# Topology Through Inquiry Self-Study

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# 1 Cardinality: To Infinity and Beyond

## 1.1 Sets and Functions

**Theorem 1.2 (DeMorgan's Laws).** Let $X$ be a set and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then

$$X - \left(\bigcup_{k=1}^{N} A_k\right) = \bigcap_{k=1}^{N}(X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^{N} A_k\right) = \bigcup_{k=1}^{N}(X - A_k).$$

*Proof.* Let $a \in X - \left(\bigcup_{k=1}^{N} A_k\right)$ be arbitrary. Then $a \notin \bigcup_{k=1}^{N} A_k$, so for all $k$, $a \notin A_k$, which means that $a \in X - A_k$ for all $k$. Therefore $x \in \bigcap_{k=1}^{N}(X - A_k)$ and so $X - \left(\bigcup_{k=1}^{N} A_k\right) \subset \bigcap_{k=1}^{N}(X - A_k)$. Now let $a \in \bigcap_{k=1}^{N}(X - A_k)$ be arbitrary. Then we have that $a \in X - A_k$ for all $k$, which means that $a \notin A_k$ for all $k$. Therefore $a \notin \bigcup_{k=1}^{N} A_k$, so we have that $a \in X - \left(\bigcup_{k=1}^{N} A_k\right)$ and therefore $\bigcap_{k=1}^{N}(X - A_k) \subset X - \left(\bigcup_{k=1}^{N} A_k\right)$. Therefore

$$X - \left(\bigcup_{k=1}^{N} A_k\right) = \bigcap_{k=1}^{N}(X - A_k).$$

Let $a \in X - \left(\bigcap_{k=1}^{N}\right)$ be arbitrary. Then $a \notin \bigcap_{k=1} A_k$, so there exists some $j$ such that $a \notin A_j$, which means that $a \in X - A_j$. Therefore $a \in \bigcup_{k=1}^{N}(X - A_k)$ and so $X - \left(\bigcap_{k=1}^{N} A_k\right) \subset$

$\bigcup_{k=1}^{N}(X - A_k)$. Now let $a \in \bigcup_{k=1}^{N}(X - A_k)$. Then there exists some $j$ such that $a \in X - A_j$, so $a \notin A_j$ and therefore $a \notin \bigcap_{k=1}^{N}$. This means that $a \in X - \left(\bigcap_{k=1}^{N} A_k\right)$ and so $\bigcup_{k=1}^{N}(X - A_k) \subset X - \left(\bigcap_{k=1}^{N} A_k\right)$. Therefore we have that

$$X - \left(\bigcap_{k=1}^{N} A_k\right) = \bigcup_{k=1}^{N}(X - A_k).$$

$\square$

**Exercise 1.3.** For a function $f : X \to Y$ and sets $A, B \subset Y$, we have that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

*Proof.* See MATH 200 final exam review sheet notes in the graph paper notebook. $\square$

**Exercise 1.4.** If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.

*Proof.* Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y)$ contains more than one point. Then there exist $x_1, x_2 \in X$ such that $x_1, x_2 \in f^{-1}(y)$ and $x_1 \neq x_2$. By the definition of $f^{-1}(y)$, we have that $f(x_1), f(x_2) \in \{y\}$, so $f(x_1) = y = f(x_2)$ and $f$ is therefore not injective since $x_1 \neq x_2$. We have shown the contrapositive of the claim. $\square$

**Exercise 1.5.** If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.

*Proof.* Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y) = \emptyset$. Then for all $x \in X$, $f(x) \notin f^{-1}(y)$, and by the definition of $f^{-1}(y)$, this means that for all $x \in X$, $f(x) \neq y$, so $f$ is not surjective. We have shown the contrapositive of the claim. $\square$

## 1.2 Cardinality and Countable Sets

**Theorem 1.8.** Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

*Proof.* Let $S \subset \mathbb{N}$ be an arbitrary subset of $\mathbb{N}$. If $S$ is finite, then we are done. If $S$ is not finite, then it is infinite. Let $s_0 = \min S$, let $s_1 = \min(S - \{s_0\})$, and let $s_i = \min(S - \{s_0, \ldots, s_{i-1}\})$. Define the function $f : \mathbb{N} \to S$ by the following: $f(n) = s_n$. Suppose $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. Without loss of generality, assume that $n_1 < n_2$. Then $f(n_2) = \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\})$. Since $f(n_1) = s_{n_1} \notin S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}$, we have that $f(n_1) \neq \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}) = f(n_2)$, which means that $f$ is injective. Let $s \in S \subset \mathbb{N}$ be arbitrary. Then set $j = |\{r \in S : r < s\}| + 1 \in \mathbb{N}$. Then we have that

$$f(j) = s_j = \min(S - \{s_0, \ldots, s_{|\{r \in S : r < s\}|}\}) = s$$

where the last equality follows from the fact that $s$ must be the smallest element of the subset of $S$ from which all elements smaller than $s$ have been removed. Therefore $f$ is surjective, and since it is also injective, $f$ is a bijection, meaning that the cardinality of $S$ is the same as the cardinality of $\mathbb{N}$. $\square$

**Theorem 1.9.** Every infinite set has a countable subset.

*Proof.* I think I need the axiom of choice here? I will return in the future! $\square$

**Theorem 1.10.** A set is infinite if and only if there is an injection from the set into a proper subset of itself.

*Proof.* I think I also need the axiom of choice here... $\square$

**Theorem 1.11.** The union of two countable sets is countable.

*Proof.* Let $A$ and $B$ be countable sets. There are two cases to consider. In the first case, the intersection of $A$ and $B$ is finite, and so there exists a bijection $h : A \cap B \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ where $n$ is the size of the set $A \cap B$ If $A \cap B$ is empty, we instead use $n = 0$. Since $A \cap B$ is finite, $A - (A \cap B)$ is infinite, so there exists a bijection $f : A - (A \cap B) \to \mathbb{E}$, where $\mathbb{E}$ is the countable set containing all positive even integers greater than $n$. Similarly, there exists a bijection $g : B - (A \cap B) \to \mathbb{O}$, where $\mathbb{O}$ is the countable set containing all positive odd integers greater than $n$. Then we have that the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the second case, $A \cap B$ is infinite and there are three subcases. In the first subcase, one of $A - (A \cap B)$ or $B - (A \cap B)$ (assume without loss of generality that this is $A - (A \cap B)$) is finite. In this case, we use the same construction as earlier, since now $h : A - (A \cap B) \to \{1, \ldots, n\}$

is a bijection for some $n \in \mathbb{N}$, $f : A \cap B \to \mathbb{E}$ is a bijection, and $g : B - (A \cap B) \to \mathbb{O}$ is a bijection. Then the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A \cap B \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A - (A \cap B) \end{cases}.$$

In the second subcase, $A - (A \cap B)$ and $B - (A \cap B)$ are finite. Then there are bijections $f : A - (A \cap B) \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $g : B - (A \cap B) \to \{n+1, \ldots n+m\}$ for some $m \in \mathbb{N}$. Since $A \cap B$ is countably infinite, there is a bijection $h : A \cap B \to \{n+m+1, n+m+2, \ldots\}$, the set of positive integers greater than $n+m$. Therefore $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the third subcase, all three sets are countably infinite. In this case, there is a bijection $f : A - (A \cap B) \to \{n \in \mathbb{N} : n = 3k, k \in \mathbb{Z}\}$, a bijection $g : B - (A \cap B) \to \{n \in \mathbb{N} : n = 3k+1, k \in \mathbb{Z}\}$, and a bijection $h : A \cap B \to \{n \in \mathbb{N} : n = 3k+2, k \in \mathbb{Z}\}$. Then we have that $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In all cases, we have that there exists a bijection $\varphi : A \cup B \to \mathbb{N}$, so $A \cup B$ is countable. □

**Theorem 1.12.** The union of countably many sets is countable.

*Vague idea.* Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of countably manycountable sets. Arrange the elements of $\{A_i\}$ like so:

$$\begin{array}{lllll} A_1: & a_{1_1} & a_{1_2} & a_{1_3} & \ldots \\ A_2: & a_{2_1} & a_{2_2} & a_{2_3} & \ldots \\ A_3: & a_{3_1} & a_{3_2} & a_{3_3} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then define a bijection $h : \bigcup_{i=1}^{\infty} A_i \to \mathbb{N}$ where $h(a_{1_1}) = 1$, $h(a_{1_2}) = 2$ if $a_{1_2} \neq a_{1_1}$, and $h(a_{2_1}) = 3$ if $a_{2_1} \notin \{a_{1_1}, a_{1_2}\}$. The function $h$ orders the elements of $\bigcup_{i=1}^{\infty}$ starting in the upper left and working down diagonals, skipping elements that have already been mapped. ♥

## 1.3 Uncountable Sets and Power Sets

**Exercise 1.17.** If $A = \{a, b, c\}$, then $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

**Theorem 1.18.** If the set $A$ is finite, then its power set has cardinality $2^{|A|}$ ($|2^A| = 2^{|A|}$).

*Proof.* If a set $A$ is finite, then it has cardinality $|A| = n \in \mathbb{N}$. We argue by induction on $n$ that its power set $2^A$ has cardinality $2^n$. For the base case $n = 1$, let $A_1$ be a set with one element. Then $A_1 = \{a\}$ and $2^{A_1} = \{\emptyset, a\}$, which has two elements, so indeed $|2^{A_1}| = 2^{|A_1|}$. Now for the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that all sets with cardinality $k$ have power sets with cardinality $2^k$. Then let $A_{k+1}$ be an arbitrary set with cardinality $k+1$ so that $A_{k+1} = \{a_1, \ldots, a_{k+1}\}$. Note that the set $A_k = \{a_1, \ldots, a_k\}$ has cardinality $k$, and therefore has $2^k$ subsets. For each subset of $A_k$, call them $A_{k_i}$ for $1 \leq i \leq 2^k$, the sets $A_{k_i}$ and $A_{k_i} \cup \{a_{k+1}\}$ are subsets of $A_{k+1}$. Therefore there are $2 \cdot 2^k = 2^{k+1}$ subsets of $A_{k+1}$, and so $|2^{A_{k+1}}| = 2^{|A_{k+1}|}$. Therefore by induction, all finite sets $A$ have $|2^A| = 2^{|A|}$. □

**Theorem 1.19.** For any set $A$, there is an injection from $A$ to $2^A$.

*Proof.* Define $f : A \to 2^A$ as $f(a) = \{a\} \in 2^A$ since $\{a\} \subset A$. Let $x, y \in A$ such that $f(x) = f(y)$. Then we have that $\{x\} = \{y\}$, so $x = y$ and $f$ is an injection, as required. □

**Theorem 1.20.** If $P$ is the set of all functions from a set $A$ to the set $\{0, 1\}$, then $|P| = |2^A|$.

*Proof.* Define $\varphi : P \to 2^A$ as $\varphi(h) = h^{-1}(1)$. Let $f, g \in P$ be arbitrary such that $f \neq g$. Then there exists an $a \in A$ such that $f(a) \neq g(a)$. Without loss of generality, assume that $f(a) = 0$ and $g(a) = 1$. Then we have that $a \notin f^{-1}(1) = \varphi(f)$ and $a \in g^{-1}(1) = \varphi(g)$, so $\varphi(f) \neq \varphi(g)$ and therefore $\varphi$ is injective. Now let $Y \in 2^A$ be arbitrary. Then define the function $f : A \to \{0, 1\}$ such that for $a \in A$,

$$f(a) = \begin{cases} 0 & a \notin Y \\ 1 & a \in Y \end{cases}.$$

Then since $Y \in 2^A$ means that $Y \subset A$, we have that

$$\varphi(f) = f^{-1}(1) = \{a \in A : f(a) = 1\} = \{a \in A : a \in Y\} = Y,$$

so $\varphi$ is also surjective, making it a bijection. Since there is a bijection from $P$ to $2^A$, we have that $|P| = |2^A|$, as required. □

**Theorem 1.22.** There is no surjection between a set $A$ and its power set $2^A$ ($|A| \neq |2^A|$)

*Proof.* Let $A$ be an arbitrary set and suppose for contradiction that $f : A \to 2^A$ is a surjection. Consider the set $X = \{a \in A : a \notin f(a)\}$. Since $X \subset A$, $X \in 2^A$, and since $f$ is surjective, there exists an $a_0 \in A$ such that $f(a_0) = X$. There are two cases, both of which lead to a contradiction. In the first case, $a_0 \in X$, and so we have that $a_0 \notin f(a_0) = X$, a contradiction. In the second case, $a_0 \notin X$, so $a_0 \in f(a_0) = X$, also a contradiction. Since both cases lead to a contradiction, we see that the assumption that $f$ is surjective was false, so there can be no surjection between $A$ and $2^A$, which means $|A| \neq |2^A|$. □

## 1.4 The Schroeder-Bernstein Theorem

**Theorem 1.28.** The unit square and the unit interval have the same cardinality.

*Proof.* Let $a \in [0, 1]$ be arbitrary. Then it can be represented as

$$a = \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers with $a_n \in \{0, 1, \ldots, 9\}$ that does not end all in 9s (that is, if $a_i = 9$, then there exists $j > i$ such that $a_j \neq 9$). Now we claim that the function $f : [0, 1] \to [0, 1] \times [0, 1]$ is surjective, where

$$f\left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n\right) = \left(\sum_{n=1}^{\infty} a_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} a_{2n} \left(\frac{1}{10}\right)^n\right).$$

To show surjectivity, let $y \in [0, 1] \times [0, 1]$ be arbitrary. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ such that

$$y = \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right).$$

Let $x \in [0,1]$ be the point

$$x = \sum_{n=1}^{\infty} \left[ a_n \left(\frac{1}{10}\right)^{2n-1} + b_n \left(\frac{1}{10}\right)^{2n} \right] = 0.a_1b_1a_2b_2a_3b_3 \cdots = 0.c_1c_2c_3c_4c_5c_6 \ldots.$$

Then

$$\begin{aligned} f(x) = f(0.c_1c_2c_3\ldots) &= \left( \sum_{n=1}^{\infty} c_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} c_{2n} \left(\frac{1}{10}\right)^n \right) \\ &= \left( \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n \right) = y. \end{aligned}$$

Therefore $f$ is surjective. Now consider $g : [0,1] \times [0,1] \to [0,1]$ where $g(a,b) = a$. Let $y \in [0,1]$ be arbitrary and set $x = (y,0) \in [0,1] \times [0,1]$. Then $g(x) = g(y,0) = y$, and so $g$ is surjective. Since there is a surjection from $[0,1]$ to $[0,1] \times [0,1]$ and a surjection from $[0,1] \times [0,1]$ to $[0,1]$, by the surjective version of the Schroeder Bernstein Theorem (1.26), we have that $|[0,1] \times [0,1]| = |[0,1]|$ as required. $\square$

## 1.5 The Axiom of Choice

**Exercise 1.32.** Let $X$ be a set and let $P$ be the poset of all subsets of $X$ partially ordered by inclusion. Let $p \in P$ be an element of the poset with $X \leq p$. Then we have that $X \subset p$. We also have that $p \subset X$ since $p \in P$ and $P$ is the poset of all subsets of $X$. Therefore $p = X$, and so $X$ is by definition a maximal element of $P$. Suppose there exists a set $Y \in P$ such that $Y$ is also a maximal element of $P$. Then we have that $X \subset Y$ since $Y$ is a maximal element, and also that $Y \subset X$ since $Y \in P$. Therefore $X = Y$, which shows that $X$ is the unique maximal element of $P$. Now let $p \in P$ be an elemnet of the poset with $p \leq \emptyset$. Then we have that $p \subset \emptyset$, but also that $\emptyset \subset p$ since $\emptyset \subset x$ for all sets $x$. Therefore $p = \emptyset$, and so by definition we have that $\emptyset$ is a least element of the poset $P$. Now let $Z \in P$ be a least element. Then $Z \subset \emptyset$ since it is a least element, but as before, $\emptyset \subset Z$, which means $Z = \emptyset$ and so $\emptyset$ is the unique least element of the poset $P$.

**Exercise 1.33.** Let $P$ be the poset ordered by cardinality with $P = \{\{0\},\{1\},\{0,1\},\{1,2\}\}$. Then $\{0\}$ and $\{1\}$ are least elements and $\{0,1\}$ and $\{1,2\}$ are maximal elements.

**Exercise 1.34.** Consider $\mathbb{R}$ with the $\leq$ relation. The relation is reflexive, transitive, and antisymmetric, so it is a partial order on $\mathbb{R}$. We also have that for any two elements $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$, so they are comparable. This means that $\mathbb{R}$ is totally ordered with the relation $\leq$. However, $\mathbb{R}$ is not well-ordered, because there exist nonempty subsets of $\mathbb{R}$ that do not have least elements, for example, $(0,1) \subset \mathbb{R}$.

# 2 Topological Spaces: Fundamentals

## 2.2 Open Sets and the Definition of a Topological Space

**Theorem 2.1.** If $\{U_i\}_{i=1}^n$ is a finite collection of open sets in a topological space $(X, \mathcal{T})$, then $\bigcap_{i=1}^n U_i$ is open.

*Proof.* Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. We argue by induction on $n$ that $\bigcap_{i=1}^n U_i$ is open. For the base case $n = 1$, we have that $U_1$ is open. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $\bigcap_{i=1}^k U_i$ is open. Then since $U_{k+1}$ is open, we have that $\bigcap_{i=1}^{k+1} U_i = \left(\bigcap_{i=1}^k U_i\right) \cap U_{k+1}$ is the intersection of two open sets and is therefore open. By induction, $\bigcap_{i=1}^n U_i$ is open for all $n \in \mathbb{N}$, in otherwords, the intersection of finitely many open sets is open. $\square$

**Exercise 2.2.** The above theorem does not show that the intersection of infinitely many open sets is open since the intersection of infinitely many open sets cannot be represented as $\bigcap_{i=1}^n U_i$ for any $n \in \mathbb{N}$.

**Theorem 2.3.** A set $U$ is open in a topological space $(X, \mathcal{T})$ iff for every $x \in U$, there exists an open set $U_x$ such that $x \in U_x \subset U$.

*Proof.* Let $(X, \mathcal{T})$ be a topological space with $U \in \mathcal{T}$. Let $x \in U$ be arbitrary and set $U_x = U$. Then $U_x$ is open and $x \in U_x \subset U$, as required. Now let $U$ be a set such that for every $x \in U$, there exists an open set $U_x$ such that $x \in U_x \subset U$. Then we have that $\bigcup_{x \in U} U_x$ is open, since the union of a collection of open sets is open. We also have that $y \in U$ implies that $y \in \bigcup_{x \in U} U_x$ since $y \in U_y$, so $U \subset \bigcup_{x \in U} U_x$. If $y \in \bigcup_{x \in U} U_x$, then $y \in U_z \subset U$ for some $z \in U$, so $\bigcup_{x \in U} U_x \subset U$, which means that the open set $\bigcup_{x \in U} U_x$ is the set $U$. Therefore $U$ is open iff for every $x \in U$, there exists an open set $U_x$ such that $x \in U_x \subset U$. $\square$

**Exercise 2.4.** First note that, vacuously, $\emptyset \in \mathcal{T}_{\text{std}}$, so the first property is satisfied. Next, consider the set $\mathbb{R}^n \subset \mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be an arbitrary point, and since $B(p, 1) \subset \mathbb{R}^n$ and $p$ was arbitrary, we have that $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$, so the second property is satisfied. For the third property, let $U, V \in \mathcal{T}_{\text{std}}$ be arbitrary open sets in $\mathbb{R}^n$. Let $p$ be an arbitrary point in $U \cap V$. Then because $p \in U$, there exists an $\varepsilon_1$ such that $B(p, \varepsilon_1) \subset U$, and because $p \in V$, there exists an $\varepsilon_2$ such that $B(p, \varepsilon_2) \subset V$. Set $\varepsilon_p = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B(p, \varepsilon_p) \subset B(p, \varepsilon_1) \subset U$ and $B(p, \varepsilon_p) \subset B(p, \varepsilon_2) \subset V$, so we have that $B(p, \varepsilon_p) \subset U \cap V$. Therefore $U \cap V \in \mathcal{T}_{\text{std}}$ and the third property is satisfied. For the fourth property, let $\{U_i\}_{i\in\lambda}$ be a collection of sets $U_i \in \mathcal{T}_{\text{std}}$. Then let $p \in \bigcup_{i\in\lambda} U_i$ be arbitrary. Since $p$ is in the union of all the $U_i$, we have that $p \in U_j$ for some $j \in \lambda$. Then since $U_j \in \mathcal{T}_{\text{std}}$, there exists an $\varepsilon_p$ such that $B(p, \varepsilon_p) \subset U_j \subset \bigcup_{i\in\lambda} U_i$, and therefore $\bigcup_{i\in\lambda} U_i \in \mathcal{T}_{\text{std}}$, so the fourth property is satisfied and we have that $\mathcal{T}_{\text{std}}$ is indeed a topology on $\mathbb{R}^n$.

**Exercise 2.6.** The unit interval $(0, 1) \subset \mathbb{R}$ is open in the standard topology on $\mathbb{R}$, open in the discrete topology, not open in the indiscrete topology, not open in the finite complement topology, and not open in the countable complement topology.

**Exercise 2.7.** In the topological space $(\mathbb{R}, \mathcal{T}_{\text{std}})$, the interval $(0, 1)$ is open and for all $n \geq 0$, the set $U_n \subset (0, 1)$ where

$$U_n = \left(\frac{2^n - 1}{2^{n+1}}, \frac{2^n + 1}{2^{n+1}}\right).$$

Then $\frac{1}{2} \in U_n$ for all $n$ and therefore $\frac{1}{2} \in \bigcap_{n=0}^{\infty} U_n$. Let $x \in \mathbb{R}$ such that $x \neq \frac{1}{2}$. Then there exists some $m \geq 0$ such that $|x - \frac{1}{2}| > \frac{1}{2^{m+1}}$, which means that

$$x \notin \left(\frac{1}{2} - \frac{1}{2^{m+1}}, \frac{1}{2} + \frac{1}{2^{m+1}}\right) = \left(\frac{2^m - 1}{2^{m+1}}, \frac{2^m + 1}{2^{m+1}}\right) = U_m.$$

Since there exists an $m \geq 0$ such that $x \notin U_m$, we have that $x \notin \bigcap_{n=0}^{\infty} U_n$, and since $x$ was arbitrary, we have that $\bigcap_{n=0}^{\infty} U_n = \{\frac{1}{2}\} \notin \mathcal{T}_{\text{std}}$. Therefore the infinite intersection of open sets is not necessarily open.

## 2.3 Limit Points and Closed Sets

**Exercise 2.8.** In the indiscrete topology on $\mathbb{R}$, the point 0 is in $\mathbb{R}$ but not $\emptyset$, so since $(\mathbb{R} - \{0\}) \cap (1, 2) = (1, 2) \neq \emptyset$, we have that $(U - \{0\}) \cap (1, 2) \neq \emptyset$ for all open sets $U$ containing 0. Therefore 0 is a limit point of $(1, 2)$ in the indiscrete topology. In the finite complement topology, let $U \subset \mathbb{R}$ be an open set containing 0. Then suppose for contradiction that $(U - \{0\}) \cap (1, 2) = \emptyset$. Then for all $p \in (1, 2) \subset \mathbb{R}$, we have that $p \notin U - \{0\}$, which means that $p \in \mathbb{R} - (U - \{0\}) = (\mathbb{R} - U) \cup \{0\}$. Therefore $(1, 2) \subset (\mathbb{R} - U) \cup \{0\}$, but this is a contradiction since $U$ is open and therefore we have that an infinite set is a subset of a finite set. Hence it must be the case that $(U - \{0\} \cap (1,2)) \neq \emptyset$ for all $U$ containing 0, and therefore 0 is a limit point of $(1,2)$. In the standard topology and in the discrete topology, the set $(-1,1)$ is an open set containing 0, but $(-1,1) \cap (1,2) = \emptyset$, so 0 is not a limit point of $(1,2)$.

**Theorem 2.9.** Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then $p$ is not a limit point of $A$ iff there exists a neighborhood $U$ of $p$ such that $A \cap U = \emptyset$.

*Proof.* Suppose $p \notin A$ is not a limit point of $A$. Then there exists an open set $U$ containing $p$ (a neighborhood $U$ of $p$) such that $(U - \{p\}) \cap A = \emptyset$. Since $p \notin A$, we have that $p \notin A \cap U$, and therefore $A \cap U = \emptyset$, as required. Now suppose there exists a neighborhood $U$ of $p$ such that $A \cap U = \emptyset$. Then since $p \notin A$, we have that $(U - \{p\}) \cap A = \emptyset$ as well, and therefore $p$ is not a limit point of $A$. $\square$

**Exercise 2.10.** If $p$ is an isolated point of $A$ in a topological space $(X, \mathcal{T})$, then by the definition of an isolated point, we have that $p \in A$ but $p$ is not a limit point of $A$. Therefore there exists an open set $U$ such that $(U - \{p\}) \cap A = \emptyset$, and since $p \in A, U$, we have that

$$U \cap A = ((U - \{p\}) \cap A) \cup \{p\} \cap A = \emptyset \cup \{p\} = \{p\}.$$

Therefore if $p$ is an isolated point of $A$, there exists an open set $U$ such that $A \cap U = \{p\}$.

**Exercise 2.11.** Let $X = \{a, b, c, d\}$ be a set, let $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$ be a topology on $X$, and let $A = \{b, c\}$ be a set.

(1) $c \in A$ is a limit point of $A$ since the only open set containing $c$ is $X$, and we have that $(X - \{c\}) \cap A = \{b\} \neq \emptyset$.

(2) $d \notin A$ is a limit point of $A$ also because $X$ is the only open set containing $d$ and $(X - \{d\}) \cap A = \{b, c\} \neq \emptyset$.

(3) $b \in A$ is an isolated point of $A$ because it is in $A$ but is not a limit point since $\{a, b\}$ is open but $(\{a, b\} - \{b\}) \cap A = \emptyset$.

(4) $a \notin A$ is not a limit point of $A$ since $\{a\}$ is an open set but $(\{a\} - \{a\}) \cap A = \emptyset$.

**Exercise 2.12.** Let $X$ be a set and let $\mathcal{T}$ be a topology on $X$. If the set $X$ has a limit point $p$, then $p \in X$, so $\overline{X} = X$ and $X$ is closed. If Now let $p \in X$ and let $U$ be an open set containing $p$. Then $(U - \{p\}) \cap \emptyset = \emptyset$, so there are no limit points of the empty set, and therefore, vacuously, $\overline{\emptyset} = \emptyset$ and the empty set is closed.

(1) In the discrete topology, all sets are closed. Let $A$ be a nonempty proper subset of $X$ and let $p \in X$ be arbitrary. Then there exists a point $q \in X$ with $q \notin A$, and since all sets are open in the discrete topology, then set $\{p, q\}$ is a neighborhood of $p$ that satisfies $(\{p, q\} - \{p\}) \cap A = \emptyset$. Thus we have shown that there are no limit points of $A$, and so vacuously, $\overline{A} = A$.

(2) In the indiscrete topology, only $X$ and $\emptyset$ are closed. Let $A$ be a nonempty propoer subset of $X$ and let $p \in X$ such that $p \notin A$. Then $p$ is a limit point of $A$ since $X$ is the only open set containing $p$ and it satisfies $(X - \{p\}) \cap A \neq \emptyset$. However, $p \notin A$, so $\overline{A} \neq A$.

**Theorem 2.13.** For any topological space $(X, \mathcal{T})$, and $A \subset X$, the set $\overline{A}$ is closed, that is, for any set $A$ in a topological space, $\overline{\overline{A}} = \overline{A}$.

*Proof.* Let $A$ be a set in a topological space $(X, \mathcal{T})$. Since the closure of a set contains all the points in the set, we have that $\overline{A} \subset \overline{\overline{A}}$. Now let $x \in \overline{\overline{A}}$ and let $U$ be an arbitrary neighborhood of $x$. Either $x \in \overline{A}$ (in which case we're done), or $x$ is a limit point of $\overline{A}$, in which case we have that $(U - \{x\}) \cap \overline{A} \neq \emptyset$. Since $(U - \{x\}) \cap \overline{A} \subset U \cap \overline{A}$, so $U \cap \overline{A}$ is nonempty in either case. Therefore there exists some $y \in U \cap \overline{A}$, which means we have a neighborhood $U$ of $y$ such that $y \in \overline{A}$. As before, either $y \in A$ or $y$ is a limit point of $A$, in which case we have that $\emptyset \neq (U - \{y\}) \cap A \subset U \cap A$. Therefore for an arbitrary neighborhood $U$ of $x \in \overline{\overline{A}}$, we have shown that $U \cap A$ is nonempty. Either $x \in A \subset \overline{A}$, or $x \notin A$ and therefore $x \notin U \cap A \neq \emptyset$. Since this intersection is nonempty, we have also that $(U - \{x\}) \cap A \neq 0$, but $U$ was an arbitrary neighborhood of $x$, so we have shown that $(U - \{x\}) \cap A$ is nonempty for all neighborhoods $U$ of $x$. Therefore $x$ is a limit point of $A$, and so $x \in \overline{A}$. In both cases, $x \in \overline{A}$ and so $\overline{\overline{A}} \subset \overline{A}$, and since $\overline{A} \subset \overline{\overline{A}}$ also, we have that $\overline{\overline{A}} = \overline{A}$, as required. □

**Theorem 2.14.** For any topological space $(X, \mathcal{T})$, a subset $A \subset X$ is closed if and only if $X - A$ is open.

*Proof.* Suppose $A$ is closed and let $x \in X - A$. Then since $A$ is closed, it contains all its limit points and therefore $x$ is not a limit point. This means there exists an open set $U$ such that $U - \{x\} \cap A = \emptyset$. Let $y \in U - \{x\}$. Then $y \notin A$, so $y \in X - A$, which means $U - \{x\} \subset X$. Since $x$ was an arbitrary element of $X - A$, by Theorem 2.3, we have that $X - A$ is open. Now suppose that $X - A$ is open. $A \subset \overline{A}$, and we will show that $\overline{A} \subset A$. Suppose for contradiction that $\overline{A} \not\subset A$. Then there exists an $a \in \overline{A} - A$, which means that $a$ is a limit point of $A$, so all open intervals $U$ containing $a$ satisfy $U \cap A \neq \emptyset$ (by Theorem 2.9).

In particular, since $X - A$ is open, we have that $(X - A) \cap A \neq \emptyset$, which is a contradiction. Therefore $\overline{A} = A$ and $A$ is closed. $\square$

**Corollary 2.14.** If $A$ is an open set, then $X - (X - A)$ is open, which means $X - A$ is closed.

**Theorem 2.15.** For any topological space $(X, \mathcal{T})$ with an open set $U \in \mathcal{T}$ and a closed set $A \in \mathcal{T}$, $U - A$ is open and $A - U$ is closed.

*Proof.* Since $A$ is closed, $X - A$ is open, and so $U \cap (X - A)$ is also open since the intersection of two open sets is open. Then we have that $X - (U \cap (X - A))$ is closed by the corollary to Theorem 2.14. $X - (U \cap (X - A)) = X - (U - A)$ is closed, so $U - A$ is open, as claimed. The union of open sets is open, so we also have that $U \cup (X - A)$ is open. $U \cup (X - A) = X - (A - U)$ is open, so $A - U$ is closed, as claimed. $\square$

**Theorem 2.16.** Let $(X, \mathcal{T})$ be a topological space. Then:
(i) $\emptyset$ is closed.
(ii) $X$ is closed.
(iii) The union of finitely many sets is closed.
(iv) Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

*Proof.* For (i) and (ii), see exercise 2.12. For (iii), let $\{A_i\}$, $1 \leq i \leq n$ for some $n \in \mathbb{N}$. Then for each $A_i$, $X - A_i$ is open, and we have that the intersection of finitely many open sets is open, so $\bigcap_{i=1}^{n}(X - A_i)$ is open. By the corollary to Theorem 2.14 and DeMorgan's Laws,

$$X - \left(\bigcap_{i=1}^{n}(X - A_i)\right) = X - \left(X - \bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} A_i$$

is closed. For (iv), let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then for each $A_\alpha$, $X - A_\alpha$ is open, and we have that the union of a collection of open sets is open, so $\bigcup_{\alpha \in \lambda}(X - A_\alpha)$ is open. Again by DeMorgan's Laws,

$$X - \left(\bigcup_{\alpha \in \lambda}(X - A_\alpha)\right) = X - \left(X - \bigcap_{\alpha \in \lambda} A_\alpha\right) = \bigcap_{\alpha \in \lambda} A_\alpha$$

is closed. $\square$

**Exercise 2.19.** (1) In $\mathbb{Z}$ with the finite complement topology, the set $\{0,1,2\}$ is not open, since $\mathbb{Z}-\{0,1,2\}$ is infinite. $\mathbb{Z}-\{0,1,2\}$, however, is open since $\{0,1,2\}$ is finite, and therefore $\mathbb{Z}-(\mathbb{Z}-\{0,1,2\})=\{0,1,2\}$ is closed. The set of prime numbers has an infinite complement, so it is not open, but there are infinitely many prime numbers so it is also not closed. The set $\{n:|n|>10\}$ has a finite complement, so it is open, but the set itself is infinite and therefore is not closed.

(2) In $\mathbb{R}$ with the standard topology, the set $(0,1)$ is open, and its limit points are 0 and 1, neither of which are in $(0,1)$, so it is not closed. The set $(0,1]$ is neither closed nor open, since it contains one of its limit points but not both. The set $[0,1]$ contains both limit points and is therefore closed, and it is not open. The set $\{0,1\}$ has no limit points so is vacuously closed, and it is not open. The set $\{\frac{1}{n}:n\in\mathbb{N}\}$ is not open since there is no $\varepsilon>0$ such that $(1-\varepsilon,1+\varepsilon)\subset\{\frac{1}{n}:n\in\mathbb{N}\}$. Note that 0 is a limit point of this set since for any $\varepsilon>0$, there exists an $n_0\in\mathbb{N}$ such that $\frac{1}{n_0}<\varepsilon$ and therefore $\frac{1}{n_0}\in((-\varepsilon,\varepsilon)-\{0\})\cap\{\frac{1}{n}:n\in\mathbb{N}\}\neq\emptyset$. Since $0\notin\{\frac{1}{n}:n\in\mathbb{N}\}$, the set is not closed in $(\mathbb{R},\mathcal{T}_{\text{std}})$.

(3) In $\mathbb{R}^2$ with the standard topology, the set $C=\{(x,y):x^2+y^2=1\}$ is not open since if $p\in C$ and $\varepsilon_p>0$, then $p-\left(\frac{\varepsilon_p}{2},\frac{\varepsilon_p}{2}\right)$ is in $B(p,\varepsilon_p)$ but not in $C$. If $p=(\cos\theta_0,\sin\theta_0)\in C$, then $p$ is a limit point of $C$ since the point $(\cos\theta_1,\sin\theta_1)\in C$ and if $|\theta_1-\theta_0|<\arccos\left(1-\frac{\varepsilon_p^2}{2}\right)$, then $(\cos\theta_1,\sin\theta_1)\in(B(p,\varepsilon_p)-\{p\})\cap C\neq\emptyset$. If $p\notin C$, then set $\varepsilon$ to be the distance from $p$ to the nearest point of $C$. We have that $(B(p,\frac{\varepsilon}{2})-\{p\})\cap C=\emptyset$, so all points of $C$ are limit points and all points not in $C$ are not limit points, which means $\overline{C}=C$ and therefore $C$ is closed. Let $D=\{(x,y):x^2+y^2<1\}$. Then $C$ is the set of all the limit points of $D$, and so $C\cup D=\overline{D}$ is closed. Therefore, $\{(x,y):x^2+y^2>1\}=\mathbb{R}^2-\overline{D}$ is open, and its limit points are also all in the set $C$ and therefore this set is not closed. The set $D$ is open since $D=B(0,1)\in\mathcal{T}_{\text{std}}$, and so the set $\{(x,y):x^2+y^2\geq 1\}=\mathbb{R}^2-D$ is closed. This set is not open since there is no $\varepsilon$ such that $B(p,\varepsilon)\subset\{(x,y):x^2+y^2\geq 1\}$ for $p\in\{(x,y):x^2+y^2\geq 1\}\cap C$.

**Theorem 2.20.** For any set $A$ in a topological space $(X,\mathcal{T})$, the closure of $A$ is the intersection of all closed sets containing $A$, that is, $\overline{A}=\bigcap_{B\supset A,B\in\mathfrak{C}}B$, where $\mathfrak{C}$ is the set of all closed sets in $(X,\mathcal{T})$.

*Proof.* Let $\mathfrak{C}$ be the set of all closed sets in $(X,\mathcal{T})$ and let $A$ be a subset of $X$. Since its closure $\overline{A}$ is closed and $A\subset\overline{A}$, we have that $\overline{A}\in\mathfrak{C}$, and since for any sets $M$ and $N$ we have that $M\cap N\subset M,N$, we have that $\bigcap_{B\supset A,B\in\mathfrak{C}}B\subset\overline{A}$. To show equality, then, we need only show that $\overline{A}\subset\bigcap_{B\supset A,B\in\mathfrak{C}}B$. Let $a\in\overline{A}$. There are two cases. In the first case, if

$a \in A$, then $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$, which means that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$. In the second case, we have that $a \notin A$, which means that $a \in \overline{A} - A$ is a limit point of $a$. This means that for all open sets $U$ with $a \in U$, we have that $(U - \{a\}) \cap A \neq \emptyset$. Since $a \notin A$, we have that $U \cap A = (U - \{a\}) \cap A \neq \emptyset$. Let $B_0 \in \mathfrak{C}$ with $A \subset B_0$ and suppose for contradiction that $a \notin B_0$. Since $B_0$ is closed, $X - B_0$ is open, and since $a \notin B_0$, $a \in X - B_0$, which means that $(X - B_0) \cap A \neq \emptyset$. Therefore there exists some $y \in (X - B_0) \cap A$. We have that $y \in A \subset B_0$, which means that $y \notin X - B_0$, a contradiction. We have reached a contradiction by assuming that there exists a set $B_0 \in \mathfrak{C}$ such that $A \subset B_0$ and $a \notin B_0$, which means that $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$. Therefore in both cases $a \in A$ and $a \in \overline{A} - A$, we have that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$, which means that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. This shows that $\overline{A}$ is indeed the intersection of all closed sets containing $A$, as required. $\square$

**Exercise 2.21.** Consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. In the discrete topology, this set is already closed and so is its own closure. In the indiscrete topology, only $\mathbb{R}$ and $\emptyset$ are closed, so the closure of $H$ in the indiscrete topology is $\mathbb{R}$ itself. In the finite complement topology, let $p \in \mathbb{R}$ be an arbitrary point. Then let $U$ be an arbitrary open set containing $p$. Since $U$ is open, its complement is finite, which means there are only finitely many points not in $U - \{p\}$, so there must be infinitely many points in $(U - \{p\}) \cap H \neq \emptyset$. Since $U$ was an arbitrary open set, $p$ is a limit point, and since $p$ was an arbitrary point in $\mathbb{R}$, we see that all points are limit points of $H$, which means that $\overline{H} = \mathbb{R}$. In the countable complement topology, $H$ is closed since it contains countably many elements, so it is its own closure. In the standard topology, the only limit point of $H$ is 0, so the closure of $H$ is $\overline{H} = H \cup \{0\}$.

**Theorem 2.22.** Let $A$ and $B$ be subsets of a topological space $X$ with topology $\mathcal{T}$. Then:

(1) $A \subset B$ implies $\overline{A} \subset \overline{B}$.

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

*Proof.* (1) We have that $A \subset B \subset \overline{B}$, and by Theorem 2.20, we have that $\overline{A}$ is a subset of all closed sets containing $A$. Since $\overline{B}$ is a closed set containing $A$, we have that $\overline{A} \subset \overline{B}$, as required.

(2) Let $c \in \overline{A} \cup \overline{B}$. Without loss of generality, assume $c \in \overline{A}$. There are two cases. For the first case $c \in A$, we have that $c \in A \subset A \cup B \subset \overline{A \cup B}$. For the second case $c \in \overline{A} - A$, $c$ is a limit point of $A$ and therefore for all open sets $U$ containing $c$ satisfy $\emptyset \neq (U - \{c\}) \cap A \subset (U - \{c\}) \cap (A \cup B)$. Since $U$ was an arbitrary open set, $c$ is a limit point of $A \cup B$ and therefore $c \in \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Now let $d \in \overline{A \cup B}$ be arbitrary. Again there are two cases. In the first case, $d \in A \cup B$, so without loss of

generality assume $d \in A \subset \overline{A} \subset \overline{A} \cup \overline{B}$. In the second case, $d \in \overline{A \cup B} - (A \cup B)$, so $d$ is a limit point of $A \cup B$. This means that for all open sets $U$ containing $d$, we have that $\emptyset \neq (U - \{d\}) \cap (A \cup B) = ((U - \{d\}) \cap A) \cup ((U - \{d\}) \cap B)$, so one of the sets on the right hand side is nonempty. Without loss of generality, assume $(U - \{d\}) \cap A \neq \emptyset$. Then $d$ is a limit point of $A$, so $d \in \overline{A} \subset \overline{A} \cup \overline{B}$. Therefore $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ as well, and so the two sets are equal. $\square$

**Exercise 2.25.** I'm not entirely sure but I believe the Cantor Set fits this description.

## 2.4 Interior and Boundary

**Theorem 2.26.** Let $A$ be a subset of a topological space $(X, \mathcal{T})$. Then $p$ is an interior point if and only if there exists an open set $U$ such that $p \in U \subset A$.

*Proof.* Let $A \subset X$ be arbitrary and let $p \in A$ be an interior point. Then since $A^\circ$ is the union of all open subsets of $A$, $A^\circ$ is open as well, and so by Theorem 2.3, there exists an open set $U$ such that $p \in U \subset A^\circ \subset A$ where $A^\circ \subset A$ follows from the fact that if $a \in A^\circ$, then $a \in \bigcup_{U \subset A, U \in \mathcal{T}} U$ and therefore there exists an open set $U_0$ such that $a \in U_0 \subset A$. Now let $p \in A$ be an arbitrary point such that there is an open set $U$ with $a \in U \subset A$. Since $U$ is open, $U \in \mathcal{T}$ and so $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U = A^\circ$. Thus, $p$ is an interior point, as required. $\square$

**Exercise 2.27.** If $U$ is open in a topological space, then by Theorem 2.3, for every point $x \in U$, there exists an open set $U_x$ such that $x \in U_x \subset U$, which means $x$ is an interior point of $U$. If $x$ is an interior point of $U$, then by Theorem 2.26, there exists a set $U_x$ such that $x \in U_x \subset U$ and so $U$ is open. Therefore $U$ is open in a topological space if and only if every point of $U$ is an interior point.

**Lemma 2.28.** Given a set $A$ in a topological space $(X, \mathcal{T})$, the closure of $A$ is $\overline{A} = X - (X - A)^\circ$ and the interior of $A$ is $A^\circ = X - \overline{X - A}$.

*Proof.* Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$. For all $B \in \mathfrak{C}$ such that $A \subset B$, we have that $X - B \subset X - A$, and since $B$ is closed, $X - B$ is open. Given an open set $U$ with $U \subset X - A$, $X - U \supset A$ is closed. This means that $\{B \in \mathfrak{C} : B \supset A\} = \{X - U : U \in$

$\mathcal{T}$ s.t. $U \subset X - A\}$. Therefore we have that

$$\begin{aligned}\overline{A} &= \bigcap_{B\supset A, B\in\mathfrak{C}} B = X - \left(X - \bigcap_{B\supset A, B\in\mathfrak{C}} B\right) = X - \left(\bigcup_{B\supset A, B\in\mathfrak{C}} (X - B)\right) \\ &= X - \left(\bigcup_{U\subset X-A, U\in\mathcal{T}} (X - (X - U))\right) = X - \left(\bigcup_{U\subset X-A, U\in\mathcal{T}} U\right) = X - (X - A)^\circ,\end{aligned}$$

as required. The proof that $A^\circ = X - \overline{X - A}$ is similar $\square$

**Theorem 2.28.** Let $A$ be a subset of a topological space $(X, \mathcal{T})$. Then $A^\circ$, $\partial A$, and $(X-A)^\circ$ are all disjoint and their union is $X$.

*Proof.* Let $a \in A^\circ$ and suppose for contradiction that $a \in \partial A = \overline{A} \cap \overline{X - A}$. Then either $a \in X - A$ or $a$ is a limit point of $X - A$. In the first case, we have a clear contradiction since $a \in A^\circ \subset A$ cannot be in $X - A$. In the second case, every open set $U$ containing $a$ satisfies $(U - \{a\}) \cap (X - A) \neq \emptyset$. However, $a \in A^\circ$, so there exists an open set $V$ such that $a \in V \subset A$, so we have that $\emptyset \neq (V - \{a\}) \cap (X - A) \subset A \cap (X - A) = \emptyset$, a contradiction. Therefore the sets $A^\circ$ and $\partial A$ are disjoint, and a similar argument shows that $\partial A$ and $(X - A)^\circ$ are disjoint as well. It remains to check that $A^\circ$ and $(X - A)^\circ$ are disjoint. Suppose not, so that there exists an $a \in A^\circ$ such that $a \in (X - A)^\circ$. By Theorem 2.26, there exists an open set $U_1$ containing $a$ such that $U_1 \subset A$ and an open set $U_2$ containing $a$ such that $U_2 \subset X - A$. Therefore we have that $a \in U_1 \cap U_2 \subset A \cap (X - A) = \emptyset$, a contradiction showing that $A^\circ$ and $(X - A)^\circ$ must be disjoint. Since $A^\circ = X - \overline{X - A}$ by Lemma 2.28, we have that

$$\begin{aligned}\overline{A} - A^\circ &= \overline{A} - (X - \overline{X - A}) \\ &= \overline{A} \cap (X - (X - \overline{X - A})) \\ &= \overline{A} \cap \overline{X - A} = \partial A,\end{aligned}$$

so we see that the closure of $A$ is the disjoint union of the boundary of $A$ and the interior of $A$. Again using Lemma 2.28 ($\overline{A} = X - (X - A)^\circ$), we also have that

$$\begin{aligned}X &= \overline{A} \cup (X - \overline{A}) = A^\circ \cup \partial A \cup (X - \overline{A}) \\ &= A^\circ \cup \partial A \cup (X - (X - (X - A)^\circ)) \\ &= A^\circ \cup \partial A \cup (X - A)^\circ.\end{aligned}$$

Therefore we have that $A^\circ$, $\partial A$, $(X-A)^\circ$ are disjoint sets whose union is $X$. $\square$

**Exercise 2.29.** Again consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$. In the discrete topology, we have that $\overline{H} = H$ and $\overline{\mathbb{R}-H} = \mathbb{R}-H$, and so we have that $\partial N = \emptyset$. The interior of $N$ is $N^\circ = \overline{H} - \partial H = H - \emptyset = H$. In the indiscrete topology, we have that $\overline{H} = \mathbb{R} = \overline{\mathbb{R}-H}$, so $\partial H = \mathbb{R}$ and $H^\circ = \overline{H} - \partial H = \mathbb{R} - \mathbb{R} = \emptyset$. In the finite complement topology, we saw in Exercise 2.21 that the closure of $H$ was $\overline{H} = \mathbb{R}$ and by the same reasoning, $\overline{\mathbb{R}-H} = \mathbb{R}$. Therefore once again we have that $\partial H = \mathbb{R}$ and $H^\circ = \emptyset$, which makes sense since $\emptyset$ is the largest open set contained within $H$ since $H$ is countably infinite. In the countable complement topology, $H = \overline{H}$ is the disjoint union of $H^\circ$ and $\partial H$. If $V$ is a nonempty open set in the countable complement topology, then $\mathbb{R} - V$ is countable, meaning that $V$ is uncountable, so we cannot have $V \subset H$. The only open set $U$ with $U \subset H$ is $U = \emptyset$. Therefore $H^\circ = \emptyset$ and $\partial H = H$. In the standard topology, no open set contains 1 and $\frac{1}{2}$ without also containing all points in the open set $(\frac{1}{2}, 1)$, which means that no open set is a subset of $H$. Therefore we have that $H^\circ = \emptyset$, and so $\partial H = \overline{H} = H \cup \{0\}$.

## 2.5 Convergence of Sequences

**Theorem 2.30.** Let $A$ be a set in a topological space $(X, \mathcal{T})$, and let $p$ be a point in $X$. If $\{x_i\}_{i\in\mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$.

*Proof.* Suppose for contradiction that $\{x_i\}_{i\in\mathbb{N}} \subset A$, $x_i \to p$, but $p \notin \overline{A}$. Then $p$ is not a limit point of $A$, so there exists an open set $U_0$ containing $p$ such that $(U_0 - \{p\}) \cap A = \emptyset$. Since $x_i \to p$ and $p \in U_0$, we have that there exists an $N \in \mathbb{N}$ such that $x_i \in U_0$ for all $i > N$. Since $x_i \in A$ for all $i \in \mathbb{N}$, $x_i \neq p$ for all $i \in \mathbb{N}$. Therefore, we have that for all $i > N$, $x_i \in (U_0 - \{p\}) \cap A = \emptyset$, a contradiction. Therefore if $\{x_i\}_{i\in\mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$. In particular, $p$ is a limit point of $A$. $\square$

**Theorem 2.31.** In the standard topology on $\mathbb{R}^n$, if $p$ is a limit point of a set $A$, then there is a sequence of points in $A$ that converges to $p$.

*Proof.* If $p$ is a limit point of $A$, then $A \neq \emptyset$, so there exists a point $a \in A$. Set $\varepsilon = d(a,p) > 0$. Then for all $n \in \mathbb{N}$, $B(p, \frac{\varepsilon}{n})$ is open. Since $p$ is a limit point and $p \in B(p, \frac{\varepsilon}{n})$, we have that $(B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A \neq \emptyset$. Since each of these sets is nonempty there exists an $a_n \in (B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A$ for all $n$. Define a sequence $(x_n)_{n\in N}$ such that $x_n = a_n$. Therefore $\{x_n\}_{n\in\mathbb{N}} \subset A$, so it only remains to show that the sequence converges to $p$. Let $U$ be an open set containing $p$. Since $U$ is open in the standard topology, $U = B(p, \varepsilon_p)$ for some $\varepsilon_p > 0$, and there exists and $N \in \mathbb{N}$ such that $\frac{\varepsilon}{N} < \varepsilon_p$. Let $i > N$. We have that

$$x_i \in B\left(p, \frac{\varepsilon}{i}\right) \subset B\left(p, \frac{\varepsilon}{N}\right) \subset B\left(p, \varepsilon_p\right),$$

which means that $x_i \in U$ for all $i > N$, and therefore $x_i \to p$, as required. □

**Exercise 2.32.** Consider $\mathbb{R}$ with the indiscrete topology. Let $p$ be a point in $\mathbb{R}$ and set $x_n = n$. This sequence converges to $p$ since if $U$ is an open set containing $p$, then $U = \mathbb{R}$ and so we have that $x_n \in U$ for all $n \in \mathbb{N}$. Therefore $x_n \to p$, but $p$ was arbitrary, so we see that this sequence converges to every point in $\mathbb{R}$.

# 3 Bases, Subspaces, Products: Creating New Spaces

## 3.1 Bases

**Theorem 3.1.** Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a collection of subsets of $X$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

(1) $\mathcal{B} \subset \mathcal{T}$, and

(2) for every open set $U$, and point $p \in U$, there exists a set $V \in \mathcal{B}$ such that $p \in V \subset U$.

*Proof.* Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then by definition, (1) is satisfied. Now let $U$ be an arbitrary open set and let $p \in U$ be an arbitrary point. Since $\mathcal{B}$ is a basis, we have that $U = \bigcup_{V \in \lambda} V$ for some collection of sets $\lambda \subset \mathcal{B}$, and since $p \in U$, we have that $p \in V_0 \subset U$ for some set $V_0 \in \lambda$, so (2) is satisfied as well. Now suppose that $\mathcal{B}$ is a collection of sets satisfying (1) and (2). Let $U_0$ be an arbitrary open set and define $\lambda$ to be the collection of sets $\lambda = \{V \in \mathcal{B} : V \subset U_0\} \subset \mathcal{B}$. Now let $p \in U_0$ be an arbitrary point in $U_0$. By (2), there exists a $V_0 \in \mathcal{B}$ such that $p \in V_0 \subset U_0$, which means $V_0 \in \lambda$. Therefore $p \in \bigcup_{V \in \lambda} V$, and so $U_0 \subset \bigcup_{V \in \lambda} V$. Now let $p$ be an arbitrary point in $\bigcup_{V \in \lambda} V$. Then there exists a $V_0 \in \lambda \subset \mathcal{B}$ such that $p \in V_0$, and since $V_0 \in \lambda$, $V_0 \subset U_0$, so $p \in U_0$. Therefore $U_0 = \bigcup_{V \in \lambda} V$ since both sets are subsets of the other. Since any arbitrary open set $U$ is the union of sets in the collection $\mathcal{B}$, $\mathcal{B}$ is a basis for $\mathcal{T}$. □

**Exercise 3.2.** $\mathcal{B}_1$ satisfies Theorem 3.1(1) since it consists only of open intervals and therefore $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Since $U$ is open in $\mathcal{T}_{\text{std}}$, $U = (c - \varepsilon, c + \varepsilon)$ for some $c \in \mathbb{R}$ and $\varepsilon > 0$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $c - \varepsilon < a < p$ and $p < b < \varepsilon$. We have that $p \in (a, b) \subset U$ and $(a, b) \in \mathcal{B}_1$, so Theorem 3.2(2) is satisfied and therefore $\mathcal{B}_1$ is a basis.

**Theorem 3.3.** Let $X$ be a set and let $\mathcal{B}$ be a collection of subsets of $X$. Then $\mathcal{B}$ is a basis for some topology on $X$ if and only if

(1) each point of $X$ is in some element of $\mathcal{B}$, and

(2) if $U$ and $V$ are sets in $\mathcal{B}$ and $p$ is a point in $U \cap V$, there is a set $W$ in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

*Proof.* If $\mathcal{B}$ is a basis for some topology on $X$, then $X = \bigcup_{B \in \mathcal{B}} B$, so if $p \in X$, then $p \in B_0$ for some $B_0 \in \mathcal{B}$. Therefore (1) is satisfied. Now let $U$ and $V$ be sets in $\mathcal{B}$ and let $p$ be an arbitrary point in $U \cap V$. Since $U$ and $V$ and in $\mathcal{B}$, they are open, and therefore $U \cap V$ is open, so by Theorem 3.1, there exists a $W \in \mathcal{B}$ such that $p \in W \subset (U \cap V)$, satisfying (2). Now suppose $\mathcal{B}$ is a collection of subsets of $X$ satisfying (1) and (2). Then $\emptyset$ is the empty union of sets in $\mathcal{B}$, and $X = \bigcup_{B \in \mathcal{B}} B$ by (1). Suppose $U$ and $V$ are in $\mathcal{B}$ and set $\lambda = \{W \in \mathcal{B} : W \subset (U \cap V)\} \subset \mathcal{B}$. If $p$ is an arbitrary point of $U \cap V$, then by (2) there exists a $W_0$ such that $p \in W_0 \subset (U \cap V)$, so $p \in \bigcup_{W \in \lambda} W$. If $p$ is an arbitrary point of $\bigcup_{W \in \lambda} W$, then there exists a $W_0 \in \mathcal{B}$ such that $p \in W_0 \subset (U \cap V)$. Therefore $U \cap V = \bigcup_{W \in \lambda} W$, so the intersection of two sets that are the unions of sets in $\mathcal{B}$ is also the union of sets in $\mathcal{B}$. Now let $\alpha$ be a collection of sets in $\mathcal{B}$. Then for each $\beta \in \alpha$, $\beta = \bigcup_{B \in \lambda} B$ for some collection of sets $\lambda \subset \mathcal{B}$. Then we have that $\bigcup_{B \in \alpha} B$ is the union of a collection of unions of sets in $\mathcal{B}$ and is therefore itself the union of sets in $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for some topology on $X$ since the collection of sets that are unions of sets in $\mathcal{B}$ satisfies all four properties of a topology. $\square$

**Exercise 3.4.** Let $\mathcal{B}_{\text{LL}}$ be the set of subsets of $\mathbb{R}$ of the form $[a, b)$. Then if $x \in \mathbb{R}$, $x \in [x, x+1)$, so Theorem 3.3(1) is satisfied. Let $U$ and $V$ be arbitrary sets in $\mathcal{B}_{\text{LL}}$. Then there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $U = [a_1, b_1)$ and $V = [a_2, b_2)$. Let $x \in U \cap V$ be arbitrary. Then $a_1, a_2 \leq x < b_1, b_2$, so $p \in W = [\max\{a_1, a_2\}, \min\{b_1, b_2\}) = U \cap V$. Therefore Theorem 3.3(2) is satisfied and so $\mathcal{B}_{\text{LL}}$ is a basis for a topology on $\mathbb{R}$. ($\mathbb{R}$ together with this topology is the Sorgenfrey Line, $\mathbb{R}_{\text{LL}}$.)

**Exercise 3.6.** As discussed in Exercise 2.21, the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed in $\mathbb{R}$ with the standard topology, which means that $\mathbb{R} - H$ is not open in the standard topology. However, $H$ contains countably many points, so $\mathbb{R} - H$ is open in the countable complement topology. Therefore we have that the standard topology on $\mathbb{R}$ is not finer than the countable complement topology. On the other hand, the set $(0, 1)$ is open in the standard topology, but not open in the countable complement topology, so we also have that the countable complement topology is not finer than the standard topology.

**Exercise 3.7.** Let $\mathbb{R}_{+00}$ be the set of all positive real numbers ($\mathbb{R}_+$) together with the points $0'$ and $0''$, and let $\mathcal{B}$ be the set of all intervals of the form $(a, b)$, $(0, b) \cup \{0'\}$, or $(0, b) \cup \{0''\}$ for $a, b \in \mathbb{R}_+$.. We claim that $\mathcal{B}$ is the basis for some topology $\mathcal{T}$. For any $x \in \mathbb{R}_+$, $x \in (\frac{x}{2}, x+1)$, $0' \in (0, 1) \cup \{0'\}$, and $0'' \in (0, 1) \cup \{0''\}$, so every point in $\mathbb{R}_{+00}$ is in some element of $\mathcal{B}$. Now let $U$ and $V$ be arbitrary elements of $\mathcal{B}$ and let $p \in U \cap V$ be arbitrary. If one of $U$, $V$ is of the form $(0, b_1) \cup \{0'\}$ and the other is of the form $(0, b_2) \cup \{0''\}$, and $p$ is an arbitrary point in $U \cap V$, then set $W = (\frac{p}{2}, \min\{b_1, b_2\})$. Then we have that $p \in W \subset (U \cap V)$ and $W \in \mathcal{B}$. Otherwise, Set $W = (U \cap V)$. Then $p \in W \subset (U \cap V)$, so to show that $\mathcal{B}$ is the basis for some topology, it remains only to show that $W \in \mathcal{B}$. If one of $U$, $V$ is of the form $(a, b)$ for some $a, b \in \mathbb{R}_+$, then $U \cap V$ is of the same form and therefore in $\mathcal{B}$. For the remaining case to check, assume without loss of generality that $U = (0, b_1) \cup \{0'\}$ and $V = (0, b_2) \cup \{0'\}$ (the case for $0''$ is the same). Then $U \cap V = (0, \min\{b_1, b_2\}) \cup \{0'\} \in \mathcal{B}$. Therefore by Theorem 3.3, $\mathcal{B}$ is a basis for some topology on $\mathbb{R}_{+00}$. (This topological space is called the Double Headed Snake and will also be written as $\mathbb{R}_{+00}$.)

**Exercise 3.8.** In the Double Headed Snake, $\mathbb{R}_{+00}$, let $p$ be an arbitrary point. If $p = 0'$ or $0''$, we claim that $\{p\}$ is closed. Without loss of generality, assume $p = 0'$. Then

$$\mathbb{R}_{+00} - \{p\} = (0, \infty) \cup \{0''\} = \bigcup_{n \in \mathbb{N}} ((0, n) \cup \{0''\}).$$

Since $(0, n) \cup \{0''\} \in \mathcal{B}$, the basis for the Double Headed Snake given in Exercise 3.7, we have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and is therefore open in the Double Headed Snake, which means that $\{p\}$ is closed. Now if $p \neq 0', 0''$, we have that $p \in \mathbb{R}_+$. We have that

$$\begin{gathered}\mathbb{R}_{+00} - \{p\} = (0, p) \cup (p, \infty) \cup \{0', 0''\} \\ = ((0, p) \cup \{0'\}) \cup ((0, p) \cup \{0''\}) \cup \left( \bigcup_{p \in \mathbb{N}} (p, p+n) \right).\end{gathered}$$

Since $(p, p+n) \in \mathcal{B}$ for all $n \in \mathbb{N}$, we again have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and therefore is open in the Double Headed Snake, so $\{p\}$ is closed.

Suppose for contradiction that $U$ and $V$ are disjoint open sets in the Double Headed Snake such that $0' \in U$ and $0'' \in V$. Since $0' \in U$, and $U$ is open, it is the union of sets in $\mathcal{B}$, so there exists a set of the form $(0, b_1) \cup \{0'\} \subset U$. Similarly, there exists a set of the form $(0, b_2) \cup \{0''\} \subset V$. Therefore $(0, \min\{b_1, b_2\}) \subset U \cap V$, so the sets are not disjoint since

$b_1, b_2 \in \mathbb{R}_+$. This means it is impossible to have disjoint open sets each containing a different head of the snake.

**Exercise 3.9.** (1) In the topological space $\mathbb{R}_{\text{har}}$, the set $\mathbb{R} - H$ is open since it is the union of sets in the basis for $\mathbb{R}_{\text{har}}$:

$$\mathbb{R} - H = \bigcup_{n \in \mathbb{N}} ((-n, n) - H)\,.$$

Since $\mathbb{R} - H$ is open in $\mathbb{R}_{\text{har}}$, $H$ is closed and therefore $\overline{H} = H$.

(2) If $H^- = \{-\frac{1}{n} : n \in \mathbb{N}\}$, then $\overline{H^-} = H^- \cup \{0\}$.

(3) Nope!

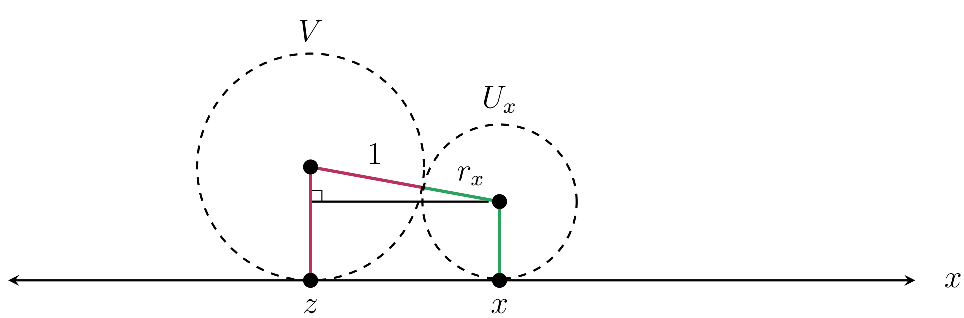
**Exercise 3.10.** (1) Let $\mathbb{H}_{\text{bub}}$ be the upper half plane with the Sticky Bubble Topology. Let $Q = \{(x, 0) : x \in \mathbb{Q}\}$. Then

$$\mathbb{H} - Q = \bigcup_{x \in (\mathbb{R} - \mathbb{Q})} \left( \bigcup_{n \in \mathbb{N}} (B((x, n), n) \cup \{(x, 0)\}) \right)$$

is the union of sets in the basis for the Sticky Bubble Topology, so it is open. Therefore $Q$ is closed and $\overline{Q} = Q$.

(2) This is similar to (1) since any subset of the $x$-axis can be treated the way $\mathbb{Q}$ was in the previous example.

(3) If $A$ is a countable subset of the $x$-axis, and $z$ is a point on the $x$-axis not in $A$, we wish to show that there are disjoint open sets $U$ and $V$ such that $A \subset U$ and $z \in V$. Set $V = B((z, 1), 1) \cup \{(z, 0)\}$. Now for all $x \in A$, set $r_x = (x - z)^2/4$ and $U_x = B((x, r_x), r_x) \cup \{(x, 0)\}$. $U_x$ is open for all $x$, so the set $U = \bigcup_{x \in A} U_x$ is also open. Clearly $A \subset U$, so it remains to check that $U \cap V \neq \emptyset$, which is the case provided none of the $U_x$ bubbles overlap with the bubble $V$. Consider an arbitrary $x$ and corresponding bubble $U_x$:



The distance from $(z,1)$ to $(x,r_x)$ is $((1-r_x)^2+(x-z)^2)^{\frac{1}{2}}$, and since $U_x$ and $V$ are open, we have that $U_x \cap V \neq \emptyset$ since:

$$\begin{aligned} r_x &= \frac{(x-z)^2}{4} \\ \iff (x-z)^2 &= 4r_x \\ \iff 1+r_x^2+(x-z)^2 &= 1+4r_x+r_x^2 \\ \iff 1-2r_x+r_x^2+(x-z)^2 &= 1+2r_x+r_x^2 \\ \iff (1-r_x)^2+(x-z)^2 &= (1+r_x)^2 \end{aligned}$$

Taking square roots shows that the distance between the centers of $U_x$ and $V$ is the same as the sum of the radii of $U_x$ and $V$, so the bubbles overlap at a single point that is not contained in either open set. This was the case for an arbitrary $x \in A$, so it is true for all $x \in A$. Therefore we have found two open sets $U$ and $V$ such that $A \subset U$, $z \in V$, and $U \cap V = \emptyset$. We did not use the fact that $A$ contains countably many points, so this holds for all subsets of the $x$-axis.

**Exercise 3.11.** Let $\mathbb{Z}_{\text{arith}}$ be the integers $\mathbb{Z}$ together with a topology generated by a basis of arithmetic progressions (the basis $\mathcal{B}$ is the collection of all sets of the form $\{az+b : z \in \mathbb{Z}\}$ for $a,b \in \mathbb{Z}$ with $a \neq 0$). To show that this is indeed a basis for some topology on $\mathbb{Z}$, note that $\mathbb{Z} = \{1 \cdot z + 0 : z \in \mathbb{Z}\}$ is itself in $\mathcal{B}$, so all points in $\mathbb{Z}$ are in some element of $\mathcal{B}$. Now let $U$ and $V$ be arithmetic progressions in $\mathcal{B}$ and let $z_0 \in U \cap V$ be arbitrary. Then $U = \{a_1 z + b_2 : z \in \mathbb{Z}\}$ and $V = \{a_2 z + b_2 : z \in \mathbb{Z}\}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Now $z_0 \in U \cap V$, then we have that $a_1 k_1 + b_1 = z_0 = a_2 k_2 + b_2$ for some $k_1, k_2 \in \mathbb{Z}$. Then $z_0 \in W = \{\operatorname{lcm}(a_1,a_2)z + z_0 : z \in \mathbb{Z}\} \in \mathcal{B}$. We also have that $W \subset U \cap V$ since if $w_0 = \operatorname{lcm}(a_1,a_2)k_0 + z_0 \in W$, then

$$\begin{aligned} a_1\left(k_1 + \frac{a_2 k_0}{\gcd(a_1,a_2)}\right) + b_1 = a_1 k_1 + b_1 + \frac{a_1 a_2}{\gcd(a_1,a_2)} k_0 = \operatorname{lcm}(a_1,a_2)k_0 + z_0 \\ = w_0 = \operatorname{lcm}(a_1,a_2)k_0 + z_0 = \frac{a_1 a_2}{\gcd(a_1,a_2)} k_0 + a_2 k_2 + b_2 = a_2\left(k_2 + \frac{a_1 k_0}{\gcd(a_1,a_2)}\right) + b_2, \end{aligned}$$

and $w_0$ is in both arithmetic progressions. Therefore $\mathbb{Z}_{\text{arith}}$ is a topological space generated by the basis $\mathcal{B}$.

**Exercise 3.12.** Consider the topological space $\mathbb{Z}_{\text{arith}}$ and note that if $U$ is an open set, then since it is the union of infinite arithmetic progressions, it is itself infinite. Note also that for a prime $p$ and $a = 1, 2, \ldots, p-1$, we have that $\{pz + a : z \in \mathbb{Z}\} \in \mathcal{B}$ is an arithmetic progression. Let $p\mathbb{Z}$ denote the set $\{pz : z \in \mathbb{Z}\}$. Then we have that

$$\mathbb{Z} - p\mathbb{Z} \ = \bigcup_{a=1,\ldots,p} \{pz + a : z \in \mathbb{Z}\}$$

is the union of open sets and is therefore open, meaning that $p\mathbb{Z}$ itself is closed. Let $P$ denote the set $P = \bigcup_{\text{primes } p} p\mathbb{Z}$, and suppose for contradiction that there are finitely many primes. Then $P$ is the union of finitely many closed sets and is therefore itself closed. Note that $P$ contains all numbers that are integer multiples of a prime, so the only numbers not contained in $P$ are $-1$ and $1$, that is, $\mathbb{Z} - P = \{-1, 1\}$. But this set is open since $P$ is closed, which is a contradiction since $\mathbb{Z} - P$ is finite. Therefore there are infinitely many primes.

## 3.2 Subbases

**Exercise 3.13.** Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Then if $U$ is in an open set in $(X, \mathcal{T})$, it is the union of sets in $\lambda \subset \mathcal{B}$. For each set $S \in \lambda$, $S$ is the trivial intersection of itself, and this intersection is finite. Since $\mathcal{B}$ is a basis for a the topology $\mathcal{T}$, the finite intersections of sets in $\mathcal{B}$ is in $\mathcal{T}$. Therefore every open set in $\mathcal{T}$ can be generated by taking the union of sets that are themselves the finite intersections of sets in $\mathcal{B}$ and so $\mathcal{B}$ is a subbasis.

**Exercise 3.14.** Consider $\mathbb{R}$ together with the standard topology. Recall that $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis and note that $(a, b) = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x < b\}$ and this is a finite intersection. Therefore every set in the basis is the finite intersection of sets in $\mathcal{S}$, the set of rays of the form $\{x \in \mathbb{R} : x < x_0\}$ and $\{x \in \mathbb{R} : x_0 > x\}$ for some $x_0 \in \mathbb{R}$. Therefore $\mathcal{S}$ is a subbasis for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

**Theorem 3.16.** Let $X$ be a set and let $\mathcal{S}$ be a collection of subsets of $X$. Then $\mathcal{S}$ is a subbasis for some topology $\mathcal{T}$ on $X$ if and only if every point of $X$ is contained in some element of $\mathcal{S}$.

*Proof.* Suppose $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on $X$ and let $x \in X$ be arbitrary. Since $\mathcal{S}$ is a subbasis, the set $\mathcal{B}$ of finite intersections of elements of $\mathcal{S}$ is a basis, and therefore $x \in B_0$ for some $B_0 \in \mathcal{B}$. Then since $B_0$ is the finite intersection of elements in $\mathcal{S}$, $x \in S_0$ for some $S_0 \in \mathcal{S}$. Now suppose that for all $x \in X$, there exists some $S_0 \in \mathcal{S}$ such that $x \in S_0$. Then if $\mathcal{B}$ is the set of finite intersections of elements of $\mathcal{S}$, $x \in S_0 \in \mathcal{B}$, so Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary and let $p \in U \cap V$. Then $U = \bigcap_{S \in \lambda_1} S$ and $V = \bigcap_{S \in \lambda_2} S$ where $\lambda_1$ and $\lambda_2$ are collections of sets in $\mathcal{S}$. Therefore

$$p \in \bigcap_{S \in \lambda_1 \cap \lambda_2} S \subset U \cap V$$

since $\lambda_1 \cap \lambda_2 \subset \lambda_1, \lambda_2$. Therefore Theorem 3.3(2) is satisfied as well since $\bigcap_{S \in \lambda_1 \cap \lambda_2} S \in \mathcal{B}$, and so $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, which also means that $\mathcal{S}$ is a subbasis for this topology. $\square$

**Exercise 3.17.** Let $\mathcal{S}$ be the set of all subsets of $\mathbb{R}$ of the form $\{x \in \mathbb{R} : x < a\}$ or $\{x \in \mathbb{R} : a \leq x\}$. We claim that this is a subbasis for the lower limit topology on $\mathbb{R}$, $\mathcal{T}_{\text{LL}}$. If $U \in \mathcal{S}$, then there are two cases. In the first case, $U = \{x \in \mathbb{R} : x < a\} = \bigcup_{n \in \mathbb{N}} [a - n, a) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. In the second case, $U = \{x \in \mathbb{R} : a \leq x\} = \bigcup_{n \in \mathbb{N}} [a, a + n) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. Both inclusions in $\mathcal{T}_{\text{LL}}$ come from the fact that all open sets of the Sorgenfrey Line are the union of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Since any arbitrary $U \in \mathcal{S}$ has $U \in \mathcal{T}_{\text{LL}}$, we have that $S \subset \mathcal{T}_{\text{LL}}$ and therefore Theorem 3.15(1) is satisfied. Now suppose $U$ is an open set of the Sorgenfrey Line and let $p \in U$ be arbitrary. Then $U = \bigcup_{W \in \lambda} W$ where $\lambda \subset \mathcal{B}_{\text{LL}}$ is a collection of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Therefore we have that $p \in W_0 = [a_0, b_0)$ for some $W_0 \in \lambda$. Since $W_0 = \{x \in \mathbb{R} : a_0 \leq x\} \cap \{x \in \mathbb{R} : x < b_0\}$ is the finite intersection of sets in $\mathcal{S}$ and $W_0 \subset U$, we have satisfied Theorem 3.15(2), meaning that $\mathcal{S}$ is indeed a subbasis for the lower limit topology on $\mathbb{R}$, as claimed.

## 3.3 Order Topology

**Exercise 3.19.** Consider $\mathbb{R}$ with the order topology from $\leq$. It has a basis $\mathcal{B}$ containing sets of the form $\{x \in \mathbb{R} : x < a\}$, $\{x \in \mathbb{R} : a < x\}$, and $\{x \in \mathbb{R} : a < x < b\}$. A set $U$ is open in the standard topology on $\mathbb{R}$ if for each point $p \in U$, there is an $\varepsilon_p > 0$ such that $(p - \varepsilon_p, p + \varepsilon_p) \subset U$. Note that this is the case for every set of the forms contained in $\mathcal{B}$ (for the first and second forms, $\varepsilon_p = |a - p|$, and for the third form, $\varepsilon_p = \min\{p - a, b - p\}$), so $\mathcal{B} \subset \mathcal{T}_{\text{std}}$, satisfying Theorem 3.1(1). If $U \subset \mathbb{R}$ is open in $\mathcal{T}_{\text{std}}$ and $p \in U$, then since there exists $\varepsilon_p > 0$ such that $p \in (p - \varepsilon_p, p + \varepsilon_p) \subset U$ and $(p - \varepsilon_p, p + \varepsilon_p) \in \mathcal{B}$, Theorem 3.1(2) is satisfied as well and therefore $\mathcal{B}$ is a basis for the standard topology as well, so the order topology on $\mathbb{R}$ with $\leq$ is the standard topology.

**Exercise 3.21.** $A = \{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\}$ has closure $\overline{A} = A \cup \{(0, 1)\}$.

$B = \{\left(1 - \frac{1}{n}, \frac{1}{2}\right) : n \in \mathbb{N}\}$ has closure $\overline{B} = B \cup \{(1, 0)\}$.

$C = \{(x, 0) : 0 < x < 1\}$ has closure $\overline{C} = \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$.

$D = \{(x, \frac{1}{2}) : 0 < x < 1\}$ has closure $\overline{D} = D \cup \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$.
$E = \{(\frac{1}{2}, y) : 0 < y < 1\}$ has closure $\overline{E} = E \cup \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$

## 3.4 Subspaces

**Theorem 3.25.** Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $\mathcal{T}_Y$ is indeed a topology on $Y$.

*Proof.* Note that $\emptyset \in \mathcal{T}_Y$ since $\emptyset = \emptyset \cap Y$ and $\emptyset \in \mathcal{T}$. Similarly, $Y \in \mathcal{T}_Y$ since $Y = X \cap Y$ and $X \in \mathcal{T}$. Now let $A, B \in \mathcal{T}_Y$ be arbitrary. Then there exist sets $V_A, V_B \in \mathcal{T}$ such that $A = V_A \cap Y$ and $B = V_B \cap Y$ and we have that

$$A \cap B = (V_A \cap Y) \cap (V_B \cap Y) = (V_A \cap V_B) \cap Y \in \mathcal{T}_Y$$

since $V_A \cap V_B$ is the (finite) intersection of sets in the topology $\mathcal{T}$ and is therefore itself in the topology $\mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in \lambda}$ be a collection of sets in $\mathcal{T}_Y$. Then for each $\alpha \in \lambda$, there exists a set $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$ and we have that

$$\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (V_\alpha \cap Y) = \left( \bigcup_{\alpha \in \lambda} V_\alpha \right) \cap Y \in \mathcal{T}_Y$$

since $\bigcup_{\alpha \in \lambda} V_\alpha$ is the union of sets in $\mathcal{T}$ and is therefore itself in $\mathcal{T}$. Thus we have that $\emptyset, Y \in \mathcal{T}_Y$ and that $\mathcal{T}_Y$ contains the finite intersections of sets in $\mathcal{T}_Y$ and the (possibly infinite) unions of sets in $\mathcal{T}_Y$ and is therefore a topology on $Y$. $\square$

**Exercise 3.26.** Taking $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$, we see that the set $\left[\frac{1}{2}, 1\right)$ is closed in $Y$. This is because the set $\left(-1, \frac{1}{2}\right)$ is open in $\mathbb{R}_{\text{std}}$ and therefore $\left(-1, \frac{1}{2}\right) \cap Y = \left[0, \frac{1}{2}\right)$ is open in $Y$, so $Y - \left[0, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right)$ is closed in $Y$.

**Exercise 3.27.** If $Y$ is a subspace of a topological space $(X, \mathcal{T})$, it is not necessarily the case that every subset of $Y$ that is open in $Y$ is open in $(X, \mathcal{T})$. As in Exercise 3.26, $Y = [0, 1)$ is a subspace of $\mathbb{R}_{\text{std}}$ and $\left[0, \frac{1}{2}\right)$ is open in $Y$. However, it is neither open nor closed in $\mathbb{R}_{\text{std}}$.

**Theorem 3.28.** Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

*Proof.* Let $C \subset Y$ be closed in $(Y, \mathcal{T}_Y)$. This is the case if and only if $Y - C$ is open in $(Y, \mathcal{T}_Y)$, which is the case if and only if there exists a set $V \subset X$, open in $(X, \mathcal{T})$, such that

$Y - C = V \cap Y$. This then is the case if and only if

$$C = Y - (V \cap Y) = Y - V = Y \cap (X - V) = Y \cap D$$

where $D = X - V$ is closed in $(X, \mathcal{T})$ since $V$ is open in $(X, \mathcal{T})$. All implications here go in both directions, so both directions of the proof are complete. $\square$

**Theorem 3.30.** Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$ that has basis $\mathcal{B}$. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

*Proof.* Let $A_0 \in \mathcal{B}_Y$ be arbitrary. Then there exists $B_0 \in \mathcal{B}$ such that $A_0 = B_0 \cap Y$. Since $B_0 \in \mathcal{B}$, it is open in the topological space $(X, \mathcal{T})$. Since $\mathcal{T}_Y = \{U : U = V \cap Y \text{ for some } V \in \mathcal{T}\}$, we have that $A_0 \in \mathcal{T}_Y$ and so Theorem 3.1(1) is satisfied. Now let $U$ be an open set in $(Y, \mathcal{T}_Y)$ and let $p \in U$ be an arbitrary point. Since $U \in \mathcal{T}_Y$, there exists a set $W \in \mathcal{T}$ such that $U = W \cap Y$, which means that $p \in Y$ and $p \in W$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, by Theorem 3.1(2), there exists a set $V \in \mathcal{B}$ such that $p \in V \subset W$. Then since $V \in \mathcal{B}$, $V \cap Y \in \mathcal{B}_Y$ and we have that

$$p \in V \cap Y \subset W \cap Y = U,$$

which satisfies Theorem 3.1(2), meaning that $\mathcal{B}_Y$ is indeed a basis for $\mathcal{T}_Y$. $\square$

## 3.5 Product Spaces

**Exercise 3.32.** Let $X$ and $Y$ be topological spaces and let $\mathcal{B}$ be the set of all cartesian products of open sets $U \subset X$ and $V \subset Y$. Then $p = (a, b)$ is a point of $X \times Y$, then $p$ is in some element of $\mathcal{B}$ since $X$ is open in $X$ and $Y$ is open in $Y$ (that is, $X \times Y \in \mathcal{B}$). This means that Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary. Then there exist sets $X_1$, $X_2$ open in $X$ and $Y_1$, $Y_2$ such that $U = X_1 \times Y_1$ and $V = X_2 \times Y_2$. Now let $p \in U \cap V$ be arbitrary. Then we have that $p \in U \cap V \subset U \cap V$ and also that

$$U \cap V = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2) \in \mathcal{B}$$

since $X_1 \cap X_2$ is open in $X$ and $Y_1 \cap Y_2$ is open in $Y$. Therefore Theorem 3.3(2) is also satisfied and we have that $\mathcal{B}$ is indeed the basis for some topology on $X \times Y$.

**Exercise 3.34.** The product of closed sets is closed in the product topology. Let $X$ and $Y$ be topological spaces, let $A$ be closed in $X$ and let $B$ be closed in $Y$. Then there exists an open set $A_0 \subset X$ such that $A = X - A_0$ and there exists an open set $B_0 \subset Y$ such that $B = Y - B_0$ and we have that

$$\begin{aligned}
A \times B &= (X - A_0) \times (Y - B_0) \\
&= (X \times (Y - B_0)) - (A_0 \times (Y - B_0)) \\
&= ((X \times Y) - (X \times B_0)) - ((A_0 \times Y) - (A_0 \times B_0)) \\
&= ((X \times Y) \cup (A_0 \times B_0)) - ((X \times B_0) \cup (A_0 \times Y)) \\
&= (X \times Y) - ((X \times B_0) \cup (A_0 \times Y)).
\end{aligned}$$

The sets $X \times B_0$ and $A_0 \times Y$ are both basic open sets in $X \times Y$ since $A_0$ is open in $X$ and $B_0$ is open in $Y$. Since the union of open sets is open, we have that $A \times B$ is the complement of open sets and is therefore closed. $A$ and $B$ were arbitrary closed sets in arbitrary topological spaces, so we have shown that the product of closed sets is closed in the product topology in general.

**Theorem 3.35.** The product topology on $X \times Y$ has a subbasis $\mathcal{S}$ that contains the inverse images of open sets under the projection functions, that is,

$$\mathcal{S} = \{\pi_X^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ is open in } Y\}.$$

*Proof.* Let $S_0$ be a set in $\mathcal{S}$. There are two cases. In the first, $S_0$ is of the form $S_0 = \{\pi_X^{-1}(U) : U \text{ is open in } X\} = U \times Y$ and is therefore a basic open set in the product space. In the second, $S_0$ is of the form $S_0 = \{\pi_Y^{-1}(V) : V \text{ is open in } Y\} = X \times V$ and is also a basic open set in the product space. Therefore $S_0$ is open in all cases and so $\mathcal{S} \subset \mathcal{T}$ where $\mathcal{T}$ is the product topology. Thus Theorem 3.15(1) is satisfied. Now let $W$ be an open set in the product space and let $p \in W$ be an arbitrary point. Then since $W$ is open, it is the union of sets of the form $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$. This means there are sets $U_0 \subset X$ and $V_0 \subset Y$ such that $p \in U_0 \times V_0 \subset W$. All that remains to satisfy Theorem 3.15(2) is to show that $U_0 \times V_0$ is the finite intersection of sets in $\mathcal{S}$, which is the case since $U_0 \times V_0 = \pi_X^{-1}(U_0) \cap \pi_Y^{-1}(V_0)$. Therefore $\mathcal{S}$ is indeed a subbasis for the product topology on $X \times Y$. $\square$

**Exercise 3.36.** Let $W$ be an open set in $\mathbb{R}^2_{\text{std}}$. Then $W = B(p, \varepsilon_p)$ for some $p \in \mathbb{R}^2$ and $\varepsilon_p > 0$. Now let $x \in W$ be arbitrary. Suppose $p = (p_1, p_2)$ and $x = (x_1, x_2)$. Set

$$U = \left( x_1 - \left| x_1 - \left( \frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1 \right) \right|, x_1 + \left| x_1 - \left( \frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1 \right) \right| \right)$$

and

$$V = \left( x_2 - \left| x_2 - \left( \frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2 \right) \right|, x_2 + \left| x_2 - \left( \frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2 \right) \right| \right).$$

(See Desmos sketch) Then we have that $x \in U \times V \subset W$, and since $U \times V$ is an open set with the product topology on $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$, we have that $W$ is an open set in this topology as well by Theorem 2.3 since $x$ was an arbitrary point of $W$. Then since $W$ was an arbitrary open set under the standard topology, it follows that the standard topology on $\mathbb{R}^2$ is a subset of the product topology on $\mathbb{R}^2$.

Now let $W$ be an open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology and let $p \in W$ be arbitrary. Then since $W$ is open, it is the union of sets of the form $U \times V$ where $U$ and $V$ are open sets in $\mathbb{R}_{\text{std}}$. This means that there exist some $U_0$ and $V_0$ open in $\mathbb{R}_{\text{std}}$ such that $p \in U_0 \times V_0 \subset W$. Then there exist intervals $(a_x, b_x)$ and $(a_y, b_y)$ such that $\pi_{U_0}(p) \in (a_x, b_x)$ and $\pi_{V_0}(p) \in (a_y, b_y)$. Therefore we have that $p \in B(p, \varepsilon) \subset U_0 \times V_0 \subset W$ where

$$\varepsilon = \min\{|a_x - \pi_{U_0}(p)|, |b_x - \pi_{U_0}(p)|, |a_y - \pi_{V_0}(p)|, |b_y - \pi_{V_0}(p)|\}.$$

Since this $B(p, \varepsilon)$ is an open set in $\mathbb{R}^2_{\text{std}}$, by Theorem 2.3 it follows that $W$ is also open in $\mathbb{R}^2_{\text{std}}$, and since $W$ was an arbitrary open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology, we have that this topology on $\mathbb{R}^2$ is a subset of the standard topology on $\mathbb{R}^2$. Since each topology is a subset of the other, they are equal, meaning that product topology on $\mathbb{R}^2$ from $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is the same as the standard topology on $\mathbb{R}^2$.

**Theorem 3.37.** The product topology on $\prod_{\alpha \in \lambda} X_\alpha$ has a basis containing all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$ where $U_\alpha$ is open in $X_\alpha$ for each $\alpha$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha$.

*Proof.* Since $\mathcal{S}$, the collection of sets of the form $\pi_\beta^{-1}(U_\beta)$ where $U_\beta$ is open in $(X_\beta, \mathcal{T}_\beta)$, is a subbasis for the product topology $\prod_{\alpha \in \lambda} X_\alpha$, the set of finite intersections of elements of $\mathcal{S}$ is therefore a basis for the product topology. Let $W$ be a set in the collection described in the theorem statement. Then there exists a finite index set $\gamma \subset \lambda$ such that $W = \prod_{\alpha \in \lambda} U_\alpha$ where $U_\alpha$ is open in $X_\alpha$ for each $\alpha$ and $U_\alpha = X_\alpha$ for all $\alpha \in \lambda - \gamma$. Then since for some $\beta \in \gamma$,

$\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha\in\lambda-\{\beta\}} X_\alpha$, we have that $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) = U_{\beta_1} \times U_{\beta_2} \times \prod_{\alpha\in\lambda-\{\beta_1,\beta_2\}} X_\alpha$ and therefore $W = \prod_{\alpha\in\gamma} U_\alpha \times \prod_{\alpha\in\lambda-\gamma} X_\alpha = \bigcap_{\alpha\in\gamma} \pi_\alpha^{-1}(U_\alpha)$ is the finite intersection of elements of $\mathcal{S}$. Similarly, any finite intersection of elements of $\mathcal{S}$ is a product of the form described in the theorem statement, and so by the definition of a subbasis, the collection described in the theorem statement is indeed a basis for the product topology. $\square$

**Exercise 3.42.** The set $2^{\mathbb{N}} = \prod_{n\in\mathbb{N}}\{0,1\}$ with the box topology has the discretet topology since all singletons are open. Let $\{a_n\}_{n\in\mathbb{N}}$ be a binary sequence (an arbitrary element of $2^{\mathbb{N}}$). Then the singleton containing $\{a_n\}_{n\in\mathbb{N}}$ is $\{\{a_n\}_{n\in\mathbb{N}}\} = \prod_{n\in\mathbb{N}}\{a_n\} = \{a_1\} \times \{a_2\} \times \cdots$, so it is a basic open set. Compare this to $2^{\mathbb{N}}$ with the product topology. Since with this topology, if $p \in 2^{\mathbb{N}}$ is a point in the space and $U$ an open set containing $p$, then $(U - \{p\}) \cap 2^{\mathbb{N}} \neq \emptyset$ since any open set $U$ contains infinitely many different sequences. Therefore $2^{\mathbb{N}}$ under the product topology has no isolated points since every point is a limit point of the set.

# 4 Separation Properties: Separating This From That

## 4.1 Hausdorff, Regular, and Normal Spaces

**Theorem 4.1.** A space $(X, \mathcal{T})$ is $T_1$ if and only if every point in $X$ is a closed set.

*Proof.* Suppose $(X, \mathcal{T})$ is $T_1$ and let $x \in X$ be arbitrary. Let $y \in X - \{x\}$ be arbitrary. Then we have that $x \neq y$, so since the space is $T_1$, we have that there exist open sets $U$ and $V$ such that $x \in U - V$ and $y \in V - U$. Since $V$ is open, $V \subset X$, and since $x \notin V$, $V \subset X - \{x\}$. Therefore $y \in V \subset X - \{x\}$ and by Theorem 2.3, we have that $X - \{x\}$ is open, meaning that the singleton $\{x\}$ is closed. But $x$ was an arbitrary point in this $T_1$ space, so we have that points are closed in $T_1$ spaces. Now suppose $(X, \mathcal{T})$ is a topological space in which all points are closed and let $x, y \in X$ be arbitrary points such that $x \neq y$. Since $\{x\}$ and $\{y\}$ are closed, we have that $x \in X - \{y\}$, an open set, and $y \in X - \{x\}$, another open set. Since $x \notin X - \{x\}$ and $y \notin X - \{y\}$, and $x$ and $y$ were arbitrary elements of $X$, we have that the space $(X, \mathcal{T})$ is $T_1$. $\square$

**Exercise 4.2.** Let $X$ be a set with the finite complement topology. Then let $x \in X$ be arbitrary. We have that $X - \{x\}$ is open since its complement, $\{x\}$, is finite. This means $\{x\}$ is closed, and since the point $x$ was arbitrary, we have that all singletons are closed and therefore by Theorem 4.1, all sets with the finite complement topology are $T_1$.

**Exercise 4.3.** Let $x, y \in \mathbb{R}$ be points in the space $\mathbb{R}_{\text{std}}$. Then the sets $A = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$ and $B = \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$ are open and disjoint with $x \in A$ and $y \in B$. Therefore $\mathbb{R}_{\text{std}}$ is Hausdorff.

**Exercise 4.5.** Let $A$ and $B$ be disjoint, closed sets in $\mathbb{R}_{\text{LL}}$. For every $a \in A$ and $b \in B$, set

$$\delta_a = \frac{\inf\{b \in B : b > a\} - a}{2} \quad \text{and} \quad \delta_b = \frac{\inf\{a \in A : a > b\} - b}{2}.$$

Since for all $a \in A$, $a$ is in the basic open set $U_a = [a, a+\delta_a)$, we have that $A \subset U = \bigcup_{a\in A} U_a$. Similarly, we have that $B \subset V = \bigcup_{b\in B} V_b$ where $V_b = [b, b+\delta_b)$ is a basic open set. Suppose for contradiction that there exists and $x \in U \cap V$. Then we have that there exist $\alpha \in A$ and $\beta \in B$ such that $x \in U_\alpha \cap V_\beta$. Without loss of generality, assume that $\alpha < \beta$. Then we have that $x \in U_\alpha = [\alpha, \alpha + \delta_\alpha)$ and $x \in V_\beta = [\beta, \beta + \delta_\beta)$. Then

$$\delta_\alpha = \frac{\inf\{b \in B : b > \alpha\} - \alpha}{2} \implies \alpha + \delta_\alpha = \frac{\alpha + \inf\{b \in B : b > \alpha\}}{2} \leq \inf\{b \in B : b > \alpha\}$$

since $\alpha \leq \inf\{b \in B : b > \alpha\}$. Since $\beta \in B$ and $\beta > \alpha$, $\beta \in \{b \in B : b > \alpha\}$ and therefore we have that

$$x < \alpha + \delta_\alpha \leq \inf\{b \in B : b > \alpha\} \leq \beta \leq x.$$

But $x < x$ is a contradiction and so we have that $U \cap V = \emptyset$ and therefore we have found disjoint, open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$. This means that $\mathbb{R}_{\text{LL}}$ is normal.

**Exercise 4.6.** (1) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Suppose for contradiction that $\inf\{d(a,p) : a \in A\} = 0$. This means that there exists a sequence $(x_i)_{i\in\mathbb{N}} \subset A$ such that $d(x_i, p) \to 0$. This means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies that $|d(x_i, p) - 0| < \varepsilon$, which is equivalent to the statement that

$$\varepsilon > |\parallel x_i - p \parallel - 0| = \parallel x_i - p \parallel .$$

We now have that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies $\parallel x_i - p \parallel < \varepsilon$, which in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ means that $x_i \to p$. By Theorem 2.30, we have that $p \in \overline{A} = A$ since $A$ is closed. But this is a contradiction, so the assumption that $\inf\{d(a,p) : a \in A\} = 0$ was false. We have either that this is greater than 0 or less than 0, but it cannot be less than 0 since the distance between two points is always nonnegative. Therefore we have that $\inf\{d(a,p) : a \in A\} > 0$.

(2) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Then by (1), there exists an $\varepsilon > 0$ such that $\inf\{d(a,p) : a \in A\} = \varepsilon$. Set $U = B(p, \frac{\varepsilon}{2})$ and $V = \bigcup_{a \in A} B(a, \frac{\varepsilon}{2})$. Then $U$ is a basic open set, and $V$ is the union of basic open sets so both $U$ and $V$ are open. We also have that $p \in U$ and $A \subset V$ since if $a \in A$, $a \in B(a, \frac{\varepsilon}{2}) \subset V$. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means that $d(p,x) < \varepsilon/2$ and $x \in V$ means that there exists an $a \in A$ such that $d(a,x) < \varepsilon/2$. Therefore we have that $d(a,p) \leq d(a,x) + d(x,p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction since $\inf\{d(a,p) : a \in A\} = \varepsilon$. Therefore $U \cap V = \emptyset$ and $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is regular.

(3) The sets $A = \{(x,0) : x \in \mathbb{R}\}$ and $B = \{(x,y) : x > 0, y \geq \frac{1}{x}\}$. Since for any points $(x_a, y_a) \in A$ and $(x_b, y_b) \in B$, $y_b \geq \frac{1}{x_b} > 0 = y_a$, we have that $A \cap B = \emptyset$. All limit points of $A$ have $y$-coordinate equal to 0 and are therefore in $A$, meaning $A$ is closed. Similarly, all limit points in $B$ have $y$-coordinate equal to $\frac{1}{x_0}$ for some $x_0 > 0$ and are therefore in $B$, meaning $B$ is closed. Hence $A$ and $B$ are disjoint, closed subsets of $\mathbb{R}^2$. However, $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$. To see this, let $\varepsilon > 0$. Then we have that $a_\varepsilon = (\frac{1}{\varepsilon} + 1, 0) \in A$ and $b_\varepsilon = (\frac{1}{\varepsilon} + 1, \frac{1}{\frac{1}{\varepsilon}+1}) \in B$. Then $d(a_\varepsilon, b_\varepsilon) = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$, so $\inf\{d(a,b) : a \in A \text{ and } b \in B\} < \varepsilon$. However, $\varepsilon > 0$ was an arbitrary, so we have that $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$ since distance is nonnegative.

(4) Let $A$ and $B$ be disjoint, closed sets in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$. For $a_0 \in A$ and $b_0 \in B$, define $\varepsilon_{a_0} = \frac{1}{2}\inf\{d(a_0, b) : b \in B\} > 0$ and $\varepsilon_{b_0} = \frac{1}{2}\inf\{d(a, b_0) : a \in A\} > 0$. Now set $U = \bigcup_{a \in A} B(a, \varepsilon_a)$ and $V = \bigcup_{b \in B} B(b, \varepsilon_b)$. Then since $U$ and $V$ are the unions of open balls in $\mathbb{R}^2$, they are open. Since if $a \in A$, then $a \in B(a, \varepsilon_a) \subset U$ and if $b \in B$, then $b \in B(b, \varepsilon_b) \subset V$, we have that $A \subset U$ and $B \subset V$. To show $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal, it only remains to show that $U$ and $V$ are disjoint. Suppose for contradiction that there exists $p \in \mathbb{R}^2$ such that $p \in U \cap V$. Then $p \in U$, so there exists an $\alpha \in A$ such that $p \in B(\alpha, \varepsilon_\alpha)$ and similarly, there exists a $\beta \in B$ such that $p \in B(\beta, \varepsilon_\beta)$. Since $\alpha \in A$, $d(\alpha, \beta) \in \{d(a, \beta) : a \in A\}$ and therefore $d(\alpha, \beta) \geq \inf\{d(a, \beta) : a \in A\} = 2\varepsilon_\alpha$. Similarly, $d(\alpha, \beta) \geq 2\varepsilon_\beta$, and so we have that $\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta)$. Since $p \in B(\alpha, \varepsilon_\alpha)$, we have that $d(\alpha, p) < \varepsilon_\alpha$, and since $p \in B(\beta, \varepsilon_\beta)$, we have that $d(p, \beta) < \varepsilon_\beta$. Putting this all together using the triangle inequality, we see that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta) \leq d(\alpha, p) + d(p, \beta) < \varepsilon_\alpha + \varepsilon_\beta.$$

This is a contradiction, so we have that $U \cap V = \emptyset$ and therefore $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal.

**Theorem 4.7.** (1) A $T_2$-space (Hausdorff) is a $T_1$-space.

(2) A $T_3$-space (regular and $T_1$) is a Hausdorff space, that is, a $T_2$-space.

(3) A $T_4$-space (normal and $T_1$) is regular and $T_1$, that is, a $T_3$-space.

*Proof.* (1) Let $(X, \mathcal{T})$ be a Hausdorff space and let $x, y \in X$ be distinct, arbitrary points. Then there exist disjoint, open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$, we have that $x \notin V$ and $y \notin U$, so $(X, \mathcal{T})$ is a $T_1$-space. $\square$

*Proof.* (2) Let $(X, \mathcal{T})$ be a $T_3$-space and let $x, y \in X$ be distinct, arbitrary points. Since this space is $T_1$, by Theorem 4.1 we have that $\{y\}$ is closed. Since this space is regular, we have that there exist disjoint, open sets such that $U$ and $V$ such that $x \in U$ and $\{y\} \subset V$. But $\{y\} \subset V$ means $y \in V$, so we have found disjoint, open sets separating the arbitrary points $x$ and $y$, so $(X, \mathcal{T})$ is Hausdorff. $\square$

*Proof.* (3) Let $(X, \mathcal{T})$ be a $T_4$-space, let $x \in X$ be arbitrary, and let $A$ be a closed set with $x \notin A$. Since this space is $T_1$, by Theorem 4.1 we have that $\{x\}$ is closed. Since this space is normal, there exist disjoint, open sets $U$ and $V$ such that $\{x\} \subset U$ and $A \subset V$. But $\{x\} \subset U$ means $x \in U$, so we have found disjoint, open sets separating the arbitrary point $x$ from the arbitrary closed set $A$, so $(X, \mathcal{T})$ is $T_3$ since it is normal and $T_1$. $\square$

**Theorem 4.8.** A topological space is regular if and only if for each point $p$ in $X$ and open set $U$ containing $p$ there exists an open set $V$ such that $p \in V$ and $\overline{V} \subset U$.

*Proof.* ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a regular topological space and let $U$ be an open set containing the point $p$. Then we have that $X - U$ is closed and since $p \in U$, $p \notin X - U$. Since this space is regular, there exist disjoint open sets $V$ and $W$ such that $p \in V$ and $X - U \subset W$. Therefore we have that $X - W \subset U$ since $X - U \subset W$, and that $X - W$ is closed since $W$ is open. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$) and therefore $x \in X - W$. Since $x$ was arbitrary, we have that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$p \in V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since $U$ and $p$ were arbitary, there exists an open set $V$ containing $p$ such that $\overline{V} \subset U$ for all $p \in X$ and open sets $U$ containing $p$.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all $p \in X$ and $W \in \mathcal{T}$ with $p \in W$, there exists an open set $U$ such that $p \in U$ and $\overline{U} \subset W$. Let $p \in X$ be arbitrary and let $A$ be a closed subset of $X$ such that $p \notin A$. Then we have that $p \in X - A$, which is open, and therefore there exists an open set $U$ such that $p \in U$ and $\overline{U} \subset X - A$, which implies that $A \subset V$ where $V$ is the open set $X - \overline{U}$. Let $x \in U$ be arbitary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since $x$ was arbitary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets $U$ and $V$ such that $p \in U$ and $A \subset V$, so $(X, \mathcal{T})$ is regular, as required. □

**Theorem 4.9.** A topological space is normal if and only if for each closed set $A$ in $(X, \mathcal{T})$ and open set $U$ containing $A$ there exists an open set $V$ such that $A \subset V$ and $\overline{V} \subset U$.

*Proof.* ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space, let $A$ be a closed set, and let $U$ be an open set such that $A \subset U$. Then $X - U$ closed and $A \cap (X - U) = \emptyset$, so since this is a normal space, there exist disjoint open sets $V$ and $W$ such that $A \subset V$ and $X - U \subset W$. Therefore $X - W \subset U$ is a closed set. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$), so $x \in X - W$, which means that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$A \subset V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since $A$ and $U$ were arbitrary, there exists an open set $V$ with $A \subset V$ and $\overline{V} \subset U$ for all open sets $U$ containing closed sets $A$.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all closed sets $A$ and open sets $W$ with $A \subset W$, there exists an open set $U$ such that $A \subset U$ and $\overline{U} \subset W$. Let $A$ and $B$ be arbitrary disjoint sets. Then we have that $A \subset X - B$, and since $X - B$ is open, there exists an open set $U$ such that $A \subset U$ and $\overline{U} \subset X - B$, which implies that $B \subset V$ where $V$ is the open set $X - \overline{U}$. Let $x \in U$ be arbitrary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since $x$ was arbitrary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$, so $(X, \mathcal{T})$ is normal, as required. □

**Theorem 4.10.** A topological space is normal if and only if for each pair of disjoint closed sets $A$ and $B$, there exist open sets $U$ and $V$ such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

*Proof.* ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let $A$ and $B$ be disjoint closed sets. Then since $X - B$ is open and $A \subset X - B$, by Theorem 4.9 there exists an open set $U$ such that $A \subset U$ and $\overline{U} \subset X - B$. Therefore $\overline{U} \cap B = \emptyset$, so there exists an open set $V$ such that $B \subset V$ and $\overline{V} \subset X - \overline{U}$, which means that $\overline{U} \cap \overline{V} = \emptyset$, as required.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for all disjoint closed sets $A$ and $B$, there exist open sets $U$ and $V$ such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$. But if $\overline{U}$ and $\bar{(}V)$ are disjoint, so are $U$ and $V$, and therefore this space is normal. $\square$

**Theorem 4.11 (The Incredible Shrinking Theorem).** A topological space is normal if and only if for each pair of open sets $U$ and $V$ such that $U \cup V = X$, there exist open sets $U'$ and $V'$ such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$.

*Proof.* ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let $U$ and $V$ be open sets with $U \cup V = X$. Then $X - U$ and $X - V$ are closed sets and

$$(X - U) \cap (X - V) = X - (U \cup V) = X - X = \emptyset.$$

Since these sets are closed and disjoint, by Theorem 4.10, there exist open sets $W_1$ and $W_2$ such that $X - U \subset W_1$, $X - V \subset W_2$, and $\overline{W_1} \cap \overline{W_2} = \emptyset$. Define $U' = X - \overline{W_1}$ and $V' = X - \overline{W_2}$. Then

$$U' \cup V' = (X - \overline{W_1}) \cup (X - \overline{W_2}) = X - (\overline{W_1} \cap \overline{W_2}) = X - \emptyset = X.$$

By Lemma 2.28, $\overline{W_1} = X - (X - W_1)^\circ$, so

$$U' = X - \overline{W_1} = X - (X - (X - W_1)^\circ) = (X - W_1)^\circ.$$

Since $(X - W_1)^\circ \subset X - W_1$ and $X - U \subset W_1$, we have that

$$\overline{U'} = \overline{(X - W_1)^\circ} \subset \overline{X - W_1} = X - W_1 \subset U$$

where we have used the fact that $X - W_1$ is closed. Similarly, $\overline{V'} \subset V$, and so we have shown the forward direction of the implication.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for any open sets $U$ and $V$ with $U \cup V = X$, there exist open sets $U'$ and $V'$ such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$. Let $A$ and $B$ be arbitrary closed sets. Then $X - A$ and $X - B$ are open sets such that

$$(X - A) \cup (X - B) = X - (A \cap B) = X - \emptyset = X,$$

so there exist open sets $U'$ and $V'$ such that $\overline{U'} \subset X - A$, $\overline{V'} \subset X - B$, and $U' \cup V' = X$. Define open sets $U = X - \overline{U'}$ and $V = X - \overline{V'}$. Then since $\overline{U'} \subset X - A$, $A \subset X - \overline{U'} = U$, and similarly, $B \subset V$. Then using Theorem 2.22(2), we have that

$$U \cap V = (X - \overline{U'}) \cap (X - \overline{V'}) = X - (\overline{U'} \cup \overline{V'}) = X - \overline{U' \cup V'} = X - \overline{X} = \emptyset$$