# Math 1231 Summer 2024
# Practice Final Exam

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made ahead of time.
- You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.
- The exam has 6 required problems, one on each major topic we've covered, and two on each secondary topic covered in the last two weeks. The exam has 10 pages total.
- Each part of each topic is worth ten points. The whole test is scored out of 100 points, with the opportunity to gain 4 bonus points by answering up to two of the earlier secondary topics (S1–S6).
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all. When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

| | | | | | |
|---|---|---|---|---|---|
| M1: | | | M2: | | |
| M3: | | | M4: | | |
| S7: | | S8: | | | /100 |
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**Name:** Solutions

Name: Solutions

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**Problem 1** (M1). Compute the following limits if they exist. Show enough work to justify your computation.

(a) $\displaystyle \lim_{x\to+\infty} \frac{\sqrt{3x^5+2x}}{x^{5/2}-x^{3/2}+1}$

**Solution.**

$$\lim_{x\to+\infty} \frac{\sqrt{3x^5+2x}}{x^{5/2}-x^{3/2}+1} = \lim_{x\to+\infty} \frac{\sqrt{3+2/x^4}}{1-1/x+1/x^{5/2}} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

(b) $\displaystyle \lim_{x\to3} \frac{1}{x-3} - \frac{3}{x^2-3x}$

**Solution.**

$$\begin{aligned}\lim_{x\to3} \frac{1}{x-3} + \frac{3}{x^2-3x} &= \lim_{x\to3} \frac{x^2-3x-3(x-3)}{(x-3)(x^2-3x)}\\ &= \lim_{x\to3} \frac{x^2-6x+9}{x(x-3)^2}\\ &= \lim_{x\to3} \frac{1}{x} = \frac{1}{3}.\end{aligned}$$

Name: Solutions

**Problem 2** (M2). Compute the following derivatives.

(a) $\dfrac{d}{dx}\csc^4\left(\dfrac{x^3+4}{\sqrt[3]{x^2-3}}\right)$

**Solution.**

$$= 4\csc^3\left(\frac{x^3+4}{\sqrt[3]{x^2-3}}\right)\cdot -\csc\left(\frac{x^3+4}{\sqrt[3]{x^2-3}}\right)\cot\left(\frac{x^3+4}{\sqrt[3]{x^2-3}}\right)$$
$$\cdot\frac{3x^2(x^2-3)^{\frac{1}{3}}-(x^3+4)\cdot\frac{1}{3}(x^2-3)^{-\frac{2}{3}}\cdot 2x}{(x^2-3)^{\frac{2}{3}}}$$

(b) $\dfrac{d}{dx}x^2\sin\Big(\sec(\cos(x))+x^7\Big)$

**Solution.**

$$= 2x\sin(\sec(\cos(x))+x^7)+x^2\cos(\sec(\cos(x))+x^7)\cdot(\sec(\cos(x))\tan(\cos(x))\cdot -\sin(x)+7x^6)$$

Name: Solutions

**Problem 3** (M3).

(a) The function $f(x) = \dfrac{x^2+5}{x+2}$ has absolute extrema either on the interval $[-3,0]$ or on the interval $[0,3]$. Pick one of those intervals, explain why $f$ has extrema on that interval, and find the absolute extrema.

**Solution.** $f$ is continuous on the closed interval $[0,3]$, so it must have extrema there. (It is not continuous on $[-3,0]$ because it is undefined at $-2$.)

We compute

$$\begin{aligned} f'(x) &= \frac{2x(x+2)-(x^2+5)}{(x+2)^2} = \frac{x^2+4x-5}{(x+2)^2} \\ &= \frac{(x+5)(x-1)}{(x+2)^2} \end{aligned}$$

is zero for $x = 1, -5$ and is undefined for $x = -2$, so those are the critical points. The only one we have to care about is $x = 1$.

$$\begin{aligned} f(0) &= 5/2 \\ f(1) &= 2 \\ f(2) &= 9/4 \end{aligned}$$

so the absolute minimum is 2 at 1, and the absolute maximum is 5/2 at 0.

(b) Find and classify the critical points of $g(x) = x^4 + 4x^3 + 2$.

**Solution.** We see that $g'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$ so the critical points are $0, -3$.

The second derivative test doesn't work:

$$\begin{aligned} g''(x) &= 12x^2 + 24x \\ g''(0) &= 0 \\ g''(-3) &= 108 - 72 = 36 \end{aligned}$$

So we have a relative minimum at $-3$ but this doesn't tell us anything about 0.

Instead we make a chart:

| | $4x^2$ | $x+3$ | $g'$ |
|---|---|---|---|
| $x < -3$ | $+$ | $-$ | $-$ |
| $-3 < x < 0$ | $+$ | $+$ | $+$ |
| $0 < x$ | $+$ | $+$ | $+$ |

thus we see we have a local minimum at $-3$ and neither a max nor a min at 0.

Name: Solutions

**Problem 4** (M4). Compute the following integrals.

(a) **By changing the bounds of the integral** compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

**Solution.** Take $u = x^2$ so $du = 2x\,dx$ and $dx = du/2x$. We then have $g(0) = 0^2 = 0$ and $g(\sqrt{\pi}) = \sqrt{\pi}^2 = \pi$ so we get

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx &= \int_0^{\pi} x\sin(u)\frac{du}{2x}\\ &= \frac{1}{2}\int_0^{\pi} -\cos(u)\,du = \frac{-1}{2}\cos(u)\big|_0^{\pi} = \frac{-1}{2}(-1-1) = 1.\end{aligned}$$

(b) Compute $\int \frac{x^2}{(x^3-3)^3}\,dx$

**Solution.** We take $u = x^3 - 3$ so $du = 3x^2\,dx$ and then get

$$\begin{aligned}\int \frac{x^2}{(x^3-3)^3}\,dx = \int \frac{x^2}{u^3}\frac{du}{3x^2} &= \int \frac{1}{3}u^{-3}\,du\\ &= \frac{1}{3}\frac{1}{-2}u^{-2} + C = \frac{-1}{6u^2} + C = \frac{-1}{6(x^3-3)^2} + C.\end{aligned}$$

Name: Solutions

**Problem 5** (S7). Using **only the definition of Riemann sum** and your knowledge of limits, compute the exact (signed) area under the curve $x^3 + 2x$ on the interval $[-1, 2]$.

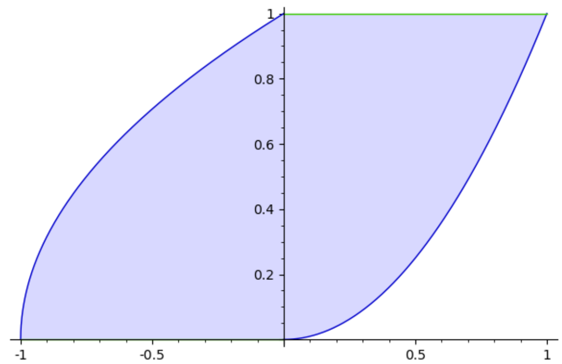
**Solution.** We compute

$$
\begin{aligned}
R_n &= \sum_{k=1}^{n} \frac{3}{n} f\left(-1 + \frac{3k}{n}\right) = \frac{3}{n} \sum_{k=1}^{n} (-1 + 3i/n)^3 + 2(-1 + 3k/n) \\
&= \frac{3}{n} \sum_{k=1}^{n} 27k^3/n^3 - 27k^2/n^2 + 9k/n - 1 - 2 + 6k/n \\
&= \frac{3}{n} \sum_{k=1}^{n} -3 + 15k/n - 27k^2/n^2 + 27k^3/n^3 \\
&= \frac{-9}{n} \sum_{k=1}^{n} 1 + \frac{45}{n^2} \sum_{k=1}^{n} k + \frac{-81}{n^3} \sum_{k=1}^{n} k^2 + \frac{81}{n^4} \sum_{k=1}^{n} k^3 \\
&= \frac{-9}{n} \cdot n + \frac{45}{n^2} \cdot \frac{n(n+1)}{2} - \frac{81}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{-9}{n} \cdot n + \frac{45}{n^2} \cdot \frac{n(n+1)}{2} - \frac{81}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= -9 + \frac{45}{2} - 27 + \frac{81}{4} = \frac{27}{4}.
\end{aligned}
$$

Name: Solutions

**Problem 6** (S8). Sketch and clearly label the region bounded by the curves $x = y^2 - 1$, $y = 0$, $y = 1$, and $x = \sqrt{y}$, and find the area of that region.

**Solution.** We sketch the region:




We really want to integrate this with respect to $y$. So we have

$$\int_0^1 \sqrt{y} - (y^2 - 1)\, dy = \int 1 + \sqrt{y} - y^2\, dy$$
$$= y + \frac{2}{3} y^{3/2} - \frac{y^3}{3}\Big|_0^1 = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}.$$

If we really want to, though, we can integrate with respect to $x$. We compute

$$\begin{aligned} A &= \int_{-1}^0 \sqrt{x+1}\, dx + \int_0^1 1 - x^2\, dx \\ &= \frac{2}{3}(x+1)^{3/2}\Big|_{-1}^0 + x - \frac{x^3}{3}\Big|_0^1 \\ &= \frac{2}{3} - 0 + 1 - \frac{1}{3} - 0 + 0 = \frac{4}{3}. \end{aligned}$$

Name: Solutions

**Problems 7 & 8** (S1–S6). Choose **two** of the following problems, and solve them on the blank page attached to this exam (front and back). The two problems you pick are both worth up to two bonus points on this exam, and will also count as mastery points on the chosen topics. For example, if you currently have a 0/2 or a 1/2 on S4, it might be a good idea to choose the S4 problem below. If you get a 2/2 on it on this exam, your exam grade will increase by 2 points, and your final mastery grade for S4 will be a 2/2.

**(S1)** If $f(x) = x^2 + \sqrt{x}$, compute $f'(4)$ directly from the definition of the derivative.

**Solution.** With the $h \to 0$ version we get

$$\begin{aligned}
f'(4) &= \lim_{h\to 0} \frac{(4+h)^2 + \sqrt{4+h} - 4^2 - \sqrt{4}}{h} \\
&= \lim_{h\to 0} \frac{(4+h)^2 - 16}{h} + \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \\
&= \lim_{h\to 0} \frac{16 + 8h + h^2 - 16}{h} + \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\
&= \lim_{h\to 0} \frac{8h + h^2}{h} + \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\
&\overset{\text{AIF}}{=} \lim_{h\to 0} 8 + h + \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = 8 + \frac{1}{\sqrt{4} + 2} = \frac{33}{4}.
\end{aligned}$$

Using the $t \to 4$ version instead we have

$$\begin{aligned}
f'(4) &= \lim_{t\to 4} \frac{t^2 + \sqrt{t} - 4^2 - \sqrt{4}}{t - 4} \\
&= \lim_{t\to 4} \frac{t^2 - 16}{t - 4} + \lim_{t\to 4} \frac{\sqrt{t} - 2}{t - 4} \\
&= \lim_{t\to 4} \frac{(t-4)(t+4)}{t - 4} + \lim_{t\to 4} \frac{\sqrt{t} - 2}{(\sqrt{t} - 2)(\sqrt{t} + 2)} \\
&\overset{\text{AIF}}{=} \lim_{t\to 4} t + 4 + \lim_{t\to 4} \frac{1}{\sqrt{t} + 2} = 4 + 4 + \frac{1}{\sqrt{4} + 2} = \frac{33}{4}.
\end{aligned}$$

Name: Solutions

**(S2)** If $f(x) = \frac{x^2+3}{x-2}$, use a linear approximation to estimate $f(2.9)$.

**Solution.**

$$\begin{aligned}
f(3) &= \frac{12}{1} = 12 \\
f'(x) &= \frac{2x(x-2) - (x^2+3)}{(x-2)^2} \\
f'(3) &= \frac{6 \cdot 1 - 12}{1^2} = -6 \\
f(x) &\approx f(3) + f'(3)(x-3) = 12 - 6(x-3) = 30 - 6x \\
f(2.9) &\approx 12 - 6(2.9-3) = 12 - (-0.6) = 12.6.
\end{aligned}$$

**(S3)** Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

**Solution.** We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$\begin{aligned}
4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} \\
\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx}
\end{aligned}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is $y - 1 = \sqrt{5}(x - \sqrt{5})$.

**(S4)** The surface area of a cube is given by the formula $A = 6s^2$ where $s$ is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

**Solution.** We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find $s$. But when $A = 54$ we have $54 = 6s^2$, meaning $s = 3$. Taking derivatives gives $A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$ so the area is increasing at 72 square inches per second.

Name: Solutions

**(S5)** Let $f(x) = \sqrt[3]{x^2 - 2x} = \sqrt[3]{x(x-2)}$. We compute that

$$f'(x) = \frac{2(x-1)}{3x^{2/3} \cdot (x-2)^{2/3}}$$
$$f''(x) = \frac{-2(x^2 - 2x + 4)}{9(x-2)^{5/3} \cdot x^{5/3}}.$$

Sketch a graph of $f$ and find all of the following:

(a) the domain of the function
(b) any horizontal or vertical asymptotes
(c) the roots of the function
(d) the critical points of the function
(e) intervals on which the function is increasing or decreasing
(f) any relative minima or maxima
(g) intervals on which the function is concave up or concave down
(h) any inflection points.

**Solution.** (a) The function is defined everywhere.

(b) Because the domain includes all real numbers, there are no vertical asymptotes, and $\lim_{x\to\pm\infty} f(x) = +\infty$ so there are no horizontal asymptotes.

(c) We see there are roots at $x = 0, 2$.

(d) We see that $f'(x)$ is undefined at $x = 0, 2$, and is zero when $x = 1$. So our critical points occur at $0, 1, 2$. We calculate $f(0) = f(2) = 0$, and $f(1) = \sqrt[3]{-1} = -1$.

(e) By making a chart, we get

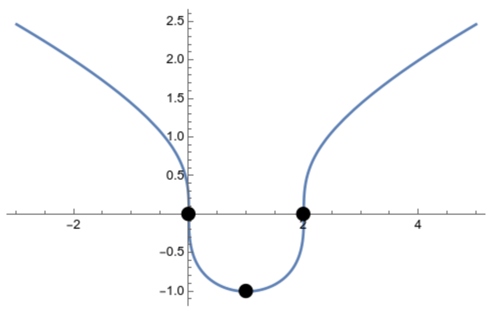
| | $2(x-1)$ | $3x^{2/3}$ | $(x-2)^{-2/3}$ | $f'$ |
|---|---|---|---|---|
| $x < 0$ | $-$ | $+$ | $+$ | $-$ |
| $0 < x < 1$ | $-$ | $+$ | $+$ | $-$ |
| $1 < x < 2$ | $+$ | $+$ | $+$ | $+$ |
| $2 < x$ | $+$ | $+$ | $+$ | $+$ |

so $f$ is decreasing on $(-\infty, 1)$ and it's increasing on $(1, +\infty)$.

The second derivative is undefined at $0, 2$ and is zero when $x^2 - 2x + 4 = 0$, which never happens. We still have $f(0) = f(2) = 0$. We can again make a chart:

| | $-2(x^2 - 2x + 4)$ | $(x-2)^{-5/3}/9$ | $x^{-5/3}$ | $f''$ |
|---|---|---|---|---|
| $x < 0$ | $-$ | $-$ | $-$ | $-$ |
| $0 < x < 2$ | $-$ | $-$ | $+$ | $+$ |
| $2 < x$ | $-$ | $+$ | $+$ | $-$ |

so the function is concave up for $0 < x < 2$, and it's concave down for $x < 0$ and $x > 2$. Thus we have the graph




**(S6)** Find the point on the line $y = 2x + 5$ that is closest to the origin.

**Solution.** Our objective function is $D = \sqrt{x^2 + y^2}$, and our constraint is that $y = 2x + 3$. However, we will get the same point whether we use the function $D$ or the function $D^2$—the point that minimizes distance will also minimize squared distance, so we can take $S = D^2 = x^2 + y^2$ as our objective function instead (this will be much easier to differentiate). Using the constraint, we have

$$S(x) = x^2 + (2x + 5)^2 = x^2 + 4x^2 + 20x + 25 = 5x^2 + 20x + 25$$

and therefore $S'(x) = 10x + 20$. Hence $S$ has a critical point only at $x = -2$, which gives us the point $(-2, 1)$.

To check this is really a minimum, we can't really use the EVT since $x$ can be infinitely big or small on the line. However, we can observe that for $x < -2$, $S'(x) = 10x + 20 < 0$ and for $x > -2$, $S'(x) > 0$, so the derivative goes from negative to positive and we have a min. We could also note that the second derivative $S''(x) = 10$ is always positive, meaning any critical point we find is a min.