Math 1231 Course Notes

Summer 2024

0 Introduction

These notes are written to follow the six-week summer course Math 1231. On the whole, we'll be following the OpenStax textbook, but I'll sometimes treat topics in slightly different orders or emphasize different things. These notes have been written with the aim of presenting the material closer to how it will be presented in the course lectures.

1 Limits

1.1 Functions

We begin by recalling what functions are: rules for assigning outputs to inputs. Anything that takes in a piece of input and returns a specific output according to some rule is a function. The rule might be expressed algebraically, as in $f(x) = 3x+1$, which says that the output $f(x)$ assigned to the input x is $3x+1$. This is an example of a linear equation—if we plot all of the pairs (x, y) satisfying the rule $y = f(x) = 3x + 1$, we get a line in the plane. We are using the letter f here to refer to the function itself—f is the rule, and we apply it to an input x to get an output. For instance, we could have an input of 4, and applying the function f we get $f(4) = 13$. The point $(4, 13)$ is then a point on the line that is the graph of this function.

The rule for a function doesn't always have to look like this though. Sometimes we might combine multiple rules into one overall rule, where the kind of input determines which rule we use. This happens when we define *piecewise functions*, e.g.

$$f(x) = \begin{cases} x-2 & x \leq 4 \\ 10-2x & x > 4 \end{cases} \qquad \text{or} \qquad g(x) = \begin{cases} 3x+1 & x \text{ is odd} \\ x/2 & x \text{ is even.} \end{cases}$$

Note that this second piecewise funtion (which is quite famous!) has a rule that we can really only use with integers, since while it makes sense to ask whether a number like 10 is odd or even (so we can say $g(10) = 5$), it doesn't make sense to ask whether a number like 10.1 is odd or even. We would say that the **domain** of this function is the set of all integers (usually denoted $\mathbb{Z}$). The first function is defined for any real number, so it has domain we could write either as $(-\infty, \infty)$, which tells us any real number strictly between $-\infty$ and ∞ is in the domain (so, all of the real numbers), or we could also use the symbol $\mathbb{R}$. The rule

$$h(x) = \frac{(x-2)(x+1)}{x-3}$$

also defines a function where it makes sense to try plugging in real numbers, but here we have to be a little bit careful. Most real numbers work just fine, but if we try to ask what output the rule assigns to the input $x = 3$, we run into a problem. When x is 3, the denominator of the expression we've given for y becomes 0, and division by zero is undefined, meaning the rule we have doesn't yield any output for $x = 3$. We would say the domain of this function is therefore all real numbers except for 3, which we can write as $(-\infty, 3) \cup (3, \infty)$. The first interval here tells us anything strictly between $-\infty$ and 3 is included in the domain, and the second interval says anything strictly between 3 and ∞ is in the domain. The symbol $\cup$ means *union*, so what this is saying is that the domain of the function is both of these two intervals taken together.

This isn't the only way to write down the domain of a function, of course. A domain is just a set of numbers, and we have lots of ways of writing down sets. A common one is to use notation like $\{x :$ -some condition-$\}$, which means the set of all numbers x such that the specified condition is true. For example, if we wanted a different way to write the domain of the function above, we could say $\{x : x \neq 3\}$, i.e., all numbers x such that x is not 3. The interval (a, b) could also be written as $\{x : a < x < b\}$ (called an **open** interval), and the interval $[a, b]$ would be $\{x : a \leq x \leq b\}$ (called a **closed** interval—square brackets mean a and b are included). An interval like $[6, 7)$ is the set $\{x : 6 \leq x < 7\}$, and this is **neither open nor closed**.

Functions and their domains can also be a lot more abstract. Inputs and outputs don't necessarily have to be things we would usually think of as being mathematical objects. For instance, GW assigns students a GWID—this is a function whose inputs are people and outputs are numbers. The domain of this function is the set of GW students, $\{x : x$ is a GW student$\}$.

In this course, we'll be focusing on functions whose inputs and outputs are mathematical objects of some kind, usually real numbers (although when we start talking about derivatives, we'll be talking about a function whose inputs and outputs are themselves functions!). Here are some familiar kinds of functions we'll be spending a lot of time with this summer:

Polynomial functions are functions like $f(x) = 5x^6 + x^2 + 4x - 2$, sums of powers of a variable x with some coefficients, written very generally like

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the $a_0, \ldots, a_n$ are some numbers and n is called the **degree** of the polynomial. When $n = 1$ we have a linear equation, $n = 2$ is a quadratic equation, and $n = 3$ is a cubic equation. You should remember the quadratic formula, which tells you how to find the inputs x that get assigned outputs $f(x) = 0$. It says that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also useful to recall that

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$
- $(a^2 + ab + b^2)(a-b) = a^3 - b^3$.

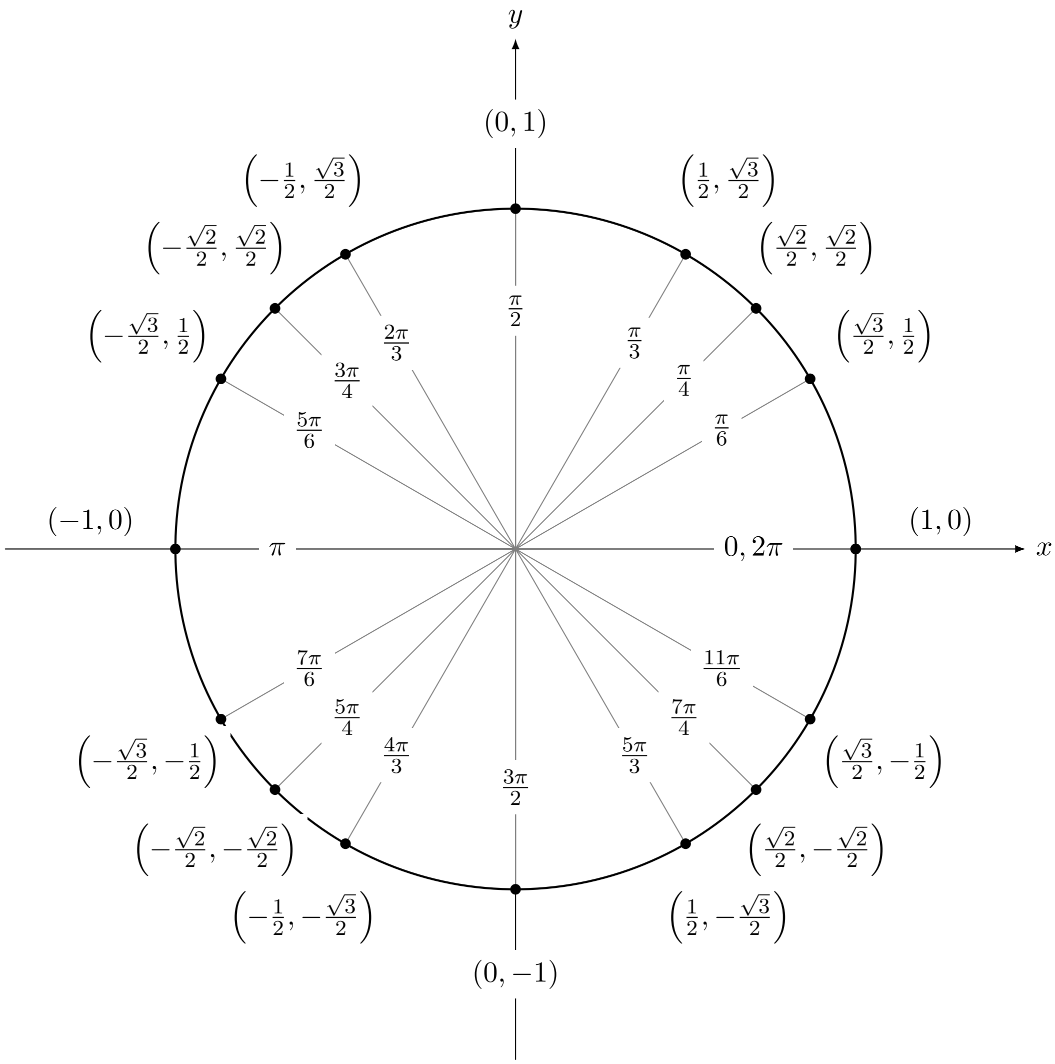
Rational functions are the ratio of two polynomials, e.g.

$$f(x) = \frac{x^2 + 2x}{x^3 - 6}.$$

Trigonometric functions: In this course we will *always* use radians, because they are unitless and thus easier to track (especially when using the chain rule). Useful facts include:

- The most important trigonometric identity, and really the only one you probably need to remember, is $\cos^2(x) + \sin^2(x) = 1$.
- From this you can derive the fact that $1 + \tan^2(x) = \sec^2(x)$.
- $\sin(-x) = -\sin(x)$. We call functions like this "odd".
- $\cos(-x) = \cos(x)$. We call functions like this "even."
- $\sin(x + \pi/2) = \sin(\pi/2 - x) = \cos(x)$
- A fact that we will probably use exactly twice is the sum of angles formula for sine: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- Similarly, $\cos(x+y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

Throughout the course you will be asked to recall various values of the trig functions, that is, you will have to be able to say that, e.g., $\cos(\pi/6) = \sqrt{3}/2$. For this reason, you should always have the unit circle written down or committed to memory: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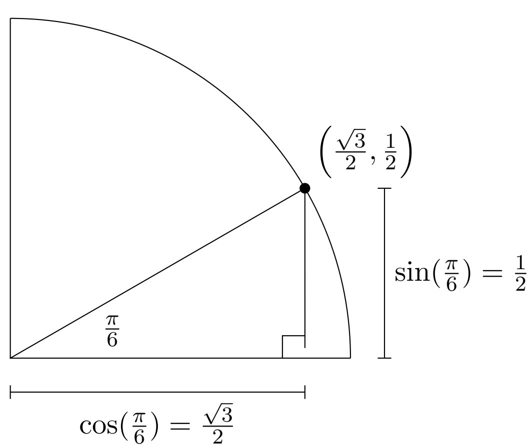

$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\sin(\frac{\pi}{6}) = \frac{1}{2}$
$\frac{\pi}{6}$
$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

1.2 Limits, intuitively

Intuitively, we know what it means for the value of a function to approach or get closer to some particular value, and we'll be relying on this intuitive understanding to develop the tools around limits.

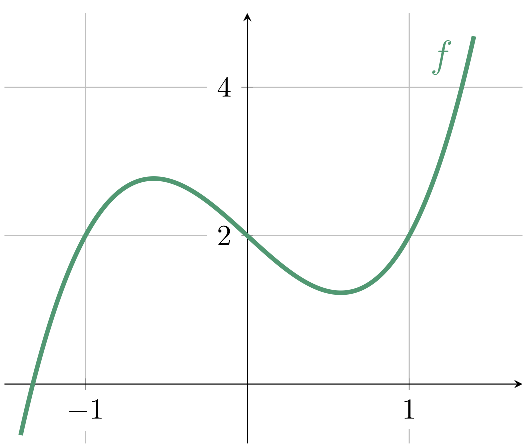
Given a function f that is defined at all the points *near* some point a, we can define the limit of f at a, should it exist. Here's the definition given in OpenStax Section 2.2:

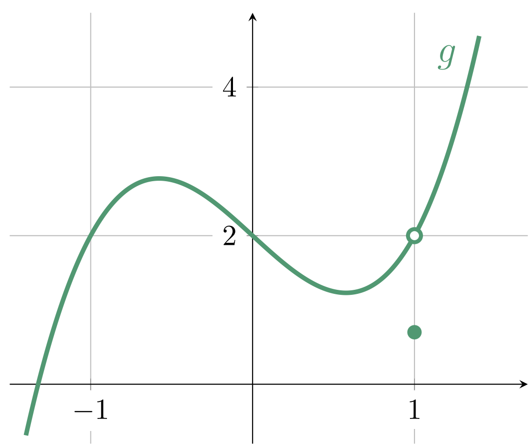
Definition. Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of x approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x\to a} f(x) = L.$$

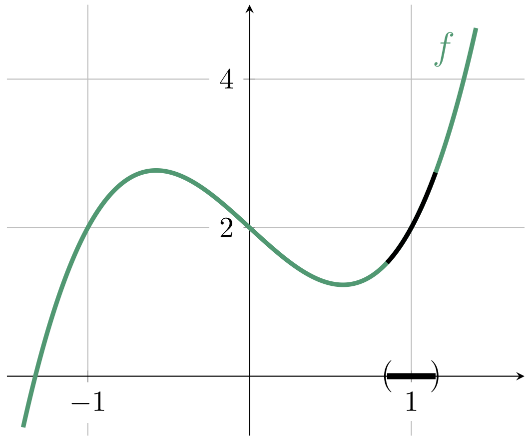
Note that this really does rely on our intuition—we're not formally defining what it means to "get closer" and "stay close," even though these are key to understanding limits. This is okay though, and in fact, the formal way we might define limits (in say, section 2.5 of OpenStax) didn't show up until almost 200 years after Newton and Leibniz began their work on the main ideas of calculus.

The kind of intuition we should have about things getting close and staying close is a visual intuition, so for all of our first examples, we'll consider graphs of some functions without actually saying what equations define the functions. We should be able to use our intuitive definition right away on the two functions pictured below:





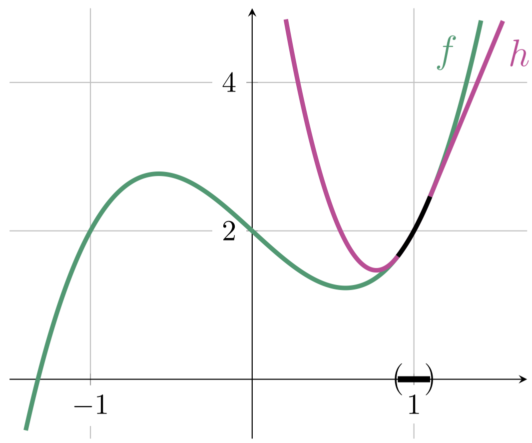
Let's see what we can say about $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$. Per our intuitive definition, the question we need to be asking is the following: when x "gets close" to 1, is there some value L such that $f(x)$ "gets close" and "stays close" to L? The picture suggests that when x is close to 1, $f(x)$ looks pretty close to 2, so we should have in mind that $L = 2$ is a good candidate for the limit. Let's check both parts of the definition. If we consider the points close to $x = 1$ (say, points in a small open interval around 1) on the x axis and then look at their corresponding outputs, we get the following picture:



The outputs $f(x)$ *get close* to $L = 2$ because no matter how close to 2 we get, say, 2.1, there's some input x' in this little interval around $x = 1$ with an output even closer to 2 than 2.1 is. The outputs $f(x)$ also *stay close* to $L = 2$ since the outputs we get from this small interval around $x = 1$ are all contained in their own nice little interval around $L = 2$—they don't suddenly wander off away from $L = 2$ with some outputs close to 2 and some farther away. If this is true for all the intervals we could draw around $L = 2$ and $x = 1$, then we know that $f(x)$ really does get close and stay close to $L = 2$ when x gets close to 1, and looking at the picture it should be easy to convince ourselves that this really is the case: there's nothing special about the little interval drawn here—it could get much smaller and the same "closeness" considerations would apply. Therefore we can say with confidence that $\lim_{x\to 1} f(x) = 2$.

The discussion here with these small intervals around the input $x = 1$ and the limit $L = 2$ is already pushing us in the direction of a more formal definition of limits, so we won't pursue this any further. Looking at the first drawing of the function f, we can already say just based on the graph that the limit as x approaches 1 should clearly be 2, and this business with small intervals can help give us a little bit more to back up our intuition about what "closeness" should mean, but we don't need it. It'll be more helpful in a moment when we see some limits that fail to exist.

Before we do that, let's look for a second at the function g. It looks nearly identical to f, except we have these little dots telling us that even though the function g looks like it should have $g(1) = 2$, the function is actually defined so that $g(1) \neq 2$. Instead, $g(1)$ looks like something maybe a bit less than 1. And in fact, when we say that g "looks like it should have $g(1) = 2$" this is appealing to exactly the fact that as x gets close to 1, $g(x)$ gets close to 2 and stays close to 2, except at the point $x = 1$ itself. And by our intuitive definition, this is all we need to say that we also have $\lim_{x\to 1} g(x) = 2$. (Recall that in our definition, the point $x = a$ is explicitly excluded from consideration when we're deciding if outputs are getting close and staying close to L.) Because we really only care about the points that are close to $x = a$ in general, this same argument means that if there's some interval around $x = a$ on which two functions agree, except maybe at a itself, then the two functions have to have the same limit at a, if the limit exists. For instance, we've already seen that f and g agree everywhere except $x = 1$, so they have the same limit at $x = 1$. Here's another function h, graphed now on top of f:



We see that they agree on an open interval around $x = 1$, and since we only care about what's happening close to $x = 1$, we can ignore the fact that they are different functions—as far as $\lim_{x\to 1}$ is concerned, f and h are exactly the same:

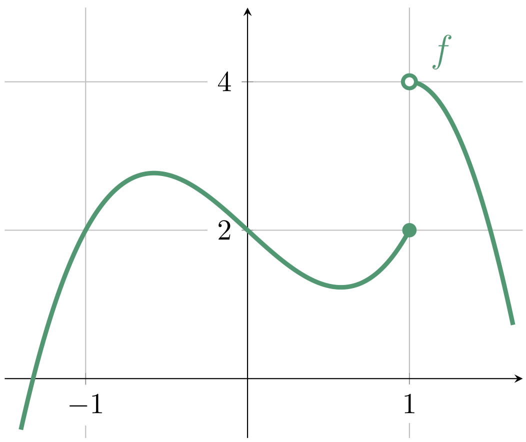
$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) = 2.$$

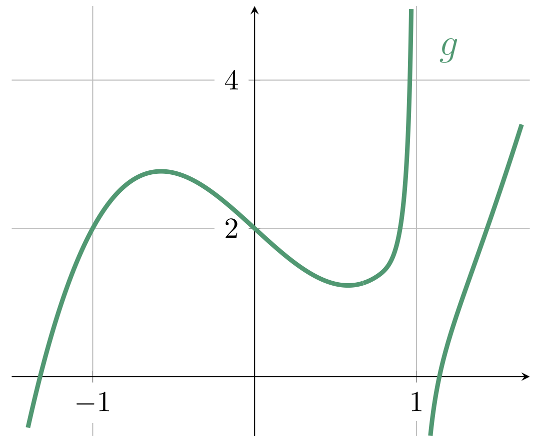
Stating this a little more formally, we get the following definition and fact about limits:

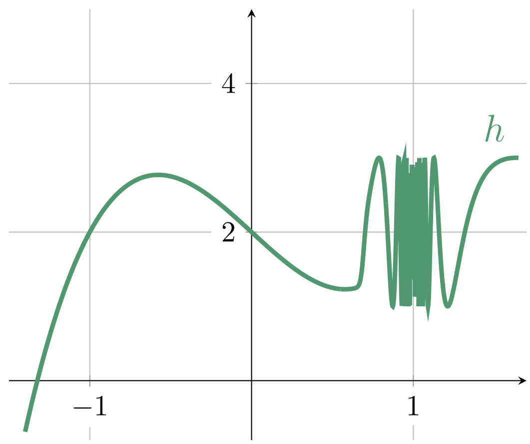
Definition. Two functions f and h are said to be **almost identical near** a if there exists some open interval around $x = a$ such that $f(x) = h(x)$ for all x in the interval, except possibly at $x = a$ itself.

Almost Identical Functions Property. If f and h are almost identical near a, then whenever the limit exists, we have that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$.

This property will be key to several algebraic tricks we'll discuss later on, but before we do this, let's look at some of the ways limits can fail to exist. The pictures below show three different ways this can happen:







The function f is very close to having a limit at $x = 1$. If we approach 1 from the left, as x gets closer to 1, $f(x)$ looks to be getting closer and staying close to $L = 2$. However, if we approach from the right, x getting closer to 1 means $f(x)$ getting closer and staying close to $L = 4$. Our definition requires that there is a single number L that *all* the values $f(x)$ approach, whereas we have two numbers that are approached depending on the direction of approach. We must therefore conclude that the limit $\lim_{x\to 1} f(x)$ does not exist. However, we can still say that the **one-sided limits** exist, and we have the notation

$$\lim_{x\to 1^-} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = 4,$$

where the minus sign superscript next to the 1 means approaching from the left and the plus sign superscript means approaching from the right. The (two-sided) limit does not exist because even though the one-sided limits exist, they are not the same.

The function g exhibits similar behavior in that the one-sided limits do not agree, but the situation here is even worse, because the one-sided limits aren't even real numbers. Approaching from the left, $g(x)$ looks to be getting arbitrarily large and positive, and so we would say that $\lim_{x\to 1^-} g(x) = \infty$, and similarly from the right we have $\lim_{x\to 1^+} g(x) = -\infty$ since the values $g(x)$

are getting arbitrarily large in the negative direction. We'll discuss this sort of example more later, but one thing to note here is that because ∞ and $-\infty$ aren't real numbers, it wouldn't be incorrect to say that these one-sided limits don't exist. This is true, but it doesn't tell the full story—we can be more specific about how the limits fail to exist, and so any time we can say what sort of infinite limit we're dealing with, we should do that. Since g is approaching both infinities from different directions, the most specific answer for what the two-sided limit is should be $\lim_{x\to 1} g(x) = \pm\infty$, rather than simply saying it doesn't exist.

For h, the best possible answer really is that the limit just doesn't exist. This is a case where we see why it matters that part of our intuitive definition requires the function to *stay* close to a particular value. As x approaches 1, the outputs $h(x)$ begin to oscillate wildly, repeatedly getting very close to all sorts of different values between about 1 and 3, but it doesn't stay close to any of them. For instance, it's true that as x goes to 1, $h(x)$ gets close to $L = 2$ infinitely often, but because it never stays close, we cannot say that the limit is 2. Instead, $\lim_{x\to 1} h(x)$ does not exist.

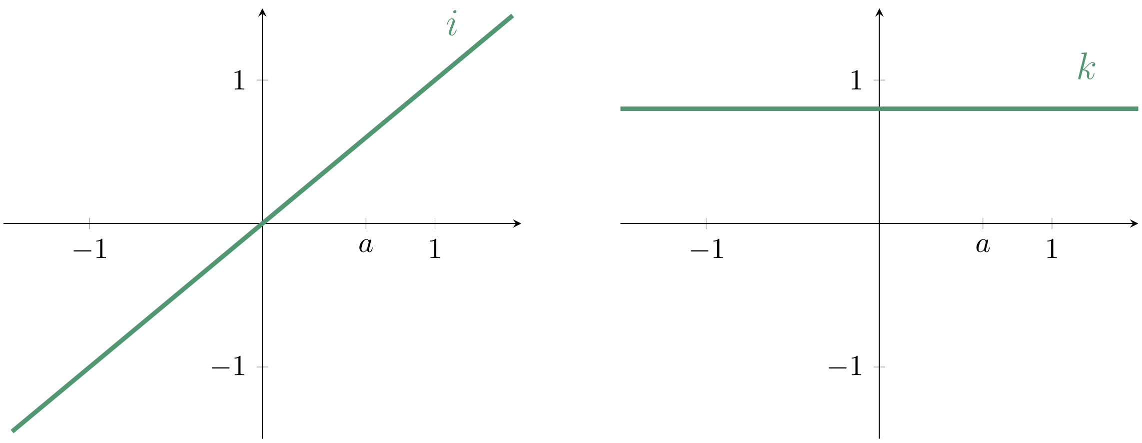
1.3 Limits laws and continuity

So far the only tool we have for computing limits beside looking at the graph of a function is the Almost Identical Functions (AIF) property. If we have a graph to look at, this is usually fine—our definition of a limit is essentially based on visual intuition about "closeness"—but in this course, the more common situation will be that we are given a rule or equation defining a function and asked to find some limit directly from the equation. We therefore introduce the following *limit laws*, rules for breaking a function down into smaller pieces whose limits we know. For instance, given a complicated limit like

$$\lim_{x\to\pi} \frac{x^2(x-\pi)+7x}{\sqrt{7x/\pi}},$$

we'd like some way of splitting this into more manageable pieces.

We begin by considering some of the smallest, most manageable pieces: the function $i(x) = x$ and the function $k(x) = c$ where c is some real number. The function i is called the identity function, and the function k is called a constant function. These functions are shown below, and have easy limits to understand:



Looking at $\lim_{x\to a} i(x)$, we see that as x gets close to a, the outputs $i(x)$ are by definition just the values x, so these are also getting close and staying close to a. Therefore the limit is $L = a$. Looking at $\lim_{x\to a} k(x)$, no matter where x is close to, the outputs $k(x)$ are always exactly the number c, whatever it is, so in particular as x gets close to a, $k(x)$ gets close and stays close to $L = c$. This gives us the first two limit laws:

$$\lim_{x\to a} x = a \qquad \text{and} \qquad \lim_{x\to a} c = c.$$

For the rest of the limit laws, let f and g be functions defined near the point $x = a$ (we can think there's some open interval around a on which the functions are defined) such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. We'll just state the rest of the laws without proof or much analysis—they should mostly seem quite natural. That said, you should think a little bit about *why* these are things we expect to be true (e.g. for the first one here, the claim is that if we add $f(x)+g(x)$, we're adding something close to L to something close to M, so we should get something close to $L+M$). Here's the full list:

Proposition. *For functions f and g defined near a with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ for real numbers L and M, we have the following:*

- ***Sum law:*** $\lim_{x\to a} f(x) + g(x) = L + M$
- ***Difference law:*** $\lim_{x\to a} f(x) - g(x) = L - M$
- ***Constant multiple law:*** *for a real number c,* $\lim_{x\to a} cf(x) = cL$
- ***Product law:*** $\lim_{x\to a} f(x)g(x) = LM$
- ***Quotient law:*** *if $M \neq 0$, then* $\lim_{x\to a} f(x)/g(x) = L/M$
- ***Power law:*** *for all positive integers n,* $\lim_{x\to a}(f(x))^n = L^n$

- ***Root law:*** *for odd positive integers n, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, and also for even n when $L \geq 0$*

- ***Comparison law:*** *if $f(x) \leq g(x)$ for all x near a (except possibly at a itself), then $L \leq M$.*

Let's now use these laws to compute the more complicated limit from earlier,

$$L = \lim_{x\to\pi} \frac{x^2(x-\pi)+7x}{\sqrt{7x/\pi}}.$$

First, the quotient law tells us this is the same as

$$L = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\lim_{x\to\pi}\sqrt{7x/\pi}}.$$

To use this, we're assuming that the denominator has a nonzero limit, and this is justified by the comparison law: we know that for x near π, $7x/\pi$ is near 7, which is greater than 1, so the comparison law says the limit must be greater than or equal to 1. Then $\lim_{x\to\pi} 7x/\pi \geq 1 > 0$. Since this limit is positive, the root law says that the limit of the root is the root of the limit, and so the limit of the denominator is therefore also positive, and so cannot be equal to zero, justifying our use of the quotient law. Writing out what we've just said, we now have that

$$L = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\sqrt{\lim_{x\to\pi} 7x/\pi}}.$$

Continuing to work on the denominator, the limit inside the square root is the limit of the function $7x/\pi = (7/\pi)x$, a constant multiple of the identity function. Using the constant multiple law and our rule about the limit of the identity function, we get that

$$L = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\sqrt{\lim_{x\to\pi}(7/\pi)x}} = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\sqrt{(7/\pi)\lim_{x\to\pi} x}} = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\sqrt{(7/\pi)\cdot\pi}}.$$

Inside the square root, we are both dividing and multiplying by π, so this cancels out and we get

$$L = \frac{\lim_{x\to\pi}(x^2(x-\pi)+7x)}{\sqrt{7}}.$$

Let's look now at the numerator. Since the numerator is the limit of a sum of two functions, the sum law tells us this is the same as the sum of the limits. Then we can use the product rule on the first term and the scalar multiple law on the second term to get

$$L = \frac{\lim_{x\to\pi} x^2(x-\pi) + \lim_{x\to\pi} 7x}{\sqrt{7}} = \frac{\lim_{x\to\pi} x^2 \cdot \lim_{x\to\pi}(x-\pi) + 7\lim_{x\to\pi} x}{\sqrt{7}}.$$

Using the power law on the first term and the difference law on the second gives us

$$\frac{(\lim_{x\to\pi} x)^2\,(\lim_{x\to\pi} x - \lim_{x\to\pi} \pi) + 7\lim_{x\to\pi} x}{\sqrt{7}}.$$

At this point, the only limits left to calculate are the two easy ones that became our very first limit laws: the identity law and the constant law. Using these, we get

$$L = \frac{(\pi)^2(\pi - \pi) + 7\pi}{\sqrt{7}} = \frac{\pi^2 \cdot 0 + 7\pi}{\sqrt{7}} = \frac{7\pi}{\sqrt{7}}.$$

Recalling some exponent rules, this is

$$L = \frac{7}{\sqrt{7}} \cdot \pi = \frac{7^1}{7^{\frac{1}{2}}} \cdot \pi = 7^{1-\frac{1}{2}} \cdot \pi = 7^{\frac{1}{2}} \cdot \pi = \sqrt{7}\pi.$$

As is readily apparent, actually writing out and explaining all the various limit laws we use is sort of a pain. **You will not have to do this unless explicitly prompted.** In fact, for this particular limit, there's an easier property to appeal to. You may have noticed that $\sqrt{7}\pi$ is exactly what we get if we plug in the number we approach (π) into our original function:

$$\frac{x^2(x-\pi) + 7x}{\sqrt{7x/\pi}} \text{ at } x = \pi : \qquad \frac{(\pi)^2((\pi) - \pi) + 7(\pi)}{\sqrt{7(\pi)/\pi}} = \frac{7\pi}{\sqrt{7}} = \sqrt{7}\pi$$

This reflects the fact that this function is *continuous* at $x = \pi$. Informally, we can think of a function f being continuous if we can draw its graph without picking up our pencil. What this means is that every value of the function is equal to the value you'd expect based on the values of the points nearby—you don't need to pick up the pencil at the value $f(a)$ for some a if $f(a)$ is already really close to all the values $f(x)$ you've just drawn. To make this formal, we just need to complete this translation into the language of limits:

Definition. A function f is said to be **continuous at the point** a if all of the following hold:

(i) f is defined at a, that is, a is in the domain of f;
(ii) $\lim_{x\to a} f(x)$ exists;
(iii) $\lim_{x\to a} f(x) = f(a)$.

Otherwise, we say f is **discontinuous at** a. A function f is said to be **continuous** if it is continuous at every point in its domain.

The calculation we just did above showed that the function

$$\frac{x^2(x-\pi) + 7x}{\sqrt{7x/\pi}}$$

is defined at π, has a limit as x approaches π, and that the limit is equal to the value the function takes at π. That is, we have just checked all the conditions for the function to be continuous at π. This was a somewhat laborious process, and as we've already seen, it would have been a lot easier to calculate the limit by just plugging in π to the function. Had we known in advance that this function was continuous at π, we could have been completely justified in doing exactly this. In general, if we have some function f and we happen to know in advance that f is continuous at some point a, then by definition we know that the limit as x approaches a exists and that it's equal to $f(a)$.

In order to make the most use out of this, we'd like to know which kinds of functions are continuous. When we have a function we know is continuous, we can compute its limits very easily. Fortunately, a lot of the functions we're used to working with are, in fact, continuous!

Fact. Any function built out of algebra and trig is continuous (i.e. continuous at any point where it's defined).

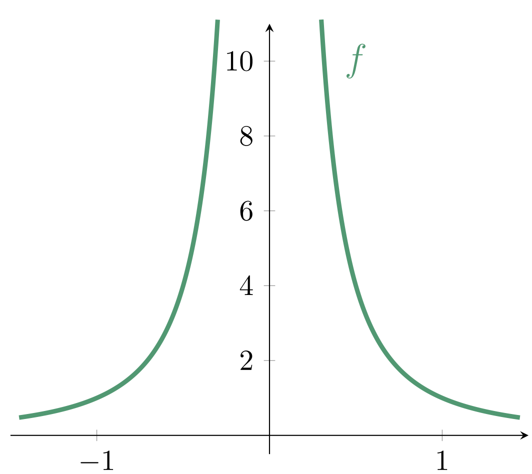
Here are some examples of such functions:

- All trigonometric functions
- All polynomial and rational functions
- Root and power functions
- Any function that is the sum, difference, product, quotient, or composition of these kinds of functions—this is what is meant by "built out of algebra and trig"

More concretely, the functions

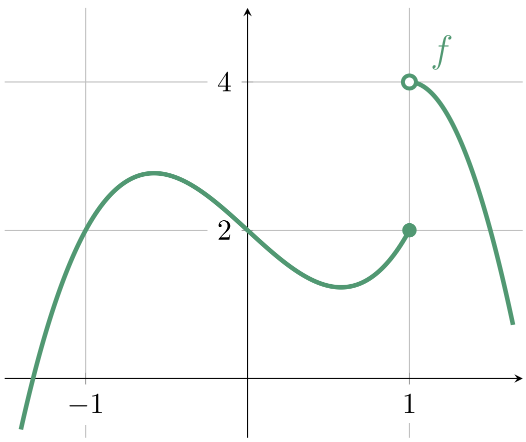
$$\sqrt{\sin(x)}, \qquad \frac{x^2-1}{x^2+1}, \qquad \text{and} \qquad \frac{1}{x^2}$$

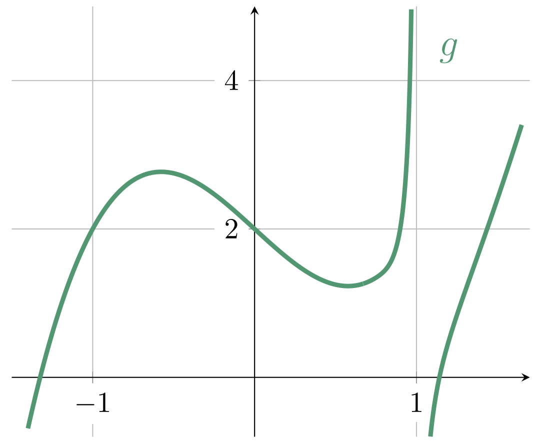
are all continuous where they are defined. Looking at a graph of $f(x) = 1/x^2$, we have the following:

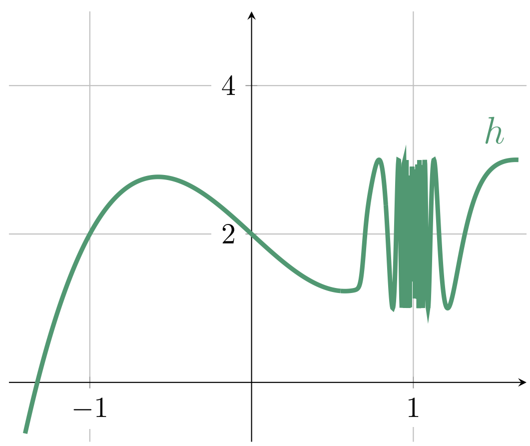


We can clearly see there's a discontinuity here, even though we just said the function is continuous. This is where it's important to recall that being continuous only means being continuous at every point in the domain. The problem with this function then is that the point $x = 0$ is not in the domain (why?), and this allows the function to have a discontinuity there even though it still qualifies as being "built out of algebra and trig." Hence it's accurate to say both that this function is continuous (on its domain) and that it has a discontinuity at $x = 0$.

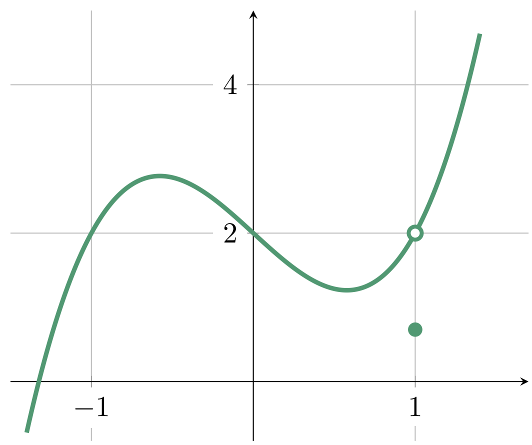
Since the problem here is that the function suddenly shoots off to infinity near $x = 0$, this kind of discontinuity is called an **infinite discontinuity**. The three functions







we looked at earlier also exhibit discontinuities, this time at $x = 1$. The function f has what is called a **jump discontinuity** (which happens when both one-sided limits exist but aren't equal), and g again has an infinite discontinuity. The function h fails to have one-sided limits existing since again it's not getting close and staying close to any one real number, and we call this behavior an **essential discontinuity**. (Since ∞ and $-\infty$ also aren't real numbers, all infinite discontinuities are also technically essential discontinuities—the one-sided limits aren't real numbers.) One more kind of discontinuity we've seen is called a **removable discontinuity**, which is when the limit does exist but isn't equal to the value of the function. Here's that example again:



We said the limit of this function as $x \to 1$ was 2, but we see that this limit isn't the same as the value the function actually takes at $x = 1$, so it's not continuous there.

Functions with non-infinite essential discontinuities can look especially crazy. Here are a couple examples you might want to think about to convince yourself that not all functions look so nice:

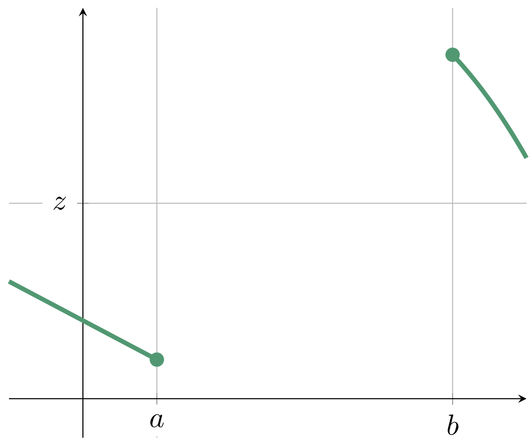
$$d(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases} \qquad t(x) = \begin{cases} 1/q & x = p/q \text{ is rational in lowest terms} \\ 0 & x \text{ is irrational} \end{cases}$$

Again, these really are both functions! They have a clearly defined rule for how to turn input into a single output. For instance, $d(1/2) = 1$ because $1/2$ is a rational number, and $t(4/6) = t(2/3) = 1/3$, while $d(\pi) = t(\pi) = 0$ because π is irrational. It turns out that d is discontinuous *at every single point*, and t is discontinuous at every rational number, but continuous at every irrational number. We won't worry too much about functions like this for this course, but they're still out there, and the truth is that our intuitive definition of limits doesn't help us much when it comes to determining where these functions are continuous and discontinuous—there is a good reason to want a very formal version of our intuitive definition, and if any of this has piqued your interest, you can find a definition in OpenStax 2.5 that actually can help us see why these functions are discontinuous where they are (and then by all means come talk to me about it in office hours)! But to reiterate, there's no requirement that you look at the formal definition or spend any time with these kinds of functions.

Now that we've introduced continuity, we have a limit law that generalizes the power and root laws. The power and root laws both involve the composition of two functions, and tell us that the limit of a power or root is the power or root of the limit. What this is using really is that powers and roots are continuous functions, and so it turns out that the same holds whenever the outside function of a composition is continuous.

Composition Limit Law: If f is defined near a with limit $\lim_{x\to a} f(x) = L$ and g is continuous at L, then $\lim_{x\to a} g(f(x)) = g(L)$.

Another nice property of continuous functions is that the Intermediate Value Theorem (IVT) holds. This says that if we have a continuous function f and some values $f(a) \neq f(b)$, f has to take on every value between $f(a)$ and $f(b)$ at some point. If we have the following setup



with $f(a) < z < f(b)$, there's no way for us to fill in the missing part of the graph without crossing this line $y = z$ if I require the function I draw to be continuous. Try it! What you'll find is that you have to cross the line, and this means there's at least one point, call it c, between a and b such that $f(c)$ is exactly z. Formally, we have the following:

Intermediate Value Theorem: Let f be continuous on the closed interval $[a, b]$. If z is any real number between the numbers $f(a)$ and $f(b)$, then there exists some point c with $a \leq c \leq b$ such that $f(c) = z$.

Previously we've probably learned a lot of ways to solve certain polynomial equations algebraically, for instance with the quadratic formula. But some polynomials don't have roots that can be written down in a nice algebraic way (e.g. it's a theorem that there's no "quintic formula" for finding the zeros of a degree five polynomial using just the usual algebraic operations and roots). For example, $f(x) = x^5 + x^3 - 1$ is unsolvable (in this algebraic sense), but we can use the IVT to show it's got at least one real root. We have

$$f(0) = (0)^5 + (0)^3 - 1 = -1$$
$$f(1) = (1)^5 + (1)^3 - 1 = 1,$$

so $f(0) < 0 < f(1)$ implies that f has at least one real root on the interval $[0, 1]$ since f is a polynomial and is therefore continuous.

1.4 Beyond the limit laws–algebraic tricks, trig limits

Now that we have the continuity of algebraic and trigonometric functions established, we're ready to address the main sorts of limits we'll be encountering in the course. We can look at a graph of a function and determine a limit that way, but if the function is presented as an equation, we've only discussed how to compute a limit at a point where the function is continuous. We'd like to have a way of computing more interesting limits, and this is where the Almost Identical Function property helps us tremendously. Consider the following: what is

$$\lim_{x\to 1} \frac{x^2-1}{x-1}?$$

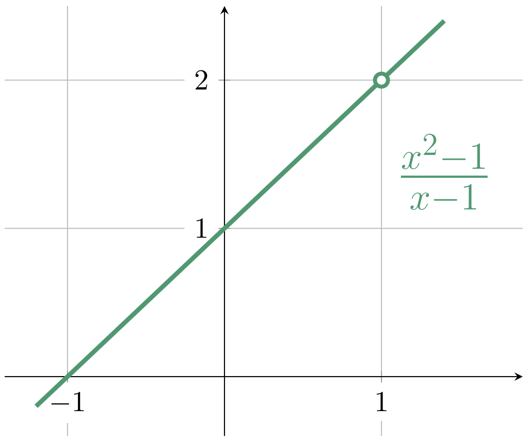
We're taking the limit as x approaches 1, so the first thing we should do is ask if this function is continuous at 1. If it is, we can just plug 1 in and the limit is whatever this comes out to be. However, this is a function that fits into the category "built out of algebra and trig," so we can determine continuity at $x = 1$ by checking if 1 is in the domain of the function. We see that it unfortunately isn't, since the expression is undefined due to division by zero. Hence this isn't continuous at $x = 1$, and so we have to work harder to compute the limit.

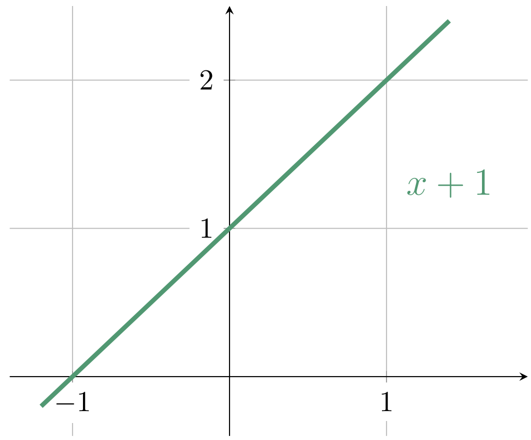
What to try? One thing to note is that in plugging in $x = 1$ to check if the function is defined, not only is the denominator 0, so is the numerator. We have a $0/0$ situation, which is an example of an **indeterminate form**. That is to say, we can't really figure anything out about the limit just knowing what the function looks like right at $x = 1$—a $0/0$ could be anything at all. This is in contrast to something like a function we looked at earlier, $1/x^2$ as $x \to 0$. Here we say we had an infinite limit, and this is because when we try and plug in $x = 0$, only the denominator is zero, we get a $1/0$, which is not an indeterminate form—we always get some kind of infinite limit when we have something nonzero divided by zero (more on this in the next section). When we have a genuine indeterminate form like $0/0$, the trick is usually to do some kind of algebraic manipulation to find a function that is almost identical near where we're taking the limit. In this case, we're looking for a function that agrees with $(x^2-1)/(x-1)$ near 1, with the exception of 1 itself. Because continuous functions are the only functions we can really take limits of directly, our goal will be to find an almost identical function that is continuous.

Something that we should always give a try whenever we have a rational function like this is to factor the numerator and denominator and look for anything that will cancel out. If we do this, we get that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}.$$

If $x \neq 1$, then $x - 1$ is nonzero and we can divide both numerator and denominator to cancel the $(x - 1)$ factor, leaving us with the new function $x + 1$. This really is a different function! The function we started with wasn't defined at $x = 1$, but this new function certainly is. Compare their graphs below:





Even though they are different functions, the Almost Identical Functions property guarantees they have the same limit at $x = 1$, so the full solution to the limit problem would look like the following:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \overset{\text{AIF}}{=} \lim_{x\to 1} x+1 = 1+1 = 2,$$

where we put AIF above the equals sign to remind us that this equality holds because we're using the AIF property. Once we've done this, we're left with the continuous function $x + 1$ and so we can evaluate the limit just by evaluating the function at 1.

Here are a few more examples of this type:

$$\lim_{x\to 3} \frac{x^2-4x+3}{x-3} = \lim_{x\to 3} \frac{(x-1)(x-3)}{x-3} \overset{\text{AIF}}{=} \lim_{x\to 3} x-1 = 3-1 = 2.$$
$$\lim_{x\to -2} \frac{x^2+5x+6}{x^2-4x-12} = \lim_{x\to -2} \frac{(x+2)(x+3)}{(x+2)(x-6)} \overset{\text{AIF}}{=} \lim_{x\to -2} \frac{x+3}{x-6} = \frac{-2+3}{-2-6} = -\frac{1}{8}.$$
$$\lim_{x\to 2} \frac{x^3-64}{x-4} = \lim_{x\to 2} \frac{(x-4)(x^2+4x+16)}{x-4} \overset{\text{AIF}}{=} \lim_{x\to 4} x^2+4x+16 = (4)^2+4(4)+16 = 48.$$

Sometimes when we encounter a $0/0$ situation in computing a limit, there is no way to factor and cancel to find an almost identical function, and we have to employ some clever tricks. For instance, if we were to try and plug in a zero to the numerator and denominator of the limit below, we'll get a $0/0$ with no way to factor:

$$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}.$$

The tool we need here is to take advantage of the way a difference of squares factors: $a^2 - b^2 = (a-b)(a+b)$. Our numerator has the form $a - b$, so we multiply by what's called the *conjugate*,

$a+b$, on both numerator and denominator:

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}.$$

The point is that the numerator now has the form $(a-b)(a+b)$, which we know is the same as a^2-b^2 (double check this by FOILing it out). Our limit is then

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}=\lim_{x\to 0}\frac{(\sqrt{x+9})^2-(3)^2}{x(\sqrt{x+9}+3)}.$$

Simplifying the numerator, we have $(x+9)-(9)=x$. Hence

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\frac{x}{x(\sqrt{x+9}+3)}\overset{\text{AIF}}{=}\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}$$

where we notice for $x\neq 0$ that we can cancel the x in the numerator and denominator, producing a function that is almost identical to the original. This new function $1/(\sqrt{x+9}+3)$ is built out of algebra and trig, so it's continuous where it's defined. Since it's defined at $x=0$ (having cancelled the x, we're no longer dividing by zero), it's continuous there, and so we can now finish off the computation by plugging zero into the new function. Putting all of this work together, our work looks like this:

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}&=\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}=\lim_{x\to 0}\frac{(\sqrt{x+9})^2-(3)^2}{x(\sqrt{x+9}+3)}\\&=\lim_{x\to 0}\frac{x}{x(\sqrt{x+9}+3)}\overset{\text{AIF}}{=}\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}=\frac{1}{\sqrt{0+9}+3}=\frac{1}{3+3}=\frac{1}{6}.\end{aligned}$$

Even though we're multiplying *and dividing* by the conjugate to avoid completely changing the function, this trick is usually just referred to as "multiplying by the conjugate." It comes up frequently, so here are a couple more examples of the trick in action:

$$\begin{aligned}\lim_{x\to 6}\frac{2-\sqrt{x-2}}{x-6}&=\lim_{x\to 6}\frac{2-\sqrt{x-2}}{x-6}\cdot\frac{2+\sqrt{x-2}}{2+\sqrt{x-2}}=\lim_{x\to 6}\frac{2^2-(x-2)}{(x-6)(2+\sqrt{x-2})}\\&=\lim_{x\to 6}\frac{6-x}{(x-6)(2+\sqrt{x-2}}\overset{\text{AIF}}{=}\lim_{x\to 6}\frac{-1}{2+\sqrt{x-2}}=\frac{-1}{\sqrt{6-2}+2}=-\frac{1}{4}.\\\lim_{x\to 14}\frac{x-14}{\sqrt{x-5}-3}&=\lim_{x\to 14}\frac{x-14}{\sqrt{x-5}-3}\cdot\frac{\sqrt{x-5}+3}{\sqrt{x-5}+3}=\lim_{x\to 14}\frac{(x-14)(\sqrt{x-5}+3)}{(x-5)-3^2}\\&=\lim_{x\to 14}\frac{(x-14)(\sqrt{x-5}+3)}{x-14}\overset{\text{AIF}}{=}\lim_{x\to 14}\sqrt{x-5}+3=\sqrt{14-5}+3=6.\end{aligned}$$

Note that in the first example here, we've used that $6-x=-(x-6)$ and then cancelled to get an almost identical function.

Here's another limit that results in a 0/0 indeterminate form:

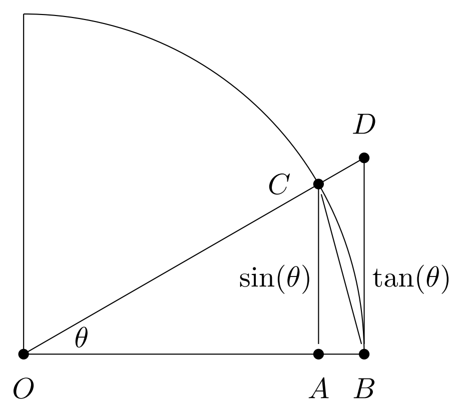
$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

This is a tough limit to evaluate since there's no factoring we could do and no conjugates we could multiply by. Instead, we'll need to use some geometry. The following proposition gives what we'll refer to as the **small angle approximation**. You won't need to recreate the proof, but you will need to know how to use the result.

Proposition. *When θ is very small, $\sin(\theta)$ is approximately θ. Formally,*

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Proof. Consider the following portion of the unit circle:



Recall that since we're working with radians, θ is the measure of the angle, but also the length of the curved arc between points B and C. The full circle has length (i.e. circumference) 2π, so the wedge BOC must have area $\theta/2\pi$ of the area of the full circle. The unit circle has radius 1, hence area π, so the wedge BOC has area

$$\frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}.$$

This wedge completely contains the triangle BOC, so the area of the wedge must be greater than the area of the triangle. BOC has the line segment BO as its base, and the line segment AC as its height. BO has length 1 because it is a radius of the unit circle, and AC is the side opposite to the angle θ and therefore has length $\sin(\theta)$. Hence

$$\frac{\theta}{2} = \text{ area of wedge } \geq \text{ area of triangle } = \frac{1}{2} \cdot \text{ base } \cdot \text{height} = \frac{\sin(\theta)}{2}.$$

Multiplying through by 2, we get that $\theta \geq \sin(\theta)$, meaning $\frac{\sin(\theta)}{\theta} \leq 1$. By the limit comparison law, we have that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \leq \lim_{\theta \to 0} 1 = 1.$$

Of course, our goal is to show that the limit is *equal* to 1, not just less than or equal to 1, so our next step is to show that the limit is also greater than or equal to 1. This will rule out the possibility that the limit is strictly less than 1, allowing us to conclude the limit is exactly 1. To do this, we'll need to use the fact that the segment BD has length $\tan(\theta)$. This is because the triangle BOD is similar to triangle BOC (both triangles share the same angle θ at the tip and then have right angles next to it), so the ratio of their side lengths are the same. In particular, this means

$$DB = \frac{DB}{1} = \frac{DB}{OB} = \frac{CA}{AC} = \frac{\text{opp.}}{\text{adj.}} = \tan(\theta).$$

Now to get our inequality, we just need to notice that the wedge BOC, which we said has area $\theta/2$, is completely contained within triangle BOD, meaning

$$\frac{\theta}{2} \leq \text{area of } BOD = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{\tan(\theta)}{2}.$$

Again multiplying by 2 we get that $\theta \leq \tan(\theta)$. But $\tan(\theta)$ is the same as $\sin(\theta)/\cos(\theta)$, so we have the inequality

$$\theta \leq \frac{\sin(\theta)}{\cos(\theta)}.$$

If we then divide both sides by θ and multiply both sides by $\cos(\theta)$, we get that

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta}.$$

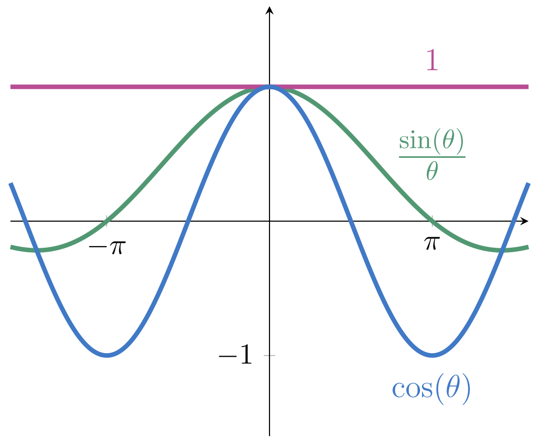
Therefore, again using the comparison law for limits,

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \geq \lim_{\theta \to 0} \cos(\theta) = \cos(0) = 1$$

where we use the fact that cosine is continuous, and so in particular continuous at zero. Since we have shown that the limit is less than or equal to 1 and *also* greater than or equal to 1, the only possibility is that the limit is exactly 1. $\square$

In the proof here, we're really only showing that $\lim_{\theta \to 0^+} \sin(\theta)/\theta = 1$ since we've only considered angles θ in the first quadrant ($\theta > 0$), but flipping our picture upside down and giving the same argument works just as well and gives the other one-sided limit.

The proof isn't something you need to memorize, but there's one aspect of it that's worth knowing about. We wanted to show a limit was equal to a certain number (1), and so to do this, we found a function ($\cos(\theta)$) always less than the function we were interested. and a function (1) always greater than the function we were interested in. Because the smaller function and the bigger function both had the same limit, we could conclude by the comparison law that the middle function ($\sin(\theta)/\theta$) also had the same limit. Here's the picture of all these different functions:



That this approach to proving a limit works is called the **Squeeze Theorem** for the way that we find two functions (in pink and blue) whose limit we know and that squeeze the function we're actually interested in (green). Here's the full statement:

Theorem. *Let f, g, and h be functions defined over some open interval containing $x = a$, except possibly at a itself. If $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ in this interval and there is some real number L such that*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x),$$

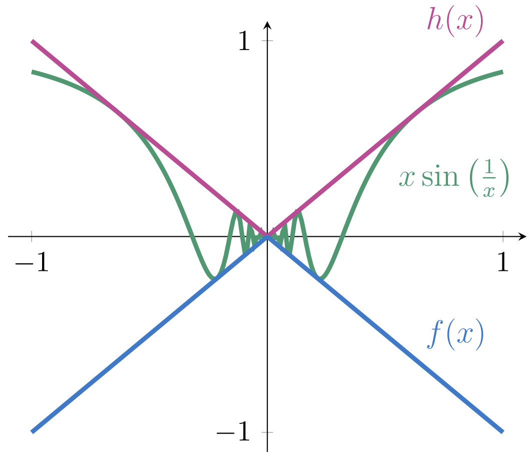
then $\lim_{x\to a} g(x) = L$ also.

As another example of the usefullness of this technique, we said earlier that the function $\sin(1/x)$ has an essential discontinuity at $x = 0$. As x gets close to 0, $\sin(1/x)$ oscillates wildly and gets close to every value between -1 and 1. If we multiply by x to get $g(x) = x\sin(1/x)$, the function still oscillates wildly, but now we can use the Squeeze Theorem to show that it does actually have a limit that exists as x approaches 0.

No matter what the input is to the sine function, the output is always between -1 and 1. Therefore if we multiply the inequality $-1 \leq \sin(1/x) \leq 1$ by x, we get that $-x \leq x\sin(1/x) \leq x$ when $x \geq 0$, and since multiplying by a negative reverses inequalities, also that $x \leq x\sin(1/x) \leq -x$ when $x \leq 0$. The functions we need to bound $g(x) = x\sin(1/x)$ are then the piecewise functions

$$f(x) = \begin{cases} -x & x \geq 0 \\ x & x \leq 0 \end{cases} \qquad \leq \qquad g(x) \qquad \leq \qquad h(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

Here's our picture of the situation:



Hopefully from the picture we recognize $h(x)$ as the absolute value function and $f(x)$ as its negative. They may not be smooth with the sharp point at $x = 0$, but they're still continuous there—we can find the one-sided limits from their piecewise description and see that they are all equal to 0. This means the limit at 0 of f and h is 0, and so the Squeeze Theorem then says that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

So even though both functions oscillate infinitely often as x approaches 0, only $\sin(1/x)$ has an essential discontinuity. For $x\sin(1/x)$, the limit does exist, the only problem is that the function isn't defined there—we say $x\sin(1/x)$ has a removable discontinuity at $x = 0$. The discontinuity is removable in the sense that we can define a new function that is almost identical but doesn't have a discontinuity:

$$\begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is a continuous function almost identical to the one we started with.

Not only is the Squeeze Theorem a powerful tool, so is the limit we initially used it to compute, the Small Angle Approximation (SAA):

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

In this limit, θ is just a number approaching zero, but we can take θ also to be some function $f(x)$ that approaches zero by appealing to our limit composition law. The function $\sin(\theta)/\theta$ has a removable discontinuity at 0 since the limit does exist, so if we patch that hole, we get the function

$$g(\theta) = \begin{cases} \frac{\sin(\theta)}{\theta} & \theta \neq 0 \\ 1 & \theta = 0, \end{cases}$$

which is continuous. Since this is continuous at $\theta = 0$, we can use the limit composition law to say that if f is any function with $\lim_{x\to a} f(x) = 0$, then

$$\lim_{x\to a} \frac{\sin(f(x))}{f(x)} = 1.$$

For instance, we showed above that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0, \quad \text{meaning by SAA} \quad \lim_{x\to 0} \frac{\sin(x\sin(1/x))}{x\sin(1/x)} = 1.$$

The point is that the Small Angle Approximation allows us to calculate more complicated limits involving trig functions. Frequently the way we'll do this is by first doing some algebraic manipulation to introduce something that looks like $\sin(f(x))/f(x)$ for some function f that goes to 0 as x approaches whatever point we're interested in. For example, if we want to compute a limit like

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)},$$

we should expect the answer to be $2/3$ since $\sin(2x)$ is roughly $2x$ and $\sin(3x)$ is roughly $3x$ when x is very small. To actually show this is the case and use the SAA formally, we'll need to do a little algebraic manipulation:

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0} \frac{\frac{\sin(2x)}{2x}\cdot 2x}{\frac{\sin(3x)}{3x}\cdot 3x} = \frac{\lim_{x\to 0}\frac{\sin(2x)}{2x}}{\lim_{x\to 0}\frac{\sin(3x)}{3x}} \cdot \lim_{x\to 0} \frac{2x}{3x} = \frac{1}{1}\cdot \lim_{x\to 0} \frac{2x}{3x} \overset{\text{AIF}}{=} \lim_{x\to 0} \frac{2}{3} = \frac{2}{3}.$$

Another important trig limit we can derive from the Small Angle Approximation concerns the cosine function:

$$\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} &= \lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2(\theta)}{\theta(1+\cos(\theta))} \\
&= \lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta(1+\cos(\theta))} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot \lim_{\theta\to 0} \frac{\sin(\theta)}{1+\cos(\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{\sin(0)}{1+\cos(0)} = \frac{0}{1+1} = 0.
\end{aligned}$$

This says that $\cos(\theta)$ and 1 are extremely close to one another near zero—even closer than θ itself is to zero when we let θ approach zero.

1.5 Infinite limits and limits at infinity

We've already seen that some functions have infinite discontinuities. For example, $f(x) = 1/x$ has an infinite discontinuity at $x = 0$. This means that if we take a limit as x approaches 0, we get some kind of infinity. Previously, we were looking at a graph to see this, but we can just as well find the limits from the equation $1/x$ itself. This is the function that takes the reciprocal of a number, and the key things we need to remember are the following: a) the reciprocal preserves the sign of a number, and b) the reciprocal of something large (i.e. something greater than 1) is small (something less than 1). This helps us think about what the one-sided limits should be. If we let x approach zero from the right, all of our x values are positive. This means all of their reciprocals $1/x$ are also positive. Moreover, the values x are getting arbitrarily small since we're letting x approach zero, the smallest number (in absolute value). This means the reciprocals are getting arbitrarily large, and we just said they all still have to be positive, so what we've argued here is that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

A similar argument gives us the other one-sided limit: $x \to 0^-$ means our inputs are getting arbitrarily small, but are now negative, meaning our outputs are getting arbitrarily large and negative:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Putting these together, it would be correct to say that the limit does not exist, but the best answer here is the following:

$$\lim_{x \to 0} \frac{1}{x} = \pm\infty,$$

which says that as x approaches 0, the function $1/x$ approaches both kinds of infinities, positive and negative, depending on the direction of approach.

If we look instead at the function $1/x^2$, we have a slightly different situation. This time, whether we approach from right or left doesn't matter. If x is approaching 0, it's some number getting increasingly small, so x^2 is also getting increasingly small, and $1/x^2$ is therefore getting increasingly large. As for the sign, if x is a little bit less than 0 or a little bit greater than 0, when we square it, we always have $x^2 > 0$, so $1/x^2$ is also *always positive*. This argument tells us that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

If we were to plug in $x = 0$ to these functions directly, in both cases we would get the expression $1/0$, which we know doesn't make sense. We saw earlier that when we had an expression $0/0$, the answer to the limit could conceivably be any number, so we called this an *indeterminate form*. Limits of the form $1/0$ are **not indeterminate forms**. These are the prototypical examples of infinite limits, because whenever we try to plug in the number we're approaching to a function and get something like $1/0$ or $c/0$ for some real number $c \neq 0$, the answer will always be some kind of infinity. More formally, we have the following proposition:

Proposition. *Let f and g be functions such that $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \pm\infty.$$

If $c > 0$ and then the limit is ∞ if and only if $g(x) \geq 0$ near a and is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$, then the opposite is true.

Here are a few examples of how to use this proposition. The notation we'll be using to show our work needs some explanation. Always with limits, we want to start by plugging in the number we're approaching (so in the first example below, we want to try plugging in 3). Either the function is defined and continuous at this point (so we can stop), or we'll get something undefined, often due to division by zero. In this latter case, the proposition above says that when we are dealing with division by zero and the numerator is approaching something *nonzero*, we must have some kind of infinity. Our task then is to determine whether the answer is $\pm\infty$ (approaching both kinds of infinity), ∞ (approaching positive infinity from both directions), or $-\infty$ (approaching negative infinity from both directions). To show our work on a problem like this, we use arrows in the numerator and denominator showing what each part of the fraction approaches as x approaches whatever the limit suggests it should approach (3 in the case of the first example below), and when the denominator is approaching zero, we want to put a superscript to denote how it's approaching zero ($\pm$ for both sides, $+$ for only from the positive direction, or $-$ for only from the negative direction). In the first example below, we put a $\pm$ superscript next to the zero, because if x is a bit larger than 3, the denominator is positive, but if x is a bit less than 3, the denominator is negative. In the second example, we put a $+$ superscript because the denominator gets squared and so is always positive. For the first example, our arrows indicate we have a positive number (6) divided by a number approaching zero from both directions, so our answer is $\pm\infty$. On the left, you'll see what the problem statement would look like, and on the right, you'll see how the notation works for completely showing work solving the problem:

$$\lim_{x\to 3} \frac{2x}{x-3}: \qquad \lim_{x\to 3} \frac{2x^{\nearrow 6}}{x-3_{\searrow 0^{\pm}}} = \pm\infty$$

$$\lim_{x\to 3} \frac{2x}{(x-3)^2}: \qquad \lim_{x\to 3} \frac{2x^{\nearrow 6}}{(x-3)^2_{\searrow 0^{+}}} = \infty$$

$$\lim_{x\to -3} \frac{2x}{x+3}: \qquad \lim_{x\to 3} \frac{2x^{\nearrow -6}}{x+3_{\searrow 0^{\pm}}} = \pm\infty$$

$$\lim_{x\to -3} \frac{2x}{(x+3)^2}: \qquad \lim_{x\to 3} \frac{2x^{\nearrow -6}}{(x+3)^2_{\searrow 0^{+}}} = -\infty$$

Here's another example we'll need a little bit more detail for:

$$\lim_{x \to 2} \frac{x-1}{x^3 - 12x^2 + 36x - 32}.$$

Trying to plug in $x = 2$ to get a first idea, we see the numerator is approaching $2 - 1 = 1$, and the denominator is approaching

$$2^3 - 12(2^2) + 36(2) - 32 = 8 - 48 + 72 - 32 = 80 - 80 = 0.$$

Since the denominator approaches zero and the numerator approaches something nonzero, we know it's going to have an infinite limit of some kind. However, the way our function is written at the moment makes it difficult to reason about which direction the denominator approaches zero from. What we'd like to do is be able to easily determine what sign the denominator has near $x = 2$, and for this, our best tool is to factor. Usually factoring a cubic like this would be difficult, but notice we already know one of the roots—we just calculated that the cubic is zero when $x = 2$. This means $(x - 2)$ is a factor, and we know when we divide by it, we should be left with some quadratic equation. To find this quadratic, we could use any polynomial division method. Here's synthetic division:

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

This tells us $x^3 - 12x^2 + 36x - 32 = (x-2)(x^2 - 10x + 16)$, and to completely factor this thing, we can now use whatever approach we'd like on the quadratic term to see that $x^2 - 10x + 16 = (x-2)(x-8)$. We're therefore trying to compute the limit

$$\lim_{x \to 2} \frac{x-1}{x^3 - 12x^2 + 36x - 32} = \lim_{x \to 2} \frac{x-1}{(x-2)^2(x-8)}.$$

If x is approaching 2, we know that the numerator is approaching 1, and now we can see how the denominator works more clearly: the $(x-2)^2$ term is the part that makes the whole thing approach zero, and because we're squaring, this term is always positive. The $(x - 8)$ term, however, is approaching -6 when x approaches 2, so when x is near 2, the denominator as a whole is negative. Now we can fill in our arrows to complete this solution:

$$\lim_{x \to 2} \frac{x-1}{x^3 - 12x^2 + 36x - 32} = \lim_{x \to 2} \frac{x-1 \nearrow^{1}}{(x-2)^2(x-8) \searrow_{0^-}} = -\infty.$$

One last thing to note about this proposition is how it compares to our previous limit laws. Earlier, we had a limit law for quotients, but using this required the denominator to have a nonzero limit. Now we have a tool that covers this case, provided the numerator has nonzero limit. When both numerator and denominator have limit equal to zero, we have to look for some almost identical function and hope its limit is easier to handle, as in the previous section.

Recall also that all the limit laws we gave previously required the functions f and g we were working with to have *real number* limits L and M. When f and g have infinite limits, the limit laws can break down. For a simple example, consider the function $1/x - 1/x$. Everywhere this is defined (i.e. for $x \neq 0$, we are subtracting $1/x$ from itself, and so we get 0. Hence

$$\lim_{x\to 0} 1/x - 1/x = 0.$$

The limit law we had for the difference of two functions, if applied here, would suggest the limit should be $\pm\infty \pm \infty$, which is another example of an indeterminate form—any sum or difference of infinities is indeterminate, and so we often need to rewrite the functions involved to avoid this situation. For instance, another difference of infinities is

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2}.$$

We said $\lim_{x\to 0} 1/x = \pm\infty$ and that $\lim_{x\to 0} 1/x^2 = \infty$, but the answer to this question is not $\pm\infty - \infty$. Instead, we need to rewrite the equation by finding a common denominator to see what's really going on:

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2} = \lim_{x\to 0} \frac{x}{x^2} - \frac{1}{x^2} \lim_{x\to 0} \frac{x - 1^{\nearrow -1}}{x^2_{\searrow 0^+}} = -\infty.$$

Here are a couple more examples:

$$\begin{aligned}
&\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0^\pm}} = \pm\infty \quad \text{and} \quad \lim_{x\to 0} \frac{2^{\nearrow 2}}{x(x-2)_{\searrow 0^\pm}} = \pm\infty, \\
&\qquad \text{but} \quad \lim_{x\to 0} \frac{1}{x} + \frac{3}{x(x-3)} = \lim_{x\to 0} \frac{x-3}{x(x-3)} + \frac{3}{x(x-3)} = \lim_{x\to 0} \frac{x}{x(x-3)} \overset{\text{AIF}}{=} \lim_{x\to 0} \frac{1}{x-3} = -\frac{1}{3} \\
&\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0^\pm}} = \pm\infty \quad \text{and} \quad \lim_{x\to 0} \frac{x^2 - 3x + 2^{\nearrow 2}}{x^2 - 2x_{\searrow 0^\pm}} = \pm\infty, \\
&\qquad \text{but} \quad \lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x-2}{x^2-2x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x^2-2x}{x^2-2x} \overset{\text{AIF}}{=} \lim_{x\to 0} 1 = 1.
\end{aligned}$$

We've seen that a limit of the form $\pm\infty \pm \infty$ can equal 0, $-\infty$, $-1/3$, or 1, and in fact, it can equal any real number or any kind of infinity. It all depends on what the function we're taking the limit of is, which is why this is an example of an indeterminate form. The best approach usually is trying to find a common denominator and simplifying what we can.

Not only can we let x approach some number a and get outputs that get increasingly large, we can also ask about what happens when x itself gets increasingly large. You may recall from a high school math class the idea of "end behavior" of polynomial functions. There you might have learned that if a polynomial f has a positive leading coefficient, then $f(x)$ goes to infinity as x goes to infinity, and goes to ∞ or $-\infty$ as x goes to negative infinity, depending on the degree of the polynomial. This is the idea of taking a limit *at infinity*, and with the limit notation, we would denote this as $\lim_{x\to\infty} f(x) = \infty$.

It can also happen that $f(x)$ approaches some real number when x approaches infinity, an example of this being the function $1/x$. We have the following two limits, both of which we can see by looking at a graph of the function:

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0.$$

This says that the function $1/x$ gets arbitrarily small as x gets arbitrarily large in either direction. Another way you might have talked about this fact in a high school math class is that this function has a *horizontal asymptote* that it approaches.

The problems we'll look at in this class concerning limits at infinity will all boil down to this limit, or something that looks a lot like it:

Fact. For any positive integer n, we have

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0.$$

With this fact, we can use our original limit laws to take limits like the following, where the tool we have is to divide everything by the highest power of x appearing and employ the fact above:

$$\lim_{x\to-\infty} \frac{x}{x^2+1} \overset{\text{AIF}}{=} \lim_{x\to-\infty} \frac{x/x^2}{x^2/x^2+1/x^2} = \lim_{x\to-\infty} \frac{1/x}{1+1/x^2} = \frac{\lim_{x\to-\infty} 1/x}{1+\lim_{x\to-\infty} 1/x^2} = \frac{0}{1+0} = 0$$

$$\lim_{x\to\infty} \frac{2x^3-2x+1}{3x^3+x^2} \overset{\text{AIF}}{=} \lim_{x\to\infty} \frac{2-2/x^2+1/x^3}{3+1/x} = \frac{2-2\lim_{x\to\infty} 1/x^2+\lim_{x\to\infty} 1/x^3}{3+\lim_{x\to\infty} 1/x} = \frac{2-2\cdot 0+0}{3+0} = \frac{2}{3}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2-2}}{x-2} &\overset{\text{AIF}}{=} \lim_{x\to\infty} \frac{\sqrt{x^2-2}/x}{1-2/x} \overset{\text{AIF}}{=} \lim_{x\to\infty} \frac{\sqrt{x^2-2}/\sqrt{x^2}}{1-2/x} = \lim_{x\to\infty} \frac{\sqrt{1-2/x^2}}{1-2/x} \\ &= \frac{\sqrt{1-2\lim_{x\to\infty} 1/x^2}}{1-2\lim_{x\to\infty} 1/x} = \frac{\sqrt{1-2\cdot 0}}{1-2\cdot 0} = \frac{1}{1} = 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{x^2-2}}{x-2} &\overset{\text{AIF}}{=} \lim_{x\to-\infty} \frac{\sqrt{x^2-2}/x}{1-2/x} \overset{\text{AIF}}{=} \lim_{x\to-\infty} \frac{\sqrt{x^2-2}/-\sqrt{x^2}}{1-2/x} = \lim_{x\to-\infty} \frac{-\sqrt{1-2/x^2}}{1-2/x} \\ &= \frac{-\sqrt{1-2\lim_{x\to-\infty} 1/x^2}}{1-2\lim_{x\to-\infty} 1/x} = \frac{-\sqrt{1-2\cdot 0}}{1-2\cdot 0} = \frac{-1}{1} = -1\end{aligned}$$

The first $=$ of each line has an 'AIF' over it to remind us that we're changing the function by dividing by whatever power of x is necessary. The new function is almost identical, but now undefined at $x=0$. In the last two examples, there's a much more subtle use of the Almost Identical Functions property being used. Here, we have a square root of something whose highest power of x is 2, so the "highest" power in the numerator is therefore a half of this (because of the square root). Similarly, the highest power in the denominator is 1, so we know we need to divide top and bottom by x^1. To move the x inside the square root, though, we need to figure out some equivalent expression that involves a square root. This is because we have the rule

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \quad \text{and it is not true that} \quad \frac{\sqrt{a}}{b} = \sqrt{\frac{a}{b}}.$$

Therefore we replace x by $\sqrt{x^2}$, since the squaring cancels the square root. However, while it's true that $\sqrt{3^2} = \sqrt{9} = 3$, we can't do this with negative numbers: $\sqrt{(-3)^2} = \sqrt{9} \neq -3$. So it turns out that replacing x by $\sqrt{x^2}$ is only valid for $x \geq 0$. Fortunately, we are taking the limit as $x \to \infty$, so we only care about positive x anyway. Nonetheless, we are changing the function and using the AIF property to say that the limit stays the same, so we denote this we another use of $\overset{\text{AIF}}{=}$. For the final example, we cannot replace x by $\sqrt{x^2}$ since we are taking the limit as $x \to -\infty$, meaning we instead use $x = -\sqrt{x^2}$ for $x \leq 0$. The limit is then otherwise the same, but because of this we end up with a -1 at the end instead of a 1.

2 Derivatives

3 Optimization

4 Integration