Math 1231 Summer 2024
Midterm 2

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.
- The exam has 5 problems: one review problem, and one problem on each mastery topic we've covered since the first midterm. The exam has 6 pages total.
- The whole test is scored out of 100 points, with the points for individual questions indicated on the exam.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all. When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

M1:		M2:	
M3a:		M3b:	
S4:		S5:	
S6:		Tot:	/100

Name: Solutions

Name: Solution

Problem 1 (Midterm 1 Review). These are very similar to the trickier limit and derivative problems on the first midterm. Part (a) is worth 10 points, and part (b) 15 points.

(a) Evaluate $\lim_{x\to 0} \frac{\sin(x^2)\sin(4x)}{x^3}$ **without** using any shortcuts or L'Hôpital's rule.

Solution. We use the small angle approximation:

$$\boxed{\lim_{x\to 0} \frac{\sin(x^2)\sin(4x)}{x^3} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \cdot \lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} 4 \cdot \frac{\sin(4x)}{4x} = 4.}$$

(b) Compute $\frac{d}{dx} \csc\left(\frac{\tan(x^2)}{4x\sin(x)}\right)$.

Solution.

$$\boxed{= -\csc\left(\frac{\tan(x^2)}{4x\sin(x)}\right)\cot\left(\frac{\tan(x^2)}{4x\sin(x)}\right) \cdot \frac{\sec^2(x^2)\cdot 2x \cdot 4x\sin(x) - \tan(x^2)\cdot(4\sin(x) + 4x\cos(x))}{(4x\sin(x))^2}}$$

Name: Solution

Problem 2 (M3). Each part is worth 20 points.

(a) The function $f(x) = \dfrac{(x-1)^{\frac{2}{3}}}{x+1}$ has absolute extrema either on $[-2,0]$ or on $[0,2]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution. This function is built out of algebra, so it's continuous exactly where it's defined. However, it's not defined at $x = -1$ due to division by zero, so it's not continuous at -1. Since -1 is in the interval $[-2,0]$, we cannot apply the Extreme Value Theorem to this interval. We *can* apply it to $[0,2]$, however:

We choose $[0,2]$ since this is a closed interval on which f is continuous. By the EVT, there must exist absolute extrema on this interval.

Now that we know they exist, to find the absolute extrema we begin by looking for critical points. The derivative is

$$\boxed{f'(x) = \frac{\frac{2}{3}(x-1)^{-\frac{1}{3}}(x+1) - (x-1)^{\frac{2}{3}}}{(x+1)^2} = \frac{2(x+1) - 3(x-1)}{3(x+1)^2(x-1)^{\frac{1}{3}}} = \frac{5-x}{3(x+1)^2(x-1)^{\frac{1}{3}}}}$$

where we multiply the numerator and denominator by $3(x-1)^{\frac{1}{3}}$ to get the first equality. Since $x = 1$ makes the denominator zero and $x = 5$ makes the numerator zero, $\boxed{\text{the critical points are } x = 1 \text{ and } x = 5.}$ Since we are looking for the absolute extrema on $[0,2]$ and $x = 5$ is not in this interval, it is not a point we need to consider. Instead, 0, 1, and 2 are the points any extrema could occur at, so we check them:

$$\boxed{\begin{aligned} f(0) &= \frac{(-1)^{\frac{2}{3}}}{1} = 1 \\ f(1) &= \frac{0^{\frac{2}{3}}}{1+1} = 0 \\ f(2) &= \frac{1^{\frac{2}{3}}}{2+1} = \frac{1}{3} \end{aligned}}$$

We see that the largest value is 1, so $\boxed{f \text{ has an absolute maximum of 1 at 0}}$, and the smallest value is 0, so $\boxed{f \text{ has an absolute minimum of 0 at 1}}$.

Name: Solution

(b) Find and classify all the critical points of $g(x) = \dfrac{x^2 - 3x - 4}{x + 5}$. That is, for each critical point you find, say whether it is a relative maximum, a relative minimum, or neither.

Solution. To find the critical points, we take a derivative:

$$\boxed{g'(x) = \frac{(2x-3)(x+5) - (x^2 - 3x - 4)}{(x+5)^2} = \frac{x^2 + 10x - 11}{(x+5)^2} = \frac{(x+11)(x-1)}{(x+5)^2}.}$$

Then g' is undefined at $x = -5$, but since we require a critical point to be a point in the domain of the original function, the fact that -5 is not in the domain of g means it is not technically a critical point. Therefore the critical points come just from when $g'(x) = 0$, which we can see from the numerator happens at $\boxed{x = -11 \text{ and } x = 1.}$ To classify, we can use the second derivative test, calculating that

$$\boxed{g''(x) = \frac{(x - 1 + x + 11)(x+5)^2 - 2(x+5)(x+11)(x-1)}{(x+5)^4}.}$$

At either of the critical points, the second term in the numerator on the right of the minus sign is zero, and both $(x+5)^2$ and $(x+5)^4$ are always positive. Hence the sign on g'' at the critical points depends only on the very first $2x + 10$ term. Plugging in -11 and 1, we see that $\boxed{g''(-11) < 0, \text{ so } g(-11) \text{ is a maximum, and } g''(1) > 0, \text{ so } g(0) \text{ is a minimum.}}$

If we don't want to take a second derivative, we can also use the first derivative test. Something to note is that even though -5 is not technically a critical point, we still have to include it in our table since the sign of g' could theoretically change there:

	$(x+11)$	$(x-1)$	$(x+5)^2$	$g'(x)$
$(-\infty, -11)$	$-$	$-$	$+$	$+$
$(-11, -5)$	$+$	$-$	$+$	$-$
$(-5, 1)$	$+$	$-$	$+$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

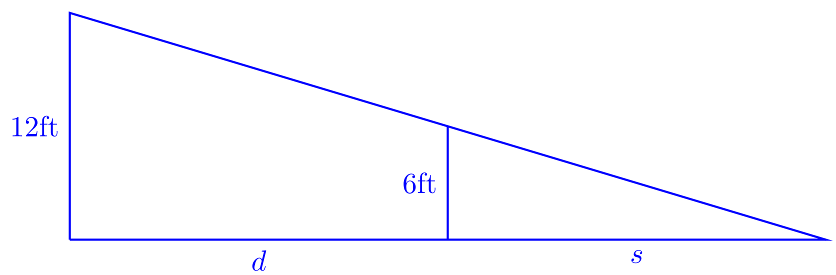
So again we have the same conclusion based on g' switching signs at -11 and 1 that $\boxed{g(-11) \text{ is a maximum and } g(1) \text{ is a minimum}}$.

It's important to note here that this is a nice example of why we can't classify critical points the same way we do when we find absolute extrema: $g(-11) = -25$ and $g(1) = -1$. If we tried to draw any conclusions about whether these are minima or maxima based on the fact that -1 is greater than -25, we would end up with the wrong answer, since we've just shown that in this case, the larger output is actually a minimum, and the smaller output is actually a maximum!

Name: Solution

Problem 3 (S4). A street light is mounted at the top of a 12-foot-tall pole. A six-foot-tall man walks straight away from the pole at 4 feet per second. How fast is the length of his shadow changing when he is twenty feet from the pole?

Solution. After drawing a picture, we see we have two triangles in the same shape: we know how one triangle is changing, and we want to figure out how the other is changing, so we should relate those similar triangles.



Let d be the distance of the man from the pole. Then $d = 20$ and $d' = 4$. If s is the length of the shadow, then we have $s/6 = (d+s)/12$ so we get

$$
\begin{aligned}
s &= \frac{d+s}{2} \\
s' &= d'/2 + s'/2 \\
s'/2 &= d'/2 \\
s' &= d' = 4.
\end{aligned}
$$

Thus the length of the shadow is growing at 4 feet per second.

Name: Solution

Problem 4 (S5). Sketch the graph of $f(x) = \dfrac{x^3}{(x-1)^2}$. We have $f'(x) = \dfrac{x^2(x-3)}{(x-1)^3}$ and $f''(x) = \dfrac{6x}{(x-1)^4}$. Your answer should state

(a) the domain of the function
(b) any horizontal or vertical asymptotes
(c) the roots of the function
(d) the critical points of the function
(e) intervals on which the function is increasing or decreasing
(f) any relative minima or maxima
(g) intervals on which the function is concave up or concave down
(h) any inflection points.

Solution. (a) The function is defined everywhere except at $x = 1$, so the domain is $(-\infty, 1) \cup (1, \infty)$.

(b) Since the function is undefined at $x = 1$, we check for vertical asymptotes here:

$$\lim_{x \to 1} \frac{x^{3} \nearrow^{1}}{(x-1)^2 \searrow_{0+}} = \infty,$$

so there is indeed a vertical asymptote at $x = 1$. Since the degree of the polynomial in the numerator of f is greater than the one in the denominator, there are no horizontal asymptotes. We have $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

(c) For roots, we see that $f(x) = 0$ at $x = 0$, so the point $(0, 0)$ is on the curve.

(d) The critical points come from f' being zero or undefined, so the critical points are $x = 0$ and $x = 3$ (we don't include $x = 1$ here because it's not in the domain of f, but we *do* include it as a point where the sign of the derivative can change).

(e) We make a sign table:

	x^2	$x-3$	$(x-1)^3$	$f'(x)$
$(-\infty,0)$	$+$	$-$	$-$	$+$
$(0,1)$	$+$	$-$	$-$	$+$
$(1,3)$	$+$	$-$	$+$	$-$
$(3,\infty)$	$+$	$+$	$+$	$+$

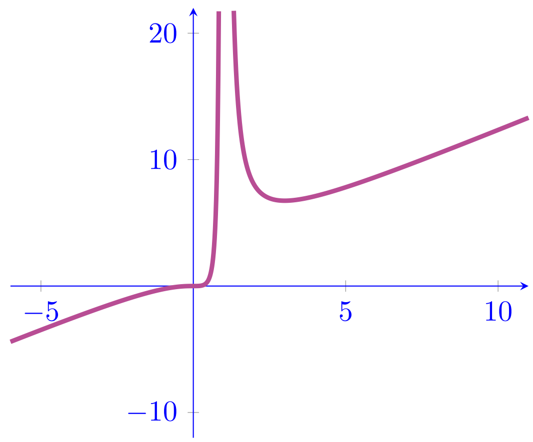
Therefore f is increasing on $(-\infty, 0) \cup (0, 1) \cup (3, \infty)$ and decreasing on $(1, 3)$.

(f) Since the sign of f' doesn't change at $x = 0$, $f(0) = 0$ is neither a min nor a max. Since the sign *does* change at $x = 3$, the first derivative test says that $f(3) = \frac{27}{4}$ is a relative minimum.

(g) Since the denominator of f'' is always positive, the sign of f'' is determined just by the numerator. Since f'' is positive on $(0, \infty)$, f is concave up there, and on $(-\infty, 0)$, f is concave down.

(h) Since f goes from concave down to concave up at $x = 0$, this is an inflection point.

Based on all this information, we have the following sketch:



(I'm not requiring that you check for this, but when the limits at infinity are themselves infinite, we rule out horizontal asymptotes, but there's really another sort of asymptote that's clear from the picture I've drawn, which are *slant asymptotes,* which we can find by doing polynomial division and rewriting

$$f(x) = \frac{x^3}{(x-1)^2} = x + 2 + \frac{3x+2}{(x-1)^2}.$$

Then we see that as x gets increasingly large, $f(x)$ is getting increasingly close to the line $x + 2$, as is evident from the graph. Again, this isn't a detail you have to include—drawing anything going off to infinity is good enough.)

Name: Solution

Problem 5 (S6). To check a bag on a certain airplane, the length plus width plus height must be less than or equal to 63in. Assuming the suitcase should be twice as long as it is wide, what height maximizes the volume of the suitcase? **Your answer should include some justification for how you know this is really a maximum.**

Solution. Our objective function is $V = \ell w h$. We know that $\ell = 2w$ and that $\ell + w + h = 63$, thus that $3w + h = 63$ and so $h = 63 - 3w$. Then we have $V = 2w \cdot w \cdot (63 - 3w) = 126w^2 - 6w^3$.

$$V' = 252w - 18w^2 = 18w(14 - w)$$

so the critical points are $w = 0$ and $w = 14$.

We have three options for proving this is a maximum (we only need one):

1. Extreme Value Theorem: The function $V(w) = 126w^2 - 6w^3$ is a continuous function, defined on the interval $[0, 21]$ (I'd also accept $[0, 63]$ here). Thus by the extreme value theorem there is an absolute maximum, which happens at a critical point or an endpoint. $V(0) = 0 = V(21)$ and $V(14) => 0$ so that value is a maximum.

2. First Derivative Test: For $w < 14$ we have $V'(w) = 18w(14 - w) > 0$ so the function is increasing, and for $w > 14$ we have $V'(w) < 0$ so the function is decreasing. Thus we have a unique maximum at 14.

3. Second derivative test: $V''(w) = 252 - 36w$. Then $V''(14) = 252 - 504 = -252 < 0$, which implies there is a relative maximum at 14. This doesn't really rigorously prove that this is an absolute maximum but I'll take it.

Finally, we asked about the *height* that maximizes the volume. This height is $h = 63 - 3w = 63 - 42 = 21$in.