Math 1231 Summer 2024
Mastery Quiz 4
Due Monday, July 15

This week's mastery quiz has four topics. Everyone should submit M2, S2, and S3 but if you already have a 2/2 on S1 (check Blackboard!) you don't need to submit it again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 2: Computing derivatives
- Secondary Topic 1: Definition of the derivative
- Secondary Topic 2: Linear approximation
- Secondary Topic 3: Implicit differentiation

Name:

Name:

Major Topic 2: Computing derivatives

(a) Compute $\dfrac{d}{dx}\sec^3\left(\cot(x^3+x)\right)$

Solution.

$$\boxed{\begin{aligned}&= 3\sec^2\left(\cot(x^3+x)\right)\cdot\sec\left(\cot(x^3+x)\right)\tan\left(\cot(x^3+x)\right)\\&\cdot -\csc^2\left(x^3+x\right)\cdot(3x^2+1)\end{aligned}}$$

(b) Compute $\dfrac{d}{dx}\dfrac{\sin^{3/7}(x^2)+x}{x^3\tan(x^3)}$

Solution.

$$\boxed{=\frac{\left(\frac{3}{7}\sin^{-4/7}(x^2)\cdot\cos(x^2)\cdot 2x+1\right)x^3\tan(x^3)-\left(3x^2\tan(x^3)+x^3\sec^2(x^3)\cdot 3x^2\right)\left(\sin^{3/7}(x^2)+x\right)}{(x^3\tan(x^3))^2}}$$

Name:

Secondary Topic 1: Definition of the derivative

(a) If $f(x) = \frac{x+1}{x-1}$, find $f'(2)$ directly from the limit definition of the derivative.

Solution. Using the $h \to 0$ formulation, we have

$$\boxed{\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{3+h}{1+h}-3}{h} \\ &= \lim_{h\to 0} \frac{\frac{3+h-3(1+h)}{1+h}}{h} = \lim_{h\to 0} \frac{-2h}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-2}{1+h} = -2. \end{aligned}}$$

If we want instead to use the $t \to 2$ version, we get

$$\boxed{\begin{aligned} f'(2) &= \lim_{t\to 2} \frac{f(t)-f(2)}{t-2} = \lim_{t\to 2} \frac{\frac{t+1}{t-1}-3}{t-2} \\ &= \lim_{t\to 2} \frac{\frac{t+1-3(t-1)}{t-1}}{t-2} = \lim_{t\to 2} \frac{-2t+4}{(t-2)(t-1)} \\ &= \lim_{t\to 2} \frac{-2(t-2)}{(t-2)(t-1)} = \lim_{t\to 2} \frac{-2}{t-1} = -2. \end{aligned}}$$

(b) If $g(x) = \sqrt{2x-3}$, find $g'(x)$ directly from the limit definition of the derivative.

Solution. Using the $h \to 0$ limit, we get

$$\boxed{\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-3}-\sqrt{2x-3}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2x-3+2h}-\sqrt{2x-3}}{h} \cdot \frac{\sqrt{2x-3+2h}+\sqrt{2x-3}}{\sqrt{2x-3+2h}+\sqrt{2x-3}} \\ &= \lim_{h\to 0} \frac{(2x-3+2h)-(2x-3)}{h(\sqrt{2x-3+2h}+\sqrt{2x-3})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{2x-3+2h}+\sqrt{2x-3})} \\ &\overset{\text{AIF}}{=} \lim_{h\to 0} \frac{2}{\sqrt{2x-3+2h}+\sqrt{2x-3}} = \frac{2}{2\sqrt{2x-3}} = \frac{1}{\sqrt{2x-3}}. \end{aligned}}$$

Name:

And we get the same result with the $t \to x$ limit:

$$\boxed{\begin{aligned} g'(x) &= \lim_{t\to x} \frac{g(t)-g(x)}{t-x} = \lim_{t\to x} \frac{\sqrt{2t-3}-\sqrt{2x-3}}{t-x} \\ &= \lim_{t\to x} \frac{\sqrt{2t-3}-\sqrt{2x-3}}{t-x} \cdot \frac{\sqrt{2t-3}+\sqrt{2x-3}}{\sqrt{2t-3}+\sqrt{2x-3}} \\ &= \lim_{t\to x} \frac{(2t-3)-(2x-3)}{(t-x)(\sqrt{2t-3}+\sqrt{2x-3})} \\ &= \lim_{t\to x} \frac{2t-2x}{(t-x)(\sqrt{2t-3}+\sqrt{2x-3})} \\ &= \lim_{t\to x} \frac{2}{\sqrt{2t-3}+\sqrt{2x-3}} = \frac{2}{2\sqrt{2x-3}} = \frac{1}{\sqrt{2x-3}}. \end{aligned}}$$

Name:

Secondary Topic 2: Linear approximation

(a) Give a formula for a linear approximation of $f(x) = x\sqrt{x+1}$ near the point $a = 3$, and use this to estimate $f(2.8)$.

Solution. We find that $f(a) = 3\sqrt{4} = 6$, and we use the product rule to take the derivative:

$$\boxed{\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\ f'(3) &= 2 + \frac{3}{4} = \frac{11}{4} \\ f(x) &\approx f(a) + f'(a)(x-a) = 6 + \frac{11}{4}(x-3). \end{aligned}}$$

Therefore $f(2.8) \approx 6 + \frac{11}{4}(-.2) = \frac{109}{20}$.

(b) Use linear approximation to estimate $\sqrt[4]{14}$.

Solution. We take $f(x) = \sqrt[4]{x}$, and take $a = 16$. Then

$$\boxed{\begin{aligned} f'(x) &= \frac{1}{4}x^{-3/4} \\ f'(16) &= \frac{1}{4}(16)^{-3/4} = \frac{1}{4 \cdot 8} = \frac{1}{32} \\ f(x) &\approx f(a) + f'(a)(x-a) = 2 + \frac{1}{32}(14-16) = \frac{31}{16}. \end{aligned}}$$

Name:

Secondary Topic 3: Implicit differentiation

(a) Find a formula for $\frac{dy}{dx}$ in terms of x and y if $x^8 + x^4 + y^4 + y^6 = 1$.

Solution. We use the power rule for each term, and the chain rule on terms involving y (because we think of y as being a function of x):

$$\boxed{\begin{aligned} 8x^7 + 4x^3 + 4y^3\frac{dy}{dx} + 6y^5\frac{dy}{dx} &= 0 \\ 8x^7 + 4x^3 &= -(4y^3 + 6y^5)\frac{dy}{dx} \\ -\frac{4x^7 + 2x^3}{2y^3 + 3y^5} &= \frac{dy}{dx}. \end{aligned}}$$

(b) Find a line tangent to the curve given by $y^3 + xy + \frac{x}{y} = 5$ at the point $(2, 1)$.

Solution. We use implicit differentiation, and find that

$$\begin{aligned} 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} + \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} &= 0 \\ 3\frac{dy}{dx} + 1 + 2\frac{dy}{dx} + 1 - 2\frac{dy}{dx} &= 0 \\ 3\frac{dy}{dx} &= -2 \end{aligned}$$

So that $\frac{dy}{dx} = -2/3$. Alternatively we could compute

$$\begin{aligned} 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} + \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} &= 0 \\ \left(3y^2 + x - \frac{x}{y^2}\right)\frac{dy}{dx} &= -y - \frac{1}{y} \\ \frac{dy}{dx} &= \frac{-y - 1/y}{3y^2 + x - \frac{x}{y^2}} \end{aligned}$$

so at the point $(2, 1)$ we get $\frac{dy}{dx} = \frac{-2}{3+2-2} = \frac{-2}{3}$ and so either way, the equation of our tangent line is

$$y - 1 = \frac{-2}{3}(x - 2).$$