1.6 Discrete memoryless Channel:

A channel is defined as the medium through which the coded signals are generated by an information source are transmitted. In general, the input to the channel is a symbol belonging to an alphabet ‘A’ with ‘r’ symbols, the output of a channel is a symbol belonging to an alphabet ‘B’ with ‘s’ symbols.

Due to errors in the channel, the output symbols may differ from input symbols.

1.6.1 Representation of a channel:

A communication channel may be represented by a set of input alphabets A=$(a_1, a_2, a_3 \dots \dots .. a_r)$ consisting of ‘r’ symbols and set of output alphabets B=$(b_1, b_2, b_3 \dots \dots .. b_s)$ consisting of s symbols and a set of conditional probability P($^{b_j}/_{a_i}$) with i=1,2,….r and j=1,2,…….s

$$A \left.\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{matrix}\right\} \rightarrow \mathrm{P}(^{b_j}/_{a_i}) \rightarrow \left\{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{matrix}\right. B$$

The conditional probabilities come in to the existence due to the presence of noise in the channel. Because of noise there will be some amount of uncertainty in the reception of any symbols. For this reason there are ‘s’ number of symbols at the receiver from ‘r’ symbols at transmitter. Totally there are r * s conditional probabilities represented in a form of matrix which is called as Channel Matrix or Noise Matrix.

$$P(b_j/a_i) = \begin{array}{cc} & \begin{array}{ccccc} \quad b_1 \quad & \quad b_2 \quad & \quad b_3 \quad & & \quad b_s \quad \end{array} \\ a_1 & \begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & P(b_3/a_2) & \cdots & P(b_s/a_2) \\ P(b_1/a_3) & P(b_2/a_3) & P(b_3/a_3) & \cdots & P(b_s/a_3) \\ P(b_1/a_4) & P(b_2/a_4) & P(b_3/a_4) & \cdots & P(b_s/a_4) \\ \vdots & & & & \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & \cdots & P(b_s/a_r) \end{bmatrix} \end{array}$$

When a_1 is transmitted, it can be received as any one of the output symbols $(b_1, b_2, b_3 \dots \dots .. b_s)$

Therefore $P_{11} + P_{12} + P_{13} + \cdots \ldots P_S$=1

=>P($^{b_1}/_{a_1}$)+ P($^{b_2}/_{a_1}$)+ P($^{b_3}/_{a_1}$)+…………….. P($^{b_s}/_{a_1}$)=1

In general, $\sum_{j=1}^{s}$ P($^{b_j}/_{a_i}$) $= 1$ for i= 1 to r

Thus the sum of all the elements in any row of the channel matrix is equal to UNITY.

1.6.2 Joint Probability:

Joint probability between any input symbol a_i and any output symbol b_j is given by

P($a_i \cap b_j$) $= \boldsymbol{P(a_i, b_j)} =$ **P**($^{\boldsymbol{b_j}}/_{\boldsymbol{a_i}}$)$\boldsymbol{P(a_i)}$

$$\boldsymbol{P(a_i, b_j)} = \mathbf{P}(^{\boldsymbol{a_i}}/_{\boldsymbol{b_j}})\boldsymbol{P(b_j)}$$

Properties:

Consider the source alphabet A=($a_{1,} a_{2,} a_3 \ldots \ldots \ldots a_r$) and output alphabet B=($b_{1,} b_{2,} b_3 \ldots \ldots \ldots b_s$)

- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$
- The entropy of the receiver or output is given by $H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$
- If all the symbols are equi-probable, then maximum source entropy is $H(A)_{max} = log_2 r$
- Conditional Entropy: The entropy of input symbols $a_{1,} a_{2,} a_3 \ldots \ldots \ldots a_r$ after the transmission and reception of particular output symbol b_j is defined as conditional entropy, denoted by $H(^{\boldsymbol{A}}/_{\boldsymbol{b_j}})$

$$H\left(^{A}/_{b_j}\right) = \sum_{i=1}^{r} \mathrm{P}\left(^{a_i}/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left(^{a_i}/_{b_j}\right)}.$$

- If the average value of all the conditional probability is taken as j varies from 1 to s denoted by $H(^{A}/_{B}) = \sum_{j=1}^{s} P(b_j)\ H\left(^{A}/_{b_j}\right)$

$$=\sum_{j=1}^{s} \sum_{i=1}^{r} P(b_j)\ \mathrm{P}\left(^{a_i}/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left(^{a_i}/_{b_j}\right)}$$

$$H\left({}^{A}/_{B}\right) = \sum_{j=1}^{s}\sum_{i=1}^{r} P(a_i, b_j)\, log_2 \frac{1}{P\left({}^{a_i}/_{b_j}\right)}$$ is conditional entropy of transmitter

Similarly $$H\left({}^{B}/_{A}\right) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P\left({}^{b_j}/_{a_i}\right)}$$ is conditional entropy of receiver.

- $H(A, B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\, log_2 \frac{1}{P(a_i, b_j)}$ is joint conditional probability.

1.6.3 Mutual Information:

When an average amount of information H(x) is transmitted over a noisy channel, then an amount of information $H\left({}^{x}/_{y}\right)$ is last in the channel. The balance of the information at the receiver is defined as Mutual Information I(x.y)

I(x,y)= H(x)- $H\left({}^{x}/_{y}\right)$

= H(y)- $H\left({}^{y}/_{x}\right)$

$I(x_i) = \log\left(\frac{1}{P(x_i)}\right)$ and $I\left({}^{x_i}/_{y_j}\right) = \log\left(\frac{1}{P\left({}^{x_i}/_{y_j}\right)}\right.$

The difference between the above 2 is the information gained through the channel.

$$I(x_i, y_j) = \log\left(\frac{1}{P(x_i)}\right) - \log\left(\frac{1}{P\left({}^{x_i}/_{y_j}\right)}\right)$$

$$I(x_i, y_j) = \log\frac{P\left({}^{x_i}/_{y_j}\right)}{P(x_i)}$$

$$I(x_i, y_j) = \log\frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Properties:

- The Mutual Information is symmetric. $I(x_i, y_j) = I(y_j, x_i)$
- $I(X, Y) = H(X) + H(Y) - H(X, Y)$
- $I(X, Y) = H(X) - H\left({}^{X}/_{Y}\right)$
- $I(X, Y) = H(Y) - H\left({}^{Y}/_{X}\right)$

1.6.4 Channel Capacity

It is known that average information content of the source is $H(X) = \sum_{i=1}^{M} p(x_i) \log_2 \left(\frac{1}{p(x_i)}\right)$. Average information per symbol going in to the channel is $R_{in} = r_s * H(X)$. Due to the error, it is not possible to reconstruct the input symbol sequence with certainty on the recovered sequence. Therefore source information is lost due to the errors.

- Therefore average rate of information transmission is given by $R_t = I(X,Y) \,.\, r_s$. Bits/sec.
- The capacity of a discrete memoryless noisy channel is defined a s maximum possible rate of maximum rate of transmission occurs when the source is matched to the channel.
- $\therefore C = Max(R_t)$
- $= \text{Max}[I(X,Y) \,.\, r_s]$
- $C = \text{Max}\{[H(X) - H(X/Y)]\, r_s\}$

1.6.5 Channel Efficiency

$\%\eta_{ch} = \frac{R_t}{C} * 100$

$= \frac{I(X,Y) \,.\, r_s}{\text{Max}[I(X,Y) \,.\, r_s]} * 100$

$\%\eta_{ch} = \frac{H(X) - H(X/Y)}{Max[H(X) - H(X/Y)]} * 100$

Redundancy =1-$\boldsymbol{\eta_{ch}}$

1.6.6 Symmetry Channel

Symmetry channel is defined as the channel in which the channel matrix has 2nd and subsequent rows, the same elements as the first row, but in different order.

$\therefore H(Y/X) = h$, where →entropy of any single row. The channel capacity with r_s=1 bits/sec is given by,

$C = Max(R_t)$

$= \text{Max}[I(X,Y)]\, r_s$

$= \text{Max}[I(X,Y)]$

$= \text{Max}\Big(H(Y) - H(Y/X)\Big)$

$= \text{Max}[H(Y)] - \text{Max}[H(Y/X)]$

$= \text{Max}[H(Y)]$ - $\text{Max}(h)$

$C = \text{Max}[H(Y)] - h$

H(Y) is the entropy of symbol which becomes maximum if and only if all the receive symbols become equi-probable.

Since there are ‘s’ output symbols

$\text{Max}[H(Y)] = log_2 s$

$\therefore$ $\boldsymbol{C = log_2 s - h}$

Ex.1: A transmitter has an alphabet containing of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and the receiver has an alphabet of four letters $\{b_1, b_2, b_3, b_4\}$. The joint probabilities of the system are given below. Compute different entropies of this channel.

$$P(A, B) = \begin{array}{c c} & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} & \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.10 & 0.30 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0 \\ 0 & 0 & 0.05 & 0.1 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \end{array}$$

Solution:

$P(b_1) = 0.35$, $P(b_2) = 0.35$, $P(b_3) = 0.2$ $P(b_4) = 0.1$

$P(a_1) = 0.25$, $P(a_2) = 0.4$ $P(a_3) = 0.15$, $P(a_4) = 0.15$, $P(a_5) = 0.05$

$$H(A) = \sum_{i=1}^{5} P(a_i) \log \frac{1}{P(a_i)}$$

$$= 0.25 \log \frac{1}{0.25} + 0.4 \log \frac{1}{0.4} + 0.15 \log \frac{1}{0.15} + 0.15 \log \frac{1}{0.15}$$

$$+ 0.05 \log \frac{1}{0.05}$$

H(A) = 2.066 bits/message-symbol

$$H(B) = \sum_{j=1}^{4} P(b_j) \log \frac{1}{P(b_j)}$$

$$= 0.35 \log \frac{1}{0.35} + 0.35 \log \frac{1}{0.35} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1}$$

H(B) = 1.857 bits/message-symbol

$$H(A, B) = \sum_{i=1}^{5}\sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

$$H(A, B) = 0.25 \log \frac{1}{0.25} + 0.1 \log \frac{1}{0.1} + 0.3 \log \frac{1}{0.3} + 0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1} + 0.05 \log \frac{1}{0.05} + 0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1}$$

$$\mathbf{H(A, B) = 2.666 \text{ bits/message-symbol}}$$

$$\begin{aligned} H(B/A) &= H(A, B) - H(A) \\ &= 2.666 - 2.066 \\ \mathbf{H(B/A)} &= \mathbf{0.6 \text{ bits/message-symbol}} \end{aligned}$$

$$\begin{aligned} H(A/B) &= H(A, B) - H(B) \\ &= 2.666 - 1.857 \\ \mathbf{H(A/B)} &= \mathbf{0.809 \text{ bits/message-symbol}} \end{aligned}$$

$$\begin{aligned} I(A, B) &= H(A) - H(A/B) \\ &= 2.066 - 0.809 \\ \mathbf{I(A, B)} &= \mathbf{1.257 \text{ bits/message-symbol}} \end{aligned}$$

$$\begin{aligned} I(A, B) &= H(B) - H(B/A) \\ &= 1.857 - 0.6 \\ \mathbf{I(A, B)} &= \mathbf{1.257 \text{ bits/message-symbol}} \end{aligned}$$

Ex.2: A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P\left({}^{B}/_{A}\right)$, Calculate H(B) and H(A,B)

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:

We know that, $P(a_i, b_j) = P(a_i)P(b_j/a_i)$

The JPM may now be constructed by multiplying 1st row elements by $P(a_1) = 0.2 = \frac{1}{5}$, 2nd row by $P(a_2) = 0.3 = \frac{3}{10}$, 3rd row by $P(a_3) = 0.2 = \frac{1}{5}$, 4th row by $P(a_4) = 0.1 = \frac{1}{10}$ and 5th row by $P(a_5) = 0.2 = \frac{1}{5}$.

$$P(A, B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \end{array}$$

Adding the element pf each column, we get

$$P(b_1) = \frac{1}{5} + \frac{3}{40} = \frac{11}{40}$$

$$P(b_2) = \frac{9}{40} + \frac{1}{15} = \frac{7}{24}$$

$$P(b_3) = \frac{2}{15} + \frac{1}{30} + \frac{1}{5} = \frac{11}{30}$$

$$P(b_4) = \frac{1}{15}$$

$$H(B) = \sum_{j=1}^{4} P(b_j) \log \frac{1}{P(b_j)}$$

$$= \frac{11}{40} \log \frac{40}{11} + \frac{7}{24} \log \frac{24}{7} + \frac{11}{30} \log \frac{30}{11} + \frac{1}{15} \log 15$$

$$\mathbf{H(B) = 1.822 \text{ bits/message-symbol}}$$

$$H(A, B) = \sum_{i=1}^{5} \sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

$$= \frac{1}{5} \log 5 + \frac{3}{40} \log \frac{40}{3} + \frac{9}{40} \log \frac{40}{9} + \frac{1}{15} \log 15 + \frac{2}{15} \log \frac{15}{2}$$

$$+ \frac{1}{30} \log 30 + \frac{1}{5} \log 5 + \frac{1}{15} \log 15$$

$$\mathbf{H(A, B) = 2.7653 \text{ bits/message-symbol}}$$

Ex.3: For the JPM given below, compute individually H(X), H(Y), H(X,Y), H(X/Y), H(Y/X), I(X,Y) and channel Capacity if r=1000 symbols/sec. Verify the relationship among these entropies.

$$P(X, Y) = \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix}$$

Solution:

The given JPM can be rewritten with input and output symbols as below:

$$P(X, Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix} \end{array}$$

From property-1 of JPM, addition of elements of JPM columnwise results in probabil of output symbols.

$$\therefore \quad P(y_1) = 0.05 + 0.05 = 0.10$$
$$P(y_2) = 0.10 + 0.05 = 0.15$$
$$P(y_3) = 0.2 + 0.1 + 0.2 = 0.50$$
$$P(y_4) = 0.05 + 0.1 + 0.1 = 0.25$$

From property-2 of JPM, addition of elements of JPM rowwise results in probability of input symbols.

$$\therefore \quad P(x_1) = 0.05 + 0.2 + 0.05 = 0.30$$
$$P(x_2) = 0.10 + 0.10 = 0.20$$
$$P(x_3) = 0.20 + 0.10 = 0.30$$
$$P(x_4) = 0.05 + 0.05 + 0.1 = 0.20$$

$$H(X) = \sum_{i=1}^{4} P(x_i) \log \frac{1}{P(x_i)}$$
$$= \left(0.3 \log \frac{1}{0.3}\right)(2) + \left(0.2 \log \frac{1}{0.2}\right)(2)$$

H(X) = 1.971 bits/message-symbol

$$H(Y) = \sum_{j=1}^{4} P(y_j) \log \frac{1}{P(y_j)}$$
$$= 0.1 \log \frac{1}{0.1} + 0.15 \log \frac{1}{0.15} + 0.5 \log \frac{1}{0.5} + 0.25 \log \frac{1}{0.25}$$

H(Y) = 1.743 bits/message-symbol

$$H(X/Y) = \sum_{i=1}^{4} \sum_{j=1}^{4} P(x_i, y_j) \log \frac{1}{P(x_i/y_j)} \text{ bits/message symbol.}$$

Using the relationship $P(x_i/y_j) = \dfrac{P(x_i, y_j)}{P(y_j)}$, the matrix P(X/Y) is constructed as below:

$$
P(X/Y) = \begin{bmatrix} \frac{0.05}{0.10} & 0 & \frac{0.2}{0.5} & \frac{0.05}{0.25} \\ 0 & \frac{0.10}{0.15} & \frac{0.1}{0.5} & 0 \\ 0 & 0 & \frac{0.2}{0.5} & \frac{0.10}{0.25} \\ \frac{0.05}{0.10} & \frac{0.05}{0.15} & 0 & \frac{0.10}{0.25} \end{bmatrix}
$$

$\therefore$

$$
P(X/Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}
\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 2/5 & 1/5 \\ 0 & 2/3 & 1/5 & 0 \\ 0 & 0 & 2/5 & 2/5 \\ 1/2 & 1/3 & 0 & 2/5 \end{bmatrix} \end{array}
$$

$$
\begin{aligned}
\therefore \quad H(X/Y) &= 0.05 \log 2 + 0.05 \log 2 + 0.1 \log 3/2 + 0.05 \log 3 + 0.2 \log 5/2 \\
&\quad + 0.1 \log 5 + 0.2 \log 5/2 + 0.05 \log 5 + 0.1 \log 5/2 + 0.1 \log 5/2
\end{aligned}
$$

$\therefore$ **$H(X/Y) = 1.379$ bits/message symbol**

$$
H(Y/X) = \sum_{i=1}^{4}\sum_{j=1}^{4} P(x_i, y_j) \log \frac{1}{P(y_j/x_i)} \text{ bits/message-symbol.}
$$

Using the relationship $P(y_j/x_i) = \dfrac{P(x_i, y_j)}{P(x_i)}$, the channel matrix $P(Y/X)$ is constructed below:

$$
P(Y/X) = \begin{bmatrix} \frac{0.05}{0.30} & 0 & \frac{0.20}{0.30} & \frac{0.05}{0.30} \\ 0 & \frac{0.10}{0.20} & \frac{0.10}{0.20} & 0 \\ 0 & 0 & \frac{0.20}{0.30} & \frac{0.10}{0.30} \\ \frac{0.05}{0.20} & \frac{0.05}{0.20} & 0 & \frac{0.10}{0.20} \end{bmatrix}
$$

$$
P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}
\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1/6 & 0 & 2/3 & 1/6 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix} \end{array}
$$

$$
\begin{aligned}
H(Y/X) &= 0.05 \log 6 + 0.2 \log 3/2 + 0.05 \log 6 + 0.1 \log 2 + 0.1 \log 2 \\
&\quad + 0.2 \log 3/2 + 0.1 \log 3 + 0.05 \log 4 + 0.05 \log 4 + 0.1 \log 2
\end{aligned}
$$

$H(Y/X) = 1.151$ bits/message-symbol

Verification:

$$
\begin{aligned}
H(Y/X) &= H(X, Y) - H(X) \\
&= 3.122 - 1.971
\end{aligned}
$$

$H(Y/X) = 1.151$ bits/message-symbol as before

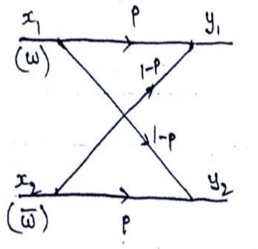
$$H(X/Y) = H(X, Y) - H(Y)$$
$$= 3.122 - 1.743$$
$$\mathbf{H(Y/X) = 1.379 \text{ bits/message-symbol as before}}$$

$$I(X, Y) = H(X) - H(X/Y)$$
$$= 1.971 - 1.379$$
$$\mathbf{I(X, Y) = 0.592 \text{ bits/message-symbol}}$$

$$I(X, Y) = H(Y) - H(Y/X)$$
$$= 1.743 - 1.151$$
$$\mathbf{I(X, Y) = 0.592 \text{ bits/message-symbol}} \quad \text{– verified.}$$

1.7 Binary Symmetric Channel:

The binary symmetric channel is one of the most commonly and widely used channel whose channel diagram is given below



From the above diagram, channel matrix can be written as

$$P(X/Y) = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix} = \begin{bmatrix} P & \bar{P} \\ \bar{P} & P \end{bmatrix}$$

The matrix is a symmetric matrix. Hence the channel is binary symmetric channel.

1.8 Channel Capacity

It is known that C = Max$\{[H(Y) - H(Y/X)]\, r_s\}$.

For symmetry channel, $H(Y/X) = h = P\log\frac{1}{P} + \bar{P}\log\frac{1}{\bar{P}}$

Since it is a binary symmetric channel, $H(Y)_{max} = log_2 s = log_2 2 = 1$

$\therefore C = 1 - h$ bits/sec.

Ex.1: A binary symmetric channel has the following noise matrix with source probabilities of $P(x_1)$=2/3 and $P(x_2)$=1/3. $P\left(Y/_X\right) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$. Determine H(X), H(Y), H(X,Y), H(Y/X), H(X/Y), I(X,Y), Channel Capacity, Channel efficiency and redundancy.

Solution:

$$p = \frac{1}{4},\ \bar{p} = \frac{3}{4},\ w = \frac{2}{3}\ \text{and}\ \bar{w} = \frac{1}{3}$$

$$H(X) = \sum_{i=1}^{2} P(x_i) \log \frac{1}{P(x_i)}$$

$$= \frac{2}{3} \log \frac{3}{2} + \frac{1}{3} \log 3$$

$$\therefore \quad \mathbf{H(X) = 0.9183\ bits/message\text{-}symbol}$$

$$\text{We have} \quad \bar{p}w + p\bar{w} = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{7}{12}$$

$$\text{and} \quad pw + \bar{p}\bar{w} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = \frac{5}{12}$$

H(Y)=7/12 log(12/7)+5/12 log(12/5)

H(Y)=0.9799 bits/symbol

$$H\left(Y/_X\right) = h = P \log \frac{1}{P} + \bar{P} \log \frac{1}{\bar{P}}$$

= ¾ log(4/3)+1/4 log(4/1)

=0.8113 bits/symbols

$$H(X, Y) = H(X) + H(Y/X)$$

$$= 0.9183 + 0.8113$$

$$\mathbf{H(X, Y) = 1.7296\ bits/message\text{-}symbol}$$

$$H(Y, X) = H(X, Y) = H(Y) + H(X/Y)$$

$$H(X/Y) = H(X, Y) - H(Y)$$

$$= 1.7296 - 0.9799$$

$$\mathbf{H(X/Y) = 0.7497\ bits/message\text{-}symbol}$$

$$I(X, Y) = H(X) - H(X/Y)\ [\text{or } H(Y) - H(Y/X)]$$

$$= 0.9183 - 0.7497$$

$$\mathbf{I(X, Y) = 0.1686\ bits/message\text{-}symbol}$$

$$C = 1 - h = 1 - H(Y/X)$$
$$= 1 - 0.8113$$
$$C = \mathbf{0.1887 \text{ bits/message-symbol}}$$

$$\text{Channel efficiency} = \frac{I(X,Y)}{C}$$
$$= \frac{0.1686}{0.1887}$$
$$\therefore \quad \eta_{ch} = 89.35\%$$

Channel Redundancy =$\eta_{ch} = 10.65\%$

1. 9 Shannon Hartley Theorem and its Implication

The waveform received at the receiver may be accompanied by a waveform which varies with respect to time in an entirely unpredictable manner. This unpredictable voltage waveform is a random process called "noise".

The noise introduced due to thermal motion of electrons is called ***"Johnson Noise"*** or ***"White Noise"*** and the noise resulting due to the flow of electrons across semiconductor junction is called ***"Shot Noise"***. When the noise adds to the signal it is called ***"Additive Noise"*** and if it multiplies, then it is called ***"Fading"***.

In channels, the noise is almost white and it has a distribution which resembles Gaussian (or normal) distribution with zero mean and some variance. Hence, this noise is also called "additive white Gaussian-noise (AWGN)".

Statement of Shannon-Hartley Law :

Shannon-Hartley law also called Shannon's third theorem, states that the capacity of a band-limited Gaussian channel with AWGN is given by

$$C = B \log\left(1 + \frac{S}{N}\right) \text{ bits/sec} \qquad \text{..... (4.99)}$$

Where B = Channel bandwidth in Hz

S = Signal power in watts

N = Noise power in watts $= \eta B$

where the two sided power spectral density of noise is $(\eta/2)$ watts/Hz.

Proof :

From equation (4.85), the channel capacity C is given by

$$C = [H(Y) - H(N)]_{max} \qquad \text{...... (4.100)}$$

If the noise is additive, white and Gaussian having a power N in a bandwidth of B Hz, then from equation (4.66), we have

$$H(N)_{max} = B \log 2\pi eN \text{ bits/sec} \qquad \text{...... (4.101)}$$

When the input signal is limited to an average power S, over the same bandwidth of B Hz, and when the signal at the receiver Y = X + N with X and N being independent, then the received signal power is nothing but variance given by

$$\sigma_Y^2 = (S + N) \qquad \text{...... (4.102)}$$

We have seen in the derivation of equation (4.65) that for a given mean square value, the entropy will be maximum if the signal is Gaussian and the maximum entropy is given by

$$H(Y)_{max} = B \log 2\pi e\sigma_Y^2 \text{ bits/sec} \qquad \text{...... (4.103)}$$

Using equation (4.102) in (4.103), we get

$$H(Y)_{max} = B \log 2\pi e(S + N) \text{ bits/sec} \qquad \text{...... (4.104)}$$

Using equations (4.101) and (4.104) in equation (4.100), we get

$$C = B \log 2\pi e(S + N) - B \log 2\pi eN$$

$$= B \log \frac{2\pi e(S+N)}{2\pi eN}$$

$$\text{or} \quad \mathbf{C = B \log \left(1+\frac{S}{N}\right) \text{ bits/sec}} \qquad \text{...... (4.105)}$$

1st Implication :

From Shannon-Hartley law, we have

$$C = B \log_2\left(1+\frac{S}{N}\right) \text{ bits/sec} \qquad \text{...... (4.107)}$$

It looks, from the above equation that when B is increased, channel capacity C also increases and since $R_{max} = C$, the maximum rate of information transmission can be enhanced to any large value as we please. However, the channel capacity does not become infinite when the bandwidth is made infinite. This is because, as B increases, the noise power N which is dependent on B, also increases thereby reducing $\left(\frac{S}{N}\right)$. Thus the product of B and $\log_2\left(1+\frac{S}{N}\right)$ will increase only upto a certain value and becomes constant with increasing B. This value is denoted by C_∞. Let us calculate that value.

Substituting $N = \eta B$ in equation (4.107), we get

$$C = B \log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\left(\frac{\eta B}{S}\right)\log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{\left(\eta B/S\right)}$$

Let $x = \frac{S}{\eta B}$

$$\text{Then} \quad C = \frac{S}{\eta}\log_2 (1+x)^{(1/x)}$$

Accordingly when $B \to \infty$, $x \to 0$

$$\therefore \quad \lim_{\substack{B\to\infty \\ x\to 0}} C = \lim_{\substack{B\to\infty \\ x\to 0}} \frac{S}{\eta}\log_2 (1+x)^{(1/x)}$$

$$\therefore \quad C_\infty = \frac{S}{\eta}\log_2 \lim_{x\to 0}\left[(1+x)^{(1/x)}\right]$$

$$\therefore \quad C_\infty = \frac{S}{\eta}\log_2 e$$

$$\text{or} \quad \mathbf{C_\infty = B\frac{S}{N}\log_2 e \text{ bits/sec}} \qquad \text{...... (4.108)}$$

SHANNON'S LIMIT :

We define an ***"ideal system"*** as one that transmits data at a bit rate R_t equal to the channel capacity C. We may then express the average transmitted power as

$$S = E_b C \qquad \text{...... (4.109)}$$

Where, E_b = transmitted energy per bit in joules.

Using $N = \eta B$ and $S = E_b C$ in equation (4.107), we get for an ideal system

$$C = B\log_2\left(1+\frac{E_b}{\eta}\frac{C}{B}\right)$$

$$\text{or} \quad \frac{C}{B} = \log_2\left(1+\frac{E_b}{\eta}\frac{C}{B}\right) \qquad \text{...... (4.110)}$$

The quantity $\left(\frac{C}{B}\right)$ is called ***"Bandwidth-efficiency"*** and the quantity (E_b/η) is given by

$$\frac{E_b}{\eta} = \frac{2^{C/B}-1}{(C/B)} \qquad \text{...... (4.111)}$$

When (R_t/B) is plotted as a function of (E_b/η), we get the bandwidth-efficiency diagram which is shown in figure 4.3. The resulting curve represents the capacity boundary for which

$R_s = C$ corresponding to equation (4.111). Based on this diagram, the following observations are made:

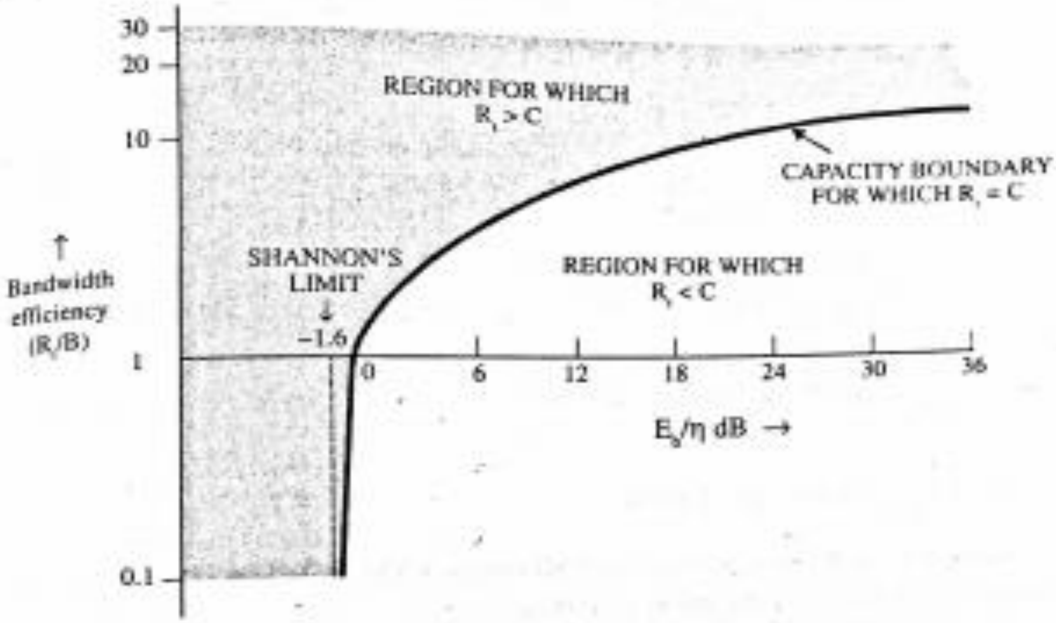


Fig. 4.3 : Illustrating Bandwidth-efficiency diagram

1. For infinite bandwidth, the signal energy-to-noise ratio E_b/η approaches the limiting value.

$$\left(\frac{E_b}{\eta}\right)_\infty = \lim_{B\to\infty}\left(\frac{E_b}{\eta}\right) = \lim_{B\to\infty}\left[\frac{2^{C/B}-1}{(C/B)}\right]$$

Let $\frac{C}{B} = x$. As $B \to \infty$, $x \to 0$

$$\therefore \quad \left(\frac{E_b}{\eta}\right)_\infty = \lim_{x\to 0}\left[\frac{2^x-1}{x}\right] \qquad \text{...... (4.112)}$$

Using L'Hospital Rule, the above limit can be evaluated as below:

$$\text{Let} \quad y = 2^x$$

Taking ln on both sides

$$ln\, y = x\, ln2$$

Differentiating, $\frac{1}{y}\, dy = (ln2)\, dx$ ...... (4.113)

$$\therefore \quad \frac{dy}{dx} = y\,(ln2) = 2^x\,(ln2)$$

Differentiating both numerator and denominator of the RHS of equation (4.112) with respect to 'x', we get

$$\left(\frac{E_b}{\eta}\right)_\infty = \lim_{x\to 0}\left[\frac{\frac{d}{dx}(2^x-1)}{\frac{d}{dx}(x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2^x(\ln 2)}{1}\right] \text{ by using equation (4.113)}$$

$$= 2^0 \ln 2$$

$$\therefore \left(\frac{E_b}{\eta}\right)_\infty = \ln 2 = 0.693$$

or $$\left(\frac{E_b}{\eta}\right)_\infty \text{ in dB} = 10\log_{10}(0.693)$$

$$\therefore \left(\frac{E_b}{\eta}\right)_\infty \text{ in dB} \cong -1.6 \text{ dB} \qquad \text{...... (4.114)}$$

This value of – 1.6 dB is called the **"Shannon's Limit"**. The corresponding value of channel capacity is given by equation (4.108) as

$$C_\infty = B\frac{S}{N}\log_2 e \text{ bits/sec}$$

2. The capacity boundary, defined by the curve for critical bit rate $R_t = C$, separates combinations of system parameters that have the potential for supporting error free transmission ($R_t < C$) from those for which error-free transmission is not possible ($R_t > C$). The latter region is shown using dots in figure 4.3.
3. The diagram of figure 4.3 highlights trade-offs between (E_b/η) and (R_t/B). This is discussed in a difficult aspect in the 2nd implication of Shannon-Hartley law.

2nd Implication :

Bandwidth - (S/N) Trade Off :

An important implication of Shannon-Hartley law is the exchange of bandwidth with signal to noise power ratio and vice-versa as given below :

$$\text{Suppose } \left(\frac{S_1}{N_1}\right) = 7 \text{ and } B_1 = 4 \text{ KHz.}$$

$$\therefore \text{ Channel capacity } C_1 = B_1 \log\left(1+\frac{S_1}{N_1}\right)$$

$$= 4\times 10^3 \log(1+7)$$

$$= 12\times 10^3 \text{ bits/sec.}$$

Keeping the channel capacity C_2 same as C_1 and if signal-to-noise ratio is increased to 15, then

$$C_2 = C_1 = 12 \times 10^3 = B_2 \log\left(1 + \frac{S_2}{N_2}\right)$$
$$= B_2 \log(1 + 15)$$
$$\therefore \quad \mathbf{B_2 = 3\ KHz}$$

Since the noise power $N = \eta B$, as the bandwidth gets reduced from 4 to 3 KHz, the noise also decreases indicating an increase in signal power as shown below.

We have $N_1 = \eta B_1 = (\eta)(4\ KHz)$

and $N_2 = \eta B_2 = (\eta)(3\ KHz)$

$$\text{Consider } \frac{(S_2/N_2)}{(S_1/N_1)} = \frac{15}{7}$$

$$\therefore \quad \frac{S_2}{S_1} = \frac{15 N_2}{7 N_1} = \frac{(15)(\eta)(3KHz)}{(7)(\eta)(4KHz)} = \frac{45}{28} = 1.6$$

Thus a 25% reduction in bandwidth from 4 KHz to 3 KHz requires a 60% approximate increase in signal power for maintaining the same channel capacity. Let us look into the exact significance by drawing the ***"trade-off curve"***.

From Shannon-Hartley law

$$\frac{B}{C} = \frac{1}{\log_2\left(1 + \frac{S}{N}\right)} \qquad \text{...... (4.115)}$$

The values of (B/C) for different values of (S/N) are listed in table 4.1 below :

$\frac{S}{N}$	0.5	1	2	5	10	15	20	30
$\frac{B}{C}$	1.71	1	0.63	0.37	0.289	0.25	0.23	0.2

Table 4.1 : Table of values of (B/C) for different values of (S/N)

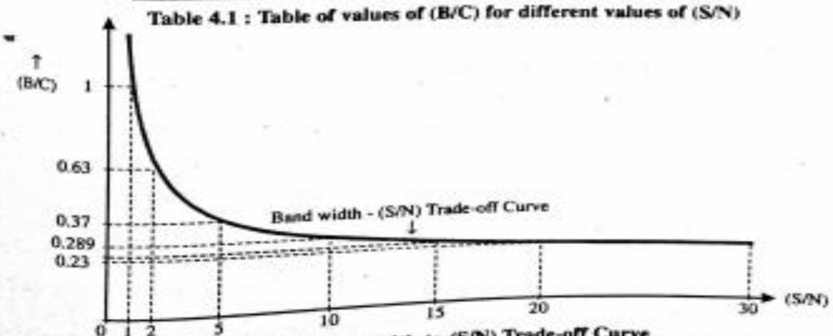


Fig. 4.4 : Bandwidth-to-(S/N) Trade-off Curve

Figure 4.4 above shows a plot of (B/C) as a function of (S/N). Using this trade-off curve the same channel capacity may be obtained by increasing bandwidth if (S/N) is small. Furthermore, the curve also indicates that there exists a threshold point at around (S/N) = 10 up to which the exchange rate of bandwidth with (S/N) is advantageous. Beyond (S/N) = 10, the reduction in B with increasing (S/N) is very poor. FM, PM and PCM systems including DM and ADM systems require larger bandwidths with reasonably good (S/N) ratio.

Ex.1:

Alphanumeric data are entered into a computer from a remote terminal through a voice-grade telephone channel. The channel has a bandwidth of 3.4 KHz, and output signal-to-noise ratio of 20 dB. The terminal has a total of 128 symbols. Assume that the symbols are equiprobable and the successive transmissions are statistically independent.

(a) Calculate channel capacity.

(b) Find the average information content per character.

(c) Calculate the maximum symbol rate for which error-free transmission over the channel is possible.

Solution

Given $B = 3.4 \text{ KHz} = 3400 \text{ Hz}$

$$10 \log_{10}\frac{S}{N} = 20 \text{ dB} \quad \therefore \quad \frac{S}{N} = 100$$

Number of characters $= q = 128$ equiprobable characters.

(a) Channel capacity from equation (4.107)

$$C = B \log_2\left(1+\frac{S}{N}\right)$$

$$= 3400 \log_2(1 + 100)$$

$$\therefore \quad \mathbf{C = 22638 \text{ bits/sec}}$$

(b) Average information content per character [it is maximum since all the characters are equiprobable] is

$$H = H_{max} = \log_2 q = \log_2 128$$

$$\therefore \quad \mathbf{H = 7 \text{ bits/character}}$$

(c) Average information rate $= R_s = r_s H$

For error-free transmission we must have $R_s < C$

$$\therefore \quad r_s H < C$$

$$\therefore \quad r_s < \frac{C}{H}$$

$$\therefore \quad r_s < \frac{22638}{7}$$

$$\therefore \quad \mathbf{r_s < 3234 \text{ symbols/sec}}$$

$\therefore$ The maximum symbol rate for which error-free transmission over the channel is possible = 3234 symbols/sec.

Ex.2: A CRT terminal is used to enter alphanumeric data into a chamber. The CRT is connected through a voice-grade telephone line having usable band width of 3 KHz and an output (S/N) of 10 dB. Assume that the terminal has 128 characters and data is sent in an independent manner with equal probability.

(i) Find the average information per characters.

(ii) Find capacity of the channel.

(iii) Find maximum rate at which data can be sent from terminal to the computer error.

(i) Since all the characters are equiprobable, the average information content per character is

$$H = H_{max} = \log_2 q = \log_2 128$$

or $$\mathbf{H = 7 \text{ bits/character}}$$

(ii) Given, $$B = 3 \text{ KHz} = 3000 \text{ Hz}$$

$$10 \log_{10} \frac{S}{N} = 10 \quad \therefore \quad \frac{S}{N} = 10$$

From equation (4.107), channel capacity is given by

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$= 3000 \log_2 (1 + 10)$$

$\therefore$ $$\mathbf{C = 10378.295 \text{ bits/sec}}$$

(iii) Average information rate = $R_s = r_s H$

For error-free transmission we must have

$$R_s = C$$

i.e., $$r_s H < C$$

$\therefore$ $$r_s < \frac{C}{H}$$

$\therefore$ $$r_s < \frac{10378.295}{7}$$

$\therefore$ $$\mathbf{r_s < 1482.614 \text{ symbols/sec}}$$

$\therefore$ The maximum rate at which data can be sent from terminal to the computer without error = 1482.614 symbols/sec.

Ex.3: A voice-grade channel of the telephone network has a bandwidth of 3.4 KHz.

(a) Calculate channel capacity of the telephone channel foe a signal-to-noise ratio of 30dB.

(b) Calculate the minimum signal-to-noise ratio required to support information transmission through the telephone channel at the rate of 4800 bits/sec.

Solution

$$\text{Given} \quad B = 3.4 \text{ KHz} = 3400 \text{ Hz}$$

$$10 \log_{10} \frac{S}{N} = 30 \text{ dB} \quad \therefore \frac{S}{N} = 1000$$

(a) Channel capacity from equation (4.107)

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$= 3400 \log_2(1 + 1000)$$

$$\therefore \quad \mathbf{C = 33889 \text{ bits/sec}}$$

(b) $$\text{Given} \quad C = 4800 \text{ bits/sec}, \frac{S}{N} = ?$$

$$\text{We have} \quad C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$\therefore \quad 4800 = 3400 \log_2 \left(1 + \frac{S}{N}\right)$$

$$\therefore \quad \frac{S}{N} = 2^{48/34} - 1$$

$$\therefore \quad \frac{S}{N} = \mathbf{1.66}$$

$$\text{or} \quad \frac{S}{N} \text{ in dB} = 10 \log_{10} 1.66 = 2.2 \text{ dB}$$

$$\therefore \quad \frac{S}{N} = \mathbf{2.2 \text{ dB}}$$

Ex.4:

A black and white television picture may be viewed as consisting of approximately 3×10^5 elements, each one of which may occupy one of 10 distinct brightness levels with equal probability. Assume (a) the rate of transmission is 30 picture frames per second and (b) the signal-to-noise ratio is 30 dB.

Using the channel capacity theorem (Shannon-Hartley law), calculate the minimum bandwidth required to support the transmission of the resultant video signal.

Solution:

Given number of elements/picture frame = 3×10^5

Number of brightness levels = 10

$\therefore$ Number of different frames possible = $10^{3 \times 10^5}$ frames.

Since all the levels are equiprobable, the maximum average information content per frame is given by

$$I = \log_2 10^{3\times10^5} \text{ bits/frame}$$
$$= 3\times10^5 \log_2 10 \text{ bits/frame}$$
$$\mathbf{I = 9.96\times10^5 \text{ bits/frame}}$$

The maximum rate of information is given by

$$R_{s_{max}} = r_s I$$
$$= (30 \text{ frames/sec})\,(9.96\times10^5 \text{ bits/frame})$$
$$= 29.88\times10^6 \text{ bits/sec}$$

According to Shannon's second theorem, $R_{s_{max}}$ is equal to channel capacity C. And according to Shannon-Hartley law.

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

Given $\quad 10\log_{10}\frac{S}{N} = 30 \text{ dB}$

$$\therefore \quad \frac{S}{N} = 1000$$

$$\therefore \quad B = \frac{C}{\log_2\left(1+\frac{S}{N}\right)} = \frac{29.88\times10^6}{\log_2(1+1000)}$$

$$\therefore \quad \mathbf{B = 3\ MHz}$$

Ex.5:

An analog signal has a 4 KHz bandwidth. The signal is sampled at 2.5 times the Nyquist rate and each sample quantized into 256 equally likely levels. Assume that the successive samples are statistically independent.

(i) Find the information rate of this source.

(ii) Can the output of this source be transmitted without errors over a Gaussian channel of bandwidth 50 KHz and (S/N) ratio of 20 dB?

(iii) If the output of this source is to be transmitted without errors over an analog channel having (S/N) of 10 dB, compute the bandwidth requirement of the channel.

Solution

$$\text{Given } B = 4000 \text{ Hz}$$
$$\therefore \text{ Nyquist rate} = 2B = 8000 \text{ Hz}$$
$$\therefore \text{ Sampling rate } r_s = (2.5)\,(\text{Nyquist rate})$$
$$= (2.5)\,(8000)$$
$$= 20,000 \text{ samples/sec}$$

Since all the 256 quantization levels are equally likely, the maximum information content is

$$I = \log_2 q = \log_2 256 = 8 \text{ bits/sample}$$

(i) Information Rate $R_s = r_s I$

$$= (20{,}000 \text{ samples/sec})\ (8 \text{ bits/sample})$$

$$\therefore\ \mathbf{R_s = 1{,}60{,}000\ bits/sec}$$

(ii) From Shannon-Hartley law, we have

$$C = B \log\left(1+\frac{S}{N}\right)$$

Given $10 \log_{10}\frac{S}{N} = 20 \text{ dB}$

$$\therefore\ \frac{S}{N} = 100 \text{ and } B = 50 \times 10^3 \text{ Hz}$$

$$\therefore\ C = 50 \times 10^3 \log(1 + 100)$$

$$\therefore\ \mathbf{C = 332.91 \times 10^3\ bits/sec}$$

Since $R_s < C$, according to Shannon's second theorem, it is possible to transmit over t given channel without errors.

(iii) Given $10 \log_{10}\frac{S}{N} = 10 \text{ dB}$ $\therefore$ $\frac{S}{N} = 10$

From Shannon-Hartley law,

$$C = B \log\left(1+\frac{S}{N}\right) = R_s$$

$$\therefore\ B = \frac{R_s}{\log\left(1+\frac{S}{N}\right)} = \frac{160\times10^3}{\log(1+10)}$$

$$\therefore\ \mathbf{B = 46.25\ KHz}$$