Math 1231 Summer 2024
Midterm 2

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.
- The exam has 5 problems: one review problem, and one problem on each mastery topic we've covered since the first midterm. The exam has 6 pages total.
- The whole test is scored out of 100 points, with the points for individual questions indicated on the exam.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all. When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

M1:		M2:	
M3a:		M3b:	
S4:		S5:	
S6:		Tot:	/100

Name: Solutions

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Problem 1 (Midterm 1 Review). These are very similar to the trickier limit and derivative problems on the first midterm. Part (a) is worth 10 points, and part (b) 15 points.

(a) Evaluate $\lim_{x\to 0} \frac{\sin(6x)\sin(2x)}{x\sin(3x)}$ **without** using any shortcuts or L'Hôpital's rule.

Solution. We use the Small Angle Approximation, which says that $\lim_{\theta\to 0}\sin(\theta)/\theta = 1$. We multiply and divide by whatever we need to make each part of the limit that involves the sine function look like the limit in the SAA:

$$\boxed{\lim_{x\to 0}\frac{\sin(6x)\sin(2x)}{x\sin(3x)} = \lim_{x\to 0}\frac{\sin(6x)}{6x}\cdot\frac{\sin(2x)}{2x}\cdot\frac{1}{\frac{\sin(3x)}{3x}}\cdot\frac{1}{x}\cdot 6x\cdot 2x\cdot\frac{1}{3x} = \lim_{x\to 0}\frac{12x}{3x} = 4.}$$

(b) Compute $\frac{d}{dx}\sec\left(\frac{x^3\cos(x)}{\sin(\sqrt{x})}\right)$.

Solution.

$$\boxed{= \sec\left(\frac{x^3\cos(x)}{\sin(\sqrt{x})}\right)\tan\left(\frac{x^3\cos(x)}{\sin(\sqrt{x})}\right)\cdot\frac{(3x^2\cos(x) - x^3\sin(x))\sin(\sqrt{x}) - x^3\cos(x)\cdot\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}}{\sin^2(\sqrt{x})}}$$

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Problem 2 (M3). Each part is worth 20 points.

(a) Find and classify the critical points of the function $f(x) = \dfrac{2x-1}{x^2+2}$. That is, for each critical point you find, say whether it is a relative maximum, a relative minimum, or neither.

Solution. We have

$$\begin{aligned} f'(x) &= \frac{2(x^2+2) - 2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2+2x+4}{(x^2+2)^2} \\ &= -2\frac{x^2-x-2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2} \end{aligned}$$

so the critical points are 2 and -1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$f'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$\begin{aligned} f''(x) &= -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4} \\ f''(-1) &= -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0 \\ f''(2) &= -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0. \end{aligned}$$

Thus $f''(-1) > 0$ so g has a minimum at -1; and $f''(2) < 0$ so f has a maximum at 2.

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(b) The function $g(x) = x^{\frac{2}{3}}(x-3)$ has absolute extrema either on the interval $(-1,1)$ or on the interval $[-1,1]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution. Since g is continuous on the closed interval $[-1,1]$, by the Extreme Value Theorem, g has absolute extrema on this interval. To find the extrema, we find critical points:

$$g'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x-3) + x^{\frac{2}{3}} = \frac{2(x-3)+x}{3x^{\frac{1}{3}}} = \frac{5x-6}{3x^{\frac{1}{3}}},$$

so the critical points are $x = 0$ (where g' is undefined) and $x = 6/5$ (where g' is zero). Of these, we care only about $x = 0$ since $x = 6/5$ is not in $[-1,1]$. We check

$$\begin{aligned} g(-1) &= (-1)^{\frac{2}{3}}(-1-3) = -4 \\ g(0) &= (0)^{\frac{2}{3}}(0-3) = 0 \\ g(1) &= (1)^{\frac{2}{3}}(1-3) = -2 \end{aligned}$$

Therefore the absolute max is 0 at 0 and the absolute min is -4 at -1.

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Problem 3 (S4). 10 points. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Solution. We know how quickly the bottom is sliding away from the wall, and we want to know how fast the top is sliding down. The ladder forms a triangle against the wall, and we want to relate the lengths of two sides, so we use the Pythagorean theorem $h^2 + b^2 = c^2$.

We know $b = 6$ and $c = 10$; by the Pythagorean theorem, we know that $h = 8$. Further we know that $b' = 2$ and $c' = 0$. So taking a derivative gives

$$\begin{aligned} 2hh' + 2bb' &= 2cc' \\ 2 \cdot 8h' + 2 \cdot 6 \cdot 2 &= 0 \\ 16h' &= -24 \\ h' &= -3/2. \end{aligned}$$

Thus the top of the ladder is sliding down the wall at 1.5 feet per second.

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Problem 4 (S5). 15 points. Sketch the graph of $f(x) = x^5 - 5x^4 + 5x^3 = x^3(x^2 - 5x + 5)$. We have $f'(x) = 5x^2(x-3)(x-1)$ and $f''(x) = 10x(2x^2 - 6x + 3)$. Your answer should state

(a) the domain of the function
(b) any horizontal or vertical asymptotes
(c) the roots of the function
(d) the critical points of the function
(e) intervals on which the function is increasing or decreasing
(f) any relative minima or maxima
(g) intervals on which the function is concave up or concave down
(h) any inflection points.

Solution. (a) The domain of f is all reals.

(b) We see that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, so there are no horizontal asymptotes. There are also no vertical asymptotes because f is defined everywhere.

(c) There are roots at 0 and at $5/2 \pm \sqrt{5}/2$.

(d) The critical points are $0, 1, 3$. We compute $f(0) = 0, f(1) = 1, f(3) = -27$.

(e) For increase and decrease we make a chart:

	$5x^2$	$(x-3)$	$(x-1)$	$f'(x)$
$x < 0$	$+$	$-$	$-$	$+$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 1)$ and on $(3, +\infty)$, and is decreasing on $(1, 3)$.

(f) The above table lets us use the first derivative test to say that $f(0)$ is neither max nor min, that $f(1)$ is a relative maximum, and that $f(3)$ is a relative minimum.

(g) The possible points of inflection are 0 and $\frac{6\pm\sqrt{36-24}}{4} = \frac{3\pm\sqrt{3}}{2}$. For concavity, we can make a chart for the sign of the second derivative:

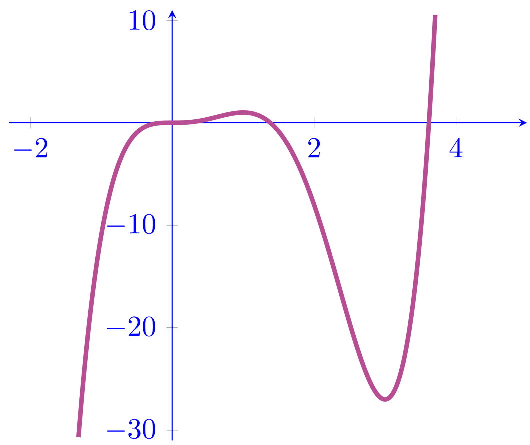
	$10x$	$2x^2 - 6x + 3$	$f''(x)$
$x < 0$	$-$	$+$	$-$
$0 < x < (3-\sqrt{3})/2$	$+$	$+$	$+$
$(3-\sqrt{3})/2 < x < (3+\sqrt{3}))/2$	$+$	$-$	$-$
$(3+\sqrt{3})/2 < x$	$+$	$+$	$+$

And this shows

$$f \text{ concave up on } \left(0, \frac{3-\sqrt{3}}{2}\right) \cup \left(\frac{3+\sqrt{3}}{2}, \infty\right) \quad \text{down on } (-\infty, 0) \cup \left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right).$$

(h) Because f *does* change signs at each one of the possible points of inflection we identified in (g), each of these is really an inflection point.

Based on all this information, we have the following sketch:



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Problem 5 (S6). 10 points. Suppose that a company that produces and sells x units of a product each day makes a revenue of $R(x) = 260x - 9x^2/10$ dollars per day and has costs given by $C(x) = 1000 + 100x + x^2/10$ dollars per day. What is the maximum profit that can be made (where profit is revenues minus costs)? **Your answer should include some justification for how you know this is really a maximum.**

Solution. Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$\begin{aligned} P'(x) &= 160 - 2x \\ 160 &= 2x \\ 80 &= x \end{aligned}$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

Note that the EVT doesn't apply here since we're not really working on a closed interval. If we knew there were some maximum number of machines that could be produced, then the EVT could work, but without this we're working on the domain $[0, \infty)$.

In any case, the first or second derivative tests both still work at showing $x = 80$ is a maximum, and the profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

Thus our maximum profit is 5400 dollars per day.