Math 310 Practice Problems

Ben Clingenpeel

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1 The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.7. (a) If $f(x) = x^2$, then $f(A) = [0, 4]$ and $f(B) = [1, 16]$. Therefore

$$f(A \cap B) = f([1, 2]) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$$

and

$$f(A \cup B) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B).$$

(b) If $f(x) = x^2$, $A = [0, 1]$ and $B = [-1, 0]$, then $A \cap B = \{0\}$ and so $f(A \cap B) = \{0\}$. However, $f(A) = [0, 1] = f(B)$, so $f(A) \cap f(B) = [0, 1]$. In general, $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let A and B be subsets of $\mathbb{R}$. Let $y \in g(A \cap B)$ be arbitrary as well. Then there exists an $x \in A \cap B$ such that $g(x) = y$, so $x \in A$ and $x \in B$. Therefore $y = g(x) \in g(A)$ and $y = g(x) \in g(B)$, meaning $y \in g(A) \cap g(B)$. Since y was an arbitrary element of $g(A \cap B)$, we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let A and B be subsets of $\mathbb{R}$ as before. Again let $y \in g(A \cup B)$ be arbitrary. Then there exists an $x \in A \cup B$ such that $g(x) = y$, so $x \in A$ or $x \in B$. Without loss of generality, assume $x \in A$. Then $y = g(x) \in g(A) \subseteq g(A) \cup g(B)$, so we have that $g(A \cup B) \subseteq g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$. Then without loss of generality, assume $y \in g(A)$. This means there exists and $x \in A$ such that $g(x) = y$. Since $x \in A \subseteq A \cup B$, we have that $y = g(x) \in g(A \cup B)$, and so $g(A) \cup g(B) \subseteq g(A \cup B)$. Therefore $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.9. (a) We have that $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. Therefore

$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = [-2, 2] \cap [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$$

and

$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = [-2, 2] \cup [-1, 1] = f^{-1}(A) \cup f^{-1}(B).$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be arbitrary and let $y \in g^{-1}(A \cap B)$. Then there exists an $x \in A \cap B$ such that $g(x) = y$. Since $x \in A$, $y = g(x) \in g^{-1}(A)$, and since $x \in B$, $y = g(x) \in g^{-1}(B)$. Therefore $y \in g^{-1}(A) \cap g^{-1}(B)$. Follow the argument backwards and you get the other inclusion, so $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$. The argument for $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is similar.

Exercise 1.2.10. (a) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(b) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(c) True. Certainly $a < b$ implies that $a < b + \varepsilon$ for all $\varepsilon > 0$. Suppose $a < b + \varepsilon$ for all $\varepsilon > 0$ and suppose for contradiction that $a > b$. Then $a - b > 0$, so we have that $a < b + (a - b) = a$, a contradiction. Therefore $a \not> b$, i.e. $a \leq b$.

1.3 The Axiom of Completeness

Exercise 1.3.6. (a) Let $c \in A + B$. Then c can be written as $c = a + b$ for some $a \in A$ and $b \in B$. Since $s = \sup A$ and $t = \sup B$, we have that $a \leq s$ and $b \leq t$, meaning $c = a + b \leq s + t$. Therefore $s + t$ is an upper bound for $A + B$.

(b) Let u be an upper bound of $A + B$ and let $a \in A$. Then we have that $a + b \leq u$ for all $b \in B$, so also $b \leq u - a$ for all $b \in B$. Therefore $u - a$ is an upper bound for B and so $t = \sup B \leq u - a$.

(c) The element $a \in A$ in (b) was arbitrary, so we have that $t \leq u - a$ for all $a \in A$, that is, $a \leq u - t$ for all $a \in A$. Therefore $u - t$ is an upper bound of A, so $s = \sup A \leq u - t$. Hence $s + t \leq u$, and so since u was an arbitrary upper bound of $A + B$, we have that $\sup(A + B) = s + t$.

(d) Let $\varepsilon > 0$. Since $s = \sup A$, there exists an $a \in A$ such that $s - (\varepsilon/2) < a$. Similarly, $t = \sup B$ means that there exists a $b \in B$ such that $t - (\varepsilon/2) < b$. Therefore $s + t - \varepsilon < a + b \in A + B$, so since ε was arbitrary, $\sup(A + B) = s + t$.

Exercise 1.3.11. (a) True. Let $A \subseteq B$ for nonempty, bounded sets A and B. Let $a \in A \subseteq B$. Then $a \leq \sup B$, so $\sup B$ is an upper bound for A. Since $\sup A$ is the least upper bound for A, $\sup A \leq \sup B$.

(b) True. Let A and B be sets such that $\sup A < \inf B$ and set

$$c = \frac{\sup A + \inf B}{2}.$$

Then let $a \in A$ and $b \in B$. We have that

$$a \leq \sup A = \frac{\sup A + \sup A}{2} < c < \frac{\inf B + \inf B}{2} \leq b.$$

(c) False. Let $A = \{-1/n \mid n \in \mathbb{N}\}$ and $B = \{1/n \mid n \in \mathbb{N}\}$. Then $a < 0 < b$ for all $a \in A$ and $b \in B$, but $\sup A = 0 = \inf B$, so we do not have that $\sup A < \inf B$. In general, the statement should be that $\sup A \leq \inf B$.

1.4 Consequences of Completeness

Exercise 1.4.2. Assume the hypotheses of the claim. Suppose for contradiction that $s < \sup A$. Then $\sup A - s > 0$, so there exists an $n_0 \in \mathbb{N}$ such that $\sup A - s > 1/n_0 > 0$ and therefore $s + 1/n_0 < \sup A$. However, $s + 1/n_0$ is an upper bound for A by hypothesis, so this implies that $\sup A$ is not the least upper bound of A, a contradiction. Now suppose for contradiction that $s > \sup A$. Then there exists an $m_0 \in \mathbb{N}$ such that $s - \sup A > 1/m_0 > 0$, which means $\sup A < 1 - 1/m_0$. Hence $1 - 1/m_0$ is an upper bound for A, but this contradicts our assumption that for all $n \in \mathbb{N}$, $1 - 1/n$ is not an upper bound. Therefore we must have that $s = \sup A$.

Exercise 1.4.8. (a) Consider $A = (0,1) \cap \mathbb{Q}$ and $B = (0,1) \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$, $\sup A = \sup B = 1$, $\sup A = 1 \notin A$, and $\sup B = 1 \notin B$.

(c) Define $L_n = [n, \infty)$. Then $L_i \supset L_{i+1}$ as required, and for all $x \in \mathbb{R}$, there exists an $n_0 \in \mathbb{N}$ such that $n_0 > x$, and therefore $x \notin L_{n_0}$, meaning $x \notin \bigcap_{n=1}^{\infty} L_n$. Therefore $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

1.5 Cardinality

Exercise 1.5.6. (a) For $n \in \mathbb{N}$, define $U_n = (n, n+1)$. Then the U_n are pairwise disjoint, and since $\mathbb{N}$ is countable, the collection $\{U_n\}_{n \in \mathbb{N}}$ is countable as well.

(b) Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \lambda}$ be a collection of pairwise disjoint open intervals. Then since $\mathbb{Q}$ is dense in $\mathbb{R}$, each U_α contains some $q_\alpha \in U_\alpha \cap \mathbb{Q}$. Define the map $f : \mathcal{U} \to \mathbb{Q}$ by $f(U_\alpha) = q_\alpha$. This map is injective: let U_β and U_γ be distinct open intervals and suppose for contradiction that $f(U_\beta) = f(U_\gamma)$. Then $q_\beta = q_\gamma \in U_\gamma$, so $q_\beta \in U_\beta \cap U_\gamma = \emptyset$, a contradiction since elements of $\mathcal{U}$ are pairwise disjoint. Hence it is not the case that $f(U_\beta) = f(U_\gamma)$, so f is injective. Now define $g : \mathcal{U} \to \mathrm{im} f$ by $g(U_\alpha) = f(U_\alpha)$. Then since f is injective, g is injective, and if $q \in \mathrm{im} f$, there exists an $\alpha \in \lambda$ such that $f(U_\alpha) = q$, meaning also $g(U_\alpha) = q$. Therefore g is also onto, meaning $\mathcal{U}$ has the same cardinality as $\mathrm{im} f$, and since $\mathrm{im} f \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable, $\mathcal{U}$ is either finite or countable. In particular, $\mathcal{U}$ is not uncountable.

Exercise 1.5.9. (a) Since $\sqrt{2}$ is a root of $x^2 - 2$ it is algebraic; since $\sqrt[3]{2}$ is a root of $x^3 - 2$ it is algebraic; and since $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$.

(b) There are finitely manny choices for the coefficients $a_1, \dots, a_n$ such that $|a_1| + \cdots + |a_n| = m$ for a fixed $m \in \mathbb{N}$, and each polynomial has finitely many roots, so first write down all of the roots corresponding to $m = 1$, then $m = 2$, and so on. There are finitely many roots corresponding to each m, so eventually every polynomial is included in the list, and therefore A_n is countable.

(c) The union of countably many sets is countable, so there are countably many algebraic numbers.

2 Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.3. (a) We would need to find a college in the United States where there is not a single student at least seven feet tall.

(b) We would need to find a college in the United States where for every professor there exists a student they give a grade other than A or B.

(c) We would need to show that every college has a student who is under six feet.

Exercise 2.2.4. (a) Define (a_n) to be the sequence that is 1 at every odd input and 0 at every even input.

(b) This is not possible. To see this, suppose for contradiction that there were such a sequence (a_n) and let L be the limit it converges to. Then set $\varepsilon = |1 - L|$. Since (a_n) converges, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. However, since this sequences has an infinite number of ones and it cannot therefore be the case that no term after N is equal to one, there is an $n \geq N$ such that $a_n = 1$, and therefore $|a_n - L| = |1 - L| = \varepsilon \not< \varepsilon$, a contradiction showing such a sequence does not exist.

(c) Define (a_n) to be one everywhere except at perfect squares. At perfect squares $a_n = n$. Then since the gaps between perfect squares grow without bound, there are consecutive ones of arbitrary length. Also, because the squares themselves grow without bound, the sequence does not converge.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.9. (a) Let $a_n \to 0$ and let (b_n) be bounded. Then there exists $M \in \mathbb{R}$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then because $a_n \to 0$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|a_n| < \varepsilon/M$. Then we have that

$$|a_n b_n| \leq M|a_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

so $(a_n b_n) \to 0$ as claimed. We cannot use the ALT to prove this because we do not know that (b_n) converges.

(b) No. Suppose $a_n = 1$ for all n. Then $a_n b_n = b_n$, which is bounded, but may or may not converge.

(c) To prove (iii) when $a_n \to 0$, note that (b_n) converging means that it is bounded. By (a), $a_n b_n \to 0$.

Exercise 2.3.12. (a) Suppose a_n is an upper bound for B and that $a_n \to a \in \mathbb{R}$. Then define the sequence $b_n = \sup B$ for all $n \in \mathbb{N}$. Then $a_n \geq b_n$ for all $n \in \mathbb{N}$, so by the Order Limit Theorem, $a \geq \sup B$ since $b_n \to \sup B$. Therefore a is also an upper bound of B.

(b) If $a_n \notin (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \to a$, then there exists an $N \in \mathbb{N}$ such that either $n > N$ implies $a_n \geq 1$ or $n > N$ implies $a_n \leq 0$. Then the Order Limit Theorem implies that either $a \geq 1$ or $a \leq 0$, but in either case $a \notin (0, 1)$.

(c) This is false. Consider the sequence of truncated decimal expansions for $\sqrt{2}$ ($a_1 = 1, a_2 = 1.4, a_3 = 1.41, \ldots$). Then $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ but $a_n \to \sqrt{2} \notin \mathbb{Q}$.