Math 1231 Summer 2024
Mastery Quiz 6
Due Monday, July 22

This week's mastery quiz has two topics. Everyone should submit M3 but if you already have a 2/2 on S4 (check Blackboard—grades may have changed after the midterm) you don't need to submit those topics again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn in this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name is clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on this quiz:

- Major Topic 3: Extrema and optimization
- Secondary Topic 4: Related rates

Name: Solutions

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Major Topic 3: Extrema and optimization

(a) The function $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ has absolute extrema either on the interval $(-1, 2)$, or on the interval $[-1, 2]$. Pick one of those intervals, explain why g has absolute extrema on that interval, and find the absolute extrema.

Solution. We begin by determining the interval we need to work over. The tool we have to make this determination is the Extreme Value Theorem (EVT), and we *must* include something about this in our answer. The EVT requires a function to be continuous on a closed interval, so we check these requirements. The function g that we are given is continuous everywhere, which means the choice of interval comes down to picking whichever one is closed.

> We choose $[-1, 2]$ since this is a closed interval on which g is continuous. By the EVT, there must exist absolute extrema on this interval.

Now that we know g has absolute extrema on $[-1, 2]$, we look for the points at which they occur. By Fermat's Theorem, we know we should look for critical points as well as the endpoints of the interval we're working on. To find the critical points, we take a derivative:

$$\boxed{g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x+1)(x-1).}$$

This is always defined, so the crititcal points are the inputs that make the derivative zero, i.e., $\boxed{\text{the critical points are } -1/2,\ 0, \text{ and } 1.}$ The absolute extrema occur either at these points or at the endpoints -1 and 2, so we plug all of these points into the *original* function g to see which have the biggest and smallest outputs:

$$\boxed{\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}}$$

The biggest output is 25 from the input 2 and the smallest output is 3 from the input 1. $\boxed{\text{Therefore } g \text{ has an absolute maximum of 25 at 2 and an absolute minimum of 3 at 1.}}$

Name: Solutions

(b) Find and classify the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$, i.e. for each critical point you find, say whether it is a relative minimum, a relative maximum, or neither.

Solution. Since we are asked to find and classify all critical points, it won't be enough to compare the different outputs at each critical point and see which are the biggest and smallest. We still begin by finding the critical points, which means computing $f'(x)$:

$$\boxed{f'(x) = \frac{1}{3}(x^3 - 3x)^{-\frac{2}{3}} \cdot (3x^2 - 3) = \frac{3(x+1)(x-1)}{\left(\sqrt[3]{x(x^2-3)}\right)^2}}$$

which is zero when $x = \pm 1$ and undefined when $x = 0$ or $x = \pm\sqrt{3}$.

$$\boxed{\text{Hence the critical points are: } -\sqrt{3},\ -1,\ 0,\ 1,\ \sqrt{3}.}$$

We can use a first derivative test to determine where f is increasing and decreasing, which will tell us which critical points are minima, maxima, or possibly neither. We can make the following table:

	$x+1$	$x-1$	$\left(\sqrt[3]{x(x^2-3)}\right)^2$	$f'(x)$
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$-$	$+$	$+$
$-1 < x < 0$	$+$	$-$	$+$	$-$
$0 < x < 1$	$+$	$-$	$+$	$-$
$1 < x < \sqrt{3}$	$+$	$+$	$+$	$+$
$\sqrt{3} < x$	$+$	$+$	$+$	$+$

This tells us that f is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.

Since f changes from increasing to decreasing at $x = -1$, this is a maximum, and since f changes from decreasing to increasing at $x = 1$, this is a minimum. Because there is no change at $x = \pm\sqrt{3}$ or $x = 0$, these points are neither maxima nor minima.

Note that a second derivative test wouldn't be so helpful here, since the first derivative being undefined at several of the critical points guarantees the second derivative would also fail to exist at these points.

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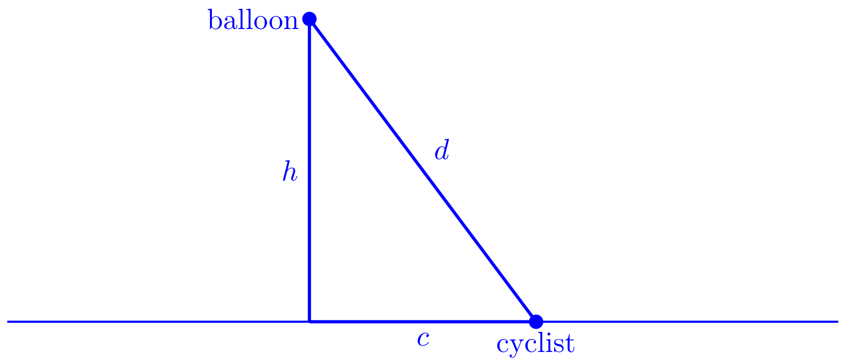
Secondary Topic 4: Related rates

A balloon is rising at a constant speed of 5 feet per second. A cyclist is moving along a straight road at a speed of 15 feet per second. When she passes under the balloon, it is 45 feet above her. How quickly is the distance between the cyclist and the balloon changing 3 seconds after she passes under it?

Solution. We begin by drawing a picture:



We know two distances and how fast they're changing. We want to know how fast another distance is changing, so the Pythagorean theorem, which relates all these distances to each other, seems like a reasonable choice. So we use the equation $d^2 = c^2 + h^2$, where c is the ground distance from cyclist to balloon, h is the distance from balloon to ground, and d is the distance we're interested in understanding.

After 3 seconds have passed, we see that the height of the balloon is $h = 60\text{ft}$, and the derivative is $h' = 5\text{ft/s}$. The distance between the cyclist and the point under the balloon is $c = 45\text{ft}$, with derivative $c' = 15\text{ft/s}$. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the distance at 3 seconds after passing under is 75ft, and then that

$$\begin{aligned}
2dd' &= 2cc' + 2hh' \\
dd' &= cc' + hh' \\
75\text{ft}d' &= 45\text{ft} \cdot 15\text{ft/s} + 60\text{ft} \cdot 5\text{ft/s} = 675\text{ft}^2/\text{s} + 300\text{ft}^2/\text{s} = 975\text{ft}^2/\text{s} \\
d' &= \frac{975}{75}\text{ft/s} = \frac{325}{25}\text{ft/s} = 13\text{ft/s}.
\end{aligned}$$

Thus the distance between the cyclist and the balloon is increasing by 13 feet per second.