Topology Through Inquiry Self-Study

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1 Cardinality: To Infinity and Beyond

1.1 Sets and Functions

Theorem 1.2 (DeMorgan's Laws). Let X be a set and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then

$$X - \left(\bigcup_{k=1}^{N} A_k\right) = \bigcap_{k=1}^{N}(X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^{N} A_k\right) = \bigcup_{k=1}^{N}(X - A_k).$$

Proof. Let $a \in X - \left(\bigcup_{k=1}^N A_k\right)$ be arbitrary. Then $a \notin \bigcup_{k=1}^N A_k$, so for all k, $a \notin A_k$, which means that $a \in X - A_k$ for all k. Therefore $x \in \bigcap_{k=1}^N (X - A_k)$ and so $X - \left(\bigcup_{k=1}^N A_k\right) \subset \bigcap_{k=1}^N (X - A_k)$. Now let $a \in \bigcap_{k=1}^N (X - A_k)$ be arbitrary. Then we have that $a \in X - A_k$ for all k, which means that $a \notin A_k$ for all k. Therefore $a \notin \bigcup_{k=1}^N A_k$, so we have that $a \in X - \left(\bigcup_{k=1}^N A_k\right)$ and therefore $\bigcap_{k=1}^N (X - A_k) \subset X - \left(\bigcup_{k=1}^N A_k\right)$. Therefore

$$X - \left(\bigcup_{k=1}^{N} A_k\right) = \bigcap_{k=1}^{N}(X - A_k).$$

Let $a \in X - \left(\bigcap_{k=1}^N\right)$ be arbitrary. Then $a \notin \bigcap_{k=1} A_k$, so there exists some j such that $a \notin A_j$, which means that $a \in X - A_j$. Therefore $a \in \bigcup_{k=1}^N (X - A_k)$ and so $X - \left(\bigcap_{k=1}^N A_k\right) \subset$

$\bigcup_{k=1}^N (X - A_k)$. Now let $a \in \bigcup_{k=1}^N (X - A_k)$. Then there exists some j such that $a \in X - A_j$, so $a \notin A_j$ and therefore $a \notin \bigcap_{k=1}^N$. This means that $a \in X - \left(\bigcap_{k=1}^N A_k\right)$ and so $\bigcup_{k=1}^N (X - A_k) \subset X - \left(\bigcap_{k=1}^N A_k\right)$. Therefore we have that

$$X - \left(\bigcap_{k=1}^N A_k\right) = \bigcup_{k=1}^N (X - A_k).$$

$\square$

Exercise 1.3. For a function $f : X \to Y$ and sets $A, B \subset Y$, we have that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proof. See MATH 200 final exam review sheet notes in the graph paper notebook. $\square$

Exercise 1.4. If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y)$ contains more than one point. Then there exist $x_1, x_2 \in X$ such that $x_1, x_2 \in f^{-1}(y)$ and $x_1 \neq x_2$. By the definition of $f^{-1}(y)$, we have that $f(x_1), f(x_2) \in \{y\}$, so $f(x_1) = y = f(x_2)$ and f is therefore not injective since $x_1 \neq x_2$. We have shown the contrapositive of the claim. $\square$

Exercise 1.5. If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y) = \emptyset$. Then for all $x \in X$, $f(x) \notin f^{-1}(y)$, and by the definition of $f^{-1}(y)$, this means that for all $x \in X$, $f(x) \neq y$, so f is not surjective. We have shown the contrapositive of the claim. $\square$

1.2 Cardinality and Countable Sets

Theorem 1.8. Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Proof. Let $S \subset \mathbb{N}$ be an arbitrary subset of $\mathbb{N}$. If S is finite, then we are done. If S is not finite, then it is infinite. Let $s_0 = \min S$, let $s_1 = \min(S - \{s_0\})$, and let $s_i = \min(S - \{s_0, \ldots, s_{i-1}\})$. Define the function $f : \mathbb{N} \to S$ by the following: $f(n) = s_n$. Suppose $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. Without loss of generality, assume that $n_1 < n_2$. Then $f(n_2) = \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\})$. Since $f(n_1) = s_{n_1} \notin S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}$, we have that $f(n_1) \neq \min(S - \{s_0, \dots, s_{n_1}, \dots, s_{n_2-1}\}) = f(n_2)$, which means that f is injective. Let $s \in S \subset \mathbb{N}$ be arbitrary. Then set $j = |\{r \in S : r < s\}| + 1 \in \mathbb{N}$. Then we have that

$$f(j) = s_j = \min(S - \{s_0, \dots, s_{|\{r \in S : r < s\}|}\}) = s$$

where the last equality follows from the fact that s must be the smallest element of the subset of S from which all elements smaller than s have been removed. Therefore f is surjective, and since it is also injective, f is a bijection, meaning that the cardinality of S is the same as the cardinality of $\mathbb{N}$. $\square$

Theorem 1.9. Every infinite set has a countable subset.

Proof. I think I need the axiom of choice here? I will return in the future! $\square$

Theorem 1.10. A set is infinite if and only if there is an injection from the set into a proper subset of itself.

Proof. I think I also need the axiom of choice here... $\square$

Theorem 1.11. The union of two countable sets is countable.

Proof. Let A and B be countable sets. There are two cases to consider. In the first case, the intersection of A and B is finite, and so there exists a bijection $h : A \cap B \to \{1, \dots, n\}$ for some $n \in \mathbb{N}$ where n is the size of the set $A \cap B$ If $A \cap B$ is empty, we instead use $n = 0$. Since $A \cap B$ is finite, $A - (A \cap B)$ is infinite, so there exists a bijection $f : A - (A \cap B) \to \mathbb{E}$, where $\mathbb{E}$ is the countable set containing all positive even integers greater than n. Similarly, there exists a bijection $g : B - (A \cap B) \to \mathbb{O}$, where $\mathbb{O}$ is the countable set containing all positive odd integers greater than n. Then we have that the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the second case, $A \cap B$ is infinite and there are three subcases. In the first subcase, one of $A - (A \cap B)$ or $B - (A \cap B)$ (assume without loss of generality that this is $A - (A \cap B)$) is finite. In this case, we use the same construction as earlier, since now $h : A - (A \cap B) \to \{1, \dots, n\}$

is a bijection for some $n \in \mathbb{N}$, $f : A \cap B \to \mathbb{E}$ is a bijection, and $g : B - (A \cap B) \to \mathbb{O}$ is a bijection. Then the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A \cap B \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A - (A \cap B) \end{cases}.$$

In the second subcase, $A - (A \cap B)$ and $B - (A \cap B)$ are finite. Then there are bijections $f : A - (A \cap B) \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $g : B - (A \cap B) \to \{n+1, \ldots n+m\}$ for some $m \in \mathbb{N}$. Since $A \cap B$ is countably infinite, there is a bijection $h : A \cap B \to \{n+m+1, n+m+2, \ldots\}$, the set of positive integers greater than $n+m$. Therefore $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the third subcase, all three sets are countably infinite. In this case, there is a bijection $f : A - (A \cap B) \to \{n \in \mathbb{N} : n = 3k, k \in \mathbb{Z}\}$, a bijection $g : B - (A \cap B) \to \{n \in \mathbb{N} : n = 3k+1, k \in \mathbb{Z}\}$, and a bijection $h : A \cap B \to \{n \in \mathbb{N} : n = 3k+2, k \in \mathbb{Z}\}$. Then we have that $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In all cases, we have that there exists a bijection $\varphi : A \cup B \to \mathbb{N}$, so $A \cup B$ is countable. $\square$

Theorem 1.12. The union of countably many countable sets is countable.

Proof. Let $\{B_i\}_{i \in \mathbb{N}}$ be a collection of countable sets. Then for $i \in \mathbb{N}$, define A_i recursively as $A_1 = B_1$ and $A_i = B_i - \bigcup_{k<i} B_k$ so that the union $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} B_i$ but the A_i are all disjoint. Now since the set of primes is countable, there exists a bijection $f : \mathbb{N} \to \{\text{primes}\}$. Define the map $g : \bigcup_{i \in \mathbb{N}} A_i \to \mathbb{N}$ by $g(a_{ij}) = f(i)^j$ where $a_{ij} \in A_i$ is the jth element of the set A_i. This map is well-defined since the sets A_i are disjoint. Suppose $g(a_{ij}) = g(a_{mn})$

for some $a_{ij}, a_{mn} \in \bigcup_{i\in\mathbb{N}} A_i$. Then $f(i)^j = f(m)^n$, so they must also divide each other, and because the $f(i)$ and $f(m)$ are prime, they must be equal and have the same powers, so $j = n$. Then $f(i) = f(m)$ implies that $i = m$ since f is a bijection. Therefore $a_{ij} = a_{mn}$, so g is an injection, meaning the cardinality of $\bigcup_{i\in\mathbb{N}} A_i$ is less than or equal to the cardinality of $\mathbb{N}$, meaning the countable union of countable sets is countable. $\square$

Theorem 1.14. The set of all finite subsets of a countable set is countable.

Proof. Let $A = \{a_i \mid i \in \mathbb{N}\}$ be a countable set. Then let N be a finite subset of $\mathbb{N}$ and define $C_N = \{a_i \mid i \in N\}$. Since N is finite, is has a maximum. Then for all $k \in \mathbb{N}$, define A_k to be the set $A_k = \{C_M \mid \max M = k\}$. Then $A_k \subset 2^{\{a_1,\dots a_k\}}$, the power set of a finite set, which is finite. Therefore for all $k \in \mathbb{N}$, A_k is finite, and if C is the set of all finite subset of A, then $C = \bigcup_{k\in\mathbb{N}} A_k$. Because the A_k are finite, they are countable, so by Theorem 1.12, C is countable. $\square$

1.3 Uncountable Sets and Power Sets

Exercise 1.17. If $A = \{a, b, c\}$, then $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 1.18. If the set A is finite, then its power set has cardinality $2^{|A|}$ ($|2^A| = 2^{|A|}$).

Proof. If a set A is finite, then it has cardinality $|A| = n \in \mathbb{N}$. We argue by induction on n that its power set 2^A has cardinality 2^n. For the base case $n = 1$, let A_1 be a set with one element. Then $A_1 = \{a\}$ and $2^{A_1} = \{\emptyset, a\}$, which has two elements, so indeed $|2^{A_1}| = 2^{|A_1|}$. Now for the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that all sets with cardinality k have power sets with cardinality 2^k. Then let A_{k+1} be an arbitrary set with cardinality $k+1$ so that $A_{k+1} = \{a_1, \dots, a_{k+1}\}$. Note that the set $A_k = \{a_1, \dots, a_k\}$ has cardinality k, and therefore has 2^k subsets. For each subset of A_k, call them A_{k_i} for $1 \le i \le 2^k$, the sets A_{k_i} and $A_{k_i} \cup \{a_{k+1}\}$ are subsets of A_{k+1}. Therefore there are $2 \cdot 2^k = 2^{k+1}$ subsets of A_{k+1}, and so $|2^{A_{k+1}}| = 2^{|A_{k+1}|}$. Therefore by induction, all finite sets A have $|2^A| = 2^{|A|}$. $\square$

Theorem 1.19. For any set A, there is an injection from A to 2^A.

Proof. Define $f : A \to 2^A$ as $f(a) = \{a\} \in 2^A$ since $\{a\} \subset A$. Let $x, y \in A$ such that $f(x) = f(y)$. Then we have that $\{x\} = \{y\}$, so $x = y$ and f is an injection, as required. $\square$

Theorem 1.20. If P is the set of all functions from a set A to the set $\{0, 1\}$, then $|P| = |2^A|$.

Proof. Define $\varphi : P \to 2^A$ as $\varphi(h) = h^{-1}(1)$. Let $f, g \in P$ be arbitrary such that $f \neq g$. Then there exists an $a \in A$ such that $f(a) \neq g(a)$. Without loss of generality, assume that $f(a) = 0$ and $g(a) = 1$. Then we have that $a \notin f^{-1}(1) = \varphi(f)$ and $a \in g^{-1}(1) = \varphi(g)$, so $\varphi(f) \neq \varphi(g)$ and therefore φ is injective. Now let $Y \in 2^A$ be arbitrary. Then define the function $f : A \to \{0, 1\}$ such that for $a \in A$,

$$f(a) = \begin{cases} 0 & a \notin Y \\ 1 & a \in Y \end{cases}.$$

Then since $Y \in 2^A$ means that $Y \subset A$, we have that

$$\varphi(f) = f^{-1}(1) = \{a \in A : f(a) = 1\} = \{a \in A : a \in Y\} = Y,$$

so φ is also surjective, making it a bijection. Since there is a bijection from P to 2^A, we have that $|P| = |2^A|$, as required. $\square$

Theorem 1.22. There is no surjection between a set A and its power set 2^A ($|A| \neq |2^A|$)

Proof. Let A be an arbitrary set and suppose for contradiction that $f : A \to 2^A$ is a surjection. Consider the set $X = \{a \in A : a \notin f(a)\}$. Since $X \subset A$, $X \in 2^A$, and since f is surjective, there exists an $a_0 \in A$ such that $f(a_0) = X$. There are two cases, both of which lead to a contradiction. In the first case, $a_0 \in X$, and so we have that $a_0 \notin f(a_0) = X$, a contradiction. In the second case, $a_0 \notin X$, so $a_0 \in f(a_0) = X$, also a contradiction. Since both cases lead to a contradiction, we see that the assumption that f is surjective was false, so there can be no surjection between A and 2^A, which means $|A| \neq |2^A|$. $\square$

1.4 The Schroeder-Bernstein Theorem

Theorem 1.28. The unit square and the unit interval have the same cardinality.

Proof. Let $a \in [0, 1]$ be arbitrary. Then it can be represented as

$$a = \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers with $a_n \in \{0, 1, \ldots, 9\}$ that does not end all in 9s (that is, if $a_i = 9$, then there exists $j > i$ such that $a_j \neq 9$). Now we claim that the function

$f : [0,1] \to [0,1] \times [0,1]$ is surjective, where

$$f\left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n\right) = \left(\sum_{n=1}^{\infty} a_{2n-1}\left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} a_{2n}\left(\frac{1}{10}\right)^n\right).$$

To show surjectivity, let $y \in [0,1] \times [0,1]$ be arbitrary. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ such that

$$y = \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right).$$

Let $x \in [0,1]$ be the point

$$x = \sum_{n=1}^{\infty}\left[a_n\left(\frac{1}{10}\right)^{2n-1} + b_n\left(\frac{1}{10}\right)^{2n}\right] = 0.a_1b_1a_2b_2a_3b_3\cdots = 0.c_1c_2c_3c_4c_5c_6\ldots.$$

Then

$$\begin{aligned} f(x) = f(0.c_1c_2c_3\ldots) &= \left(\sum_{n=1}^{\infty} c_{2n-1}\left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} c_{2n}\left(\frac{1}{10}\right)^n\right) \\ &= \left(\sum_{n=1}^{\infty} a_n\left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n\left(\frac{1}{10}\right)^n\right) = y. \end{aligned}$$

Therefore f is surjective. Now consider $g : [0,1] \times [0,1] \to [0,1]$ where $g(a,b) = a$. Let $y \in [0,1]$ be arbitrary and set $x = (y,0) \in [0,1] \times [0,1]$. Then $g(x) = g(y,0) = y$, and so g is surjective. Since there is a surjection from $[0,1]$ to $[0,1] \times [0,1]$ and a surjection from $[0,1] \times [0,1]$ to $[0,1]$, by the surjective version of the Schroeder Bernstein Theorem (1.26), we have that $|[0,1] \times [0,1]| = |[0,1]|$ as required. □

1.5 The Axiom of Choice

Exercise 1.32. Let X be a set and let P be the poset of all subsets of X partially ordered by inclusion. Let $p \in P$ be an element of the poset with $X \leq p$. Then we have that $X \subset p$. We also have that $p \subset X$ since $p \in P$ and P is the poset of all subsets of X. Therefore $p = X$, and so X is by definition a maximal element of P. Suppose there exists a set $Y \in P$ such that Y is also a maximal element of P. Then we have that $X \subset Y$ since Y is a maximal element, and also that $Y \subset X$ since $Y \in P$. Therefore $X = Y$, which shows that X is the unique maximal element of P. Now let $p \in P$ be an elemnet of the poset with $p \leq \emptyset$. Then we have that $p \subset \emptyset$, but also that $\emptyset \subset p$ since $\emptyset \subset x$ for all sets x. Therefore $p = \emptyset$, and so by definition we have that $\emptyset$ is a least element of the poset P. Now let $Z \in P$ be a least element. Then $Z \subset \emptyset$ since it is a least element, but as before, $\emptyset \subset Z$, which means $Z = \emptyset$ and so $\emptyset$ is the unique least element of the poset P.

Exercise 1.33. Let P be the poset ordered by cardinality with $P = \{\{0\}, \{1\}, \{0,1\}, \{1,2\}\}$. Then $\{0\}$ and $\{1\}$ are least elements and $\{0,1\}$ and $\{1,2\}$ are maximal elements.

Exercise 1.34. Consider $\mathbb{R}$ with the $\leq$ relation. The relation is reflexive, transitive, and antisymmetric, so it is a partial order on $\mathbb{R}$. We also have that for any two elements $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$, so they are comparable. This means that $\mathbb{R}$ is totally ordered with the relation $\leq$. However, $\mathbb{R}$ is not well-ordered, because there exist nonempty subsets of $\mathbb{R}$ that do not have least elements, for example, $(0,1) \subset \mathbb{R}$.

2 Topological Spaces: Fundamentals

2.2 Open Sets and the Definition of a Topological Space

Theorem 2.1. If $\{U_i\}_{i=1}^n$ is a finite collection of open sets in a topological space $(X, \mathcal{T})$, then $\bigcap_{i=1}^n U_i$ is open.

Proof. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. We argue by induction on n that $\bigcap_{i=1}^n U_i$ is open. For the base case $n = 1$, we have that U_1 is open. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $\bigcap_{i=1}^k U_i$ is open. Then since U_{k+1} is open, we have that $\bigcap_{i=1}^{k+1} U_i = \left(\bigcap_{i=1}^k U_i\right) \cap U_{k+1}$ is the intersection of two open sets and is therefore open. By induction, $\bigcap_{i=1}^n U_i$ is open for all $n \in \mathbb{N}$, in otherwords, the intersection of finitely many open sets is open. $\square$

Exercise 2.2. The above theorem does not show that the intersection of infinitely many open sets is open since the intersection of infinitely many open sets cannot be represented as $\bigcap_{i=1}^n U_i$ for any $n \in \mathbb{N}$.

Theorem 2.3. A set U is open in a topological space $(X, \mathcal{T})$ iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.

Proof. Let $(X, \mathcal{T})$ be a topological space with $U \in \mathcal{T}$. Let $x \in U$ be arbitrary and set $U_x = U$. Then U_x is open and $x \in U_x \subset U$, as required. Now let U be a set such that for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Then we have that $\bigcup_{x\in U} U_x$ is open, since the union of a collection of open sets is open. We also have that $y \in U$ implies that $y \in \bigcup_{x\in U} U_x$ since $y \in U_y$, so $U \subset \bigcup_{x\in U} U_x$. If $y \in \bigcup_{x\in U} U_x$, then $y \in U_z \subset U$ for some $z \in U$, so $\bigcup_{x\in U} U_x \subset U$, which means that the open set $\bigcup_{x\in U} U_x$ is the set U. Therefore U is open iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. $\square$

Exercise 2.4. First note that, vacuously, $\emptyset \in \mathcal{T}_{\text{std}}$, so the first property is satisfied. Next, consider the set $\mathbb{R}^n \subset \mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be an arbitrary point, and since $B(p,1) \subset \mathbb{R}^n$ and p was arbitrary, we have that $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$, so the second property is satisfied. For the third property, let $U, V \in \mathcal{T}_{\text{std}}$ be arbitrary open sets in $\mathbb{R}^n$. Let p be an arbitrary point in $U \cap V$. Then because $p \in U$, there exists an ε_1 such that $B(p,\varepsilon_1) \subset U$, and because $p \in V$, there exists an ε_2 such that $B(p,\varepsilon_2) \subset V$. Set $\varepsilon_p = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B(p,\varepsilon_p) \subset B(p,\varepsilon_1) \subset U$ and $B(p,\varepsilon_p) \subset B(p,\varepsilon_2) \subset V$, so we have that $B(p,\varepsilon_p) \subset U \cap V$. Therefore $U \cap V \in \mathcal{T}_{\text{std}}$ and the third property is satisfied. For the fourth property, let $\{U_i\}_{i\in\lambda}$ be a collection of sets $U_i \in \mathcal{T}_{\text{std}}$. Then let $p \in \bigcup_{i\in\lambda} U_i$ be arbitrary. Since p is in the union of all the U_i, we have that $p \in U_j$ for some $j \in \lambda$. Then since $U_j \in \mathcal{T}_{\text{std}}$, there exists an ε_p such that $B(p,\varepsilon_p) \subset U_j \subset \bigcup_{i\in\lambda} U_i$, and therefore $\bigcup_{i\in\lambda} U_i \in \mathcal{T}_{\text{std}}$, so the fourth property is satisfied and we have that $\mathcal{T}_{\text{std}}$ is indeed a topology on $\mathbb{R}^n$.

Exercise 2.6. The unit interval $(0,1) \subset \mathbb{R}$ is open in the standard topology on $\mathbb{R}$, open in the discrete topology, not open in the indiscrete topology, not open in the finite complement topology, and not open in the countable complement topology.

Exercise 2.7. In the topological space $(\mathbb{R}, \mathcal{T}_{\text{std}})$, the interval $(0,1)$ is open and for all $n \geq 0$, the set $U_n \subset (0,1)$ where

$$U_n = \left(\frac{2^n-1}{2^{n+1}}, \frac{2^n+1}{2^{n+1}}\right).$$

Then $\frac{1}{2} \in U_n$ for all n and therefore $\frac{1}{2} \in \bigcap_{n=0}^{\infty} U_n$. Let $x \in \mathbb{R}$ such that $x \neq \frac{1}{2}$. Then there exists some $m \geq 0$ such that $|x - \frac{1}{2}| > \frac{1}{2^{m+1}}$, which means that

$$x \notin \left(\frac{1}{2} - \frac{1}{2^{m+1}}, \frac{1}{2} + \frac{1}{2^{m+1}}\right) = \left(\frac{2^m-1}{2^{m+1}}, \frac{2^m+1}{2^{m+1}}\right) = U_m.$$

Since there exists an $m \geq 0$ such that $x \notin U_m$, we have that $x \notin \bigcap_{n=0}^{\infty} U_n$, and since x was arbitrary, we have that $\bigcap_{n=0}^{\infty} U_n = \{\frac{1}{2}\} \notin \mathcal{T}_{\text{std}}$. Therefore the infinite intersection of open sets is not necessarily open.

2.3 Limit Points and Closed Sets

Exercise 2.8. In the indiscrete topology on $\mathbb{R}$, the point 0 is in $\mathbb{R}$ but not $\emptyset$, so since $(\mathbb{R} - \{0\}) \cap (1,2) = (1,2) \neq \emptyset$, we have that $(U - \{0\}) \cap (1,2) \neq \emptyset$ for all open sets U containing 0. Therefore 0 is a limit point of $(1,2)$ in the indiscrete topology. In the finite complement topology, let $U \subset \mathbb{R}$ be an open set containing 0. Then suppose for contradiction that $(U - \{0\}) \cap (1,2) = \emptyset$. Then for all $p \in (1,2) \subset \mathbb{R}$, we have that $p \notin U - \{0\}$, which means that $p \in \mathbb{R} - (U - \{0\}) = (\mathbb{R} - U) \cup \{0\}$. Therefore $(1,2) \subset (\mathbb{R} - U) \cup \{0\}$, but this is a contradiction since U is open and therefore we have that an infinite set is a subset of a finite set. Hence it must be the case that $(U - \{0\} \cap (1,2)) \neq \emptyset$ for all U containing 0, and therefore 0 is a limit point of $(1,2)$. In the standard topology and in the discrete topology, the set $(-1,1)$ is an open set containing 0, but $(-1,1) \cap (1,2) = \emptyset$, so 0 is not a limit point of $(1,2)$.

Theorem 2.9. Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A iff there exists a neighborhood U of p such that $A \cap U = \emptyset$.

Proof. Suppose $p \notin A$ is not a limit point of A. Then there exists an open set U containing p (a neighborhood U of p) such that $(U - \{p\}) \cap A = \emptyset$. Since $p \notin A$, we have that $p \notin A \cap U$, and therefore $A \cap U = \emptyset$, as required. Now suppose there exists a neighborhood U of p such that $A \cap U = \emptyset$. Then since $p \notin A$, we have that $(U - \{p\}) \cap A = \emptyset$ as well, and therefore p is not a limit point of A. $\square$

Exercise 2.10. If p is an isolated point of A in a topological space $(X, \mathcal{T})$, then by the definition of an isolated point, we have that $p \in A$ but p is not a limit point of A. Therefore there exists an open set U such that $(U - \{p\}) \cap A = \emptyset$, and since $p \in A, U$, we have that

$$U \cap A = ((U - \{p\}) \cap A) \cup \{p\} \cap A = \emptyset \cup \{p\} = \{p\}.$$

Therefore if p is an isolated point of A, there exists an open set U such that $A \cap U = \{p\}$.

Exercise 2.11. Let $X = \{a, b, c, d\}$ be a set, let $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$ be a topology on X, and let $A = \{b, c\}$ be a set.

(1) $c \in A$ is a limit point of A since the only open set containing c is X, and we have that $(X - \{c\}) \cap A = \{b\} \neq \emptyset$.

(2) $d \notin A$ is a limit point of A also because X is the only open set containing d and $(X - \{d\}) \cap A = \{b, c\} \neq \emptyset$.

(3) $b \in A$ is an isolated point of A because it is in A but is not a limit point since $\{a, b\}$ is open but $(\{a, b\} - \{b\}) \cap A = \emptyset$.

(4) $a \notin A$ is not a limit point of A since $\{a\}$ is an open set but $(\{a\} - \{a\}) \cap A = \emptyset$.

Exercise 2.12. Let X be a set and let $\mathcal{T}$ be a topology on X. If the set X has a limit point p, then $p \in X$, so $\overline{X} = X$ and X is closed. If Now let $p \in X$ and let U be an open set containing p. Then $(U - \{p\}) \cap \emptyset = \emptyset$, so there are no limit points of the empty set, and therefore, vacuously, $\overline{\emptyset} = \emptyset$ and the empty set is closed.

(1) In the discrete topology, all sets are closed. Let A be a nonempty proper subset of X and let $p \in X$ be arbitrary. Then there exists a point $q \in X$ with $q \notin A$, and since all sets are open in the discrete topology, then set $\{p, q\}$ is a neighborhood of p that satisfies $(\{p, q\} - \{p\}) \cap A = \emptyset$. Thus we have shown that there are no limit points of A, and so vacuously, $\overline{A} = A$.

(2) In the indiscrete topology, only X and $\emptyset$ are closed. Let A be a nonempty propoer subset of X and let $p \in X$ such that $p \notin A$. Then p is a limit point of A since X is the only open set containing p and it satisfies $(X - \{p\}) \cap A \neq \emptyset$. However, $p \notin A$, so $\overline{A} \neq A$.

Theorem 2.13. For any topological space $(X, \mathcal{T})$, and $A \subset X$, the set $\overline{A}$ is closed, that is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

Proof. Let A be a set in a topological space $(X, \mathcal{T})$. Since the closure of a set contains all the points in the set, we have that $\overline{A} \subset \overline{\overline{A}}$. Now let $x \in \overline{\overline{A}}$ and let U be an arbitrary neighborhood of x. Either $x \in \overline{A}$ (in which case we're done), or x is a limit point of $\overline{A}$, in which case we have that $(U - \{x\}) \cap \overline{A} \neq \emptyset$. Since $(U - \{x\}) \cap \overline{A} \subset U \cap \overline{A}$, so $U \cap \overline{A}$ is nonempty in either case. Therefore there exists some $y \in U \cap \overline{A}$, which means we have a neighborhood U of y such that $y \in \overline{A}$. As before, either $y \in A$ or y is a limit point of A, in which case we have that $\emptyset \neq (U - \{y\}) \cap A \subset U \cap A$. Therefore for an arbitrary neighborhood U of $x \in \overline{\overline{A}}$, we have shown that $U \cap A$ is nonempty. Either $x \in A \subset \overline{A}$, or $x \notin A$ and therefore $x \notin U \cap A \neq \emptyset$. Since this intersection is nonempty, we have also that $(U - \{x\}) \cap A \neq 0$, but U was an arbitrary neighborhood of x, so we have shown that $(U - \{x\}) \cap A$ is nonempty for all neighborhoods U of x. Therefore x is a limit point of A, and so $x \in \overline{A}$. In both cases, $x \in \overline{A}$ and so $\overline{\overline{A}} \subset \overline{A}$, and since $\overline{A} \subset \overline{\overline{A}}$ also, we have that $\overline{\overline{A}} = \overline{A}$, as required. □

Theorem 2.14. For any topological space $(X, \mathcal{T})$, a subset $A \subset X$ is closed if and only if $X - A$ is open.

Proof. Suppose A is closed and let $x \in X - A$. Then since A is closed, it contains all its limit points and therefore x is not a limit point. This means there exists an open set U such that $U - \{x\} \cap A = \emptyset$. Let $y \in U - \{x\}$. Then $y \notin A$, so $y \in X - A$, which means $U - \{x\} \subset X$. Since x was an arbitrary element of $X - A$, by Theorem 2.3, we have that $X - A$ is open. Now suppose that $X - A$ is open. $A \subset \overline{A}$, and we will show that $\overline{A} \subset A$. Suppose for contradiction that $\overline{A} \not\subset A$. Then there exists an $a \in \overline{A} - A$, which means that a is a limit point of A, so all open intervals U containing a satisfy $U \cap A \neq \emptyset$ (by Theorem 2.9). In particular, since $X - A$ is open, we have that $(X - A) \cap A \neq \emptyset$, which is a contradiction. Therefore $\overline{A} = A$ and A is closed. $\square$

Corollary 2.14. If A is an open set, then $X - (X - A)$ is open, which means $X - A$ is closed.

Theorem 2.15. For any topological space $(X, \mathcal{T})$ with an open set $U \in \mathcal{T}$ and a closed set $A \in \mathcal{T}$, $U - A$ is open and $A - U$ is closed.

Proof. Since A is closed, $X - A$ is open, and so $U \cap (X - A)$ is also open since the intersection of two open sets is open. Then we have that $X - (U \cap (X - A))$ is closed by the corollary to Theorem 2.14. $X - (U \cap (X - A)) = X - (U - A)$ is closed, so $U - A$ is open, as claimed. The union of open sets is open, so we also have that $U \cup (X - A)$ is open. $U \cup (X - A) = X - (A - U)$ is open, so $A - U$ is closed, as claimed. $\square$

Theorem 2.16. Let $(X, \mathcal{T})$ be a topological space. Then:

(i) $\emptyset$ is closed.

(ii) X is closed.

(iii) The union of finitely many sets is closed.

(iv) Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

Proof. For (i) and (ii), see exercise 2.12. For (iii), let $\{A_i\}$, $1 \leq i \leq n$ for some $n \in \mathbb{N}$. Then for each A_i, $X - A_i$ is open, and we have that the intersection of finitely many open sets is open, so $\bigcap_{i=1}^{n}(X - A_i)$ is open. By the corollary to Theorem 2.14 and DeMorgan's Laws,

$$X - \left(\bigcap_{i=1}^{n} (X - A_i) \right) = X - \left(X - \bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} A_i$$

is closed. For (iv), let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then for each A_α, $X - A_\alpha$ is open, and we have that the union of a collection of open sets is open, so

$\bigcup_{\alpha\in\lambda}(X - A_\alpha)$ is open. Again by DeMorgan's Laws,

$$X - \left(\bigcup_{\alpha\in\lambda}(X - A_\alpha)\right) = X - \left(X - \bigcap_{\alpha\in\lambda} A_\alpha\right) = \bigcap_{\alpha\in\lambda} A_\alpha$$

is closed. $\square$

Exercise 2.19. (1) In $\mathbb{Z}$ with the finite complement topology, the set $\{0,1,2\}$ is not open, since $\mathbb{Z} - \{0,1,2\}$ is infinite. $\mathbb{Z} - \{0,1,2\}$, however, is open since $\{0,1,2\}$ is finite, and therefore $\mathbb{Z} - (\mathbb{Z} - \{0,1,2\}) = \{0,1,2\}$ is closed. The set of prime numbers has an infinite complement, so it is not open, but there are infinitely many prime numbers so it is also not closed. The set $\{n : |n| > 10\}$ has a finite complement, so it is open, but the set itself is infinite and therefore is not closed.

(2) In $\mathbb{R}$ with the standard topology, the set $(0,1)$ is open, and its limit points are 0 and 1, neither of which are in $(0,1)$, so it is not closed. The set $(0,1]$ is neither closed nor open, since it contains one of its limit points but not both. The set $[0,1]$ contains both limit points and is therefore closed, and it is not open. The set $\{0,1\}$ has no limit points so is vacuously closed, and it is not open. The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not open since there is no $\varepsilon > 0$ such that $(1-\varepsilon, 1+\varepsilon) \subset \{\frac{1}{n} : n \in \mathbb{N}\}$. Note that 0 is a limit point of this set since for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ and therefore $\frac{1}{n_0} \in ((-\varepsilon, \varepsilon) - \{0\}) \cap \{\frac{1}{n} : n \in \mathbb{N}\} \neq \emptyset$. Since $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, the set is not closed in $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

(3) In $\mathbb{R}^2$ with the standard topology, the set $C = \{(x,y) : x^2+y^2 = 1\}$ is not open since if $p \in C$ and $\varepsilon_p > 0$, then $p - \left(\frac{\varepsilon_p}{2}, \frac{\varepsilon_p}{2}\right)$ is in $B(p, \varepsilon_p)$ but not in C. If $p = (\cos\theta_0, \sin\theta_0) \in C$, then p is a limit point of C since the point $(\cos\theta_1, \sin\theta_1) \in C$ and if $|\theta_1 - \theta_0| < \arccos\left(1 - \frac{\varepsilon_p^2}{2}\right)$, then $(\cos\theta_1, \sin\theta_1) \in (B(p, \varepsilon_p) - \{p\}) \cap C \neq \emptyset$. If $p \notin C$, then set ε to be the distance from p to the nearest point of C. We have that $(B(p, \frac{\varepsilon}{2}) - \{p\}) \cap C = \emptyset$, so all points of C are limit points and all points not in C are not limit points, which means $\overline{C} = C$ and therefore C is closed. Let $D = \{(x,y) : x^2 + y^2 < 1\}$. Then C is the set of all the limit points of D, and so $C \cup D = \overline{D}$ is closed. Therefore, $\{(x,y) : x^2 + y^2 > 1\} = \mathbb{R}^2 - \overline{D}$ is open, and its limit points are also all in the set C and therefore this set is not closed. The set D is open since $D = B(0,1) \in \mathcal{T}_{\text{std}}$, and so the set $\{(x,y) : x^2 + y^2 \geq 1\} = \mathbb{R}^2 - D$ is closed. This set is not open since there is no ε such that $B(p, \varepsilon) \subset \{(x,y) : x^2 + y^2 \geq 1\}$ for $p \in \{(x,y) : x^2 + y^2 \geq 1\} \cap C$.

Theorem 2.20. For any set A in a topological space $(X, \mathcal{T})$, the closure of A is the intersection of all closed sets containing A, that is, $\overline{A} = \bigcap_{B \supset A, B \in \mathfrak{C}} B$, where $\mathfrak{C}$ is the set of all closed sets in $(X, \mathcal{T})$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$ and let A be a subset of X. Since its closure $\overline{A}$ is closed and $A \subset \overline{A}$, we have that $\overline{A} \in \mathfrak{C}$, and since for any sets M and N we have that $M \cap N \subset M, N$, we have that $\bigcap_{B \supset A, B \in \mathfrak{C}} B \subset \overline{A}$. To show equality, then, we need only show that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. Let $a \in \overline{A}$. There are two cases. In the first case, if $a \in A$, then $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$, which means that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$. In the second case, we have that $a \notin A$, which means that $a \in \overline{A} - A$ is a limit point of a. This means that for all open sets U with $a \in U$, we have that $(U - \{a\}) \cap A \neq \emptyset$. Since $a \notin A$, we have that $U \cap A = (U - \{a\}) \cap A \neq \emptyset$. Let $B_0 \in \mathfrak{C}$ with $A \subset B_0$ and suppose for contradiction that $a \notin B_0$. Since B_0 is closed, $X - B_0$ is open, and since $a \notin B_0$, $a \in X - B_0$, which means that $(X - B_0) \cap A \neq \emptyset$. Therefore there exists some $y \in (X - B_0) \cap A$. We have that $y \in A \subset B_0$, which means that $y \notin X - B_0$, a contradiction. We have reached a contradiction by assuming that there exists a set $B_0 \in \mathfrak{C}$ such that $A \subset B_0$ and $a \notin B_0$, which means that $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$. Therefore in both cases $a \in A$ and $a \in \overline{A} - A$, we have that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$, which means that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. This shows that $\overline{A}$ is indeed the intersection of all closed sets containing A, as required. $\square$

Exercise 2.21. Consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. In the discrete topology, this set is already closed and so is its own closure. In the indiscrete topology, only $\mathbb{R}$ and $\emptyset$ are closed, so the closure of H in the indiscrete topology is $\mathbb{R}$ itself. In the finite complement topology, let $p \in \mathbb{R}$ be an arbitrary point. Then let U be an arbitrary open set containing p. Since U is open, its complement is finite, which means there are only finitely many points not in $U - \{p\}$, so there must be infinitely many points in $(U - \{p\}) \cap H \neq \emptyset$. Since U was an arbitrary open set, p is a limit point, and since p was an arbitrary point in $\mathbb{R}$, we see that all points are limit points of H, which means that $\overline{H} = \mathbb{R}$. In the countable complement topology, H is closed since it contains countably many elements, so it is its own closure. In the standard topology, the only limit point of H is 0, so the closure of H is $\overline{H} = H \cup \{0\}$.

Theorem 2.22. Let A and B be subsets of a topological space X with topology $\mathcal{T}$. Then:

(1) $A \subset B$ implies $\overline{A} \subset \overline{B}$.

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) We have that $A \subset B \subset \overline{B}$, and by Theorem 2.20, we have that $\overline{A}$ is a subset of all closed sets containing A. Since $\overline{B}$ is a closed set containing A, we have that $\overline{A} \subset \overline{B}$, as required.

(2) Let $c \in \overline{A} \cup \overline{B}$. Without loss of generality, assume $c \in \overline{A}$. There are two cases. For the first case $c \in A$, we have that $c \in A \subset A \cup B \subset \overline{A \cup B}$. For the second case $c \in \overline{A} - A$, c is a limit point of A and therefore for all open sets U containing c satisfy $\emptyset \neq (U - \{c\}) \cap A \subset (U - \{c\}) \cap (A \cup B)$. Since U was an arbitrary open set, c is a limit point of $A \cup B$ and therefore $c \in \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Now let $d \in \overline{A \cup B}$ be arbitrary. Again there are two cases. In the first case, $d \in A \cup B$, so without loss of generality assume $d \in A \subset \overline{A} \subset \overline{A} \cup \overline{B}$. In the second case, $d \in \overline{A \cup B} - (A \cup B)$, so d is a limit point of $A \cup B$. This means that for all open sets U containing d, we have that $\emptyset \neq (U - \{d\}) \cap (A \cup B) = ((U - \{d\}) \cap A) \cup ((U - \{d\}) \cap B)$, so one of the sets on the right hand side is nonempty. Without loss of generality, assume $(U - \{d\}) \cap A \neq \emptyset$. Then d is a limit point of A, so $d \in \overline{A} \subset \overline{A} \cup \overline{B}$. Therefore $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ as well, and so the two sets are equal. □

Exercise 2.25. I'm not entirely sure but I believe the Cantor Set fits this description.

2.4 Interior and Boundary

Theorem 2.26. Let A be a subset of a topological space $(X, \mathcal{T})$. Then p is an interior point if and only if there exists an open set U such that $p \in U \subset A$.

Proof. Let $A \subset X$ be arbitrary and let $p \in A$ be an interior point. Then since A° is the union of all open subsets of A, A° is open as well, and so by Theorem 2.3, there exists an open set U such that $p \in U \subset A^\circ \subset A$ where $A^\circ \subset A$ follows from the fact that if $a \in A^\circ$, then $a \in \bigcup_{U \subset A, U \in \mathcal{T}} U$ and therefore there exists an open set U_0 such that $a \in U_0 \subset A$. Now let $p \in A$ be an arbitrary point such that there is an open set U with $a \in U \subset A$. Since U is open, $U \in \mathcal{T}$ and so $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U = A^\circ$. Thus, p is an interior point, as required. □

Exercise 2.27. If U is open in a topological space, then by Theorem 2.3, for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$, which means x is an interior point of U. If x is an interior point of U, then by Theorem 2.26, there exists a set U_x such that $x \in U_x \subset U$ and so U is open. Therefore U is open in a topological space if and only if every point of U is an interior point.

Lemma 2.28. Given a set A in a topological space $(X, \mathcal{T})$, the closure of A is $\overline{A} = X - (X - A)^\circ$ and the interior of A is $A^\circ = X - \overline{X - A}$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$. For all $B \in \mathfrak{C}$ such that $A \subset B$, we have that $X - B \subset X - A$, and since B is closed, $X - B$ is open. Given an open set U with $U \subset X - A$, $X - U \supset A$ is closed. This means that $\{B \in \mathfrak{C} : B \supset A\} = \{X - U : U \in \mathcal{T} \text{ s.t. } U \subset X - A\}$. Therefore we have that

$$\begin{aligned}\overline{A} = \bigcap_{B \supset A, B \in \mathfrak{C}} B = X - \left(X - \bigcap_{B \supset A, B \in \mathfrak{C}} B\right) = X - \left(\bigcup_{B \supset A, B \in \mathfrak{C}} (X - B)\right) \\ = X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} (X - (X - U))\right) = X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} U\right) = X - (X - A)^\circ,\end{aligned}$$

as required. The proof that $A^\circ = X - \overline{X - A}$ is similar □

Theorem 2.28. Let A be a subset of a topological space $(X, \mathcal{T})$. Then A°, ∂A, and $(X - A)^\circ$ are all disjoint and their union is X.

Proof. Let $a \in A^\circ$ and suppose for contradiction that $a \in \partial A = \overline{A} \cap \overline{X - A}$. Then either $a \in X - A$ or a is a limit point of $X - A$. In the first case, we have a clear contradiction since $a \in A^\circ \subset A$ cannot be in $X - A$. In the second case, every open set U containing a satisfies $(U - \{a\}) \cap (X - A) \neq \emptyset$. However, $a \in A^\circ$, so there exists an open set V such that $a \in V \subset A$, so we have that $\emptyset \neq (V - \{a\}) \cap (X - A) \subset A \cap (X - A) = \emptyset$, a contradiction. Therefore the sets A° and ∂A are disjoint, and a similar argument shows that ∂A and $(X - A)^\circ$ are disjoint as well. It remains to check that A° and $(X - A)^\circ$ are disjoint. Suppose not, so that there exists an $a \in A^\circ$ such that $a \in (X - A)^\circ$. By Theorem 2.26, there exists an open set U_1 containing a such that $U_1 \subset A$ and an open set U_2 containing a such that $U_2 \subset X - A$. Therefore we have that $a \in U_1 \cap U_2 \subset A \cap (X - A) = \emptyset$, a contradiction showing that A° and $(X - A)^\circ$ must be disjoint. Since $A^\circ = X - \overline{X - A}$ by Lemma 2.28, we have that

$$\begin{aligned}\overline{A} - A^\circ &= \overline{A} - (X - \overline{X - A}) \\ &= \overline{A} \cap (X - (X - \overline{X - A})) \\ &= \overline{A} \cap \overline{X - A} = \partial A,\end{aligned}$$

so we see that the closure of A is the disjoint union of the boundary of A and the interior of

A. Again using Lemma 2.28 ($\overline{A} = X - (X - A)^\circ$), we also have that

$$\begin{aligned} X &= \overline{A} \cup (X - \overline{A}) = A^\circ \cup \partial A \cup (X - \overline{A}) \\ &= A^\circ \cup \partial A \cup (X - (X - (X - A)^\circ)) \\ &= A^\circ \cup \partial A \cup (X - A)^\circ. \end{aligned}$$

Therefore we have that A°, ∂A, $(X - A)^\circ$ are disjoint sets whose union is X. □

Exercise 2.29. Again consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$. In the discrete topology, we have that $\overline{H} = H$ and $\overline{\mathbb{R} - H} = \mathbb{R} - H$, and so we have that $\partial N = \emptyset$. The interior of N is $N^\circ = \overline{H} - \partial H = H - \emptyset = H$. In the indiscrete topology, we have that $\overline{H} = \mathbb{R} = \overline{\mathbb{R} - H}$, so $\partial H = \mathbb{R}$ and $H^\circ = \overline{H} - \partial H = \mathbb{R} - \mathbb{R} = \emptyset$. In the finite complement topology, we saw in Exercise 2.21 that the closure of H was $\overline{H} = \mathbb{R}$ and by the same reasoning, $\overline{\mathbb{R} - H} = \mathbb{R}$. Therefore once again we have that $\partial H = \mathbb{R}$ and $H^\circ = \emptyset$, which makes sense since $\emptyset$ is the largest open set contained within H since H is countably infinite. In the countable complement topology, $H = \overline{H}$ is the disjoint union of H° and ∂H. If V is a nonempty open set in the countable complement topology, then $\mathbb{R} - V$ is countable, meaning that V is uncountable, so we cannot have $V \subset H$. The only open set U with $U \subset H$ is $U = \emptyset$. Therefore $H^\circ = \emptyset$ and $\partial H = H$. In the standard topology, no open set contains 1 and $\frac{1}{2}$ without also containing all points in the open set $(\frac{1}{2}, 1)$, which means that no open set is a subset of H. Therefore we have that $H^\circ = \emptyset$, and so $\partial H = \overline{H} = H \cup \{0\}$.

2.5 Convergence of Sequences

Theorem 2.30. Let A be a set in a topological space $(X, \mathcal{T})$, and let p be a point in X. If $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$.

Proof. Suppose for contradiction that $\{x_i\}_{i \in \mathbb{N}} \subset A$, $x_i \to p$, but $p \notin \overline{A}$. Then p is not a limit point of A, so there exists an open set U_0 containing p such that $(U_0 - \{p\}) \cap A = \emptyset$. Since $x_i \to p$ and $p \in U_0$, we have that there exists an $N \in \mathbb{N}$ such that $x_i \in U_0$ for all $i > N$. Since $x_i \in A$ for all $i \in \mathbb{N}$, $x_i \neq p$ for all $i \in \mathbb{N}$. Therefore, we have that for all $i > N$, $x_i \in (U_0 - \{p\}) \cap A = \emptyset$, a contradiction. Therefore if $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$. In particular, p is a limit point of A. □

Theorem 2.31. In the standard topology on $\mathbb{R}^n$, if p is a limit point of a set A, then there is a sequence of points in A that converges to p.

Proof. If p is a limit point of A, then $A \neq \emptyset$, so there exists a point $a \in A$. Set $\varepsilon = d(a,p) > 0$. Then for all $n \in \mathbb{N}$, $B(p, \frac{\varepsilon}{n})$ is open. Since p is a limit point and $p \in B(p, \frac{\varepsilon}{n})$, we have that $(B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A \neq \emptyset$. Since each of these sets is nonempty there exists an $a_n \in (B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A$ for all n. Define a sequence $(x_n)_{n \in N}$ such that $x_n = a_n$. Therefore $\{x_n\}_{n \in \mathbb{N}} \subset A$, so it only remains to show that the sequence converges to p. Let U be an open set containing p. Since U is open in the standard topology, $U = B(p, \varepsilon_p)$ for some $\varepsilon_p > 0$, and there exists and $N \in \mathbb{N}$ such that $\frac{\varepsilon}{N} < \varepsilon_p$. Let $i > N$. We have that

$$x_i \in B\left(p, \frac{\varepsilon}{i}\right) \subset B\left(p, \frac{\varepsilon}{N}\right) \subset B\left(p, \varepsilon_p\right),$$

which means that $x_i \in U$ for all $i > N$, and therefore $x_i \to p$, as required. □

Exercise 2.32. Consider $\mathbb{R}$ with the indiscrete topology. Let p be a point in $\mathbb{R}$ and set $x_n = n$. This sequence converges to p since if U is an open set containing p, then $U = \mathbb{R}$ and so we have that $x_n \in U$ for all $n \in \mathbb{N}$. Therefore $x_n \to p$, but p was arbitrary, so we see that this sequence converges to every point in $\mathbb{R}$.

3 Bases, Subspaces, Products: Creating New Spaces

3.1 Bases

Theorem 3.1. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

(1) $\mathcal{B} \subset \mathcal{T}$, and

(2) for every open set U, and point $p \in U$, there exists a set $V \in \mathcal{B}$ such that $p \in V \subset U$.

Proof. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then by definition, (1) is satisfied. Now let U be an arbitrary open set and let $p \in U$ be an arbitrary point. Since $\mathcal{B}$ is a basis, we have that $U = \bigcup_{V \in \lambda} V$ for some collection of sets $\lambda \subset \mathcal{B}$, and since $p \in U$, we have that $p \in V_0 \subset U$ for some set $V_0 \in \lambda$, so (2) is satisfied as well. Now suppose that $\mathcal{B}$ is a collection of sets satisfying (1) and (2). Let U_0 be an arbitrary open set and define λ to be the collection of sets $\lambda = \{V \in \mathcal{B} : V \subset U_0\} \subset \mathcal{B}$. Now let $p \in U_0$ be an arbitrary point in U_0. By (2), there exists a $V_0 \in \mathcal{B}$ such that $p \in V_0 \subset U_0$, which means $V_0 \in \lambda$. Therefore $p \in \bigcup_{V \in \lambda} V$, and so $U_0 \subset \bigcup_{V \in \lambda} V$. Now let p be an arbitrary point in $\bigcup_{V \in \lambda} V$. Then there exists a $V_0 \in \lambda \subset \mathcal{B}$ such that $p \in V_0$, and since $V_0 \in \lambda$, $V_0 \subset U_0$, so $p \in U_0$. Therefore $U_0 = \bigcup_{V \in \lambda} V$ since both sets are subsets of the other. Since any arbitrary open set U is the union of sets in the collection $\mathcal{B}$, $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

Exercise 3.2. $\mathcal{B}_1$ satisfies Theorem 3.1(1) since it consists only of open intervals and therefore $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Since U is open in $\mathcal{T}_{\text{std}}$, $U = (c-\varepsilon, c+\varepsilon)$ for some $c \in \mathbb{R}$ and $\varepsilon > 0$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $c - \varepsilon < a < p$ and $p < b < \varepsilon$. We have that $p \in (a, b) \subset U$ and $(a, b) \in \mathcal{B}_1$, so Theorem 3.2(2) is satisfied and therefore $\mathcal{B}_1$ is a basis.

Theorem 3.3. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if

(1) each point of X is in some element of $\mathcal{B}$, and

(2) if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof. If $\mathcal{B}$ is a basis for some topology on X, then $X = \bigcup_{B\in\mathcal{B}} B$, so if $p \in X$, then $p \in B_0$ for some $B_0 \in \mathcal{B}$. Therefore (1) is satisfied. Now let U and V be sets in $\mathcal{B}$ and let p be an arbitrary point in $U \cap V$. Since U and V and in $\mathcal{B}$, they are open, and therefore $U \cap V$ is open, so by Theorem 3.1, there exists a $W \in \mathcal{B}$ such that $p \in W \subset (U \cap V)$, satisfying (2). Now suppose $\mathcal{B}$ is a collection of subsets of X satisfying (1) and (2). Then $\emptyset$ is the empty union of sets in $\mathcal{B}$, and $X = \bigcup_{B\in\mathcal{B}} B$ by (1). Suppose U and V are in $\mathcal{B}$ and set $\lambda = \{W \in \mathcal{B} : W \subset (U \cap V)\} \subset \mathcal{B}$. If p is an arbitrary point of $U \cap V$, then by (2) there exists a W_0 such that $p \in W_0 \subset (U \cap V)$, so $p \in \bigcup_{W\in\lambda} W$. If p is an arbitrary point of $\bigcup_{W\in\lambda} W$, then there exists a $W_0 \in \mathcal{B}$ such that $p \in W_0 \subset (U \cap V)$. Therefore $U \cap V = \bigcup_{W\in\lambda} W$, so the intersection of two sets that are the unions of sets in $\mathcal{B}$ is also the union of sets in $\mathcal{B}$. Now let α be a collection of sets in $\mathcal{B}$. Then for each $\beta \in \alpha$, $\beta = \bigcup_{B\in\lambda} B$ for some collection of sets $\lambda \subset \mathcal{B}$. Then we have that $\bigcup_{B\in\alpha} B$ is the union of a collection of unions of sets in $\mathcal{B}$ and is therefore itself the union of sets in $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for some topology on X since the collection of sets that are unions of sets in $\mathcal{B}$ satisfies all four properties of a topology. $\square$

Exercise 3.4. Let $\mathcal{B}_{\text{LL}}$ be the set of subsets of $\mathbb{R}$ of the form $[a, b)$. Then if $x \in \mathbb{R}$, $x \in [x, x+1)$, so Theorem 3.3(1) is satisfied. Let U and V be arbitrary sets in $\mathcal{B}_{\text{LL}}$. Then there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $U = [a_1, b_1)$ and $V = [a_2, b_2)$. Let $x \in U \cap V$ be arbitrary. Then $a_1, a_2 \leq x < b_1, b_2$, so $p \in W = [\max\{a_1, a_2\}, \min\{b_1, b_2\}) = U \cap V$. Therefore Theorem 3.3(2) is satisfied and so $\mathcal{B}_{\text{LL}}$ is a basis for a topology on $\mathbb{R}$. ($\mathbb{R}$ together with this topology is the Sorgenfrey Line, $\mathbb{R}_{\text{LL}}$.)

Exercise 3.6. As discussed in Exercise 2.21, the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed in $\mathbb{R}$ with the standard topology, which means that $\mathbb{R} - H$ is not open in the standard topology. However, H contains countably many points, so $\mathbb{R} - H$ is open in the countable complement topology. Therefore we have that the standard topology on $\mathbb{R}$ is not finer than the countable complement topology. On the other hand, the set $(0, 1)$ is open in the standard topology, but not open in the countable complement topology, so we also have that the countable complement topology is not finer than the standard topology.

Exercise 3.7. Let $\mathbb{R}_{+00}$ be the set of all positive real numbers ($\mathbb{R}_+$) together with the points $0'$ and $0''$, and let $\mathcal{B}$ be the set of all intervals of the form (a, b), $(0, b) \cup \{0'\}$, or $(0, b) \cup \{0''\}$ for $a, b \in \mathbb{R}_+$.. We claim that $\mathcal{B}$ is the basis for some topology $\mathcal{T}$. For any $x \in \mathbb{R}_+$, $x \in (\frac{x}{2}, x+1)$, $0' \in (0, 1) \cup \{0'\}$, and $0'' \in (0, 1) \cup \{0''\}$, so every point in $\mathbb{R}_{+00}$ is in some element of $\mathcal{B}$. Now let U and V be arbitrary elements of $\mathcal{B}$ and let $p \in U \cap V$ be arbitrary. If one of U, V is of the form $(0, b_1) \cup \{0'\}$ and the other is of the form $(0, b_2) \cup \{0''\}$, and p is an arbitrary point in $U \cap V$, then set $W = (\frac{p}{2}, \min\{b_1, b_2\})$. Then we have that $p \in W \subset (U \cap V)$ and $W \in \mathcal{B}$. Otherwise, Set $W = (U \cap V)$. Then $p \in W \subset (U \cap V)$, so to show that $\mathcal{B}$ is the basis for some topology, it remains only to show that $W \in \mathcal{B}$. If one of U, V is of the form (a, b) for some $a, b \in \mathbb{R}_+$, then $U \cap V$ is of the same form and therefore in $\mathcal{B}$. For the remaining case to check, assume without loss of generality that $U = (0, b_1) \cup \{0'\}$ and $V = (0, b_2) \cup \{0'\}$ (the case for $0''$ is the same). Then $U \cap V = (0, \min\{b_1, b_2\}) \cup \{0'\} \in \mathcal{B}$. Therefore by Theorem 3.3, $\mathcal{B}$ is a basis for some topology on $\mathbb{R}_{+00}$. (This topological space is called the Double Headed Snake and will also be written as $\mathbb{R}_{+00}$.)

Exercise 3.8. In the Double Headed Snake, $\mathbb{R}_{+00}$, let p be an arbitrary point. If $p = 0'$ or $0''$, we claim that $\{p\}$ is closed. Without loss of generality, assume $p = 0'$. Then

$$\mathbb{R}_{+00} - \{p\} = (0, \infty) \cup \{0''\} = \bigcup_{n \in \mathbb{N}} \left((0, n) \cup \{0''\}\right).$$

Since $(0, n) \cup \{0''\} \in \mathcal{B}$, the basis for the Double Headed Snake given in Exercise 3.7, we have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and is therefore open in the Double Headed Snake, which means that $\{p\}$ is closed. Now if $p \neq 0', 0''$, we have that $p \in \mathbb{R}_+$. We have that

$$\begin{aligned}\mathbb{R}_{+00} - \{p\} &= (0,p) \cup (p,\infty) \cup \{0', 0''\} \\ &= ((0,p) \cup \{0'\}) \cup ((0,p) \cup \{0''\}) \cup \left(\bigcup_{p \in \mathbb{N}} (p, p+n)\right).\end{aligned}$$

Since $(p, p+n) \in \mathcal{B}$ for all $n \in \mathbb{N}$, we again have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and therefore is open in the Double Headed Snake, so $\{p\}$ is closed.

Suppose for contradiction that U and V are disjoint open sets in the Double Headed Snake such that $0' \in U$ and $0'' \in V$. Since $0' \in U$, and U is open, it is the union of sets in $\mathcal{B}$, so there exists a set of the form $(0, b_1) \cup \{0'\} \subset U$. Similarly, there exists a set of the form $(0, b_2) \cup \{0''\} \subset V$. Therefore $(0, \min\{b_1, b_2\}) \subset U \cap V$, so the sets are not disjoint since $b_1, b_2 \in \mathbb{R}_+$. This means it is impossible to have disjoint open sets each containing a different head of the snake.

Exercise 3.9. (1) In the topological space $\mathbb{R}_{\text{har}}$, the set $\mathbb{R} - H$ is open since it is the union of sets in the basis for $\mathbb{R}_{\text{har}}$:

$$\mathbb{R} - H = \bigcup_{n \in \mathbb{N}} ((-n, n) - H).$$

Since $\mathbb{R} - H$ is open in $\mathbb{R}_{\text{har}}$, H is closed and therefore $\overline{H} = H$.

(2) If $H^- = \{-\frac{1}{n} : n \in \mathbb{N}\}$, then $\overline{H^-} = H^- \cup \{0\}$.

(3) Nope!

Exercise 3.10. (1) Let $\mathbb{H}_{\text{bub}}$ be the upper half plane with the Sticky Bubble Topology. Let $Q = \{(x, 0) : x \in \mathbb{Q}\}$. Then

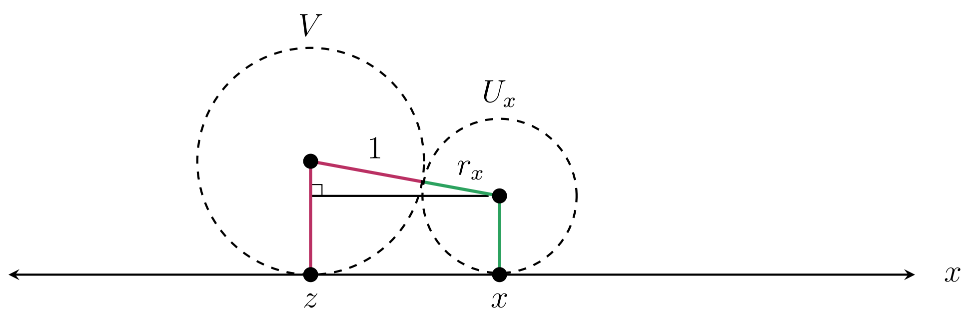
$$\mathbb{H} - Q = \bigcup_{x \in (\mathbb{R} - \mathbb{Q})} \left(\bigcup_{n \in \mathbb{N}} (B((x,n),n) \cup \{(x,0)\})\right)$$

is the union of sets in the basis for the Sticky Bubble Topology, so it is open. Therefore Q is closed and $\overline{Q} = Q$.

(2) This is similar to (1) since any subset of the x-axis can be treated the way $\mathbb{Q}$ was in the previous example.

(3) If A is a countable subset of the x-axis, and z is a point on the x-axis not in A, we wish to show that there are disjoint open sets U and V such that $A \subset U$ and $z \in V$. Set $V =$

$B((z,1),1) \cup \{(z,0)\}$. Now for all $x \in A$, set $r_x = (x-z)^2/4$ and $U_x = B((x,r_x),r_x) \cup \{(x,0)\}$. U_x is open for all x, so the set $U = \bigcup_{x\in A} U_x$ is also open. Clearly $A \subset U$, so it remains to check that $U \cap V \neq \emptyset$, which is the case provided none of the U_x bubbles overlap with the bubble V. Consider an arbitrary x and corresponding bubble U_x:



The distance from $(z,1)$ to (x,r_x) is $((1-r_x)^2 + (x-z)^2)^{\frac{1}{2}}$, and since U_x and V are open, we have that $U_x \cap V \neq \emptyset$ since:

$$\begin{aligned}
r_x &= \frac{(x-z)^2}{4} \\
\iff (x-z)^2 &= 4r_x \\
\iff 1 + r_x^2 + (x-z)^2 &= 1 + 4r_x + r_x^2 \\
\iff 1 - 2r_x + r_x^2 + (x-z)^2 &= 1 + 2r_x + r_x^2 \\
\iff (1-r_x)^2 + (x-z)^2 &= (1+r_x)^2
\end{aligned}$$

Taking square roots shows that the distance between the centers of U_x and V is the same as the sum of the radii of U_x and V, so the bubbles overlap at a single point that is not contained in either open set. This was the case for an arbitrary $x \in A$, so it is true for all $x \in A$. Therefore we have found two open sets U and V such that $A \subset U$, $z \in V$, and $U \cap V = \emptyset$. We did not use the fact that A contains countably many points, so this holds for all subsets of the x-axis.

Exercise 3.11. Let $\mathbb{Z}_{\text{arith}}$ be the integers $\mathbb{Z}$ together with a topology generated by a basis of arithmetic progressions (the basis $\mathcal{B}$ is the collection of all sets of the form $\{az+b : z \in \mathbb{Z}\}$ for $a, b \in \mathbb{Z}$ with $a \neq 0$). To show that this is indeed a basis for some topology on $\mathbb{Z}$, note that $\mathbb{Z} = \{1 \cdot z + 0 : z \in \mathbb{Z}\}$ is itself in $\mathcal{B}$, so all points in $\mathbb{Z}$ are in some element of $\mathcal{B}$. Now let U and V be arithmetic progressions in $\mathcal{B}$ and let $z_0 \in U \cap V$ be arbitrary. Then $U = \{a_1 z + b_2 : z \in \mathbb{Z}\}$ and $V = \{a_2 z + b_2 : z \in \mathbb{Z}\}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Now

$z_0 \in U \cap V$, then we have that $a_1 k_1 + b_1 = z_0 = a_2 k_2 + b_2$ for some $k_1, k_2 \in \mathbb{Z}$. Then $z_0 \in W = \{\text{lcm}(a_1, a_2)z + z_0 : z \in \mathbb{Z}\} \in \mathcal{B}$. We also have that $W \subset U \cap V$ since if $w_0 = \text{lcm}(a_1, a_2)k_0 + z_0 \in W$, then

$$a_1 \left(k_1 + \frac{a_2 k_0}{\gcd(a_1, a_2)} \right) + b_1 = a_1 k_1 + b_1 + \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 = \text{lcm}(a_1, a_2)k_0 + z_0$$
$$= w_0 = \text{lcm}(a_1, a_2)k_0 + z_0 = \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 + a_2 k_2 + b_2 = a_2 \left(k_2 + \frac{a_1 k_0}{\gcd(a_1, a_2)} \right) + b_2,$$

and w_0 is in both arithmetic progressions. Therefore $\mathbb{Z}_{\text{arith}}$ is a topological space generated by the basis $\mathcal{B}$.

Exercise 3.12. Consider the topological space $\mathbb{Z}_{\text{arith}}$ and note that if U is an open set, then since it is the union of infinite arithmetic progressions, it is itself infinite. Note also that for a prime p and $a = 1, 2, \ldots, p-1$, we have that $\{pz + a : z \in \mathbb{Z}\} \in \mathcal{B}$ is an arithmetic progression. Let $p\mathbb{Z}$ denote the set $\{pz : z \in \mathbb{Z}\}$. Then we have that

$$\mathbb{Z} - p\mathbb{Z} \;=\; \bigcup_{a=1,\ldots,p-1} \{pz + a : z \in \mathbb{Z}\}$$

is the union of open sets and is therefore open, meaning that $p\mathbb{Z}$ itself is closed. Let P denote the set $P = \bigcup_{\text{primes } p} p\mathbb{Z}$, and suppose for contradiction that there are finitely many primes. Then P is the union of finitely many closed sets and is therefore itself closed. Note that P contains all numbers that are integer multiples of a prime, so the only numbers not contained in P are -1 and 1, that is, $\mathbb{Z} - P = \{-1, 1\}$. But this set is open since P is closed, which is a contradiction since $\mathbb{Z} - P$ is finite. Therefore there are infinitely many primes.

3.2 Subbases

Exercise 3.13. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Then if U is in an open set in $(X, \mathcal{T})$, it is the union of sets in $\lambda \subset \mathcal{B}$. For each set $S \in \lambda$, S is the trivial intersection of itself, and this intersection is finite. Since $\mathcal{B}$ is a basis for a the topology $\mathcal{T}$, the finite intersections of sets in $\mathcal{B}$ is in $\mathcal{T}$. Therefore every open set in $\mathcal{T}$ can be generated by taking the union of sets that are themselves the finite intersections of sets in $\mathcal{B}$ and so $\mathcal{B}$ is a subbasis.

Exercise 3.14. Consider $\mathbb{R}$ together with the standard topology. Recall that $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis and note that $(a, b) = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x < b\}$ and this is a finite intersection. Therefore every set in the basis is the finite intersection of sets in $\mathscr{S}$, the set of rays of the form $\{x \in \mathbb{R} : x < x_0\}$ and $\{x \in \mathbb{R} : x_0 > x\}$ for some $x_0 \in \mathbb{R}$. Therefore $\mathscr{S}$ is a subbasis for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

Theorem 3.16. Let X be a set and let $\mathscr{S}$ be a collection of subsets of X. Then $\mathscr{S}$ is a subbasis for some topology $\mathcal{T}$ on X if and only if every point of X is contained in some element of $\mathscr{S}$.

Proof. Suppose $\mathscr{S}$ is a subbasis for a topology $\mathcal{T}$ on X and let $x \in X$ be arbitrary. Since $\mathscr{S}$ is a subbasis, the set $\mathcal{B}$ of finite intersections of elements of $\mathscr{S}$ is a basis, and therefore $x \in B_0$ for some $B_0 \in \mathcal{B}$. Then since B_0 is the finite intersection of elements in $\mathscr{S}$, $x \in S_0$ for some $S_0 \in \mathscr{S}$. Now suppose that for all $x \in X$, there exists some $S_0 \in \mathscr{S}$ such that $x \in S_0$. Then if $\mathcal{B}$ is the set of finite intersections of elements of $\mathscr{S}$, $x \in S_0 \in \mathcal{B}$, so Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary and let $p \in U \cap V$. Then $U = \bigcap_{S \in \lambda_1} S$ and $V = \bigcap_{S \in \lambda_2} S$ where λ_1 and λ_2 are collections of sets in $\mathscr{S}$. Therefore

$$p \in \bigcap_{S \in \lambda_1 \cap \lambda_2} S \subset U \cap V$$

since $\lambda_1 \cap \lambda_2 \subset \lambda_1, \lambda_2$. Therefore Theorem 3.3(2) is satisfied as well since $\bigcap_{S \in \lambda_1 \cap \lambda_2} S \in \mathcal{B}$, and so $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, which also means that $\mathscr{S}$ is a subbasis for this topology. $\square$

Exercise 3.17. Let $\mathscr{S}$ be the set of all subsets of $\mathbb{R}$ of the form $\{x \in \mathbb{R} : x < a\}$ or $\{x \in \mathbb{R} : a \leq x\}$. We claim that this is a subbasis for the lower limit topology on $\mathbb{R}$, $\mathcal{T}_{\text{LL}}$. If $U \in \mathscr{S}$, then there are two cases. In the first case, $U = \{x \in \mathbb{R} : x < a\} = \bigcup_{n \in \mathbb{N}} [a - n, a) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. In the second case, $U = \{x \in \mathbb{R} : a \leq x\} = \bigcup_{n \in \mathbb{N}} [a, a + n) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. Both inclusions in $\mathcal{T}_{\text{LL}}$ come from the fact that all open sets of the Sorgenfrey Line are the union of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Since any arbitrary $U \in \mathscr{S}$ has $U \in \mathcal{T}_{\text{LL}}$, we have that $S \subset \mathcal{T}_{\text{LL}}$ and therefore Theorem 3.15(1) is satisfied. Now suppose U is an open set of the Sorgenfrey Line and let $p \in U$ be arbitrary. Then $U = \bigcup_{W \in \lambda} W$ where $\lambda \subset \mathcal{B}_{\text{LL}}$ is a collection of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Therefore we have that $p \in W_0 = [a_0, b_0)$ for some $W_0 \in \lambda$. Since $W_0 = \{x \in \mathbb{R} : a_0 \leq x\} \cap \{x \in \mathbb{R} : x < b_0\}$ is the finite intersection of sets in $\mathscr{S}$ and $W_0 \subset U$, we have satisfied Theorem 3.15(2), meaning that $\mathscr{S}$ is indeed a subbasis for the lower limit topology on $\mathbb{R}$, as claimed.

3.3 Order Topology

Exercise 3.19. Consider $\mathbb{R}$ with the order topology from $\leq$. It has a basis $\mathcal{B}$ containing sets of the form $\{x \in \mathbb{R} : x < a\}$, $\{x \in \mathbb{R} : a < x\}$, and $\{x \in \mathbb{R} : a < x < b\}$. A set U is open in the standard topology on $\mathbb{R}$ if for each point $p \in U$, there is an $\varepsilon_p > 0$ such that $(p-\varepsilon_p, p+\varepsilon_p) \subset U$. Note that this is the case for every set of the forms contained in $\mathcal{B}$ (for the first and second forms, $\varepsilon_p = |a-p|$, and for the third form, $\varepsilon_p = \min\{p-a, b-p\}$), so $\mathcal{B} \subset \mathcal{T}_{\text{std}}$, satisfying Theorem 3.1(1). If $U \subset \mathbb{R}$ is open in $\mathcal{T}_{\text{std}}$ and $p \in U$, then since there exists $\varepsilon_p > 0$ such that $p \in (p-\varepsilon_p, p+\varepsilon_p) \subset U$ and $(p-\varepsilon_p, p+\varepsilon_p) \in \mathcal{B}$, Theorem 3.1(2) is satisfied as well and therefore $\mathcal{B}$ is a basis for the standard topology as well, so the order topology on $\mathbb{R}$ with $\leq$ is the standard topology.

Exercise 3.21. $A = \{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\}$ has closure $\overline{A} = A \cup \{(0,1)\}$.
$B = \{\left(1-\frac{1}{n}, \frac{1}{2}\right) : n \in \mathbb{N}\}$ has closure $\overline{B} = B \cup \{(1,0)\}$.
$C = \{(x,0) : 0 < x < 1\}$ has closure $\overline{C} = \{(x,0) : 0 < x \leq 1\} \cup \{(x,1) : 0 \leq x < 1\}$.
$D = \{\left(x, \frac{1}{2}\right) : 0 < x < 1\}$ has closure $\overline{D} = D \cup \{(x,0) : 0 < x \leq 1\} \cup \{(x,1) : 0 \leq x < 1\}$.
$E = \{\left(\frac{1}{2}, y\right) : 0 < y < 1\}$ has closure $\overline{E} = E \cup \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right)\right\}$

3.4 Subspaces

Theorem 3.25. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $\mathcal{T}_Y$ is indeed a topology on Y.

Proof. Note that $\emptyset \in \mathcal{T}_Y$ since $\emptyset = \emptyset \cap Y$ and $\emptyset \in \mathcal{T}$. Similarly, $Y \in \mathcal{T}_Y$ since $Y = X \cap Y$ and $X \in \mathcal{T}$. Now let $A, B \in \mathcal{T}_Y$ be arbitrary. Then there exist sets $V_A, V_B \in \mathcal{T}$ such that $A = V_A \cap Y$ and $B = V_B \cap Y$ and we have that

$$A \cap B = (V_A \cap Y) \cap (V_B \cap Y) = (V_A \cap V_B) \cap Y \in \mathcal{T}_Y$$

since $V_A \cap V_B$ is the (finite) intersection of sets in the topology $\mathcal{T}$ and is therefore itself in the topology $\mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in \lambda}$ be a collection of sets in $\mathcal{T}_Y$. Then for each $\alpha \in \lambda$, there exists a set $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$ and we have that

$$\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (V_\alpha \cap Y) = \left(\bigcup_{\alpha \in \lambda} V_\alpha\right) \cap Y \in \mathcal{T}_Y$$

since $\bigcup_{\alpha \in \lambda} V_\alpha$ is the union of sets in $\mathcal{T}$ and is therefore itself in $\mathcal{T}$. Thus we have that $\emptyset, Y \in \mathcal{T}_Y$ and that $\mathcal{T}_Y$ contains the finite intersections of sets in $\mathcal{T}_Y$ and the (possibly infinite) unions of sets in $\mathcal{T}_Y$ and is therefore a topology on Y. $\square$

Exercise 3.26. Taking $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$, we see that the set $\left[\frac{1}{2}, 1\right)$ is closed in Y. This is because the set $\left(-1, \frac{1}{2}\right)$ is open in $\mathbb{R}_{\text{std}}$ and therefore $\left(-1, \frac{1}{2}\right) \cap Y = \left[0, \frac{1}{2}\right)$ is open in Y, so $Y - \left[0, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right)$ is closed in Y.

Exercise 3.27. If Y is a subspace of a topological space $(X, \mathcal{T})$, it is not necessarily the case that every subset of Y that is open in Y is open in $(X, \mathcal{T})$. As in Exercise 3.26, $Y = [0, 1)$ is a subspace of $\mathbb{R}_{\text{std}}$ and $\left[0, \frac{1}{2}\right)$ is open in Y. However, it is neither open nor closed in $\mathbb{R}_{\text{std}}$.

Theorem 3.28. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

Proof. Let $C \subset Y$ be closed in $(Y, \mathcal{T}_Y)$. This is the case if and only if $Y - C$ is open in $(Y, \mathcal{T}_Y)$, which is the case if and only if there exists a set $V \subset X$, open in $(X, \mathcal{T})$, such that $Y - C = V \cap Y$. This then is the case if and only if

$$C = Y - (V \cap Y) = Y - V = Y \cap (X - V) = Y \cap D$$

where $D = X - V$ is closed in $(X, \mathcal{T})$ since V is open in $(X, \mathcal{T})$. All implications here go in both directions, so both directions of the proof are complete. $\square$

Theorem 3.30. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$ that has basis $\mathcal{B}$. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Proof. Let $A_0 \in \mathcal{B}_Y$ be arbitrary. Then there exists $B_0 \in \mathcal{B}$ such that $A_0 = B_0 \cap Y$. Since $B_0 \in \mathcal{B}$, it is open in the topological space $(X, \mathcal{T})$. Since $\mathcal{T}_Y = \{U : U = V \cap Y \text{ for some } V \in \mathcal{T}\}$, we have that $A_0 \in \mathcal{T}_Y$ and so Theorem 3.1(1) is satisfied. Now let U be an open set in $(Y, \mathcal{T}_Y)$ and let $p \in U$ be an arbitrary point. Since $U \in \mathcal{T}_Y$, there exists a set $W \in \mathcal{T}$ such that $U = W \cap Y$, which means that $p \in Y$ and $p \in W$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, by Theorem 3.1(2), there exists a set $V \in \mathcal{B}$ such that $p \in V \subset W$. Then since $V \in \mathcal{B}$, $V \cap Y \in \mathcal{B}_Y$ and we have that

$$p \in V \cap Y \subset W \cap Y = U,$$

which satisfies Theorem 3.1(2), meaning that $\mathcal{B}_Y$ is indeed a basis for $\mathcal{T}_Y$. $\square$

3.5 Product Spaces

Exercise 3.32. Let X and Y be topological spaces and let $\mathcal{B}$ be the set of all cartesian products of open sets $U \subset X$ and $V \subset Y$. Then $p = (a, b)$ is a point of $X \times Y$, then p is in some element of $\mathcal{B}$ since X is open in X and Y is open in Y (that is, $X \times Y \in \mathcal{B}$). This means that Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary. Then there exist sets X_1, X_2 open in X and Y_1, Y_2 such that $U = X_1 \times Y_1$ and $V = X_2 \times Y_2$. Now let $p \in U \cap V$ be arbitrary. Then we have that $p \in U \cap V \subset U \cap V$ and also that

$$U \cap V = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2) \in \mathcal{B}$$

since $X_1 \cap X_2$ is open in X and $Y_1 \cap Y_2$ is open in Y. Therefore Theorem 3.3(2) is also satisfied and we have that $\mathcal{B}$ is indeed the basis for some topology on $X \times Y$.

Exercise 3.34. The product of closed sets is closed in the product topology. Let X and Y be topological spaces, let A be closed in X and let B be closed in Y. Then there exists an open set $A_0 \subset X$ such that $A = X - A_0$ and there exists an open set $B_0 \subset Y$ such that $B = Y - B_0$ and we have that

$$\begin{aligned}
A \times B &= (X - A_0) \times (Y - B_0) \\
&= (X \times (Y - B_0)) - (A_0 \times (Y - B_0)) \\
&= ((X \times Y) - (X \times B_0)) - ((A_0 \times Y) - (A_0 \times B_0)) \\
&= ((X \times Y) \cup (A_0 \times B_0)) - ((X \times B_0) \cup (A_0 \times Y)) \\
&= (X \times Y) - ((X \times B_0) \cup (A_0 \times Y)).
\end{aligned}$$

The sets $X \times B_0$ and $A_0 \times Y$ are both basic open sets in $X \times Y$ since A_0 is open in X and B_0 is open in Y. Since the union of open sets is open, we have that $A \times B$ is the complement of open sets and is therefore closed. A and B were arbitrary closed sets in arbitrary topological spaces, so we have shown that the product of closed sets is closed in the product topology in general.

Theorem 3.35. The product topology on $X \times Y$ has a subbasis $\mathscr{S}$ that contains the inverse images of open sets under the projection functions, that is,

$$\mathscr{S} = \{\pi_X^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ is open in } Y\}.$$

Proof. Let S_0 be a set in $\mathscr{S}$. There are two cases. In the first, S_0 is of the form $S_0 = \{\pi_X^{-1}(U) : U \text{ is open in } X\} = U \times Y$ and is therefore a basic open set in the product space. In the second, S_0 is of the form $S_0 = \{\pi_Y^{-1}(V) : V \text{ is open in } Y\} = X \times V$ and is also a basic open set in the product space. Therefore S_0 is open in all cases and so $\mathscr{S} \subset \mathcal{T}$ where $\mathcal{T}$ is the product topology. Thus Theorem 3.15(1) is satisfied. Now let W be an open set in the product space and let $p \in W$ be an arbitrary point. Then since W is open, it is the union of sets of the form $U \times V$ where U is open in X and V is open in Y. This means there are sets $U_0 \subset X$ and $V_0 \subset Y$ such that $p \in U_0 \times V_0 \subset W$. All that remains to satisfy Theorem 3.15(2) is to show that $U_0 \times V_0$ is the finite intersection of sets in $\mathscr{S}$, which is the case since $U_0 \times V_0 = \pi_X^{-1}(U_0) \cap \pi_Y^{-1}(V_0)$. Therefore $\mathscr{S}$ is indeed a subbasis for the product topology on $X \times Y$. $\square$

Exercise 3.36. Let W be an open set in $\mathbb{R}^2_{\text{std}}$. Then $W = B(p, \varepsilon_p)$ for some $p \in \mathbb{R}^2$ and $\varepsilon_p > 0$. Now let $x \in W$ be arbitrary. Suppose $p = (p_1, p_2)$ and $x = (x_1, x_2)$. Set

$$U = \left(x_1 - \left|x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1\right)\right|, x_1 + \left|x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1\right)\right|\right)$$

and

$$V = \left(x_2 - \left|x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2\right)\right|, x_2 + \left|x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2\right)\right|\right).$$

(See Desmos sketch) Then we have that $x \in U \times V \subset W$, and since $U \times V$ is an open set with the product topology on $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$, we have that W is an open set in this topology as well by Theorem 2.3 since x was an arbitrary point of W. Then since W was an arbitrary open set under the standard topology, it follows that the standard topology on $\mathbb{R}^2$ is a subset of the product topology on $\mathbb{R}^2$.

Now let W be an open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology and let $p \in W$ be arbitrary. Then since W is open, it is the union of sets of the form $U \times V$ where U and V are open sets in $\mathbb{R}_{\text{std}}$. This means that there exist some U_0 and V_0 open in $\mathbb{R}_{\text{std}}$ such that $p \in U_0 \times V_0 \subset W$. Then there exist intervals (a_x, b_x) and (a_y, b_y) such that $\pi_{U_0}(p) \in (a_x, b_x)$ and $\pi_{V_0}(p) \in (a_y, b_y)$. Therefore we have that $p \in B(p, \varepsilon) \subset U_0 \times V_0 \subset W$ where

$$\varepsilon = \min\{|a_x - \pi_{U_0}(p)|, |b_x - \pi_{U_0}(p)|, |a_y - \pi_{V_0}(p)|, |b_y - \pi_{V_0}(p)|\}.$$

Since this $B(p, \varepsilon)$ is an open set in $\mathbb{R}^2_{\text{std}}$, by Theorem 2.3 it follows that W is also open in $\mathbb{R}^2_{\text{std}}$, and since W was an arbitrary open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology, we have that this topology on $\mathbb{R}^2$ is a subset of the standard topology on $\mathbb{R}^2$. Since each topology is a subset of the other, they are equal, meaning that product topology on $\mathbb{R}^2$ from $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is the same as the standard topology on $\mathbb{R}^2$.

Theorem 3.37. The product topology on $\prod_{\alpha \in \lambda} X_\alpha$ has a basis containing all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.

Proof. Since $\mathscr{S}$, the collection of sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in $(X_\beta, \mathcal{T}_\beta)$, is a subbasis for the product topology $\prod_{\alpha \in \lambda} X_\alpha$, the set of finite intersections of elements of $\mathscr{S}$ is therefore a basis for the product topology. Let W be a set in the collection described in the theorem statement. Then there exists a finite index set $\gamma \subset \lambda$ such that $W = \prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all $\alpha \in \lambda - \gamma$. Then since for some $\beta \in \gamma$,

$\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha\in\lambda-\{\beta\}} X_\alpha$, we have that $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) = U_{\beta_1} \times U_{\beta_2} \times \prod_{\alpha\in\lambda-\{\beta_1,\beta_2\}} X_\alpha$ and therefore $W = \prod_{\alpha\in\gamma} U_\alpha \times \prod_{\alpha\in\lambda-\gamma} X_\alpha = \bigcap_{\alpha\in\gamma} \pi_\alpha^{-1}(U_\alpha)$ is the finite intersection of elements of $\mathscr{S}$. Similarly, any finite intersection of elements of $\mathscr{S}$ is a product of the form described in the theorem statement, and so by the definition of a subbasis, the collection described in the theorem statement is indeed a basis for the product topology. $\square$

Exercise 3.41. Let U be an open set in the box topology (or product topology) on $\mathbb{R}^\omega$ containing $\mathbf{0} := (0, 0, 0, \dots)$. Then there is a basic open set V such that $\mathbf{0} \in V \subset U$. Since V is a basic open set, $V = \prod_{i\in\mathbb{N}} V_i$ where V_i is open in $\mathbb{R}_{\text{std}}$ for all $i \in \mathbb{N}$ (and in the case of the product topology, $V_i = \mathbb{R}_{\text{std}}$ for all but finitely many $i \in \mathbb{N}$). Since $\mathbf{0} \in V$, $0 \in V_i$ for all $i \in \mathbb{N}$. $V_i \in \mathcal{T}_{\text{std}}$ means there exists an $\varepsilon_i > 0$ such that $(-\varepsilon_i, \varepsilon_i) \subset V_i$ for all $i \in \mathbb{N}$. Define $a_i = \varepsilon_i/2 \in V_i$. Therefore the sequence $(a_i)_{i\in\mathbb{N}}$ is in $\prod_{i\in\mathbb{N}} V_i = V \subset U$. We also have that $a_i = \varepsilon_i/2 > 0$ for all $i \in \mathbb{N}$, which means $(a_i)_{i\in\mathbb{N}} \neq \mathbf{0}$ and $(a_i)_{i\in\mathbb{N}} \in A$. Therefore $(a_i)_{i\in\mathbb{N}} \in (U - \{\mathbf{0}\}) \cap A \neq \emptyset$, and since U was an arbitrary open set, $\mathbf{0}$ is a limit point of A in both the box topology and product topology on $\mathbb{R}^\omega$.

We claim that the sequence $(a_j)_{j\in\mathbb{N}}$ defined by $a_j = (1/j, 1/j, \dots)$ converges to $\mathbf{0}$ in $\mathbb{R}^\omega$ with the product topology. Let U be an open set containing $\mathbf{0}$, and let V be a basic open set such that $\mathbf{0} \in V \subset U$. Then $V = \prod_{i\in\mathbb{N}} V_i$ where V_i is open in $\mathbb{R}_{\text{std}}$ for all $i \in \mathbb{N}$ and $V_i = \mathbb{R}_{\text{std}}$ for all but finitely many $i \in \mathbb{N}$. Define $F = \{i \in \mathbb{N} \mid V_i \neq \mathbb{R}_{\text{std}}\}$. If F is empty, then V is the entire space $\mathbb{R}^\omega$, and since $V \subset U$, U is also the entire space $\mathbb{R}^\omega$. Set $N = 1$. Then for all $n > N$, $a_n \in U = \mathbb{R}^\omega$. Now if F is nonempty, $0 \in V_i$ for all $i \in F$ means there exists an ε_i such that $(-\varepsilon_i, \varepsilon_i) \subset V_i$. Define $s = \min\{\varepsilon_i | i \in F\}$, which exists because F is finite and nonempty. Since $\mathbb{N}$ is not bounded above, there exists an $N \in \mathbb{N}$ such that $N > 1/s$. Therefore if $n > N$, we have that

$$0 < \frac{1}{n} < \frac{1}{N} < s \leq \varepsilon_i \implies \frac{1}{n} \in (0, \varepsilon_i) \subset V_i$$

for all $i \in F$. Also, $1/n \in \mathbb{R} = V_i$ for all $i \in \mathbb{N} - F$, so the sequence $a_n = (1/n, 1/n, \dots) \in V \subset U$ for all $n > N$. In both cases (F empty or nonempty), there exists an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \in U$. Since U was an arbitrary open set containing $\mathbf{0}$, this is true of all such sets, and therefore $(a_j)_{j\in\mathbb{N}}$ is a sequence of points in A that converges to $\mathbf{0}$, as required.

Now let $(a_k)_{k\in\mathbb{N}}$ be a sequence of points in $A \subset \mathbb{R}^\omega$ with the box topology. We claim that this sequence does not converge to $\mathbf{0}$. Elements in $\mathbb{R}^\omega$ are sequences, so we can write each point a_k as $a_k = (a_{k1}, a_{k2}, a_{k3}, \dots)$. We will construct an open set $U \in \mathbb{R}^\omega$ containing $\mathbf{0}$ such that $a_k \notin U$ for all $k \in \mathbb{N}$, no matter how large. Define $U_i \subset \mathbb{R}$ to be the interval $(-1, a_{ii})$ and define $U = \prod_{i \in \mathbb{N}} U_i$. Since $(a_k)_{k \in \mathbb{N}}$ is a sequence of points in A, every coordinate of each point is positive, which means $0 \in U_i$ for all $i \in \mathbb{N}$ and therefore $\mathbf{0} \in U$. Each U_i is an open interval in $\mathbb{R}_{\text{std}}$, which means U is a basic open set in the box topology since it is the product of all the U_i. Note that $a_{nn} \notin U_n = (-1, a_{nn})$, and therefore

$$a_n = (a_{n1}, \ldots, a_{nn}, \ldots) \notin \prod_{i \in \mathbb{N}} U_i = U$$

for all $n \in \mathbb{N}$. This means that there does not exist an $N \in \mathbb{N}$ such that $i > N$ implies $a_i \in U$, and therefore $(a_k)_{k \in \mathbb{N}}$ does not converge to $\mathbf{0}$. Since $(a_k)_{k \in \mathbb{N}}$ was an arbitrary sequence of points in A, no such sequence converges to $\mathbf{0}$.

Exercise 3.42. The set $2^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \{0, 1\}$ with the box topology has the discrete topology since all singletons are open. Let $\{a_n\}_{n \in \mathbb{N}}$ be a binary sequence (an arbitrary element of $2^{\mathbb{N}}$). Then the singleton containing $\{a_n\}_{n \in \mathbb{N}}$ is $\{\{a_n\}_{n \in \mathbb{N}}\} = \prod_{n \in \mathbb{N}} \{a_n\} = \{a_1\} \times \{a_2\} \times \cdots$, so it is a basic open set. Compare this to $2^{\mathbb{N}}$ with the product topology. Since with this topology, if $p \in 2^{\mathbb{N}}$ is a point in the space and U an open set containing p, then $(U - \{p\}) \cap 2^{\mathbb{N}} \neq \emptyset$ since any open set U contains infinitely many different sequences. Therefore $2^{\mathbb{N}}$ under the product topology has no isolated points since every point is a limit point of the set.

4 Separation Properties: Separating This From That

4.1 Hausdorff, Regular, and Normal Spaces

Theorem 4.1. A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.

Proof. Suppose $(X, \mathcal{T})$ is T_1 and let $x \in X$ be arbitrary. Let $y \in X - \{x\}$ be arbitrary. Then we have that $x \neq y$, so since the space is T_1, we have that there exist open sets U and V such that $x \in U - V$ and $y \in V - U$. Since V is open, $V \subset X$, and since $x \notin V$, $V \subset X - \{x\}$. Therefore $y \in V \subset X - \{x\}$ and by Theorem 2.3, we have that $X - \{x\}$ is open, meaning that the singleton $\{x\}$ is closed. But x was an arbitrary point in this T_1 space, so we have that points are closed in T_1 spaces. Now suppose $(X, \mathcal{T})$ is a topological space in which all points are closed and let $x, y \in X$ be arbitrary points such that $x \neq y$. Since $\{x\}$ and $\{y\}$ are closed, we have that $x \in X - \{y\}$, an open set, and $y \in X - \{x\}$, another open set. Since $x \notin X - \{x\}$ and $y \notin X - \{y\}$, and x and y were arbitrary elements of X, we have that the space $(X, \mathcal{T})$ is T_1. $\square$

Exercise 4.2. Let X be a set with the finite complement topology. Then let $x \in X$ be arbitrary. We have that $X - \{x\}$ is open since its complement, $\{x\}$, is finite. This means $\{x\}$ is closed, and since the point x was arbitrary, we have that all singletons are closed and therefore by Theorem 4.1, all sets with the finite complement topology are T_1.

Exercise 4.3. Let $x, y \in \mathbb{R}$ be points in the space $\mathbb{R}_{\text{std}}$. Then the sets $A = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$ and $B = \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$ are open and disjoint with $x \in A$ and $y \in B$. Therefore $\mathbb{R}_{\text{std}}$ is Hausdorff.

Exercise 4.5. Let A and B be disjoint, closed sets in $\mathbb{R}_{\text{LL}}$. For every $a \in A$ and $b \in B$, set

$$\delta_a = \frac{\inf\{b \in B : b > a\} - a}{2} \quad \text{and} \quad \delta_b = \frac{\inf\{a \in A : a > b\} - b}{2}.$$

Since for all $a \in A$, a is in the basic open set $U_a = [a, a+\delta_a)$, we have that $A \subset U = \bigcup_{a \in A} U_a$. Similarly, we have that $B \subset V = \bigcup_{b \in B} V_b$ where $V_b = [b, b + \delta_b)$ is a basic open set. Suppose for contradiction that there exists and $x \in U \cap V$. Then we have that there exist $\alpha \in A$ and $\beta \in B$ such that $x \in U_\alpha \cap V_\beta$. Without loss of generality, assume that $\alpha < \beta$. Then we have that $x \in U_\alpha = [\alpha, \alpha + \delta_\alpha)$ and $x \in V_\beta = [\beta, \beta + \delta_\beta)$. Then

$$\delta_\alpha = \frac{\inf\{b \in B : b > \alpha\} - \alpha}{2} \implies \alpha + \delta_\alpha = \frac{\alpha + \inf\{b \in B : b > \alpha\}}{2} \leq \inf\{b \in B : b > \alpha\}$$

since $\alpha \leq \inf\{b \in B : b > \alpha\}$. Since $\beta \in B$ and $\beta > \alpha$, $\beta \in \{b \in B : b > \alpha\}$ and therefore we have that

$$x < \alpha + \delta_\alpha \leq \inf\{b \in B : b > \alpha\} \leq \beta \leq x.$$

But $x < x$ is a contradiction and so we have that $U \cap V = \emptyset$ and therefore we have found disjoint, open sets U and V such that $A \subset U$ and $B \subset V$. This means that $\mathbb{R}_{\text{LL}}$ is normal.

Exercise 4.6. (1) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Suppose for contradiction that $\inf\{d(a,p) : a \in A\} = 0$. This means that there exists a sequence $(x_i)_{i \in \mathbb{N}} \subset A$ such that $d(x_i, p) \to 0$. This means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies that $|d(x_i, p) - 0| < \varepsilon$, which is equivalent to the statement that

$$\varepsilon > |\parallel x_i - p \parallel - 0| = \parallel x_i - p \parallel.$$

We now have that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies $\parallel x_i - p \parallel < \varepsilon$, which in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ means that $x_i \to p$. By Theorem 2.30, we have that $p \in \overline{A} = A$ since

A is closed. But this is a contradiction, so the assumption that $\inf\{d(a,p) : a \in A\} = 0$ was false. We have either that this is greater than 0 or less than 0, but it cannot be less than 0 since the distance between two points is always nonnegative. Therefore we have that $\inf\{d(a,p) : a \in A\} > 0$.

(2) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Then by (1), there exists an $\varepsilon > 0$ such that $\inf\{d(a,p) : a \in A\} = \varepsilon$. Set $U = B(p, \frac{\varepsilon}{2})$ and $V = \bigcup_{a \in A} B(a, \frac{\varepsilon}{2})$. Then U is a basic open set, and V is the union of basic open sets so both U and V are open. We also have that $p \in U$ and $A \subset V$ since if $a \in A$, $a \in B(a, \frac{\varepsilon}{2}) \subset V$. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means that $d(p,x) < \varepsilon/2$ and $x \in V$ means that there exists an $a \in A$ such that $d(a,x) < \varepsilon/2$. Therefore we have that $d(a,p) \leq d(a,x) + d(x,p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction since $\inf\{d(a,p) : a \in A\} = \varepsilon$. Therefore $U \cap V = \emptyset$ and $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is regular.

(3) The sets $A = \{(x,0) : x \in \mathbb{R}\}$ and $B = \{(x,y) : x > 0, y \geq \frac{1}{x}\}$. Since for any points $(x_a, y_a) \in A$ and $(x_b, y_b) \in B$, $y_b \geq \frac{1}{x_b} > 0 = y_a$, we have that $A \cap B = \emptyset$. All limit points of A have y-coordinate equal to 0 and are therefore in A, meaning A is closed. Similarly, all limit points in B have y-coordinate equal to $\frac{1}{x_0}$ for some $x_0 > 0$ and are therefore in B, meaning B is closed. Hence A and B are disjoint, closed subsets of $\mathbb{R}^2$. However, $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$. To see this, let $\varepsilon > 0$. Then we have that $a_\varepsilon = (\frac{1}{\varepsilon} + 1, 0) \in A$ and $b_\varepsilon = (\frac{1}{\varepsilon} + 1, \frac{1}{\frac{1}{\varepsilon}+1}) \in B$. Then $d(a_\varepsilon, b_\varepsilon) = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$, so $\inf\{d(a,b) : a \in A \text{ and } b \in B\} < \varepsilon$. However, $\varepsilon > 0$ was an arbitrary, so we have that $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$ since distance is nonnegative.

(4) Let A and B be disjoint, closed sets in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$. For $a_0 \in A$ and $b_0 \in B$, define $\varepsilon_{a_0} = \frac{1}{2}\inf\{d(a_0, b) : b \in B\} > 0$ and $\varepsilon_{b_0} = \frac{1}{2}\inf\{d(a, b_0) : a \in A\} > 0$. Now set $U = \bigcup_{a \in A} B(a, \varepsilon_a)$ and $V = \bigcup_{b \in B} B(b, \varepsilon_b)$. Then since U and V are the unions of open balls in $\mathbb{R}^2$, they are open. Since if $a \in A$, then $a \in B(a, \varepsilon_a) \subset U$ and if $b \in B$, then $b \in B(b, \varepsilon_b) \subset V$, we have that $A \subset U$ and $B \subset V$. To show $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal, it only remains to show that U and V are disjoint. Suppose for contradiction that there exists $p \in \mathbb{R}^2$ such that $p \in U \cap V$. Then $p \in U$, so there exists an $\alpha \in A$ such that $p \in B(\alpha, \varepsilon_\alpha)$ and similarly, there exists a $\beta \in B$ such that $p \in B(\beta, \varepsilon_\beta)$. Since $\alpha \in A$, $d(\alpha, \beta) \in \{d(a, \beta) : a \in A\}$ and therefore $d(\alpha, \beta) \geq \inf\{d(a, \beta) : a \in A\} = 2\varepsilon_\alpha$. Similarly, $d(\alpha, \beta) \geq 2\varepsilon_\beta$, and so we have that $\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta)$. Since $p \in B(\alpha, \varepsilon_\alpha)$, we have that $d(\alpha, p) < \varepsilon_\alpha$, and since $p \in B(\beta, \varepsilon_\beta)$, we have that $d(p, \beta) < \varepsilon_\beta$. Putting this all together using the triangle inequality, we see that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta) \leq d(\alpha, p) + d(p, \beta) < \varepsilon_\alpha + \varepsilon_\beta.$$

This is a contradiction, so we have that $U \cap V = \emptyset$ and therefore $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal.

Theorem 4.7. (1) A T_2-space (Hausdorff) is a T_1-space.

(2) A T_3-space (regular and T_1) is a Hausdorff space, that is, a T_2-space.

(3) A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof. (1) Let $(X, \mathcal{T})$ be a Hausdorff space and let $x, y \in X$ be distinct, arbitrary points. Then there exist disjoint, open sets U and V such that $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$, we have that $x \notin V$ and $y \notin U$, so $(X, \mathcal{T})$ is a T_1-space. □

Proof. (2) Let $(X, \mathcal{T})$ be a T_3-space and let $x, y \in X$ be distinct, arbitrary points. Since this space is T_1, by Theorem 4.1 we have that $\{y\}$ is closed. Since this space is regular, we have that there exist disjoint, open sets such that U and V such that $x \in U$ and $\{y\} \subset V$. But $\{y\} \subset V$ means $y \in V$, so we have found disjoint, open sets separating the arbitrary points x and y, so $(X, \mathcal{T})$ is Hausdorff. □

Proof. (3) Let $(X, \mathcal{T})$ be a T_4-space, let $x \in X$ be arbitrary, and let A be a closed set with $x \notin A$. Since this space is T_1, by Theorem 4.1 we have that $\{x\}$ is closed. Since this space is normal, there exist disjoint, open sets U and V such that $\{x\} \subset U$ and $A \subset V$. But $\{x\} \subset U$ means $x \in U$, so we have found disjoint, open sets separating the arbitrary point x from the arbitrary closed set A, so $(X, \mathcal{T})$ is T_3 since it is normal and T_1. □

Theorem 4.8. A topological space is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a regular topological space and let U be an open set containing the point p. Then we have that $X - U$ is closed and since $p \in U$, $p \notin X - U$. Since this space is regular, there exist disjoint open sets V and W such that $p \in V$ and $X - U \subset W$. Therefore we have that $X - W \subset U$ since $X - U \subset W$, and that $X - W$ is closed since W is open. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$) and therefore $x \in X - W$. Since x was arbitrary, we have that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$p \in V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since U and p were arbitary, there exists an open set V containing p such that $\overline{V} \subset U$ for all $p \in X$ and open sets U containing p.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all $p \in X$ and $W \in \mathcal{T}$ with $p \in W$, there exists an open set U such that $p \in U$ and $\overline{U} \subset W$. Let $p \in X$ be arbitrary and let A be a closed subset of X such that $p \notin A$. Then we have that $p \in X - A$, which is open, and therefore there exists an open set U such that $p \in U$ and $\overline{U} \subset X - A$, which implies that $A \subset V$ where V is the open set $X - \overline{U}$. Let $x \in U$ be arbitary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since x was arbitary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets U and V such that $p \in U$ and $A \subset V$, so $(X, \mathcal{T})$ is regular, as required. $\square$

Theorem 4.9. A topological space is normal if and only if for each closed set A in $(X, \mathcal{T})$ and open set U containing A there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space, let A be a closed set, and let U be an open set such that $A \subset U$. Then $X - U$ closed and $A \cap (X - U) = \emptyset$, so since this is a normal space, there exist disjoint open sets V and W such that $A \subset V$ and $X - U \subset W$. Therefore $X - W \subset U$ is a closed set. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$), so $x \in X - W$, which means that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$A \subset V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since A and U were arbitrary, there exists an open set V with $A \subset V$ and $\overline{V} \subset U$ for all open sets U containing closed sets A.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all closed sets A and open sets W with $A \subset W$, there exists an open set U such that $A \subset U$ and $\overline{U} \subset W$. Let A and B be arbitrary disjoint sets. Then we have that $A \subset X - B$, and since $X - B$ is open, there exists an open set U such that $A \subset U$ and $\overline{U} \subset X - B$, which implies that $B \subset V$ where V is the open set $X - \overline{U}$. Let $x \in U$ be arbitrary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since x was arbitrary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so $(X, \mathcal{T})$ is normal, as required. $\square$

Theorem 4.10. A topological space is normal if and only if for each pair of disjoint closed sets A and B, there exist open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let A and B be disjoint closed sets. Then since $X - B$ is open and $A \subset X - B$, by Theorem 4.9 there exists an open set U

such that $A \subset U$ and $\overline{U} \subset X - B$. Therefore $\overline{U} \cap B = \emptyset$, so there exists an open set V such that $B \subset V$ and $\overline{V} \subset X - \overline{U}$, which means that $\overline{U} \cap \overline{V} = \emptyset$, as required.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for all disjoint closed sets A and B, there exist open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$. But if $\overline{U}$ and $\bar{(}V)$ are disjoint, so are U and V, and therefore this space is normal. $\square$

Theorem 4.11 (The Incredible Shrinking Theorem). A topological space is normal if and only if for each pair of open sets U and V such that $U \cup V = X$, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let U and V be open sets with $U \cup V = X$. Then $X - U$ and $X - V$ are closed sets and

$$(X - U) \cap (X - V) = X - (U \cup V) = X - X = \emptyset.$$

Since these sets are closed and disjoint, by Theorem 4.10, there exist open sets W_1 and W_2 such that $X - U \subset W_1$, $X - V \subset W_2$, and $\overline{W_1} \cap \overline{W_2} = \emptyset$. Define $U' = X - \overline{W_1}$ and $V' = X - \overline{W_2}$. Then

$$U' \cup V' = (X - \overline{W_1}) \cup (X - \overline{W_2}) = X - (\overline{W_1} \cap \overline{W_2}) = X - \emptyset = X.$$

By Lemma 2.28, $\overline{W_1} = X - (X - W_1)^\circ$, so

$$U' = X - \overline{W_1} = X - (X - (X - W_1)^\circ) = (X - W_1)^\circ.$$

Since $(X - W_1)^\circ \subset X - W_1$ and $X - U \subset W_1$, we have that

$$\overline{U'} = \overline{(X - W_1)^\circ} \subset \overline{X - W_1} = X - W_1 \subset U$$

where we have used the fact that $X - W_1$ is closed. Similarly, $\overline{V'} \subset V$, and so we have shown the forward direction of the implication.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for any open sets U and V with $U \cup V = X$, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$. Let A and B be arbitrary closed sets. Then $X - A$ and $X - B$ are open sets such that

$$(X - A) \cup (X - B) = X - (A \cap B) = X - \emptyset = X,$$

so there exist open sets U' and V' such that $\overline{U'} \subset X - A$, $\overline{V'} \subset X - B$, and $U' \cup V' = X$. Define open sets $U = X - \overline{U'}$ and $V = X - \overline{V'}$. Then since $\overline{U'} \subset X - A$, $A \subset X - \overline{U'} = U$, and similarly, $B \subset V$. Then using Theorem 2.22(2), we have that

$$U \cap V = (X - \overline{U'}) \cap (X - \overline{V'}) = X - (\overline{U'} \cup \overline{V'}) = X - \overline{U' \cup V'} = X - \overline{X} = \emptyset.$$

Therefore we have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so this space is normal. □

Exercise 4.12. (1) The Double Headed Snake, (2) $\mathbb{R}_{\text{har}}$, and (3) ... is trickier.

4.2 Separation Properties and Products

Theorem 4.16. The product of Hausdorff spaces is Hausdorff.

Proof. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ be a collection of Hausdorff spaces and let f and g be distinct elements of the product space $\prod_{\alpha \in \lambda} X_\alpha$. Since f and g are distinct, they differ at at least one coordinate, so there exists a $\beta \in \lambda$ such that $\pi_\beta(f) \neq \pi_\beta(g)$. Then in the Hausdorff space $(X_\beta, \mathcal{T}_\beta)$, the elements $\pi_\beta(f)$ and $\pi_\beta(g)$ are distinct, so there exist disjoint open sets U_β and V_β such that $\pi_\beta(f) \in U_\beta$ and $\pi_\beta(g) \in V_\beta$. Then we have that $f \in \pi_\beta^{-1}(U_\beta)$ and $g \in \pi_\beta^{-1}(V_\beta)$, and since U_β and V_β are disjoint, so are $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$, and they are subbasic open sets. Thus we have found disjoint open sets in $\prod_{\alpha \in \lambda} X_\alpha$ separating f and g, so the product of Hausdorff spaces is Hausdorff. □

Theorem 4.17. The product of regular spaces is regular.

Proof. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ be a collection of regular spaces and let p be an element of the space $\prod_{\alpha \in \lambda} X_\alpha$ under the product topology. Let U^* be an open set containing p. Recall that the product topology is defined to be the topology generated by the subbasis $\mathscr{S}$ of sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in $(X_\beta, \mathcal{T}_\beta)$. By Theorem 3.15, there exists a finite collection of subsets $\{W_i\}_{i=1}^n$ such that each $W_i \in \mathscr{S}$ and $p \in \bigcap_{i=1}^n W_i \subset U^*$. Since $W_i \in \mathscr{S}$, there exists a $\beta_i \in \lambda$ such that $W_i = \pi_{\beta_i}^{-1}(W_{\beta_i})$ where W_{β_i} is open in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$. Then $p \in \pi_{\beta_i}^{-1}(W_{\beta_i})$ for all $i = 1, \ldots, n$, which means that $\pi_{\beta_i}(p) \in W_{\beta_i}$. Since $(X_{\beta_i}, \mathcal{T}_{\beta_i})$ is regular, there exists an open V_{β_i} such that $\pi_{\beta_i}(p) \in V_{\beta_i}$ and $\overline{V_{\beta_i}} \subset W_{\beta_i}$. Define V_i to be the set $V_i = \pi_{\beta_i}^{-1}(V_{\beta_i})$, a subbasic open set. Since $\pi_{\beta_i}(p) \in V_{\beta_i}$, $p \in V_i$ for all $i = 1, \ldots, n$. Let $g \in \overline{\pi_{\beta_i}^{-1}(V_{\beta_i})}$ be arbitrary. There are two cases—either $g \in \pi_{\beta_i}^{-1}(V_{\beta_i})$ or g is a limit point of $\pi_{\beta_i}^{-1}(V_{\beta_i})$. In the first case, $\pi_{\beta_i}(g) \in V_{\beta_i} \subset \overline{V_{\beta_i}}$ and therefore $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. In the second case, let U be an arbitrary open set in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$ such that $\pi_{\beta_i}(g) \in U$. Suppose for contradiction that $(U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i} = \emptyset$. Then we have that

$$\begin{aligned}\emptyset &= \pi_{\beta_i}^{-1}\left((U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i}\right) = \pi_{\beta_i}^{-1}(U - \{\pi_{\beta_i}(g)\}) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}) \\ &= \left(\pi_{\beta_i}^{-1}(U) - \pi_{\beta_i}^{-1}(\{\pi_{\beta_i}(g)\})\right) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}) = \left(\pi_{\beta_i}^{-1}(U) - \{g\}\right) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}).\end{aligned}$$

But this is a contradiction since $\pi_{\beta_i}^{-1}(U)$ is an open set containing g and g is a limit point in this case. Therefore for all open U in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$, $(U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i} \neq \emptyset$, meaning that $\pi_{\beta_i}(g)$ is a limit point of V_{β_i}. Therefore $\pi_{\beta_i}(g) \in \overline{V_{\beta_i}}$, which implies that $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. Since g was an arbitrary element of $\overline{V_i} = \overline{\pi_{\beta_i}^{-1}(V_{\beta_i})}$ and in both cases we have that $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$, so $\overline{V_i} \subset \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. Since for each $i = 1, \ldots, n$, $\overline{V}_{\beta_i} \subset W_{\beta_i}$, we have that

$$\pi_{\beta_i}^{-1}(\overline{V_{\beta_i}}) = \left\{ f \in \prod_{\alpha \in \lambda} X_\alpha : \pi_{\beta_i}(f) \in \overline{V_{\beta_i}} \right\} \subset \left\{ f \in \prod_{\alpha \in \lambda} X_\alpha : \pi_{\beta_i}(f) \in W_{\beta_i} \right\} = \pi_{\beta_i}^{-1}(W_{\beta_i}).$$

Therefore for all $i = 1, \ldots, n$, we have that $\overline{V_i} \subset \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}}) \subset \pi_{\beta_i}^{-1}(W_{\beta_i}) = W_i$, and $p \in V_i$. Define V^* to be the set $V^* = \bigcap_{i=1}^n V_i$, which is the finite intersection of subbasic open sets and is therefore open. Finally, we have that $p \in V^*$ and

$$\overline{V^*} = \overline{\bigcap_{i=1}^n V_i} \subset \bigcap_{i=1}^n \overline{V_i} \subset \bigcap_{i=1}^n W_i \subset U^*.$$

Therefore the product topology on $\prod_{\alpha \in \lambda} X_\alpha$ is regular, and since this was the product of an arbitrary collection of regular spaces, all products of regular spaces are regular. $\square$

4.3 A Question of Heredity

Theorem 4.19. Every Hausdorff space is hereditarily Hausdorff.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space, and let $Y \subset X$ be arbitrary. Let $a, b \in Y$ be arbitrary distinct points. Since $(X, \mathcal{T})$ is Hausdorff, there exist disjoint open sets U and V such that $a \in U$ and $b \in V$. Therefore $a \in U \cap Y$ and $b \in V \cap Y$, and the sets $U \cap Y$ and $V \cap Y$ are open in the subspace by the definition of $\mathcal{T}_Y$ and disjoint since U and V are disjoint. Therefore $\mathcal{T}_Y$ is Hausdorff. $\square$

Theorem 4.20. Every regular space is hereditarily regular.

Proof. Let $(X, \mathcal{T})$ be a regular space, and let $Y \subset X$ be arbitrary. Let $p \in Y$ be an arbitrary point and let A be an arbitrary closed set in $(Y, \mathcal{T}_Y)$ with $p \notin A$. Then by Theorem 3.28, there exists a set D that is closed in $(X, \mathcal{T})$ such that $A = D \cap Y$. Since $p \in Y$ but $p \notin A \cap Y$, we have that $p \notin D$, and so by the regularity of $(X, \mathcal{T})$, there exist disjoint open sets U' and V' such that $p \in U'$, $D \subset V'$, and $U' \cap V' = \emptyset$. Define the sets U and V to be $U = U' \cap Y$ and $V = V' \cap Y$ so that they are open in $(Y, \mathcal{T}_Y)$. We have that $U \cap V \subset U' \cap V' = \emptyset$. Since $p \in U'$ and $p \in Y$, $p \in U$, and similarly, $D \subset V'$ and $D \subset Y$, so $D \subset V$. Since p, A, $(Y, \mathcal{T}_Y)$, and $(X, \mathcal{T})$ were arbitrary, all subspaces of regular spaces are regular, meaning that regular spaces are hereditarily regular. $\square$

Theorem 4.23. If Y is closed in a normal space $(X, \mathcal{T})$, then $(Y, \mathcal{T}_Y)$ is normal.

Proof. Let $(X, \mathcal{T})$ be normal and let Y be closed in $(X, \mathcal{T})$. Let A and B be disjoint closed sets in $(Y, \mathcal{T}_Y)$. Then we have that there exist sets C and D that are closed in $(X, \mathcal{T})$ such that $A = C \cap Y$ and $B = D \cap Y$. Since Y is closed in $(X, \mathcal{T})$, A and B are each the intersections of closed sets and are therefore themselves closed in $(X, \mathcal{T})$. By the normality of $(X, \mathcal{T})$, there exist disjoint open sets U' and V' such that $A \subset U'$ and $B \subset V'$. Then since $A, B \subset Y$, we have that $A \subset U := U' \cap Y$ and $B \subset V := V' \cap Y$. By the definition of $\mathcal{T}_Y$, U and V are open in $(Y, \mathcal{T}_Y)$ and we have that $U \cap V \subset U' \cap V' = \emptyset$, so we have found disjoint open sets separating A and B, meaning that $(Y, \mathcal{T}_Y)$ is normal. $\square$

Lemma 4.25 (Corollary 3.29). Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $\overline{C} \cap Y = C$ where the closure is taken in $(X, \mathcal{T})$.

Proof. ($\Longrightarrow$) Let C be closed in $(Y, \mathcal{T}_Y)$. Then by Theorem 3.28, there exists a D closed in $(X, \mathcal{T})$ such that $C = D \cap Y$. Note that $C \subset Y$, so $C \subset \overline{C} \cap Y$, so to show equality, it suffices to show that $\overline{C} \cap Y \subset C$. We have:

$$\overline{C} \cap Y = \overline{D \cap Y} \cap Y \subset \overline{D} \cap \overline{Y} \cap Y = \overline{D} \cap Y = D \cap Y = C$$

where $\overline{D} = D$ since D is closed in $(X, \mathcal{T})$.

($\Longleftarrow$) Let C be a subset of Y such that $\overline{C} \cap Y = C$, so there is a closed set $D = \overline{C}$ such that $C = D \cap Y$, and therefore C is closed in $(Y, \mathcal{T}_Y)$ by Theorem 3.28. $\square$

Theorem 4.25. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$ and let A and B be disjoint closed sets in $(Y, \mathcal{T}_Y)$. Then we have that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ where the closures are taken in $(X, \mathcal{T})$.

Proof. Let A and B be disjoint closed sets in $(Y, \mathcal{T})$. Then by the Lemma, $\overline{A} \cap B \subset \overline{A} \cap Y = A$ and therefore $\overline{A} \cap B = (\overline{A} \cap B) \cap B \subset A \cap B = \emptyset$. Similarly, $A \cap \overline{B} \subset Y \cap \overline{B} = B$ since B is closed in $(Y, \mathcal{T}_Y)$, and $A \cap \overline{B} = A \cap (A \cap \overline{B}) \subset A \cap B = \emptyset$. Hence $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. $\square$

Theorem 4.26. A topological space $(X, \mathcal{T})$ is completely normal if and only if it is hereditarily normal.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a completely normal space, let $(Y, \mathcal{T}_Y)$ be a subspace, and let A and B be closed sets in $(Y, \mathcal{T}_Y)$. By Theorem 4.25, A and B are separated in $(X, \mathcal{T}_Y)$, and so there exist disjoint open sets U' and V' such that $A \subset U'$ and $B \subset V'$. Then we also have that $A \subset U := U' \cap Y$, $B \subset V := V' \cap Y$, and $U \cap V \subset U' \cap V' = \emptyset$. By definition the definition of $\mathcal{T}_Y$, U and V are open in $(Y, \mathcal{T}_Y)$, and therefore $(X, \mathcal{T})$ is hereditarily normal since $(Y, \mathcal{T}_Y)$ was an arbitrary subspace.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a hereditarily normal space and let A and B be separated sets. Define the set $Y = X - (\overline{A} \cap \overline{B})$, an open set in $(X, \mathcal{T})$. Note that

$$\overline{A} \cap Y = \left(\bigcup_{D \supset A, D \in \mathcal{C}} D \right) \cap Y = \bigcup_{D \supset A, D \in \mathcal{C}} (D \cap Y) = \mathrm{Cl}_Y(A)$$

since $\bigcup_{D \supset A, D \in \mathcal{C}} (D \cap Y)$ is the intersection of all closed sets in $(Y, \mathcal{T}_Y)$ containing A ($\mathcal{C}$ here is the set of all closed sets in $(X, \mathcal{T})$). Similarly, we have that $\mathrm{Cl}_Y(B) = \overline{B} \cap Y$. Therefore

$$\mathrm{Cl}_Y(A) \cap \mathrm{Cl}_Y(B) = (\overline{A} \cap \overline{B}) \cap Y = (\overline{A} \cap \overline{B}) \cap (X - (\overline{A} \cap \overline{B})) = \emptyset.$$

Since $(X, \mathcal{T})$ is hereditarily normal and $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_Y(B)$ are disjoint closed sets in the subspace $(Y, \mathcal{T}_Y)$, there exist disjoint open sets U and V such that $\mathrm{Cl}_Y(A) \subset U$ and $\mathrm{Cl}_Y(B) \subset V$. These sets are open in $(Y, \mathcal{T}_Y)$, so there exist open sets in U' and V' in $(X, \mathcal{T})$ such that $U = U' \cap Y$ and $V = V' \cap Y$. Then since Y is open in $(X, \mathcal{T})$, U and V are the (finite) intersections of open sets and are therefore open. We have that $A \subset Y$, $B \subset Y$, $A \subset \overline{A}$, and $B \subset \overline{B}$, so therefore $A \subset \overline{A} \cap Y = \mathrm{Cl}_Y(A) \subset U$ and $B \subset \overline{B} \cap Y = \mathrm{Cl}_Y(B) \subset V$. We have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, and since A and B were arbitrary separated sets, $(X, \mathcal{T})$ is completely normal. $\square$

Exercise 4.27. (2) Consider the set $Y = \left\{ \left(x, \frac{1}{2}\right) : x \in [0, 1] \right\}$ as a subspace of the lexicographically ordered square $(X, \mathcal{T})$. Let $p = (x_0, y_0) \in Y$ be arbitrary. Note that in the lexicographically ordered square, the set $U_p = \{(x_0, y) : 0 < y < 1\}$ is open since the lexicographically ordered square is an order topology and $(x_0, 0) < (x_0, 1)$ Therefore U_p is open and contains the point p. Therefore $U_p \cap Y = \{p\}$ is open in the subspace $(Y, \mathcal{T}_Y)$. But this means that $\mathcal{T}_Y$ is not the order topology on Y, since by Theorem 4.15, order topologies are T_1, meaning that points are closed. This differs from $\mathcal{T}_Y$ since in $(Y, \mathcal{T}_Y)$, points are open, making this is the discrete topology on Y.

(3) Consider the set $Y = \{(x, 1) : x \in [0, 1)\}$ as a subspace of the lexicographically ordered square $(X, \mathcal{T})$. $\mathcal{T}$ has a basis consisting of sets of the form $\{x \in X : x < (x_0, y_0)\}$, $\{x \in X : (x_0, y_0) < x\}$, $\{x \in X : (x_1, y_1) < x < (x_2, y_2)\}$, so by Theorem 3.30, the subspace $(Y, \mathcal{T}_Y)$ has a basis of sets of the form $\{(x, 1) : 0 \leq x < x_0\}$, $\{(x, 1) : x_0 \leq x < 1\}$ (if $y_0 < 1$), $\{(x, 1) : x_0 < x < 1\}$ (if $y_0 = 1$), $\{(x, 1) : x_1 \leq x < x_2\}$ (if $y_0 < 1$), and $\{(x, 1) : x_1 < x < x_2\}$ (if $y_0 = 1$). Sets of forms 3 and 5 form a basis for the standard topology on Y, and sets of forms 1, 2, and 4 form a basis for the lower limit topology. Therefore $\mathcal{T}_Y$ is the union of the standard topology and the lower limit topology, which is just the lower limit topology since it is strictly finer than the standard topology. But the lower limit topology is not an order topology, so we have another example of a non-order topology subspace of an order topology.

4.4 The Normality Lemma

Theorem 4.29 (The Normality Lemma). Let A and B be sets in a topological space X and let $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ be collections of open sets such that

$$
\begin{aligned}
&1) \quad A \subset \bigcup_{i \in \mathbb{N}} U_i \\
&2) \quad B \subset \bigcup_{i \in \mathbb{N}} V_i \\
&3) \quad \overline{U_i} \cap B = \emptyset \text{ and } \overline{V_i} \cap A = \emptyset \text{ for all } i \in \mathbb{N}.
\end{aligned}
$$

Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Proof. To construct sets U and V, we build them up out of the parts of the $\{U_i\}$ and $\{V_i\}$ that don't overlap. Begin with the first set U_1 as the first building block of U and the first set V_1 excluding anything overlapping with U_1 as the first building block of V. Then add all the elements in U_2 that aren't in what we have so far of V into U, and add all the elements in V_2 that aren't in what we have so far of U into V. Continue doing this to construct the sets U and V.

More formally, we define two collections of sets $\{U_i'\}_{i \in \mathbb{N}}$ and $\{V_i'\}_{i \in \mathbb{N}}$ recursively. Let

$U_1' = U_1$ and let $V_1' = V_1 - \overline{U_1'}$. Then for $n \in \mathbb{N}$, define $U_{n+1}' = (U_n' \cup U_{n+1}) - \overline{V_n'}$, and define $V_{n+1}' = (V_n' \cup V_{n+1}) - \overline{U_{n+1}'}$. We claim that the sets

$$U = \bigcup_{i \in \mathbb{N}} U_i' \quad \text{and} \quad V = \bigcup_{i \in \mathbb{N}} V_i'$$

are open such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

We first argue by induction on n that U_n' and V_n' are open for all $n \in \mathbb{N}$. For the base case $n = 1$, $U_1' = U_1$ is open by assumption, and $V_1' = V_1 - \overline{U_1'}$ is an open set minus a closed set, so it is open by Theorem 2.15. Now suppose there is a $k \in \mathbb{N}$ such that U_k' and V_k' are open. Then $U_{k+1}' = (U_k' \cup U_{k+1}) - \overline{V_k'}$, which is the union of two open sets minus a closed set, so it is open. Similarly, $V_{k+1}' = (V_k' \cup V_{k+1}) - \overline{U_{k+1}'}$ is open. This completes the induction, and so all U_i' and V_i' are open, and since unions of open sets are open, U and V are also open.

Now we claim that if $a \in A$, $a \notin \overline{V_n'}$, and if $b \in B$, $b \notin \overline{U_n'}$ for all $n \in \mathbb{N}$ again by induction on n. For the base case $n = 1$, $a \notin \overline{V_1}$ by the hypotheses of the Theorem, so since $V_1' = V_1 - \overline{U_1'} \subset V_1$ implies that $\overline{V_1'} \subset \overline{V_1}$, we have that $a \notin \overline{V_1'}$. We also have that $b \notin \overline{U_1} = \overline{U_1'}$. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $a \notin \overline{V_k'}$ and $b \notin \overline{U_k'}$. Then since $a \notin \overline{V_{k+1}}$, $a \notin \overline{V_k'} \cup \overline{V_{k+1}} = \overline{V_k' \cup V_{k+1}}$, and since $V_{k+1}' = (V_k' \cup V_{k+1}) - \overline{U_{k+1}'} \subset V_k' \cup V_{k+1}$, we have that $\overline{V_{k+1}'} \subset \overline{V_k' \cup V_{k+1}}$, and therefore $a \notin \overline{V_{k+1}'}$. Similarly, $b \notin \overline{U_{k+1}'}$. Now since $a \in A \subset \bigcup_{i \in \mathbb{N}} U_i$, there exists a $j \in \mathbb{N}$ such that $a \in U_j$. If $j = 1$, then $a \in U_1 = U_1'$. If $j \neq 1$, then $a \in U_j \subset U_{j-1}' \cup U_j$, and since $a \notin \overline{V_n'}$ for all $n \in \mathbb{N}$, we have that $a \in (U_{j-1}' \cup U_j) - \overline{V_{j-1}} = U_j' \subset U$, so $A \subset U$. A similar argument shows that $B \subset V$.

Now we show that $U \cap V = \emptyset$. Note that $V_1' = V_1 - U_1'$ so $V_1' \cap U_1' = \emptyset$. Also if $n > 1$, then $V_n' = (V_{n-1}' \cup V_n) - \overline{U_n'}$, so $U_n' \cap V_n' = \emptyset$ for all $n \in \mathbb{N}$. Now let $m \in \mathbb{N}$ be fixed. We argue by induction on n that $U_m' \subset U_n'$ for all $n \geq m$. For the base case $n = m$, $U_m' \subset U_m' = U_n'$. Suppose there exists a $k \geq m$ such that $U_m' \subset U_k'$. Then $U_{k+1}' = (U_k' \cup U_{k+1}) - \overline{V_k'}$. If $y \in U_m' \subset U_k'$, then $y \notin V_k'$ since $U_k' \cap V_k' = \emptyset$. But since U_k' is an open set containing y, y cannot be a limit point of V_k' by Theorem 2.9, so $y \notin \overline{V_k'}$, and therefore $y \in U_{k+1}'$. We have shown that $U_m' \subset U_{k+1}'$ and this completes the induction. We also have that $V_m' \subset V_n'$ for all $n \geq m$. For the base case $n = m$, $V_m' \subset V_m' = V_n'$. Suppose there exists a $k \geq m$ such that $V_m' \subset U_k'$. Then $V_{k+1}' = (V_k' \cup V_{k+1}) - \overline{U_{k+1}'}$. If $y \in V_m' \subset V_k' \subset \overline{V_k'}$, then $y \notin (U_k' \cup U_{k+1}) - \overline{V_k'} = U_{k+1}'$. Since V_k' is an open set containing y and $V_k' \cap U_{k+1}' = \emptyset$, y cannot be a limit point of U_{k+1}', again by Theorem 2.9. Therefore $y \in (V_k' \cup V_{k+1}) - \overline{U_{k+1}'} = V_{k+1}'$, so $V_m' \subset V_{k+1}'$. Now if there exists an $x \in U \cap V$, then there exist $j, k \in \mathbb{N}$ such that $x \in U_j'$

and $x \in V'_k$. If $j \leq k$, then $x \in U'_j \cap V'_k \subset U'_k \cap V'_k = \emptyset$, a contradiction. On the other hand, if $j \geq k$, then $x \in U'_j \cap V'_k \subset U'_j \cap V'_j = \emptyset$, again a contradiction. Therefore there is no x such that $x \in U \cap V$, so $U \cap V = \emptyset$ and we have shown that there exist disjoint open sets U and V such that $A \in U$ and $B \in V$. $\square$

Theorem 4.31. Let X be regular and countable. Then X is normal.

Proof. Let A and B be disjoint closed subsets of X. Since X is countable, A and B are also countable. Then there exist surjective sequences $(a_i)_{i\in\mathbb{N}}$ with $a_i \in A$ and $(b_i)_{i\in\mathbb{N}}$ with $b_i \in B$. Since X is regular, for each $a_i \in A \subset X - B$, there exists an open U_i such that $a_i \in U_i$ and $\overline{U_i} \subset X - B$ by Theorem 4.8. That is, $\overline{U_i} \cap B = \emptyset$. Similarly, for each $b_i \in B$, there exists an open V_i such that $b_i \in V_i$ and $\overline{V_i} \cap A = \emptyset$. This means condition (3) of the Normality Lemma is satisfied, and since there is a U_i for each $a_i \in A$ such that $a_i \in U_i$, we have that $A \subset \bigcup_{i\in\mathbb{N}} U_i$. Similarly, $B \subset \bigcup_{i\in\mathbb{N}} V_i$, so conditions (1) and (2) are satisfied as well, meaning there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since A and B were arbitrary disjoint closed sets, X is normal. $\square$

Theorem 4.32. Let X be regular with a countable basis $\mathcal{B}$. Then X is normal.

Proof. Let A and B be disjoint closed subsets of X. Since X is regular, for all $a \in A$, there exists an open set U_a such that $a \in U_a$ and $\overline{U_a} \subset X - B$. Since X has a basis $\mathcal{B}$, there exists a $U'_a \in \mathcal{B}$ such that $a \in U'_a \subset U_a$. Therefore $\overline{U'_a} \subset \overline{U_a} \subset X - B$, meaning $\overline{U'_a} \cap B = \emptyset$. Since there exists such a U'_a for all $a \in A$, $A \subset \bigcup_{i\in\mathbb{N}} U'_a$. Then because $\mathcal{B}$ is countable, there are only a countable number of distinct U'_a, although A may be uncountable. Therefore there is a surjection $f : \mathbb{N} \to \{U'_a \mid a \in A\}$, so we have that

$$A \subset \bigcup_{a\in A} U'_a = \bigcup_{i\in\mathbb{N}} f(i)$$

where $\overline{f(i)} \cap B = \emptyset$. Similarly, for each $b \in B$ there exists a $V'_b \in \mathcal{B}$ defined similarly such that $\overline{V'_b} \cap B$, and there exists a surjective $g : \mathbb{N} \to \{V'_b \mid b \in B\}$ such that $B \subset \bigcup_{i\in\mathbb{N}} g(i)$ and $\overline{g(i)} \cap B = \emptyset$. Hence the conditions of the Normality Lemma are satisfied and so there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so X is normal. $\square$

5 Countable Features of Spaces: Size Restrictions

5.1 Separable Spaces, An Unfortunate Name

Exercise 5.1. A subset $A \subset X$ is dense in X if and only if every non-empty open set contains a point of A.

Proof. ($\Longrightarrow$) Suppose A is dense in X and let U be an open set. Then since $U \subset X = \overline{A}$, if $p \in U$, then either $p \in A$ and we are done or p is limit point of A. If this is the case, then $p \in U$ so $(U - \{p\}) \cap A \neq \emptyset$, which means U contains a point in A, as required.

($\Longleftarrow$) Suppose all open sets U contain a point in A. Then certainly $\overline{A} \subset X$, so to A is dense in X it suffices to show that $X \subset \overline{A}$. Let $x \in X$. Then either $x \in A \subset \overline{A}$ and we are done, or $x \notin A$. If $x \notin A$, then we claim that x is a limit point of A. Let U be an open set containing x. Since $x \notin A$, $x \notin U \cap A$, which means $(U - \{x\}) \cap A = U \cap A \neq \emptyset$. Therefore x is a limit point of A, and so in all cases, $x \in \overline{A}$, meaning $X \subset \overline{A}$ and therefore A is dense in X. $\square$

Exercise 5.2. $\mathbb{R}_{\text{std}}$ is separable, but $\mathbb{R}$ with the discrete topology is not.

Proof. The set $\mathbb{Q}$ of rational numbers is countable, and we claim it is dense in $\mathbb{R}_{\text{std}}$. Let U be a nonempty basic open set $\mathcal{B}$, the basis of open intervals in $\mathbb{R}$. Then there exist $a, b \in \mathbb{R}$ such that $U = (a, b)$, and since U is nonempty, $|a - b| > 0$. Since $\mathbb{N}$ is unbounded, there exists an $n \in \mathbb{N}$ such that $n > \frac{1}{|a-b|} > 0$. Then we have that

$$|na - nb| = n|a - b| > 1,$$

so there exists an integer $m \in \mathbb{Z}$ such that $na < m < nb$, and therefore $a < m/n < b$ $(n > 0)$. Therefore since $m/n \in \mathbb{Q}$ and the basic open set U was arbitrary, there exists an element of $\mathbb{Q}$ in all nonempty basic open sets. Because all open sets in $\mathbb{R}_{\text{std}}$ are unions of basic open sets, there exists an element of $\mathbb{Q}$ in all nonempty open sets, meaning $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$ by Exercise 5.1, and therefore $\mathbb{R}_{\text{std}}$ is separable.

However, $\mathbb{R}$ is not separable with the discrete topology. Let C be a countable subset of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable, there exists an $\alpha \in \mathbb{R} - C$, and since $\{\alpha\}$ is open in the discrete topology, $\{\alpha\}$ is a nonempty open set that does not contain a point of C, and since such a set exists, C is not dense in $\mathbb{R}$ with the discrete topology, so $\mathbb{R}$ is not separable in this case. $\square$

Exercise 5.4. Consider the x-axis X as a subset of $\mathbb{H}_{\text{bub}}$. Then the set $\{(\alpha, 0)\} = X \cap (\{(\alpha, 0)\} \cup B((\alpha, 1), 1)$ is open in the relative topology because $(\{(\alpha, 0)\} \cup B((\alpha, 1), 1)$ is a basic open set in $\mathbb{H}_{\text{bub}}$. But $\{(\alpha, 0)\}$ was an arbitrary singleton in X and is open, so the relative topology on X is the discrete topology, which means X is not separable even though it is a subspace of the separable space $\mathbb{H}_{\text{bub}}$.

Theorem 5.5. If X and Y are separable spaces, then $X \times Y$ is separable.

Proof. Since X and Y are separable, there exist countable sets A and B such that A is dense in X and B is dense in Y. Then $A \times B \subset X \times Y$ and is countable, so it remains to show that $A \times B$ is dense in $X \times Y$. Let W be an open set in $X \times Y$. Then $W = U \times V$ for some open U in X and open V in Y. Since A is dense in X, there exists an $a \in A \cap U$ and since B is dense in Y, there exists a $b \in B \cap V$. Therefore $(a, b) \in (A \times B) \cap W$, and so every open set in $X \times Y$ contains an element of the countable subset $A \times B$, which means $A \times B$ is dense in $X \times Y$ and therefore $X \times Y$ is separable. □

5.2 2nd Countable Spaces

Theorem 5.9. Let X be a 2^{nd} countable space. Then X is separable.

Proof. Let X be 2^{nd} countable. Then there exists a countable basis $\mathcal{B}$ of nonempty sets V_i, $i \in \mathbb{N}$. Since V_i is nonempty, there exists an $a_i \in V_i$ for all $i \in \mathbb{N}$. Define the set A to be the union $A = \bigcup_{i \in \mathbb{N}} \{a_i\}$. Then A is a countable subset of X. Suppose U is a nonempty open set in X. Then there exists a point $p \in U$, and since $\mathcal{B}$ is a basis, there exists a basic open set V_j for some $j \in \mathbb{N}$ such that $p \in V_j \subset U$. By construction of A, there exists a point a_j such that $a_j \in V_j \subset U$. Because U was an arbitrary nonempty open set in X, all nonempty open sets in X contain a point in A, so A is dense in X by Theorem 5.1, and since A is countable, X is separable. □

Exercise 5.10. (1) The standard topology $\mathbb{R}^n_{\text{std}}$ has a basis of open balls with rational radii centered at rational points in $\mathbb{R}^n$, and this basis is countable.

(2) Let $\mathcal{B}$ be a basis for $\mathbb{R}_{\text{LL}}$. We compare cardinalities by constructing a map from $\mathbb{R}$ to $\mathcal{B}$. Since for each $x \in \mathbb{R}$, the set $[x, x+1)$, is open, there exists a basic open set V_x such that $x \in V_x \subset [x, x+1)$ by Theorem 3.1. This defines a map from $\mathbb{R}$ to $\mathcal{B}$ that takes a real number x to this basic open set V_x. Suppose there exist $\alpha, \beta \in \mathbb{R}$ such that $V_\alpha = V_\beta$. Then $\alpha, \beta \in V_\alpha \subset [\alpha, \alpha+1)$, and $\alpha, \beta \in V_\alpha = V_\beta \subset [\beta, \beta+1)$. Since $\alpha = \min[\alpha, \alpha+1) \supset V_\alpha$,

$\alpha = \min V_\alpha$. But similarly, $\beta = \min V_\beta = \min V_\alpha = \alpha$. Therefore this map is injective, and so $|\mathbb{R}| \leq |\mathcal{B}|$, and we see that our arbitrary basis $\mathcal{B}$ is not countable. Hence $\mathbb{R}_{\mathrm{LL}}$ is not 2^{nd} countable.

(3) The argument that $\mathbb{H}_{\text{bub}}$ is not 2^{nd} countable is very similar. This time, define a map from $\mathbb{R}$ to a potential basis $\mathcal{B}$ by associating to each $x \in \mathbb{R}$ a basic open set $V_x \in \mathcal{B}$ such that $(x,0) \in V_x \subset B_x := B((x,1),1) \cup \{(x,0)\}$. Since B_x is open in $\mathbb{H}_{\text{bub}}$, Theorem 3.1 again guarantees such a V_x exists. Then as earlier if there exist $\alpha, \beta \in \mathbb{R}$ such that $V_\alpha = V_\beta$, then we have that $(\beta, 0) \in V_\alpha \subset B_\alpha$. But $(\beta, 0) \notin B((\alpha,1),1)$ since everything in this ball has y-coordinate strictly greater than 0, so $\beta \in B_\alpha$ implies that $(\beta,0) \in \{(\alpha,0)\}$, which is only the case if $\alpha = \beta$ and our map is injective. Then this shows that $|\mathbb{R}| \leq |\mathcal{B}|$, and so $\mathbb{H}_{\text{bub}}$ is not 2^{nd} countable.

Theorem 5.11. Every uncountable set in a 2^{nd} countable space has a limit point.

Proof. Let A be an uncountable set in a space X, and suppose it has no limit points. Then every point of A is an isolated point, so by Exercise 2.10, for each $x \in A$, there exists an open U_x such that $U_x \cap A = \{x\}$. Suppose $\mathcal{B}$ is a basis for the topology on X. Then there exists a $V_\alpha \in \mathcal{B}$ such that $x \in V_x \subset U_x$. Then $V_x \cap A = \{x\}$, so this defines a map from A to $\mathcal{B}$ associating a point of A with a basic open set containing it. Then if there exist $\alpha, \beta \in A$ such that $V_\alpha = V_\beta$, then there exists $\beta \in V_\beta = V_\alpha$ and so $\beta \in V_\alpha \cap A = \{\alpha\}$. Therefore $\beta = \alpha$ and this map is injective, meaning $|A| \leq |\mathcal{B}|$. This means the basis $\mathcal{B}$ is uncountable and so our space X is not 2^{nd} countable. We have shown the contrapositive of the claim. $\square$

Exercise 5.12. All 2^{nd} countable spaces are hereditarily 2^{nd} countable.

Proof. This is a corollary of Theorem 3.30, which says that if $\mathcal{B}$ is a basis for a space $(X, \mathcal{T})$, and $(Y, \mathcal{T}_Y)$ is a subspace, then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$. If $\mathcal{B}$ is countable, then so is $\mathcal{B}_Y$: the map $f : \mathcal{B} \to \mathcal{B}_Y$ given by $f(B) = B \cap Y$ is a surjection. $\square$

Exercise 5.13. If X and Y are 2^{nd} countable, then so is $X \times Y$.

Proof. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be countable bases for X and Y. Then the product $\mathcal{B} = \mathcal{B}_X \times \mathcal{B}_Y$ is countable, and since for just two sets X and Y the product topology on $X \times Y$ contains all products $U \times V$ where U is open in X and V is open in Y, all elements of $\mathcal{B}$ are in the topology, meaning condition (1) of Theorem 3.1 is satisfied. Let W be an open set in $X \times Y$ and let $p = (x,y) \in W$. Then by the definition of the product topology, it can be written as the union of sets of the for $U_\alpha \times V_\alpha$ for all $\alpha \in \lambda$ for some index set λ. Therefore there exists a particular α such that $p \in U_\alpha \times V_\alpha$ where U_α is open in X and V_α is open in Y. Since $\mathcal{B}_X$ is a basis for the topology on X, there exists a basic open set $B_{x,\alpha} \in \mathcal{B}_X$ such that $x \in B_{x,\alpha} \subset U_\alpha$. Similarly, there is a $B_{y,\alpha} \in \mathcal{B}_Y$ such that $y \in B_{y,\alpha} \subset V_\alpha$. Therefore $p = (x, y) \in B_{x,\alpha} \times B_{y,\alpha} \subset U_\alpha \times V_\alpha \subset W$. Therefore condition (2) is satisfied, and so $\mathcal{B}$ is a basis for the product topology on $X \times Y$. Since $\mathcal{B}$ is countable, $X \times Y$ is 2^{nd} countable. $\square$

5.3 1st Countable Spaces

Theorem 5.14. If X is a 2^{nd} countable space, then it is 1^{st} countable.

Proof. Let X be a 2^{nd} countable space. Then it has a countable basis $\mathcal{B}$. Let $p \in X$ and define the set $\mathcal{B}_p$ to be $\mathcal{B}_p = \{V \in \mathcal{B} \mid p \in V\}$. By definition, every set in B_p is an open set containing p, and if W is an open set containing p, then since $\mathcal{B}$ is a basis, there exists a $V_p \in \mathcal{B}$ such that $p \in V_p \subset W$, and since $p \in V_p$, $V_p \in \mathcal{B}_p$, which means both conditions for being a neighborhood basis are satisfied by $\mathcal{B}_p$. Since $\mathcal{B}_p \subset \mathcal{B}$ and $\mathcal{B}$ is countable, $\mathcal{B}_p$ is countable. Since p was an arbitrary point of X and has a countable neighborhood basis, the space X is 1^{st} countable. $\square$

Theorem 5.15. If p is a point in a space X with a countable neighborhood basis, then there exists a nested countable neighborhood basis for p.

Proof. Let $\mathcal{B} = \{U_n\}_{n\in\mathbb{N}}$ be a countable neighborhood basis for p. Define the set $V_n = \bigcap_{i=1}^{n} U_i$. Then the set $\mathcal{B}_{\text{nested}} = \{V_n \mid n \in \mathbb{N}\}$ is countable and nested, since

$$V_n = U_n \cap \bigcap_{i=1}^{n-1} = U_n \cap V_{n-1} \subset V_{n-1}$$

for all $n \in \mathbb{N}$. Since every set in $\mathcal{B}_{\text{nested}}$ is in $\mathcal{B}$ which is a neighborhood basis for p, all V_n contain p, and if W is an open set containing p, then there exists a $U_j \in \mathcal{B}$ such that $p \in U_j \subset W$. Then by the above, we have that $V_n \subset U_n$ for all $n \in \mathbb{N}$, so $p \in V_j \subset U_j \subset W$ and $\mathcal{B}_{\text{nested}}$ is also a countable neighborhood basis for p. $\square$

Theorem 5.18. Let X be a 1^{st} countable space with a subset $A \subset X$. If $x \in X$ is a limit point of A, then there exists a sequence $\{a_i\}_{i\in\mathbb{N}}$ contained in A that converges to x.

Proof. Since X is 1^{st} countable, there exists a countable neighborhood basis for x that is nested (by Theorem 5.15, $\mathcal{B} = \{V_i \mid i \in \mathbb{N}\}$. Since each V_i is open and x is a limit point of A, the sets $(V_i - \{x\}) \cap A$ are all nonempty, so for each there exists a point $a_i \in (V_i - \{x\}) \cap A$. Then the sequence $(a_i)_{i \in \mathbb{N}}$ is contained in A and converges to x, because for any open set U containing x, there exists an $N \in \mathbb{N}$ such that $x \in V_N \subset U$, and so for all $i > N$, $a_i \in V_i \subset V_N \subset U$ since the neighborhood basis we chose for x is nested. □

Exercise 5.19. All 1^{st} countable spaces are hereditarily 1^{st} countable.

Proof. Let $(X, \mathcal{T})$ be a 1^{st} countable space with subspace $(Y, \mathcal{T}_Y)$, and let $p \in Y$ be a point with countable neighborhood basis $\mathcal{B}$. Then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a countable collection of subsets of Y. It is also a neighborhood basis for p because $p \in Y$ and $p \in B$ for all $B \in \mathcal{B}$ implies p is in all elements of $\mathcal{B}_Y$. We also have that if U is an open set in $(Y, \mathcal{T}_Y)$ containing p, then there exists an open set V in $(X, \mathcal{T})$ such that $U = V \cap Y$. Therefore $p \in V$ and so there exist a $B_0 \in \mathcal{B}$ such that $p \in B_0 \subset V$, which means $p \in B_0 \cap Y \subset V \cap Y = U$. Since $B_0 \cap Y \in \mathcal{B}_Y$, $\mathcal{B}_Y$ is a countable neighborhood basis for p in $(Y, \mathcal{T}_Y)$, and because p was an arbitrary point, we have shown that $(Y, \mathcal{T}_Y)$ is 1^{st} countable, as required. □

6 Compactness: The Next Best Thing to Being Finite

6.1 Compact Sets

Theorem 6.1. Let X be a finite topological space. Then X is compact.

Proof. Let $(X, \mathcal{T})$ be a finite topological space and let $\mathscr{C}$ and open cover. Then $\mathscr{C} \subset \mathcal{T}$ because elements of $\mathscr{C}$ are open sets and are therefore in $\mathcal{T}$, and $\mathcal{T} \subset 2^X$ because elements of $\mathcal{T}$ are subsets of X. The space X is finite, so its power set 2^X is also finite, which means $\mathcal{T}$ and $\mathscr{C}$ are also finite and $\mathscr{C}$ is therefore a finite subcover of itself. □

Theorem 6.2. If C is a compact subset of $\mathbb{R}_{\text{std}}$, then C has a maximum point.

Proof. We will show the contrapositive. Let C be a nonempty subset of $\mathbb{R}_{\text{std}}$ without a maximum point. Define the function $f : C \to \mathcal{T}_{\text{std}}$ by $f(x) = (-\infty, x)$ for $x \in C$. Then the image of f is an open cover for C because if $a \in C$, then there exists a $b \in C$ such that $a < b$ (since C has no maximum), and therefore $a \in f(b)$. Let F be a finite, nonempty subset of $f(C)$. Then $f^{-1}(F)$ is a finite, nonempty subset of C (f is an injection) and hence has a maximum, α. However, there is no maximum of C, so there exists some $\beta \in C$ such that $\alpha < \beta$. Let $F_0 \in F$. Then $F_0 = (-\infty, x_0)$ for some $x_0 \in f^{-1}(F)$. Since α is the maximum of

$f^{-1}(F)$, $x_0 \leq \alpha < \beta$, which means $\beta \notin F_0$. We have shown that β is not in any element of F, which means F is not a cover of C. Since F was an arbitrary finite subset of $f(C)$, we have shown that there exists an open cover of C without a finite subcover, meaning C is not compact. $\square$

Theorem 6.3. Let X be compact. Then every infinite subset of X has a limit point.

Proof. We again show the contrapositive. Let A be an infinite subset of a topological space X such that A has no limit points. For each point $p \in A$, p is not a limit point, so there exists at least one open set U_p such that $(U_p - \{p\}) \cap A = \emptyset$. Since A has no limit points, it vacuously contains all its limit points and is therefore closed, meaning $X - A$ is open. The set $\mathscr{C} = \{U_p \mid p \in A\} \cup \{(X - A)\}$ is an open cover for X because each U_p is open and $p \in A$ means $p \in U_p$ and $p \notin A$ means $p \in X - A$, so every point in X is in some element of $\mathscr{C}$ and therefore $X \subset \bigcup_{V \in \mathscr{C}} V$. Let F be a finite subset of $\mathscr{C}$. Then there exists an $x \in A$ such that $U_x \notin F$, otherwise $F = \mathscr{C} - \{(X - A)\}$ which is infinite. Let $U_y \in F$ be arbitrary. Then $(U_y - \{y\}) \cap A = \emptyset$, and $x \in A$ therefore implies x must not be in $U_y - \{y\}$. We have that $x \neq y$, because otherwise we would have $U_x = U_y \in F$, so in order for $(U_y - \{y\}) \cap A$ to be empty, we must have $x \notin U_y$. The set U_y was arbitrary, so x does not belong to any element of F of this form, and $x \in A$ implies $x \notin (X - A)$, so there is no element of F that x belongs to, meaning F is not a cover of X. Since F was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ has no finite subcover and therefore X is not compact. $\square$

Corollary 6.4. If X is compact and $E \subset X$ has no limit points, then E is finite.

Proof. By Theorem 6.3, every infinite subset of X has a limit point, which means in order to have no limit points, E must be finite. $\square$

Theorem 6.5. A space X is compact if and only if every collection of closed sets with the finite intersection property has a nonempty intersection.

Proof. ($\Longrightarrow$) Let X be a compact space and let $\{D_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets. To show that the finite intersection property implies a nonempty intersection, we will show that $\bigcap_{\alpha \in \lambda} D_\alpha = \emptyset$ implies that this collection does not have the finite intersection property. Suppose this intersection is empty. Then

$$\bigcap_{\alpha \in \lambda} D_\alpha = \emptyset \implies X = X - \bigcap_{\alpha \in \lambda} D_\alpha = \bigcup_{\alpha \in \lambda} (X - D_\alpha),$$

and since all $X - D_\alpha$ are open, the collection $\{X - D_\alpha\}_{\alpha\in\lambda}$ is an open cover for X. Because X is compact, this open cover has a finite subcover, call it $\{X - D_i\}_{i\in N}$ where N is a finite subset of the index set λ. Therefore we have that

$$X = \bigcup_{i\in N}(X - D_i) = X - \bigcap_{i\in N} D_i \implies \bigcap_{i\in N} D_i = \emptyset,$$

and so we have found a finite subcollection of $\{D_\alpha\}_{\alpha\in\lambda}$ that does not have a nonempty intersection, meaning this collection does not have the finite intersection property.

($\Longleftarrow$) Now suppose that in a space X, every collection of closed sets with the finite intersection property has a nonempty intersection. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be an open cover of X. Then

$$X = \bigcup_{\alpha\in\lambda} C_\alpha \implies \bigcap_{\alpha\in\lambda}(X - C_\alpha) = \emptyset.$$

Therefore $\{X - C_\alpha\}_{\alpha\in\lambda}$ is a collection of closed sets with empty intersection, which means it does not have the finite intersection property by our hypotheses. Therefore there exists a finite subcollection $\{X - C_i\}_{i\in N}$ for some finite $N \subset \lambda$ such that $\bigcap_{i\in N}(X - C_i) = \emptyset$. Therefore $\bigcup_{i\in N} C_i = X$, which means $\{C_i\}_{i\in N}$ is a finite subcover of $\{C_\alpha\}_{\alpha\in\lambda}$. Since $\{C_\alpha\}_{\alpha\in\lambda}$ was an arbitrary open cover, all open covers have finite subcovers, and therefore X is compact. $\square$

Theorem 6.6. A space X is compact if and only if for any open set U in X and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$, there exists a finite subcollection of K_αs whose intersection is a subset of U.

Proof. ($\Longrightarrow$) Let X be a compact space, let U be an open set, and let $\{K_\alpha\}_{\alpha\in\lambda}$ be a collection of closed sets with $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$. Then we have that

$$X - U \subset X - \bigcap_{\alpha\in\lambda} K_\alpha = \bigcup_{\alpha\in\lambda}(X - K_\alpha) \implies X = U \cup \bigcup_{\alpha\in\lambda}(X - K_\alpha),$$

so the collection $\{U\} \cup \{X - K_\alpha\}_{\alpha\in\lambda}$ is an open cover for X, and since X is compact, there exists a finite subcover which either looks like $\{U\} \cup \{X - K_i\}_{i\in N}$ or $\{X - K_i\}_{i\in N}$ for some finite $N \subset \lambda$. Then either

$$X = U \cup \bigcup_{i\in N}(X - K_i) \quad \text{or} \quad X = \bigcup_{i\in N}(X - K_i),$$

and in both cases we have

$$X - U \subset \bigcup_{i \in N} (X - K_i) = X - \bigcap_{i \in N} K_i \implies \bigcap_{i \in N} K_i \subset U,$$

as required.

($\Longleftarrow$) Let X be a space in which every open set U and collection of closed sets with intersection in U has a finite subcollection with intersection in U. Let $\{K_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets with empty intersection. Then $\bigcap_{\alpha \in \lambda} K_\alpha \subset \emptyset = U$, an open set, so there exists a finite subcollection of the K_αs with empty intersection, meaning $\{K_\alpha\}_{\alpha \in \lambda}$ does not have the finite intersection property. Therefore all collections of closed sets with the finite intersection property have nonempty intersections, meaning X is compact by Theorem 6.5. $\square$

Exercise 6.7. The union of finitely many compact subsets of X is compact.

Proof. Let $\{A_i\}_{i=1}^n$ be compact subsets of X, and let $\mathscr{C}$ be an open cover of $\bigcup_{i=1}^n A_i$. Then $\mathscr{C}$ is an open cover of A_i for all $i = 1, \dots, n$, so there exist finite subcovers C_i such that C_i is a cover of A_i. The collection $\mathscr{C}' = \bigcup_{i=1}^n C_i$ is a subset of $\mathscr{C}$ because if $U \in \mathscr{C}'$, then $U \in C_j \subset \mathscr{C}$ for some $j = 1, \dots, n$, and it is a finite collection because it is the finite union of finite sets. If $a \in \bigcup_{i=1}^n A_i$, then $a \in A_j$ for some j, and therefore $a \in U^*$ for some $U^* \in C_j \subset \mathscr{C}'$ since C_j is a cover of A_j. Therefore all elements of $\bigcup_{i=1}^n A_i$ are in some element of $\mathscr{C}'$, so $\mathscr{C}'$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, $\bigcup_{i=1}^n A_i$ is compact. $\square$

Theorem 6.8. Let A be a closed subspace of a compact space X. Then A is compact.

Proof. Let $\mathscr{C}$ be an open cover of A. Then since A is closed, $X - A$ is open and so the collection $\mathscr{C} \cup \{X - A\}$ is an open cover for X. X is compact, so this open cover has a finite subcover $\mathscr{C}'$. Then the collection $\mathscr{C}^* = \mathscr{C}' - \{X - A\}$ is a finite subcover of $\mathscr{C}$ since every element of $\mathscr{C}^*$ is an element of $\mathscr{C}$, $|\mathscr{C}^*| \le |\mathscr{C}'|$ which is finite, and every element in A is in some element of $\mathscr{C}'$ (because it covers X) other than $X - A$. $\mathscr{C}$ was an arbitrary open cover of A, so A is compact. $\square$

Theorem 6.9. Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Proof. Suppose X is Hausdorff and that $A \subset X$ is not closed. We will show that A is not compact. A not being closed means there is a limit point p of A such that $p \notin A$. For all $a \in A$, $a \neq p$, so since X is Hausdorff, there exist disjoint open sets U_a and V_a such that

$a \in U_a$ and $p \in V_a$. Since all U_a are open, the collection $\mathscr{C} = \{U_a\}_{a \in A}$ is an open cover of A. Let $F = \{U_i\}_{i \in N}$ be a finite subcollection for some finite $N \subset A$. The set $V = \bigcap_{i \in N} V_i$ is the finite intersection of open sets and is therefore open. Additionally, $p \in V_a$ for all $a \in A$, so $p \in V$. Since p is a limit point of A, $(V - \{p\}) \cap A$ is nonempty, so there exists an $x \in V \cap A$. Because $x \in V$ means $x \in V_i$ for all $i \in N$, $x \notin U_i$ for all $i \in N$, and so we have an element $x \in A$ that is not covered by the collection $F = \{U_i\}_{i \in N}$. Therefore A is not compact because it has an open cover with no finite subcover. We have shown that A not closed implies that A is not compact or not Hausdorff, so this is the contrapositive of the theorem statement. □

Exercise 6.10. Consider the interval $(0,1)$ as a subset of $\mathbb{R}$ with the finite complement topology. There are infinitely many points in $(0,1)$, so it is not closed. Let $\mathscr{C}$ be an open cover of $(0,1)$. $(0,1)$ is nonempty, so there must be at least one set in $\mathscr{C}$. Call this set C_0. Because C_0 is open, its complement $\mathbb{R} - C_0$ is finite, which means the set $(0,1) - C_0 \subset \mathbb{R} - C_0$ is also finite, so $(0,1) - C_0 = \{a_1, \ldots, a_n\}$ for some $a_i \in (0,1)$. Since $\mathscr{C}$ is an open cover of $(0,1)$ and $a_i \in (0,1)$, there exists a $C_i \in \mathscr{C}$ such that $a_i \in C_i$ for all $i = 1, \ldots, n$. Hence the collection $\{C_i\}_{i=0}^{n}$ is a cover of $(0,1)$, and since each $C_i \in \mathscr{C}$ for $i = 0, \ldots, n$, this collection is a finite subcover of the arbitrary open cover $\mathscr{C}$. Therefore $(0,1) \subset \mathbb{R}$ with the finite complement topology is an example of a subset the is compact but not closed.

Exercise 6.11. Intersections of compact sets need not be compact (I think the Double Headed Snake should have a counterexample?). However if we add the assumption that we are working with subsets of a Hausdorff space, then we can say that arbitrary intersections of compact sets are compact. If $\{K_\alpha\}_{\alpha \in \lambda}$ is a collection of compact sets, then Theorem 6.9 implies that K_α is closed for all $\alpha \in \lambda$, and therefore the intersection $K = \bigcap_{\alpha \in \lambda} K_\alpha$ is also closed. Since K is a closed subset of the compact set K_β for any $\beta \in \lambda$, Theorem 6.8 implies that K is itself compact.

Theorem 6.12. Every compact, Hausdorff space is normal.

Proof. Let X be a compact, Hausdorff space with closed set $A \subset X$ and $p \in X - A$. We will first show that X is regular. For each $\alpha \in A$, X being Hausdorff means that there exist disjoint open sets U_α and V_α such that $p \in U_\alpha$ and $\alpha \in V_\alpha$. Therefore $\{V_\alpha\}_{\alpha \in A}$ is an open cover of A. Since A is a subset of the compact space X, by Theorem 6.8, A is compact, and so the open cover $\{V_\alpha\}_{\alpha \in A}$ has a finite subcover $\{V_i\}_{i \in N}$ for some finite subset $N \subset A$. Define the sets $U = \bigcap_{i \in N} U_i$ and $V = \bigcup_{i \in N} V_i$. U is open because it is the *finite* intersection of open sets, and V is open because it is the union of open sets. We also have that $p \in U$ because $p \in U_i$ for all $i \in N$, and $A \subset V$ because $\{V_i\}_{i\in N}$ is a cover of A. It remains to show that U and V are disjoint. Suppose for contradiction that there exists an element $x \in U \cap V$. Then $x \in V$ means there exists a $j \in N$ such that $x \in V_j$, and $x \in U$ means $x \in U_i$ for all $i \in N$, in particular, $x \in U_j$. Therefore $x \in U_j \cap V_j$, but this is a contradiction since U_j and V_j are disjoint. Therefore $U \cap V = \emptyset$, and so X is regular.

Now let A and B be closed subsets of X. Since X is regular, for each $\alpha \in A$, there exist disjoint open sets U_α and V_α such that $\alpha \in U_\alpha$ and $B \subset V_\alpha$. Therefore $\{U_\alpha\}_{\alpha\in\lambda}$ is an open cover of A, and since A is a closed subset of the compact space X, A is compact, so there exists a finite subcover $\{U_i\}_{i\in N}$ for some finite subset $N \subset A$. Define $U = \bigcup_{i\in N} U_i$ and $V = \bigcap_{i\in N} V_i$. U is open because it is the union of open sets, and V is open because it is the *finite* intersection of open sets. The collection $\{U_i\}_{i\in N}$ is a cover of A, so $A \subset U$, and $B \subset V_i$ for all $i \in N$ by construction, so $B \subset V$ as well. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means there exists a $j \in N$ such that $x \in U_j$, and $x \in V$ means $x \in V_i$ for all $i \in N$. Therefore $x \in U_j \cap V_j$, but this is a contradiction since U_j and V_j are disjoint. Therefore $U \cap V = \emptyset$, and so X is normal. □

Theorem 6.13. Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets has a finite subcover.

Proof. ($\Longrightarrow$) Let $\mathcal{B}$ be a basis for a compact space X, and let $\mathscr{C}$ be a basic open cover. A cover by basic open sets is an open cover, and since X is compact, $\mathscr{C}$ has a finite subcover.

($\Longleftarrow$) Suppose every cover by basic open sets has a finite subcover and let $\mathscr{C}$ be an open cover. Then for all $x \in X$, there exists a $C_x \in \mathscr{C}$ such that $x \in C_x$. Since C_x is open, there exists a basic open V_x such that $x \in V_x \subset C_x$. Since such a V_x exists for all $x \in X$, the collection $\mathscr{C}' = \{V_x\}_{x\in X}$ is a cover of X by basic open sets and therefore has a finite subcover $\{V_i\}_{i\in N}$ for some finite subset $N \subset X$. Define $\mathscr{C}^* = \{C_i\}_{i\in N}$. Then since $C_i \in \mathscr{C}$ for all $i \in N$, $\mathscr{C}^*$ is a finite subcollection of $\mathscr{C}$. We also have that

$$X = \bigcup_{i\in N} V_i \subset \bigcup_{i\in N} C_i$$

since $V_x \subset C_x$ for all $x \in X$. Therefore $\mathscr{C}^*$ covers X and so is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitary open cover, all open covers have finite subcovers and X is compact. □

6.2 The Heine-Borel Theorem

Theorem 6.14. For all $a \leq b$, the subspace $[a, b]$ is compact.

Proof. If $a = b$, then any open cover $\mathscr{C}$ of $[a, b]$ contains an open set C with $a \in C$, so $\{C\}$ is a finite subcover of $[a, b]$. Assume $a < b$ and let $\mathscr{C}$ be an open cover of $[a, b]$. Then the set

$$A = \{x \in [a, b] \mid [a, x] \text{ is covered by a finite subcover of } \mathscr{C}\}$$

is nonempty ($a \in A$) and is bounded above (for example by b). Therefore $s = \sup A$ exists. Since b is an upper bound of A by construction, $s \leq b$. Suppose for contradiction that $s < b$. Then since $s \in [a, b]$ and $\mathscr{C}$ is an open cover of $[a, b]$, there exists a $C_0 \in \mathscr{C}$ such that $s \in C_0$. Therefore there exists a basic open set $B_0 = (s - \varepsilon_0, s + \varepsilon_0)$ such that $s \in B_0 \subset C_0$. Set $\varepsilon = \min\{s - a, b - s, \varepsilon_0\}$. Then $a \leq s - \varepsilon < s$ so $s - \varepsilon$ is not an upper bound of A, which means there exists a $y \in A$ such that $s - \varepsilon < y < s$. Since $y \in A$, the set $[a, y]$ is covered by a finite subcover of $\mathscr{C}$, call it $\mathscr{C}'$. Define $\mathscr{C}^* = \mathscr{C}' \cup C_0$. Then $\mathscr{C}^*$ is a finite subcollection of $\mathscr{C}$, and since $[a, y] \subset \bigcup_{C \in \mathscr{C}'} C$, we have that

$$\left[a, s + \frac{\varepsilon}{2}\right] \subset [a, y] \cup (s - \varepsilon, s + \varepsilon) \subset [a, y] \cup B_0 \subset \left(\bigcup_{C \in \mathscr{C}'} C\right) \cup C_0 = \bigcup_{C \in \mathscr{C}^*} C.$$

Therefore $[a, s + \varepsilon/2]$ is covered by a finite subcover of $\mathscr{C}$, and so $s + \varepsilon/2 \in A$, contradicting $s = \sup A$. Since $s \not< b$, $s = b$. Since $b \in [a, b]$, there exists a $C_1 \in C$ such that $b \in C_1$, and so there also exists a basic open set $B_1 = (b - \varepsilon_1, b + \varepsilon_1)$ such that $b \in B_1 \subset C_1$. If $b - \varepsilon_1 < a$, then $[a, b] \subset B_1 \subset C_1$ and $\{C_1\}$ is a finite subcover of $\mathscr{C}$ for $[a, b]$. If $a \leq b - \varepsilon_1$, then $b - \varepsilon_1$ is not an upper bound of A since $b = \sup A$, so there exists a $z \in (b - \varepsilon_1, b)$ such that $z \in A$. Then $z \in A$ means $[a, z]$ is finitely covered by some subcollection of $\mathscr{C}$, call it $\mathscr{C}'$, and so we have that

$$[a, b] \subset [a, z] \cup (b - \varepsilon_1, b + \varepsilon_1) \subset \left(\bigcup_{C \in \mathscr{C}'} C\right) \cup C_1.$$

Therefore $[a, b]$ is covered by a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover of $[a, b]$, all open covers have finite subcovers and $[a, b]$ is compact. $\square$

Theorem 6.15. Let A be a subset of $\mathbb{R}_{\text{std}}$. Then A is compact if and only if A is closed and bounded.

Proof. ($\Longrightarrow$) Let A be a compact subset of $\mathbb{R}_{\text{std}}$. Then since $\mathbb{R}_{\text{std}}$ is Hausdorff, by Theorem 6.9 we have that A is closed. Suppose for contradiction that A is unbounded, and without loss of generality, assume it is unbounded above. Then the set $\mathscr{C} = \{(-\infty, a) \mid a \in A\}$ is an open cover for A, and if $\mathscr{C}'$ is a finite subset of $\mathscr{C}$, then the set $\{a \in A \mid (-\infty, a) \in \mathscr{C}'\}$ has a maximum. Since A is unbounded above, there exists an $\alpha \in A$ such that $\alpha > \max\{a \in A \mid (-\infty, a) \in \mathscr{C}'\}$, and therefore $\alpha \notin \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. Since $\mathscr{C}'$ was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ is an open cover of A without a finite subcover, and therefore A is not compact, a contradiction. Therefore A being compact means A is closed and bounded.

($\Longleftarrow$) Let A be a closed and bounded subset of $\mathbb{R}_{\text{std}}$. Then there exists an $M \in \mathbb{R}$ such that $A \subset [-M, M]$, and since $[-M, M]$ is compact by Theorem 6.14, the set A is also compact by Theorem 6.8. $\square$

Exercise 6.16. Consider the set $A = [0, \sqrt{2}] \cap \mathbb{Q}$. Since $[0, \sqrt{2}]$ is closed in $\mathbb{R}_{\text{std}}$, A is closed in $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$ by Theorem 3.28, and note that A is bounded below by 0 and above by 2. Define $\mathscr{C} = \{(-\infty, q) \cap \mathbb{Q} \mid q \in A\}$ and let $\mathscr{C}'$ be a finite subset of $\mathscr{C}$. Then the set $\{a \in A \mid (-\infty, a) \cap \mathbb{Q} \in \mathscr{C}'\}$ is finite and therefore has a maximum. Note that A has no maximum since $\sqrt{2} \notin A \subset \mathbb{Q}$, so there exists an $\alpha \in A$ such that $\alpha > \max\{a \in A \mid (-\infty, a) \cap \mathbb{Q} \in \mathscr{C}'\}$. Therefore $\alpha \notin \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. Since $\mathscr{C}$ is therefore an open cover of A without a finite subcover, A is not compact in $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$.

Theorem 6.17. Every compact subset $C \subset \mathbb{R}_{\text{std}}$ has a maximum.

Proof. Assume C is nonempty By the Heine-Borel Theorem, C is closed and bounded. Since C is bounded and nonempty, $\alpha = \sup C$ exists. We claim that $\alpha \in C$. Let U be an open set containing α. Then there exists a basic open set $V = (\alpha - \varepsilon, \alpha + \varepsilon)$ for some $\varepsilon > 0$ such that $\alpha \in V \subset U$. Since $\alpha = \sup C$, the point $\alpha - \varepsilon$ is not an upper bound for C, which means there exists a point $c \in C$ such that $\alpha - \varepsilon < c \leq \alpha = \sup C$. Therefore $c \in V \subset U$. If $c = \alpha$, then $\alpha \in C$ and we are done. If $c \neq \alpha$, then $c \in U - \{\alpha\}$, so $(U - \{\alpha\}) \cap C$ is nonempty. For an arbitrary open set U containing α, we have that shown that either $\alpha \in C$ or $(U - \{\alpha\}) \cap C \neq \emptyset$. Therefore if there exists a choice of U such that $\alpha \in C$, we are done, and if not, then $(U - \{\alpha\}) \cap C \neq \emptyset$ for all open sets U containing α, which means α is a limit point of C. Becuase C is closed, it contains all of its limit points, so $\alpha \in C$ in this case as well. Therefore $\alpha \in C$ and $\alpha = \sup C \geq c$ for all $c \in C$, which means α is a maximum of C. $\square$

6.3 Compactness and Products

Lemma 6.18. Let Y be compact, let $x_0 \in X$, and let U be an open set containing the slice $x_0 \times Y$. Let $y \in Y$ be arbitrary. Then $(x_0, y) \in x_0 \times Y \subset U$, so there exists a basic open set $W_y \times V_y$ (where W_y is open in X and V_y is open in Y) such that $(x_0, y) \in W_y \times V_y \in U$. Then since y was arbitary, there exists such an open V_y for all $y \in Y$, and therefore $\mathscr{C} = \{V_y\}_{y \in Y}$ is an open cover of Y. Since Y is compact, there exists a finite subcover $\mathscr{C}' = \{V_i\}_{i \in N}$ for some finite $N \subset Y$. Define the set $W = \bigcap_{i \in N} W_i$. Since N is finite, W is the intersection of a finite number of sets that are open in X and is therefore also open in X. W also contains x_0 because $x_0 \in W_y$ for all $y \in Y$ and $N \subset Y$. Therefore the set $W \times Y$ is an open tube in $X \times Y$ containing the slice $x_0 \times Y$. Let (α, β) be a point in this tube. Then $\alpha \in W$ and $\beta \in Y$. Recall that $\mathscr{C}' = \{V_i\}_{i \in N}$ is a cover of Y. Therefore $\beta \in Y$ implies that there exists a $j \in N$ such that $\beta \in V_j$. Then because $\alpha \in W$, $\alpha \in W_i$ for all $i \in N$, and in particular, $\alpha \in W_j$. Therefore $(\alpha, \beta) \in W_j \times V_j$, which is a basic open subset of U, meaning we also have $(\alpha, \beta) \in U$. Therefore W is open in X, contains x_0, and $W \times Y \subset U$, as required.

Theorem 6.19. Let X and Y be compact spaces. Then $X \times Y$ is compact.

Proof. Let X and Y be compact spaces and let $\mathscr{C} = \{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of $X \times Y$ by basic open sets. By Theorem 6.13, it is sufficient to show that this cover has a finite subcover in order to show that $X \times Y$ is compact. Since U_α is a basic open set for all $\alpha \in \lambda$, we can rewrite $\mathscr{C}$ as $\mathscr{C} = \{S_\alpha \times T_\alpha\}_{\alpha \in \lambda}$ such that S_α is open in X for all $\alpha \in \lambda$ and T_α is open in Y for all $\alpha \in \lambda$. For $x \in X$, define $\gamma_x = \{\alpha \in \lambda \mid x \in S_\alpha\}$. Then $\{S_\alpha \times T_\alpha\}_{\alpha \in \gamma_x}$ covers the slice $x \times Y$ for all $x \in X$. Therefore $\{T_\alpha\}_{\alpha \in \gamma_x}$ is an open cover for Y, and since Y is compact, $\{T_\alpha\}_{\alpha \in \gamma_x}$ has a finite subcover, call it $\{T_i\}_{i \in N_x}$ for some finite subset $N_x \subset \gamma_x$. This implies that $x \times Y$ is covered by the finite collection $\{S_i \times T_i\}_{i \in N_x}$, meaning $x \times Y$ subset $\bigcup_{i \in N_x}(S_i \times T_i)$. Since Y is compact, there exists an open set $W_x \subset X$ containing x such that $x \times Y \subset W_x \times Y \subset \bigcup_{i \in N_x}(S_i \times T_i)$ by the tube lemma. Since there exists such a W_x for all $x \in X$, the collection $\{W_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{W_x\}_{x \in X}$, call it $\{W_j\}_{j \in M}$ for some finite subset $M \subset X$. Define $N = \bigcup_{j \in M} N_j$. We claim that $\{S_i \times T_i\}_{i \in N}$ is a finite subcover of $\mathscr{C}$. Since N_j is finite for all $j \in M$ and M is also finite, the union N is finite. Since $N_j \subset \gamma_j \subset \lambda$ for all $j \in M$, the union N is also a subset of λ. Therefore $\{S_i \times T_i\}_{i \in N}$ is a finite subcollection of $\mathscr{C}$ and it only remains to show that this subcollection covers $X \times Y$. To show this, let $(x_0, y_0) \in X \times Y$ be arbitrary. Then $x_0 \in X$ and X is covered by $\{W_j\}_{j \in M}$, so there exists a $j_0 \in M$ such that $x_0 \in W_{j_0}$. Therefore the point (x_0, y_0) is in the tube $W_{j_0} \times Y$ since $y_0 \in Y$, and this tube is covered by $\{S_i \times T_i\}_{i \in N_{j_0}}$. Therefore there exists an $i_0 \in N_{j_0} \subset N$ such that $(x_0, y_0) \in S_{i_0} \times T_{i_0} \in \{S_i \times T_i\}_{i \in N}$. Therefore $X \times Y \subset \bigcup_{i \in N}(S_i \times T_i)$, so $\{S_i \times T_i\}_{i \in N}$ is a finite subcover of $\mathscr{C}$. $\square$

Theorem 6.20. A subset $A \subset \mathbb{R}^n_{\text{std}}$ is compact if and only if A is closed and bounded.

Proof. ($\Longrightarrow$) Suppose $A \subset \mathbb{R}^n_{\text{std}}$ is compact. Then since $\mathbb{R}_{\text{std}}$ is Hausdorff, A is closed by Theorem 6.9. Suppose for contradiction that A is unbounded. Then for every open ball $B(\mathbf{0}, M)$, we have that $A \not\subset B(\mathbf{0}, M)$, otherwise A would be bounded. However, the set $\mathscr{C} = \{B(\mathbf{0}, M) \mid M \in \mathbb{R}\}$ is an open cover for $\mathbb{R}^n$, so it is also an open cover for A. Let $\mathscr{C}'$ be a finite subcollection of $\mathscr{C}$. Then the set $\{M \in \mathbb{R} \mid B(\mathbf{0}, M) \in \mathscr{C}'\}$ is finite and so it has a maximum, call it M'. Then $\bigcup_{C \in \mathscr{C}'} C = B(\mathbf{0}, M')$, and so since A is unbounded, $A \not\subset \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. But this means A is not compact, so we have a contradiction and therefore have that A is bounded in addition to being closed.

($\Longleftarrow$) Suppose $A \subset \mathbb{R}^n_{\text{std}}$ is closed and bounded. Then there exists some ball $B(\mathbf{0}, M)$ such that $A \subset B(\mathbf{0}, M)$, and this ball is in turn a subset of the product of intervals $\prod_{i=1}^n [-M, M]$. By Theorem 6.14, $[-M, M] \subset \mathbb{R}_{\text{std}}$ is compact, which means that by Theorem 6.19, $\prod_{i=1}^n$ is also compact. Since A is a subset of a compact set, A is compact by Theorem 6.8. $\square$

Theorem 6.21. Let $\mathscr{S}$ be a subbasis for a space X. Then X is compact if and only if every subbasic open cover has a finite subcover.

Proof. ($\Longrightarrow$) Suppose X is compact, let $\mathscr{S}$ be a subbasis for X, and let $\mathscr{C}$ be an arbitrary subbasic open cover of X. Then since subbasic open sets are open sets, $\mathscr{C}$ is an open cover and therefore has a finite subcover since X is compact.

($\Longleftarrow$) Let X be a space with a subbasis $\mathscr{S}$ satisfing the property that every subbasic open cover has a finite subcover, and suppose for contradiction that X is not compact. Define the set $\mathscr{C}$ to be the set of all open covers of X that do not have finite subcovers. Because X is not compact, $\mathscr{C}$ is not empty since there is at least one open cover without a finite subcover. The elements of $\mathscr{C}$ are collections of open covers, so we may partially order them by set inclusion: if $C \in \mathscr{C}$, then $C \subset C$; if $A, B \in \mathscr{C}$ with $A \subset B$ and $B \subset A$, then $A = B$; and if $A, B, C \in \mathscr{C}$ with $A \subset B$ and $B \subset C$, then $A \subset C$, so this does define a partial order. Let $\mathfrak{T} \subset \mathscr{C}$ be an arbitary totally ordered subset. Define $C_{\mathfrak{T}} = \bigcup_{T \in \mathfrak{T}} T$ and note that for all $T \in \mathfrak{T}$, $T \subset C_{\mathfrak{T}}$, so $C_{\mathfrak{T}}$ is an upper bound for $\mathfrak{T}$. In order to apply Zorn's Lemma to $\mathscr{C}$, we claim that $C_{\mathfrak{T}} \in \mathscr{C}$, that is, we claim $C_{\mathfrak{T}}$ is an open cover of X with no finite subcover. Since $\mathfrak{T}$ is a subset of $\mathscr{C}$, each $T \in \mathfrak{T} \subset \mathscr{C}$ is an open cover of X, so $C_{\mathfrak{T}}$ is also an open cover of

X. Suppose for contradiction that $C_{\mathfrak{T}}$ had a finite subcover, call it $\{C_i\}_{i\in F}$ for some finite index set F. Since $C_{\mathfrak{T}} = \bigcup_{T\in\mathfrak{T}} T$, for all $i \in F$, there exists some $T_i \in \mathfrak{T}$ such that $C_i \in T_i$. Then since the set $\{T_i \mid i \in F\}$ is a subset of $\mathfrak{T}$, it is finite and totally ordered by inclusion (so all elements can be compared), and therefore this set has a maximum, call it T_j. Then $C_i \in T_j$ for all $i \in F$, and since $\{C_i\}_{i\in F}$ covers X, it is a finite subcover of $T_j \in \mathfrak{T} \subset \mathscr{C}$, a contradiction since elements of $\mathscr{C}$ are exactly those open covers that do not have finite subcovers. Therefore the upper bound $C_{\mathfrak{T}}$ of $\mathfrak{T}$ is an open cover without a finite subcover, and so $C_{\mathfrak{T}} \in \mathscr{C}$, as claimed. Since $\mathfrak{T}$ as an arbitrary totally ordere subset of $\mathscr{C}$, all such subsets have upper bounds in $\mathscr{C}$, and so we may apply Zorn's Lemma to $\mathscr{C}$ to say that $\mathscr{C}$ itself has a maximum, call it $M \in \mathscr{C}$.

We claim that $M \cap \mathscr{S}$ is not an open cover of X. If it were, $M \cap \mathscr{S} \subset \mathscr{S}$ would imply that it has a finite subcover since all subbasic open covers of X have finite subcovers. However, this cannot be the case because also $M \cap \mathscr{S} \subset M$, and so any finite subcover of $M \cap \mathscr{S}$ is a finite subcover of M, an open cover with no finite subcover. Therefore $M \cap \mathscr{S}$ is not an open cover of X, so there exists an $x \in X$ such that for all $V \in M \cap \mathscr{S}$, $x \notin N$. Since M itself is a cover of X, there exists a $U \in M$ such that $x \in U$. Because $\mathscr{S}$ is a subbasis for the topology on X, there exists a finite collection of subbasic open sets $S_1, \ldots, S_n$ such that $x \in \bigcap_{i=1}^n S_i \subset U$. For all $i = 1, \ldots, n$, we have that $S_i \notin M$. This is because $x \in \bigcap_{i=1}^n S_i$ implies that $x \in S_i$ for all $i = 1, \ldots n$, and therefore that $x \in U \cap S_i$. If S_i were an element of M, $U \cap S_i$ would be an element of $M \cap \mathscr{S}$ containing x, which would contradict our choice of x as an element not covered by $M \cap \mathscr{S}$.

We now have a finite number of subbasic open sets $S_1, \ldots, S_n$ such that $S_i \notin M$. Therefore for all $i = 1, \ldots, n$, M is a subset of $M \cup \{S_i\}$. Recall that M was defined to be the maximum of the set $\mathscr{C}$ with respect to the ordering by set inclusion, so M being a proper subset of $M \cup \{S_i\}$ means that for all $i = 1, \ldots, n$, $M \cup \{S_i\} \notin \mathscr{C}$. Since M on its own is an open cover of X, so is $M \cup \{S_i\}$. Therefore $M \cup \{S_i\}$ is an open cover of X not in $\mathscr{C}$, meaning it has a finite subcover, and this finite subcover must be of the form $M_i \cup \{S_i\}$ for some $M_i \subset M$. Otherwise, the finite subcover of $M \cup \{S_i\}$ would also be a finite subcover of $M \in \mathscr{C}$, which is known to have no finite subcover.

Define the set $\overline{M} = \bigcup_{i=1}^n M_i$. Since $M_i \cup \{S_i\}$ is finite for all $i = 1, \ldots, n$, each M_i is also finite, and therefore so is $\overline{M} \subset M$. Suppose now for contradiction that there exists some $x_0 \in X$ that is not covered by the finite collection $\overline{M} \cup \{U\}$. Then for all $M' \in \overline{M}$, we have that $x_0 \notin M'$, and that $x_0 \notin U$, otherwise x_0 would be covered. Since $x_0 \notin U$, also $x_0 \notin \bigcap_{i=1}^n S_i \subset U$. This means there exists a $j = 1, \ldots, n$ such that $x_0 \notin S_j$. However, the collection $M_j \cup \{S_j\}$ covers X, so $x_0 \notin S_j$ means there exists an $M_0 \in M_j$ such that $x_0 \in M_0$. But $M_0 \in M_j$ and $M_j \subset \overline{M}$, so $x_0 \in M_0 \in \overline{M} \cup \{U\}$, which was assumed to not cover x_0. Therefore there does not exists such an x_0, and so $\overline{M} \cup \{U\}$ covers X. But $\overline{M} \subset M$ and $U \in M$, so $\overline{M} \cup \{U\}$ is a finite subcover of M, a contradiction. Therefore X is compact since in assuming otherwise, we have reached a contradiction. $\square$

Theorem 6.23. Products of compact spaces are compact.

Proof. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of compact spaces and suppose for contradiction that the product space $\prod_{\alpha\in\lambda} X_\alpha$ is not compact. Recall that the product topology has a subbasis $\mathscr{S}$ consisting of sets of the form $\pi_\alpha^{-1}(U_\alpha)$ for sets U_α open in X_α. By Theorem 6.21, $\prod_{\alpha\in\lambda} X_\alpha$ not being compact means it has a subbasic open cover $\mathscr{C} = \{S_\beta\}_{\beta\in\gamma}$ without a finite subcover. Since for all $\beta \in \gamma$, S_β is a subbasic open set, each S_β is identified with an open set U_β in some factor space X_{α_β} for an index $\alpha_\beta \in \lambda$, and so it can be written as $\pi_{\alpha_\beta}^{-1}(U_\beta)$. Define C_α to be the collection of open sets in X_α given by $C_\alpha = \{U_\beta \mid \alpha_\beta = \alpha\}$.

Let $\delta \in \lambda$ be an arbitrary index. We claim that C_δ does not cover X_δ. X_δ is compact, so C_δ being an open cover would mean it would have to have a finite subcover, call it $C'_\delta = \{U_i\}_{i\in F}$ for some finite index set $F \subset \gamma$. Then define $\mathscr{C}' = \{S_i\}_{i\in F}$ and let $(x_\alpha)_{\alpha\in\lambda}$ be an arbitrary point in $\prod_{\alpha\in\lambda} X_\alpha$. Then $x_\delta \in X_\delta$ which is covered by C'_δ, and so there exists a $j \in F$ such that $x_\delta \in U_j$, and therefore that $(x_\alpha)_{\alpha\in\lambda} \in \pi_\delta^{-1}(U_j) = S_j \in \mathscr{C}'$. This implies that $\mathscr{C}'$ is a finite subcover of $\mathscr{C}$, which is a contradiction meaning that C_δ does not cover X_δ.

Since $\delta \in \lambda$ was arbitrary, we have that for all $\alpha \in \lambda$, C_α does not cover X_α, so for all $\alpha \in \lambda$, there exists an $x'_\alpha \in X_\alpha$ such that x'_α is not covered by C_α. Consider the point $(x'_\alpha)_{\alpha\in\lambda} \in \prod_{\alpha\in\lambda} X_\alpha$. Because we have assumed that $\mathscr{C}$ is a cover for the product space that contains this point, there exists some $S_{\beta'} \in \mathscr{C}$ such that $(x'_\alpha)_{\alpha\in\lambda} \in S_{\beta'} = \pi_{\alpha_{\beta'}}^{-1}(U_{\beta'})$. But this then implies that $x'_{\alpha_{\beta'}} \in X_{\alpha_{\beta'}}$ is an element of $U_{\beta'} \in C_{\alpha_{\beta'}}$, and therefore that $x'_{\alpha_{\beta'}}$ is covered by $C_{\alpha_{\beta'}}$, a contradiction since $(x'_\alpha)_{\alpha\in\lambda}$ was constructed so that each coordinate was not covered by its corresponding C_α. Therefore the original $\mathscr{C}$ does not actually cover $\prod_{\alpha\in\lambda} X_\alpha$, so our assumption that such a subbasic open cover without a finite subcover existed was false, meaning $\prod_{\alpha\in\lambda} X_\alpha$ is compact. $\square$

Exercise 6.24. Consider the space $\{0,1\}^\omega$ as a subspace of $[0,1]^\omega$. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence not in $\{0,1\}^\omega$. Then there exists an $m \in \mathbb{N}$ such that $a_m \in (0,1)$. Since the sets $[0,1]$ and $(0,1)$ are both open in $[0,1]$, the set $U = \underbrace{[0,1] \times \cdots \times [0,1]}_{(m-1)\text{ times}} \times (0,1) \times [0,1] \times [0,1] \times \ldots$ is open in $[0,1]^\omega$ and contains the sequence $(a_i)_{i\in\mathbb{N}}$. If $(x_i)_{i\in\mathbb{N}}$ is a sequence in $\{0,1\}^\omega$, then

$(x_i)_{i\in\mathbb{N}} \notin U$, because $x_m = \{0,1\}$, meaning $x_m \notin (0,1)$. Therefore U is an open set containing $(a_i)_{i\in\mathbb{N}}$ such that $(U-(a_i)_{i\in\mathbb{N}})\cap\{0,1\}^\omega = \emptyset$, and so $(a_i)_{i\in\mathbb{N}}$ is not a limit point of $\{0,1\}^\omega$. We have shown that all elements of $[0,1]^\omega - \{0,1\}^\omega$ are not limit points, and so if an element of $[0,1]^\omega$ is a limit point of $\{0,1\}^\omega$, it must also be am element of $\{0,1\}^\omega$. Therefore $\{0,1\}^\omega$ is a closed subset of $[0,1]^\omega$. As a subspace of $[0,1]^\omega$ with the box topology, $\{0,1\}^\omega$ also has the box topology, so by Exercise 3.42, $\{0,1\}^\omega$ has the discrete topology, meaning singletons are open. Therefore the set $\mathscr{C} = \{\{p\}\}_{p\in\{0,1\}^\omega}$ is an open cover of $\{0,1\}^\omega$. If $\mathscr{C}'$ is a finite subset of $\mathscr{C}$, then there exists a $p_0 \in \{0,1\}^\omega$ such that $\{p_0\} \notin \mathscr{C}'$, and therefore $p_0 \notin \bigcup_{C\in\mathscr{C}'} C$, so $\mathscr{C}'$ does not cover $\{0,1\}^\omega$. Therefore the subspace $\{0,1\}^\omega$ is not compact, so applying the contrapositive of Theorem 6.8, we have that $[0,1]^\omega$ is not compact or that $\{0,1\}^\omega$ is not closed. Since we have shown that $\{0,1\}^\omega$ is a closed subset of $[0,1]^\omega$, we have that $[0,1]^\omega$ is not compact with the box topology.

6.4 Countably Compact, Lindelöf Spaces

Theorem 6.25. Every countably compact and Lindelöf space is compact.

Proof. Let X be countably compact and Lindelöf, and let $\mathscr{C}$ be an open cover of X. Then because X is Lindelöf, $\mathscr{C}$ has a countable subcover, call it $\mathscr{C}^*$. Since $\mathscr{C}^*$ is a countable open cover of X and X is countably compact, $\mathscr{C}^*$ has a finite subcover, call it $\mathscr{C}'$. Therefore $\mathscr{C}' \subset \mathscr{C}^* \subset \mathscr{C}$, $\mathscr{C}'$ is finite, and and it covers X. Since $\mathscr{C}$ was an arbitrary open cover and it has a finite subcover, X is compact. □

Theorem 6.26. Let X be a T_1 space. Then X is countably compact if and only if every infinite subset has a limit point.

Proof. ($\Longrightarrow$) We will show the contrapositive. Suppose $A^* \subset X$ is an infinite subset with no limit point. Then by Theorem 1.9, A^* has a countably infinite subset, call it A. We claim that A also has no limit points. If p is a limit point of A, then for all open U containing p, we have $(U - \{p\}) \cap A \neq \emptyset$, so also $(U - \{p\}) \cap A^* \neq \emptyset$, since $A \subset A^*$. Therefore if p is a limit point of A, p is also a limit point of A^* which has no limit points. Hence A has no limit points, and we proceed as in Theorem 6.3: for each point $p \in A$, there is an open set U_p containing p such that $(U_p - \{p\}) \cap A = \emptyset$. Since A vacuously contains all of its limit points, A is closed and $X - A$ is open. Therefore $\mathscr{C} = \{U_p\}_{p\in A} \cup \{X - A\}$ is a countable open cover of X. Let F be a finite subset of $\mathscr{C}$. Then there exists an $x \in A$ such that $U_x \notin F$, otherwise $F = \mathscr{C} - \{(X - A)\}$ which is infinite. Let $U_y \in F$ be arbitrary. Then

$(U_y - \{y\}) \cap A = \emptyset$, and $x \in A$ therefore implies x must not be in $U_y - \{y\}$. We have that $x \neq y$, because otherwise we would have $U_x = U_y \in F$, so in order for $(U_y - \{y\}) \cap A$ to be empty, we must have $x \notin U_y$. The set U_y was arbitrary, so x does not belong to any element of F of this form, and $x \in A$ implies $x \notin (X - A)$, so there is no element of F that x belongs to, meaning F is not a cover of X. Since F was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ has no finite subcover and therefore X is not countably compact.

($\Longleftarrow$) We will again show the contrapositive. Suppose X is not countably compact. Then there exists a countable open cover $\mathscr{C} = \{C_n\}_{n\in\mathbb{N}}$ of X with no finite subcover. This means that for each $n \in \mathbb{N}$, the collection $\{C_i\}_{i=1,\ldots,n}$ is a finite subcollection of $\mathscr{C}$ and therefore does not cover X, so there exists an $a_n \notin \bigcup_{i=1}^{n} C_i$. Define the set $A = \{a_n \mid n \in \mathbb{N}\}$. Now suppose for contradiction that A is finite. Then $A = \{a_n \mid n \in N\}$ for some finite $N \subset \mathbb{N}$. Since $\mathscr{C}$ covers X, for each $n \in N$, there exists an $m_n \in \mathbb{N}$ such that $a_n \in C_{m_n}$. Define $M = \max\{m_n \mid n \in N\}$, which exists because $\{m_n \mid n \in N\}$ is a finite set. Then we have that

$$A = \{a_n \mid n \in N\} \subset \bigcup_{n\in N} C_{m_n} \subset \bigcup_{i=1}^{M} C_i.$$

However, $a_M \in A$, and by definition, $a_M \notin \bigcup_{i=1}^{M} C_i$, contradiction the assumption that A is finite. Therefore A is an infinite subset of X. Now let $p \in X$ be arbitrary. Then because $\mathscr{C}$ covers X, there exists an $m \in \mathbb{N}$ such that $p \in C_m$. If $(C_m - \{p\}) \cap A = \emptyset$, we are done. Otherwise, set $A' = (C_m - \{p\}) \cap A$. If $a_n \in A'$, then $a_n \neq p$ and $a_n \in C_m$, so $n < m$ since $a_n \notin \bigcup_{i=1}^{n} C_i$ means for all $n \geq m$, $C_m \subset \bigcup_{i=1}^{n} C_i$. Therefore $a_n \in \bigcup_{i=1}^{m-1}\{a_i\}$, so $A' \subset \bigcup_{i=1}^{m-1}\{a_i\}$ and is therefore finite. A' being finite means it is the union of singletons, all of which are closed because X is a T_1 space. Therefore A' is closed, so the set $C_m - A'$ is open by Theorem 2.15. By definition, $p \notin A'$, so $C_m - A'$ is an open set containing p and

$$((C_m - A') - \{p\}) \cap A = ((C_m - \{p\}) \cap A) - A' = A' - A' = \emptyset.$$

Since we have found such an open set $C_m - A'$, the point $p \in X$ is not a limit point of A, and since p was arbitrary, A has no limit points despite being an infinite set. □

Theorem 6.27. Let X be a Lindelöf space. Then every uncountable set has a limit point.

Proof. Let X be a space with uncountable subset A such that A does not have a limit point. We will show that X is not Lindelöf. For all $p \in A$, p is not a limit point of A, so there exists an open set U_p containing p such that $(U_p - \{p\}) \cap A = \emptyset$. Additionally, A vacuously contains all of its limit points, so A is closed and $X - A$ is open. Consider the collection $\mathscr{C} = \{U_p\}_{p \in A} \cup \{X - A\}$. Note that if $p \neq p'$, for $p, p' \in A$ then $p' \notin U_p$, and therefore $U_p \neq U_p'$, meaning the map $f : A \to \mathscr{C}$ defined by $f(p) = U_p$ is an injection and $\mathscr{C}$ is therefore an uncountable open cover of X. Suppose $N \subset \mathscr{C}$ is countable. Then there exists an $x \in A$ such that $U_x \notin N$, otherwise $\{U_p\}_{p \in A} \subset N$ and N is uncountable. Let $U_y \in N$ be arbitrary. Then $(U_y - \{y\}) \cap A = \emptyset$, and since $x \in A$, we have that $x \notin (U_y - \{y\})$. We also have that $x \neq y$ since otherwise $U_x = U_y \in N$. Therefore $(U_y - \{y\}) \cap A$ implies that $x \notin U_y$. However, U_y was an arbitrary element of $N - \{X - A\}$, and because $x \in A$, also $x \notin X - A$, meaning there is no element of N that contains x. Therefore N does not cover X, and since N was an arbitrary countable subcollection of $\mathscr{C}$, no such subcollection covers X, so X is not Lindelöf. $\square$

7 Continuity: When Nearby Points Stay Together

7.1 Continuous Functions

Theorem 7.1. Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:

(1) The function f is continuous.

(2) For every closed set K in Y, the inverse image $f^{-1}(K)$ is closed in X.

(3) For every limit point p of a set A in X, the image $f(p) \in \overline{f(A)}$.

(4) For every $x \in X$ and open set V with $f(x) \in V$, there exists an open set U containing x such that $f(U) \subset V$.

Proof. (1) $\Longrightarrow$ (2): Let $f : X \to Y$ be continuous and let K be closed in Y. Then $Y - K$ is open in Y, so $f^{-1}(Y - K)$ is open in X. We also have that

$$f^{-1}(K) = f^{-1}(Y - (Y - K)) = f^{-1}(Y) - f^{-1}(Y - K) = X - f^{-1}(Y - K),$$

so $f^{-1}(K)$ is a closed set minus an open set and is therefore closed by Theorem 2.15.

(2) $\Longrightarrow$ (3) Let p be a limit point of A and let $f : X \to Y$ be a function satisfying property (2). Since $f(A) \subset \overline{f(A)}$, we have that

$$A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)}).$$

Since $\overline{f(A)}$ is closed in Y, $f^{-1}(f(\overline{A}))$ is closed in X, and since it contains A, it contains $\overline{A}$

by Theorem 2.20. Therefore if p is a limit point of A, $p \in \overline{A} \subset f^{-1}(\overline{f(A)})$, so

$$f(p) \in f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$$

as required.

(3) $\implies$ (4) Let $f : X \to Y$ satisfy (3), let $x \in X$ be a point with $f(x) \in V$ for some set V open in Y, and define $U = X - \overline{f^{-1}(Y - V)}$. Then $x \in U$, since otherwise, we would have $x \in \overline{f^{-1}(Y - V)}$, which means either $x \in f^{-1}(Y - V)$, implying that $f(x) \in Y - V$, a contradiction, or that x is a limit point of $f^{-1}(Y - V)$, implying by (3) that

$$f(x) \in \overline{f(f^{-1}(Y - V))} \subset \overline{Y - V} = Y - V,$$

giving the same contradiction (the last equality follows from V being open in Y and $Y - V$ therefore already being closed). Therefore $x \in U$, and since U is the complement of the closed set $\overline{f^{-1}(Y - V)}$, U is open in X. To show $f(U) \subset V$, let $y_0 \in f(U)$ be arbitrary. Then there exists some $x_0 \in U$ such that $y_0 = f(x_0)$. Since $x_0 \in U$, $x_0 \notin \overline{f^{-1}(Y - V)}$ and therefore also $x_0 \notin f^{-1}(Y - V) \subset \overline{f^{-1}(Y - V)}$. This means that $y_0 = f(x_0) \notin Y - V$, and therefore we have that $y_0 \in V$, so indeed $f(U) \subset V$. Therefore for all $x \in X$ and open sets V containing $f(x)$, there exists an open U containing x such that $f(U) \subset X$.

(4) $\implies$ (1) Let $f : X \to Y$ be a function satisfying (4) and let V be open in Y. Then for all $x \in f^{-1}(V)$, there exists a U_x open in X and containing x such that $f(U_x) \subset V$. Define the set U as $U = \bigcup_{x \in f^{-1}(V)} U_x$. Then U is the union of open sets in X and is therefore itself open in X. We claim that $U = f^{-1}(V)$. To show $U \subset f^{-1}(V)$, let $x_0 \in U$. Then $x_0 \in \bigcup_{x \in f^{-1}(V)} U_x$, so there exists an $x' \in f^{-1}(V)$ such that $x_0 \in U_{x'}$, meaning $f(x_0) \in f(U_{x'}) \subset V$. Therefore $x \in f^{-1}(V)$. To show $f^{-1}(V) \subset U$, let $x_0 \in f^{-1}(V)$. Then $x_0 \in U_{x_0} \subset U$. Therefore $f^{-1}(V) = U$ is open in X and so f is continuous. $\square$

Theorem 7.2. Let $y_0 \in Y$ and define the map $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$. Then f is continuous.

Proof. Let $x \in X$ be arbitrary and let V be an open set containing $f(x)$. Then X is open and contains x, and we have that $f(X) = \{y_0\} = \{f(x)\}$, so $f(X) \subset V$. Therefore f is continuous by (4) in Theorem 7.1. $\square$

Theorem 7.3. Let $X \subset Y$ be topological spaces and define the inclusion map $i : X \to Y$ by $i(x) = x$. Then i is continuous.

Proof. Let $x \in X$ be arbitrary and let V be an open set containing $i(x) = x$. Define $U = X \cap V \subset V$. Then U is open in X as a subspace of Y and contains x. Then since $i(U) = U$, $i(U) \subset V$ and so i is continuous by (4) in Theorem 7.1. $\square$

Theorem 7.4. Let $f : X \to Y$ be a continuous map and let A be a subset of X. Then the restriction map $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.

Proof. Let A' be an arbitrary subset of $A \subset X$. Since $A' \subset X$, if $p \in A$ is a limit point of A', then $f(p) \in \overline{f(A')}$. Now since $A' \subset A$, $a \in A'$ means that $f|_A(a) = f(a)$, so $f|_A(A') = f(A')$. Therefore we have that $f|_A(p) = f(p) \in \overline{f(A')} = \overline{f|_A(A')}$, meaning $f|_A$ is continuous by (3) in Theorem 7.1. $\square$

Theorem 7.5. A function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous if and only if for every point $x \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Proof. ($\Longrightarrow$) Suppose $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Then the interval $V = (f(x) - \varepsilon, f(x) + \varepsilon)$ is open in $\mathbb{R}_{\text{std}}$ and contains $f(x)$, so by (4) of Theorem 7.1, there exists an open set U in $\mathbb{R}_{\text{std}}$ such that U contains x and $f(U) \subset V$. Since U contains x, there exists a basic open set in $\mathbb{R}_{\text{std}}$ that contains x and is a subset of U, that is, there exists a $\delta > 0$ such that $x \in (x - \delta, x + \delta) \subset U$. Let $y \in \mathbb{R}$ be a point such that $|x - y| < \delta$. Then we have that $y \in (x - \delta, x + \delta) \subset U$, so $f(y) \in f(U) \subset V = (f(x) - \varepsilon, f(x) + \varepsilon)$. Therefore $|f(x) - f(y)| < \varepsilon$.

($\Longleftarrow$) Suppose $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is a function such that for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $x \in X$ and let V be an open set in $\mathbb{R}_{\text{std}}$ containing $f(x)$. We will show that there exists an open U containing X such that $f(U) \subset V$. Since $f(x) \in V$, there exists a basic open set in $\mathbb{R}_{\text{std}}$ containing $f(x)$ that is a subset of V. That is, there exists an $\varepsilon > 0$ such that $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$. Since $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Define $U = (x - \delta, x + \delta)$. Then U is open in $\mathbb{R}_{\text{std}}$ and contains x. Now let $z_0 \in f(U)$. Then there exists some $y_0 \in U$ such that $z_0 = f(y_0)$. Since $y_0 \in U$, we have that $|x - y_0| < \delta$, meaning $|f(x) - f(y_0)| < \varepsilon$. Therefore $z_0 = f(y_0) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$, so we have $f(U) \subset V$. By (4) of Theorem 7.1, f is continuous. $\square$

Theorem 7.6. Let X be 1^{st} countable. Then a function $f : X \to Y$ for some topological space Y is continuous if and only if for all convergent sequences $x_n \to x$ in X, the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$ in Y.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be a continuous map. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence converging to $x \in X$ and consider the sequence $(f(x_n))_{n\in\mathbb{N}}$. We claim that $f(x_n) \to f(x)$. Let V be an open set containing $f(x)$. By (4) of Theorem 7.1, there exists an open set U in X containing x such that $f(U) \subset V$. Since $x_n \to x$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$. Therefore for all $n > N$, $f(x_n) \in f(U) \subset V$, so since such an N exists, $f(x_n) \to f(x)$, as required.

($\Longleftarrow$) Let X be 1^{st} countable and let $f : X \to Y$ be a function such that for all convergent sequences $x_n \to x$ in X, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in Y. Let p be a limit point of a set A in X. Then by Theorem 5.18, X being 1^{st} countable implies that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A that converges to p. By assumption, this means $f(a_n) \to f(p)$. Since $a_n \in A$ for all $n \in \mathbb{N}$, we have that $f(a_n) \in f(A)$. By Theorem 2.30, this means that the limit of the sequence $(f(a_n))_{n\in\mathbb{N}}$ is in the closure $\overline{f(A)}$. That is, $f(p) \in \overline{f(A)}$, and since A was an arbitrary subset of X and p an arbitrary limit point of A, we have that f is continuous by (3) of Theorem 7.1. $\square$

Theorem 7.7. Let X be a space with a dense set D and let Y be Hausdorff. Then if $f, g : X \to Y$ are continuous functions such that $f(d) = g(d)$ for all $d \in D$, then $f(x) = g(x)$ for all $x \in X$.

Proof. Assume the hypotheses of the claim and suppose for contradiction that there exists some $x \in X$ such that $f(x) \neq g(x)$. Then because Y is Hausdorff, there exist two disjoint open sets V_f and V_g such that $f(x) \in V_f$ and $g(x) \in V_g$. Because both f and g are continuous, the preimages $U_f = f^{-1}(V_f)$ and $U_g = g^{-1}(V_g)$ are open in X and contain x. Therefore $U_f \cap U_g$ is a nonempty open set, and since D is dense in X, there exists a $d \in D$ such that $d \in U_f \cap U_g = f^{-1}(V_f) \cap g^{-1}(V_g)$. Therefore $f(d) \in V_f$ and $f(d) = g(d) \in V_g$, so $f(d) \in V_f \cap V_g$, but this is a contradiction since $V_f \cap V_g = \emptyset$. Therefore $f = g$. $\square$

Theorem 7.9. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions, and let V be an open set in Z. Then $g^{-1}(V)$ is open in Y, so $f^{-1}(g^{-1}(V))$ is open in X. We claim that $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Let $x \in f^{-1}(g^{-1}(V))$. Then $f(x) \in g^{-1}(V)$, meaning $(g \circ f)(x) = g(f(x)) \in V$, so $x \in (g \circ f)^{-1}(V)$. Now let $x \in (g \circ f)^{-1}(V)$. Then $g(f(x)) \in V$, meaning $f(x) \in g^{-1}(V)$ and therefore $x \in f^{-1}(g^{-1}(V))$. Since both inclusions hold, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X, and since V was an arbitrary open set in Z, $g \circ f$ is continuous. $\square$

Theorem 7.10. Let $X = A \cup B$ for closed subsets $A, B \subset X$. Then if $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined to be f on A and g on B is continuous.

Proof. Assume the hypotheses of the claim and let V be an arbitrary closed set in Y. Then

$$h^{-1}(V) \cap A = \{x \in X \mid h(x) \in V\} \cap A = \{x \in A \mid f(x) = h(x) \in V\} = f^{-1}(V).$$

Similarly, $h^{-1}(V) \cap B = g^{-1}(V)$. Since V is closed in Y, $f^{-1}(V)$ is closed in A and $g^{-1}(V)$ is closed in B. Then by Corollary 3.29, there exist sets C and D closed in X such that $f^{-1}(V) = A \cap C$ and $g^{-1}(V) = B \cap D$. The sets A and B are also closed in X, so $f^{-1}(V)$ and $g^{-1}(V)$ are the intersections of closed sets and are therefore closed. Therefore

$$h^{-1}(V) = (h^{-1}(V) \cap A) \cup (h^{-1}(V) \cap B) = f^{-1}(V) \cup g^{-1}(V)$$

is the union of sets that are closed in X, so also $h^{-1}(V)$ is closed in X. By (2) in Theorem 7.1, h is continuous. $\square$

Theorem 7.11. Let $X = A \cup B$ for open subsets $A, B \subset X$. Then if $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined to be f on A and g on B is continuous.

Proof. Assume the hypotheses of the claim and let V be an arbitrary open set in Y. Then

$$h^{-1}(V) \cap A = \{x \in X \mid h(x) \in V\} \cap A = \{x \in A \mid f(x) = h(x) \in V\} = f^{-1}(V).$$

Similarly, $h^{-1}(V) \cap B = g^{-1}(V)$. Since V is open in Y, $f^{-1}(V)$ is open in A and $g^{-1}(V)$ is open in B. Then by the definition of being open in the relative topology on a subspace, there exist sets C and D open in X such that $f^{-1}(V) = A \cap C$ and $g^{-1}(V) = B \cap D$. The sets A and B are also open in X, so $f^{-1}(V)$ and $g^{-1}(V)$ are the intersections of open sets and are therefore open. Therefore

$$h^{-1}(V) = (h^{-1}(V) \cap A) \cup (h^{-1}(V) \cap B) = f^{-1}(V) \cup g^{-1}(V)$$

is the union of sets that are open in X, so also $h^{-1}(V)$ is open in X. Therefore h is continuous. $\square$

Exercise 7.12. The pasting lemma does not work on arbitrary sets A and B. For instance, let $A = (0, 1)$, let $B = [1, 2]$, let $f : A \to \mathbb{R}_{\text{std}}$ be given by $f(x) = 0$, and let $g : B \to \mathbb{R}_{\text{std}}$ be given by $g(x) = 1$. Since f and g are constant functions, they are continuous by Theorem 7.2, and they agree on $A \cap B = \emptyset$. However, the function $h : (0, 2] \to \mathbb{R}_{\text{std}}$ such that $h = f$ on A and $h = g$ on B is not continuous, since 1 is a limit point of the set $A = (0, 1) \subset (0, 2]$ but $h(1) = 1 \notin \{0\} = \overline{h(A)}$, violating (3) of Theorem 7.1.

Theorem 7.13. Let $f : X \to Y$ be a function and let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if for every open set $V \in \mathcal{B}$, the preimage $f^{-1}(V)$ is open in X.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be continuous and let $\mathcal{B}$ be a basis for Y. If $V \in \mathcal{B}$, then V is open in Y and so $f^{-1}(V)$ is open in X since f is continuous.

($\Longleftarrow$) Let $f : X \to Y$ be a function such that for all $V \in \mathcal{B}$, a basis for Y, $f^{-1}(V)$ is open in X. Let U be an arbitrary open set in Y. Then for all $y \in U$, there exists a $V_y \in \mathcal{B}$ such that $y \in V_y \subset U$. Then we have that $f^{-1}(V_y)$ is open in X. We claim that $f^{-1}(U) = \bigcup_{y \in U} f^{-1}(V_y)$ is the union of open sets and is therefore open in X. Let $x \in f^{-1}(U)$. Then $f(x) \in U$, so there is a corresponding $V_{f(x)}$ containing $f(x)$, and we have that $x \in f^{-1}(V_{f(x)}) \subset \bigcup_{y \in U} f^{-1}(V_y)$. Now let $x \in \bigcup_{y \in U} f^{-1}(V_y)$. Then there exists a $y_0 \in U$ such that $x \in f^{-1}(V_{y_0})$, so $f(x) \in V_{y_0} \subset U$, meaning $x \in f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in X, and since U was an arbitrary open set in Y, the function $f : X \to Y$ is continuous. $\square$

Theorem 7.14. Let $f : X \to Y$ be a function and let $\mathscr{S}$ be a subbasis for Y. Then f is continuous if and only if for every open set $V \in \mathscr{S}$, the preimage $f^{-1}(V)$ is open in X.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be continuous and let $\mathscr{S}$ be a subbasis for Y. If $V \in \mathscr{S}$, then V is open in Y and so $f^{-1}(V)$ is open in X since f is continuous.

($\Longleftarrow$) Let $f : X \to Y$ be a function such that for all $V \in \mathscr{S}$, a subbasis for Y, $f^{-1}(V)$ is open in X. Recall that $\mathcal{B}$, the set of all finite intersections of elements in $\mathscr{S}$, is a basis for the topology on Y, and let $V \in \mathcal{B}$. Then we have that $V = \bigcap_{i=1}^{n} V_i$ for some $V_i \in \mathscr{S}$, so

$$f^{-1}(V) = f^{-1}\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} f^{-1}(V_i).$$

Since $V_i \in \mathscr{S}$, $f^{-1}(V_i)$ is open in X. Since $f^{-1}(V)$ is therefore the finite intersection of open sets in X, it is also open in X. Since V was an arbitrary basic open set, by Theorem 7.13, f is continuous. $\square$

7.2 Properties Preserved by Continuous Functions

Theorem 7.15. If X is compact and $f : X \to Y$ is continuous and surjective, then Y is compact.

Proof. Let X be compact with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be an open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_x \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x\in X}$ is an open cover of X and therefore has a finite subcover, call it $\{f^{-1}(C_i)\}_{i\in F}$ for some finite subset $F \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i\in F}$ covers X, there exists an $i_0 \in F$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in F$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i\in F}$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, Y is compact. □

Theorem 7.16. If X is Lindelöf and $f : X \to Y$ is continuous and surjective, then Y is Lindelöf.

Proof. Let X be Lindelöf with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be an open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_X \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x\in X}$ is an open cover of X and therefore has a countable subcover, call it $\{f^{-1}(C_i)\}_{i\in N}$ for some countable subset $N \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i\in N}$ covers X, there exists an $i_0 \in N$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in N$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i\in N}$ is a countable subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, Y is Lindelöf. □

Theorem 7.17. If X is countably compact and $f : X \to Y$ is continuous and surjective, then Y is countably compact.

Proof. Let X be countably compact with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be a countable open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_x \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x\in X}$ is a countable open cover of X and therefore has a finite subcover, call it $\{f^{-1}(C_i)\}_{i\in F}$ for some finite subset $F \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i\in F}$ covers X, there exists an $i_0 \in F$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in F$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i\in F}$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary countable open cover, Y is countably compact. □

Theorem 7.18. If $f : X \to Y$ is continuous and surjective and X has a dense subset D, then $f(D)$ is dense in Y.

Proof. Let $f : X \to Y$ be continuous and surjective and let V be a nonempty open set in Y. Then $f^{-1}(V)$ is open in X, and because f is surjective and V is nonempty, there exists a $y \in V$ and therefore also an $x \in f^{-1}(V)$. Because D is dense in X and $f^{-1}(V)$ is open and nonempty, there exists a $d \in D \cap f^{-1}(V)$. Then $d \in f^{-1}(V)$ means $f(d) \in V$. Since V was an arbitrary open set in Y and we have found an element $f(d) \in f(D) \cap V$, the set $f(D)$ is dense in Y by Theorem 5.1. □

Corollary 7.19. If X is separable and $f : X \to Y$ is continuous and surjective, then Y is separable.

Proof. Suppose X is separable. Then it has a countable dense subset D, and by Theorem 7.18, $f(D)$ is dense in Y. Since $f : X \to Y$ is surjective, the cardinality of D is greater than or equal to the cardinality of $f(D)$, meaning $f(D)$ is also countable. This means Y has a countable dense subset and is therefore separable. □

Exercise 7.20. (1) Let f be the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology. Then U is open in $\mathbb{R}_{\text{std}}$, $f(U) \subset \mathbb{R}$ and because $\mathbb{R}$ here has the discrete topology, $f(U)$ is open. Therefore f is an open function. However, f is not continuous, because $\{0\} \subset \mathbb{R}$ is open in the discrete topology but $f^{-1}(\{0\}) = \{0\} \subset \mathbb{R}_{\text{std}}$. If $\varepsilon > 0$, then $(-\varepsilon, \varepsilon) \not\subset \{0\}$, so $\{0\}$ is not open in $\mathbb{R}_{\text{std}}$.

(2) The same function as in (1) is also closed: Let $A \subset \mathbb{R}_{\text{std}}$ be a closed subset. Then $X - f(A) \subset \mathbb{R}$ is open in the discrete topology, so $f(A)$ is closed in $\mathbb{R}$ with the discrete topology. However, f is not continuous.

(3) Now let f be the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the indiscrete topology. Then if $(0, 1)$ is open in $\mathbb{R}_{\text{std}}$ but not in $\mathbb{R}$ with the indiscrete topology, and similarly, $[0, 1]$ is closed in $\mathbb{R}_{\text{std}}$ but not in $\mathbb{R}$ with the indiscrete topology, so f is neither closed nor open. However,

f is continuous. The open sets in $\mathbb{R}$ with the indiscrete topology are $\mathbb{R}$ and $\emptyset$ with preimages $f^{-1}(\mathbb{R}) = \mathbb{R}$ and $f^{-1}(\emptyset) = \emptyset$, both of which are open in $\mathbb{R}_{\text{std}}$.

(4) Consider $(0,1)$ as a subspace of $\mathbb{R}_{\text{std}}$. Then by Theorem 7.3, the inclusion map $i : (0,1) \to \mathbb{R}_{\text{std}}$ is continuous. If $U \subset (0,1)$ is open in the relative topology on $(0,1)$, then there exists a V open in $\mathbb{R}_{\text{std}}$ such that $U = (0,1) \cap V$. The set $(0,1)$ is open in $\mathbb{R}_{\text{std}}$, so $i(U) = U$ is the intersection of two open sets and is therefore open, meaning i is an open function. However, i is not closed. Note that $(0,1)$ is closed in $(0,1)$ (it is the whole space), but $i\left((0,1)\right) = (0,1)$ is not closed in $\mathbb{R}_{\text{std}}$ because it does not contain the limit points at 0 and 1.

(5) Consider $[0,1]$ as a subspace of $\mathbb{R}_{\text{std}}$. As in (4), $i : [0,1] \to \mathbb{R}_{\text{std}}$ is continuous, and if $A \subset [0,1]$ is closed, by Theorem 3.28 there exists a closed D in $\mathbb{R}_{\text{std}}$ such that $A = [0,1] \cap D$. Then $i(A) = A$ is the intersection of two closed sets and is therefore closed, so i is a closed map. However, i is not open. Note that $[0,1]$ is open in $[0,1]$ (it is the whole space), but $i([0,1]) = [0,1]$ is not open in $\mathbb{R}_{\text{std}}$ because for all $\varepsilon > 0$, the interval $(-\varepsilon, \varepsilon)$ contains 0 but is not a subset of $[0,1]$.

Theorem 7.21. If X is normal and $f : X \to Y$ is continuous, surjective, and closed, then Y is normal.

Proof. Assume the hypotheses of the claim and let A and B be disjoint closed subsets of Y. Then because f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are closed in X, and we have that $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$. Since $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X and X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset f^{-1}(B)$. Since U is open, $X - U$ is closed in X, and therefore $f(X - U)$ is closed in Y since f is a closed function. We claim that $f(X - U) \subset Y - A$. Let $y \in f(X - U)$. Then there exists an $x \in X - U$ such that $y = f(x)$. Then $x \notin U$, and therefore $x \notin f^{-1}(A) \subset U$. Since $x \notin f^{-1}(A) = \{x \in X \mid f(x) \in A\}$, we have that $y = f(x) \notin A$, so $y \in Y - A$. Therefore $f(X - U) \subset Y - A$, meaning also $A \subset Y - f(X - U)$. Since $f(X - U)$ is closed, $Y - f(X - U)$ is an open set containing A. Similarly, $Y - f(X - V)$ is an open set containing B. To show Y is normal, we show that these two open sets are disjoint. Suppose for contradiction that there exists some $y \in (Y - f(X - U)) \cap (Y - f(X - V))$. Then $y \notin f(X - U)$ and $y \notin f(X - V)$. Since f is surjective, there exists an $x \in X$ such that $y = f(x)$, and therefore $x \notin X - U$ and $x \notin X - V$. This means $x \in U \cap V = \emptyset$, a contradiction. Therefore A and B can be put in disjoint open sets, so Y is normal. □

Theorem 7.22. If $\{B_\alpha\}_{\alpha \in \lambda}$ is a basis for X and $f : X \to Y$ is continuous, surjective, and open, then $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.

Proof. Assume the hypotheses of the claim. Since B_α is open in X for all α and f is open, $f(B_\alpha)$ is open in Y, satisfying (1) of Theorem 3.1. Now let $y \in Y$ be contained in some open set V. Then because f is surjective, there exists an $x \in X$ such that $y = f(x)$. Since f is continuous, there exists an open U in X containing x such that $f(U) \subset V$. Because $\{B_\alpha\}_{\alpha\in\lambda}$ is a basis for X, there exists a B_β such that $x \in B_\beta \subset U$. Therefore $y = f(x) \in f(B_\beta) \subset f(U) \subset V$. Because $y \in Y$ was arbitrary with V an arbitrary open set containing y and we have shown there exists an $f(B_\beta)$ such that $y \in f(B_\beta) \subset V$, the collection $\{f(B_\alpha)\}_{\alpha\in\lambda}$ satisfies condition (2) of Theorem 3.1 and is therefore a basis for the topology on Y. $\square$

Corollary 7.23. If X is 2^{nd} countable and $f : X \to Y$ is continuous, surjective, and open, then Y is 2^{nd} countable.

Proof. Assume the hypotheses of the claim. Since X is 2^{nd} countable, X has a countable basis $\{B_i\}_{i\in\mathbb{N}}$. Then by Theorem 7.22, $\{f(B_i)\}_{i\in\mathbb{N}}$ is a countable basis for Y and Y is therefore also 2^{nd} countable. $\square$

Theorem 7.24. Let X be compact and Y be Hausdorff. If $f : X \to Y$ is continuous, then it is closed.

Proof. Let X be compact, let Y be Hausdorff, let $f : X \to Y$ be continuous, and let $A \subset X$ be closed. Then by Theorem 6.8, A is compact, and by Theorem 7.4, $f|_A : A \to Y$ is continuous. Therefore $f|'_A : A \to f(A)$ is continuous as well and is also surjective, so by Theorem 7.15, $f(A) \subset Y$ is compact. Since Y is Hausdorff, Theorem 6.9 implies $f(A)$ is closed. Since A was an arbitrary closed subset in X, $f(A)$ being closed in Y implies the function f is closed. $\square$

Theorem 7.25. Let X be compact and 2^{nd} countable, and let Y be Hausdorff. Then if $f : X \to Y$ is continuous and surjective, Y is 2^{nd} countable.

Proof. Assume the hypotheses of the claim. Then X has a countable basis, call it $\mathcal{B}$. Denote by $\mathcal{B}^*$ the set of all finite subsets of $\mathcal{B}$, which is countable by Theorem 1.14, and denote by $\mathcal{B}'$ the set of all finite unions of elements of $\mathcal{B}$, that is

$$\mathcal{B}' = \left\{ \bigcup_{B\in B^*} B \mid B^* \in \mathcal{B}^* \right\}.$$

Then the cardinality of $\mathcal{B}'$ is less than or equal to that of $\mathcal{B}^*$ and so is countable. Therefore we write it as $\mathcal{B}' = \{B_i\}_{i\in\mathbb{N}}$. We claim that the collection $\mathcal{B}_Y = \{Y - f(X - B_i)\}_{i\in\mathbb{N}}$ is a countable basis for Y and that Y is therefore 2^{nd} countable. That this proposed basis is countable follows from $\mathcal{B}'$ being countable. Now note that every element of $B_i \in \mathcal{B}'$ is the union of basic open sets and is therefore open in X, meaning the sets $X - B_i$ are all closed in X. Because X is compact, Y is Hausdorff, and f is continuous, Theorem 7.24 implies that f is also closed. Therefore for all $i \in \mathbb{N}$, $f(X - B_i)$ is closed in Y, making $Y - f(X - B_i)$ open in Y. Since all elements of $\mathcal{B}_Y$ are open in Y, condition (1) of Theorem 3.1 is satisfied.

To show condition (2), let U be open in Y containing a point y. Since $\{y\} \subset U$, we have that $f^{-1}(\{y\}) \subset f^{-1}(U)$, which is open because f is continuous. Because f is surjective, $f^{-1}(\{y\})$ is nonempty. Then for all $x \in f^{-1}(\{y\})$, x is contained in the open set $f^{-1}(U)$, and since $\mathcal{B}$ is a basis for X, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset f^{-1}(U)$. Then we have that the collection $\{B_x\}_{x\in f^{-1}(\{y\})}$ is an open cover for $f^{-1}(\{y\})$. Since Y is Hausdorff, the singleton $\{y\}$ is closed, and therefore $f^{-1}(\{y\})$ is closed in X because f is continuous and nonempty because f is surjective. Since $f^{-1}(\{y\})$ is closed in X and X is compact, by Theorem 6.8, $f^{-1}(\{y\})$ is also compact. This means the open cover $\{B_x\}_{x\in f^{-1}(\{y\})}$ has a finite subcover, call it $\mathcal{B}_y$. Since it is a cover, we have

$$f^{-1}(\{y\}) \subset \bigcup_{B\in\mathcal{B}_y} B,$$

and since $\mathcal{B}_y$ is a finite subset of $\mathcal{B}$, $\mathcal{B}_y \in \mathcal{B}^*$, and therefore $\bigcup_{B\in\mathcal{B}_y} B = B_k \in \mathcal{B}'$ for some $k \in \mathbb{N}$. We now wish to show that $y \in Y - f(X - B_k) \subset U$.

Suppose $x \in X$ such that $f(x) \notin Y - f(X - B_k)$. Then $f(x) \in f(X - B_k)$, which means there exists an $x' \in X - B_k$ (not necessarily equal to x) such that $f(x') = f(x)$. Then $x' \in X - B_k$ means $x' \notin B_k$, and since $f^{-1}(\{y\}) \subset B_k$, we have that $x' \notin f^{-1}(\{y\})$. Therefore $f(x) = f(x') \neq y$. Since $f(x) \notin Y - f(X - B_k)$ implies that $f(x) \neq y$, we have that if $y = f(x)$, then $y = f(x) \in Y - f(X - B_k)$. Since y can indeed be written as $f(x)$ for some $x \in X$ by the surjectivity of f, we have $y \in Y - f(X - B_k)$ as desired. Now recall that for all $x \in f^{-1}(\{y\})$, $B_x \subset f^{-1}(U)$. Therefore all elements of the finite subcover $\mathcal{B}_y$ are subsets of $f^{-1}(U)$, and so the union B_k of the sets in $\mathcal{B}_y$ is also a subset of $f^{-1}(U)$. Then $B_k \subset f^{-1}(U)$ implies that $X - f^{-1}(U) \subset X - B_k$, so also $f(X - f^{-1}(U)) \subset f(X - B_k)$. Now because for all $C, D \subset X$, $f(C) - f(D) \subset f(C - D)$, we have that

$$f(X) - f(f^{-1}(U)) \subset f(X - f^{-1}(U)) \subset f(X - B_k).$$

Since f is surjective, $f(X) = Y$ and $f(f^{-1}(U)) = U$. This means we have $Y - U \subset f(X - B_k)$, and therefore that $Y - f(X - B_k) \subset U$. We have taken an arbitrary y contained in an arbitrary open set U in Y, and have shown that there exists a $k \in \mathbb{N}$ such that $y \in Y - f(X - B_k) \subset U$, so (2) of Theorem 3.1 is satisfied in addition to (1), and therefore the set $\mathcal{B}_Y = \{Y - f(X - B_i)\}_{i \in \mathbb{N}}$ is a basis for Y. Because it is a countable basis, Y is 2^{nd} countable. □

7.3 Homeomorphisms

Theorem 7.26. Being homeomorphic is an equivalence relation on topological spaces.

Proof. Let X be a topological space and define $i : X \to X$ to be the inclusion map $i(x) = x$. Then i is a bijection and $i^{-1} = i$. Theorem 7.3 guarantees i and i^{-1} are continuous, so i is a homeomorphism, meaning X is homeomorphic to itself, so the relation is reflexive.

Let X and Y be topological spaces such that X is homeomorphic to Y. Then there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Then the map $f^{-1} : Y \to X$ is also bijective, and $(f^{-1})^{-1} = f$, so both directions are continuous, meaning Y is also homeomorphic to X and the relation is therefore symmetric.

Let X, Y, and Z be topological spaces such that X is homeomorphic to Y and Y is homeomorphic to Z. Then there exist homeomorphisms $f : X \to Y$ and $g : Y \to Z$. Then $f \circ g$ is a bijection since both f and g are bijections, and $f \circ g$ is continuous by Theorem 7.9. Sine $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and g^{-1} and f^{-1} are both continuous (since f and g are homeomorphisms), $(f \circ g)^{-1}$ is continuous as well, again by Theorem 7.9. Therefore $f \circ g : X \to Z$ is a homeomorphism, and so the relation is transitive, meaning it is an equivalence relation on the set of topological spaces. □

Exercise 7.27. If $a < b$, then $\mathbb{R}_{\text{std}}$ is homeomorphic to (a, b) as a subspace of $\mathbb{R}_{\text{std}}$.

Proof. The function $f : \mathbb{R} \to (a, b)$ given by

$$f(x) = \frac{b - a}{1 + e^{-x}} + a$$

is a homeomorphism. □

Theorem 7.28. If $f : X \to Y$ is continuous, then the following are equivalent:

(a) f is a homeomorphism.
(b) f is a closed bijection.
(c) f is an open bijection.

Proof. (a) $\implies$ (b): Suppose $f : X \to Y$ is a homeomorphism and let $K \subset X$ be closed. Then f is a bijection so f^{-1} exists, and since it is a homeomorphism, $f^{-1} : Y \to X$ is continuous. Therefore K closed in X means $(f^{-1})^{-1}(K)$ is closed in Y, but $(f^{-1})^{-1} = f$ since f is bijective, and therefore $f(K)$ is closed in Y, meaning f is a closed bijection.

(b) $\implies$ (c): Suppose $f : X \to Y$ is a closed bijection and let $U \subset X$ be open. Then $X - U$ is closed in X, which means $f(X - U)$ is closed in Y. Since f is a bijection, $f(X - U) = f(X) - f(U) = Y - f(U)$, and since this is closed in Y, $f(U)$ must be open in Y. Therefore f is an open bijection.

(c) $\implies$ (a): Suppose $f : X \to Y$ is an open bijection. Then $f^{-1} : Y \to X$ exists, and if U is an open set in X, $(f^{-1})^{-1}(U) = f(U)$ is open in Y since f is open. Therefore f^{-1} is continuous, and so f is a continuous bijection such that f^{-1} is also continuous. Therefore f is a homeomorphism. □

Theorem 7.29. Let X be compact and Y be Hausdorff. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof. If X is compact and Y is Hausdorff, $f : X \to Y$ being continuous means it is closed by Theorem 7.24. Since f is also a bijection, it is a homeomorphism by Theorem 7.28. □

Exercise 7.30. Recall that by (3) of Exercise 7.20, the identity function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}$ where the codomain has the indiscrete topology is continuous but neither closed nor open. This remains the case if we use the identity function $f : A \to A$ where the domain $A \subset \mathbb{R}_{\text{std}}$ has the relative topology from $\mathbb{R}_{\text{std}}$ and the codomain has the indiscrete topology. Here, f is a continuous bijection and the domain is compact by Theorem 6.15, but f is neither closed nor open, so f is not a homeomorphism, which shows that the codomain being Hausdorff is necessary.

Recall that by (1) of Exercise 7.20, the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology is not continuous, and consider the identity function $f : \mathbb{R} \to \mathbb{R}_{\text{std}}$ where the domain has the discrete topology. This is a bijection, and it is continuous because if U is open in $\mathbb{R}_{\text{std}}$, then $f^{-1}(U) = U$ is open in $\mathbb{R}$ with the discrete topology. However, f^{-1} is the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology and is therefore not continuous, meaning f is not a homeomorphism. $\mathbb{R}_{\text{std}}$ is Hausdorff, but $\mathbb{R}$ with the discrete topology is not compact, so this shows the necessity of the domain being compact.

Corollary 7.31. Let X be compact and Y be Hausdorff. Then if $f : X \to Y$ is continuous and injective, f is an embedding.

Proof. Since $f : X \to Y$ is injective, $f : X \to f(X)$ is a continuous bijection. Since Y is Hausdorff and all Hausdorff spaces are hereditarily Hausdorff, $f(X)$ is Hausdorff and so by Theorem 7.29, $f : X \to f(X)$ is a homeomorphism and therefore f is an embedding. $\square$