Questions on solving degree

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We copy the relevant definitions from [CG20] here:

Definition 6 (page 15). *Let $\mathcal{F} = \{f_1, \ldots, f_r\} \subseteq R$ and let τ be a term order on R. The* solving degree *of $\mathcal{F}$ is the least degree d such that Gaussian elimination on the Macaulay matrix $M_{\leq d}$ produces a Gröbner basis of $\mathcal{F}$ with respect to τ. We denote it by* $\text{solv.deg}_\tau(\mathcal{F})$. *When the term order is clear from the context, we omit the subscript τ.*

If $\mathcal{F}$ is homogeneous, we consider the homogeneous Macaulay matrix M_d and let the solving degree of $\mathcal{F}$ be the least degree d such that Gaussian elimination on $M_0, \ldots, M_d$ produces a Gröbner basis of $\mathcal{F}$ with respect to τ.

Definition 7 (page 16). *Let $I \subseteq R$ be an ideal and let τ be a term order on R. We denote by* $\text{max.GB.deg}_\tau I$ *the maximum degree of a polynomial appearing in the reduced τ Gröbner basis of I. If $I = (\mathcal{F})$, we sometimes write* $\text{max.GB.deg}_\tau(\mathcal{F})$ *in place of* $\text{max.GB.deg}_\tau(I)$.

We walk through Example 6 of [CG20] to see what these Macaulay matrices look like. Here, $\mathcal{F} = \{f_1, f_2, f_3, f_4\} = \{x^2+x, xy, y^2+y, x^2y+x^2+x\} \subseteq \mathbb{F}_2[x, y]$. Following the constructions on page 15 with the term order $\tau = DRL$, we have that

$$M_{\leq 2} = \begin{array}{c c} & \begin{array}{cccccc} x^2 & xy & y^2 & x & y & 1 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} & \begin{pmatrix} \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \end{pmatrix} \end{array}$$

(f_4 is not included since it has degree 3). Since this matrix is already row reduced, we get the collection $\{f_1, f_2, f_3\}$, which is a (reduced) Gröbner basis for $(\mathcal{F})$ ($f_4 = f_1 + xf_2$). Since $M_{\leq d}$ is the empty matrix for $d < 2$, $d = 2$ is the first degree for which row reduction on $M_{\leq d}$ produces a Gröbner basis with respect to our chosen term order (DRL), and so $\text{solv.deg}_{DRL}(\mathcal{F}) = 2$.

For a more complicated example, we walk through Example 5 of [CG20]. This time we use the LEX order, and our system is $\mathcal{F} = \{f_1, f_2\} = \{x_3^2 - x_2, x_2^3 - x_1\} \subset \mathbb{F}_5[x_1, x_2, x_3]$. Then we have that

$$M_{\leq 2} = \begin{array}{c c} & \begin{array}{cccccccccc} x_1^2 & x_1y_1 & x_1x_3 & x_1 & y_2^2 & x_2x_3 & x_2 & x_3^2 & x_3 & 1 \end{array} \\ f_1 & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & \mathbf{1} & 0 & 0 \end{pmatrix} \end{array},$$

on which row reduction does not produce a Gröbner basis, and also that $M_{\leq 3}$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{4} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

where the columns are indexed by the monomials

$$x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1^2,\ x_1x_2^2,\ x_1x_2x_3,\ x_1x_2,\ x_1x_3^2,\ x_1x_3,\ x_1,\ x_2^3,\ x_2^2x_3,\ x_2^2,\ x_2x_3^2,\ x_2x_3,\ x_2,\ x_3^3,\ x_3^2,\ x_3,\ 1$$

and the rows by f_1, x_1f_1, x_2f_1, x_3f_1, and f_2. Row reduction on this matrix doesn't change the set of polynomials we're working with—we still have $\{f_1, x_1f_1, x_2f_1, x_3f_1, f_2\}$—but this set of polynomials is now a Gröbner basis for $\mathcal{F}$, so $\text{solv.deg}_{LEX}(\mathcal{F}) = 3$, as in [CG20].

Since the *reduced* Gröbner basis for $\mathcal{F}$ is $\{x_1 - x_3^6, x_2 - x_3^2\}$ and this reduced basis contains a degree 6 polynomial, we have that $\text{max.GB.deg}_{LEX}(\mathcal{F}) = 6 \not\leq 3 = \text{solv.deg}_{LEX}(\mathcal{F})$. This would appear to contradict the remark after Definition 7, stating that $\text{max.GB.deg}_\tau(\mathcal{F}) \leq \text{solv.deg}_\tau(\mathcal{F})$ for any term order, but we note that the system in Example 5 has infinitely many solutions over $\overline{\mathbb{F}_5}$. On pages 10-11 it is stated that the assumption is always made that there are only finitely many solutions over the algebraic closure (at least for Section 2), and indeed, the inequality appears to hold in this case.

1. Does this finiteness assumption also apply in Section 3?

2. If so, does this imply the inequality $\text{max.GB.deg}_\tau(\mathcal{F}) \leq \text{solv.deg}_\tau(\mathcal{F})$?

References

[CG20] Alessio Caminata and Elisa Gorla. Solving multivariate polynomial systems and an invariant from commutative algebra. In *International Workshop on the Arithmetic of Finite Fields*, pages 3–36. Springer, 2020.