Math 1231 Summer 2024
Mastery Quiz 9
Due Thursday, August 1

This week's mastery quiz has two topics, and everyone should submit both.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 4: Computing integrals
- Secondary Topic 7: Riemann sums

Name: Solutions

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Major Topic 4: Computing integrals

(a) Let $F(x) = \displaystyle\int_2^{\sqrt{5x+1}} t\sin(t)\,dt$. What is $F'(x)$?

Solution. We use the FTC with the chain rule. Taking $G(x) = \int_2^x t\sin(t)\,dt$, we see that $F(x) = G(\sqrt{5x+1})$, so that we do need to use the chain rule. The FTC tells us that $G'(x) = x\sin(x)$, so putting these together gives

$$\boxed{\frac{d}{dx}\int_2^{\sqrt{5x+1}} t\sin(t)\,dt = G'(\sqrt{5x+1})\cdot\frac{d}{dx}\sqrt{5x+1} = \sqrt{5x+1}\sin(\sqrt{5x+1})\cdot\frac{1}{2\sqrt{5x+1}}\cdot 5.}$$

(b) Find $\displaystyle\int \cos(x) + 2x\,dx$

Solution. We give the collection of *all* antiderivatives of the function $\cos(x) + 2x$, which is

$$\boxed{\sin(x) + x^2 + C.}$$

(c) Compute $\displaystyle\int_{-2}^4 x^3 - 3x\,dx$

Solution. We find an antiderivative and then evaluate at the endpoints:

$$\boxed{= \frac{x^4}{4} - \frac{3x^2}{2}\Big|_{-2}^4 = (64-24) - (4-6) = 40 - (-2) = 42.}$$

Name: Solutions

Secondary Topic 7: Riemann sums

Let $f(x) = x^2 - x$ be defined on the interval $[-3, 0]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution. (a) $R_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 6 + 2 + 0 = 8$.

(b) $L_3 = 1 \cdot f(-3) + 1 \cdot f(-2) + 1 \cdot f(-1) = 12 + 6 + 2 = 20$.

(c)

$$R_n = \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right)$$

(d)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right) \\
&= \sum_{i=1}^{n} \frac{3}{n}\left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9 - \frac{3i}{n} + 3\right) = \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{63i}{n^2} + \frac{36}{n} \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{63}{n^2}\sum_{i=1}^{n} i + \frac{36}{n}\sum_{i=1}^{n} 1 = \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n
\end{aligned}$$

(e) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n \\
&= \lim_{n\to+\infty} \frac{27 \cdot 1(1+1/n)(2+1/n)}{6} - \frac{63 \cdot 1(1+1/n)}{2} + 36 \\
&= 9 - \frac{63}{2} + 36 = \frac{27}{2}.
\end{aligned}$$