Math 1231 Summer 2024
Mastery Quiz 5
Due Wednesday, July 17

This week's mastery quiz has four topics. Everyone should submit S4 but if you already have a 4/4 on M2, or a 2/2 on S2 or S3 (check Blackboard—grades may have changed after the midterm) you don't need to submit those topics again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 2: Computing derivatives
- Secondary Topic 2: Linear approximation
- Secondary Topic 3: Implicit differentiation
- Secondary Topic 4: Related rates

Name: Solutions

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Major Topic 2: Computing derivatives

(a) Compute $\dfrac{d}{dt}\sqrt[5]{\dfrac{\tan^2(t^2+1)+2}{\sin(2t)-2t}}$

Solution. We get

$$\boxed{\frac{1}{5}\left(\frac{\tan^2(t^2+1)+2}{\sin(2t)-2t}\right)^{-\frac{4}{5}}\cdot\frac{2\tan(t^2+1)\cdot\sec^2(t^2+1)\cdot 2t\cdot(\sin(2t)-2t)-(\tan^2(t^2+1)+2)(\cos(2t)\cdot 2-2)}{(\sin(2t)-2t)^2}}$$

(b) Compute $\dfrac{d}{dx}x^2\sin\left(\sqrt{x^3+x}\right)$

Solution. We get

$$\boxed{2x\sin\left(\sqrt{x^3+x}\right)+x^2\cos\left(\sqrt{x^3+x}\right)\cdot\frac{1}{2}(x^3+x)^{-\frac{1}{2}}\cdot(3x^2+1)}$$

Name: Solutions

Secondary Topic 2: Linear approximation

(a) Give a formula for a linear approximation to $f(x) = \sin(x^2 - 3x)$ near $a = 0$.

Solution. Our linear approximation formula is $f(x) \approx f(a) + f'(a)(x - a)$, so we need to calculate $g(0)$ and $g'(0)$. We get $\boxed{f(0) = \sin(0) = 0}$, and then

$$\boxed{\begin{aligned} f'(x) &= \cos(x^2 - x)(2x - 3) \\ f'(0) &= 1 \cdot (0 - 3) = -3 \\ f(x) &\approx 0 - 3(x - 0) = -3x. \end{aligned}}$$

(b) Give a formula for a linear approximation of $g(x) = \sqrt{x^3 + 1}$ near the point $a = 2$, and use your answer to estimate $g(2.1)$.

Solution. Again we will need to know $g'(a)$ and $g(a)$. We get $g(2) = \sqrt{8 + 1} = 3$, and

$$\begin{aligned} g'(x) &= \frac{1}{2}(x^3 + 1)^{-1/2} 3x^2 \\ g'(a) &= \frac{1}{2}(9)^{-1/2} \cdot 12 = 2 \\ g(x) &= g(a) + g'(a)(x - a) = 3 + 2(x - 2). \end{aligned}$$

Which tells us that $g(2.1) \approx 3 + 2(.1) = 3.2$.

Name: Solutions

Secondary Topic 3: Implicit differentiation

(a) Find a formula for y' in terms of x and y if $xy^3 = \sqrt{x^2+y^2}$.

Solution. Using implicit differentiation, we have

$$\boxed{\begin{aligned} y^3 + 3xy^2y' &= \frac{2x+2yy'}{2\sqrt{x^2+y^2}} = \frac{x+yy'}{\sqrt{x^2+y^2}} \\ y^3 - \frac{x}{\sqrt{x^2+y^2}} &= \frac{yy'}{\sqrt{x^2+y^2}} - 3xy^2y' \\ y' = \frac{y^3 - \frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}} - 3xy^2} &= \frac{y^3\sqrt{x^2+y^2}-x}{y-3xy^2\sqrt{x^2+y^2}}. \end{aligned}}$$

(b) Find a formula for the second derivative y'' in terms of x and y if $\sin(y) = x^2 + y$. (Your answer should not contain y'.)

Solution. Using implicit differentiation to find the first derivative, we have

$$\boxed{\begin{aligned} \cos(y)y' &= 2x + y' \\ (\cos(y)-1)y' &= 2x \\ y' &= \frac{2x}{\cos(y)-1} \end{aligned}}$$

To take a second derivative, we use the quotient rule:

$$\boxed{\begin{aligned} y'' &= \frac{2(\cos(y)-1) - 2x(-\sin(y)y')}{(\cos(y)-1)^2} \\ &= \frac{2(\cos(y)-1) + 2x\sin(y)\left(\frac{2x}{\cos(y)-1}\right)}{(\cos(y)-1)^2} \end{aligned}}$$

Name: Solutions

Secondary Topic 4: Related rates

A snowball is melting such that its surface area is decreasing at $1\text{cm}^2/\text{min}$. When the radius is 8cm, how quickly is the radius decreasing? (It might be useful to recall that the surface area of a sphere of radius r is $4\pi r^2$.)

Solution. We have $S = 4\pi r^2$, so $S' = 8\pi r r'$. When the radius is 8cm we have

$$\begin{aligned} S' &= 8\pi \cdot 8\text{cm} \cdot r' \\ -1\text{cm}^2/\text{min} &= 64\pi r' \\ r' &= \frac{-1}{64\pi}\text{cm}/\text{min}. \end{aligned}$$

Thus when the radius of the snowball is 8cm, it is decreasing by $\frac{1}{64\pi}$ centimeters per minute.