# Math 310 Practice Problems

Ben Clingenpeel

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# 1 The Real Numbers

## 1.2 Some Preliminaries

**Exercise 1.2.7.** (a) If $f(x) = x^2$, then $f(A) = [0, 4]$ and $f(B) = [1, 16]$. Therefore

$$f(A \cap B) = f([1, 2]) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$$

and

$$f(A \cup B) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B).$$

(b) If $f(x) = x^2$, $A = [0, 1]$ and $B = [-1, 0]$, then $A \cap B = \{0\}$ and so $f(A \cap B) = \{0\}$. However, $f(A) = [0, 1] = f(B)$, so $f(A) \cap f(B) = [0, 1]$. In general, $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let $A$ and $B$ be subsets of $\mathbb{R}$. Let $y \in g(A \cap B)$ be arbitrary as well. Then there exists an $x \in A \cap B$ such that $g(x) = y$, so $x \in A$ and $x \in B$. Therefore $y = g(x) \in g(A)$ and $y = g(x) \in g(B)$, meaning $y \in g(A) \cap g(B)$. Since $y$ was an arbitrary element of $g(A \cap B)$, we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let $A$ and $B$ be subsets of $\mathbb{R}$ as before. Again let $y \in g(A \cup B)$ be arbitrary. Then there exists an $x \in A \cup B$ such that $g(x) = y$, so $x \in A$ or $x \in B$. Without loss of generality, assume $x \in A$. Then $y = g(x) \in g(A) \subseteq g(A) \cup g(B)$, so we have that $g(A \cup B) \subseteq g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$. Then without loss of generality, assume $y \in g(A)$. This means there exists and $x \in A$ such that $g(x) = y$. Since $x \in A \subseteq A \cup B$, we have that $y = g(x) \in g(A \cup B)$, and so $g(A) \cup g(B) \subseteq g(A \cup B)$. Therefore $g(A \cup B) = g(A) \cup g(B)$.

**Exercise 1.2.9.** (a) We have that $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. Therefore

$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = [-2, 2] \cap [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$$

and

$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = [-2, 2] \cup [-1, 1] = f^{-1}(A) \cup f^{-1}(B).$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be arbitrary and let $y \in g^{-1}(A \cap B)$. Then there exists an $x \in A \cap B$ such that $g(x) = y$. Since $x \in A$, $y = g(x) \in g^{-1}(A)$, and since $x \in B$, $y = g(x) \in g^{-1}(B)$. Therefore $y \in g^{-1}(A) \cap g^{-1}(B)$. Follow the argument backwards and you get the other inclusion, so $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$. The argument for $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is similar.

**Exercise 1.2.10.** (a) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(b) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(c) True. Certainly $a < b$ implies that $a < b + \varepsilon$ for all $\varepsilon > 0$. Suppose $a < b + \varepsilon$ for all $\varepsilon > 0$ and suppose for contradiction that $a > b$. Then $a - b > 0$, so we have that $a < b + (a - b) = a$, a contradiction. Therefore $a \not> b$, i.e. $a \leq b$.

## 1.3 The Axiom of Completeness

**Exercise 1.3.6.** (a) Let $c \in A + B$. Then $c$ can be written as $c = a + b$ for some $a \in A$ and $b \in B$. Since $s = \sup A$ and $t = \sup B$, we have that $a \leq s$ and $b \leq t$, meaning $c = a + b \leq s + t$. Therefore $s + t$ is an upper bound for $A + B$.

(b) Let $u$ be an upper bound of $A + B$ and let $a \in A$. Then we have that $a + b \leq u$ for all $b \in B$, so also $b \leq u - a$ for all $b \in B$. Therefore $u - a$ is an upper bound for $B$ and so $t = \sup B \leq u - a$.

(c) The element $a \in A$ in (b) was arbitrary, so we have that $t \leq u - a$ for all $a \in A$, that is, $a \leq u - t$ for all $a \in A$. Therefore $u - t$ is an upper bound of $A$, so $s = \sup A \leq u - t$. Hence $s + t \leq u$, and so since $u$ was an arbitrary upper bound of $A + B$, we have that $\sup(A + B) = s + t$.

(d) Let $\varepsilon > 0$. Since $s = \sup A$, there exists an $a \in A$ such that $s - (\varepsilon/2) < a$. Similarly, $t = \sup B$ means that there exists a $b \in B$ such that $t - (\varepsilon/2) < b$. Therefore $s + t - \varepsilon < a + b \in A + B$, so since $\varepsilon$ was arbitrary, $\sup(A + B) = s + t$.

**Exercise 1.3.11.** (a) True. Let $A \subseteq B$ for nonempty, bounded sets $A$ and $B$. Let $a \in A \subseteq B$. Then $a \leq \sup B$, so $\sup B$ is an upper bound for $A$. Since $\sup A$ is the least upper bound for $A$, $\sup A \leq \sup B$.

(b) True. Let $A$ and $B$ be sets such that $\sup A < \inf B$ and set

$$c = \frac{\sup A + \inf B}{2}.$$

Then let $a \in A$ and $b \in B$. We have that

$$a \leq \sup A = \frac{\sup A + \sup A}{2} < c < \frac{\inf B + \inf B}{2} \leq b.$$

(c) False. Let $A = \{-1/n \mid n \in \mathbb{N}\}$ and $B = \{1/n \mid n \in \mathbb{N}\}$. Then $a < 0 < b$ for all $a \in A$ and $b \in B$, but $\sup A = 0 = \inf B$, so we do not have that $\sup A < \inf B$. In general, the statement should be that $\sup A \leq \inf B$.

## 1.4 Consequences of Completeness

**Exercise 1.4.2.** Assume the hypotheses of the claim. Suppose for contradiction that $s < \sup A$. Then $\sup A - s > 0$, so there exists an $n_0 \in \mathbb{N}$ such that $\sup A - s > 1/n_0 > 0$ and therefore $s + 1/n_0 < \sup A$. However, $s + 1/n_0$ is an upper bound for $A$ by hypothesis, so this implies that $\sup A$ is not the least upper bound of $A$, a contradiction. Now suppose for contradiction that $s > \sup A$. Then there exists an $m_0 \in \mathbb{N}$ such that $s - \sup A > 1/m_0 > 0$, which means $\sup A < 1 - 1/m_0$. Hence $1 - 1/m_0$ is an upper bound for $A$, but this contradicts our assumption that for all $n \in \mathbb{N}$, $1 - 1/n$ is not an upper bound. Therefore we must have that $s = \sup A$.

**Exercise 1.4.8.** (a) Consider $A = (0, 1) \cap \mathbb{Q}$ and $B = (0, 1) \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$, $\sup A = \sup B = 1$, $\sup A = 1 \notin A$, and $\sup B = 1 \notin B$.

(c) Define $L_n = [n, \infty)$. Then $L_i \supset L_{i+1}$ as required, and for all $x \in \mathbb{R}$, there exists an $n_0 \in \mathbb{N}$ such that $n_0 > x$, and therefore $x \notin L_{n_0}$, meaning $x \notin \bigcap_{n=1}^{\infty} L_n$. Therefore $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

## 1.5 Cardinality

**Exercise 1.5.6.** (a) For $n \in \mathbb{N}$, define $U_n = (n, n+1)$. Then the $U_n$ are pairwise disjoint, and since $\mathbb{N}$ is countable, the collection $\{U_n\}_{n \in \mathbb{N}}$ is countable as well.

(b) Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\lambda}$ be a collection of pairwise disjoint open intervals. Then since $\mathbb{Q}$ is dense in $\mathbb{R}$, each $U_\alpha$ contains some $q_\alpha \in U_\alpha \cap \mathbb{Q}$. Define the map $f : \mathcal{U} \to \mathbb{Q}$ by $f(U_\alpha) = q_\alpha$. This map is injective: let $U_\beta$ and $U_\gamma$ be distinct open intervals and suppose for contradiction that $f(U_\beta) = f(U_\gamma)$. Then $q_\beta = q_\gamma \in U_\gamma$, so $q_\beta \in U_\beta \cap U_\gamma = \emptyset$, a contradiction since elements of $\mathcal{U}$ are pairwise disjoint. Hence it is not the case that $f(U_\beta) = f(U_\gamma)$, so $f$ is injective. Now define $g : \mathcal{U} \to \mathrm{im} f$ by $g(U_\alpha) = f(U_\alpha)$. Then since $f$ is injective, $g$ is injective, and if $q \in \mathrm{im} f$, there exists an $\alpha \in \lambda$ such that $f(U_\alpha) = q$, meaning also $g(U_\alpha) = q$. Therefore $g$ is also onto, meaning $\mathcal{U}$ has the same cardinality as $\mathrm{im} f$, and since $\mathrm{im} f \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable, $\mathcal{U}$ is either finite or countable. In particular, $\mathcal{U}$ is not uncountable.

**Exercise 1.5.9.** (a) Since $\sqrt{2}$ is a root of $x^2 - 2$ it is algebraic; since $\sqrt[3]{2}$ is a root of $x^3 - 2$ it is algebraic; and since $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$.

(b) There are finitely manny choices for the coefficients $a_1, \ldots, a_n$ such that $|a_1| + \cdots + |a_n| = m$ for a fixed $m \in \mathbb{N}$, and each polynomial has finitely many roots, so first write down all of the roots corresponding to $m = 1$, then $m = 2$, and so on. There are finitely many roots corresponding to each $m$, so eventually every polynomial is included in the list, and therefore $A_n$ is countable.

(c) The union of countably many sets is countable, so there are countably many algebraic numbers.

# 2 Sequences and Series

## 2.2 The Limit of a Sequence

**Exercise 2.2.3.** (a) We would need to find a college in the United States where there is not a single student at least seven feet tall.

(b) We would need to find a college in the United States where for every professor there exists a student they give a grade other than A or B.

(c) We would need to show that every college has a student who is under six feet.

**Exercise 2.2.4.** (a) Define $(a_n)$ to be the sequence that is 1 at every odd input and 0 at every even input.

(b) This is not possible. To see this, suppose for contradiction that there were such a sequence $(a_n)$ and let $L$ be the limit it converges to. Then set $\varepsilon = |1 - L|$. Since $(a_n)$ converges, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. However, since this sequences has an infinite number of ones and it cannot therefore be the case that no term after $N$ is equal to one, there is an $n \geq N$ such that $a_n = 1$, and therefore $|a_n - L| = |1 - L| = \varepsilon \not< \varepsilon$, a contradiction showing such a sequence does not exist.

(c) Define $(a_n)$ to be one everywhere except at perfect squares. At perfect squares $a_n = n$. Then since the gaps between perfect squares grow without bound, there are consecutive ones of arbitrary length. Also, because the squares themselves grow without bound, the sequence does not converge.