Math 1231 Summer 2024
Mastery Quiz 7
Due Wednesday, July 24

This week's mastery quiz has three topics, and everyone should submit all of them.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 3: Extrema and optimization
- Secondary Topic 5: Curve sketching
- Secondary Topic 6: Applied optimization

Name: Solutions

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Major Topic 3: Extrema and optimization

(a) Find and classify all the critical points of $f(x) = x^4 + 3x^3 + x^2 - 3$, that is, for each critical point you find, say whether it is a maximum, minimum, or neither.

Solution. We compute

$$f'(x) = 4x^3 + 9x^2 + 2x = x(4x^2 + 9x + 2) = x(4x+1)(x+2).$$

This is equal to zero when $x = -2, -1/4, 0$ and is never undefined, so the three critical points are $x = -2, -1/4, 0$.

We have two options here. We can use the second derivative test:

$$\begin{aligned} f''(x) &= 12x^2 + 18x + 2 \\ f''(-2) &= 48 - 36 + 2 = 14 > 0 \\ f''(-1/4) &= \frac{3}{4} - \frac{9}{2} + 2 = \frac{-7}{4} < 0 \\ f''(0) &= 2 > 0 \end{aligned}$$

so f has a local maximum at $x = -1/4$ and has local minima at $x = -2$ and $x = 0$.

Alternatively we can make a chart:

	x	$4x+1$	$x+2$	$f'(x)$
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < -1/4$	$-$	$-$	$+$	$+$
$-1/4 < x < 0$	$-$	$+$	$+$	$-$
$0 < x$	$+$	$+$	$+$	$+$

Thus f has a local maximum at $-1/4$ where the derivative switches from positive to negative, and has local minima at -2 and 0 where the derivative switches from negative to positive.

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(b) The function $g(x) = \dfrac{x^2 - 4x + 8}{2x - 1}$ has absolute extrema either on the interval $[-3, 0]$ or on the interval $[0, 3]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution. Since g is continuous on the closed interval $[-3, 0]$, by the Extreme Value Theorem, g has absolute extrema on this interval (and note g is *not* continuous on $[0, 3]$). To find the extrema, we find critical points:

$$g'(x) = \frac{(2x-4)(2x-1) - (x^2 - 4x + 8)(2)}{(2x-1)^2} = \frac{2(x-3)(x+2)}{(2x-1)^2},$$

so the critical points are $x = -2$ and $x = 3$ (the point $x = 1/2$ is not in the domain of g, hence not a critical point). Of these, we care only about $x = -2$ since $x = 3$ is not in $[-3, 0]$. We check

$$\begin{aligned} g(-3) &= \frac{29}{-7} = -4 - \frac{1}{7} \\ g(-2) &= \frac{20}{-5} = -4 \\ g(0) &= \frac{8}{-1} = -8. \end{aligned}$$

Therefore the absolute max is -4 at -2 and the absolute min is -8 at 0.

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Secondary Topic 5: Curve sketching

Sketch the graph of $g(x) = 3x^4 - 4x^3 - 36x^2 + 64 = (x+2)^2(3x-4)(x-4)$. We have $g'(x) = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$ and $g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$. Your answer should state

(a) the domain of the function
(b) any horizontal or vertical asymptotes
(c) the roots of the function
(d) the critical points of the function
(e) intervals on which the function is increasing or decreasing
(f) any relative minima or maxima
(g) intervals on which the function is concave up or concave down
(h) any inflection points.

Solution. (a) The domain is all real numbers.

(b) There aren't any vertical asymptotes since the function is defined anywhere, and there aren't horizontal asymptotes either since a fourth degree polynomial with positive leading coefficient has $\lim_{x\to\pm\infty} = +\infty$.

(c) We can see from the factored form that g has roots at $x = -2$, $x = 4/3$, and $x = 4$.

(d) To find critical points, we take a derivative:

$$g'(x) = 12x^3 - 12x^2 - 72x = 12x(x^2 - x - 6) = 12x(x-3)(x+2).$$

Therefore the critical points are $x = -2$, $x = 0$, and $x = 3$.

(e) For the intervals of increase and decrease, we make a sign table for the derivative:

	$12x$	$(x-3)$	$(x+2)$	$g'(x)$
$(-\infty, -2)$	$-$	$-$	$-$	$-$
$(-2, 0)$	$-$	$-$	$+$	$+$
$(0, 3)$	$+$	$-$	$+$	$-$
$(3, \infty)$	$+$	$+$	$+$	$+$

so that g is increasing on $(-2, 0) \cup (3, \infty)$ and decreasing on $(-\infty, -2) \cup (0, 3)$.

(f) Since g' goes from negative to positive at $x = -2$ and $x = 3$, we see that $g(-2) = 0$ and $g(3) = -125$ are relative minima. Since g' goes from positive to negative at $x = 0$, $g(0) = 64$ is a relative maximum.

(g) To find the concavity information, we take another derivative:

$$g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6).$$

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Solving for the roots of the quadratic, we get

$$x = \frac{2 \pm \sqrt{4 - 4(3)(-6)}}{6} = \frac{2 \pm 2\sqrt{1-(3)(-6)}}{6} = \frac{1 \pm \sqrt{19}}{3}.$$

These are the potential points of inflection, and divide the real line into three intervals. To find the concavity on each, we test a point. The interval $(-\infty, (1-\sqrt{19})/3)$ contains -2, so we plug in -2 and get that $g''(-2) = 12(3(-2)^2 - 2(-2) - 6) = 12(12+4-6) > 0$, so g is concave up on this interval. The point 0 is in the interval $\big((1-\sqrt{19})/3, (1+\sqrt{19})/3\big)$, and $g''(0) = 12(0-0-6) < 0$, so g is concave down on this interval. Finally, 2 is in the interval $\big((1+\sqrt{19})/3, \infty\big)$, and $g''(2) = 12(3(2)^2 - 2(2) - 6) = 12(12-4-6) > 0$, so g is concave up on this interval. Hence g is concave up on

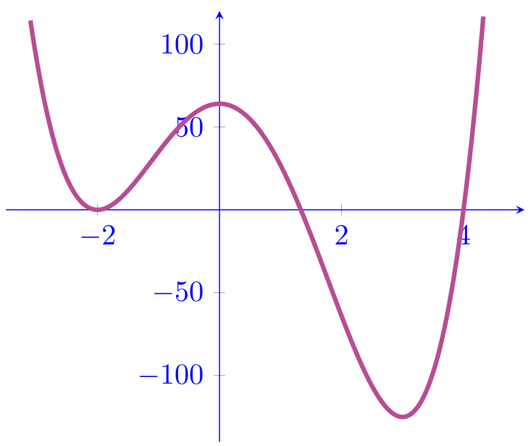
$$\left(-\infty, \frac{1-\sqrt{19}}{3}\right) \cup \left(\frac{1+\sqrt{19}}{3}, \infty\right)$$

and concave down on

$$\left(\frac{1-\sqrt{19}}{3}, \frac{1+\sqrt{19}}{3}\right).$$

(h) Since g does change concavity at $(1 \pm \sqrt{19})/3$, both potential points of inflection are indeed inflection points.

Based on all this information, we have the following sketch:



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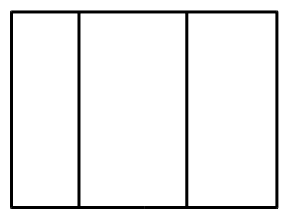
Secondary Topic 6: Applied optimization

We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. We want to maximize the total area of the pen.

(a) What is your objective function, and why?
(b) What constraint equation(s) can you use?
(c) What dimensions maximize the total area of the pen? (Prove this is a max.)



Solution. (a) Our objective function is $A = \ell w$, because this is the area we want to maximize.

(b) We see also that we have the constraint $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$.

(c) Using this constraint, we have

$$A = (500 - 2w)w = 500w - 2w^2$$
$$A' = 500 - 4w$$

has a critical point when $w = 125$. To prove this is really gives a maximum, we can do any of the following:

- Using the Extreme Value Theorem, we can say that the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$, whereas if we plug in $w = 125$, we get something greater than 0.

- With the first derivative test, we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

- With the second derivative test, we get $A'' = -4 < 0$ so we have a local maximum.

However way we want to prove this is a max, the area is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.)