Math 1231 Summer 2024
Mastery Quiz 10
Due Monday, August 5

This week's mastery quiz has three topics. Everyone should submit M4 and S8, but if you already have a 2/2 on S7 (check Blackboard!) there's no need to submit it again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 4: Computing integrals
- Secondary Topic 7: Riemann sums
- Secondary Topic 8: Applications of integration

Name: Solutions

Name: Solutions

Major Topic 4: Computing integrals

(a) Compute $\displaystyle\int \sin(x) + 5x^2 - 2\,dx$

Solution.

$$\int \sin(x) + 5x^2 - 2\,dx = -\cos(x) + \frac{5}{3}x^3 - 2x + C.$$

(b) Compute $\displaystyle\int x\cos(3x^2 - 2)\,dx$

Solution. Set $u = 3x^2 - 2$ so $du = 6xdx$ and $dx = \frac{du}{6x}$. Then

$$\begin{aligned}\int x\cos(3x^2-2)\,dx &= \int x\cos(u)\frac{du}{6x} = \frac{1}{6}\int \cos(u)\,du \\ &= \frac{1}{6}\sin(u) + C = \frac{1}{6}\sin(3x^2-2) + C.\end{aligned}$$

(c) By explicitly changing the bounds of integration, compute $\displaystyle\int_1^2 x^5\sqrt{x^3+8}\,dx$

Solution. We take $u = x^3 + 8$, so we have $du = 3x^2\,dx, dx = \frac{du}{3x^2}$, and we have $g(1) = 9$ and $g(2) = 16$. Then we compute

$$\begin{aligned}\int_1^2 x^5\sqrt{x^3+8}\,dx &= \int_9^{16} x^5\sqrt{u}\frac{du}{3x^2} \\ &= \frac{1}{3}\int_9^{16} x^3\sqrt{u}\,du = \frac{1}{3}\int_9^{16}(u-8)\sqrt{u}\,du \\ &= \frac{1}{3}\int_9^{16} u^{3/2} - 8u^{1/2}\,du \\ &= \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - \frac{8u^{3/2}}{3/2}\right)\Bigg|_9^{16} \\ &= \frac{1}{3}\left(\left(\frac{2\cdot 4^5}{5} - \frac{8\cdot 2\cdot 4^3}{3}\right) - \left(\frac{2\cdot 3^5}{5} - \frac{8\cdot 2\cdot 3^3}{3}\right)\right) \\ &= \frac{2048}{15} - \frac{1024}{9} - \frac{162}{5} + 48 = \frac{1726}{45} = 38.355\ldots.\end{aligned}$$

Name: Solutions

Secondary Topic 7: Riemann sums

Let $f(x) = 2x^3$ be defined on the interval $[0, 4]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (d) to compute the area exactly. (Do not use any other methods to compute the definite integral—we're only using Riemann sums here!)

Solution. (a) $R_4 = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) = 2 + 16 + 54 + 128 = 200$

(b) $L_4 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) = 0 + 2 + 16 + 54 = 72.$

(c) $R_n = \sum_{i=1}^{n} \frac{4}{n} f\left(0 + i\frac{4}{n}\right) = \sum_{i=1}^{n} \frac{4}{n} \cdot 2\left(\frac{4}{n}i\right)^3$

(d)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{8}{n}\frac{64i^3}{n^3} \\
&= \sum_{i=1}^{n} \frac{512i^3}{n^4} \\
&= \frac{512}{n^4}\sum_{i=1}^{n} i^3 \\
&= \frac{512}{n^4}\frac{n^2(n+1)^2}{4} \\
&= \frac{128n^2(n+1)^2}{n^4}
\end{aligned}$$

(e) We can compute

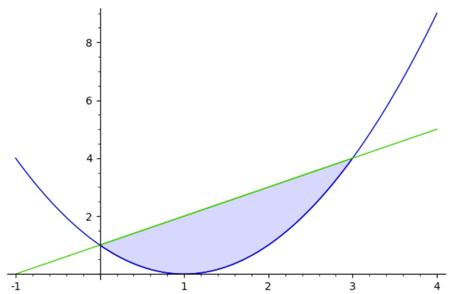
$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \frac{128n^2(n+1)^2}{n^4} = 128.$$

Name: Solutions

Secondary Topic 8: Applications of integration

(a) **Sketch the region** bounded by the curves $y = (x-1)^2$ and $y = x+1$. Find the area of the region.

Solution. We sketch the region, and see that it will be much easier to integrate with respect to x. Setting the two equations equal, we see the curves intersect when $x+1 = x^2 - 2x + 1$, and thus when $0 = x^2 - 3x = x(x-3)$, and so when $x = 0, 3$. So the x coordinates vary from 0 to 3.



$$\begin{aligned} A &= \int_0^3 x + 1 - (x-1)^2\,dx = \int_0^3 x + 1 - x^2 + 2x - 1\,dx \\ &= \int_0^3 -x^2 + 3x\,dx = -x^3/3 + 3x^2/2\big|_0^3 \\ &= -9 + 27/2 = 9/2. \end{aligned}$$

(b) Suppose your velocity is given by $v(t) = 3t - t^2 + \sin(\pi t)$ miles per hour, where t is in hours. How far in total do you travel between noon (at time 0) and 3 pm? Don't forget to include units in your answer.

Solution. Our velocity is given in miles per hour, as a function of hours. So we can compute the distance traveled by integrating, using a substitution $u = \pi x$ for the last term:

$$\begin{aligned} D &= \int_0^3 3x - x^2 + \sin(\pi x)\,dx = 3\int_0^3 x\,dx - \int_0^3 x^2\,dx + \int_0^3 \sin(\pi x)\,dx \\ &= 3\int_0^3 x\,dx - \int_0^3 x^2\,dx + \int_0^{3\pi} \sin(u)\frac{du}{\pi} = \frac{3}{2}x^2\big|_0^3 - \frac{1}{3}x^3\big|_0^3 + \frac{1}{\pi}(-\cos(u))\big|_0^{3\pi} \\ &= \frac{27}{2} - \frac{27}{3} + \frac{1}{\pi}(1+1) = \frac{9}{2} + \frac{2}{\pi}. \end{aligned}$$

This answer will be in miles, since we are integrating miles per hour with respect to hours.