Math 1231 Summer 2024
Mastery Quiz 3
Due Wednesday, July 10

This week's mastery quiz has three topics. Everyone should submit M2 and S1, but if you already have a 4/4 on M1 (check Blackboard!) you don't need to submit it again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 1: Limits
- Major Topic 2: Computing derivatives
- Secondary Topic 1: Definition of the derivative

Name: Solutions

Name: Solutions

Major Topic 1: Limits

(a) $\lim_{x\to 2} \frac{1}{x-2} - \frac{3}{(x+1)(x-2)}$

Solution. Somehow I managed to include two different limit problems here depending on which version of the assignment you looked at. If you used the "mq3_single_sheet" file, then you got the problem above, which we can solve as follows: initially, we want to try plugging in 2, and we get something that looks like 1/0 - 3/0, suggesting this might be some kind of infinite limit. However, the sum limit law does not work with infinite limits, so we should first get a common denominator:

$$\boxed{\begin{aligned} \lim_{x\to 2} \frac{1}{x-2} - \frac{3}{(x+1)(x-2)} &= \lim_{x\to 2} \frac{x+1}{(x+1)(x-2)} - \frac{3}{(x+1)(x-2)} \\ &= \lim_{x\to 2} \frac{x-2}{(x-2)(x+1)} \overset{\text{AIF}}{=} \lim_{x\to 2} \frac{1}{x+1} = \frac{1}{2+1} = \frac{1}{3}. \end{aligned}}$$

If you instead looked at the "mq3_answer_blanks" file, the problem was a different one:

$$\boxed{\begin{aligned} \lim_{x\to -3} \frac{\sqrt{x+4}-1}{(x+3)^2} &= \lim_{x\to -3} \frac{\sqrt{x+4}-1}{(x+3)^2} \cdot \frac{\sqrt{x+4}+1}{\sqrt{x+4}+1} \\ &= \lim_{x\to -3} \frac{x+3}{(x+3)^2(\sqrt{x+4}+1)} \\ &= \lim_{x\to -3} \frac{1^{\nearrow 1}}{(x+3)(\sqrt{x+4}+1)_{\searrow 0^\pm}} = \pm\infty. \end{aligned}}$$

So we see that even after multiplying by the conjugate, we might still end up in a situation where there's more work to be done. In this case, we reduce things to another limit of the form $c/0$ for some nonzero real number c, and so we know this is some kind of infinite limit. The denominator contains an $(x+3)$ term, which is either a little bit less than zero or a little bit more than zero, depending on whether x is a little more than -3 or a little less. Since the denominator can approach zero from either direction (denoted by $0^\pm$), the limit is $\pm\infty$.

Name: Solutions

(b) $\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 - 1}$

Solution. The largest exponent appearing is 2, so we divide top and bottom by x^2:

$$\boxed{\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 - 1} \overset{\text{AIF}}{=} \lim_{x\to\infty} \frac{3 - 2/x + 1/x^2}{1 - 1/x^2} = \frac{3 - 0 + 0}{1 - 0} = 3.}$$

(c) $\lim_{x\to 1} \frac{3x^2 - 2x + 1}{x^2 - 1}$

Solution. If we try plugging in 1, we see that the numerator approaches 2 and the denominator approaches 0, so this must be some sort of infinite limit. However, it's hard to tell at the moment how the denominator is approaching 0 as x approaches 1, so we should factor it as $(x-1)(x+1)$. We see that the $(x+1)$ term is always positive when x is close to 1, but $(x-1)$ can be positive or negative depending on whether x is a little bit greater than 1 or a little bit less than 1.

$$\boxed{\lim_{x\to 1} \frac{3x^2 - 2x + 1}{x^2 - 1} = \lim_{x\to 1} \frac{3x^2 - 2x + 1^{\nearrow 2}}{(x-1)(x+1)_{\searrow 0\pm}} = \pm\infty.}$$

Name: Solutions

Major Topic 2: Computing derivatives

(a) Compute the derivative of $f(x) = (x^4 + 2x)(1 - 2\sqrt{x})$.

Solution. It's helpful to rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ so we can use the power rule for that part of the derivative:

$$\boxed{f'(x) = (4x^3 + 2)(1 - 2x^{\frac{1}{2}}) + (x^4 + 2x)(-2 \cdot \frac{1}{2}x^{-\frac{1}{2}}).}$$

(b) Compute the derivative of $g(x) = \dfrac{5x^2 + 7x}{x^3 - \sqrt[3]{x} + 8}$.

Solution. Again we will make use of fractional exponents $\sqrt[3]{x} = x^{\frac{1}{3}}$:

$$\boxed{g'(x) = \frac{(10x + 7)(x^3 - x^{\frac{1}{3}} + 8) - (5x^2 + 7x)(3x^2 - \frac{1}{3}x^{-\frac{2}{3}})}{(x^3 - x^{\frac{1}{3}} + 8)^2}}$$

(c) Compute the derivative of $h(x) = \dfrac{6}{x^5}$.

Solution. There are a couple ways we could do this, one would be to use the quotient rule:

$$\boxed{h'(x) = \frac{0 \cdot x^5 - 6 \cdot 5x^4}{(x^5)^2} = -\frac{30x^4}{x^{10}} = -\frac{30}{x^6}.}$$

Alternatively, we can rewrite $h(x)$ as $6x^{-5}$ and then use the power rule:

$$\boxed{h'(x) = 6 \cdot -5x^{-6} = -30x^{-6},}$$

which is the same result.

Name: Solutions

Secondary Topic 1: Definition of the derivative

(a) If $f(x) = 2x^2 - 5x$, compute $f'(2)$ directly from the limit definition.

Solution. Using the $h \to 0$ formulation, we have

$$\boxed{\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{2(2+h)^2 - 5(2+h) - (2(2^2) - 5(2))}{h} \\ &= \lim_{h\to 0} \frac{8 + 8h + 2h^2 - 10 - 5h - 8 + 10}{h} = \lim_{h\to 0} \frac{3h + 2h^2}{h} \overset{\text{AIF}}{=} \lim_{h\to 0} 3 + 2h = 3. \end{aligned}}$$

Instead if we use the $t \to 2$ version, we have

$$\boxed{\begin{aligned} f'(2) &= \lim_{t\to 2} \frac{f(t)-f(2)}{t-2} = \lim_{t\to 2} \frac{2t^2 - 5t - (2(2^2) - 5(2))}{t-2} = \lim_{t\to 2} \frac{2t^2 - 5t - 8 + 10)}{t-2} \\ &= \lim_{t\to 2} \frac{2t^2 - 5t + 2}{t-2} = \lim_{t\to 2} \frac{(2t-1)(t-2)}{t-2} \overset{\text{AIF}}{=} \lim_{t\to 2} 2t - 1 = 4 - 1 = 3. \end{aligned}}$$

And either version is fine.

(b) If $g(x) = \frac{1}{x-3}$, compute $g'(a)$ directly from the limit definition.

Solution. Using the $h \to 0$ formulation, we have

$$\boxed{\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h-3} - \frac{1}{a-3}}{h} = \lim_{h\to 0} \frac{\frac{a-3}{(a-3)(a+h-3)} - \frac{a+h-3}{(a-3)(a+h-3)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(a-3)(a+h-3)}}{h} \overset{\text{AIF}}{=} \lim_{h\to 0} \frac{-1}{(a-3)(a+h-3)} = \frac{-1}{(a-3)^2}. \end{aligned}}$$

Using instead the $t \to a$ version, we have

$$\boxed{\begin{aligned} f'(a) &= \lim_{t\to a} \frac{f(t)-f(a)}{t-a} = \lim_{t\to a} \frac{\frac{1}{t-3} - \frac{1}{a-3}}{t-a} = \lim_{t\to a} \frac{\frac{a-3}{(a-3)(t-3)} - \frac{t-3}{(a-3)(t-3)}}{t-a} \\ &= \lim_{t\to a} \frac{\frac{a-t}{(a-3)(t-3)}}{t-a} \overset{\text{AIF}}{=} \lim_{t\to a} \frac{-1}{(a-3)(t-3)} = \frac{-1}{(a-3)^2}. \end{aligned}}$$

Again, either version works.