5.25\,z^{-1} + 1.375\,z^{-2}}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}} \qquad \dots (3)$$

We know that $H(z) = \frac{Y(z)}{X(z)}$. Hence above equation will be,

$$\frac{Y(z)}{X(z)} = \frac{7 - 5.25\,z^{-1} + 1.375\,z^{-2}}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}}$$

Let us rearrange the above equation as,

$$\frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{7 - 5.25\,z^{-1} + 1.375\,z^{-2}}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}}$$

i.e. $$\frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)} = \frac{7 - 5.25\,z^{-1} + 1.375\,z^{-2}}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}}$$

Let, $$H_1(z) \cdot H_2(z) = \frac{1}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}} \left(7 - 5.25\,z^{-1} + 1.375\,z^{-2}\right) \qquad \dots (4)$$

$\therefore$ Let $$H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1 - 0.75\,z^{-1} + 0.125\,z^{-2}} \qquad \dots (5)$$

and $$H_2(z) = \frac{Y(z)}{W(z)} = 7 - 5.25\,z^{-1} + 1.375\,z^{-2} \qquad \dots (6)$$

Cross multiplying in equation 5 (all pole function) we get,

$$W(z)\left[1 - 0.75\,z^{-1} + 0.125\,z^{-2}\right] = X(z)$$

$$\therefore \qquad W(z) = X(z) + 0.75\,z^{-1}\,W(z) - 0.125\,z^{-2}\,W(z)$$

Taking inverse z-transform of above equation,

$$w(n) = x(n) + 0.75\,w(n-1) - 0.125\,w(n-2) \qquad \dots (7)$$

Fig. 3 shows the direct form realization of above equation.

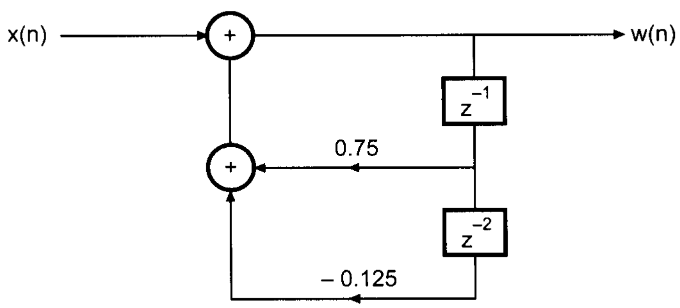


Fig. 3 Direct form realization of equation 7 i.e. $H_1(z)$

Crossmultiply the terms in equation 6, we get,

$$Y(z) = \left(7 - 5.25\, z^{-1} + 1.375\, z^{-2}\right) W(z)$$

$$= 7\, W(z) - 5.25\, z^{-1}\, W(z) + 1.375\, z^{-2}\, W(z)$$

Taking inverse z-transform of above equation,

$$y(n) = 7\, w(n) - 5.25\, w(n-1) + 1.375\, w(n-2) \qquad \ldots (8)$$

Fig. 4 shows the direct form implementation of above equation.

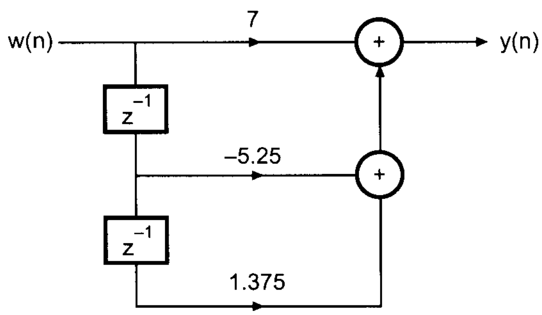


Fig. 4 Direct form realization of equation 8, i.e. $H_2(z)$

Here note that Fig. 3 is realization of $H_1(z)$ and above figure shows realization of $H_2(z)$. The realization of $H(z) = H_1(z) \cdot H_2(z)$ [i.e. equation 4] can be obtained by cascading realizations of Fig. 3 and Fig. 4. Such realization is shown in Fig. 5.

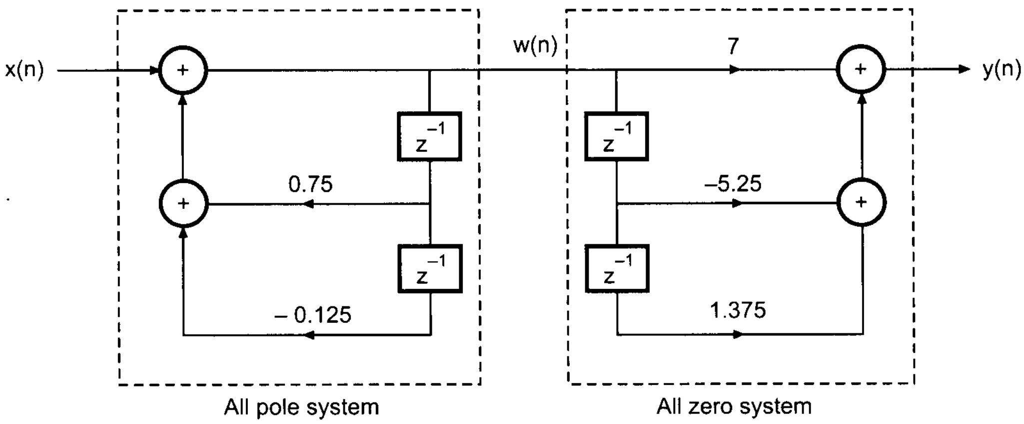


Fig. 5 Realization of $H(z) = H_1(z) \cdot H_2(z)$

In the above figure observe that the delays can be combined into two. Then the realization becomes as shown in Fig. 6.

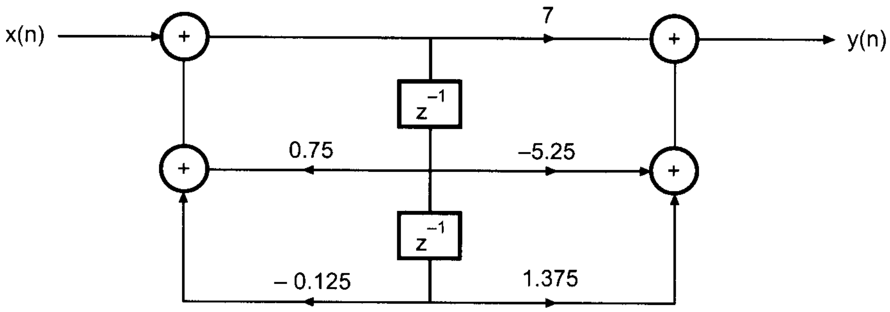


Fig. 6 Direct form-II canonic realization

In the above figure observe that there are two delay elements. The order of $H(z)$ is also two. Hence it is canonic realization. It is also called as direct form II realization of IIR filters.

Example 11 : *Realize the following system function in cascade form.*

$$H(z) = \frac{1+\frac{1}{5}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)\left(1+\frac{1}{4}z^{-2}\right)}$$

Solution : The given transfer function can be written as the product of two functions. i.e.,

$$H(z) = H_1(z)\cdot H_2(z) = \frac{1}{1+\frac{1}{4}z^{-1}}\cdot\frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}} \quad \text{... (1)}$$

Here $$H_1(z) = \frac{1}{1+\frac{1}{4}z^{-1}}$$

and $$H_2(z) = \frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$

The above two equations are in the form of equation 4.10.15 in the book. They are written as follows :

$$H_1(z) = \frac{b_{10}}{1+a_{11}z^{-1}} = \frac{1}{1+\frac{1}{4}z^{-1}} \quad \text{... (15)}$$

$$H_2(z) = \frac{b_{20}+b_{21}z^{-1}}{1+a_{21}z^{-1}+a_{22}z^{-2}} = \frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}} \quad \text{... (16)}$$

The cascade realization of $H_1(z)$ and $H_2(z)$ as given above is shown in the folowing figure.

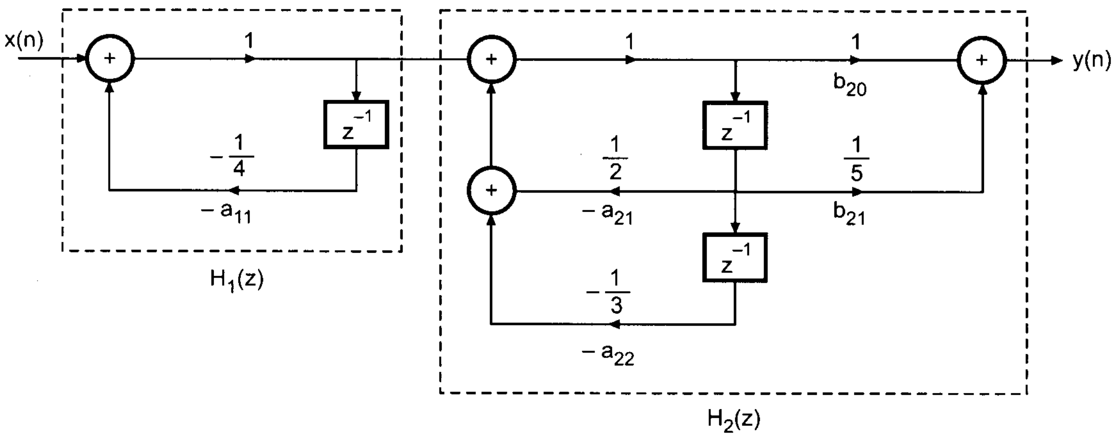


Fig. 7 Cascade realization of $H(z)$ of example 11

Example 12 : *Realize the following system function in parallel form.*

$$H(z) = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} \cdot \frac{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Solution : Let us write the given system function as,

$$H(z) = \frac{z\left(z-\frac{2}{3}\right)}{z^2-\frac{7}{8}z+\frac{3}{32}} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{z^2-z+\frac{1}{2}}$$

i.e.

$$\frac{H(z)}{z} = \frac{z-\frac{2}{3}}{z^2-\frac{7}{8}z+\frac{3}{32}} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{z^2-z+\frac{1}{2}}$$

$$= \frac{z-\frac{2}{3}}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{8}\right)} \cdot \frac{z^2-\frac{7}{4}z-\frac{1}{2}}{\left(z-\frac{1}{2}-j\frac{1}{2}\right)\left(z-\frac{1}{2}+j\frac{1}{2}\right)}$$

$$= \frac{A_1}{z-\frac{3}{4}} + \frac{A_2}{z-\frac{1}{8}} + \frac{A_3}{z-\frac{1}{2}-j\frac{1}{2}} + \frac{A_4}{z-\frac{1}{2}+j\frac{1}{2}}$$

Calculating the values of A_1, A_2, A_3 and A_4 we get,

$$\frac{H(z)}{2} = \frac{2.933}{z-\frac{3}{4}} - \frac{2.947}{z-\frac{1}{8}} + \frac{2.507-j\,10.45}{z-\frac{1}{2}-j\frac{1}{2}} + \frac{2.507+j10.45}{z-\frac{1}{2}+j\frac{1}{2}}$$

Let us combine the first two terms and last two terms. Because of this, the complex values will be combined into real coefficients. i.e.,

$$\frac{H(z)}{z} = \frac{-0.014\,z+1.843}{z^2-\frac{7}{8}z+\frac{3}{32}} + \frac{5.02z+7.743}{z^2-z+\frac{1}{2}}$$

$$\therefore \quad H(z) = \frac{-0.014\,z^2+1.843\,z}{z^2-\frac{7}{8}z+\frac{3}{32}} + \frac{5.02\,z^2+7.743\,z}{z^2-z+\frac{1}{2}}$$

The above equation can also be written as,

$$H(z) = \frac{-0.014+1.843\,z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{5.02+7.743\,z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} \qquad \text{... (1)}$$

The above equation has two terms, they can be called as,

$$H_1(z) = \frac{b_{10}+b_{11}\,z^{-1}}{1+a_{11}\,z^{-1}+a_{12}\,z^{-2}} = \frac{-0.014+1.843\,z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} \qquad \text{... (2)}$$

and
$$H_2(z) = \frac{b_{20}+b_{21}\,z^{-1}}{1+a_{21}\,z^{-1}+a_{22}\,z^{-2}} = \frac{5.02+7.743\,z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} \qquad \text{... (3)}$$

Observe that the above two equations are in the form of equation 4.10.23 in the book. The realization of equation 4.10.23 in the book is shown in Fig. 4.10.15 in the book. The realization of $H_1(z)$ and $H_2(z)$ in parallel is shown in Fig. 8.

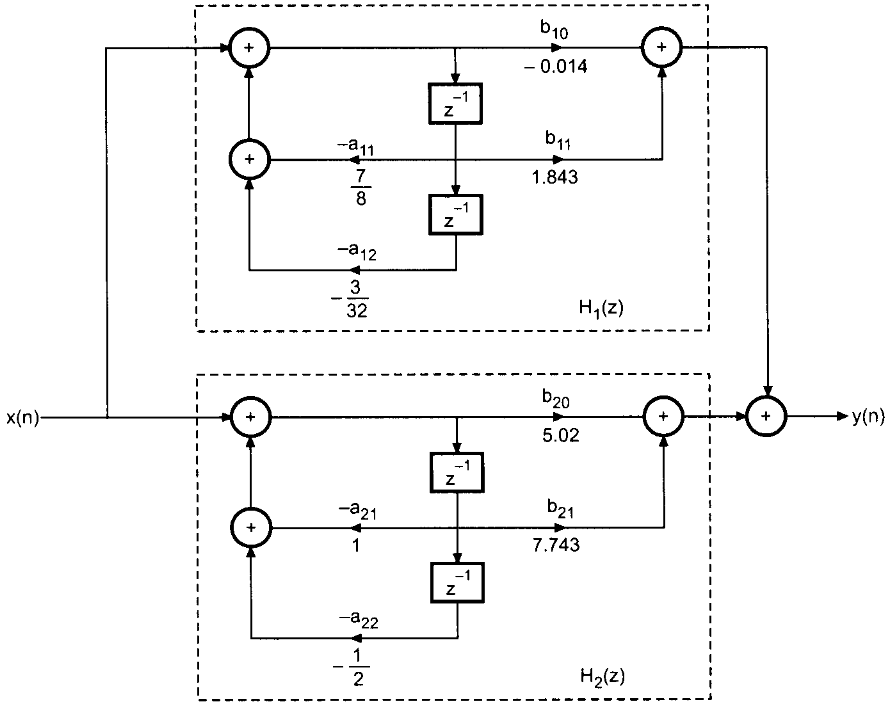


Fig. 8 Parallel realization of $H(z)$ of example 12

Example 13 : *A system function is specified by its transfer function H(z) given by,*

$$H(z) = \frac{(z-1)(z-2)(z+1)(z)}{\left[z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]\left(z-\frac{j}{4}\right)\left(z+\frac{j}{4}\right)}$$

Realize the system function in following forms :

i) Direct form-I ii) Direct form-II iii) Cascade of two biquadratic sections iv) Parallel realization in constant, linear and biquadratic sections.

Solution : i) Direct form-I

The given system function can be written in polynomial form as,

$$H(z) = \frac{z^4 - 2z^3 - z^2 + 2z}{z^4 - z^3 + 0.5625z^2 - 0.0625z + 0.0313}$$

Here $b_0 = 1,\ b_1 = -2, b_2 = -1,\ b_3 = 2$

and $a_1 = 1,\ a_2 = 0.5625,\ a_3 = -0.0625,\ a_4 = 0.0313$

Direct form-I realization is shown below :

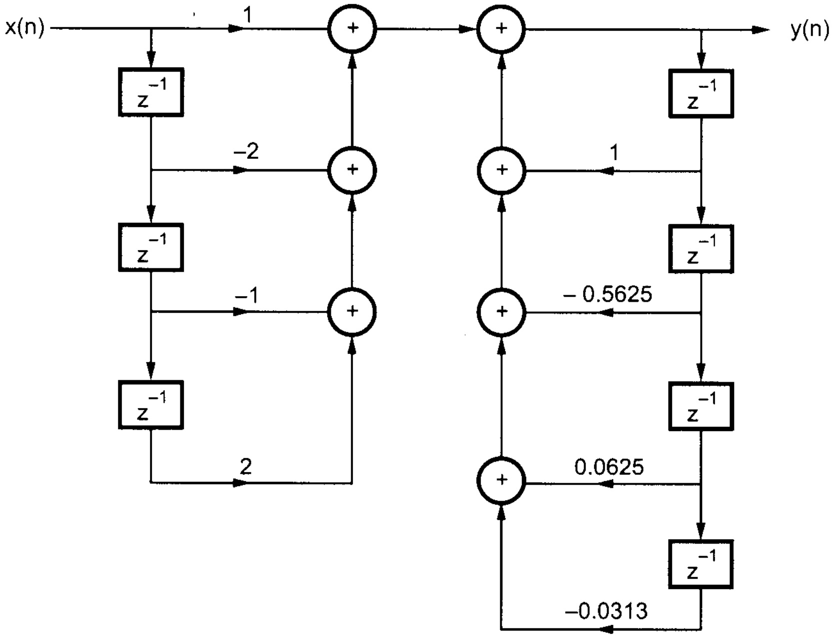


Fig. 9 Direct form-I structure of H(z) of Ex. 13

ii) Direct form-II

Fig. 10 shows the direct form-II realization of the system

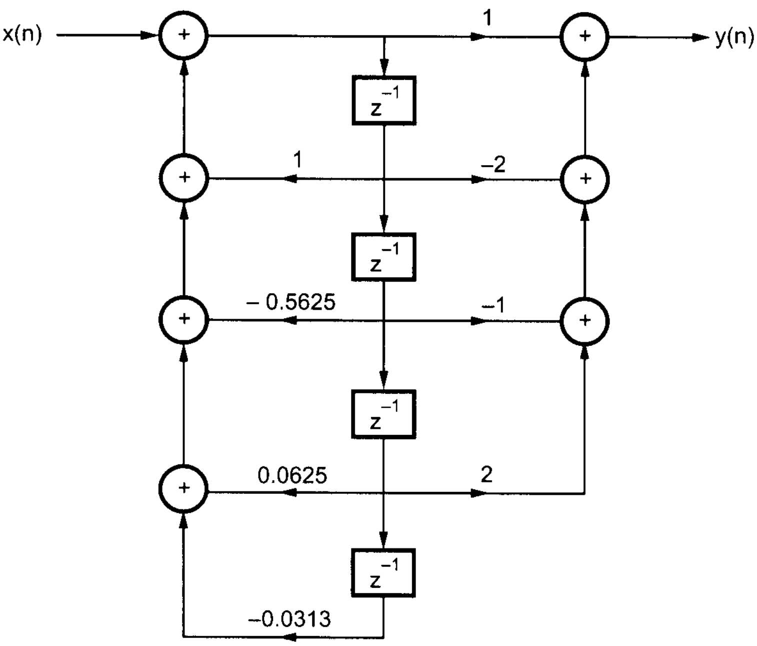


Fig. 10 Direct form-II realization of H(z) of Ex.13

iii) Cascade realization

The given H(z) can be represented as,

$$H(z) = \frac{(z-1)\,(z-2)}{[z-(0.5+j0.5)][z-(0.5-j0.5)]}\cdot\frac{(z+1)z}{(z-j0.25)(z+j0.25)}$$

$$= \frac{z^2-3z+2}{z^2-z+0.5}\cdot\frac{z^2+z}{z^2+j0.0625}$$

$$= H_1(z)\cdot H_2(z)$$

Fig. 11 shows the cascade realization of above equation.

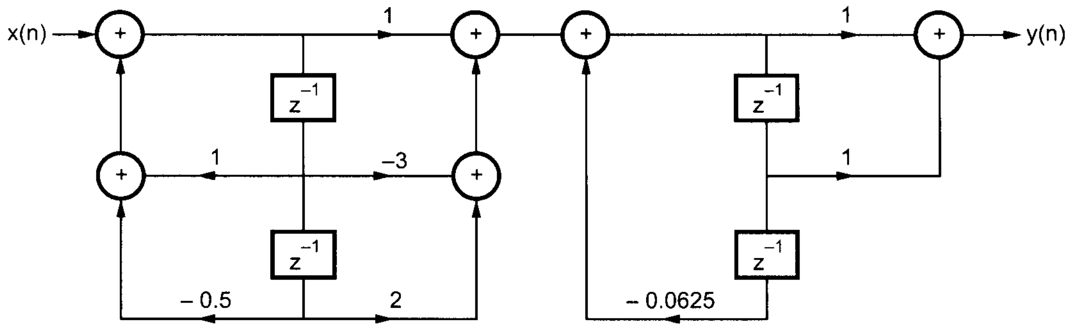


Fig. 11 Cascade realization of H(z) of Ex.13

iv) Parallel form realization

Expanding H(z) in partial fractions we get

$$\frac{H(z)}{z} = \frac{-2.7683+j2.1517}{z-(0.5+j0.5)}+\frac{-2.7683-j2.1517}{z-(0.5-j0.5)}+\frac{3.2683-j7.8371}{z-j0.25}+\frac{3.2683+j7.8371}{z+j0.25}$$

The realization using second order sections is asked. Combining complex conjugate poles in above equation,

$$\frac{H(z)}{z} = \frac{-5.5366z+0.6112}{z^2-z+0.5}+\frac{6.5366z+3.9184}{z^2+0.0625}$$

$$= \frac{-5.5366+0.6112z^{-1}}{1-z^{-1}+0.5z^{-2}}+\frac{6.5366+3.9184z^{-1}}{1+0.0625z^{-2}}$$

$$= H_1(z) + H_2(z)$$

Fig. 12 shows the parallel form realization of the above two second order sections.

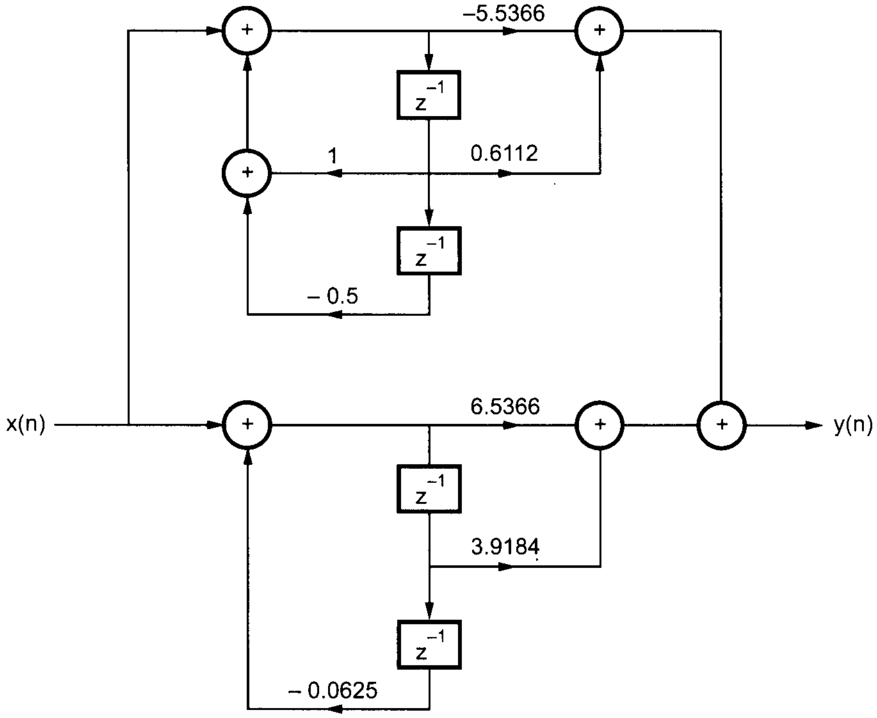


Fig. 12 Parallel form realization of H(z) of Ex.13

Example 14 : *Obtain the direct form-II (canonic) and cascade realization of*

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

The cascade section should consist of two biquadratic sections.

Solution : i) Cascade form realization

The given system function is,

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)\ (z^2-6z+8)}$$

$$= \frac{z^2-4z+3}{z^2+6z+5} \cdot \frac{z^2+5z+6}{z^2-6z+8}$$

$$= \frac{1-4z^{-1}+3z^{-2}}{1+6z^{-1}+5z^{-2}} \cdot \frac{1+5z^{-1}+6z^{-2}}{1-6z^{-1}+8z^{-2}}$$

$$= H_1(z) \cdot H_2(z)$$

Fig. 13 shows the realization of above equation.

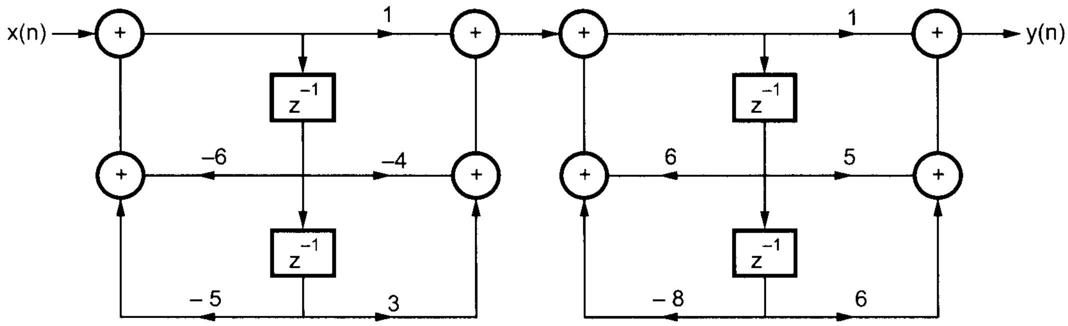


Fig. 13 Cascade form realization of H(z) of Ex. 14

ii) Direct form-II realization

The given system function is,

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

$$= \frac{z^4+z^3-11z^2-9z+18}{z^4+0z^3-23z^2+18z+40}$$

$$= \frac{1+z^{-1}-11z^{-2}-9z^{-3}+18z^{-4}}{1+0z^{-1}-23z^{-2}+18z^{-3}+40z^{-4}}$$

Fig. 14 shows the direct form-II realization of system given by above equation

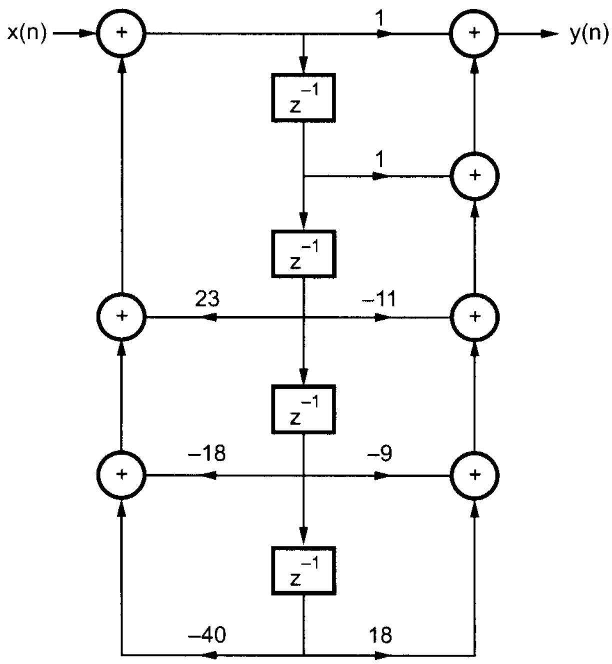


Fig. 14 Direct form-II realization of H(z) of Ex.14

Example 15 : *A discrete time system H(z) is expressed as,*

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)(1+2z^{-1})}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

i) Find the difference equation of the system

ii) Realize the system in direct form I and II

iii) Realize parallel and cascade forms using second order sections

Solution : i) To obtain difference equation

The given system can be expressed in polynomials ratio as,

$$H(z) = \frac{10+8.33z^{-1}-20z^{-2}+6.667z^{-3}}{1-1.875z^{-1}+1.14688z^{-2}-0.5313z^{-3}+0.0469z^{-4}}$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{10+8.33z^{-1}-20z^{-2}+6.667z^{-3}}{1-1.875z^{-1}+1.14688z^{-2}-0.5313z^{-3}+0.0469z^{-4}}$$

$$\therefore \quad Y(z) = 1.875\,z^{-1}\,Y(z) - 1.14688\,z^{-2}Y(z) + 0.5313\,z^{-3}\,Y(z)$$
$$- 0.0469\,z^{-4}\,Y(z) + 10\,X(z) + 8.33\,z^{-1}\,X(z)$$
$$- 20\,z^{-2}\,X(z) + 6.667\,z^{-3}\,X(z)$$

Taking inverse z-transform,

$$y(n) = 1.875\,y(n-1) - 1.14688\,y(n-2) + 0.5313\,y(n-3)$$
$$- 0.0469\,y(n-4) + 10\,x(n) + 8.33\,x(n-1)$$
$$- 20\,x(n-2) + 6.667\,x(n-3)$$

This is the required difference equation.

ii) To realize in direct form-I and II

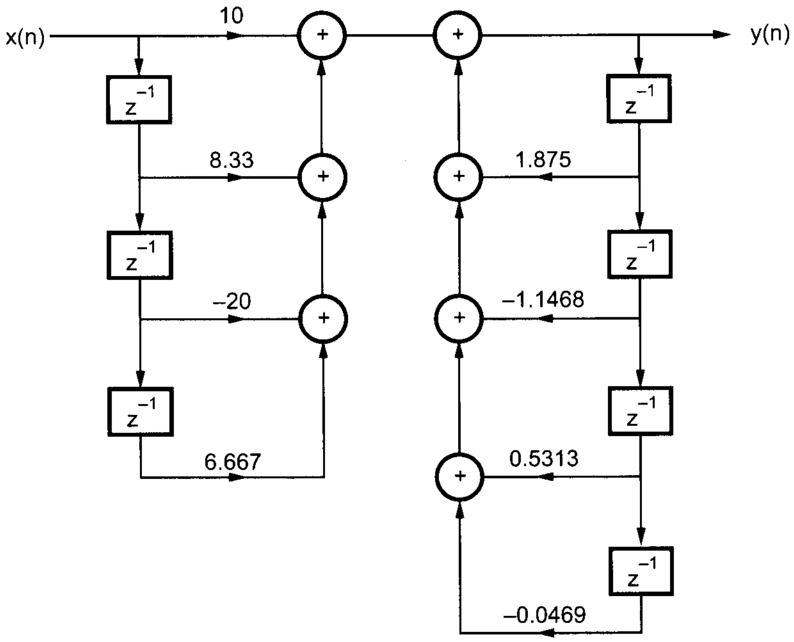
We have,

$$b_0 = 10,\ b_1 = 8.33,\ b_2 = -20,\ b_3 = 6.667$$

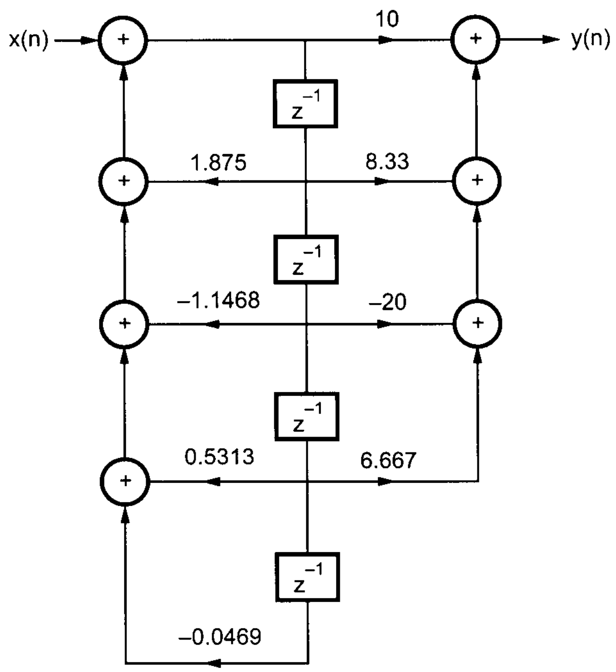
$$a_1 = -1.875,\ a_2 = 1.1468,\ a_3 = -0.5313,\ a_4 = 0.0469$$

Fig. 15 (a) shows the direct form-I realization of the system. Fig.15 (b) shows the direct form-II realization of the system.

Please see Fig. 15 on next page.



(a) Direct form - I realization



(b) Direct form - II realization

Fig. 15 Direct form realization of H(z) of Ex. 15

iii) Realization in cascade and parallel forms

cascade form realization

The given function can be expressed as,

$$H(z) = \frac{10z(z-0.5)(z-0.6667)(z+2)}{(z-0.75)(z-0.125)[z-(0.5+j0.5)][z-(0.5-j0.5)]}$$

$$= \frac{10z\,(z-0.5)}{(z-0.75)\,(z-0.125)} \cdot \frac{(z-0.6667)(z+2)}{[z-(0.5+j0.5)][z-(0.5-j0.5)]}$$

$$= \frac{10z^2-5z}{z^2-0.875z+0.0938} \cdot \frac{z^2+1.333z-1.333}{z^2-z+0.5}$$

$$= \frac{10-5z^{-1}}{1-0.875z^{-1}+0.0938z^{-2}} \cdot \frac{1+1.333z^{-1}-1.333z^{-2}}{1-z^{-1}+0.5z^{-2}}$$

$$= H_1(z) \cdot H_2(z)$$

A cascade realization of above second order sections is shown in Fig. 16

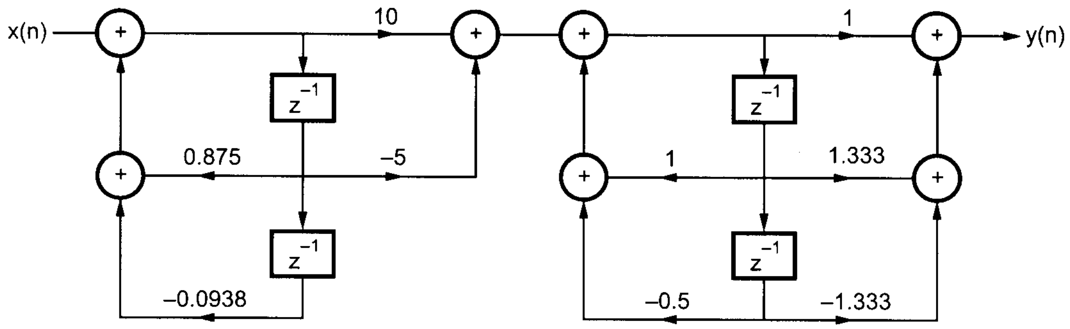


Fig. 16 Cascade realization of H(z) of Ex. 15

Parallel form realization

The system function is expressed as a rational function i.e.,

$$H(z) = \frac{10+8.33z^{-1}-20z^{-2}+6.667z^{-3}}{1-1.875z^{-1}+1.1468z^{-2}-0.5313z^{-3}+0.0469z^{-4}}$$

$$= \frac{10z^4+8.33z^3-20z^2+6.667z}{z^4-1.875z^3+1.1468z^2-0.5313z+0.0469}$$

$$\therefore \quad \frac{H(z)}{z} = \frac{10z^3+8.33z^2-20z+6.667}{z^4-1.875z^3+1.1468z^2-0.5313z+0.0469}$$

Expanding above equation in partial fractions,

$$\frac{H(z)}{z} = \frac{9.9279}{z-1.2813} + \frac{-13.4141}{z-0.11} + \frac{6.7431-j12}{z-(0.2419+j0.5238)} + \frac{6.7431+j12}{z-(0.2419-j0.5238)}$$

Let us combine the complex conjugate poles and convert to second order sections :

$$\frac{H(z)}{z} = \frac{-3.4861z+16.0953}{z^2-1.3913z+0.1409} + \frac{13.4862z+9.309}{z^2-0.4838z+0.3328}$$

$$\therefore \quad H(z) = \frac{-3.4861+16.0953z^{-1}}{1-1.3913z^{-1}+0.1409z^{-2}} + \frac{13.48622+9.304z^{-1}}{1-0.48382z^{-1}+0.3328z^{-2}}$$

$$= H_1(z) + H_2(z)$$

The parallel form realization based on above equations is shown in Fig. 17.

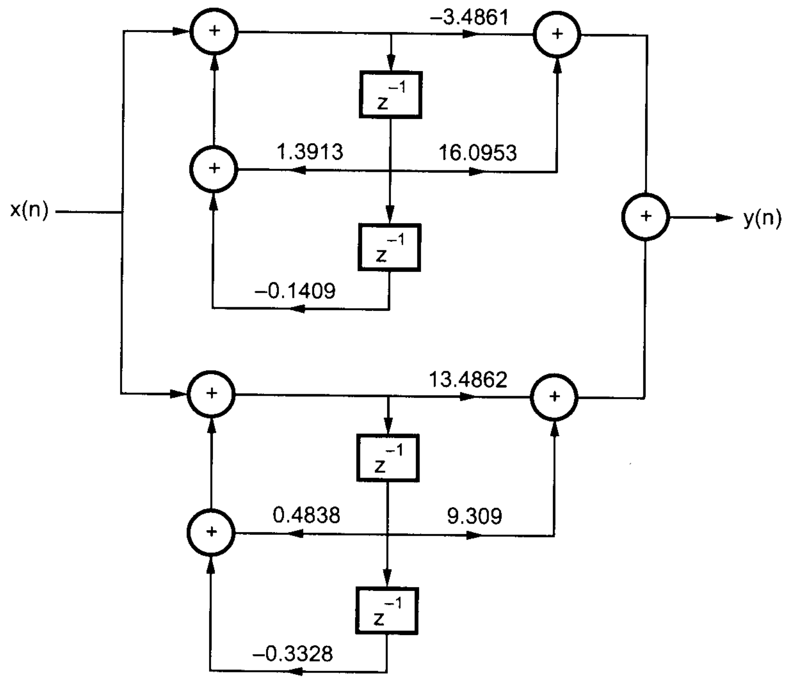


Fig. 17 Parallel form realization of H(z) of Ex. 15

Example 16 : *Design a lowpass digital filter using bilinear transformation. The filter is to be monotonic in both stop and pass-bands and has all of the following characteristics :*
i) an acceptable passband ripple of 1 dB,
ii) a passband edge of 0.3 π rad and
iii) stopband attenuation of 40 dB or greater beyond 0.6 π rad.

Solution : Given data

$$A_p = 1 \text{ dB} \quad \omega_p = 0.3\,\pi$$

$$A_s = 40 \text{ dB} \quad \omega_s = 0.6\,\pi$$

Step 1 : Prewarping

$$\Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.3\,\pi}{2} = 0.509 \text{ rad/sec}$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.6\,\pi}{2} = 1.376 \text{ rad/sec}$$

Step 2 : Order of butterworth filter (Monotonic stop and passbands) :

$$N = \frac{1}{2}\,\frac{\log\left[\dfrac{10^{0.1\,A_s\,dB}-1}{10^{0.1\,A_p\,dB}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \qquad \text{By equation 4.5.19}$$

$$= \frac{1}{2}\,\frac{\log\left[\dfrac{10^{0.1\times 40}-1}{10^{0.1\times 1}-1}\right]}{\log\left[\dfrac{1.376}{0.509}\right]} = 5.3 \approx 6$$

Step 3 : Cutoff frequency Ω_c :

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(10^{0.1\,A_p\,dB}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(10^{0.1\,A_s\,dB}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.509}{\left(10^{0.1\times 1}-1\right)^{\frac{1}{2\times 6}}} + \frac{1.376}{\left(10^{0.1\times 40}-1\right)^{\frac{1}{2\times 6}}}\right\}$$

$$= 0.6 \text{ rad/sec}$$

Step 4 : System function of normalized analog filter for N = 6 :

$$H_{an}(s) = \frac{1}{\left(s^2+0.517\,s+1\right)\left(s^2+\sqrt{2}\,s+1\right)\left(s^2+1.931\,s+1\right)} \qquad \text{from table 4.5.1}$$

Step 5 : Frequency transformation :

Lowpass to lowpass transformation is given as,

$$s \to \frac{\Omega_p}{\Omega_{LP}}\,s$$

$\Omega_p = 1$ for normalized filter and $\Omega_{LP} = \Omega_c = 0.6$

$$\therefore \qquad s \rightarrow \frac{s}{0.6}$$

$$H_a(s) = H_{an}(s)\Big|_{s \rightarrow \frac{s}{0.6}}$$

$$= \frac{1}{\left[\left(\frac{s}{0.6}\right)^2 + 0.517\left(\frac{s}{0.6}\right) + 1\right]\left[\left(\frac{s}{0.6}\right)^2 + \sqrt{2}\left(\frac{s}{0.6}\right) + 1\right]\left[\left(\frac{s}{0.6}\right)^2 + 1.931\left(\frac{s}{0.6}\right) + 1\right]}$$

$$= \frac{(0.6)^6}{(s^2 + 0.31\,s + 0.36)(s^2 + 0.848\,s + 0.36)(s^2 + 1.156\,s + 0.36)}$$

Step 6 : Applying bilinear transformation to get H(z) :

$$H(z) = H(s)\Big|_{s = \frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{(0.6)^6}{\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.31\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.36\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.848\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.36\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.156\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.36\right]}$$

$$= \frac{0.005(1 + z^{-1})^6}{(1 - 0.766\,z^{-1} + 0.628\,z^{-2})(1 - 0.58\,z^{-1} + 0.231\,z^{-2})(1 - 0.5\,z^{-1} + 0.081\,z^{-2})}$$

Example 17 : *Transform the analog filter* $H(s) = \frac{s+3}{(s+1)(s+2)}$ *to a digital filter using the matched z-transform. Let T = 0.5 sec.*

Solution : Here $(s - z_k)$ is converted to $(1 - e^{z_k T} z^{-1})$. Hence H(z) for given H(s) will be,

$$H(z) = \frac{1 - e^{-3\times0.5}\,z^{-1}}{(1 - e^{-1\times0.5}\,z^{-1})(1 - e^{-2\times0.5}\,z^{-1})}$$

$$= \frac{1 - 0.223\,z^{-1}}{(1 - 0.6\,z^{-1})(1 - 0.367\,z^{-1})}$$

Example 18 : *Using the impulse response technique, design a lowpass digital filter that is equiripple in the passband, and monotone in the stopband. The filter passband edge is at* $0.1\,\pi$ *rad with a ripple of 2.5 dB or less and the stopband edge at 0.2 π rad with attenuation 40 dB or more. Use T = 1.*

Solution : Given :

$$A_p = 2.5 \text{ dB}, \quad \omega_p = 0.1\,\pi$$

$$A_s = 40 \text{ dB}, \quad \omega_s = 0.2\,\pi$$

Step 1 : $\omega = \Omega T$ and T = 1. Hence $\omega = \Omega$. Therefore specifications of equivalent analog filters are,

$$A_p = 2.5 \text{ dB.} \qquad \Omega_p = 0.1\,\pi$$

$$A_s = 40 \text{ dB} \qquad \Omega_s = 0.2\,\pi$$

Step 2 : Type of approximation :

passband : equiripple

stopband : monotone

This is type 1 chebyshev filter approximation.

Step 3 : Normalized specifications :

Let $\Omega_p = 1$ rad/sec. Then normalized Ω_s will be,

$$\Omega_{s\,(\text{normalized})} = \frac{\Omega_s}{\Omega_p} = \frac{0.2\,\pi}{0.1\,\pi} = 2 \text{ rad/sec}$$

$$\therefore \quad A_p = 2.5 \text{ dB} \quad \Omega_p = 1$$

$$A_s = 40 \text{ dB} \quad \Omega_s = 2$$

Step 4 : Order 'N' of the filter :

$$\varepsilon = \left[10^{0.1\,A_p\,\text{dB}} - 1\right]^{\frac{1}{2}}$$

$$= \left[10^{0.1 \times 2.5} - 1\right]^{\frac{1}{2}} = 0.8822$$

$$|H_a(j\Omega)| = -20\log\varepsilon - 6(N-1) - 20\log\Omega$$

Here $\Omega = \Omega_s = 2$, $|H_a(j\Omega)| = -40$ dB,

$$-40 = -20\log 0.8822 - 6(N-1) - 20\,N\log 2$$

$$\therefore \quad N = 3.924 \approx 4$$

Step 5 : System function of normalized analog filter :

Here A_p = 2.5 dB. The table for 2.5 dB does not exist. Hence we have to calculate $H_{an}(s)$ from poles. Poles are given by,

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi \qquad K = 1, 2, \ldots\ldots N$$

$$\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \frac{1}{4}\sinh^{-1}\left(\frac{1}{0.8822}\right) = 0.2431$$

$$\therefore \quad \sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] = \sinh\ (0.2431) = 0.2455$$

and $\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] = \cosh\ (0.2431) = 1.0297$

Hence
$$\sigma_k = -0.2455\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = 1.0297\cos\left(\frac{2k-1}{2N}\right)\pi$$

Following table lists the calculations.

k	$\left(\frac{2k-1}{8}\right)\pi$	σ_k	Ω_k
1	$\frac{\pi}{8}$	– 0.094	0.951
2	$\frac{3\pi}{8}$	– 0.226	0.394
3	$\frac{5\pi}{8}$	– 0.226	– 0.394
4	$\frac{7\pi}{8}$	– 0.094	– 0.951

Thus poles are

$$s_1 = -0.094 + j\ 0.951$$

$$s_1^* = -0.094 - j\ 0.951$$

$$s_2 = 0.226 + j\ 0.394$$

$$s_2^* = -0.226 - j\ 0.394$$

$$H_{an}(s) = \frac{k}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

$$= \frac{k}{(s+0.094-j\,0.951)(s+0.094+j\,0.951)(s+0.226-j\,0.394)(s+0.226+j\,0.394)}$$

$$= \frac{k}{s^4+0.64\,s^3+1.2\,s^2+0.45\,s+0.182}$$

Here N = 4 (even). Hence k will be,

$$k = \frac{b_0}{\sqrt{1+\varepsilon^2}} \text{ for 'N' even}$$

$$= \frac{0.182}{\sqrt{1+(0.8822)}} = 0.136$$

$$\therefore \quad H_{an}(s) = \frac{0.136}{s^4+0.64\,s^3+1.2\,s^2+0.45\,s+0.182}$$

Step 6 : To apply frequency transformation :

Lowpass to lowpass transformation : $s \rightarrow \frac{\Omega_P}{\Omega_{Lp}} s$

Here $\Omega_P = 1$ and $\Omega_{LP} = 0.1\,\pi \quad \Rightarrow \quad s \rightarrow \frac{s}{0.1\,\pi}$

$$\therefore \quad H_a(s) = H_{an}(s)\big|_{s \rightarrow \frac{s}{0.1\pi}}$$

$$\therefore \quad H_a(s) = \frac{0.136}{\left(\frac{s}{0.1\,\pi}\right)^4 + 0.64\left(\frac{s}{0.1\,\pi}\right)^3 + 1.2\left(\frac{s}{0.1\,\pi}\right)^2 + 0.45\left(\frac{s}{0.1\,\pi}\right) + 0.182}$$

$$= \frac{0.00132}{s^4+0.2\,s^3+0.118\,s^2+0.014\,s+0.00177}$$

Step 7 : Applying impulse invariant transformation.

Impulse invariant transformation is applied to poles of $H_a(s)$. The poles of $H_a(s)$ are complex. Reader is advised to calculate poles and apply conventional impulse invariant transformation.

Example 19 : *Transform the analog filter* $H(s) = \frac{1}{s+\alpha}$, $\alpha > 0$ *to a digital filter using the back-ward difference mapping. Comment on the stability of the digital filter.*

Solution : The back-word difference mapping is nothing but approximation of derivatives. It is given as,

$$s = \frac{1-z^{-1}}{T}$$

$$\therefore \quad H(z) = H(s)\big|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{1}{\frac{1-z^{-1}}{T}+\alpha}$$

$$= \frac{T}{(1+\alpha\,T)-z^{-1}}$$

Stability :

$$H(z) = \frac{T}{(1+\alpha T)-z^{-1}}$$

$$= \frac{T}{1+\alpha T} \cdot \frac{z}{z-\dfrac{1}{1+\alpha T}}$$

Here pole is at $z=\frac{1}{1+\alpha T}$, which is less than 1 since $\alpha T>0$. It lies inside the unit circle. Therefore this filter is stable.

Example 20 : *Starting from a lowpass butterworth prototype analog filter, design a fourth order butterworth bandpass analog filter with upper and lower band edge frequencies 10 rad/sec and 5 rad/sec.*

Solution : Here order of the filter is not given. Let N = 2.

Step 1 : Prototype filter :

During frequency conversion from lowpass to bandpass, the order is doubled. Hence we have to select prototype filter of order 1. i.e.,

$$H_{an}(s) = \frac{1}{s+1} \quad \text{from table 4.5.1}$$

Step 2 : Frequency transformation :

Lowpass to bandpass transformation : $s \rightarrow \frac{s^2+\Omega_l \Omega_u}{s(\Omega_u-\Omega_l)}$

Here $\Omega_l = 5$ and $\Omega_u = 10$,

$$\therefore \quad s \rightarrow \frac{s^2+10\times 5}{s(10-5)} = \frac{s^2+50}{5\,s}$$

$$\therefore \quad H_a(s) = H_{an}(s)\Big|_{s\rightarrow \frac{s^2+50}{5\,s}}$$

$$= \frac{1}{\dfrac{s^2+50}{5\,s}+1}$$

$$= \frac{5\,s}{s^2+5\,s+50}$$

Example 21 : *Design a butterworth filter using the bilinear transformation for the following specifications :*

$$0.8 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1 \quad for \quad 0 \leq \omega \leq 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \leq 0.2 \quad for \quad 0.6\,\pi \leq \omega \leq \pi$$

Solution : **Given data**

$$A_p = 0.8 = 1.93 \text{ dB} \qquad \omega_p = 0.2\,\pi$$

$$A_s = 0.2 = 14 \text{ dB} \quad \omega_s = 0.6\,\pi$$

Step 1 : Prewappring :

$$\Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.2\pi}{2} = 0.325$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.6\pi}{2} = 1.376$$

Step 2 : Order of filter :

$$N = \frac{1}{2}\frac{\log\left[\frac{10^{0.1\,A_s\,dB}-1}{10^{0.1\,A_p\,dB}-1}\right]}{\log\frac{\Omega_s}{\Omega_p}}$$

$$= \frac{1}{2}\frac{\log\left[\frac{10^{0.1\times 14}-1}{10^{0.1\times 1.93}-1}\right]}{\log\left(\frac{1.376}{0.325}\right)}$$

$$= 1.3 \approx 2$$

Step 3 : Cutoff frequency Ω_c :

$$\Omega_c = \frac{1}{2}\left\{\frac{\Omega_p}{\left(10^{0.1\,A_p\,dB}-1\right)^{\frac{1}{2N}}} + \frac{\Omega_s}{\left(10^{0.1\,A_s\,dB}-1\right)^{\frac{1}{2N}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{0.325}{\left(10^{0.1\times 1.93}-1\right)^{\frac{1}{2\times 2}}} + \frac{1.376}{\left(10^{0.1\times 14}-1\right)^{\frac{1}{2\times 2}}}\right\}$$

$$= 0.5 \text{ rad/sec}$$

Step 4 : System function of prototype lowpass filter :

For N = 2, from table 4.5.1,

$$H_{an}(s) = \frac{1}{s^2+\sqrt{2}\,s+1}$$

Step 5 : To obtain $H_a(s)$ by frequency transformation :

Lowpass to lowpass transformation : $s \to \frac{\Omega_p}{\Omega_{LP}} s$

Here $\Omega_p = 1,\ \Omega_{LP} = 0.5 \Rightarrow s \to \frac{s}{0.5}$

$$\therefore \quad H_a(s) = H_{an}(s)\Big|_{s \to \frac{s}{0.5}}$$

$$= \frac{1}{\left(\frac{s}{0.5}\right)^2+\sqrt{2}\left(\frac{s}{0.5}\right)+1}$$

$$= \frac{0.25}{s^2+0.707\,s+0.25}$$

Step 6 : To apply bilinear transformation :

$$H(z) = H(s)\Big|_{s \to \frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.25}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.707\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.25}$$

$$= \frac{0.127\left(1+2\,z^{-1}+z^{-2}\right)}{1-0.766\,z^{-1}+0.277\,z^{-2}}$$

Example 22 : *Consider the system function.*

$$H(z) = \frac{1+\frac{1}{5}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

i) Realize the system in direct form-I.

ii) Realize the system in parallel form using first and second order direct form-II sections.

Solution : i) Direct form realization :

$$\frac{Y(z)}{X(z)} = \frac{1+\frac{1}{5}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$= \frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z^{-3}}$$

$$\therefore\ Y(z)-\frac{1}{4}z^{-1}Y(z)+\frac{5}{24}z^{-2}Y(z)+\frac{1}{12}z^{-3}Y(z) = X(z)+\frac{1}{5}z^{-1}X(z)$$

$$\therefore\ y(n)-\frac{1}{4}y(n-1)+\frac{5}{24}y(n-2)+\frac{1}{12}y(n-3) = x(n)+\frac{1}{5}x(n-1)$$

$$\therefore\ y(n) = \frac{1}{4}y(n-1)-\frac{5}{24}y(n-2)-\frac{1}{12}y(n-3)+x(n)+\frac{1}{5}x(n-1)$$

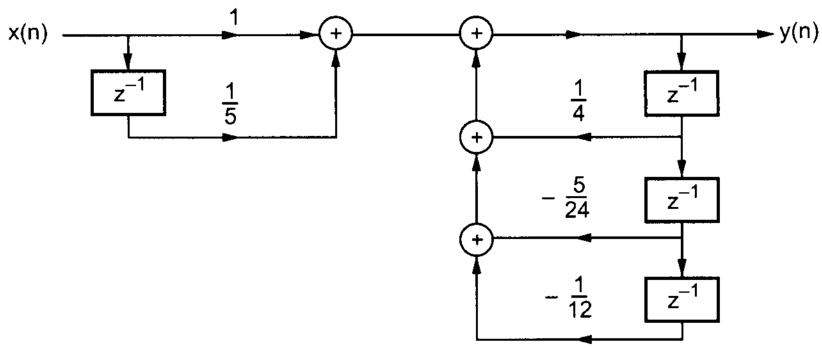


Fig. 18 Direct form-I realization

ii) Parallel form realization :

$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$\therefore\ \frac{H(z)}{z} = \frac{0.78\,z-0.2864}{z^2-\frac{1}{2}z+\frac{1}{3}} + \frac{0.216}{z+\frac{1}{4}}$$

$$\therefore \quad H(z) = \frac{0.78-0.2864\,z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}+\frac{0.216}{1-\frac{1}{4}z^{-1}}$$

$$= H_1(z)+H_2(z)$$

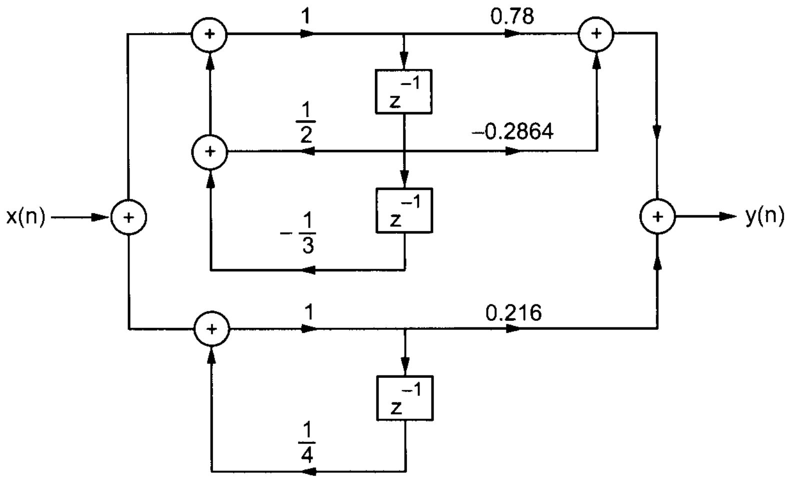


Fig. 19 Parallel form realization

Example 23 *Realize the following second order system using direct form-I of realization.*

$$y(n) = y(n-1) - 0.5y(n-3) + 0.5x(n-1)$$

Solution :

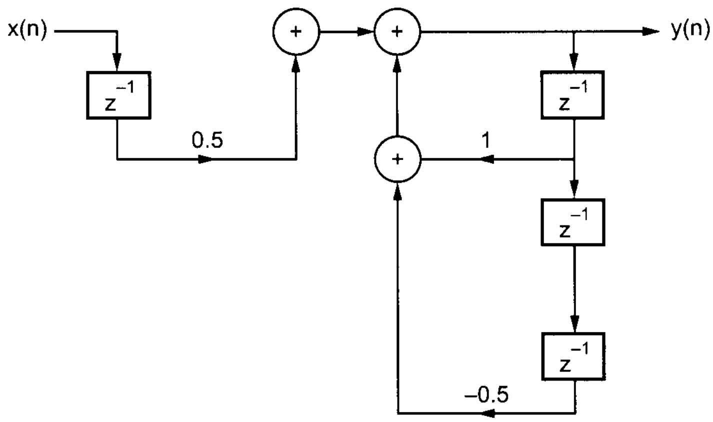


Fig. 20 Direct form-I realization

Example 24 : *Obtain the cascade and parallel form realizations for the following system.*

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

Solution : Cascade form realization

Taking z-transform of given equation,

$$Y(z) = -0.1z^{-1}Y(z) + 0.2z^{-2}Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$\therefore \quad Y(z)\left[1 + 0.1z^{-1} - 0.2z^{-2}\right] = X(z)\left[3 + 3.6z^{-1} + 0.6z^{-2}\right]$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= \frac{3\left(1 + 0.2z^{-1}\right)\left(1 + z^{-1}\right)}{\left(1 + 0.5z^{-1}\right)\left(1 - 0.4z^{-1}\right)}$$

$$= H_1(z) \cdot H_2(z)$$

Where $H_1(z) = 3 \cdot \dfrac{1 + 0.2z^{-1}}{1 + 0.5z^{-1}}$ and $H_2(z) = \dfrac{1 + z^{-1}}{1 - 0.4z^{-1}}$

Fig. 21 shows the cascade realization of $H_1(z)$ and $H_2(z)$.

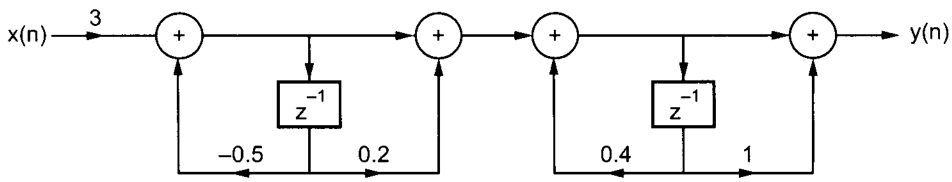


Fig. 21 Cascade realization

Parallel form realization

The given $H(z)$ is expressed as,

$$H(z) = \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$

$$\therefore \quad \frac{H(z)}{z} = \frac{3z^2 + 3.6z + 0.6}{z(z^2 - 0.1z - 0.2)}$$

$$= \frac{3z^2 + 3.6z + 0.6}{z(z + 0.5)(z - 0.4)}$$

$$= -\frac{3}{z} - \frac{1}{z + 0.5} + \frac{7}{z - 0.4}$$

$$\therefore \quad H(z) = -3 - \frac{1}{1+0.5z^{-1}} + \frac{7}{1-0.4z^{-1}}$$

$$= H_1(z) + H_2(z) + H_3(z)$$

Here $H_1(z) = -3$, $H_2(z) = -\frac{1}{1+0.5z^{-1}}$, $H_3(z) = \frac{7}{1-0.4z^{-1}}$

Fig. 22 shows the parallel form realization.

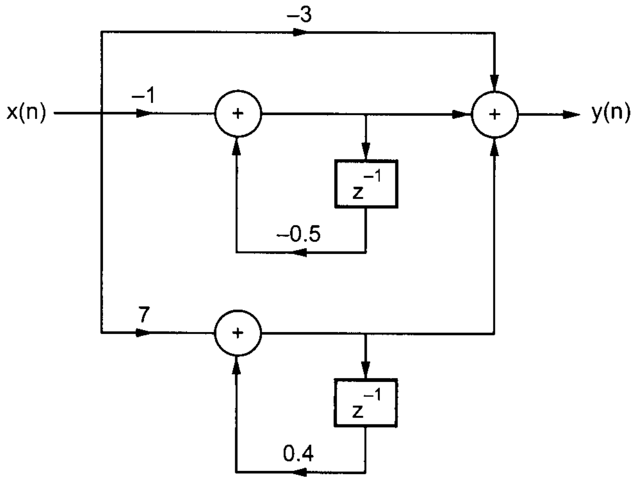


Fig. 22 Parallel form realization

Example 25 *An analog filter has a transfer function :*

$$H(s) = \frac{10}{s^2 + 7s + 10}$$

Design a digital filter equivalent to this using impulse-invariant method.

Solution : Consider the given transfer function,

$$H(s) = \frac{10}{s^2 + 7s + 10}$$

$$= \frac{10}{(s+2)(s+5)}$$

$$= \frac{\frac{10}{3}}{s+2} - \frac{\frac{10}{3}}{s+5}$$

Impulse invariant transformation is given as,

$$\frac{1}{s - p_k} \rightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

Applying this transformation to H(s),

$$H(z) = \frac{\frac{10}{3}}{1-e^{-2T}z^{-1}} - \frac{\frac{10}{3}}{1-e^{-5T}z^{-1}}$$

Example 26 : *Determine the direct form-II realization for the following system :*

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-2)$$

Solution : Here the coefficients are $a_1 = 0.1$, $a_2 = -0.72$ and $b_0 = 0.7$, $b_1 = 0$, $b_2 = -0.252$

Fig. 23 shows the direct form-II structure.

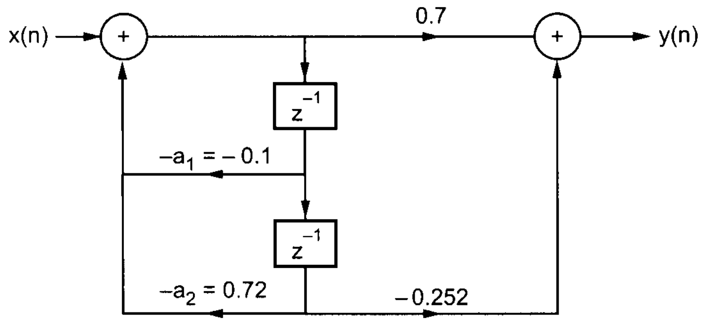


Fig. 23 Direct form-II realization

Example 27 : *Using Impulse invariant method, Find H(z) at T = 1 sec.,*

$$H(s) = \frac{2}{s^2 + 8s + 15}$$

Solution :

$$H(s) = \frac{2}{s^2+8s+15} = \frac{2}{(s+5)(s+3)}$$

$$= \frac{1}{s+3} - \frac{1}{s+5}$$

Impulse invariant transformations is given as

$$\frac{1}{s-p_k} \longrightarrow \frac{1}{1-e^{p_k T}z^{-1}}, \qquad \text{Here } T = 1$$

$$\therefore \quad H(z) = \frac{1}{1-e^{-3}z^{-1}} - \frac{1}{1-e^{-5}z^{-1}}$$

$$= \frac{1}{1-0.05z^{-1}} - \frac{1}{1-0.0067z^{-1}}$$

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