Math 310 Practice Problems

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1 The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.7. (a) If $f(x) = x^2$, then $f(A) = [0, 4]$ and $f(B) = [1, 16]$. Therefore

$$f(A \cap B) = f([1, 2]) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$$

and

$$f(A \cup B) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B).$$

(b) If $f(x) = x^2$, $A = [0, 1]$ and $B = [-1, 0]$, then $A \cap B = \{0\}$ and so $f(A \cap B) = \{0\}$. However, $f(A) = [0, 1] = f(B)$, so $f(A) \cap f(B) = [0, 1]$. In general, $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let A and B be subsets of $\mathbb{R}$. Let $y \in g(A \cap B)$ be arbitrary as well. Then there exists an $x \in A \cap B$ such that $g(x) = y$, so $x \in A$ and $x \in B$. Therefore $y = g(x) \in g(A)$ and $y = g(x) \in g(B)$, meaning $y \in g(A) \cap g(B)$. Since y was an arbitrary element of $g(A \cap B)$, we have that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, and let A and B be subsets of $\mathbb{R}$ as before. Again let $y \in g(A \cup B)$ be arbitrary. Then there exists an $x \in A \cup B$ such that $g(x) = y$, so $x \in A$ or $x \in B$. Without loss of generality, assume $x \in A$. Then $y = g(x) \in g(A) \subseteq g(A) \cup g(B)$, so we have that $g(A \cup B) \subseteq g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$. Then without loss of generality, assume $y \in g(A)$. This means there exists and $x \in A$ such that $g(x) = y$. Since $x \in A \subseteq A \cup B$, we have that $y = g(x) \in g(A \cup B)$, and so $g(A) \cup g(B) \subseteq g(A \cup B)$. Therefore $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.9. (a) We have that $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. Therefore

$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = [-2, 2] \cap [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$$

and

$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = [-2, 2] \cup [-1, 1] = f^{-1}(A) \cup f^{-1}(B).$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be arbitrary and let $y \in g^{-1}(A \cap B)$. Then there exists an $x \in A \cap B$ such that $g(x) = y$. Since $x \in A$, $y = g(x) \in g^{-1}(A)$, and since $x \in B$, $y = g(x) \in g^{-1}(B)$. Therefore $y \in g^{-1}(A) \cap g^{-1}(B)$. Follow the argument backwards and you get the other inclusion, so $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$. The argument for $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is similar.

Exercise 1.2.10. (a) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(b) False. We have that $1 < 1 + \varepsilon$ for all $\varepsilon > 0$, but $1 \not< 1$.

(c) True. Certainly $a < b$ implies that $a < b + \varepsilon$ for all $\varepsilon > 0$. Suppose $a < b + \varepsilon$ for all $\varepsilon > 0$ and suppose for contradiction that $a > b$. Then $a - b > 0$, so we have that $a < b + (a - b) = a$, a contradiction. Therefore $a \not> b$, i.e. $a \leq b$.

1.3 The Axiom of Completeness

Exercise 1.3.6. (a) Let $c \in A + B$. Then c can be written as $c = a + b$ for some $a \in A$ and $b \in B$. Since $s = \sup A$ and $t = \sup B$, we have that $a \leq s$ and $b \leq t$, meaning $c = a + b \leq s + t$. Therefore $s + t$ is an upper bound for $A + B$.

(b) Let u be an upper bound of $A + B$ and let $a \in A$. Then we have that $a + b \leq u$ for all $b \in B$, so also $b \leq u - a$ for all $b \in B$. Therefore $u - a$ is an upper bound for B and so $t = \sup B \leq u - a$.

(c) The element $a \in A$ in (b) was arbitrary, so we have that $t \leq u - a$ for all $a \in A$, that is, $a \leq u - t$ for all $a \in A$. Therefore $u - t$ is an upper bound of A, so $s = \sup A \leq u - t$. Hence $s + t \leq u$, and so since u was an arbitrary upper bound of $A + B$, we have that $\sup(A + B) = s + t$.

(d) Let $\varepsilon > 0$. Since $s = \sup A$, there exists an $a \in A$ such that $s - (\varepsilon/2) < a$. Similarly, $t = \sup B$ means that there exists a $b \in B$ such that $t - (\varepsilon/2) < b$. Therefore $s + t - \varepsilon < a + b \in A + B$, so since ε was arbitrary, $\sup(A + B) = s + t$.

Exercise 1.3.11. (a) True. Let $A \subseteq B$ for nonempty, bounded sets A and B. Let $a \in A \subseteq B$. Then $a \leq \sup B$, so $\sup B$ is an upper bound for A. Since $\sup A$ is the least upper bound for A, $\sup A \leq \sup B$.

(b) True. Let A and B be sets such that $\sup A < \inf B$ and set

$$c = \frac{\sup A + \inf B}{2}.$$

Then let $a \in A$ and $b \in B$. We have that

$$a \leq \sup A = \frac{\sup A + \sup A}{2} < c < \frac{\inf B + \inf B}{2} \leq b.$$

(c) False. Let $A = \{-1/n \mid n \in \mathbb{N}\}$ and $B = \{1/n \mid n \in \mathbb{N}\}$. Then $a < 0 < b$ for all $a \in A$ and $b \in B$, but $\sup A = 0 = \inf B$, so we do not have that $\sup A < \inf B$. In general, the statement should be that $\sup A \leq \inf B$.

1.4 Consequences of Completeness

Exercise 1.4.2. Assume the hypotheses of the claim. Suppose for contradiction that $s < \sup A$. Then $\sup A - s > 0$, so there exists an $n_0 \in \mathbb{N}$ such that $\sup A - s > 1/n_0 > 0$ and therefore $s + 1/n_0 < \sup A$. However, $s + 1/n_0$ is an upper bound for A by hypothesis, so this implies that $\sup A$ is not the least upper bound of A, a contradiction. Now suppose for contradiction that $s > \sup A$. Then there exists an $m_0 \in \mathbb{N}$ such that $s - \sup A > 1/m_0 > 0$, which means $\sup A < 1 - 1/m_0$. Hence $1 - 1/m_0$ is an upper bound for A, but this contradicts our assumption that for all $n \in \mathbb{N}$, $1 - 1/n$ is not an upper bound. Therefore we must have that $s = \sup A$.

Exercise 1.4.8. (a) Consider $A = (0, 1) \cap \mathbb{Q}$ and $B = (0, 1) \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$, $\sup A = \sup B = 1$, $\sup A = 1 \notin A$, and $\sup B = 1 \notin B$.

(c) Define $L_n = [n, \infty)$. Then $L_i \supset L_{i+1}$ as required, and for all $x \in \mathbb{R}$, there exists an $n_0 \in \mathbb{N}$ such that $n_0 > x$, and therefore $x \notin L_{n_0}$, meaning $x \notin \bigcap_{n=1}^{\infty} L_n$. Therefore $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

1.5 Cardinality

Exercise 1.5.6. (a) For $n \in \mathbb{N}$, define $U_n = (n, n + 1)$. Then the U_n are pairwise disjoint, and since $\mathbb{N}$ is countable, the collection $\{U_n\}_{n \in \mathbb{N}}$ is countable as well.

(b) Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\lambda}$ be a collection of pairwise disjoint open intervals. Then since $\mathbb{Q}$ is dense in $\mathbb{R}$, each U_α contains some $q_\alpha \in U_\alpha \cap \mathbb{Q}$. Define the map $f : \mathcal{U} \to \mathbb{Q}$ by $f(U_\alpha) = q_\alpha$. This map is injective: let U_β and U_γ be distinct open intervals and suppose for contradiction that $f(U_\beta) = f(U_\gamma)$. Then $q_\beta = q_\gamma \in U_\gamma$, so $q_\beta \in U_\beta \cap U_\gamma = \emptyset$, a contradiction since elements of $\mathcal{U}$ are pairwise disjoint. Hence it is not the case that $f(U_\beta) = f(U_\gamma)$, so f is injective. Now define $g : \mathcal{U} \to \operatorname{im} f$ by $g(U_\alpha) = f(U_\alpha)$. Then since f is injective, g is injective, and if $q \in \operatorname{im} f$, there exists an $\alpha \in \lambda$ such that $f(U_\alpha) = q$, meaning also $g(U_\alpha) = q$. Therefore g is also onto, meaning $\mathcal{U}$ has the same cardinality as $\operatorname{im} f$, and since $\operatorname{im} f \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable, $\mathcal{U}$ is either finite or countable. In particular, $\mathcal{U}$ is not uncountable.

Exercise 1.5.9. (a) Since $\sqrt{2}$ is a root of $x^2 - 2$ it is algebraic; since $\sqrt[3]{2}$ is a root of $x^3 - 2$ it is algebraic; and since $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$.

(b) There are finitely manny choices for the coefficients $a_1, \dots, a_n$ such that $|a_1| + \cdots + |a_n| = m$ for a fixed $m \in \mathbb{N}$, and each polynomial has finitely many roots, so first write down all of the roots corresponding to $m = 1$, then $m = 2$, and so on. There are finitely many roots corresponding to each m, so eventually every polynomial is included in the list, and therefore A_n is countable.

(c) The union of countably many sets is countable, so there are countably many algebraic numbers.

2 Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.3. (a) We would need to find a college in the United States where there is not a single student at least seven feet tall.

(b) We would need to find a college in the United States where for every professor there exists a student they give a grade other than A or B.

(c) We would need to show that every college has a student who is under six feet.

Exercise 2.2.4. (a) Define (a_n) to be the sequence that is 1 at every odd input and 0 at every even input.

(b) This is not possible. To see this, suppose for contradiction that there were such a sequence (a_n) and let L be the limit it converges to. Then set $\varepsilon = |1 - L|$. Since (a_n) converges, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. However, since this sequences has an infinite number of ones and it cannot therefore be the case that no term after N is equal to one, there is an $n \geq N$ such that $a_n = 1$, and therefore $|a_n - L| = |1 - L| = \varepsilon \not< \varepsilon$, a contradiction showing such a sequence does not exist.

(c) Define (a_n) to be one everywhere except at perfect squares. At perfect squares $a_n = n$. Then since the gaps between perfect squares grow without bound, there are consecutive ones of arbitrary length. Also, because the squares themselves grow without bound, the sequence does not converge.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.9. (a) Let $a_n \to 0$ and let (b_n) be bounded. Then there exists $M \in \mathbb{R}$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then because $a_n \to 0$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|a_n| < \varepsilon/M$. Then we have that

$$|a_n b_n| \leq M|a_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

so $(a_n b_n) \to 0$ as claimed. We cannot use the ALT to prove this because we do not know that (b_n) converges.

(b) No. Suppose $a_n = 1$ for all n. Then $a_n b_n = b_n$, which is bounded, but may or may not converge.

(c) To prove (iii) when $a_n \to 0$, note that (b_n) converging means that it is bounded. By (a), $a_n b_n \to 0$.

Exercise 2.3.12. (a) Suppose a_n is an upper bound for B and that $a_n \to a \in \mathbb{R}$. Then define the sequence $b_n = \sup B$ for all $n \in \mathbb{N}$. Then $a_n \geq b_n$ for all $n \in \mathbb{N}$, so by the Order Limit Theorem, $a \geq \sup B$ since $b_n \to \sup B$. Therefore a is also an upper bound of B.

(b) If $a_n \notin (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \to a$, then there exists an $N \in \mathbb{N}$ such that either $n > N$ implies $a_n \geq 1$ or $n > N$ implies $a_n \leq 0$. Then the Order Limit Theorem implies that either $a \geq 1$ or $a \leq 0$, but in either case $a \notin (0, 1)$.

(c) This is false. Consider the sequence of truncated decimal expansions for $\sqrt{2}$ ($a_1 = 1, a_2 = 1.4, a_3 = 1.41, \ldots$). Then $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ but $a_n \to \sqrt{2} \notin \mathbb{Q}$.

2.4 The Monotone Convergence Theorem

Exercise 2.4.8. (a) We argue by induction that $s_n = \sum_{i=1}^{n} 1/2^i = 1 - 1/2^n$. For the base case $n = 1$, we have that $\sum_{i=1}^{1} 1/2^i = 1/2 = 1 - 1/2^n$. For the inductive step, assume the claim holds for $k \in \mathbb{N}$. Then

$$\sum_{i=1}^{k+1} \frac{1}{2^i} = \frac{1}{2^{k+1}} + \sum_{i=1}^{k} \frac{1}{2^i} = \frac{1}{2^{k+1}} + 1 - \frac{1}{2^k} = 1 - \frac{1}{2^{k+1}}.$$

Therefore the claim holds for all n. Since s_n is increasing and bounded above by 1, the Monotone Convergence Theorem implies that $s_n \to \sup\{s_n \mid n \in \mathbb{N}\}$. Since 1 is an upper bound and for all $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that $1/2^m < \varepsilon$, meaning $1 - \varepsilon < 1 - 1/2^m = s_m$, we have that $1 = \sup\{s_n \mid n \in \mathbb{N}\}$ and therefore that the series converges to 1.

(c) We argue by induction that $s_n = \log(n+1)$. For the base case $n = 1$, we have that $\sum_{i=1}^{1} \log((i+1)/i) = \log(2/1) = \log(2)$. For the inductive step, assume the claim holds for $k \in \mathbb{N}$. Then

$$\sum_{i=1}^{k+1} \log\left(\frac{i+1}{i}\right) = \log\left(\frac{k+2}{k+1}\right) + s_k = \log\left(\frac{k+2}{k+1}\right) + \log(k+1) = \log(k+2).$$

Therefore the claim holds for all n. Since s_n is unbounded above, it does not converge.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1. (a) This is impossible. If a sequence has a subsequence that is bounded, the bounded subsequence has a subsequence that converges by The Bolzano-Weirstrass Theorem, and this subsequence is also a subsequence of the original sequence.

(b) Define

$$a_n = \begin{cases} 1/n & n \equiv 0 \pmod 2 \\ 1 - 1/n & n \equiv 1 \pmod 2 \end{cases}.$$

Then no term is equal to 0 or to 1, but the subsequence of even numbered terms converges to 0 and the subsequence of odd numbered terms converges to 1.

(c) Take a_n to be the sequence with a 1 at all odd indices, a $1/2$ at all multiples of 2 that are not multiples of a higher power of 2, a $1/3$ at all multiples of 4 that are not multiples of a higher power of 2, a $1/4$ at all multiples of 8 that are not multiples of a higher power of 2, etc. Then $1/n$ appears at all multiples M of 2^{n-1} that for all $k \geq n$, M is not a multiple of 2^k. Therefore there are infinitely many indices at which $1/n$ appears, so there is a subsequence converging to each value.

(d) Suppose a_n has a subsequence converging to each value in the list. Then the subsequence consisting of the first appearances of each $1/n$ converges to 0, which is not in the list.

Exercise 2.5.4. Assume the nested interval property holds and that the sequence $1/2^n$ converges to 0. Let A be a nonempty set that is bounded above. Since A is nonempty, there exists an element $a_1 \in A$. Since A is bounded above, there exists an element b_1 that is an upper bound for A. Define the interval I_1 to be $[a_1, b_1]$. If there exist no other elements of A greater than a_1, then a_1 is an upper bound for A since there is no other element of A greater than it, and it is also a least upper bound of A since any upper bound that is not a_1 is greater. Therefore A has a least upper bound and we are done. If there exists another element of A, call it a_2, then we proceed as earlier. If there exists no upper bound of A less than b_2, then b_2 is the least upper bound and again we are done. Now suppose there exists an $a_2 > a_1$ and a $b_2 < b_1$ such that b_2 is an upper bound of A. Define $I_2 = [a_2, b_2]$. Continue in this fashion, defining $I_n = [a_n, b_n]$ where $a_n > a_{n-1}$ and $b_n < b_{n-1}$. If such an a_n does not exist, then $a_{n-1} = \sup A$, and if such a b_n does not exist, then $b_{n-1} = \sup A$. Suppose now that a_n and b_n exist for all $n \in \mathbb{N}$. Then by the nested interval property, there exists an $x \in \mathbb{R}$ such that $x \in \bigcap_{i=1}^{\infty} I_i$. We claim that $\sup A$ exists and is equal to x. Suppose there exists an element $a \in A$ such that $a > x$. Then $a > a_n$ for all $n \in \mathbb{N}$, and because ...

2.6 The Cauchy Criterion

Exercise 2.6.2. (a) Take $a_n = (-1)^n/n$. This converges to 0 but is not monotone.

(b) This is impossible because if a subsequence is unbounded, the sequence itself is unbounded and therefore does not converge, meaning it is not Cauchy.

(c) This is impossible because if the sequence is monotone and diverges, it is unbounded, and so the subsequence is also monotone. If it were Cauchy, it would converge and be bounded, in which case there is eventually a term of the sequence greater in absolute value than the proposed upper bound, meaning there would also be a term of the subsequence greater than the proposed upper bound, a contradiction.

(d) Take (a_n) to be the sequence that is 1 at all odd coordinates and n at all even coordinates. Then the subsequence consisting of odd terms converges to 1 and therefore is Cauchy, however, the sequence itself is unbounded because of the even terms.

Exercise 2.6.4. (a) We show that this is a Cauchy sequence. Let $\varepsilon > 0$. Since (a_n) and (b_n) are Cauchy sequences, there exist N and M such that $n, m > N$ implies $|a_n - a_m| < \varepsilon/2$

and $n, m > M$ implies $|b_n - b_m| < \varepsilon/2$. Set $K = \max\{N, M\}$ and suppose $n, m > K$. Then we have that

$$\begin{aligned}|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \leq |a_n - b_n - (a_m - b_m)| \\ &= |(a_n - a_m) + (b_m - b_n)| \leq |a_n - a_m| + |b_n - b_m| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore (c_n) is a Cauchy sequence.

(b) This is not necessarily a Cauchy sequence. Take $a_n = 1$ for all n. Then $c_n = (-1)^n$ which does not converge and is therefore not Cauchy.

(c) This is not necessarily a Cauchy sequence. Define a_n as

$$a_n = \begin{cases} 1 + 1/n & n \text{ odd} \\ 1 - 1/n & n \text{ even} \end{cases}$$

Then (a_n) converges to 1 and is therefore Cauchy, but the odd terms of a_n are strictly between 1 and 2 while the even terms are strictly between 0 and 1. This means that the sequence (c_n) is

$$c_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

which does not converge and therefore is not Cauchy.

2.7 Properties of Infinite Series

Exercise 2.7.9. (a) Since $r < r' < 1$,

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r$$

implies there exists an $N \in \mathbb{N}$ corresponding to the choice $\varepsilon = r' - r > 0$. Then if $n \geq N$, we have that

$$r - \varepsilon < \left| \frac{a_{n+1}}{a_n} \right| < r + \varepsilon = r + (r' - r) = r',$$

meaning $|a_{n+1}| < |a_n| r'$, as required.

(b) Since $r' < 1$, we have that $\sum (r')^n$ must converge because it is a geometric series with common ratio less than 1. This means multiplying it by the constant $|a_N|$ doesn't change the convergence, so the series $\sum |a_N|(r')^n$ also converges.

(c) Using induction on n, we can show that $|a_n| \leq |a_N|(r')^{n-N}$ for all $n \geq N$. Since the series $\sum_{n=0}^{\infty} |a_N|(r')^n$, so does $\sum_{n=N}^{\infty} |a_N|(r')^{n-N}$. Defining $b_n = |a_n|$ for $n < N$ and $b_n = |a_N|(r')^{n-N}$ for $n \geq N$ then gives us a sequence b_n such that $|a_n| \leq b_n$ for all $n \in \mathbb{N}$. Then because

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_N|(r')^{n-N}$$

converges, we have also that $\sum_{i=1}^{\infty} |a_n|$ converges as well by the Comparison Test. Therefore a_n converges because it converges absolutely.

3 Basic Topology of R

3.2 Open and Closed Sets

Exercise 3.2.1. (a) Finiteness is used when taking the minimum of the finite set $\{\varepsilon_1, \ldots, \varepsilon_N\}$. If this set were not finite, it could fail to have a minimum.

(b) Take the collection

$$\mathscr{O} = \{(-1/n, 1/n)\}_{n \in \mathbb{N}}.$$

Then the collection is countable and its intersection is $\bigcap_{O \in \mathscr{O}} O = \{0\}$, which is nonempty, not all of $\mathbb{R}$, and is closed since $\mathbb{R}$ is T_1 (alternatively, because $\mathbb{R} - \{0\}$ is the union of the open sets $(-\infty, 0)$ and $(0, \infty)$, which makes $\mathbb{R} - \{0\}$ open and the singleton therefore closed).

Exercise 3.2.3. (a) We have that $\sqrt{2}$ is a limit point that is not in $\mathbb{Q}$ so $\mathbb{Q}$ is not closed, and also any ε interval will contain infinitely many irrationals, so also $\mathbb{Q}$ is not open.

(b) $\mathbb{N}$ is not open for the same reason, but it is closed because it has no limit points and therefore vacuously contains all of them.

(c) This is the union of two open intervals and is therefore open, but 0 is a limit point not contained in the set, so it isn't closed.

(d) and (e) These are both not open for the same reasons as in (a) and (b), and neither is closed because 0 is a limit point of both sets and isn't in either.

Exercise 3.2.6. (a) This is false. Consider the open set $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ containing all real numbers except $\sqrt{2}$. It is open and contains all rationals but is not all reals.

(b) Note that the sets $A_n = \{n, n+1, n+2, \ldots\}$ have no limit points and are therefore vacuously closed. However, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(c) This is true because every nonempty open set contains a point and therefore an ε interval around it, which then must contain a rational by the density of $\mathbb{Q}$ in $\mathbb{R}$.

(d) This is false. Consider the sequence $a_n = \sqrt{2} - \sqrt{2}/2n$. Each term is distinct, so there are infinitely many elements in its image, and every term is rational. The closure of the image of the sequence is the image together with the element $\sqrt{2}$, so all elements of this closed set are irrational. The set is also bounded, below by 0 and above by 2.

(e) This is true. Since C_n is the union of finitely many closed sets, it is also closed, and therefore C is the intersection of closed sets and is closed.

Exercise 3.2.10. The interval $[0, 1]$ is compact, so all infinite subsets have limit points, meaning (i) is impossible. For (ii), take $\mathbb{Q} \cap [0, 1]$ for a countable set with no isolated points—every point here is a limit point, because the rationals are dense in the reals and so every open set containing $q \in \mathbb{Q} \cap [0, 1]$ has to intersect at another point as well. Case (iii) is impossible because if uncountably many points of a set were isolated points, there would need to be disjoint ε intervals around them, something we know is impossible for $\mathbb{R}$ and its subsets.

3.3 Compact Sets

Exercise 3.3.2. (a) $\mathbb{N}$ is not compact because it is unbounded. Every subsequence of the sequence $a_n = n$ is also unbounded, so none converge.

(b) Because $1/\sqrt{2} \in [0, 1] \subset \mathbb{R}$, there exists a sequence of rational numbers converging to it. Because of the convergence, there exists an $N \in \mathbb{N}$ such that each term of the sequence for $n > N$ is within the interval $[0, 1]$. Define the sequence a_n to be the sequence of all of these terms after the Nth term. Then $a_n \to \sqrt{2}$ and so every subsequence must also converge to $\sqrt{2} \notin \mathbb{Q} \cap [0, 1]$.

(c) The Cantor set is compact because it is closed and bounded.

(d) This set is already indexed by the natural numbers and the resulting sequence converges to $\pi^2/6$ which is not an element of the set, so this set is not compact.

(e) This is compact because it is closed (there is one limit point, 1, which is contained in the set) and bounded, e.g. below by 0 and above by 1.