Math 1231 Summer 2024
Mastery Quiz 8
Due Monday, July 29

This week's mastery quiz has three topics. If you already have a 4/4 on M3, a 2/2 on S5, or a 2/2 on S6 (check Blackboard—grades may have changed after the midterm), you don't need to submit these topics again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 3: Extrema and optimization
- Secondary Topic 5: Curve sketching
- Secondary Topic 6: Applied optimization

Name: Solutions

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Major Topic 3: Extrema and optimization

(a) The function $f(x) = \dfrac{x^3 - 5x^2}{x+3}$ has absolute extrema either on the interval $[-4, -1]$ or on the interval $[-1, 4]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution. f is continuous on the closed interval $[-1, 4]$, so it must have extrema there. (It is not continuous on $[-4, -1]$ because it is undefined at -3.)

We compute

$$\begin{aligned} f'(x) &= \frac{(3x^2 - 10x)(x+3) - (x^3 - 5x^2) \cdot 1}{(x+3)^2} \\ &= \frac{3x^3 + 9x^2 - 10x^2 - 30x - x^3 + 5x^2}{(x+3)^2} \\ &= \frac{2x^3 + 4x^2 - 30x}{(x+3)^2} = \frac{(2x)(x^2 + 2x - 15)}{(x+3)^2} \\ &= \frac{(2x)(x+5)(x-3)}{(x+3)^2} \end{aligned}$$

is zero for $x = 0, 3, -5$ and these are the critical points. (The derivative is undefined for $x = -3$, but -3 isn't in the domain of the original function.) The only critical points we have to care about are $x = 0, 3$, since these are within the interval we're considering.

$$\begin{aligned} f(-1) &= -3 \\ f(0) &= 0 \\ f(3) &= -3 \\ f(4) &= -16/7 \end{aligned}$$

so the absolute minimum is -3 at -1 and 3, and the absolute maximum is 0 at 0.

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(b) Find and classify all the critical points of $f(x) = x^4 + 8x^3 + 10x^2 + 1$, that is, for each critical point you find, say whether it is a maximum, minimum, or neither.

Solution. We compute

$$f'(x) = 4x^3 + 24x^2 + 20x = 4x(x^2 + 6x + 5) = 4x(x+5)(x+1).$$

This is equal to zero when $x = 0, -1, -5$.

We can use the second derivative test:

$$\begin{aligned} f''(x) &= 12x^2 + 48x + 20 \\ f''(0) &= 20 > 0 \\ f''(-1) &= 16 < 0 \\ f''(-5) &= 300 - 240 + 20 = 80 > 0 \end{aligned}$$

so we see that f has local minima at $x = 0, x = -5$, and a local maximum at $x = -1$.

Alternatively, we could compute a chart

	$4x$	$x+5$	$x+1$	$f'(x)$
$x < -5$	$-$	$-$	$-$	$-$
$-5 < x < -1$	$-$	$+$	$-$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$-$
$0 < x$	$+$	$+$	$+$	$+$

Thus we conclude that f has local minima at $x = -5, 0$ and a local maximum at $x = -1$.

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Secondary Topic 5: Curve sketching

Let $g(x) = \dfrac{x^2-7}{x^2-4}$. We can compute that $g'(x) = \dfrac{6x}{(x+2)^2(x-2)^2}$ and also that $g''(x) = \dfrac{-6(3x^2+4)}{(x+2)^3(x-2)^3}$. Sketch a graph of the function $g(x)$. Your answer should state

(a) the domain of the function
(b) any horizontal or vertical asymptotes
(c) the roots of the function
(d) the critical points of the function
(e) intervals on which the function is increasing or decreasing
(f) any relative minima or maxima
(g) intervals on which the function is concave up or concave down
(h) any inflection points.

(a) The domain all reals except ± 2, so $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.
(b) Because

$$\lim_{x\to\pm\infty} \frac{x^2-7}{x^2-4} = \lim_{x\to\pm\infty} \frac{1-7/x^2}{1-4/x^2} = 1,$$

the line $y=1$ is a horizontal asymptote. We also have the points ± 2 where g is undefined, so we should check if these are places where vertical asymptotes occur. Indeed,

$$\lim_{x\to -2^-} \frac{x^2-7^{\nearrow -3}}{x^2-4_{\searrow 0^+}} = -\infty \quad \text{and} \quad \lim_{x\to -2^+} \frac{x^2-7^{\nearrow -3}}{x^2-4_{\searrow 0^-}} = \infty,$$

so there's a vertical asymptote at $x=-2$, going to $-\infty$ left of -2 and ∞ right of -2. Similarly, we compute that

$$\lim_{x\to 2^-} \frac{x^2-7^{\nearrow -3}}{x^2-4_{\searrow 0^-}} = \infty \quad \text{and} \quad \lim_{x\to 2^+} \frac{x^2-7^{\nearrow -3}}{x^2-4_{\searrow 0^+}} = -\infty,$$

so there is another vertical asymptote at $x=2$.

(c) $g(x)=0$ when $x=\pm\sqrt{7}$.

(d) The derivative $g'(x) = \frac{6x}{(x+2)^2(x-2)^2}$ is defined except at $x=\pm 2$. It is zero when $x=0$. The critical points are thus 0 and arguably ± 2 (although technically they aren't); $g(0)=7/4$.

(e) Since the denominator is a square, it is always positive. Hence the sign of g' depends only on the numerator, which is positive when $x>0$ and negative when $x<0$. Therefore g is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

(f) Above we show g' goes from negative to positive, which means that $(0, 7/4)$ is a relative minimum.

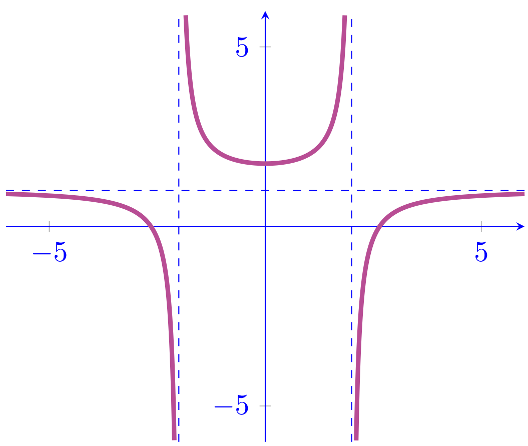
(g) We know that

$$g''(x) = \frac{-6(3x^2+4)}{(x^2-4)^3}.$$

The numerator is negative everywhere. The denominator is negative when $-2 < x < 2$ and is positive when $x < -2$ or when $2 < x$. Thus the concavity only changes at ± 2; it is concave up on $(-2, 2)$ and concave down elsewhere, i.e. on $(-\infty, -2) \cup (2, \infty)$.

(h) The concavity changes at ± 2, so we could consider these points of inflection (so if you wrote that, that's just fine). If you require inflection points to be in the domain of the original function (and probably one does), then there are no points of inflection. So either ± 2 or "none" is a good answer.

Putting all of this together, we get the following sketch:



Name: Solutions

Secondary Topic 6: Applied optimization

Suppose you are running a toy shop. It costs $C(x) = 200 + 10x$ dollars to produce x toys in a day, and you make a revenue of $R(x) = 26x - .2x^2$ dollars if you sell x toys in a day. How many toys should you produce per day to maximize your profit?

Solution. Our profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) = (26x - .2x^2) - (200 + 10x) \\ &= 16x - .2x^2 - 200 \\ P'(x) &= 16 - .4x \end{aligned}$$

To find a critical point, We set $16 - .4x = 0$, which gives $16 = .4x$ or $40 = x$.

Since this is the only critical point, we expect it to be our maximum. We can provide evidence for this by looking at either the first or second derivative. We see that $P'(x) > 0$ for $x < 40$ and $P'(x) < 0$ for $x > 40$, which implies that P has a local maximum at 40. Further, since the function is <u>always</u> increasing for $x < 40$ and <u>always</u> decreasing for $x > 40$ this must be a global maximum.

We can also look at the second derivative and see that $P''(x) = -.4 < 0$. Thus the function is concave down, and any local extremum must be a critical point.

Therefore: we should make 40 toys per day.