# Math 1231 Summer 2024
# Midterm 1

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.
- The exam has 5 problems, one on each mastery topic we've covered. The exam has 5 pages total.
- Each part of each topic is worth ten points, except the M2 questions are worth 15 points. The whole test is scored out of 100 points.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all. When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

| | a | b | c | d |
|---|---|---|---|---|
| M1: | | | | |
| M2: | | | S1: | |
| S2: | | S3: | | /100 |

**Name:** Solutions

Name: Solutions

**Problem 1** (M1). Compute the following limits if they exist. Show enough work to justify your computation.

(a) $\lim\limits_{x\to+\infty} \dfrac{3x^2+2x+1}{\sqrt{x^4-x^2+x}}$

**Solution.** Noting that since the $x^4$ term in the denominator is inside the square root, we should consider the highest power appearing to be $x^2$. We have

$$\boxed{\lim_{x\to+\infty} \frac{3x^2+2x+1}{\sqrt{x^4-x^2+x}} = \lim_{x\to\infty} \frac{3+2/x+1/x^2}{\sqrt{1-1/x^2+1/x^3}} = \frac{3+2\cdot 0+0}{1-0+1} = 3.}$$

(b) $\lim\limits_{x\to 5} \dfrac{1}{x-5} - \dfrac{5}{x^2-5x}$

**Solution.** Both terms individually appear to be reaching some kind of infinite limit, but we recall that the different limit law only works for limits that actually *exist*. Therefore we must rewrite this as a single limit and go from there, which we can do by getting a common denominator:

$$\boxed{\lim_{x\to 5} \frac{1}{x-5} - \frac{5}{x^2-5x} = \lim_{x\to 5} \frac{x}{x^2-5x} - \frac{5}{x^2-5x} = \lim_{x\to 5} \frac{x-5}{x(x-5)} \overset{\text{AIF}}{=} \lim_{x\to 5} \frac{1}{x} = \frac{1}{5}.}$$

(c) $\lim\limits_{x\to 1} \dfrac{\sin^2(x-1)}{(x-1)^2}$

**Solution.** We use the small angle approximation with $\theta = x-1$, which is small (i.e. approaches zero) when $x$ approaches 1:

$$\boxed{\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1.}$$

In the first equality we have used the power law for limits.

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(d) $\lim_{x\to 4} \dfrac{\sqrt{x+5}-3}{x-4}$

**Solution.** Plugging in 4 gives a $0/0$ situation, so we multiply by the conjugate to obtain an almost identical function that is defined at 4:

$$\boxed{\begin{aligned}\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} &= \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x+5}+3)} \\ &\overset{\text{AIF}}{=} \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{3+3} = \frac{1}{6}.\end{aligned}}$$

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**Problem 2** (M2). Compute the derivatives of the following functions using any of the methods we have learned in class.

(a) $\dfrac{d}{dx}\left(\dfrac{x\csc(x)}{\sqrt{x^3-x}}\right)^3$

**Solution.** We get

$$\boxed{=3\left(\frac{x\csc(x)}{\sqrt{x^3-x}}\right)^2\cdot\frac{(\csc(x)-x\csc(x)\cot(x))\sqrt{x^3-x}-x\csc(x)\cdot\frac{1}{2}(x^3-x)^{-\frac{1}{2}}\cdot(3x^2-1)}{x^3-x}}$$

(b) $\dfrac{d}{dx}\sec(\tan(\cos((x+1)^2)))$

**Solution.** We get

$$\boxed{\begin{aligned}&=\sec(\tan(\cos((x+1)^2)))\tan(\tan(\cos((x+1)^2)))\\&\cdot\sec^2(\cos((x+1)^2))\cdot-\sin((x+1)^2)\cdot 2(x+1).\end{aligned}}$$

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**Problem 3** (S1). If $f(x) = \sqrt{x+3}$, find $f'(6)$, directly from the limit definition of derivative.

**Solution.** Using the $h \to 0$ version of the limit, we get

$$\boxed{\begin{aligned} f'(6) &= \lim_{h\to 0} \frac{f(6+h)-f(6)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} \overset{\text{AIF}}{=} \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}. \end{aligned}}$$

If we instead use the $t \to 6$ formulation, we have

$$\boxed{\begin{aligned} f'(6) &= \lim_{t\to 6} \frac{f(t)-f(6)}{t-6} = \lim_{t\to 6} \frac{\sqrt{t+3}-3}{t-6} \cdot \frac{\sqrt{t+3}+3}{\sqrt{t+3}+3} \\ &= \lim_{t\to 6} \frac{t-6}{(t-6)(\sqrt{t+3}+3)} \overset{\text{AIF}}{=} \lim_{t\to 6} \frac{1}{\sqrt{t+3}+3} = \frac{1}{\sqrt{6+3}+3} = \frac{1}{6}. \end{aligned}}$$

**Problem 4** (S2). Give the equation for the linear approximation of $f(x) = \dfrac{x^3}{1+x}$ near the point $a = 1$. Use it to estimate $f(1.3)$.

**Solution.** We want to use the linear approximation formula $f(x) \approx f(a) + f'(a)(x-a)$ for $a = 1$, so we will need to compute $f(1)$ and $f'(1)$. We see that $\boxed{f(1) = 1/2}$, and that

$$\boxed{f'(x) = \frac{3x^2(1+x)-x^3}{(1+x)^2}, \text{ so } f'(1) = \frac{3(2)-1}{2^2} = \frac{5}{4}.}$$

Putting these together, we get our approximation that $\boxed{f(x) \approx \frac{1}{2} + \frac{5}{4}(x-1)}$. Therefore

$$\boxed{f(1.3) \approx \frac{1}{2} + \frac{5}{4} \cdot \frac{3}{10} = \frac{4}{8} + \frac{3}{8} = \frac{7}{8} = .875.}$$

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**Problem 5** (S3). Find a formula for $\frac{dy}{dx}$ in terms of $x$ and $y$ if $x^3y + x^2y^2 + y^4 = 0$.

**Solution.** We use implicit differentiation:

$$\boxed{\begin{gathered} 3x^2y + x^3\frac{dy}{dx} + 2xy^2 + x^2 \cdot 2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0 \\ (x^3 + 2x^2y + 4y^3)\frac{dy}{dx} = -(3x^2y + 2xy^2) \\ \frac{dy}{dx} = -\frac{3x^2y + 2xy^2}{x^3 + 2x^2y + 4y^3} \end{gathered}}$$