Math 1231 Summer 2024
Mastery Quiz 11
Due Wednesday, August 7

This week's mastery quiz has two topics. If you already have a 4/4 on M4 or a 2/2 on S8 (check Blackboard!) there's no need to submit them again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 4: Computing integrals
- Secondary Topic 8: Applications of integration

Name: Solutions

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Major Topic 4: Computing integrals

(a) Compute $\displaystyle\int_{\pi/12}^{3\pi/8} \csc^2(2t)\,dt$

Solution. Set $u = 2t$ so $du = 2\,dt$ and $dt = \frac{du}{2}$. Then

$$\begin{aligned}\int_{\pi/12}^{3\pi/8} \csc^2(2t)\,dt &= \int_{\pi/6}^{3\pi/4} \csc^2(u)\frac{du}{2} = -\frac{1}{2}\cot(u)\Big|_{\pi/6}^{3\pi/4} \\ &= \frac{-\cos(3\pi/4)}{2\sin(3\pi/4)} - \frac{-\cos(\pi/6)}{2\sin(\pi/6)} \\ &= \frac{\sqrt{2}/2}{2\sqrt{2}/2} + \frac{\sqrt{3}/2}{2/2} = \frac{1+\sqrt{3}}{2}.\end{aligned}$$

(b) By explicitly changing the bounds of integration, compute $\displaystyle\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

Solution. Take $u = x^3 + 3x$ so $du = (3x^2+3)dx$ and $dx = \frac{du}{3x^2+3}$. Note that $u(1) = 4$ and $u(3) = 36$. So we have

$$\begin{aligned}\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx &= \int_4^{36} \frac{x^2+1}{\sqrt{u}}\frac{du}{3x^2+3} \\ &= \frac{1}{3}\int_4^{36}\frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int_4^{36} u^{-1/2}\,du \\ &= \frac{1}{3}\frac{u^{1/2}}{1/2}\Big|_4^{36} = \frac{2}{3}\sqrt{u}\Big|_4^{36} = \frac{2}{3}\sqrt{36} - \frac{2}{3}\sqrt{4} = 4 - \frac{4}{3} = \frac{8}{3}.\end{aligned}$$

(c) Compute $\displaystyle\int_{-1}^{2} 2x^4 - 6x^2 + 3\,dx$

Solution.

$$\begin{aligned}\int_{-1}^{2} 2x^4 - 6x^2 + 3\,dx &= \frac{2}{5}x^5 - 2x^3 + 3x\Big|_{-1}^{2} \\ &= \left(\frac{64}{5} - 16 + 6\right) - \left(-\frac{2}{5} + 2 - 3\right) \\ &= \frac{64}{5} - 10 + \frac{2}{5} + 1 = \frac{66}{5} - 9 = \frac{21}{5}.\end{aligned}$$

Name: Solutions

Secondary Topic 8: Applications of integration

(a) What is the average value of the function $h(x) = x + \sqrt{x}$ on the interval $[1, 4]$?

Solution. The average value is the area under the curve over the length of the interval:

$$\begin{aligned} \frac{1}{3}\int_1^4 x + \sqrt{x}\,dx &= \frac{1}{3}x^2/2 + \frac{2}{3}x^{3/2}|_1^4 \\ &= \frac{1}{3}(8 + 16/3 - 1/2 - 2/3) = \frac{8}{3} + \frac{16}{9} - \frac{1}{6} - \frac{2}{9} = \frac{73}{18} \end{aligned}$$

(b) A spring with natural length of 20cm takes 15N of force to stretch to 30cm. How much work does it take to stretch it from 50cm to 100cm? Give units in your answer.

Solution. We know the force equation is $F(x) = k(x - x_0)$, and $x_0 = 20cm$. So $F(30) = 15$ which means that $15 = k(30 - 20)$ and thus $k = 1.5$. Then the integral is

$$\begin{aligned} \int_{50}^{100} 1.5(x - 20)\,dx = \frac{3}{4}x^2 - 30x\Big|_{50}^{100} &= \left(\frac{3}{4}(100)^2 - 30(100)\right) - \left(\frac{3}{4}(50)^2 - 30(50)\right) \\ &= (7500 - 3000) - (1875 - 1500) = 4125. \end{aligned}$$

These units are in Newton-centimeters, so this is equivalent to $41.25J$.

We could also work in meters from the start. Then we'd have $F(x) = k(x - .2)$ and then we'd have $15 = F(.3) = k(.1)$ and thus $k = 150N/m$. Then the integral is

$$\begin{aligned} \int_{.5}^{1} 150(x - .2)\,dx &= \int_{.5}^{1} 150x - 30\,dx \\ &= 75x^2 - 30x\Big|_{.5}^{1} = \Big(75 - 30\Big) - \left(\frac{75}{4} - 15\right) \\ &= 60 - \frac{75}{4} = 41.25. \end{aligned}$$

and this answer is naturally in Newton-meters or in Joules.