Math 1231 Summer 2024
Practice Midterm 1

- You will have 90 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.
- The exam has 5 problems, one on each mastery topic we've covered. The exam has 5 pages total.
- Each part of each topic is worth ten points, except the M2 questions are worth 15 points. The whole test is scored out of 100 points.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all. When in doubt, show more work and write complete sentences.
- If you need more paper to show work, I have extra at the front of the room.
- Good luck!

	a	b	c	d
M1:				
M2:			S1:	
S2:		S3:		/100

Name: Solutions

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Problem 1 (M1). Compute the following limits if they exist. Show enough work to justify your computation.

(a) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{5x^4+x+1}}{x^2+3x+4}$

Solution. We see the largest exponent is x^2, couting the x^4 under the square root as $\sqrt{x^4} = x^2$.

$$\boxed{\lim_{x\to\infty} \frac{\sqrt{5x^4+x+1}}{x^2+3x+4} \overset{\text{AIF}}{=} \lim_{x\to\infty} \frac{\frac{\sqrt{5x^4+x+1}}{x^2}}{1+3/x+4/x^2} \overset{\text{AIF}}{==} \lim_{x\to\infty} \frac{\sqrt{5+1/x^3+1/x^4}}{1+3/x+4/x^2} = \sqrt{5}.}$$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)\tan(2x)}{5x^2}$

Solution. Rewriting $\tan(2x)$ as $\sin(2x)/\cos(2x)$, we multiply and divide by whatever we need to to make the sine terms look like $\sin(\theta)/\theta$. Looking at the expression here, we'll need to do this twice, once with $\theta = 3x$ and once with $\theta = 2x$.

$$\boxed{\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)\tan(2x)}{5x^2} &= \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(2x)}{x} \cdot \frac{1}{5\cos(2x)} \cdot \frac{3}{3} \cdot \frac{2}{2} \\ &= \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{\sin(2x)}{2x} \cdot \frac{6}{5\cos(2x)} = 1 \cdot 1 \cdot \frac{6}{5\cos(0)} = \frac{6}{5}.\end{aligned}}$$

(c) $\displaystyle\lim_{x\to 3} \frac{x^2-5x+4}{x^2-2x-3}$

Solution. Trying to plug in 3, we see that the numerator approaches $9-15+4 = -2$ while the denominator approaches $9-6-3 = 0$, so we know this is some kind of infinite limit. Factoring the denominator, we see that there is an $x+1$ term which is always positive when x is near 3, and an $x-3$ term that could be positive or negative if x is near 3:

$$\boxed{\lim_{x\to 3} \frac{x^2-5x+4}{x^2-2x-3} = \lim_{x\to 3} \frac{x^2-5x+4^{\nearrow -2}}{(x+1)(x-3)_{\searrow 0\pm}} = \pm\infty.}$$

Name: Solutions

(d) $\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5}$

Solution.

$$\boxed{\begin{aligned}\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} &= \lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5} \frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} \\ &= \lim_{x\to 5} \frac{x-5}{(x-5)(\sqrt{x+4}+3)} = \lim_{x\to 5} \frac{1}{\sqrt{x+4}+3} = \frac{1}{\sqrt{5+4}+3} = \frac{1}{6}.\end{aligned}}$$

Name: Solutions

Problem 2 (M2). Compute the derivatives of the following functions using any of the methods we have learned in class. Show enough work to justify your answers.

(a) $\dfrac{d}{dx}\cos^2\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)$

Solution.

$$\begin{aligned}\frac{d}{dx}\cos^2\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right) &= 2\cos\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\\ &\cdot -\sin\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\\ &\cdot 2\tan\left(\sec^2(\sqrt{x}+x)\right)\\ &\cdot \sec^2\left(\sec^2(\sqrt{x}+x)\right)\\ &\cdot 2\sec(\sqrt{x}+x)\\ &\cdot \sec(\sqrt{x}+x)\tan(\sqrt{x}+x)\\ &\cdot \left(\frac{1}{2\sqrt{x}}+1\right).\end{aligned}$$

(b) $\dfrac{d}{dx}\dfrac{\csc(x^2+1)}{x^4+\sqrt{\cos(x)}}$

Solution.

$$\frac{d}{dx}\frac{\csc(x^2+1)}{x^4+\sqrt{\cos(x)}} = \frac{-\csc(x^2+1)\cot(x^2+1)\cdot 2x\cdot(x^4+\sqrt{\cos(x)}) - \csc(x^2+1)(4x^2-\frac{1}{2}\cos^{-\frac{1}{2}}(x)\sin(x))}{(x^4+\sqrt{\cos(x)})^2}.$$

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Problem 3 (S1). If $f(x) = (x-3)^2$, find $f'(5)$, directly from the definition of derivative.

Solution. Using the $h \to 0$ version, we get

$$\boxed{\begin{aligned} f'(5) &= \lim_{h\to 0} \frac{(5+h-3)^2 - (5-3)^2}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} \\ &= \lim_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim_{h\to 0} \frac{4h+h^2}{h} \stackrel{\text{AIF}}{=} \lim_{h\to 0} 4+h = 4. \end{aligned}}$$

If we instead use the $t \to 5$ formulation, we get

$$\boxed{\begin{aligned} f'(5) &= \lim_{t\to 5} \frac{(t-3)^2 - (5-3)^2}{t-5} = \lim_{t\to 5} \frac{t^2-6t+9-4}{t-5} \\ &= \lim_{t\to 5} \frac{t^2-6t+5}{t-5} = \lim_{t\to 5} \frac{(t-5)(t-1)}{t-5} \stackrel{\text{AIF}}{=} \lim_{t\to 5} t-1 = 4. \end{aligned}}$$

Problem 4 (S2). Give the equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$. Use it to estimate $f(1.5)$.

Solution. In general we approximate a differentiable function as $f(x) \approx L(x) = f(a) + f'(a)(x-a)$, meaning we want to calculate $f(\pi/2)$ and $f'(\pi/2)$.

$$\boxed{f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.}$$

Using the product rule, we get

$$\boxed{f'(x) = \sin(x) + x\cos(x), \text{ so } f'\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 1.}$$

Putting these together, our approximation is that

$$\boxed{x\sin(x) \approx \frac{\pi}{2} + 1\cdot\left(x - \frac{\pi}{2}\right) = x.}$$

Because 1.5 is near $\pi/2$, we can use this to say that $\boxed{1.5\sin(1.5) = f(1.5) \approx 1.5.}$

Name: Solutions

Problem 5 (S3). Find a formula for the line tangent to the curve $x^2y^2 = 5 + x + y$ at the point $(1, 3)$.

Solution. We use implicit differentiation, taking the derivative of both sides to obtain

$$\boxed{2xy^2 + 2x^2y\frac{dy}{dx} = 0 + 1 + \frac{dy}{dx}.}$$

Because we're only interested in the slope of the curve at the point $(1, 3)$ we can go ahead and plug in 1 for x and 3 for y, giving us a nicer equation to solve for $\frac{dy}{dx}$:

$$\boxed{\begin{aligned} 2(1)(3)^3 + 2(1)^2(3)\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\ 18 + 6\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\ 5\frac{dy}{dx} &= -17 \\ \frac{dy}{dx} &= -\frac{17}{5}. \end{aligned}}$$

Alternatively, we could solve for the slope of this curve at any point (x, y):

$$\boxed{\begin{aligned} (2x^2y - 1)\frac{dy}{dx} &= 1 - 2xy^2 \\ \frac{dy}{dx} &= \frac{1 - 2xy^2}{2x^2y - 1} \end{aligned}}$$

and then plug in $(1, 3)$ to get that

$$\boxed{\frac{dy}{dx} = \frac{1 - 2(1)(3)^2}{2(1)^2(3) - 1} = \frac{1 - 18}{6 - 1} = -\frac{17}{5}.}$$

Either way of getting the slope is fine, and now the last thing to do is to assemble this information into a tangent line at the point $(1, 3)$. We know a point and we know a slope, so the tangent line in point-slope form is

$$\boxed{y = 3 - \frac{17}{5}(x - 1).}$$