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Module : 3

FFT (Fast Fourier Transform)

It is an algorithm using this we can compute DFT in faster way basically, there exists two types of FFT algorithm

i) DIT FFT (Decimation In Time FFT)

ii) DIF FFT (Decimation In Frequency FFT)

★ Imp 8-10M

DIT FFT :-

Discuss DITFFT algorithm along with flow diag. (or) Derive expressions for N pt DFT using DIT-FFT

DITFFT uses divide and conquer approach algorithm

In this approach N pt DFT is divided into two $\frac{N}{2}$ pt DFT, each $\frac{N}{2}$ is divided into 2 $\frac{N}{4}$ pt.

and this process is continued till the 2 pt DFT's are formed. In other words N pt DFT is performed as several 2 pt DFT's.

WKT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \longrightarrow (1) \quad k=0,1,\ldots N-1$$

Decompose the sum into even and odd indexed samples

$$\Rightarrow X(k) = \sum_{\substack{n=0 \\ n=\text{even}}}^{n=N-2} x(n).W_N^{kn} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{n=N-1} x(n) W_N^{kn}.$$

Sub $n = 2r$ into first term & $n = (2r+1)$ into 2nd term

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{k(2r)} + \sum_{r=0}^{N/2-1} x(2r+1)\, W_N^{k(2r+1)}$$

when $n = 0$, $2r = 0$, $r = 0$ | $n = N-2$, $2r = N-2$, $r = \frac{N}{2} - 1$

when $n = 1$, $2r+1 = 1$, $2r = 0$, $r = 0$ | $n = N-1$, $2r+1 = N-1$, $2r = N-2$, $r = \frac{N}{2} - 1$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r)\, W_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{N/2-1} x(2r+1)\, W_{\frac{N}{2}}^{kr}$$

$$= G(k) + W_N^{k} H(k)$$

2 $\frac{N}{2}$ pt DFT's $\quad 0 \le k \le \frac{N}{2} - 1$

$$\because W_{\frac{N}{2}}^{rk} = e^{-j\left(\frac{2\pi k}{\frac{N}{2}}\right)r} = e^{-j\left(\frac{2\pi}{N}\right)2rk} = W_N^{2rk}$$

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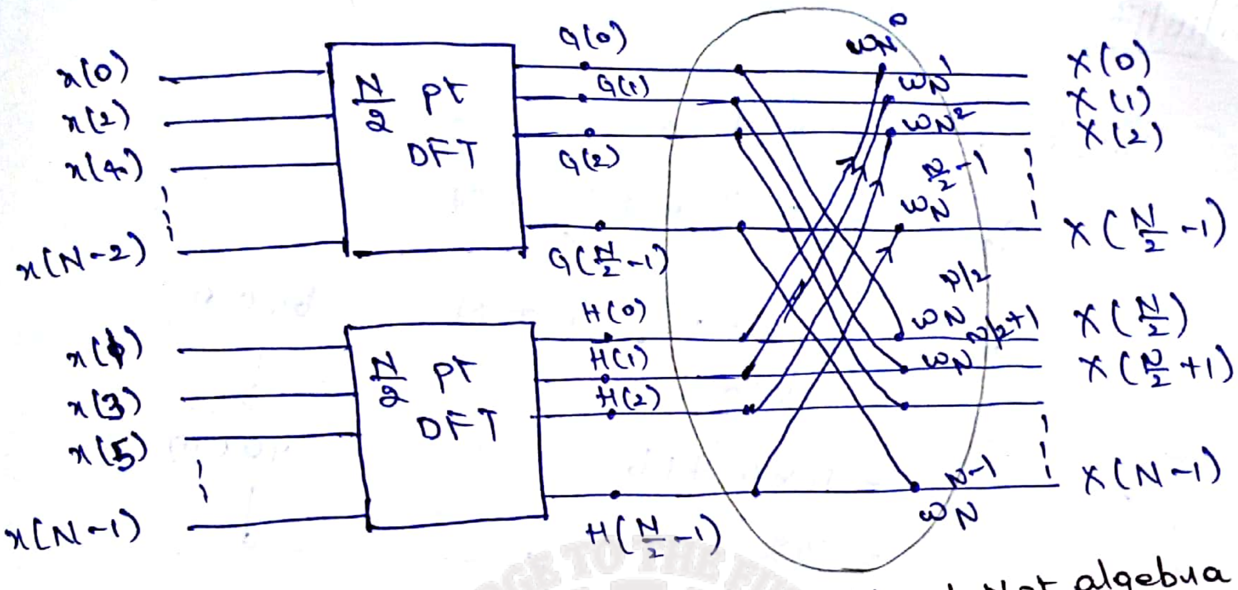
$\therefore$ N pt of DFT using $G(k)$ & $H(k)$

$$X(k) = G(k) + H(k)\, W_N^{k} \quad 0 \le k \le \frac{N}{2} - 1$$

$$X(k) = G\left(k + \frac{N}{2}\right) + H\left(k + \frac{N}{2}\right) W_N^{k + \frac{N}{2}} \quad \frac{N}{2} \le k \le N-1$$

First decimation Flow diagram for N pt DFT using DITFFT



Combined N pt algebra

∴ Total CM required is

$$\eta_1 = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = 2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$$

Total no. of CM required for N pt combined algebra.

CM required for even numbered samples.

for N = 8 ⇒ 40 CM.

CM required for odd numbered samples.

For second decimation, each $\frac{N}{2}$ pt sequences [G(K) (or) H(K)] are further decimated into 2; $\frac{N}{4}$ pt sequences

$$G(K) = \sum_{l=0}^{\frac{N}{4}-1} g(2l)\, W_{\frac{N}{2}}^{K(2l)} + \sum_{l=0}^{N/4-1} g(2l+1)\, W_{\frac{N}{2}}^{K(2l+1)}$$

$$\Rightarrow \sum_{l=0}^{N/4-1} g(2l)\, W_{\frac{N}{4}}^{Kl} + \sum_{l=0}^{N/4-1} g(2l+1)\, W_{\frac{N}{4}}^{Kl} \left(W_{\frac{N}{2}}^{K}\right)$$

$$G(K) = \begin{cases} A(K) + B(K)\, W_{N/2}^{K} & 0 \le K \le \frac{N}{4}-1 \\ A(K+\frac{N}{4}) + B(K+\frac{N}{4})\, W_{\frac{N}{2}}^{K} & \frac{N}{4} \le K \le \frac{N}{2}-1 \end{cases}$$

$$H(K) = \begin{cases} C(K) + D(K)\, W_{N/2}^{K} & 0 \le K \le \frac{N}{4}-1 \\ C(K+N/4) + D(K+\frac{N}{4})\, W_{\frac{N}{2}}^{K} & \frac{N}{4} \le K \le \frac{N}{2}-1 \end{cases}$$

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Total CM required is

$$\eta_2 = 4\left(\frac{N}{4}\right)^2 + \cancel{2}\frac{N}{\cancel{2}} + N$$

$$\eta_2 = 4\left(\frac{N^2}{16}\right) + N + N$$

$$= 4 \times 4 + 16$$

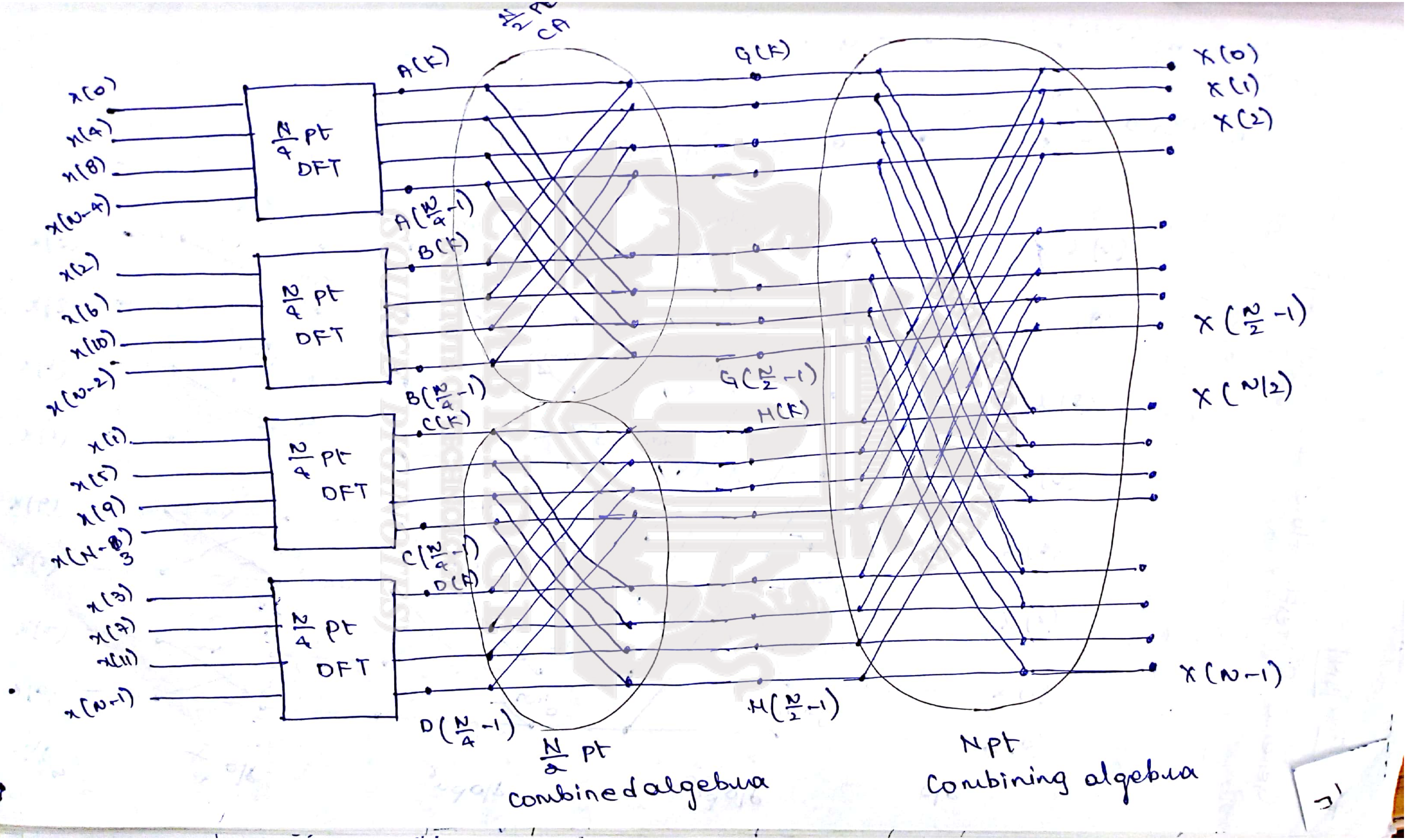
$$= 32 \text{ CM}$$

64 cm

$\downarrow$

40 cm

2nd Decimation flow Diagram for N pt DFT


x(0)
x(4)
x(8)
x(N-4)
N/4 pt DFT
x(2)
x(6)
x(10)
x(N-2)
N/4 pt DFT
x(1)
x(5)
x(9)
x(N-3)
N/4 pt DFT
x(3)
x(7)
x(11)
x(N-1)
N/4 pt DFT
A(K)
A(N/4 -1)
B(K)
B(N/4 -1)
C(K)
C(N/4 -1)
D(K)
D(N/4 -1)
N/2 pt CA
G(K)
G(N/2 -1)
H(K)
H(N/2 -1)
X(0)
X(1)
X(2)
X(N/2 -1)
X(N/2)
X(N-1)
N/2 pt
Combined algebra
N pt
Combining algebra

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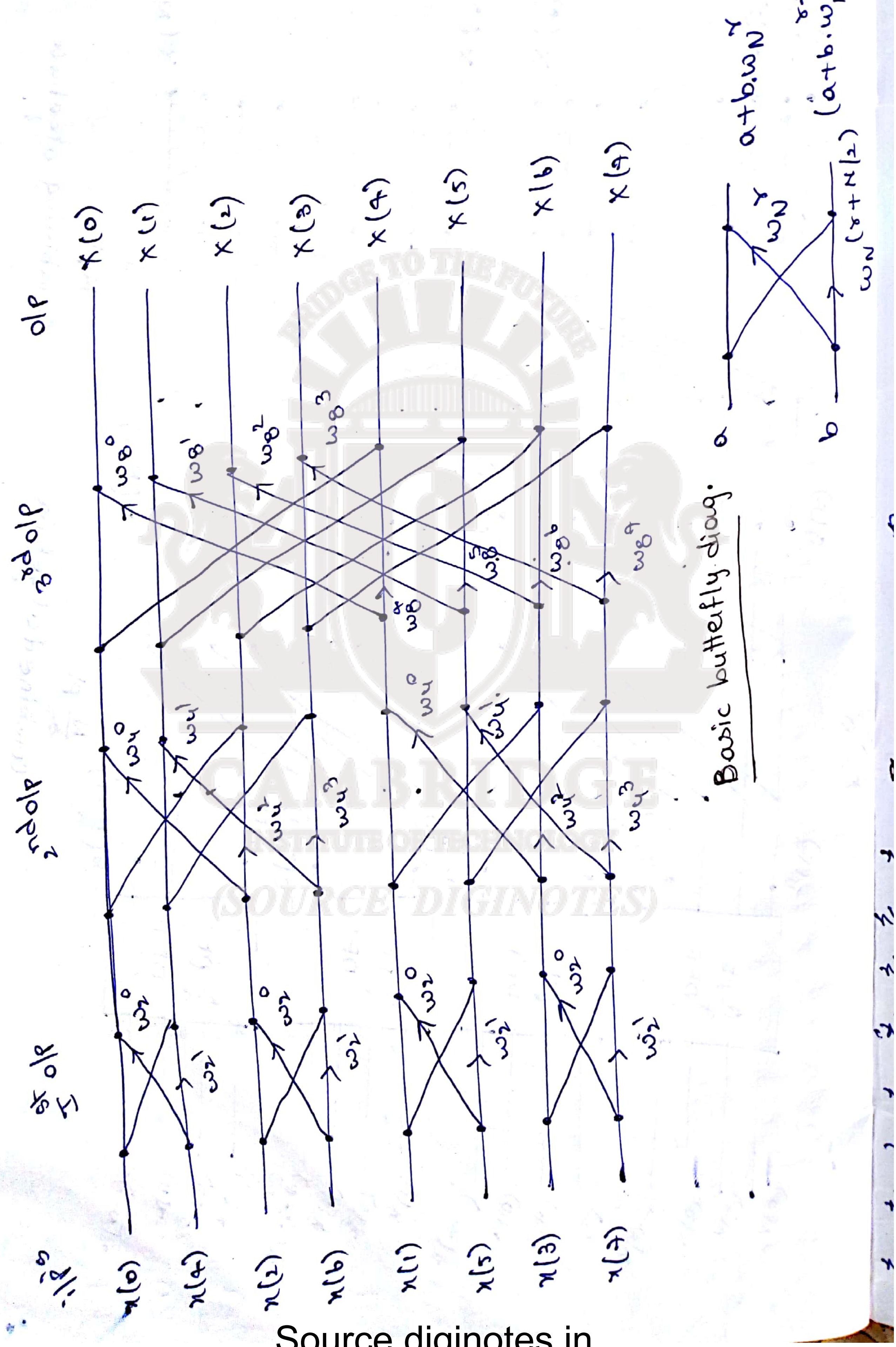
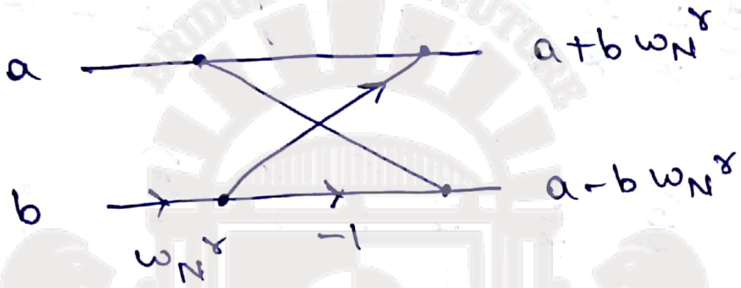
General Butterfly flow diagram 8pt DFT using DITFFT



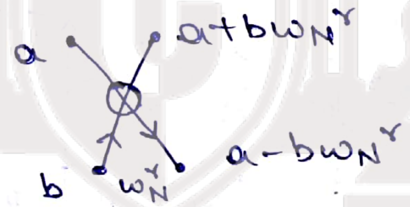
- Input data is bit reverse order
- output sequence appears in normal order
- further reductions of computations is done by cooley - Tukey algorithm

Cooley - Turkey algorithm

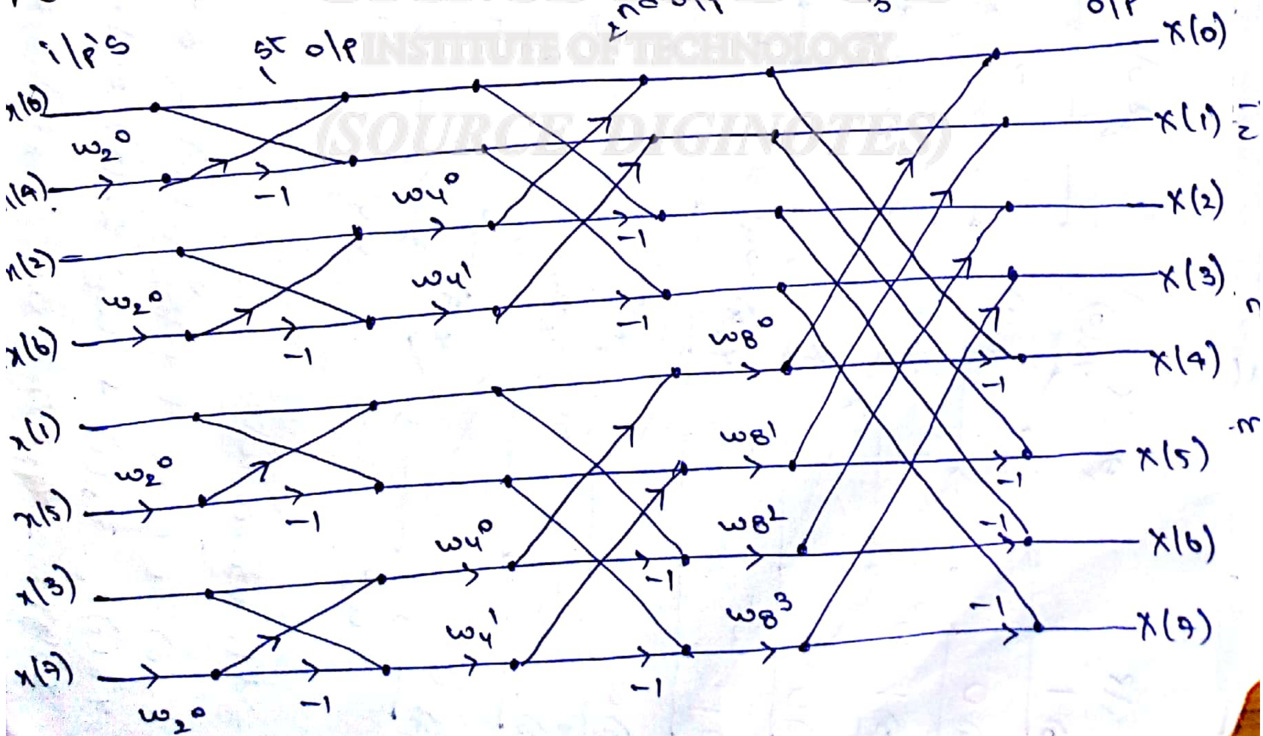


By using this algorithm we can reduce CM per Butterfly by 1

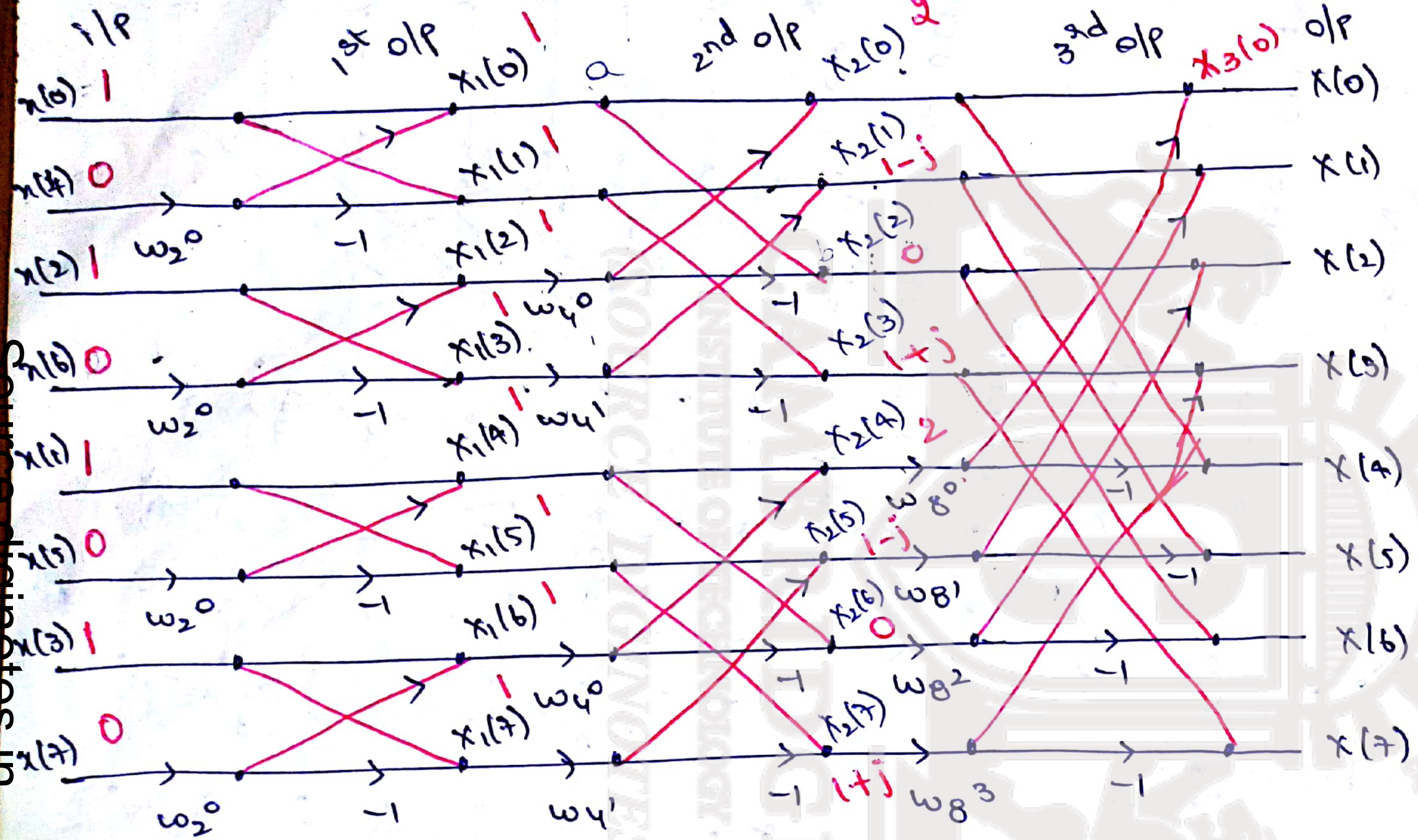
(or)



Reduced flow diagram for 8pt DFT using DITFFT



find 8pt DFT $x(n) = \{1\,1\,1\,1\,0\,0\,0\,0\}$ using DITFFT use cooley-Tukey algorithm, show all Intermediate steps



1st Intermediate

$X_1(0) = \underbrace{a + b W_2^0}_{x(0)+W_2^0 x(4)} = 1$

$X_1(1) = a - b W_2^1 = 1$

$X_1(2) = 1 + 0 \cdot 1 = 1$

$X_1(3) = 1 - 0 \cdot W_2^0 = 1$

$X_1(4) = 1$

$X_1(5) = 1$

$X_1(6) = 1$

$X_1(7) = 1$

2nd Intermediate

$X_2(0) = a + b W_4^0 = 1 + 1 = 2$

$X_2(1) = a + b W_4^1 = 1 - j$

$X_2(2) = 1 - 1 = 0$

$X_2(3) = 1 - (-j) = 1 + j$

$X_2(4) = 1 + 1 = 2$

$X_2(5) = 1 - j$

$X_2(6) = 0$

$X_2(7) = 1 + j$

3rd Intermediate

$X_3(0) = X_2(0) + W_8^0 X_2(4) = 2 + 1 \times 2 = 4$

$X_3(1) = X_2(1) + W_8^1 X_2(5) = 1 - j2.414$

$X_3(2) = X_2(2) + W_8^2 X_2(6) = 0$

$X(3) = 1 - j0.414$

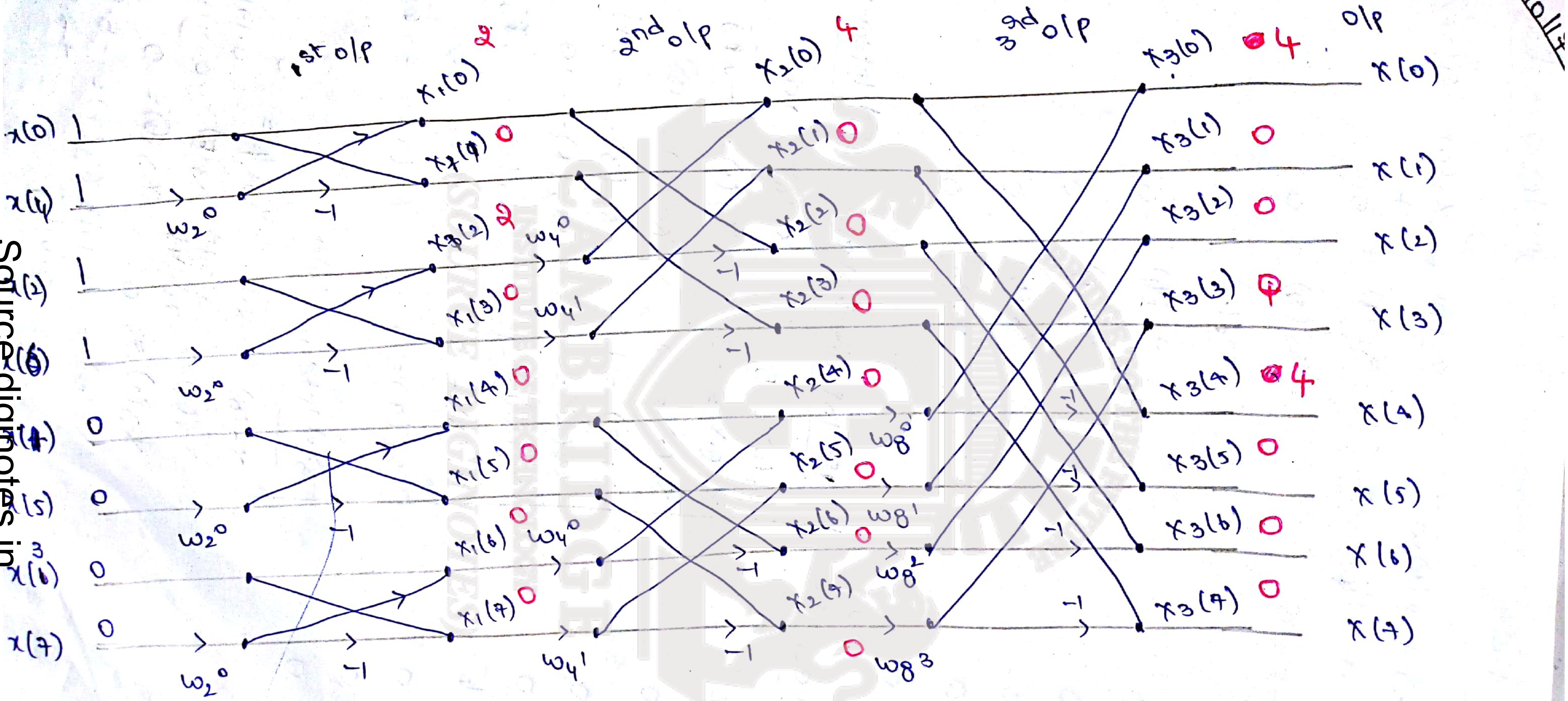
$X(4) = 0$

$X(5) = 1 + j0.414$

$X(6) = 0$

$X(7) = 1 + j2.414$

find 8pt DFT $x(n) = \{1\,0\,1\,0\,1\,0\,1\,0\}$ using DITFFT we cooley Tukey algorithm 10/10/17



1st Indermediate

$x_1(0) = x(0) + x(4) W_2^0$

$= 1 + 1(1) = 2$

$x_1(1) = 1 - 1 = 0$

$x_1(2) = 2$

$x_1(3) = 0$

$x_1(4) = 0$

$x_1(5) = 0$

$x_1(6) = 0$

$x_1(7) = 0$

2nd Intermediate

$x_2(0) = x_1(0) + x_1(2) W_4^0$

$= 4$

$x_2(1) = 0$

$x_2(2) = 0$

$x_2(3) = 0$

$x_2(4) = 0$

$x_2(5) = 0$

$x_2(6) = 0$

$x_2(7) = 0$

3rd Intermediate

$x_3(0) = 4$

$x_3(1) = 0$

$x_3(2) = 0$

$x_3(3) = 0$

$x_3(4) = 4$

$x_3(5) = 0$

$x_3(6) = 0$

$x_3(7) = 0$

1st Intermediate

$x_1(0) = 5$

$x_1(1) = -3$

$x_1(2) = 5$

$x_1(3) = 1$

$x_1(4) = 5$

$x_1(5) = -1$

$x_1(6) = 5$

$x_1(7) = 3$

2nd Intermedial

$x_2(0) = 10$

$x_2(1) = -3 + j$

$x_2(2) = 0$

$x_2(3) = -3 + j$

$x_2(4) = 10$

$x_2(5) = -3 + j - 1$

$x_2(6) = 0$

$x_2(7) = +3 + j - 1$

3rd Intemdiate

$x_3(0) = 20$

$x_3(1) = -5.828 - 2.414j$

$x_3(2) = 0$

$x_3(3) = -0.1715 - 0.414j$

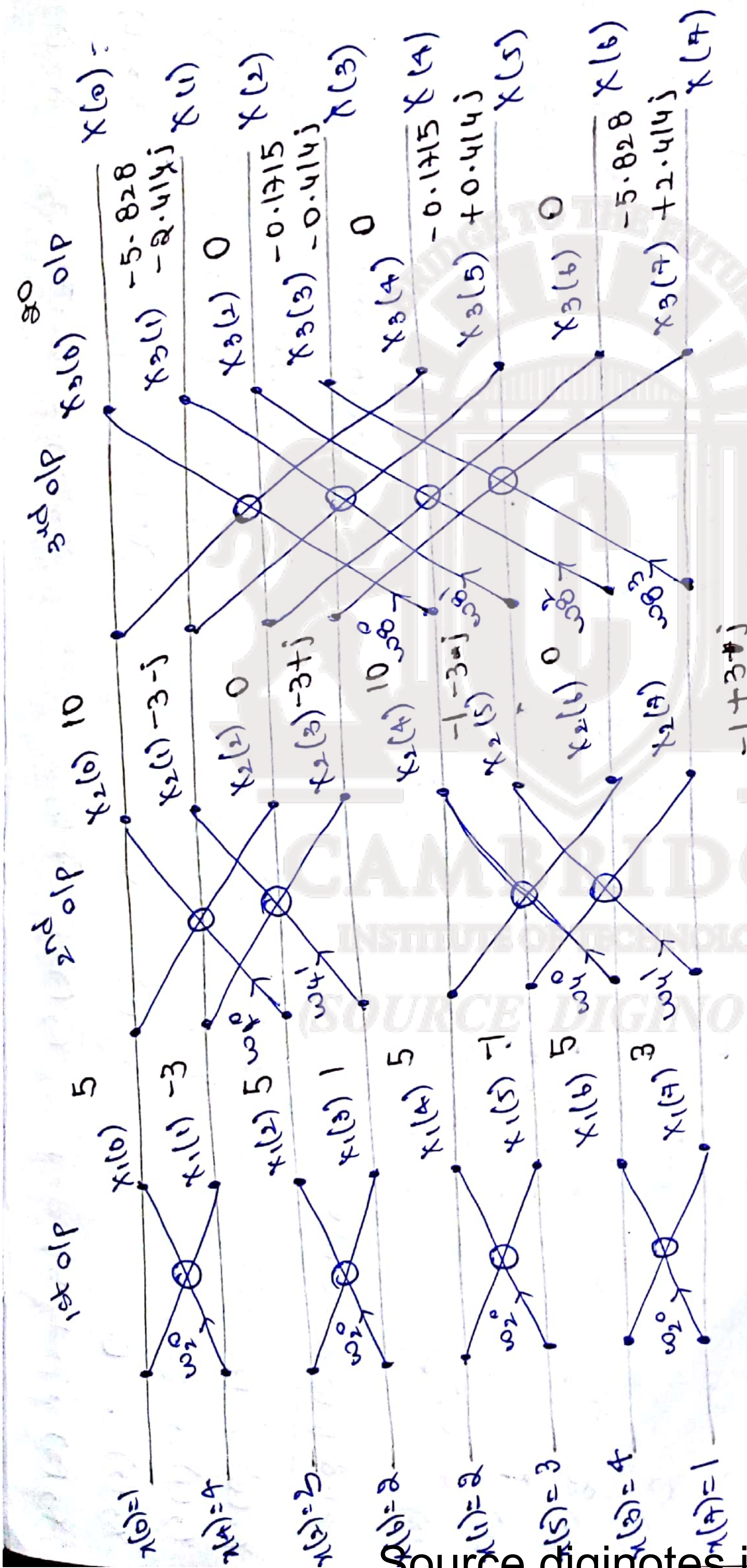
$x_3(4) = 0$

$x_3(5) = -0.1715 + 0.414j$

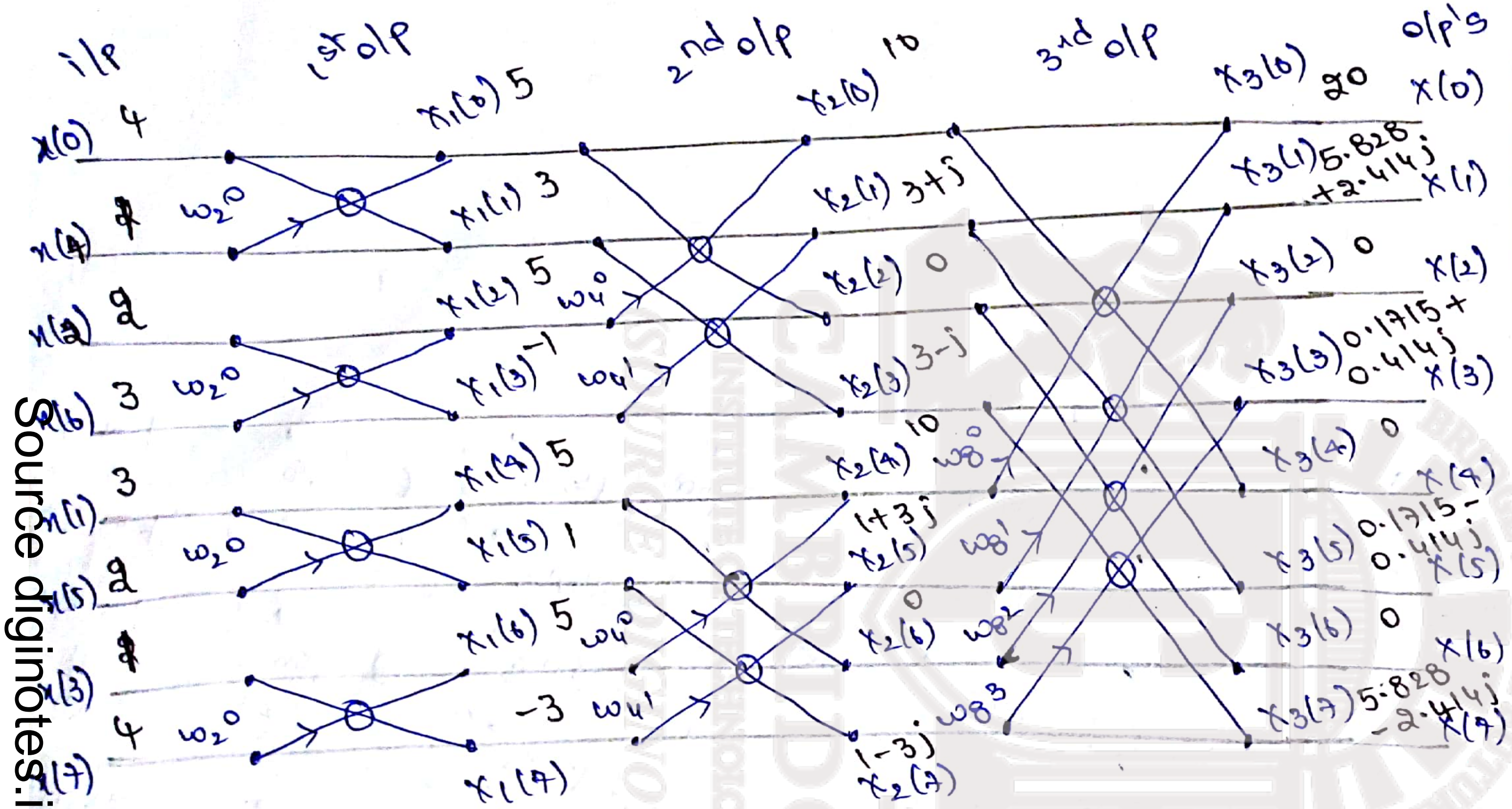
$x_3(6) = 0$

$x_3(7) = -5.828 + 2.414j$

find $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DITFFT algorithm use Butterfly ~~of~~



find DFT for $x(n) = \{4,3,2,1,1,2,3,4\}$ using DIT FFT



1st Intermediate

$X_1(0) = 5$
$X_1(1) = 3$
$X_1(2) = 5$
$X_1(3) = -1$
$X_1(4) = 5$
$X_1(5) = 1$
$X_1(6) = 5$
$X_1(7) = -3$

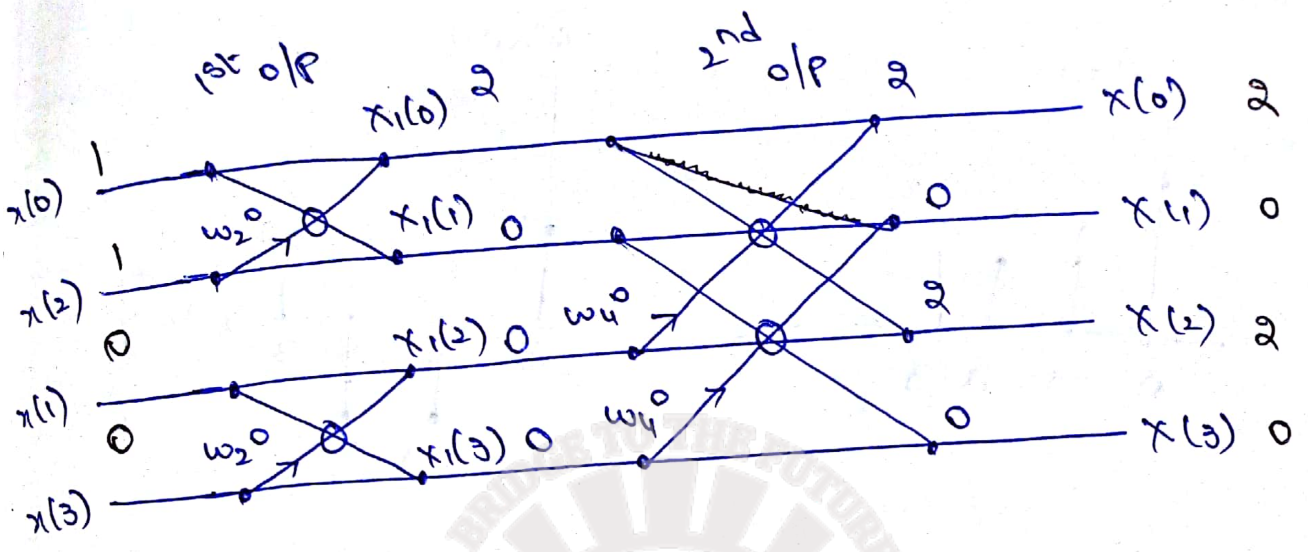
2nd Intermediate

$X_2(0) = 10$
$X_2(1) = 3+j$
$X_2(2) = 0$
$X_2(3) = \cancel{1-5j}\ 3-j$
$X_2(4) = 10$
$X_2(5) = 1+3j$
$X_2(6) = 0$
$X_2(7) = 1-3j$

3rd Intermediate

$X_3(0) = 20$
$X_3(1) = 5.828+2.414j$
$X_3(2) = 0$
$X_3(3) = 0.1715+0.414j$
$X_3(4) = 0$
$X_3(5) = 0.1715-0.414j$
$X_3(6) = 0$
$X_3(7) = 5.828-2.414j$

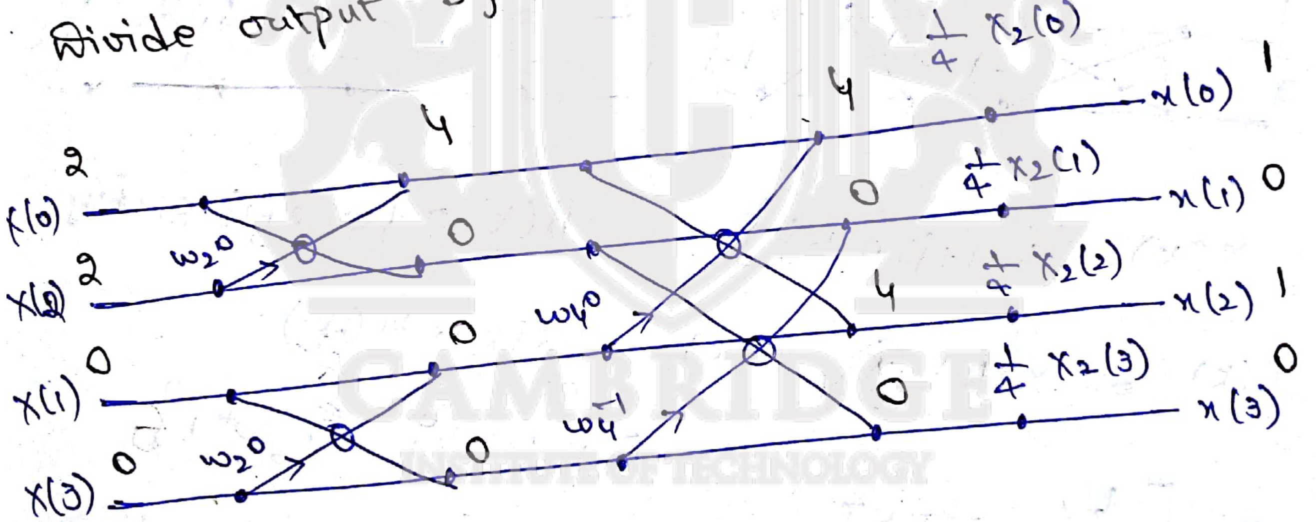
Compute 4 pt sequence DFT $x(n) = \{1\,0\,1\,0\}$ using DITFFT method also find IDFT



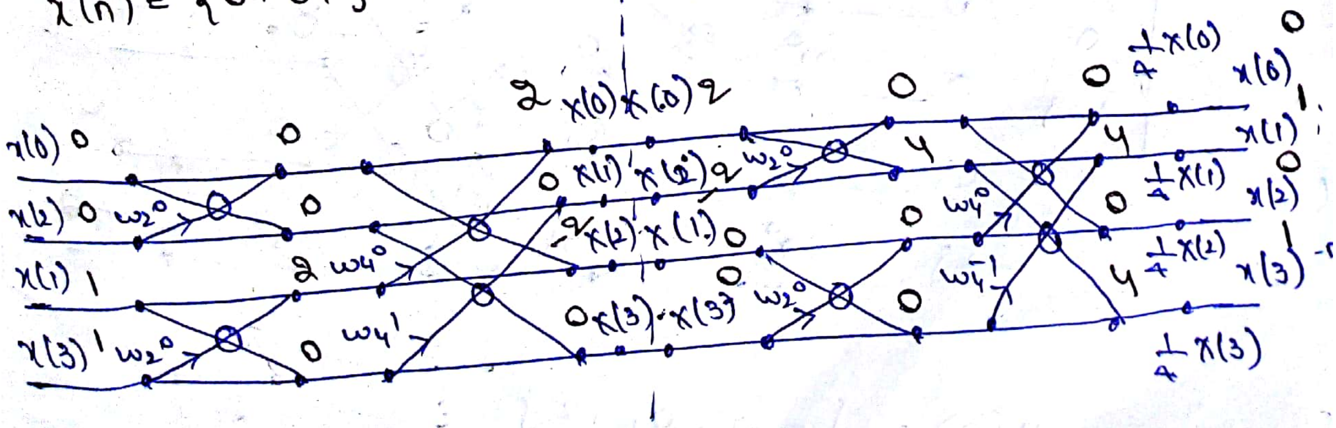
IFFT

change $W_N^{+ve} = W_N^{-ve}$

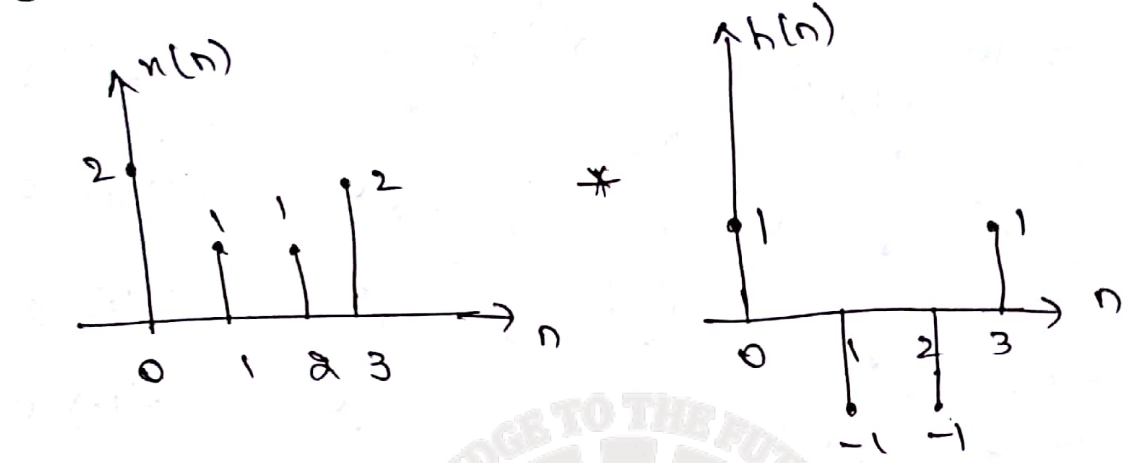
Divide output by 'N'

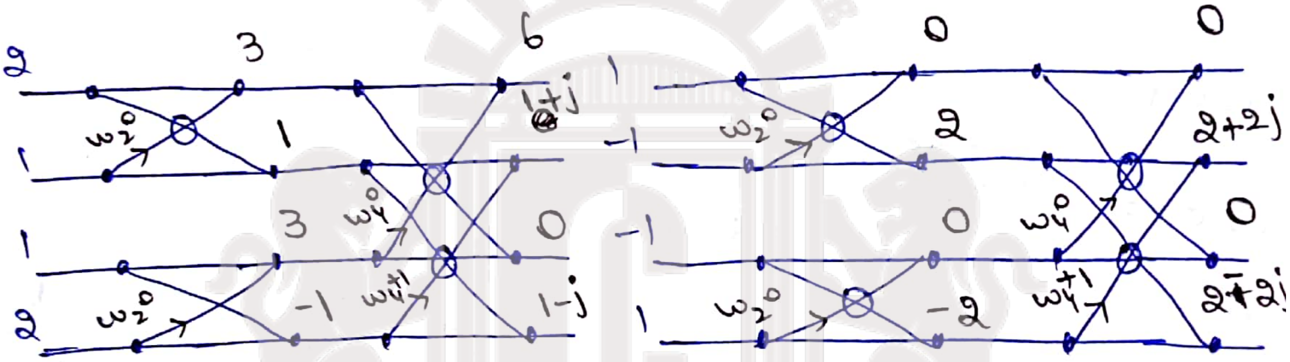


$x(n) = \{0\,1\,0\,1\}$



fo the sequences x(n) shown below compute the CC use DIT FFT method





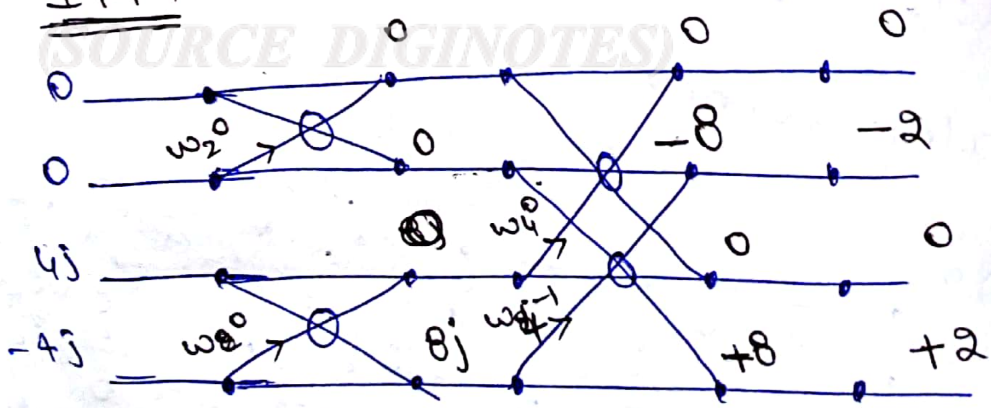
$X(k) = \{6, 1+j, 0, 1-j\}$ $H(k) = \{0, 2+2j, 0, 2-2j\}$

$Y(k) = \cancel{X(k) \circledast_N H(k)}$ $X(k) \cdot H(k)$

$= \{0, 4j, 0, -4j\}$

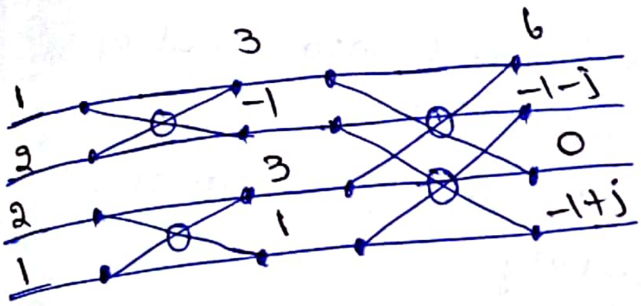
$= \cancel{\begin{bmatrix} 6 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}}$

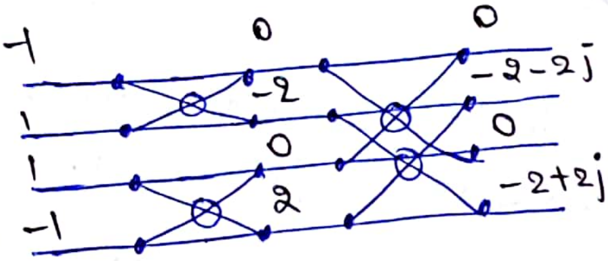
IFFT



Ans: $y(n) = \{0, -2, 0, +2\}$

$x(n) = \{1, 2, 2, 1\}$ $h(n) = \{-1, 1, 1, -1\}$





$X(k) = \{6, -1-j, 0, -1+j\}$

$H(k) = \{0, -2-2j, 0, -2+2j\}$

$Y(k) = \{0, 4j, 0, -4j\}$

$y(n) = \{0, -2, 0, 2\}$

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DIFFT

In this algorithm x(k) is divided into smaller & smaller sequences like DITFFT

Here, we evaluate odd no. of samples & even no. samples seperately

Derivation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \longrightarrow (1) \quad k = 0,1,2 \dots N-1$$

Break (1) into 2 summations

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

Sub $r = n - \frac{N}{2}$ into 2^{nd} summation

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x\left(r+\frac{N}{2}\right) W_N^{k\left(r+\frac{N}{2}\right)}$$

when $n = \frac{N}{2}$ $\quad n = N-1$

$r + \frac{N}{2} = \frac{N}{2}$ $\quad r + \frac{N}{2} = N-1$

$r = 0$ $\quad r = \frac{N}{2} - 1$

Since r is dummy variable $r \to n$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

$$\Rightarrow \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{kn} \cdot W_N^{kN/2} \quad (-1)^k$$

$$\Rightarrow \sum^{\frac{N}{2}-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} \left\{x\left(n+\frac{N}{2}\right) W_N^{kn}\right.$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} W_N^{kn} \left[x(n) + \underset{(-1)^k}{x\left(n + \frac{N}{2}\right)} \right] \rightarrow ②$$

Decimation in freq. is carried out by taking odd and even terms seperately

<u>for even terms,</u>

Substitute $k = 2r$ in ②

where $r = 0, 1, 2 \ldots N/2 - 1$

$$X(2r) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2r} x\left(n + \frac{N}{2}\right) \right] W_N^{2rn}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{rn} . \rightarrow ③$$

<u>for odd,</u>

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2r+1} x\left(n + \frac{N}{2}\right) \right] W_N^{(2r+1)n}$$

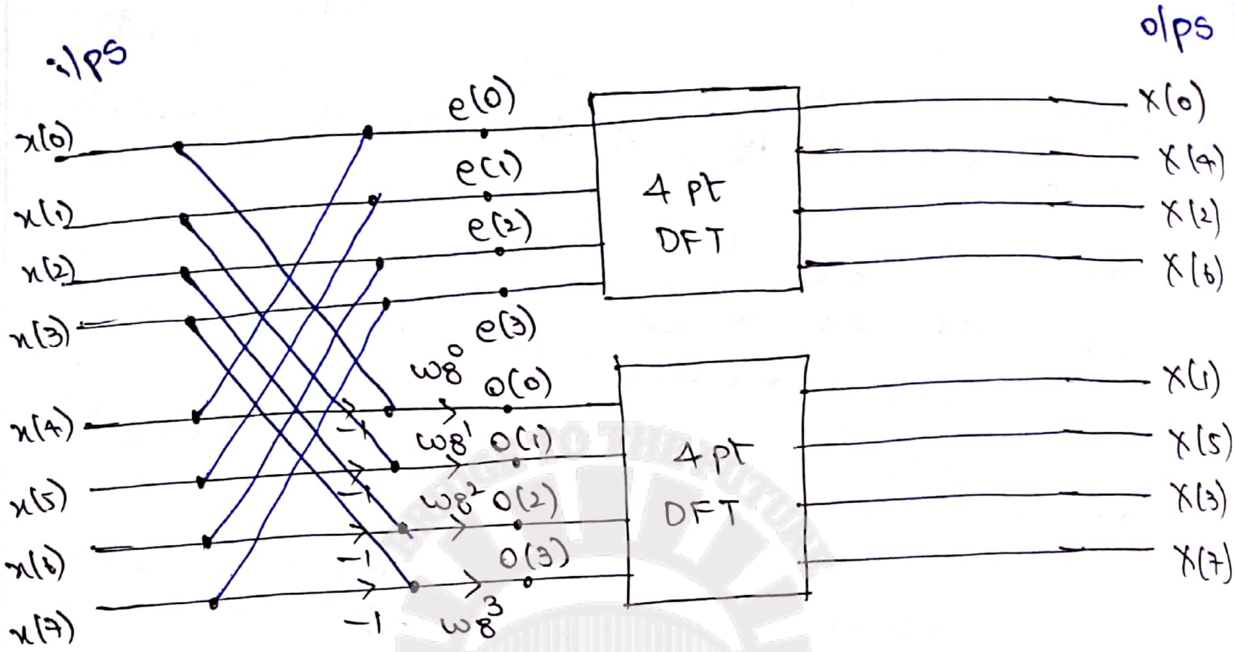
$$\Rightarrow \sum_{n=0}^{N/2-1} \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_{\frac{N}{2}}^{rn} \cdot W_N^{n} \rightarrow ④$$

<u>odd & even</u>

$$e(n) = \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_N^{n}$$

$$o(n) = \left[x(n) - x\left(n + \frac{N}{2}\right) \right]$$

flow diagram for 8pt DFT



General flow diag. of 8pt DIFFFT

Distinguish b/w DITFFT & DIFFFT

DITFFT	DIFFFT
• Input is bitreverse order	I/p is in normal order
• Twiddle factor computed Before Butterfly diag.	Twiddle factor computed after Butterfly diag.
• freq. samples are evaluated one after the other	freq. samples are evaluated seperately
• o/p is normal	o/p is bit reverse

Q.

8pt DFT $x(n) = \{1,2,2,2,1,0,0,0\}$ using DIFFFT

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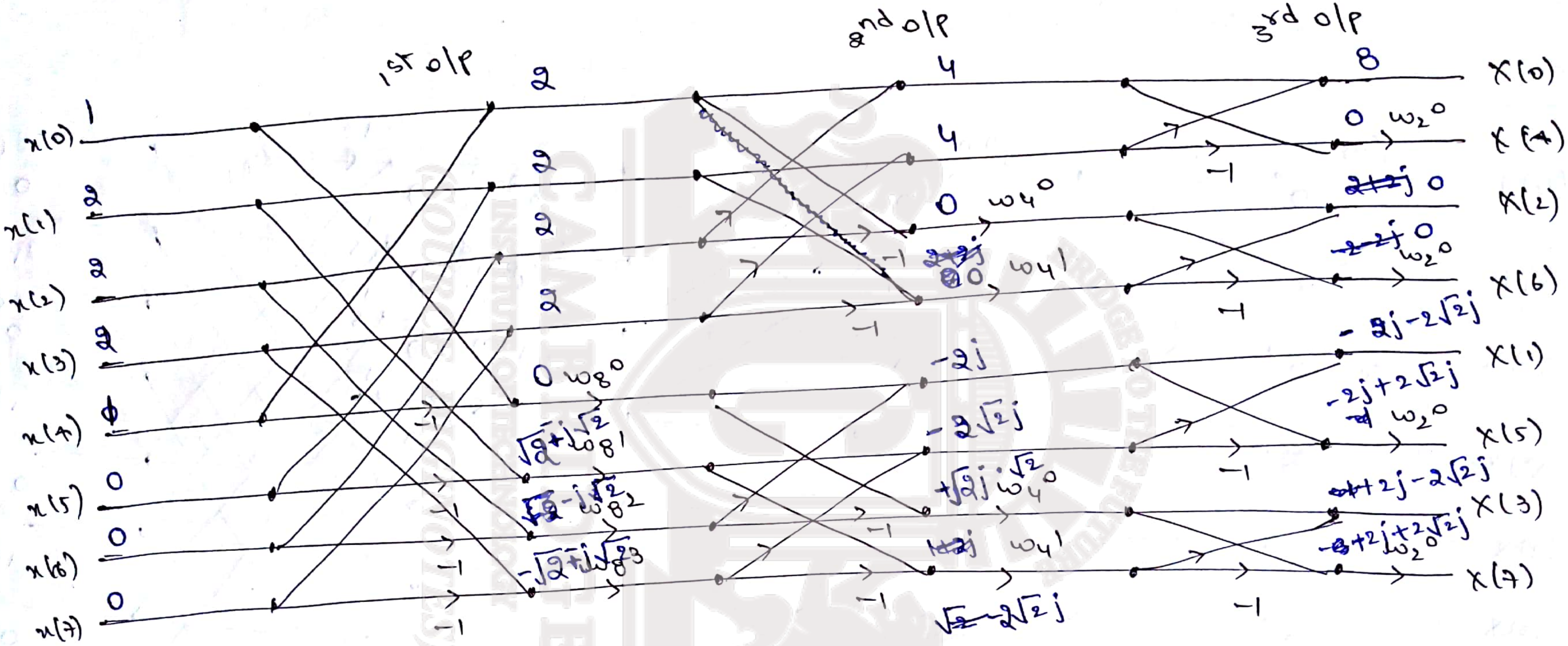
8pt DFT $x(n) = \{1,0,1,0,1,0,1,0\}$ — ″ —

— ″ — $x(n) = \{0,1,0,1,0,1,0,1\}$ — ″ —

Compute 8pt DFT $x(n) = \{2,1,2,1\}$ using DIFFFT.
Draw the signal flow graph for n=8 with Intermediate values

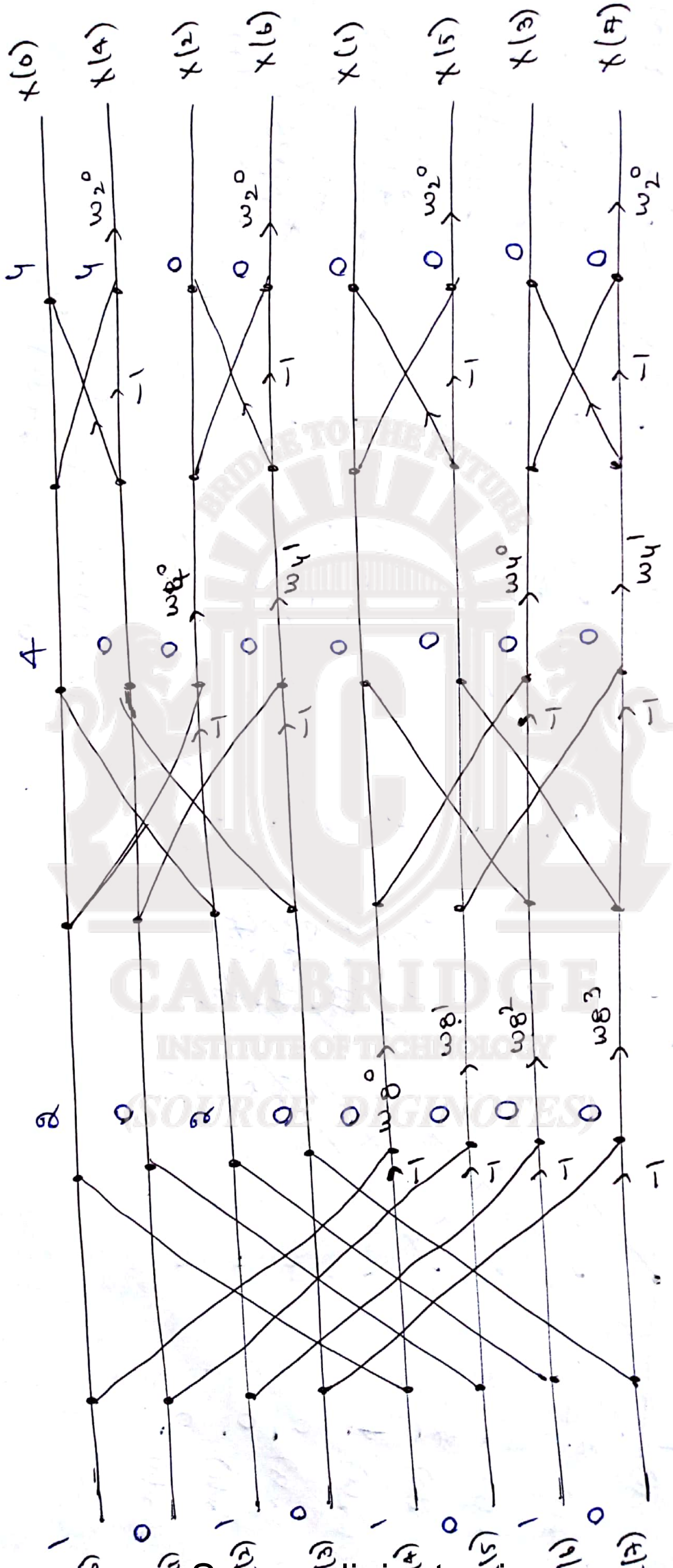


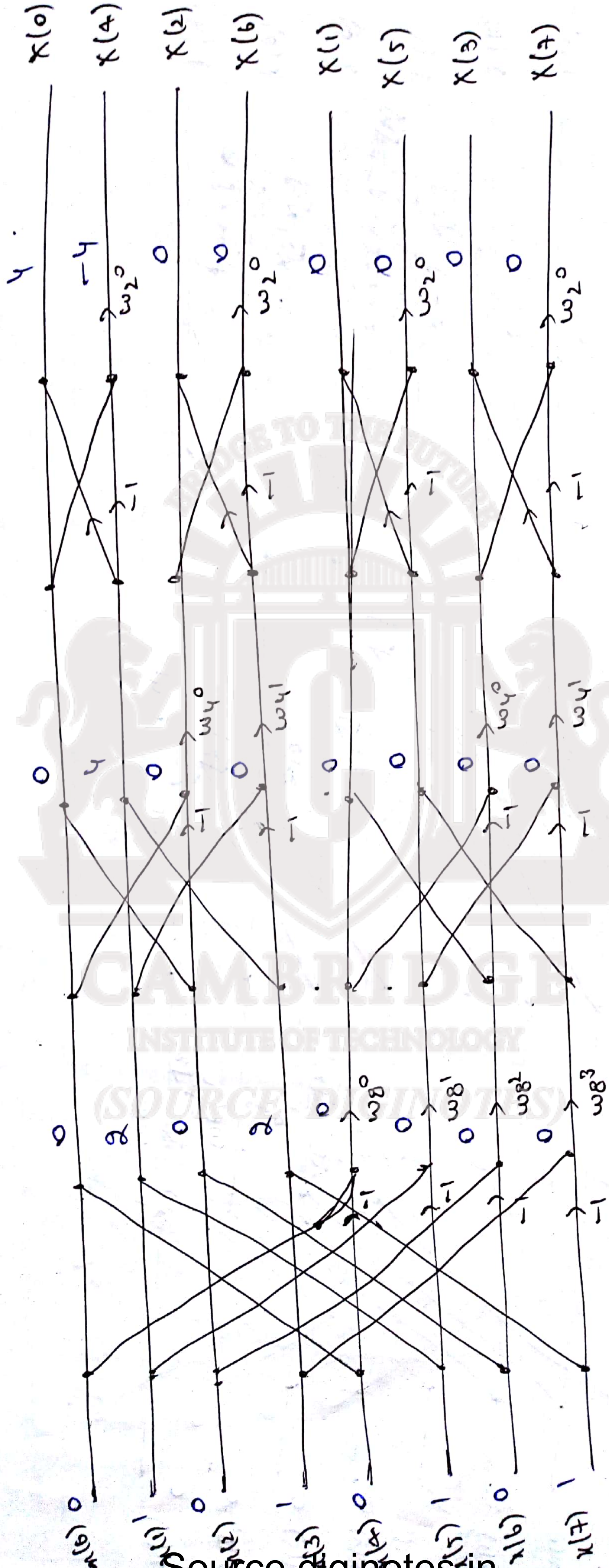
$(2-2)\,w_4^1$

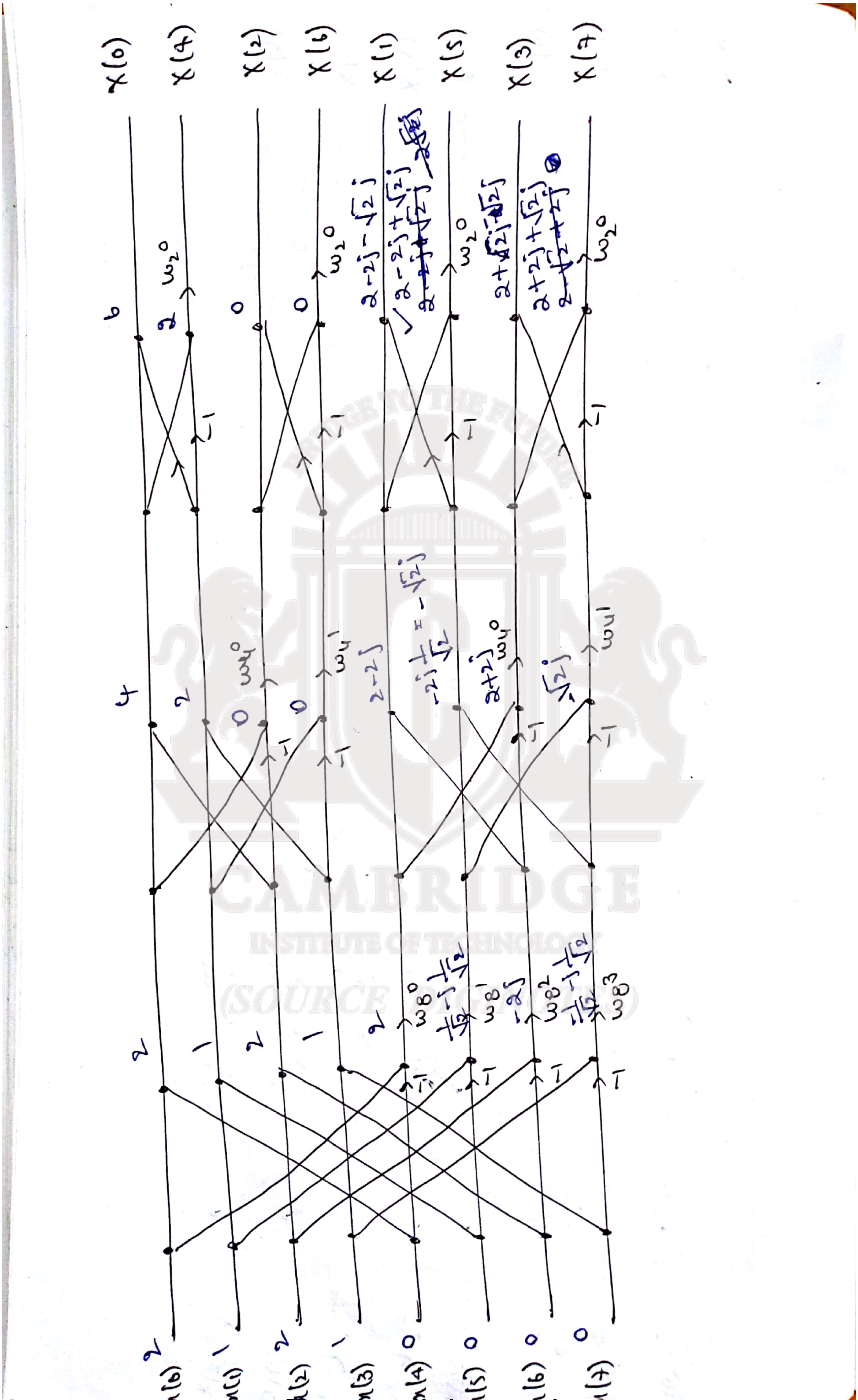
$\Rightarrow 0$

$-j2+\sqrt{2}+j\sqrt{2}$

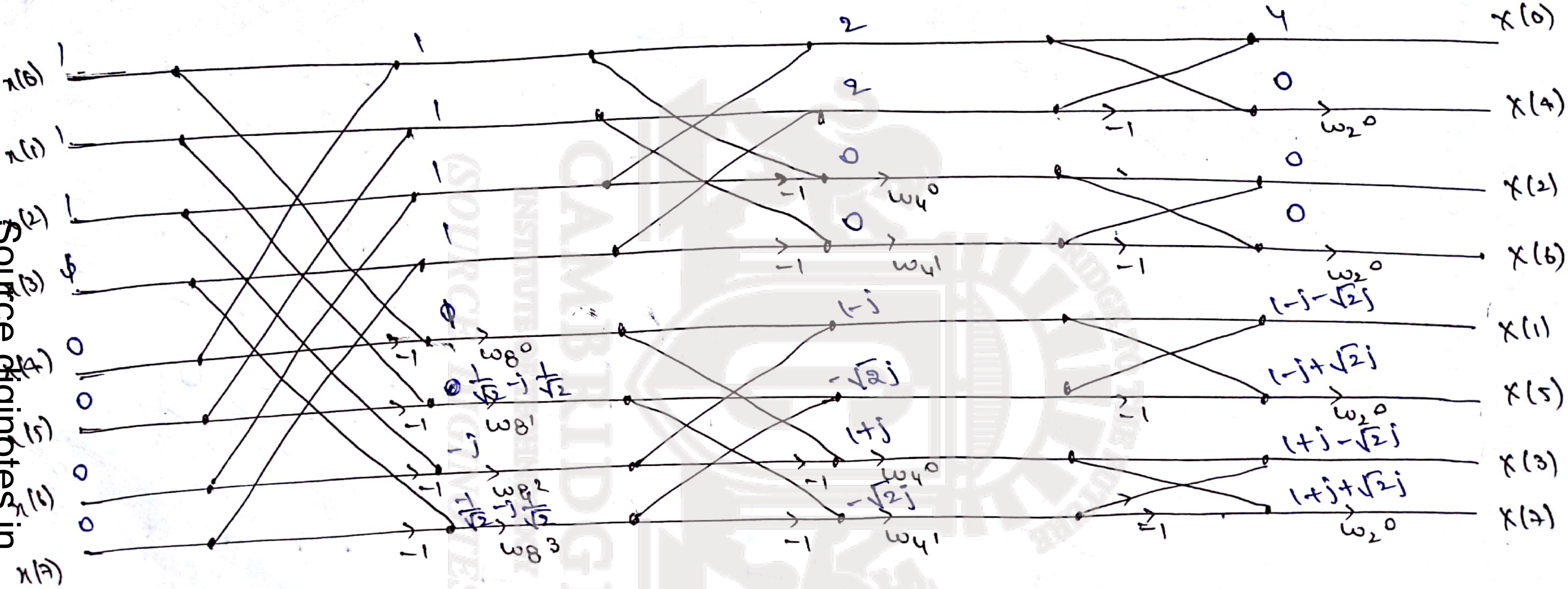
$2+\sqrt{2}j-\sqrt{2}$


x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
W_2^0
W_4^0
W_4^1
W_8^0
W_8^1
W_8^2
W_8^3
-1


x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)


x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

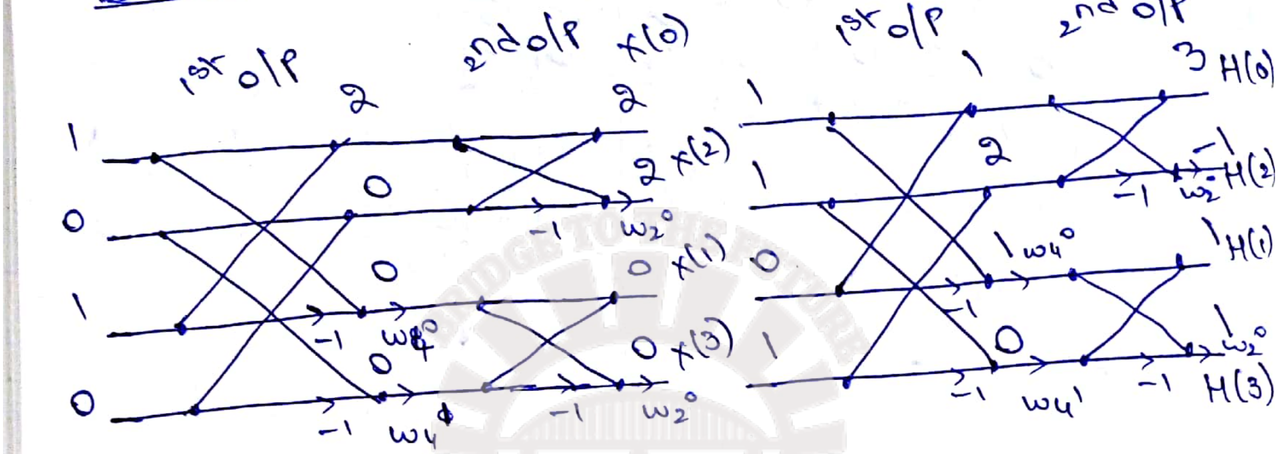
Q. $x(n) = \{1,1,1,1,0,0,0,0\}$ using DIFFFT

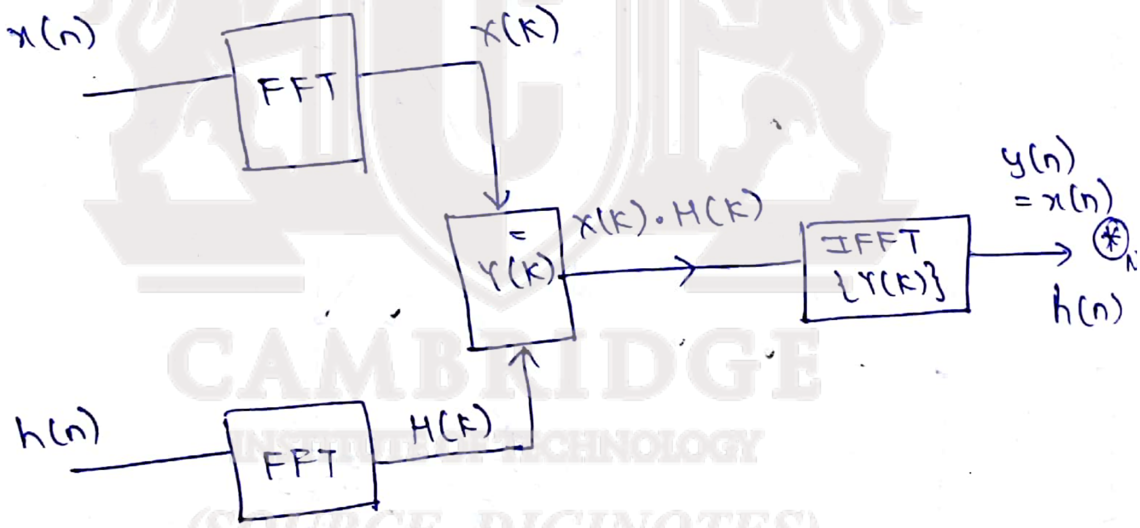


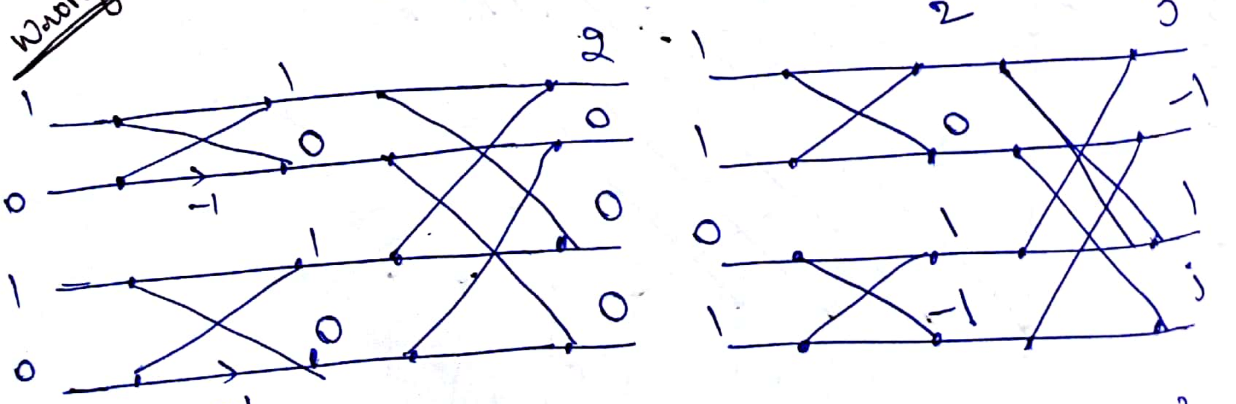
17/10/17

Perform circular convolutions for the following sequences $x(n) = \{1, 0, 1, 0\}$ and $h(n) = \{1, 1, 0, 1\}$ using

DIFFFT





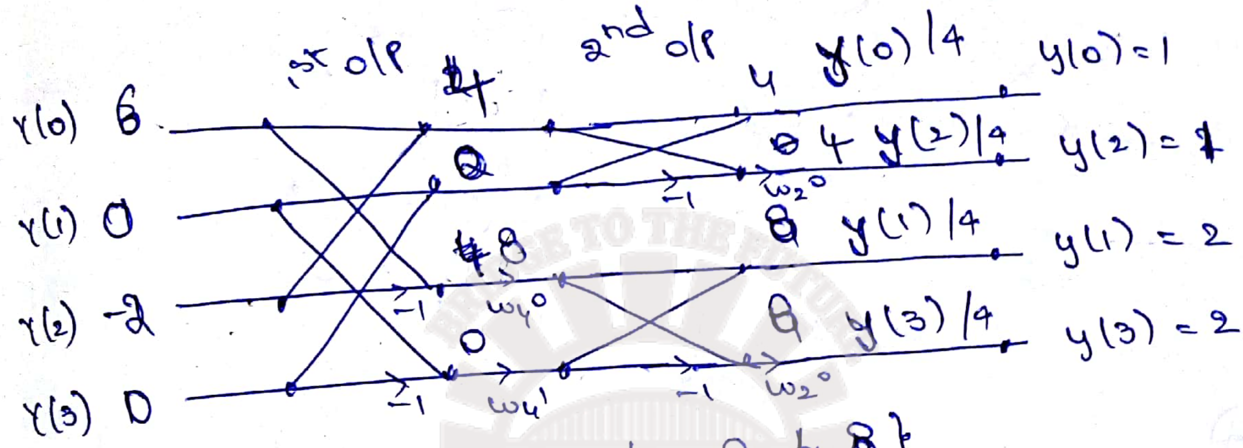


$X(K) = \{2, 0, 0, 0\}$ $\quad$ $H(K) = \{3, -1, 1, j\}$

$$Y(K) = X(K) \cdot H(K)$$

$$= \{2,0,2,0\} \cdot \{3,1,-1,1\}$$

$$= \{6,0,-2,0\}$$



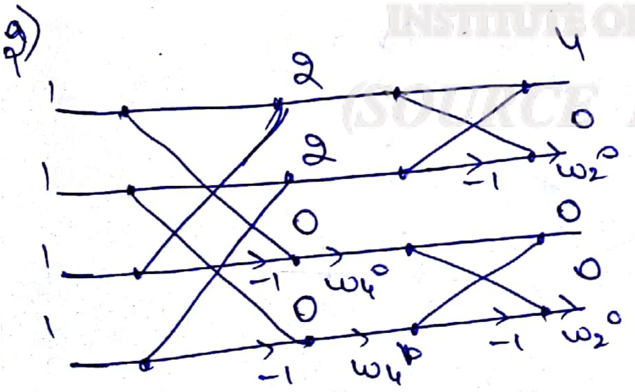
$$\Rightarrow \frac{y(n)}{4} = \left\{\frac{4}{4}, \frac{8}{4}, \frac{4}{4}, \frac{8}{4}\right\}$$

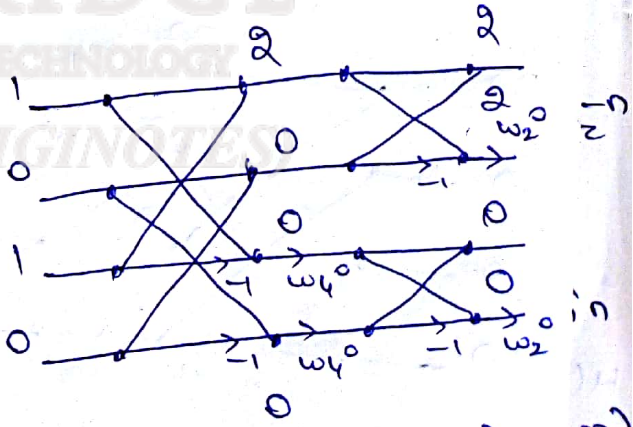
$$\Rightarrow y(n) = \{1,2,1,2\}$$

(2) For $x(n) = \{1,1,1,1\}$ and $h(n) = \{1,0,1,0\}$

(3) $x(n) = \{2,2,2,2\}$ and $h(n) = \{2,0,2,0\}$

2)

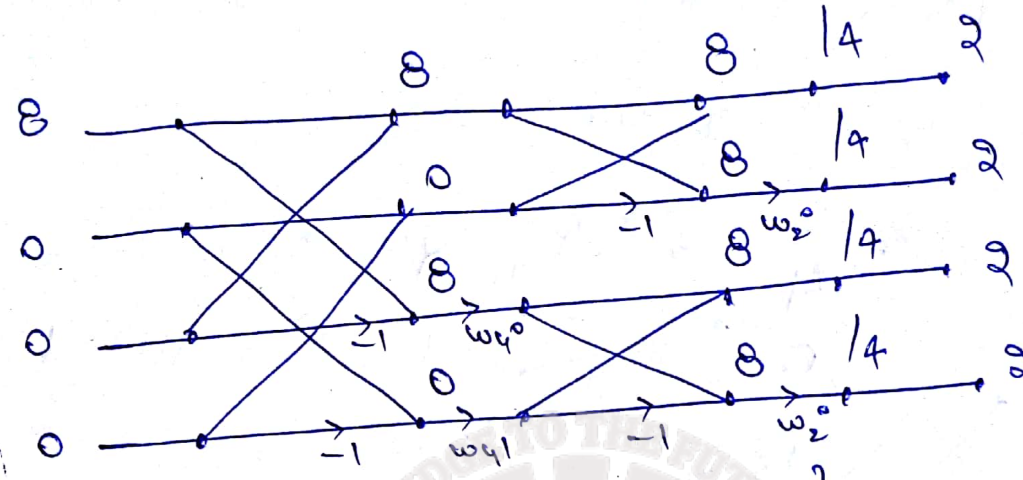




$X(K) = \{4,0,0,0\}$

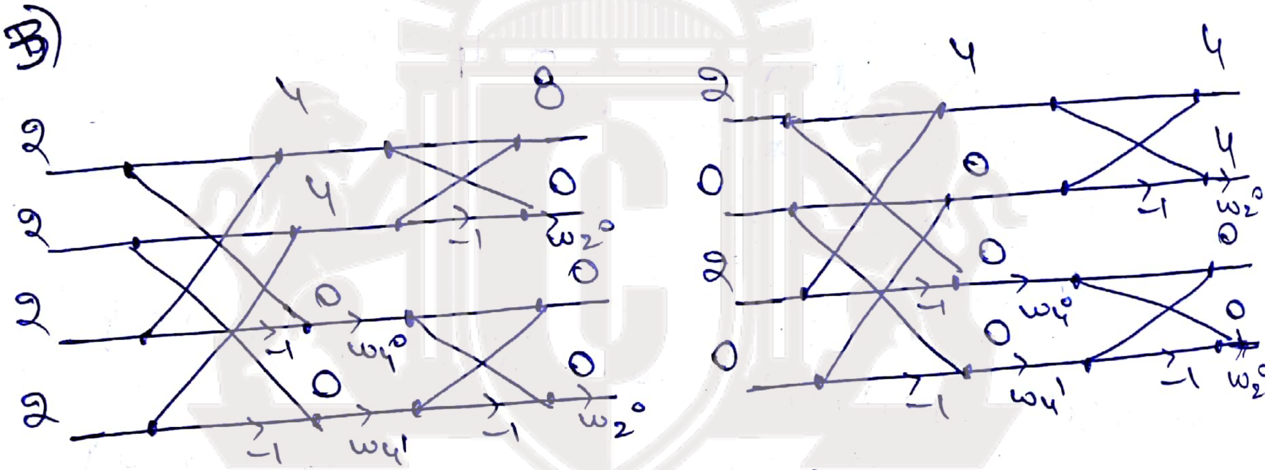
$H(K) = \{2,0,2,0\}$

$Y(k) = \{8, 0, 0, 0\}$

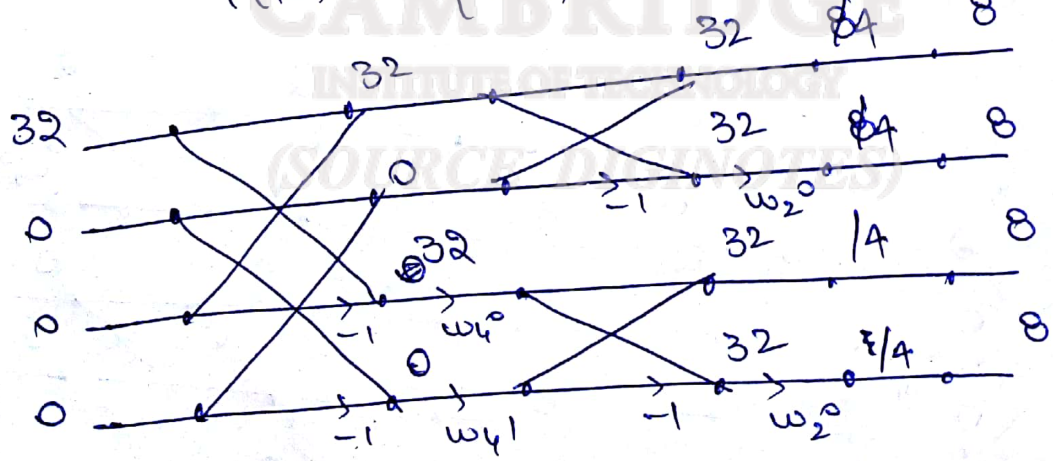


$y(n) = \{2, 2, 2, 2\}$

3)

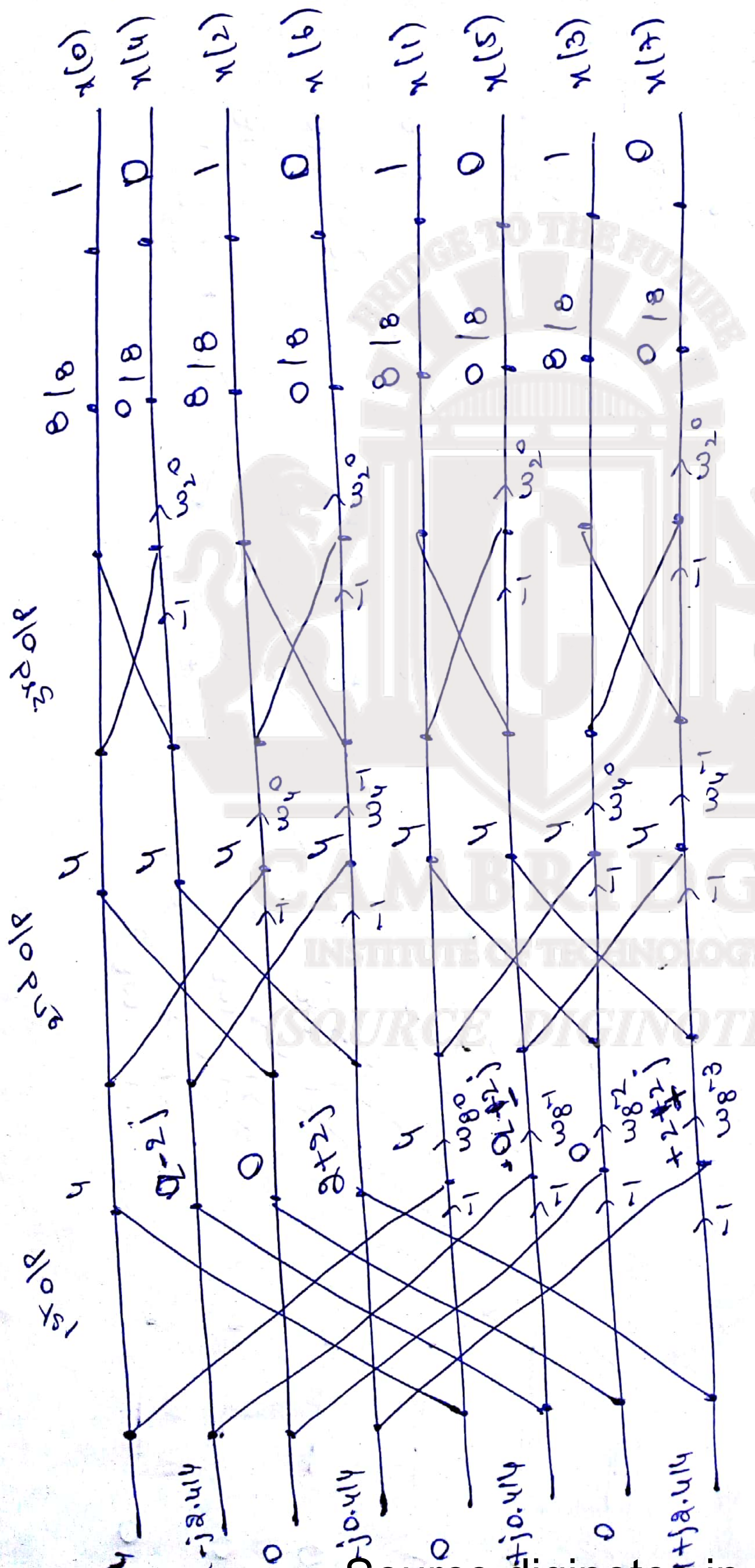


$Y(k) = \{32, 0, 0, 0\}$



$y(n) = \{8, 8, 8, 8\}$

* find IFFT of sequence $X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIFFFT



$x(n) = \{1,1,1,1,0,0,0,0\}$

find $x(n)$

$X(K) = \{4, 0, 0, 0, 4, 0, 0, 0\}$