Topology Through Inquiry Self-Study

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1 Cardinality: To Infinity and Beyond

1.1 Sets and Functions

Theorem 1.2 (DeMorgan's Laws). Let X be a set and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then

$$X - \left(\bigcup_{k=1}^{N} A_k \right) = \bigcap_{k=1}^{N} (X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^{N} A_k \right) = \bigcup_{k=1}^{N} (X - A_k).$$

Proof. Let $a \in X - \left(\bigcup_{k=1}^N A_k \right)$ be arbitrary. Then $a \notin \bigcup_{k=1}^N A_k$, so for all k, $a \notin A_k$, which means that $a \in X - A_k$ for all k. Therefore $x \in \bigcap_{k=1}^N (X - A_k)$ and so $X - \left(\bigcup_{k=1}^N A_k \right) \subset \bigcap_{k=1}^N (X - A_k)$. Now let $a \in \bigcap_{k=1}^N (X - A_k)$ be arbitrary. Then we have that $a \in X - A_k$ for all k, which means that $a \notin A_k$ for all k. Therefore $a \notin \bigcup_{k=1}^N A_k$, so we have that $a \in X - \left(\bigcup_{k=1}^N A_k \right)$ and therefore $\bigcap_{k=1}^N (X - A_k) \subset X - \left(\bigcup_{k=1}^N A_k \right)$. Therefore

$$X - \left(\bigcup_{k=1}^{N} A_k \right) = \bigcap_{k=1}^{N} (X - A_k).$$

Let $a \in X - \left(\bigcap_{k=1}^N \right)$ be arbitrary. Then $a \notin \bigcap_{k=1} A_k$, so there exists some j such that $a \notin A_j$, which means that $a \in X - A_j$. Therefore $a \in \bigcup_{k=1}^N (X - A_k)$ and so $X - \left(\bigcap_{k=1}^N A_k \right) \subset$

$\bigcup_{k=1}^{N}(X-A_k)$. Now let $a \in \bigcup_{k=1}^{N}(X-A_k)$. Then there exists some j such that $a \in X-A_j$, so $a \notin A_j$ and therefore $a \notin \bigcap_{k=1}^{N}$. This means that $a \in X-\left(\bigcap_{k=1}^{N} A_k\right)$ and so $\bigcup_{k=1}^{N}(X-A_k) \subset X-\left(\bigcap_{k=1}^{N} A_k\right)$. Therefore we have that

$$X-\left(\bigcap_{k=1}^{N} A_k\right)=\bigcup_{k=1}^{N}(X-A_k).$$

$\square$

Exercise 1.3. For a function $f : X \to Y$ and sets $A, B \subset Y$, we have that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proof. See MATH 200 final exam review sheet notes in the graph paper notebook. $\square$

Exercise 1.4. If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y)$ contains more than one point. Then there exist $x_1, x_2 \in X$ such that $x_1, x_2 \in f^{-1}(y)$ and $x_1 \neq x_2$. By the definition of $f^{-1}(y)$, we have that $f(x_1), f(x_2) \in \{y\}$, so $f(x_1) = y = f(x_2)$ and f is therefore not injective since $x_1 \neq x_2$. We have shown the contrapositive of the claim. $\square$

Exercise 1.5. If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y) = \emptyset$. Then for all $x \in X$, $f(x) \notin f^{-1}(y)$, and by the definition of $f^{-1}(y)$, this means that for all $x \in X$, $f(x) \neq y$, so f is not surjective. We have shown the contrapositive of the claim. $\square$

1.2 Cardinality and Countable Sets

Theorem 1.8. Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Proof. Let $S \subset \mathbb{N}$ be an arbitrary subset of $\mathbb{N}$. If S is finite, then we are done. If S is not finite, then it is infinite. Let $s_0 = \min S$, let $s_1 = \min(S - \{s_0\})$, and let $s_i = \min(S - \{s_0, \ldots, s_{i-1}\})$. Define the function $f : \mathbb{N} \to S$ by the following: $f(n) = s_n$. Suppose $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. Without loss of generality, assume that $n_1 < n_2$. Then $f(n_2) = \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\})$. Since $f(n_1) = s_{n_1} \notin S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}$, we have that $f(n_1) \neq \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}) = f(n_2)$, which means that f is injective. Let $s \in S \subset \mathbb{N}$ be arbitrary. Then set $j = |\{r \in S : r < s\}| + 1 \in \mathbb{N}$. Then we have that

$$f(j) = s_j = \min(S - \{s_0, \ldots, s_{|\{r \in S : r < s\}|}\}) = s$$

where the last equality follows from the fact that s must be the smallest element of the subset of S from which all elements smaller than s have been removed. Therefore f is surjective, and since it is also injective, f is a bijection, meaning that the cardinality of S is the same as the cardinality of $\mathbb{N}$. □

Theorem 1.9. Every infinite set has a countable subset.

Proof. I think I need the axiom of choice here? I will return in the future! □

Theorem 1.10. A set is infinite if and only if there is an injection from the set into a proper subset of itself.

Proof. I think I also need the axiom of choice here... □

Theorem 1.11. The union of two countable sets is countable.

Proof. Let A and B be countable sets. There are two cases to consider. In the first case, the intersection of A and B is finite, and so there exists a bijection $h : A \cap B \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ where n is the size of the set $A \cap B$ If $A \cap B$ is empty, we instead use $n = 0$. Since $A \cap B$ is finite, $A - (A \cap B)$ is infinite, so there exists a bijection $f : A - (A \cap B) \to \mathbb{E}$, where $\mathbb{E}$ is the countable set containing all positive even integers greater than n. Similarly, there exists a bijection $g : B - (A \cap B) \to \mathbb{O}$, where $\mathbb{O}$ is the countable set containing all positive odd integers greater than n. Then we have that the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the second case, $A \cap B$ is infinite and there are three subcases. In the first subcase, one of $A - (A \cap B)$ or $B - (A \cap B)$ (assume without loss of generality that this is $A - (A \cap B)$) is finite. In this case, we use the same construction as earlier, since now $h : A - (A \cap B) \to \{1, \ldots, n\}$

is a bijection for some $n \in \mathbb{N}$, $f : A \cap B \to \mathbb{E}$ is a bijection, and $g : B - (A \cap B) \to \mathbb{O}$ is a bijection. Then the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A \cap B \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A - (A \cap B) \end{cases}.$$

In the second subcase, $A - (A \cap B)$ and $B - (A \cap B)$ are finite. Then there are bijections $f : A - (A \cap B) \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $g : B - (A \cap B) \to \{n + 1, \ldots n + m\}$ for some $m \in \mathbb{N}$. Since $A \cap B$ is countably infinite, there is a bijection $h : A \cap B \to \{n + m + 1, n + m + 2, \ldots\}$, the set of positive integers greater than $n + m$. Therefore $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the third subcase, all three sets are countably infinite. In this case, there is a bijection $f : A - (A \cap B) \to \{n \in \mathbb{N} : n = 3k, k \in \mathbb{Z}\}$, a bijection $g : B - (A \cap B) \to \{n \in \mathbb{N} : n = 3k + 1, k \in \mathbb{Z}\}$, and a bijection $h : A \cap B \to \{n \in \mathbb{N} : n = 3k + 2, k \in \mathbb{Z}\}$. Then we have that $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In all cases, we have that there exists a bijection $\varphi : A \cup B \to \mathbb{N}$, so $A \cup B$ is countable. $\square$

Theorem 1.12. The union of countably many countable sets is countable.

Proof. Let $\{B_i\}_{i \in \mathbb{N}}$ be a collection of countable sets. Then for $i \in \mathbb{N}$, define A_i recursively as $A_1 = B_1$ and $A_i = B_i - \bigcup_{k<i} B_k$ so that the union $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} B_i$ but the A_i are all disjoint. Now since the set of primes is countable, there exists a bijection $f : \mathbb{N} \to \{\text{primes}\}$. Define the map $g : \bigcup_{i \in \mathbb{N}} A_i \to \mathbb{N}$ by $g(a_{ij}) = f(i)^j$ where $a_{ij} \in A_i$ is the jth element of the set A_i. This map is well-defined since the sets A_i are disjoint. Suppose $g(a_{ij}) = g(a_{mn})$

for some $a_{ij}, a_{mn} \in \bigcup_{i\in\mathbb{N}} A_i$. Then $f(i)^j = f(m)^n$, so they must also divide each other, and because the $f(i)$ and $f(m)$ are prime, they must be equal and have the same powers, so $j = n$. Then $f(i) = f(m)$ implies that $i = m$ since f is a bijection. Therefore $a_{ij} = a_{mn}$, so g is an injection, meaning the cardinality of $\bigcup_{i\in\mathbb{N}} A_i$ is less than or equal to the cardinality of $\mathbb{N}$, meaning the countable union of countable sets is countable. □

Theorem 1.14. The set of all finite subsets of a countable set is countable.

Proof. Let $A = \{a_i \mid i \in \mathbb{N}\}$ be a countable set. Then let N be a finite subset of $\mathbb{N}$ and define $C_N = \{a_i \mid i \in N\}$. Since N is finite, is has a maximum. Then for all $k \in \mathbb{N}$, define A_k to be the set $A_k = \{C_M \mid \max M = k\}$. Then $A_k \subset 2^{\{a_1,\dots a_k\}}$, the power set of a finite set, which is finite. Therefore for all $k \in \mathbb{N}$, A_k is finite, and if C is the set of all finite subset of A, then $C = \bigcup_{k\in\mathbb{N}} A_k$. Because the A_k are finite, they are countable, so by Theorem 1.12, C is countable. □

1.3 Uncountable Sets and Power Sets

Exercise 1.17. If $A = \{a, b, c\}$, then $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 1.18. If the set A is finite, then its power set has cardinality $2^{|A|}$ ($|2^A| = 2^{|A|}$).

Proof. If a set A is finite, then it has cardinality $|A| = n \in \mathbb{N}$. We argue by induction on n that its power set 2^A has cardinality 2^n. For the base case $n = 1$, let A_1 be a set with one element. Then $A_1 = \{a\}$ and $2^{A_1} = \{\emptyset, a\}$, which has two elements, so indeed $|2^{A_1}| = 2^{|A_1|}$. Now for the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that all sets with cardinality k have power sets with cardinality 2^k. Then let A_{k+1} be an arbitrary set with cardinality $k+1$ so that $A_{k+1} = \{a_1, \dots, a_{k+1}\}$. Note that the set $A_k = \{a_1, \dots, a_k\}$ has cardinality k, and therefore has 2^k subsets. For each subset of A_k, call them A_{k_i} for $1 \leq i \leq 2^k$, the sets A_{k_i} and $A_{k_i} \cup \{a_{k+1}\}$ are subsets of A_{k+1}. Therefore there are $2 \cdot 2^k = 2^{k+1}$ subsets of A_{k+1}, and so $|2^{A_{k+1}}| = 2^{|A_{k+1}|}$. Therefore by induction, all finite sets A have $|2^A| = 2^{|A|}$. □

Theorem 1.19. For any set A, there is an injection from A to 2^A.

Proof. Define $f : A \to 2^A$ as $f(a) = \{a\} \in 2^A$ since $\{a\} \subset A$. Let $x, y \in A$ such that $f(x) = f(y)$. Then we have that $\{x\} = \{y\}$, so $x = y$ and f is an injection, as required. □

Theorem 1.20. If P is the set of all functions from a set A to the set $\{0, 1\}$, then $|P| = |2^A|$.

Proof. Define $\varphi : P \to 2^A$ as $\varphi(h) = h^{-1}(1)$. Let $f, g \in P$ be arbitrary such that $f \neq g$. Then there exists an $a \in A$ such that $f(a) \neq g(a)$. Without loss of generality, assume that $f(a) = 0$ and $g(a) = 1$. Then we have that $a \notin f^{-1}(1) = \varphi(f)$ and $a \in g^{-1}(1) = \varphi(g)$, so $\varphi(f) \neq \varphi(g)$ and therefore φ is injective. Now let $Y \in 2^A$ be arbitrary. Then define the function $f : A \to \{0, 1\}$ such that for $a \in A$,

$$f(a) = \begin{cases} 0 & a \notin Y \\ 1 & a \in Y \end{cases}.$$

Then since $Y \in 2^A$ means that $Y \subset A$, we have that

$$\varphi(f) = f^{-1}(1) = \{a \in A : f(a) = 1\} = \{a \in A : a \in Y\} = Y,$$

so φ is also surjective, making it a bijection. Since there is a bijection from P to 2^A, we have that $|P| = |2^A|$, as required. $\square$

Theorem 1.22. There is no surjection between a set A and its power set 2^A ($|A| \neq |2^A|$)

Proof. Let A be an arbitrary set and suppose for contradiction that $f : A \to 2^A$ is a surjection. Consider the set $X = \{a \in A : a \notin f(a)\}$. Since $X \subset A$, $X \in 2^A$, and since f is surjective, there exists an $a_0 \in A$ such that $f(a_0) = X$. There are two cases, both of which lead to a contradiction. In the first case, $a_0 \in X$, and so we have that $a_0 \notin f(a_0) = X$, a contradiction. In the second case, $a_0 \notin X$, so $a_0 \in f(a_0) = X$, also a contradiction. Since both cases lead to a contradiction, we see that the assumption that f is surjective was false, so there can be no surjection between A and 2^A, which means $|A| \neq |2^A|$. $\square$

1.4 The Schroeder-Bernstein Theorem

Theorem 1.28. The unit square and the unit interval have the same cardinality.

Proof. Let $a \in [0, 1]$ be arbitrary. Then it can be represented as

$$a = \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers with $a_n \in \{0, 1, \dots, 9\}$ that does not end all in 9s (that is, if $a_i = 9$, then there exists $j > i$ such that $a_j \neq 9$). Now we claim that the function

$f : [0,1] \to [0,1] \times [0,1]$ is surjective, where

$$f\left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n\right) = \left(\sum_{n=1}^{\infty} a_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} a_{2n} \left(\frac{1}{10}\right)^n\right).$$

To show surjectivity, let $y \in [0,1] \times [0,1]$ be arbitrary. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ such that

$$y = \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right).$$

Let $x \in [0,1]$ be the point

$$x = \sum_{n=1}^{\infty} \left[a_n \left(\frac{1}{10}\right)^{2n-1} + b_n \left(\frac{1}{10}\right)^{2n}\right] = 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots = 0.c_1 c_2 c_3 c_4 c_5 c_6 \ldots.$$

Then

$$\begin{aligned} f(x) = f(0.c_1 c_2 c_3 \ldots) &= \left(\sum_{n=1}^{\infty} c_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} c_{2n} \left(\frac{1}{10}\right)^n\right) \\ &= \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right) = y. \end{aligned}$$

Therefore f is surjective. Now consider $g : [0,1] \times [0,1] \to [0,1]$ where $g(a,b) = a$. Let $y \in [0,1]$ be arbitrary and set $x = (y,0) \in [0,1] \times [0,1]$. Then $g(x) = g(y,0) = y$, and so g is surjective. Since there is a surjection from $[0,1]$ to $[0,1] \times [0,1]$ and a surjection from $[0,1] \times [0,1]$ to $[0,1]$, by the surjective version of the Schroeder Bernstein Theorem (1.26), we have that $|[0,1] \times [0,1]| = |[0,1]|$ as required. $\square$

1.5 The Axiom of Choice

Exercise 1.32. Let X be a set and let P be the poset of all subsets of X partially ordered by inclusion. Let $p \in P$ be an element of the poset with $X \leq p$. Then we have that $X \subset p$. We also have that $p \subset X$ since $p \in P$ and P is the poset of all subsets of X. Therefore $p = X$, and so X is by definition a maximal element of P. Suppose there exists a set $Y \in P$ such that Y is also a maximal element of P. Then we have that $X \subset Y$ since Y is a maximal element, and also that $Y \subset X$ since $Y \in P$. Therefore $X = Y$, which shows that X is the unique maximal element of P. Now let $p \in P$ be an elemnet of the poset with $p \leq \emptyset$. Then we have that $p \subset \emptyset$, but also that $\emptyset \subset p$ since $\emptyset \subset x$ for all sets x. Therefore $p = \emptyset$, and so by definition we have that $\emptyset$ is a least element of the poset P. Now let $Z \in P$ be a least element. Then $Z \subset \emptyset$ since it is a least element, but as before, $\emptyset \subset Z$, which means $Z = \emptyset$ and so $\emptyset$ is the unique least element of the poset P.

Exercise 1.33. Let P be the poset ordered by cardinality with $P = \{\{0\}, \{1\}, \{0,1\}, \{1,2\}\}$. Then $\{0\}$ and $\{1\}$ are least elements and $\{0,1\}$ and $\{1,2\}$ are maximal elements.

Exercise 1.34. Consider $\mathbb{R}$ with the $\leq$ relation. The relation is reflexive, transitive, and antisymmetric, so it is a partial order on $\mathbb{R}$. We also have that for any two elements $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$, so they are comparable. This means that $\mathbb{R}$ is totally ordered with the relation $\leq$. However, $\mathbb{R}$ is not well-ordered, because there exist nonempty subsets of $\mathbb{R}$ that do not have least elements, for example, $(0,1) \subset \mathbb{R}$.

2 Topological Spaces: Fundamentals

2.2 Open Sets and the Definition of a Topological Space

Theorem 2.1. If $\{U_i\}_{i=1}^n$ is a finite collection of open sets in a topological space $(X, \mathcal{T})$, then $\bigcap_{i=1}^n U_i$ is open.

Proof. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. We argue by induction on n that $\bigcap_{i=1}^n U_i$ is open. For the base case $n = 1$, we have that U_1 is open. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $\bigcap_{i=1}^k U_i$ is open. Then since U_{k+1} is open, we have that $\bigcap_{i=1}^{k+1} U_i = \left(\bigcap_{i=1}^k U_i\right) \cap U_{k+1}$ is the intersection of two open sets and is therefore open. By induction, $\bigcap_{i=1}^n U_i$ is open for all $n \in \mathbb{N}$, in otherwords, the intersection of finitely many open sets is open. $\square$

Exercise 2.2. The above theorem does not show that the intersection of infinitely many open sets is open since the intersection of infinitely many open sets cannot be represented as $\bigcap_{i=1}^n U_i$ for any $n \in \mathbb{N}$.

Theorem 2.3. A set U is open in a topological space $(X, \mathcal{T})$ iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.

Proof. Let $(X, \mathcal{T})$ be a topological space with $U \in \mathcal{T}$. Let $x \in U$ be arbitrary and set $U_x = U$. Then U_x is open and $x \in U_x \subset U$, as required. Now let U be a set such that for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Then we have that $\bigcup_{x \in U} U_x$ is open, since the union of a collection of open sets is open. We also have that $y \in U$ implies that $y \in \bigcup_{x \in U} U_x$ since $y \in U_y$, so $U \subset \bigcup_{x \in U} U_x$. If $y \in \bigcup_{x \in U} U_x$, then $y \in U_z \subset U$ for some $z \in U$, so $\bigcup_{x \in U} U_x \subset U$, which means that the open set $\bigcup_{x \in U} U_x$ is the set U. Therefore U is open iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. □

Exercise 2.4. First note that, vacuously, $\emptyset \in \mathcal{T}_{\text{std}}$, so the first property is satisfied. Next, consider the set $\mathbb{R}^n \subset \mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be an arbitrary point, and since $B(p, 1) \subset \mathbb{R}^n$ and p was arbitrary, we have that $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$, so the second property is satisfied. For the third property, let $U, V \in \mathcal{T}_{\text{std}}$ be arbitrary open sets in $\mathbb{R}^n$. Let p be an arbitrary point in $U \cap V$. Then because $p \in U$, there exists an ε_1 such that $B(p, \varepsilon_1) \subset U$, and because $p \in V$, there exists an ε_2 such that $B(p, \varepsilon_2) \subset V$. Set $\varepsilon_p = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B(p, \varepsilon_p) \subset B(p, \varepsilon_1) \subset U$ and $B(p, \varepsilon_p) \subset B(p, \varepsilon_2) \subset V$, so we have that $B(p, \varepsilon_p) \subset U \cap V$. Therefore $U \cap V \in \mathcal{T}_{\text{std}}$ and the third property is satisfied. For the fourth property, let $\{U_i\}_{i \in \lambda}$ be a collection of sets $U_i \in \mathcal{T}_{\text{std}}$. Then let $p \in \bigcup_{i \in \lambda} U_i$ be arbitrary. Since p is in the union of all the U_i, we have that $p \in U_j$ for some $j \in \lambda$. Then since $U_j \in \mathcal{T}_{\text{std}}$, there exists an ε_p such that $B(p, \varepsilon_p) \subset U_j \subset \bigcup_{i \in \lambda} U_i$, and therefore $\bigcup_{i \in \lambda} U_i \in \mathcal{T}_{\text{std}}$, so the fourth property is satisfied and we have that $\mathcal{T}_{\text{std}}$ is indeed a topology on $\mathbb{R}^n$.

Exercise 2.6. The unit interval $(0, 1) \subset \mathbb{R}$ is open in the standard topology on $\mathbb{R}$, open in the discrete topology, not open in the indiscrete topology, not open in the finite complement topology, and not open in the countable complement topology.

Exercise 2.7. In the topological space $(\mathbb{R}, \mathcal{T}_{\text{std}})$, the interval $(0, 1)$ is open and for all $n \geq 0$, the set $U_n \subset (0, 1)$ where

$$U_n = \left(\frac{2^n - 1}{2^{n+1}}, \frac{2^n + 1}{2^{n+1}} \right).$$

Then $\frac{1}{2} \in U_n$ for all n and therefore $\frac{1}{2} \in \bigcap_{n=0}^{\infty} U_n$. Let $x \in \mathbb{R}$ such that $x \neq \frac{1}{2}$. Then there exists some $m \geq 0$ such that $|x - \frac{1}{2}| > \frac{1}{2^{m+1}}$, which means that

$$x \notin \left(\frac{1}{2} - \frac{1}{2^{m+1}}, \frac{1}{2} + \frac{1}{2^{m+1}} \right) = \left(\frac{2^m - 1}{2^{m+1}}, \frac{2^m + 1}{2^{m+1}} \right) = U_m.$$

Since there exists an $m \geq 0$ such that $x \notin U_m$, we have that $x \notin \bigcap_{n=0}^{\infty} U_n$, and since x was arbitrary, we have that $\bigcap_{n=0}^{\infty} U_n = \{\frac{1}{2}\} \notin \mathcal{T}_{\text{std}}$. Therefore the infinite intersection of open sets is not necessarily open.

2.3 Limit Points and Closed Sets

Exercise 2.8. In the indiscrete topology on $\mathbb{R}$, the point 0 is in $\mathbb{R}$ but not $\emptyset$, so since $(\mathbb{R} - \{0\}) \cap (1, 2) = (1, 2) \neq \emptyset$, we have that $(U - \{0\}) \cap (1, 2) \neq \emptyset$ for all open sets U containing 0. Therefore 0 is a limit point of $(1, 2)$ in the indiscrete topology. In the finite complement topology, let $U \subset \mathbb{R}$ be an open set containing 0. Then suppose for contradiction that $(U - \{0\}) \cap (1, 2) = \emptyset$. Then for all $p \in (1, 2) \subset \mathbb{R}$, we have that $p \notin U - \{0\}$, which means that $p \in \mathbb{R} - (U - \{0\}) = (\mathbb{R} - U) \cup \{0\}$. Therefore $(1, 2) \subset (\mathbb{R} - U) \cup \{0\}$, but this is a contradiction since U is open and therefore we have that an infinite set is a subset of a finite set. Hence it must be the case that $(U - \{0\} \cap (1, 2)) \neq \emptyset$ for all U containing 0, and therefore 0 is a limit point of $(1, 2)$. In the standard topology and in the discrete topology, the set $(-1, 1)$ is an open set containing 0, but $(-1, 1) \cap (1, 2) = \emptyset$, so 0 is not a limit point of $(1, 2)$.

Theorem 2.9. Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A iff there exists a neighborhood U of p such that $A \cap U = \emptyset$.

Proof. Suppose $p \notin A$ is not a limit point of A. Then there exists an open set U containing p (a neighborhood U of p) such that $(U - \{p\}) \cap A = \emptyset$. Since $p \notin A$, we have that $p \notin A \cap U$, and therefore $A \cap U = \emptyset$, as required. Now suppose there exists a neighborhood U of p such that $A \cap U = \emptyset$. Then since $p \notin A$, we have that $(U - \{p\}) \cap A = \emptyset$ as well, and therefore p is not a limit point of A. $\square$

Exercise 2.10. If p is an isolated point of A in a topological space $(X, \mathcal{T})$, then by the definition of an isolated point, we have that $p \in A$ but p is not a limit point of A. Therefore there exists an open set U such that $(U - \{p\}) \cap A = \emptyset$, and since $p \in A, U$, we have that

$$U \cap A = ((U - \{p\}) \cap A) \cup \{p\} \cap A = \emptyset \cup \{p\} = \{p\}.$$

Therefore if p is an isolated point of A, there exists an open set U such that $A \cap U = \{p\}$.

Exercise 2.11. Let $X = \{a, b, c, d\}$ be a set, let $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$ be a topology on X, and let $A = \{b, c\}$ be a set.

(1) $c \in A$ is a limit point of A since the only open set containing c is X, and we have that $(X - \{c\}) \cap A = \{b\} \neq \emptyset$.

(2) $d \notin A$ is a limit point of A also because X is the only open set containing d and $(X - \{d\}) \cap A = \{b, c\} \neq \emptyset$.

(3) $b \in A$ is an isolated point of A because it is in A but is not a limit point since $\{a, b\}$ is open but $(\{a, b\} - \{b\}) \cap A = \emptyset$.

(4) $a \notin A$ is not a limit point of A since $\{a\}$ is an open set but $(\{a\} - \{a\}) \cap A = \emptyset$.

Exercise 2.12. Let X be a set and let $\mathcal{T}$ be a topology on X. If the set X has a limit point p, then $p \in X$, so $\overline{X} = X$ and X is closed. If Now let $p \in X$ and let U be an open set containing p. Then $(U - \{p\}) \cap \emptyset = \emptyset$, so there are no limit points of the empty set, and therefore, vacuously, $\overline{\emptyset} = \emptyset$ and the empty set is closed.

(1) In the discrete topology, all sets are closed. Let A be a nonempty proper subset of X and let $p \in X$ be arbitrary. Then there exists a point $q \in X$ with $q \notin A$, and since all sets are open in the discrete topology, then set $\{p, q\}$ is a neighborhood of p that satisfies $(\{p, q\} - \{p\}) \cap A = \emptyset$. Thus we have shown that there are no limit points of A, and so vacuously, $\overline{A} = A$.

(2) In the indiscrete topology, only X and $\emptyset$ are closed. Let A be a nonempty propoer subset of X and let $p \in X$ such that $p \notin A$. Then p is a limit point of A since X is the only open set containing p and it satisfies $(X - \{p\}) \cap A \neq \emptyset$. However, $p \notin A$, so $\overline{A} \neq A$.

Theorem 2.13. For any topological space $(X, \mathcal{T})$, and $A \subset X$, the set $\overline{A}$ is closed, that is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

Proof. Let A be a set in a topological space $(X, \mathcal{T})$. Since the closure of a set contains all the points in the set, we have that $\overline{A} \subset \overline{\overline{A}}$. Now let $x \in \overline{\overline{A}}$ and let U be an arbitrary neighborhood of x. Either $x \in \overline{A}$ (in which case we're done), or x is a limit point of $\overline{A}$, in which case we have that $(U - \{x\}) \cap \overline{A} \neq \emptyset$. Since $(U - \{x\}) \cap \overline{A} \subset U \cap \overline{A}$, so $U \cap \overline{A}$ is nonempty in either case. Therefore there exists some $y \in U \cap \overline{A}$, which means we have a neighborhood U of y such that $y \in \overline{A}$. As before, either $y \in A$ or y is a limit point of A, in which case we have that $\emptyset \neq (U - \{y\}) \cap A \subset U \cap A$. Therefore for an arbitrary neighborhood U of $x \in \overline{\overline{A}}$, we have shown that $U \cap A$ is nonempty. Either $x \in A \subset \overline{A}$, or $x \notin A$ and therefore $x \notin U \cap A \neq \emptyset$. Since this intersection is nonempty, we have also that $(U - \{x\}) \cap A \neq 0$, but U was an arbitrary neighborhood of x, so we have shown that $(U - \{x\}) \cap A$ is nonempty for all neighborhoods U of x. Therefore x is a limit point of A, and so $x \in \overline{A}$. In both cases, $x \in \overline{A}$ and so $\overline{\overline{A}} \subset \overline{A}$, and since $\overline{A} \subset \overline{\overline{A}}$ also, we have that $\overline{\overline{A}} = \overline{A}$, as required. □

Theorem 2.14. For any topological space $(X, \mathcal{T})$, a subset $A \subset X$ is closed if and only if $X - A$ is open.

Proof. Suppose A is closed and let $x \in X - A$. Then since A is closed, it contains all its limit points and therefore x is not a limit point. This means there exists an open set U such that $U - \{x\} \cap A = \emptyset$. Let $y \in U - \{x\}$. Then $y \notin A$, so $y \in X - A$, which means $U - \{x\} \subset X$. Since x was an arbitrary element of $X - A$, by Theorem 2.3, we have that $X - A$ is open. Now suppose that $X - A$ is open. $A \subset \overline{A}$, and we will show that $\overline{A} \subset A$. Suppose for contradiction that $\overline{A} \not\subset A$. Then there exists an $a \in \overline{A} - A$, which means that a is a limit point of A, so all open intervals U containing a satisfy $U \cap A \neq \emptyset$ (by Theorem 2.9). In particular, since $X - A$ is open, we have that $(X - A) \cap A \neq \emptyset$, which is a contradiction. Therefore $\overline{A} = A$ and A is closed. □

Corollary 2.14. If A is an open set, then $X - (X - A)$ is open, which means $X - A$ is closed.

Theorem 2.15. For any topological space $(X, \mathcal{T})$ with an open set $U \in \mathcal{T}$ and a closed set $A \in \mathcal{T}$, $U - A$ is open and $A - U$ is closed.

Proof. Since A is closed, $X - A$ is open, and so $U \cap (X - A)$ is also open since the intersection of two open sets is open. Then we have that $X - (U \cap (X - A))$ is closed by the corollary to Theorem 2.14. $X - (U \cap (X - A)) = X - (U - A)$ is closed, so $U - A$ is open, as claimed. The union of open sets is open, so we also have that $U \cup (X - A)$ is open. $U \cup (X - A) = X - (A - U)$ is open, so $A - U$ is closed, as claimed. □

Theorem 2.16. Let $(X, \mathcal{T})$ be a topological space. Then:

(i) $\emptyset$ is closed.

(ii) X is closed.

(iii) The union of finitely many sets is closed.

(iv) Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

Proof. For (i) and (ii), see exercise 2.12. For (iii), let $\{A_i\}$, $1 \leq i \leq n$ for some $n \in \mathbb{N}$. Then for each A_i, $X - A_i$ is open, and we have that the intersection of finitely many open sets is open, so $\bigcap_{i=1}^{n}(X - A_i)$ is open. By the corollary to Theorem 2.14 and DeMorgan's Laws,

$$X - \left(\bigcap_{i=1}^{n} (X - A_i) \right) = X - \left(X - \bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} A_i$$

is closed. For (iv), let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then for each A_α, $X - A_\alpha$ is open, and we have that the union of a collection of open sets is open, so

$\bigcup_{\alpha\in\lambda}(X - A_\alpha)$ is open. Again by DeMorgan's Laws,

$$X - \left(\bigcup_{\alpha\in\lambda}(X - A_\alpha)\right) = X - \left(X - \bigcap_{\alpha\in\lambda} A_\alpha\right) = \bigcap_{\alpha\in\lambda} A_\alpha$$

is closed. $\square$

Exercise 2.19. (1) In $\mathbb{Z}$ with the finite complement topology, the set $\{0,1,2\}$ is not open, since $\mathbb{Z} - \{0,1,2\}$ is infinite. $\mathbb{Z} - \{0,1,2\}$, however, is open since $\{0,1,2\}$ is finite, and therefore $\mathbb{Z} - (\mathbb{Z} - \{0,1,2\}) = \{0,1,2\}$ is closed. The set of prime numbers has an infinite complement, so it is not open, but there are infinitely many prime numbers so it is also not closed. The set $\{n : |n| > 10\}$ has a finite complement, so it is open, but the set itself is infinite and therefore is not closed.

(2) In $\mathbb{R}$ with the standard topology, the set $(0,1)$ is open, and its limit points are 0 and 1, neither of which are in $(0,1)$, so it is not closed. The set $(0,1]$ is neither closed nor open, since it contains one of its limit points but not both. The set $[0,1]$ contains both limit points and is therefore closed, and it is not open. The set $\{0,1\}$ has no limit points so is vacuously closed, and it is not open. The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not open since there is no $\varepsilon > 0$ such that $(1-\varepsilon, 1+\varepsilon) \subset \{\frac{1}{n} : n \in \mathbb{N}\}$. Note that 0 is a limit point of this set since for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ and therefore $\frac{1}{n_0} \in ((-\varepsilon,\varepsilon) - \{0\}) \cap \{\frac{1}{n} : n \in \mathbb{N}\} \neq \emptyset$. Since $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, the set is not closed in $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

(3) In $\mathbb{R}^2$ with the standard topology, the set $C = \{(x,y) : x^2+y^2 = 1\}$ is not open since if $p \in C$ and $\varepsilon_p > 0$, then $p - \left(\frac{\varepsilon_p}{2}, \frac{\varepsilon_p}{2}\right)$ is in $B(p, \varepsilon_p)$ but not in C. If $p = (\cos\theta_0, \sin\theta_0) \in C$, then p is a limit point of C since the point $(\cos\theta_1, \sin\theta_1) \in C$ and if $|\theta_1 - \theta_0| < \arccos\left(1 - \frac{\varepsilon_p^2}{2}\right)$, then $(\cos\theta_1, \sin\theta_1) \in (B(p,\varepsilon_p) - \{p\}) \cap C \neq \emptyset$. If $p \notin C$, then set ε to be the distance from p to the nearest point of C. We have that $(B(p, \frac{\varepsilon}{2}) - \{p\}) \cap C = \emptyset$, so all points of C are limit points and all points not in C are not limit points, which means $\overline{C} = C$ and therefore C is closed. Let $D = \{(x,y) : x^2 + y^2 < 1\}$. Then C is the set of all the limit points of D, and so $C \cup D = \overline{D}$ is closed. Therefore, $\{(x,y) : x^2 + y^2 > 1\} = \mathbb{R}^2 - \overline{D}$ is open, and its limit points are also all in the set C and therefore this set is not closed. The set D is open since $D = B(0,1) \in \mathcal{T}_{\text{std}}$, and so the set $\{(x,y) : x^2 + y^2 \geq 1\} = \mathbb{R}^2 - D$ is closed. This set is not open since there is no ε such that $B(p,\varepsilon) \subset \{(x,y) : x^2 + y^2 \geq 1\}$ for $p \in \{(x,y) : x^2 + y^2 \geq 1\} \cap C$.

Theorem 2.20. For any set A in a topological space $(X, \mathcal{T})$, the closure of A is the intersection of all closed sets containing A, that is, $\overline{A} = \bigcap_{B \supset A, B \in \mathfrak{C}} B$, where $\mathfrak{C}$ is the set of all closed sets in $(X, \mathcal{T})$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$ and let A be a subset of X. Since its closure $\overline{A}$ is closed and $A \subset \overline{A}$, we have that $\overline{A} \in \mathfrak{C}$, and since for any sets M and N we have that $M \cap N \subset M, N$, we have that $\bigcap_{B \supset A, B \in \mathfrak{C}} B \subset \overline{A}$. To show equality, then, we need only show that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. Let $a \in \overline{A}$. There are two cases. In the first case, if $a \in A$, then $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$, which means that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$. In the second case, we have that $a \notin A$, which means that $a \in \overline{A} - A$ is a limit point of a. This means that for all open sets U with $a \in U$, we have that $(U - \{a\}) \cap A \neq \emptyset$. Since $a \notin A$, we have that $U \cap A = (U - \{a\}) \cap A \neq \emptyset$. Let $B_0 \in \mathfrak{C}$ with $A \subset B_0$ and suppose for contradiction that $a \notin B_0$. Since B_0 is closed, $X - B_0$ is open, and since $a \notin B_0$, $a \in X - B_0$, which means that $(X - B_0) \cap A \neq \emptyset$. Therefore there exists some $y \in (X - B_0) \cap A$. We have that $y \in A \subset B_0$, which means that $y \notin X - B_0$, a contradiction. We have reached a contradiction by assuming that there exists a set $B_0 \in \mathfrak{C}$ such that $A \subset B_0$ and $a \notin B_0$, which means that $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$. Therefore in both cases $a \in A$ and $a \in \overline{A} - A$, we have that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$, which means that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. This shows that $\overline{A}$ is indeed the intersection of all closed sets containing A, as required. $\square$

Exercise 2.21. Consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. In the discrete topology, this set is already closed and so is its own closure. In the indiscrete topology, only $\mathbb{R}$ and $\emptyset$ are closed, so the closure of H in the indiscrete topology is $\mathbb{R}$ itself. In the finite complement topology, let $p \in \mathbb{R}$ be an arbitrary point. Then let U be an arbitrary open set containing p. Since U is open, its complement is finite, which means there are only finitely many points not in $U - \{p\}$, so there must be infinitely many points in $(U - \{p\}) \cap H \neq \emptyset$. Since U was an arbitrary open set, p is a limit point, and since p was an arbitrary point in $\mathbb{R}$, we see that all points are limit points of H, which means that $\overline{H} = \mathbb{R}$. In the countable complement topology, H is closed since it contains countably many elements, so it is its own closure. In the standard topology, the only limit point of H is 0, so the closure of H is $\overline{H} = H \cup \{0\}$.

Theorem 2.22. Let A and B be subsets of a topological space X with topology $\mathcal{T}$. Then:

(1) $A \subset B$ implies $\overline{A} \subset \overline{B}$.

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) We have that $A \subset B \subset \overline{B}$, and by Theorem 2.20, we have that $\overline{A}$ is a subset of all closed sets containing A. Since $\overline{B}$ is a closed set containing A, we have that $\overline{A} \subset \overline{B}$, as required.

(2) Let $c \in \overline{A} \cup \overline{B}$. Without loss of generality, assume $c \in \overline{A}$. There are two cases. For the first case $c \in A$, we have that $c \in A \subset A \cup B \subset \overline{A \cup B}$. For the second case $c \in \overline{A} - A$, c is a limit point of A and therefore for all open sets U containing c satisfy $\emptyset \neq (U - \{c\}) \cap A \subset (U - \{c\}) \cap (A \cup B)$. Since U was an arbitrary open set, c is a limit point of $A \cup B$ and therefore $c \in \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Now let $d \in \overline{A \cup B}$ be arbitrary. Again there are two cases. In the first case, $d \in A \cup B$, so without loss of generality assume $d \in A \subset \overline{A} \subset \overline{A} \cup \overline{B}$. In the second case, $d \in \overline{A \cup B} - (A \cup B)$, so d is a limit point of $A \cup B$. This means that for all open sets U containing d, we have that $\emptyset \neq (U - \{d\}) \cap (A \cup B) = ((U - \{d\}) \cap A) \cup ((U - \{d\}) \cap B)$, so one of the sets on the right hand side is nonempty. Without loss of generality, assume $(U - \{d\}) \cap A \neq \emptyset$. Then d is a limit point of A, so $d \in \overline{A} \subset \overline{A} \cup \overline{B}$. Therefore $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ as well, and so the two sets are equal. □

Exercise 2.25. I'm not entirely sure but I believe the Cantor Set fits this description.

2.4 Interior and Boundary

Theorem 2.26. Let A be a subset of a topological space $(X, \mathcal{T})$. Then p is an interior point if and only if there exists an open set U such that $p \in U \subset A$.

Proof. Let $A \subset X$ be arbitrary and let $p \in A$ be an interior point. Then since A° is the union of all open subsets of A, A° is open as well, and so by Theorem 2.3, there exists an open set U such that $p \in U \subset A^\circ \subset A$ where $A^\circ \subset A$ follows from the fact that if $a \in A^\circ$, then $a \in \bigcup_{U \subset A, U \in \mathcal{T}} U$ and therefore there exists an open set U_0 such that $a \in U_0 \subset A$. Now let $p \in A$ be an arbitrary point such that there is an open set U with $a \in U \subset A$. Since U is open, $U \in \mathcal{T}$ and so $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U = A^\circ$. Thus, p is an interior point, as required. □

Exercise 2.27. If U is open in a topological space, then by Theorem 2.3, for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$, which means x is an interior point of U. If x is an interior point of U, then by Theorem 2.26, there exists a set U_x such that $x \in U_x \subset U$ and so U is open. Therefore U is open in a topological space if and only if every point of U is an interior point.

Lemma 2.28. Given a set A in a topological space $(X, \mathcal{T})$, the closure of A is $\overline{A} = X - (X - A)^\circ$ and the interior of A is $A^\circ = X - \overline{X - A}$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$. For all $B \in \mathfrak{C}$ such that $A \subset B$, we have that $X - B \subset X - A$, and since B is closed, $X - B$ is open. Given an open set U with $U \subset X - A$, $X - U \supset A$ is closed. This means that $\{B \in \mathfrak{C} : B \supset A\} = \{X - U : U \in \mathcal{T} \text{ s.t. } U \subset X - A\}$. Therefore we have that

$$\begin{aligned}
\overline{A} &= \bigcap_{B \supset A, B \in \mathfrak{C}} B = X - \left(X - \bigcap_{B \supset A, B \in \mathfrak{C}} B\right) = X - \left(\bigcup_{B \supset A, B \in \mathfrak{C}} (X - B)\right) \\
&= X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} (X - (X - U))\right) = X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} U\right) = X - (X - A)^\circ,
\end{aligned}$$

as required. The proof that $A^\circ = X - \overline{X - A}$ is similar □

Theorem 2.28. Let A be a subset of a topological space $(X, \mathcal{T})$. Then A°, ∂A, and $(X-A)^\circ$ are all disjoint and their union is X.

Proof. Let $a \in A^\circ$ and suppose for contradiction that $a \in \partial A = \overline{A} \cap \overline{X - A}$. Then either $a \in X - A$ or a is a limit point of $X - A$. In the first case, we have a clear contradiction since $a \in A^\circ \subset A$ cannot be in $X - A$. In the second case, every open set U containing a satisfies $(U - \{a\}) \cap (X - A) \neq \emptyset$. However, $a \in A^\circ$, so there exists an open set V such that $a \in V \subset A$, so we have that $\emptyset \neq (V - \{a\}) \cap (X - A) \subset A \cap (X - A) = \emptyset$, a contradiction. Therefore the sets A° and ∂A are disjoint, and a similar argument shows that ∂A and $(X - A)^\circ$ are disjoint as well. It remains to check that A° and $(X - A)^\circ$ are disjoint. Suppose not, so that there exists an $a \in A^\circ$ such that $a \in (X - A)^\circ$. By Theorem 2.26, there exists an open set U_1 containing a such that $U_1 \subset A$ and an open set U_2 containing a such that $U_2 \subset X - A$. Therefore we have that $a \in U_1 \cap U_2 \subset A \cap (X - A) = \emptyset$, a contradiction showing that A° and $(X - A)^\circ$ must be disjoint. Since $A^\circ = X - \overline{X - A}$ by Lemma 2.28, we have that

$$\begin{aligned}
\overline{A} - A^\circ &= \overline{A} - (X - \overline{X - A}) \\
&= \overline{A} \cap (X - (X - \overline{X - A})) \\
&= \overline{A} \cap \overline{X - A} = \partial A,
\end{aligned}$$

so we see that the closure of A is the disjoint union of the boundary of A and the interior of

A. Again using Lemma 2.28 ($\overline{A} = X - (X - A)^\circ$), we also have that

$$\begin{aligned} X &= \overline{A} \cup (X - \overline{A}) = A^\circ \cup \partial A \cup (X - \overline{A}) \\ &= A^\circ \cup \partial A \cup (X - (X - (X - A)^\circ)) \\ &= A^\circ \cup \partial A \cup (X - A)^\circ. \end{aligned}$$

Therefore we have that A°, ∂A, $(X - A)^\circ$ are disjoint sets whose union is X. $\square$

Exercise 2.29. Again consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$. In the discrete topology, we have that $\overline{H} = H$ and $\overline{\mathbb{R} - H} = \mathbb{R} - H$, and so we have that $\partial N = \emptyset$. The interior of N is $N^\circ = \overline{H} - \partial H = H - \emptyset = H$. In the indiscrete topology, we have that $\overline{H} = \mathbb{R} = \overline{\mathbb{R} - H}$, so $\partial H = \mathbb{R}$ and $H^\circ = \overline{H} - \partial H = \mathbb{R} - \mathbb{R} = \emptyset$. In the finite complement topology, we saw in Exercise 2.21 that the closure of H was $\overline{H} = \mathbb{R}$ and by the same reasoning, $\overline{\mathbb{R} - H} = \mathbb{R}$. Therefore once again we have that $\partial H = \mathbb{R}$ and $H^\circ = \emptyset$, which makes sense since $\emptyset$ is the largest open set contained within H since H is countably infinite. In the countable complement topology, $H = \overline{H}$ is the disjoint union of H° and ∂H. If V is a nonempty open set in the countable complement topology, then $\mathbb{R} - V$ is countable, meaning that V is uncountable, so we cannot have $V \subset H$. The only open set U with $U \subset H$ is $U = \emptyset$. Therefore $H^\circ = \emptyset$ and $\partial H = H$. In the standard topology, no open set contains 1 and $\frac{1}{2}$ without also containing all points in the open set $(\frac{1}{2}, 1)$, which means that no open set is a subset of H. Therefore we have that $H^\circ = \emptyset$, and so $\partial H = \overline{H} = H \cup \{0\}$.

2.5 Convergence of Sequences

Theorem 2.30. Let A be a set in a topological space $(X, \mathcal{T})$, and let p be a point in X. If $\{x_i\}_{i\in\mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$.

Proof. Suppose for contradiction that $\{x_i\}_{i\in\mathbb{N}} \subset A$, $x_i \to p$, but $p \notin \overline{A}$. Then p is not a limit point of A, so there exists an open set U_0 containing p such that $(U_0 - \{p\}) \cap A = \emptyset$. Since $x_i \to p$ and $p \in U_0$, we have that there exists an $N \in \mathbb{N}$ such that $x_i \in U_0$ for all $i > N$. Since $x_i \in A$ for all $i \in \mathbb{N}$, $x_i \neq p$ for all $i \in \mathbb{N}$. Therefore, we have that for all $i > N$, $x_i \in (U_0 - \{p\}) \cap A = \emptyset$, a contradiction. Therefore if $\{x_i\}_{i\in\mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$. In particular, p is a limit point of A. $\square$

Theorem 2.31. In the standard topology on $\mathbb{R}^n$, if p is a limit point of a set A, then there is a sequence of points in A that converges to p.

Proof. If p is a limit point of A, then $A \neq \emptyset$, so there exists a point $a \in A$. Set $\varepsilon = d(a,p) > 0$. Then for all $n \in \mathbb{N}$, $B(p, \frac{\varepsilon}{n})$ is open. Since p is a limit point and $p \in B(p, \frac{\varepsilon}{n})$, we have that $(B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A \neq \emptyset$. Since each of these sets is nonempty there exists an $a_n \in (B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A$ for all n. Define a sequence $(x_n)_{n \in N}$ such that $x_n = a_n$. Therefore $\{x_n\}_{n\in\mathbb{N}} \subset A$, so it only remains to show that the sequence converges to p. Let U be an open set containing p. Since U is open in the standard topology, $U = B(p, \varepsilon_p)$ for some $\varepsilon_p > 0$, and there exists and $N \in \mathbb{N}$ such that $\frac{\varepsilon}{N} < \varepsilon_p$. Let $i > N$. We have that

$$x_i \in B\left(p, \frac{\varepsilon}{i}\right) \subset B\left(p, \frac{\varepsilon}{N}\right) \subset B\left(p, \varepsilon_p\right),$$

which means that $x_i \in U$ for all $i > N$, and therefore $x_i \to p$, as required. $\square$

Exercise 2.32. Consider $\mathbb{R}$ with the indiscrete topology. Let p be a point in $\mathbb{R}$ and set $x_n = n$. This sequence converges to p since if U is an open set containing p, then $U = \mathbb{R}$ and so we have that $x_n \in U$ for all $n \in \mathbb{N}$. Therefore $x_n \to p$, but p was arbitrary, so we see that this sequence converges to every point in $\mathbb{R}$.

3 Bases, Subspaces, Products: Creating New Spaces

3.1 Bases

Theorem 3.1. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

(1) $\mathcal{B} \subset \mathcal{T}$, and

(2) for every open set U, and point $p \in U$, there exists a set $V \in \mathcal{B}$ such that $p \in V \subset U$.

Proof. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then by definition, (1) is satisfied. Now let U be an arbitrary open set and let $p \in U$ be an arbitrary point. Since $\mathcal{B}$ is a basis, we have that $U = \bigcup_{V \in \lambda} V$ for some collection of sets $\lambda \subset \mathcal{B}$, and since $p \in U$, we have that $p \in V_0 \subset U$ for some set $V_0 \in \lambda$, so (2) is satisfied as well. Now suppose that $\mathcal{B}$ is a collection of sets satisfying (1) and (2). Let U_0 be an arbitrary open set and define λ to be the collection of sets $\lambda = \{V \in \mathcal{B} : V \subset U_0\} \subset \mathcal{B}$. Now let $p \in U_0$ be an arbitrary point in U_0. By (2), there exists a $V_0 \in \mathcal{B}$ such that $p \in V_0 \subset U_0$, which means $V_0 \in \lambda$. Therefore $p \in \bigcup_{V \in \lambda} V$, and so $U_0 \subset \bigcup_{V \in \lambda} V$. Now let p be an arbitrary point in $\bigcup_{V \in \lambda} V$. Then there exists a $V_0 \in \lambda \subset \mathcal{B}$ such that $p \in V_0$, and since $V_0 \in \lambda$, $V_0 \subset U_0$, so $p \in U_0$. Therefore $U_0 = \bigcup_{V \in \lambda} V$ since both sets are subsets of the other. Since any arbitrary open set U is the union of sets in the collection $\mathcal{B}$, $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

Exercise 3.2. $\mathcal{B}_1$ satisfies Theorem 3.1(1) since it consists only of open intervals and therefore $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Since U is open in $\mathcal{T}_{\text{std}}$, $U = (c-\varepsilon, c+\varepsilon)$ for some $c \in \mathbb{R}$ and $\varepsilon > 0$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $c - \varepsilon < a < p$ and $p < b < \varepsilon$. We have that $p \in (a,b) \subset U$ and $(a,b) \in \mathcal{B}_1$, so Theorem 3.2(2) is satisfied and therefore $\mathcal{B}_1$ is a basis.

Theorem 3.3. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if

(1) each point of X is in some element of $\mathcal{B}$, and

(2) if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof. If $\mathcal{B}$ is a basis for some topology on X, then $X = \bigcup_{B \in \mathcal{B}} B$, so if $p \in X$, then $p \in B_0$ for some $B_0 \in \mathcal{B}$. Therefore (1) is satisfied. Now let U and V be sets in $\mathcal{B}$ and let p be an arbitrary point in $U \cap V$. Since U and V and in $\mathcal{B}$, they are open, and therefore $U \cap V$ is open, so by Theorem 3.1, there exists a $W \in \mathcal{B}$ such that $p \in W \subset (U \cap V)$, satisfying (2). Now suppose $\mathcal{B}$ is a collection of subsets of X satisfying (1) and (2). Then $\emptyset$ is the empty union of sets in $\mathcal{B}$, and $X = \bigcup_{B \in \mathcal{B}} B$ by (1). Suppose U and V are in $\mathcal{B}$ and set $\lambda = \{W \in \mathcal{B} : W \subset (U \cap V)\} \subset \mathcal{B}$. If p is an arbitrary point of $U \cap V$, then by (2) there exists a W_0 such that $p \in W_0 \subset (U \cap V)$, so $p \in \bigcup_{W \in \lambda} W$. If p is an arbitrary point of $\bigcup_{W \in \lambda} W$, then there exists a $W_0 \in \mathcal{B}$ such that $p \in W_0 \subset (U \cap V)$. Therefore $U \cap V = \bigcup_{W \in \lambda} W$, so the intersection of two sets that are the unions of sets in $\mathcal{B}$ is also the union of sets in $\mathcal{B}$. Now let α be a collection of sets in $\mathcal{B}$. Then for each $\beta \in \alpha$, $\beta = \bigcup_{B \in \lambda} B$ for some collection of sets $\lambda \subset \mathcal{B}$. Then we have that $\bigcup_{B \in \alpha} B$ is the union of a collection of unions of sets in $\mathcal{B}$ and is therefore itself the union of sets in $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for some topology on X since the collection of sets that are unions of sets in $\mathcal{B}$ satisfies all four properties of a topology. $\square$

Exercise 3.4. Let $\mathcal{B}_{\text{LL}}$ be the set of subsets of $\mathbb{R}$ of the form $[a,b)$. Then if $x \in \mathbb{R}$, $x \in [x, x+1)$, so Theorem 3.3(1) is satisfied. Let U and V be arbitrary sets in $\mathcal{B}_{\text{LL}}$. Then there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $U = [a_1, b_1)$ and $V = [a_2, b_2)$. Let $x \in U \cap V$ be arbitrary. Then $a_1, a_2 \leq x < b_1, b_2$, so $p \in W = [\max\{a_1, a_2\}, \min\{b_1, b_2\}) = U \cap V$. Therefore Theorem 3.3(2) is satisfied and so $\mathcal{B}_{\text{LL}}$ is a basis for a topology on $\mathbb{R}$. ($\mathbb{R}$ together with this topology is the Sorgenfrey Line, $\mathbb{R}_{\text{LL}}$.)

Exercise 3.6. As discussed in Exercise 2.21, the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed in $\mathbb{R}$ with the standard topology, which means that $\mathbb{R} - H$ is not open in the standard topology. However, H contains countably many points, so $\mathbb{R} - H$ is open in the countable complement topology. Therefore we have that the standard topology on $\mathbb{R}$ is not finer than the countable complement topology. On the other hand, the set $(0, 1)$ is open in the standard topology, but not open in the countable complement topology, so we also have that the countable complement topology is not finer than the standard topology.

Exercise 3.7. Let $\mathbb{R}_{+00}$ be the set of all positive real numbers ($\mathbb{R}_+$) together with the points $0'$ and $0''$, and let $\mathcal{B}$ be the set of all intervals of the form (a, b), $(0, b) \cup \{0'\}$, or $(0, b) \cup \{0''\}$ for $a, b \in \mathbb{R}_+$.. We claim that $\mathcal{B}$ is the basis for some topology $\mathcal{T}$. For any $x \in \mathbb{R}_+$, $x \in (\frac{x}{2}, x+1)$, $0' \in (0, 1) \cup \{0'\}$, and $0'' \in (0, 1) \cup \{0''\}$, so every point in $\mathbb{R}_{+00}$ is in some element of $\mathcal{B}$. Now let U and V be arbitrary elements of $\mathcal{B}$ and let $p \in U \cap V$ be arbitrary. If one of U, V is of the form $(0, b_1) \cup \{0'\}$ and the other is of the form $(0, b_2) \cup \{0''\}$, and p is an arbitrary point in $U \cap V$, then set $W = (\frac{p}{2}, \min\{b_1, b_2\})$. Then we have that $p \in W \subset (U \cap V)$ and $W \in \mathcal{B}$. Otherwise, Set $W = (U \cap V)$. Then $p \in W \subset (U \cap V)$, so to show that $\mathcal{B}$ is the basis for some topology, it remains only to show that $W \in \mathcal{B}$. If one of U, V is of the form (a, b) for some $a, b \in \mathbb{R}_+$, then $U \cap V$ is of the same form and therefore in $\mathcal{B}$. For the remaining case to check, assume without loss of generality that $U = (0, b_1) \cup \{0'\}$ and $V = (0, b_2) \cup \{0'\}$ (the case for $0''$ is the same). Then $U \cap V = (0, \min\{b_1, b_2\}) \cup \{0'\} \in \mathcal{B}$. Therefore by Theorem 3.3, $\mathcal{B}$ is a basis for some topology on $\mathbb{R}_{+00}$. (This topological space is called the Double Headed Snake and will also be written as $\mathbb{R}_{+00}$.)

Exercise 3.8. In the Double Headed Snake, $\mathbb{R}_{+00}$, let p be an arbitrary point. If $p = 0'$ or $0''$, we claim that $\{p\}$ is closed. Without loss of generality, assume $p = 0'$. Then

$$\mathbb{R}_{+00} - \{p\} = (0, \infty) \cup \{0''\} = \bigcup_{n \in \mathbb{N}} \left((0, n) \cup \{0''\}\right).$$

Since $(0, n) \cup \{0''\} \in \mathcal{B}$, the basis for the Double Headed Snake given in Exercise 3.7, we have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and is therefore open in the Double Headed Snake, which means that $\{p\}$ is closed. Now if $p \neq 0', 0''$, we have that $p \in \mathbb{R}_+$. We have that

$$\mathbb{R}_{+00} - \{p\} = (0,p) \cup (p,\infty) \cup \{0', 0''\}$$
$$= ((0,p) \cup \{0'\}) \cup ((0,p) \cup \{0''\}) \cup \left(\bigcup_{p \in \mathbb{N}} (p, p+n) \right).$$

Since $(p, p+n) \in \mathcal{B}$ for all $n \in \mathbb{N}$, we again have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and therefore is open in the Double Headed Snake, so $\{p\}$ is closed.

Suppose for contradiction that U and V are disjoint open sets in the Double Headed Snake such that $0' \in U$ and $0'' \in V$. Since $0' \in U$, and U is open, it is the union of sets in $\mathcal{B}$, so there exists a set of the form $(0, b_1) \cup \{0'\} \subset U$. Similarly, there exists a set of the form $(0, b_2) \cup \{0''\} \subset V$. Therefore $(0, \min\{b_1, b_2\}) \subset U \cap V$, so the sets are not disjoint since $b_1, b_2 \in \mathbb{R}_+$. This means it is impossible to have disjoint open sets each containing a different head of the snake.

Exercise 3.9. (1) In the topological space $\mathbb{R}_{\text{har}}$, the set $\mathbb{R} - H$ is open since it is the union of sets in the basis for $\mathbb{R}_{\text{har}}$:

$$\mathbb{R} - H = \bigcup_{n \in \mathbb{N}} ((-n, n) - H).$$

Since $\mathbb{R} - H$ is open in $\mathbb{R}_{\text{har}}$, H is closed and therefore $\overline{H} = H$.

(2) If $H^- = \{-\frac{1}{n} : n \in \mathbb{N}\}$, then $\overline{H^-} = H^- \cup \{0\}$.

(3) Nope!

Exercise 3.10. (1) Let $\mathbb{H}_{\text{bub}}$ be the upper half plane with the Sticky Bubble Topology. Let $Q = \{(x, 0) : x \in \mathbb{Q}\}$. Then

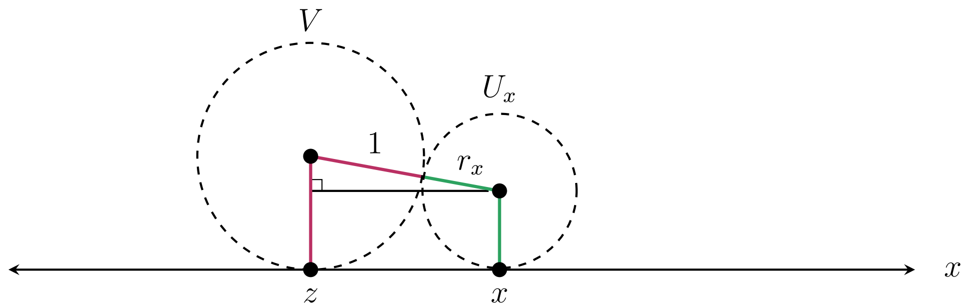
$$\mathbb{H} - Q = \bigcup_{x \in (\mathbb{R} - \mathbb{Q})} \left(\bigcup_{n \in \mathbb{N}} (B((x,n), n) \cup \{(x, 0)\}) \right)$$

is the union of sets in the basis for the Sticky Bubble Topology, so it is open. Therefore Q is closed and $\overline{Q} = Q$.

(2) This is similar to (1) since any subset of the x-axis can be treated the way $\mathbb{Q}$ was in the previous example.

(3) If A is a countable subset of the x-axis, and z is a point on the x-axis not in A, we wish to show that there are disjoint open sets U and V such that $A \subset U$ and $z \in V$. Set $V =$

$B((z,1),1) \cup \{(z,0)\}$. Now for all $x \in A$, set $r_x = (x-z)^2/4$ and $U_x = B((x,r_x),r_x) \cup \{(x,0)\}$. U_x is open for all x, so the set $U = \bigcup_{x\in A} U_x$ is also open. Clearly $A \subset U$, so it remains to check that $U \cap V \neq \emptyset$, which is the case provided none of the U_x bubbles overlap with the bubble V. Consider an arbitrary x and corresponding bubble U_x:



The distance from $(z,1)$ to (x,r_x) is $((1-r_x)^2+(x-z)^2)^{\frac{1}{2}}$, and since U_x and V are open, we have that $U_x \cap V \neq \emptyset$ since:

$$\begin{aligned} r_x &= \frac{(x-z)^2}{4} \\ \iff (x-z)^2 &= 4r_x \\ \iff 1 + r_x^2 + (x-z)^2 &= 1 + 4r_x + r_x^2 \\ \iff 1 - 2r_x + r_x^2 + (x-z)^2 &= 1 + 2r_x + r_x^2 \\ \iff (1-r_x)^2 + (x-z)^2 &= (1+r_x)^2 \end{aligned}$$

Taking square roots shows that the distance between the centers of U_x and V is the same as the sum of the radii of U_x and V, so the bubbles overlap at a single point that is not contained in either open set. This was the case for an arbitrary $x \in A$, so it is true for all $x \in A$. Therefore we have found two open sets U and V such that $A \subset U$, $z \in V$, and $U \cap V = \emptyset$. We did not use the fact that A contains countably many points, so this holds for all subsets of the x-axis.

Exercise 3.11. Let $\mathbb{Z}_{\text{arith}}$ be the integers $\mathbb{Z}$ together with a topology generated by a basis of arithmetic progressions (the basis $\mathcal{B}$ is the collection of all sets of the form $\{az+b : z \in \mathbb{Z}\}$ for $a, b \in \mathbb{Z}$ with $a \neq 0$). To show that this is indeed a basis for some topology on $\mathbb{Z}$, note that $\mathbb{Z} = \{1 \cdot z + 0 : z \in \mathbb{Z}\}$ is itself in $\mathcal{B}$, so all points in $\mathbb{Z}$ are in some element of $\mathcal{B}$. Now let U and V be arithmetic progressions in $\mathcal{B}$ and let $z_0 \in U \cap V$ be arbitrary. Then $U = \{a_1 z + b_2 : z \in \mathbb{Z}\}$ and $V = \{a_2 z + b_2 : z \in \mathbb{Z}\}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Now

$z_0 \in U \cap V$, then we have that $a_1k_1 + b_1 = z_0 = a_2k_2 + b_2$ for some $k_1, k_2 \in \mathbb{Z}$. Then $z_0 \in W = \{\text{lcm}(a_1, a_2)z + z_0 : z \in \mathbb{Z}\} \in \mathcal{B}$. We also have that $W \subset U \cap V$ since if $w_0 = \text{lcm}(a_1, a_2)k_0 + z_0 \in W$, then

$$a_1\left(k_1 + \frac{a_2k_0}{\gcd(a_1, a_2)}\right) + b_1 = a_1k_1 + b_1 + \frac{a_1a_2}{\gcd(a_1, a_2)}k_0 = \text{lcm}(a_1, a_2)k_0 + z_0$$
$$= w_0 = \text{lcm}(a_1, a_2)k_0 + z_0 = \frac{a_1a_2}{\gcd(a_1, a_2)}k_0 + a_2k_2 + b_2 = a_2\left(k_2 + \frac{a_1k_0}{\gcd(a_1, a_2)}\right) + b_2,$$

and w_0 is in both arithmetic progressions. Therefore $\mathbb{Z}_{\text{arith}}$ is a topological space generated by the basis $\mathcal{B}$.

Exercise 3.12. Consider the topological space $\mathbb{Z}_{\text{arith}}$ and note that if U is an open set, then since it is the union of infinite arithmetic progressions, it is itself infinite. Note also that for a prime p and $a = 1, 2, \ldots, p-1$, we have that $\{pz + a : z \in \mathbb{Z}\} \in \mathcal{B}$ is an arithmetic progression. Let $p\mathbb{Z}$ denote the set $\{pz : z \in \mathbb{Z}\}$. Then we have that

$$\mathbb{Z} - p\mathbb{Z} \;=\; \bigcup_{a=1,\ldots,p-1} \{pz + a : z \in \mathbb{Z}\}$$

is the union of open sets and is therefore open, meaning that $p\mathbb{Z}$ itself is closed. Let P denote the set $P = \bigcup_{\text{primes } p} p\mathbb{Z}$, and suppose for contradiction that there are finitely many primes. Then P is the union of finitely many closed sets and is therefore itself closed. Note that P contains all numbers that are integer multiples of a prime, so the only numbers not contained in P are -1 and 1, that is, $\mathbb{Z} - P = \{-1, 1\}$. But this set is open since P is closed, which is a contradiction since $\mathbb{Z} - P$ is finite. Therefore there are infinitely many primes.

3.2 Subbases

Exercise 3.13. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Then if U is in an open set in $(X, \mathcal{T})$, it is the union of sets in $\lambda \subset \mathcal{B}$. For each set $S \in \lambda$, S is the trivial intersection of itself, and this intersection is finite. Since $\mathcal{B}$ is a basis for a the topology $\mathcal{T}$, the finite intersections of sets in $\mathcal{B}$ is in $\mathcal{T}$. Therefore every open set in $\mathcal{T}$ can be generated by taking the union of sets that are themselves the finite intersections of sets in $\mathcal{B}$ and so $\mathcal{B}$ is a subbasis.

Exercise 3.14. Consider $\mathbb{R}$ together with the standard topology. Recall that $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis and note that $(a, b) = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x < b\}$ and this is a finite intersection. Therefore every set in the basis is the finite intersection of sets in $\mathscr{S}$, the set of rays of the form $\{x \in \mathbb{R} : x < x_0\}$ and $\{x \in \mathbb{R} : x_0 > x\}$ for some $x_0 \in \mathbb{R}$. Therefore $\mathscr{S}$ is a subbasis for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

Theorem 3.16. Let X be a set and let $\mathscr{S}$ be a collection of subsets of X. Then $\mathscr{S}$ is a subbasis for some topology $\mathcal{T}$ on X if and only if every point of X is contained in some element of $\mathscr{S}$.

Proof. Suppose $\mathscr{S}$ is a subbasis for a topology $\mathcal{T}$ on X and let $x \in X$ be arbitrary. Since $\mathscr{S}$ is a subbasis, the set $\mathcal{B}$ of finite intersections of elements of $\mathscr{S}$ is a basis, and therefore $x \in B_0$ for some $B_0 \in \mathcal{B}$. Then since B_0 is the finite intersection of elements in $\mathscr{S}$, $x \in S_0$ for some $S_0 \in \mathscr{S}$. Now suppose that for all $x \in X$, there exists some $S_0 \in \mathscr{S}$ such that $x \in S_0$. Then if $\mathcal{B}$ is the set of finite intersections of elements of $\mathscr{S}$, $x \in S_0 \in \mathcal{B}$, so Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary and let $p \in U \cap V$. Then $U = \bigcap_{S \in \lambda_1} S$ and $V = \bigcap_{S \in \lambda_2} S$ where λ_1 and λ_2 are collections of sets in $\mathscr{S}$. Therefore

$$p \in \bigcap_{S \in \lambda_1 \cap \lambda_2} S \subset U \cap V$$

since $\lambda_1 \cap \lambda_2 \subset \lambda_1, \lambda_2$. Therefore Theorem 3.3(2) is satisfied as well since $\bigcap_{S \in \lambda_1 \cap \lambda_2} S \in \mathcal{B}$, and so $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, which also means that $\mathscr{S}$ is a subbasis for this topology. $\square$

Exercise 3.17. Let $\mathscr{S}$ be the set of all subsets of $\mathbb{R}$ of the form $\{x \in \mathbb{R} : x < a\}$ or $\{x \in \mathbb{R} : a \leq x\}$. We claim that this is a subbasis for the lower limit topology on $\mathbb{R}$, $\mathcal{T}_{\text{LL}}$. If $U \in \mathscr{S}$, then there are two cases. In the first case, $U = \{x \in \mathbb{R} : x < a\} = \bigcup_{n \in \mathbb{N}} [a - n, a) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. In the second case, $U = \{x \in \mathbb{R} : a \leq x\} = \bigcup_{n \in \mathbb{N}} [a, a + n) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. Both inclusions in $\mathcal{T}_{\text{LL}}$ come from the fact that all open sets of the Sorgenfrey Line are the union of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Since any arbitrary $U \in \mathscr{S}$ has $U \in \mathcal{T}_{\text{LL}}$, we have that $S \subset \mathcal{T}_{\text{LL}}$ and therefore Theorem 3.15(1) is satisfied. Now suppose U is an open set of the Sorgenfrey Line and let $p \in U$ be arbitrary. Then $U = \bigcup_{W \in \lambda} W$ where $\lambda \subset \mathcal{B}_{\text{LL}}$ is a collection of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Therefore we have that $p \in W_0 = [a_0, b_0)$ for some $W_0 \in \lambda$. Since $W_0 = \{x \in \mathbb{R} : a_0 \leq x\} \cap \{x \in \mathbb{R} : x < b_0\}$ is the finite intersection of sets in $\mathscr{S}$ and $W_0 \subset U$, we have satisfied Theorem 3.15(2), meaning that $\mathscr{S}$ is indeed a subbasis for the lower limit topology on $\mathbb{R}$, as claimed.

3.3 Order Topology

Exercise 3.19. Consider $\mathbb{R}$ with the order topology from $\leq$. It has a basis $\mathcal{B}$ containing sets of the form $\{x \in \mathbb{R} : x < a\}$, $\{x \in \mathbb{R} : a < x\}$, and $\{x \in \mathbb{R} : a < x < b\}$. A set U is open in the standard topology on $\mathbb{R}$ if for each point $p \in U$, there is an $\varepsilon_p > 0$ such that $(p-\varepsilon_p, p+\varepsilon_p) \subset U$. Note that this is the case for every set of the forms contained in $\mathcal{B}$ (for the first and second forms, $\varepsilon_p = |a-p|$, and for the third form, $\varepsilon_p = \min\{p-a, b-p\}$), so $\mathcal{B} \subset \mathcal{T}_{\text{std}}$, satisfying Theorem 3.1(1). If $U \subset \mathbb{R}$ is open in $\mathcal{T}_{\text{std}}$ and $p \in U$, then since there exists $\varepsilon_p > 0$ such that $p \in (p-\varepsilon_p, p+\varepsilon_p) \subset U$ and $(p-\varepsilon_p, p+\varepsilon_p) \in \mathcal{B}$, Theorem 3.1(2) is satisfied as well and therefore $\mathcal{B}$ is a basis for the standard topology as well, so the order topology on $\mathbb{R}$ with $\leq$ is the standard topology.

Exercise 3.21. $A = \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\}$ has closure $\overline{A} = A \cup \{(0,1)\}$.
$B = \left\{\left(1-\frac{1}{n}, \frac{1}{2}\right) : n \in \mathbb{N}\right\}$ has closure $\overline{B} = B \cup \{(1,0)\}$.
$C = \{(x,0) : 0 < x < 1\}$ has closure $\overline{C} = \{(x,0) : 0 < x \leq 1\} \cup \{(x,1) : 0 \leq x < 1\}$.
$D = \left\{\left(x, \frac{1}{2}\right) : 0 < x < 1\right\}$ has closure $\overline{D} = D \cup \{(x,0) : 0 < x \leq 1\} \cup \{(x,1) : 0 \leq x < 1\}$.
$E = \left\{\left(\frac{1}{2}, y\right) : 0 < y < 1\right\}$ has closure $\overline{E} = E \cup \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right)\right\}$

3.4 Subspaces

Theorem 3.25. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $\mathcal{T}_Y$ is indeed a topology on Y.

Proof. Note that $\emptyset \in \mathcal{T}_Y$ since $\emptyset = \emptyset \cap Y$ and $\emptyset \in \mathcal{T}$. Similarly, $Y \in \mathcal{T}_Y$ since $Y = X \cap Y$ and $X \in \mathcal{T}$. Now let $A, B \in \mathcal{T}_Y$ be arbitrary. Then there exist sets $V_A, V_B \in \mathcal{T}$ such that $A = V_A \cap Y$ and $B = V_B \cap Y$ and we have that

$$A \cap B = (V_A \cap Y) \cap (V_B \cap Y) = (V_A \cap V_B) \cap Y \in \mathcal{T}_Y$$

since $V_A \cap V_B$ is the (finite) intersection of sets in the topology $\mathcal{T}$ and is therefore itself in the topology $\mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in \lambda}$ be a collection of sets in $\mathcal{T}_Y$. Then for each $\alpha \in \lambda$, there exists a set $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$ and we have that

$$\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (V_\alpha \cap Y) = \left(\bigcup_{\alpha \in \lambda} V_\alpha\right) \cap Y \in \mathcal{T}_Y$$

since $\bigcup_{\alpha \in \lambda} V_\alpha$ is the union of sets in $\mathcal{T}$ and is therefore itself in $\mathcal{T}$. Thus we have that $\emptyset, Y \in \mathcal{T}_Y$ and that $\mathcal{T}_Y$ contains the finite intersections of sets in $\mathcal{T}_Y$ and the (possibly infinite) unions of sets in $\mathcal{T}_Y$ and is therefore a topology on Y. $\square$

Exercise 3.26. Taking $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$, we see that the set $\left[\frac{1}{2}, 1\right)$ is closed in Y. This is because the set $\left(-1, \frac{1}{2}\right)$ is open in $\mathbb{R}_{\text{std}}$ and therefore $\left(-1, \frac{1}{2}\right) \cap Y = \left[0, \frac{1}{2}\right)$ is open in Y, so $Y - \left[0, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right)$ is closed in Y.

Exercise 3.27. If Y is a subspace of a topological space $(X, \mathcal{T})$, it is not necessarily the case that every subset of Y that is open in Y is open in $(X, \mathcal{T})$. As in Exercise 3.26, $Y = [0, 1)$ is a subspace of $\mathbb{R}_{\text{std}}$ and $\left[0, \frac{1}{2}\right)$ is open in Y. However, it is neither open nor closed in $\mathbb{R}_{\text{std}}$.

Theorem 3.28. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

Proof. Let $C \subset Y$ be closed in $(Y, \mathcal{T}_Y)$. This is the case if and only if $Y - C$ is open in $(Y, \mathcal{T}_Y)$, which is the case if and only if there exists a set $V \subset X$, open in $(X, \mathcal{T})$, such that $Y - C = V \cap Y$. This then is the case if and only if

$$C = Y - (V \cap Y) = Y - V = Y \cap (X - V) = Y \cap D$$

where $D = X - V$ is closed in $(X, \mathcal{T})$ since V is open in $(X, \mathcal{T})$. All implications here go in both directions, so both directions of the proof are complete. $\square$

Theorem 3.30. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$ that has basis $\mathcal{B}$. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Proof. Let $A_0 \in \mathcal{B}_Y$ be arbitrary. Then there exists $B_0 \in \mathcal{B}$ such that $A_0 = B_0 \cap Y$. Since $B_0 \in \mathcal{B}$, it is open in the topological space $(X, \mathcal{T})$. Since $\mathcal{T}_Y = \{U : U = V \cap Y \text{ for some } V \in \mathcal{T}\}$, we have that $A_0 \in \mathcal{T}_Y$ and so Theorem 3.1(1) is satisfied. Now let U be an open set in $(Y, \mathcal{T}_Y)$ and let $p \in U$ be an arbitrary point. Since $U \in \mathcal{T}_Y$, there exists a set $W \in \mathcal{T}$ such that $U = W \cap Y$, which means that $p \in Y$ and $p \in W$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, by Theorem 3.1(2), there exists a set $V \in \mathcal{B}$ such that $p \in V \subset W$. Then since $V \in \mathcal{B}$, $V \cap Y \in \mathcal{B}_Y$ and we have that

$$p \in V \cap Y \subset W \cap Y = U,$$

which satisfies Theorem 3.1(2), meaning that $\mathcal{B}_Y$ is indeed a basis for $\mathcal{T}_Y$. $\square$

3.5 Product Spaces

Exercise 3.32. Let X and Y be topological spaces and let $\mathcal{B}$ be the set of all cartesian products of open sets $U \subset X$ and $V \subset Y$. Then $p = (a, b)$ is a point of $X \times Y$, then p is in some element of $\mathcal{B}$ since X is open in X and Y is open in Y (that is, $X \times Y \in \mathcal{B}$). This means that Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary. Then there exist sets X_1, X_2 open in X and Y_1, Y_2 such that $U = X_1 \times Y_1$ and $V = X_2 \times Y_2$. Now let $p \in U \cap V$ be arbitrary. Then we have that $p \in U \cap V \subset U \cap V$ and also that

$$U \cap V = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2) \in \mathcal{B}$$

since $X_1 \cap X_2$ is open in X and $Y_1 \cap Y_2$ is open in Y. Therefore Theorem 3.3(2) is also satisfied and we have that $\mathcal{B}$ is indeed the basis for some topology on $X \times Y$.

Exercise 3.34. The product of closed sets is closed in the product topology. Let X and Y be topological spaces, let A be closed in X and let B be closed in Y. Then there exists an open set $A_0 \subset X$ such that $A = X - A_0$ and there exists an open set $B_0 \subset Y$ such that $B = Y - B_0$ and we have that

$$\begin{aligned}
A \times B &= (X - A_0) \times (Y - B_0) \\
&= (X \times (Y - B_0)) - (A_0 \times (Y - B_0)) \\
&= ((X \times Y) - (X \times B_0)) - ((A_0 \times Y) - (A_0 \times B_0)) \\
&= ((X \times Y) \cup (A_0 \times B_0)) - ((X \times B_0) \cup (A_0 \times Y)) \\
&= (X \times Y) - ((X \times B_0) \cup (A_0 \times Y)).
\end{aligned}$$

The sets $X \times B_0$ and $A_0 \times Y$ are both basic open sets in $X \times Y$ since A_0 is open in X and B_0 is open in Y. Since the union of open sets is open, we have that $A \times B$ is the complement of open sets and is therefore closed. A and B were arbitrary closed sets in arbitrary topological spaces, so we have shown that the product of closed sets is closed in the product topology in general.

Theorem 3.35. The product topology on $X \times Y$ has a subbasis $\mathscr{S}$ that contains the inverse images of open sets under the projection functions, that is,

$$\mathscr{S} = \{\pi_X^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ is open in } Y\}.$$

Proof. Let S_0 be a set in $\mathscr{S}$. There are two cases. In the first, S_0 is of the form $S_0 = \{\pi_X^{-1}(U) : U \text{ is open in } X\} = U \times Y$ and is therefore a basic open set in the product space. In the second, S_0 is of the form $S_0 = \{\pi_Y^{-1}(V) : V \text{ is open in } Y\} = X \times V$ and is also a basic open set in the product space. Therefore S_0 is open in all cases and so $\mathscr{S} \subset \mathcal{T}$ where $\mathcal{T}$ is the product topology. Thus Theorem 3.15(1) is satisfied. Now let W be an open set in the product space and let $p \in W$ be an arbitrary point. Then since W is open, it is the union of sets of the form $U \times V$ where U is open in X and V is open in Y. This means there are sets $U_0 \subset X$ and $V_0 \subset Y$ such that $p \in U_0 \times V_0 \subset W$. All that remains to satisfy Theorem 3.15(2) is to show that $U_0 \times V_0$ is the finite intersection of sets in $\mathscr{S}$, which is the case since $U_0 \times V_0 = \pi_X^{-1}(U_0) \cap \pi_Y^{-1}(V_0)$. Therefore $\mathscr{S}$ is indeed a subbasis for the product topology on $X \times Y$. $\square$

Exercise 3.36. Let W be an open set in $\mathbb{R}^2_{\text{std}}$. Then $W = B(p, \varepsilon_p)$ for some $p \in \mathbb{R}^2$ and $\varepsilon_p > 0$. Now let $x \in W$ be arbitrary. Suppose $p = (p_1, p_2)$ and $x = (x_1, x_2)$. Set

$$U = \left(x_1 - \left| x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1 \right) \right|, x_1 + \left| x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1 \right) \right| \right)$$

and

$$V = \left(x_2 - \left| x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2 \right) \right|, x_2 + \left| x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2 \right) \right| \right).$$

(See Desmos sketch) Then we have that $x \in U \times V \subset W$, and since $U \times V$ is an open set with the product topology on $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$, we have that W is an open set in this topology as well by Theorem 2.3 since x was an arbitrary point of W. Then since W was an arbitrary open set under the standard topology, it follows that the standard topology on $\mathbb{R}^2$ is a subset of the product topology on $\mathbb{R}^2$.

Now let W be an open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology and let $p \in W$ be arbitrary. Then since W is open, it is the union of sets of the form $U \times V$ where U and V are open sets in $\mathbb{R}_{\text{std}}$. This means that there exist some U_0 and V_0 open in $\mathbb{R}_{\text{std}}$ such that $p \in U_0 \times V_0 \subset W$. Then there exist intervals (a_x, b_x) and (a_y, b_y) such that $\pi_{U_0}(p) \in (a_x, b_x)$ and $\pi_{V_0}(p) \in (a_y, b_y)$. Therefore we have that $p \in B(p, \varepsilon) \subset U_0 \times V_0 \subset W$ where

$$\varepsilon = \min\{|a_x - \pi_{U_0}(p)|, |b_x - \pi_{U_0}(p)|, |a_y - \pi_{V_0}(p)|, |b_y - \pi_{V_0}(p)|\}.$$

Since this $B(p, \varepsilon)$ is an open set in $\mathbb{R}^2_{\text{std}}$, by Theorem 2.3 it follows that W is also open in $\mathbb{R}^2_{\text{std}}$, and since W was an arbitrary open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology, we have that this topology on $\mathbb{R}^2$ is a subset of the standard topology on $\mathbb{R}^2$. Since each topology is a subset of the other, they are equal, meaning that product topology on $\mathbb{R}^2$ from $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is the same as the standard topology on $\mathbb{R}^2$.

Theorem 3.37. The product topology on $\prod_{\alpha\in\lambda} X_\alpha$ has a basis containing all sets of the form $\prod_{\alpha\in\lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.

Proof. Since $\mathscr{S}$, the collection of sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in $(X_\beta, \mathcal{T}_\beta)$, is a subbasis for the product topology $\prod_{\alpha\in\lambda} X_\alpha$, the set of finite intersections of elements of $\mathscr{S}$ is therefore a basis for the product topology. Let W be a set in the collection described in the theorem statement. Then there exists a finite index set $\gamma \subset \lambda$ such that $W = \prod_{\alpha\in\lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all $\alpha \in \lambda - \gamma$. Then since for some $\beta \in \gamma$, $\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha\in\lambda-\{\beta\}} X_\alpha$, we have that $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) = U_{\beta_1} \times U_{\beta_2} \times \prod_{\alpha\in\lambda-\{\beta_1,\beta_2\}} X_\alpha$ and therefore $W = \prod_{\alpha\in\gamma} U_\alpha \times \prod_{\alpha\in\lambda-\gamma} X_\alpha = \bigcap_{\alpha\in\gamma} \pi_\alpha^{-1}(U_\alpha)$ is the finite intersection of elements of $\mathscr{S}$. Similarly, any finite intersection of elements of $\mathscr{S}$ is a product of the form described in the theorem statement, and so by the definition of a subbasis, the collection described in the theorem statement is indeed a basis for the product topology. $\square$

Exercise 3.41. Let U be an open set in the box topology (or product topology) on $\mathbb{R}^\omega$ containing $\mathbf{0} := (0, 0, 0, \dots)$. Then there is a basic open set V such that $\mathbf{0} \in V \subset U$. Since V is a basic open set, $V = \prod_{i\in\mathbb{N}} V_i$ where V_i is open in $\mathbb{R}_{\text{std}}$ for all $i \in \mathbb{N}$ (and in the case of the product topology, $V_i = \mathbb{R}_{\text{std}}$ for all but finitely many $i \in \mathbb{N}$). Since $\mathbf{0} \in V$, $0 \in V_i$ for all $i \in \mathbb{N}$. $V_i \in \mathcal{T}_{\text{std}}$ means there exists an $\varepsilon_i > 0$ such that $(-\varepsilon_i, \varepsilon_i) \subset V_i$ for all $i \in \mathbb{N}$. Define $a_i = \varepsilon_i/2 \in V_i$. Therefore the sequence $(a_i)_{i\in\mathbb{N}}$ is in $\prod_{i\in\mathbb{N}} V_i = V \subset U$. We also have that $a_i = \varepsilon_i/2 > 0$ for all $i \in \mathbb{N}$, which means $(a_i)_{i\in\mathbb{N}} \neq \mathbf{0}$ and $(a_i)_{i\in\mathbb{N}} \in A$. Therefore $(a_i)_{i\in\mathbb{N}} \in (U - \{\mathbf{0}\}) \cap A \neq \emptyset$, and since U was an arbitrary open set, $\mathbf{0}$ is a limit point of A in both the box topology and product topology on $\mathbb{R}^\omega$.

We claim that the sequence $(a_j)_{j\in\mathbb{N}}$ defined by $a_j = (1/j, 1/j, \dots)$ converges to $\mathbf{0}$ in $\mathbb{R}^\omega$ with the product topology. Let U be an open set containing $\mathbf{0}$, and let V be a basic open set such that $\mathbf{0} \in V \subset U$. Then $V = \prod_{i\in\mathbb{N}} V_i$ where V_i is open in $\mathbb{R}_{\text{std}}$ for all $i \in \mathbb{N}$ and $V_i = \mathbb{R}_{\text{std}}$ for all but finitely many $i \in \mathbb{N}$. Define $F = \{i \in \mathbb{N} \mid V_i \neq \mathbb{R}_{\text{std}}\}$. If F is empty, then V is the entire space $\mathbb{R}^\omega$, and since $V \subset U$, U is also the entire space $\mathbb{R}^\omega$. Set $N = 1$. Then for all $n > N$, $a_n \in U = \mathbb{R}^\omega$. Now if F is nonempty, $0 \in V_i$ for all $i \in F$ means there exists an ε_i such that $(-\varepsilon_i, \varepsilon_i) \subset V_i$. Define $s = \min\{\varepsilon_i | i \in F\}$, which exists because F is finite and nonempty. Since $\mathbb{N}$ is not bounded above, there exists an $N \in \mathbb{N}$ such that $N > 1/s$. Therefore if $n > N$, we have that

$$0 < \frac{1}{n} < \frac{1}{N} < s \leq \varepsilon_i \implies \frac{1}{n} \in (0, \varepsilon_i) \subset V_i$$

for all $i \in F$. Also, $1/n \in \mathbb{R} = V_i$ for all $i \in \mathbb{N} - F$, so the sequence $a_n = (1/n, 1/n, \dots) \in$

$V \subset U$ for all $n > N$. In both cases (F empty or nonempty), there exists an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \in U$. Since U was an arbitrary open set containing $\mathbf{0}$, this is true of all such sets, and therefore $(a_j)_{j\in\mathbb{N}}$ is a sequence of points in A that converges to $\mathbf{0}$, as required.

Now let $(a_k)_{k\in\mathbb{N}}$ be a sequence of points in $A \subset \mathbb{R}^\omega$ with the box topology. We claim that this sequence does not converge to $\mathbf{0}$. Elements in $\mathbb{R}^\omega$ are sequences, so we can write each point a_k as $a_k = (a_{k1}, a_{k2}, a_{k3}, \dots)$. We will construct an open set $U \in \mathbb{R}^\omega$ containing $\mathbf{0}$ such that $a_k \notin U$ for all $k \in \mathbb{N}$, no matter how large. Define $U_i \subset \mathbb{R}$ to be the interval $(-1, a_{ii})$ and define $U = \prod_{i\in\mathbb{N}} U_i$. Since $(a_k)_{k\in\mathbb{N}}$ is a sequence of points in A, every coordinate of each point is positive, which means $0 \in U_i$ for all $i \in \mathbb{N}$ and therefore $\mathbf{0} \in U$. Each U_i is an open interval in $\mathbb{R}_{\text{std}}$, which means U is a basic open set in the box topology since it is the product of all the U_i. Note that $a_{nn} \notin U_n = (-1, a_{nn})$, and therefore

$$a_n = (a_{n1}, \dots, a_{nn}, \dots) \notin \prod_{i\in\mathbb{N}} U_i = U$$

for all $n \in \mathbb{N}$. This means that there does not exist an $N \in \mathbb{N}$ such that $i > N$ implies $a_i \in U$, and therefore $(a_k)_{k\in\mathbb{N}}$ does not converge to $\mathbf{0}$. Since $(a_k)_{k\in\mathbb{N}}$ was an arbitrary sequence of points in A, no such sequence converges to $\mathbf{0}$.

Exercise 3.42. The set $2^{\mathbb{N}} = \prod_{n\in\mathbb{N}}\{0, 1\}$ with the box topology has the discrete topology since all singletons are open. Let $\{a_n\}_{n\in\mathbb{N}}$ be a binary sequence (an arbitrary element of $2^{\mathbb{N}}$). Then the singleton containing $\{a_n\}_{n\in\mathbb{N}}$ is $\{\{a_n\}_{n\in\mathbb{N}}\} = \prod_{n\in\mathbb{N}}\{a_n\} = \{a_1\} \times \{a_2\} \times \cdots$, so it is a basic open set. Compare this to $2^{\mathbb{N}}$ with the product topology. Since with this topology, if $p \in 2^{\mathbb{N}}$ is a point in the space and U an open set containing p, then $(U - \{p\}) \cap 2^{\mathbb{N}} \neq \emptyset$ since any open set U contains infinitely many different sequences. Therefore $2^{\mathbb{N}}$ under the product topology has no isolated points since every point is a limit point of the set.

4 Separation Properties: Separating This From That

4.1 Hausdorff, Regular, and Normal Spaces

Theorem 4.1. A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.

Proof. Suppose $(X, \mathcal{T})$ is T_1 and let $x \in X$ be arbitrary. Let $y \in X - \{x\}$ be arbitrary. Then we have that $x \neq y$, so since the space is T_1, we have that there exist open sets U and V such that $x \in U - V$ and $y \in V - U$. Since V is open, $V \subset X$, and since $x \notin V$, $V \subset X - \{x\}$. Therefore $y \in V \subset X - \{x\}$ and by Theorem 2.3, we have that $X - \{x\}$ is open, meaning that the singleton $\{x\}$ is closed. But x was an arbitrary point in this T_1 space, so we have that points are closed in T_1 spaces. Now suppose $(X, \mathcal{T})$ is a topological space in which all points are closed and let $x, y \in X$ be arbitrary points such that $x \neq y$. Since $\{x\}$ and $\{y\}$ are closed, we have that $x \in X - \{y\}$, an open set, and $y \in X - \{x\}$, another open set. Since $x \notin X - \{x\}$ and $y \notin X - \{y\}$, and x and y were arbitrary elements of X, we have that the space $(X, \mathcal{T})$ is T_1. $\square$

Exercise 4.2. Let X be a set with the finite complement topology. Then let $x \in X$ be arbitrary. We have that $X - \{x\}$ is open since its complement, $\{x\}$, is finite. This means $\{x\}$ is closed, and since the point x was arbitrary, we have that all singletons are closed and therefore by Theorem 4.1, all sets with the finite complement topology are T_1.

Exercise 4.3. Let $x, y \in \mathbb{R}$ be points in the space $\mathbb{R}_{\text{std}}$. Then the sets $A = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$ and $B = \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$ are open and disjoint with $x \in A$ and $y \in B$. Therefore $\mathbb{R}_{\text{std}}$ is Hausdorff.

Exercise 4.5. Let A and B be disjoint, closed sets in $\mathbb{R}_{\text{LL}}$. For every $a \in A$ and $b \in B$, set

$$\delta_a = \frac{\inf\{b \in B : b > a\} - a}{2} \quad \text{and} \quad \delta_b = \frac{\inf\{a \in A : a > b\} - b}{2}.$$

Since for all $a \in A$, a is in the basic open set $U_a = [a, a + \delta_a)$, we have that $A \subset U = \bigcup_{a \in A} U_a$. Similarly, we have that $B \subset V = \bigcup_{b \in B} V_b$ where $V_b = [b, b + \delta_b)$ is a basic open set. Suppose for contradiction that there exists and $x \in U \cap V$. Then we have that there exist $\alpha \in A$ and $\beta \in B$ such that $x \in U_\alpha \cap V_\beta$. Without loss of generality, assume that $\alpha < \beta$. Then we have that $x \in U_\alpha = [\alpha, \alpha + \delta_\alpha)$ and $x \in V_\beta = [\beta, \beta + \delta_\beta)$. Then

$$\delta_\alpha = \frac{\inf\{b \in B : b > \alpha\} - \alpha}{2} \implies \alpha + \delta_\alpha = \frac{\alpha + \inf\{b \in B : b > \alpha\}}{2} \leq \inf\{b \in B : b > \alpha\}$$

since $\alpha \leq \inf\{b \in B : b > \alpha\}$. Since $\beta \in B$ and $\beta > \alpha$, $\beta \in \{b \in B : b > \alpha\}$ and therefore we have that

$$x < \alpha + \delta_\alpha \leq \inf\{b \in B : b > \alpha\} \leq \beta \leq x.$$

But $x < x$ is a contradiction and so we have that $U \cap V = \emptyset$ and therefore we have found disjoint, open sets U and V such that $A \subset U$ and $B \subset V$. This means that $\mathbb{R}_{\text{LL}}$ is normal.

Exercise 4.6. (1) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Suppose for contradiction that $\inf\{d(a,p) : a \in A\} = 0$. This means that there exists a sequence $(x_i)_{i\in\mathbb{N}} \subset A$ such that $d(x_i, p) \to 0$. This means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies that $|d(x_i, p) - 0| < \varepsilon$, which is equivalent to the statement that

$$\varepsilon > |\parallel x_i - p \parallel - 0| = \parallel x_i - p \parallel .$$

We now have that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies $\parallel x_i - p \parallel < \varepsilon$, which in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ means that $x_i \to p$. By Theorem 2.30, we have that $p \in \overline{A} = A$ since A is closed. But this is a contradiction, so the assumption that $\inf\{d(a,p) : a \in A\} = 0$ was false. We have either that this is greater than 0 or less than 0, but it cannot be less than 0 since the distance between two points is always nonnegative. Therefore we have that $\inf\{d(a,p) : a \in A\} > 0$.

(2) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Then by (1), there exists an $\varepsilon > 0$ such that $\inf\{d(a,p) : a \in A\} = \varepsilon$. Set $U = B(p, \frac{\varepsilon}{2})$ and $V = \bigcup_{a\in A} B(a, \frac{\varepsilon}{2})$. Then U is a basic open set, and V is the union of basic open sets so both U and V are open. We also have that $p \in U$ and $A \subset V$ since if $a \in A$, $a \in B(a, \frac{\varepsilon}{2}) \subset V$. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means that $d(p, x) < \varepsilon/2$ and $x \in V$ means that there exists an $a \in A$ such that $d(a, x) < \varepsilon/2$. Therefore we have that $d(a,p) \leq d(a,x) + d(x,p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction since $\inf\{d(a,p) : a \in A\} = \varepsilon$. Therefore $U \cap V = \emptyset$ and $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is regular.

(3) The sets $A = \{(x, 0) : x \in \mathbb{R}\}$ and $B = \{(x, y) : x > 0, y \geq \frac{1}{x}\}$. Since for any points $(x_a, y_a) \in A$ and $(x_b, y_b) \in B$, $y_b \geq \frac{1}{x_b} > 0 = y_a$, we have that $A \cap B = \emptyset$. All limit points of A have y-coordinate equal to 0 and are therefore in A, meaning A is closed. Similarly, all limit points in B have y-coordinate equal to $\frac{1}{x_0}$ for some $x_0 > 0$ and are therefore in B, meaning B is closed. Hence A and B are disjoint, closed subsets of $\mathbb{R}^2$. However, $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$. To see this, let $\varepsilon > 0$. Then we have that $a_\varepsilon = (\frac{1}{\varepsilon} + 1, 0) \in A$ and $b_\varepsilon = (\frac{1}{\varepsilon} + 1, \frac{1}{\frac{1}{\varepsilon}+1}) \in B$. Then $d(a_\varepsilon, b_\varepsilon) = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$, so $\inf\{d(a,b) : a \in A \text{ and } b \in B\} < \varepsilon$. However, $\varepsilon > 0$ was an arbitrary, so we have that $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$ since distance is nonnegative.

(4) Let A and B be disjoint, closed sets in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$. For $a_0 \in A$ and $b_0 \in B$, define $\varepsilon_{a_0} = \frac{1}{2}\inf\{d(a_0, b) : b \in B\} > 0$ and $\varepsilon_{b_0} = \frac{1}{2}\inf\{d(a, b_0) : a \in A\} > 0$. Now set $U = \bigcup_{a\in A} B(a, \varepsilon_a)$ and $V = \bigcup_{b\in B} B(b, \varepsilon_b)$. Then since U and V are the unions of open balls in $\mathbb{R}^2$, they are open. Since if $a \in A$, then $a \in B(a, \varepsilon_a) \subset U$ and if $b \in B$, then $b \in B(b, \varepsilon_b) \subset V$, we have that $A \subset U$ and $B \subset V$. To show $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal, it only remains to show

that U and V are disjoint. Suppose for contradiction that there exists $p \in \mathbb{R}^2$ such that $p \in U \cap V$. Then $p \in U$, so there exists an $\alpha \in A$ such that $p \in B(\alpha, \varepsilon_\alpha)$ and similarly, there exists a $\beta \in B$ such that $p \in B(\beta, \varepsilon_\beta)$. Since $\alpha \in A$, $d(\alpha, \beta) \in \{d(a, \beta) : a \in A\}$ and therefore $d(\alpha, \beta) \geq \inf\{d(a, \beta) : a \in A\} = 2\varepsilon_\alpha$. Similarly, $d(\alpha, \beta) \geq 2\varepsilon_\beta$, and so we have that $\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta)$. Since $p \in B(\alpha, \varepsilon_\alpha)$, we have that $d(\alpha, p) < \varepsilon_\alpha$, and since $p \in B(\beta, \varepsilon_\beta)$, we have that $d(p, \beta) < \varepsilon_\beta$. Putting this all together using the triangle inequality, we see that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta) \leq d(\alpha, p) + d(p, \beta) < \varepsilon_\alpha + \varepsilon_\beta.$$

This is a contradiction, so we have that $U \cap V = \emptyset$ and therefore $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal.

Theorem 4.7. (1) A T_2-space (Hausdorff) is a T_1-space.

(2) A T_3-space (regular and T_1) is a Hausdorff space, that is, a T_2-space.

(3) A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof. (1) Let $(X, \mathcal{T})$ be a Hausdorff space and let $x, y \in X$ be distinct, arbitrary points. Then there exist disjoint, open sets U and V such that $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$, we have that $x \notin V$ and $y \notin U$, so $(X, \mathcal{T})$ is a T_1-space. $\square$

Proof. (2) Let $(X, \mathcal{T})$ be a T_3-space and let $x, y \in X$ be distinct, arbitrary points. Since this space is T_1, by Theorem 4.1 we have that $\{y\}$ is closed. Since this space is regular, we have that there exist disjoint, open sets such that U and V such that $x \in U$ and $\{y\} \subset V$. But $\{y\} \subset V$ means $y \in V$, so we have found disjoint, open sets separating the arbitrary points x and y, so $(X, \mathcal{T})$ is Hausdorff. $\square$

Proof. (3) Let $(X, \mathcal{T})$ be a T_4-space, let $x \in X$ be arbitrary, and let A be a closed set with $x \notin A$. Since this space is T_1, by Theorem 4.1 we have that $\{x\}$ is closed. Since this space is normal, there exist disjoint, open sets U and V such that $\{x\} \subset U$ and $A \subset V$. But $\{x\} \subset U$ means $x \in U$, so we have found disjoint, open sets separating the arbitrary point x from the arbitrary closed set A, so $(X, \mathcal{T})$ is T_3 since it is normal and T_1. $\square$

Theorem 4.8. A topological space is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a regular topological space and let U be an open set containing the point p. Then we have that $X - U$ is closed and since $p \in U$, $p \notin X - U$. Since this space is regular, there exist disjoint open sets V and W such that $p \in V$ and $X - U \subset W$. Therefore we have that $X - W \subset U$ since $X - U \subset W$, and that $X - W$ is closed since W is open. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$) and therefore $x \in X - W$. Since x was arbitrary, we have that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$p \in V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since U and p were arbitary, there exists an open set V containing p such that $\overline{V} \subset U$ for all $p \in X$ and open sets U containing p.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all $p \in X$ and $W \in \mathcal{T}$ with $p \in W$, there exists an open set U such that $p \in U$ and $\overline{U} \subset W$. Let $p \in X$ be arbitrary and let A be a closed subset of X such that $p \notin A$. Then we have that $p \in X - A$, which is open, and therefore there exists an open set U such that $p \in U$ and $\overline{U} \subset X - A$, which implies that $A \subset V$ where V is the open set $X - \overline{U}$. Let $x \in U$ be arbitary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since x was arbitary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets U and V such that $p \in U$ and $A \subset V$, so $(X, \mathcal{T})$ is regular, as required. □

Theorem 4.9. A topological space is normal if and only if for each closed set A in $(X, \mathcal{T})$ and open set U containing A there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space, let A be a closed set, and let U be an open set such that $A \subset U$. Then $X - U$ closed and $A \cap (X - U) = \emptyset$, so since this is a normal space, there exist disjoint open sets V and W such that $A \subset V$ and $X - U \subset W$. Therefore $X - W \subset U$ is a closed set. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$), so $x \in X - W$, which means that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$A \subset V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since A and U were arbitrary, there exists an open set V with $A \subset V$ and $\overline{V} \subset U$ for all open sets U containing closed sets A.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all closed sets A and open sets W with $A \subset W$, there exists an open set U such that $A \subset U$ and $\overline{U} \subset W$. Let A and B be arbitrary disjoint sets. Then we have that $A \subset X - B$, and since $X - B$ is open, there exists an open set U such that $A \subset U$ and $\overline{U} \subset X - B$, which implies that

$B \subset V$ where V is the open set $X - \overline{U}$. Let $x \in U$ be arbitrary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since x was arbitrary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so $(X, \mathcal{T})$ is normal, as required. □

Theorem 4.10. A topological space is normal if and only if for each pair of disjoint closed sets A and B, there exist open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let A and B be disjoint closed sets. Then since $X - B$ is open and $A \subset X - B$, by Theorem 4.9 there exists an open set U such that $A \subset U$ and $\overline{U} \subset X - B$. Therefore $\overline{U} \cap B = \emptyset$, so there exists an open set V such that $B \subset V$ and $\overline{V} \subset X - \overline{U}$, which means that $\overline{U} \cap \overline{V} = \emptyset$, as required.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for all disjoint closed sets A and B, there exist open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$. But if $\overline{U}$ and $\bar{(V)}$ are disjoint, so are U and V, and therefore this space is normal. □

Theorem 4.11 (The Incredible Shrinking Theorem). A topological space is normal if and only if for each pair of open sets U and V such that $U \cup V = X$, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a normal topological space and let U and V be open sets with $U \cup V = X$. Then $X - U$ and $X - V$ are closed sets and

$$(X - U) \cap (X - V) = X - (U \cup V) = X - X = \emptyset.$$

Since these sets are closed and disjoint, by Theorem 4.10, there exist open sets W_1 and W_2 such that $X - U \subset W_1$, $X - V \subset W_2$, and $\overline{W_1} \cap \overline{W_2} = \emptyset$. Define $U' = X - \overline{W_1}$ and $V' = X - \overline{W_2}$. Then

$$U' \cup V' = (X - \overline{W_1}) \cup (X - \overline{W_2}) = X - (\overline{W_1} \cap \overline{W_2}) = X - \emptyset = X.$$

By Lemma 2.28, $\overline{W_1} = X - (X - W_1)^\circ$, so

$$U' = X - \overline{W_1} = X - (X - (X - W_1)^\circ) = (X - W_1)^\circ.$$

Since $(X - W_1)^\circ \subset X - W_1$ and $X - U \subset W_1$, we have that

$$\overline{U'} = \overline{(X - W_1)^\circ} \subset \overline{X - W_1} = X - W_1 \subset U$$

where we have used the fact that $X - W_1$ is closed. Similarly, $\overline{V'} \subset V$, and so we have shown the forward direction of the implication.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a topological space with the property that for any open sets U and V with $U \cup V = X$, there exist open sets U' and V' such that $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$. Let A and B be arbitrary closed sets. Then $X - A$ and $X - B$ are open sets such that

$$(X - A) \cup (X - B) = X - (A \cap B) = X - \emptyset = X,$$

so there exist open sets U' and V' such that $\overline{U'} \subset X - A$, $\overline{V'} \subset X - B$, and $U' \cup V' = X$. Define open sets $U = X - \overline{U'}$ and $V = X - \overline{V'}$. Then since $\overline{U'} \subset X - A$, $A \subset X - \overline{U'} = U$, and similarly, $B \subset V$. Then using Theorem 2.22(2), we have that

$$U \cap V = (X - \overline{U'}) \cap (X - \overline{V'}) = X - (\overline{U'} \cup \overline{V'}) = X - \overline{U' \cup V'} = X - \overline{X} = \emptyset.$$

Therefore we have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so this space is normal. □

Exercise 4.12. (1) The Double Headed Snake, (2) $\mathbb{R}_{\text{har}}$, and (3) ... is trickier.

4.2 Separation Properties and Products

Theorem 4.16. The product of Hausdorff spaces is Hausdorff.

Proof. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ be a collection of Hausdorff spaces and let f and g be distinct elements of the product space $\prod_{\alpha \in \lambda} X_\alpha$. Since f and g are distinct, they differ at at least one coordinate, so there exists a $\beta \in \lambda$ such that $\pi_\beta(f) \neq \pi_\beta(g)$. Then in the Hausdorff space $(X_\beta, \mathcal{T}_\beta)$, the elements $\pi_\beta(f)$ and $\pi_\beta(g)$ are distinct, so there exist disjoint open sets U_β and V_β such that $\pi_\beta(f) \in U_\beta$ and $\pi_\beta(g) \in V_\beta$. Then we have that $f \in \pi_\beta^{-1}(U_\beta)$ and $g \in \pi_\beta^{-1}(V_\beta)$, and since U_β and V_β are disjoint, so are $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$, and they are subbasic open sets. Thus we have found disjoint open sets in $\prod_{\alpha \in \lambda} X_\alpha$ separating f and g, so the product of Hausdorff spaces is Hausdorff. □

Theorem 4.17. The product of regular spaces is regular.

Proof. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ be a collection of regular spaces and let p be an element of the space $\prod_{\alpha \in \lambda} X_\alpha$ under the product topology. Let U^* be an open set containing p. Recall that the product topology is defined to be the topology generated by the subbasis $\mathscr{S}$ of sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in $(X_\beta, \mathcal{T}_\beta)$. By Theorem 3.15, there exists a finite collection of subsets $\{W_i\}_{i=1}^n$ such that each $W_i \in \mathscr{S}$ and $p \in \bigcap_{i=1}^n W_i \subset U^*$. Since $W_i \in \mathscr{S}$, there exists a $\beta_i \in \lambda$ such that $W_i = \pi_{\beta_i}^{-1}(W_{\beta_i})$ where W_{β_i} is open in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$. Then $p \in \pi_{\beta_i}^{-1}(W_{\beta_i})$ for all $i = 1, \ldots, n$, which means that $\pi_{\beta_i}(p) \in W_{\beta_i}$. Since $(X_{\beta_i}, \mathcal{T}_{\beta_i})$ is regular, there exists an open V_{β_i} such that $\pi_{\beta_i}(p) \in V_{\beta_i}$ and $\overline{V_{\beta_i}} \subset W_{\beta_i}$. Define V_i to be the set $V_i = \pi_{\beta_i}^{-1}(V_{\beta_i})$, a subbasic open set. Since $\pi_{\beta_i}(p) \in V_{\beta_i}$, $p \in V_i$ for all $i = 1, \ldots, n$. Let $g \in \overline{\pi_{\beta_i}^{-1}(V_{\beta_i})}$ be arbitrary. There are two cases—either $g \in \pi_{\beta_i}^{-1}(V_{\beta_i})$ or g is a limit point of $\pi_{\beta_i}^{-1}(V_{\beta_i})$. In the first case, $\pi_{\beta_i}(g) \in V_{\beta_i} \subset \overline{V_{\beta_i}}$ and therefore $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. In the second case, let U be an arbitrary open set in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$ such that $\pi_{\beta_i}(g) \in U$. Suppose for contradiction that $(U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i} = \emptyset$. Then we have that

$$\begin{aligned}\emptyset &= \pi_{\beta_i}^{-1}\left((U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i}\right) = \pi_{\beta_i}^{-1}(U - \{\pi_{\beta_i}(g)\}) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}) \\ &= \left(\pi_{\beta_i}^{-1}(U) - \pi_{\beta_i}^{-1}(\{\pi_{\beta_i}(g)\})\right) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}) = \left(\pi_{\beta_i}^{-1}(U) - \{g\}\right) \cap \pi_{\beta_i}^{-1}(V_{\beta_i}).\end{aligned}$$

But this is a contradiction since $\pi_{\beta_i}^{-1}(U)$ is an open set containing g and g is a limit point in this case. Therefore for all open U in $(X_{\beta_i}, \mathcal{T}_{\beta_i})$, $(U - \{\pi_{\beta_i}(g)\}) \cap V_{\beta_i} \neq \emptyset$, meaning that $\pi_{\beta_i}(g)$ is a limit point of V_{β_i}. Therefore $\pi_{\beta_i}(g) \in \overline{V_{\beta_i}}$, which implies that $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. Since g was an arbitrary element of $\overline{V_i} = \overline{\pi_{\beta_i}^{-1}(V_{\beta_i})}$ and in both cases we have that $g \in \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$, so $\overline{V_i} \subset \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}})$. Since for each $i = 1, \ldots, n$, $\overline{V}_{\beta_i} \subset W_{\beta_i}$, we have that

$$\pi_{\beta_i}^{-1}(\overline{V_{\beta_i}}) = \left\{ f \in \prod_{\alpha \in \lambda} X_\alpha : \pi_{\beta_i}(f) \in \overline{V_{\beta_i}} \right\} \subset \left\{ f \in \prod_{\alpha \in \lambda} X_\alpha : \pi_{\beta_i}(f) \in W_{\beta_i} \right\} = \pi_{\beta_i}^{-1}(W_{\beta_i}).$$

Therefore for all $i = 1, \ldots, n$, we have that $\overline{V_i} \subset \pi_{\beta_i}^{-1}(\overline{V_{\beta_i}}) \subset \pi_{\beta_i}^{-1}(W_{\beta_i}) = W_i$, and $p \in V_i$. Define V^* to be the set $V^* = \bigcap_{i=1}^n V_i$, which is the finite intersection of subbasic open sets and is therefore open. Finally, we have that $p \in V^*$ and

$$\overline{V^*} = \overline{\bigcap_{i=1}^n V_i} \subset \bigcap_{i=1}^n \overline{V_i} \subset \bigcap_{i=1}^n W_i \subset U^*.$$

Therefore the product topology on $\prod_{\alpha \in \lambda} X_\alpha$ is regular, and since this was the product of an arbitrary collection of regular spaces, all products of regular spaces are regular. $\square$

4.3 A Question of Heredity

Theorem 4.19. Every Hausdorff space is hereditarily Hausdorff.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space, and let $Y \subset X$ be arbitrary. Let $a, b \in Y$ be arbitrary distinct points. Since $(X, \mathcal{T})$ is Hausdorff, there exist disjoint open sets U and V such that $a \in U$ and $b \in V$. Therefore $a \in U \cap Y$ and $b \in V \cap Y$, and the sets $U \cap Y$ and $V \cap Y$ are open in the subspace by the definition of $\mathcal{T}_Y$ and disjoint since U and V are disjoint. Therefore $\mathcal{T}_Y$ is Hausdorff. $\square$

Theorem 4.20. Every regular space is hereditarily regular.

Proof. Let $(X, \mathcal{T})$ be a regular space, and let $Y \subset X$ be arbitrary. Let $p \in Y$ be an arbitrary point and let A be an arbitrary closed set in $(Y, \mathcal{T}_Y)$ with $p \notin A$. Then by Theorem 3.28, there exists a set D that is closed in $(X, \mathcal{T})$ such that $A = D \cap Y$. Since $p \in Y$ but $p \notin A \cap Y$, we have that $p \notin D$, and so by the regularity of $(X, \mathcal{T})$, there exist disjoint open sets U' and V' such that $p \in U'$, $D \subset V'$, and $U' \cap V' = \emptyset$. Define the sets U and V to be $U = U' \cap Y$ and $V = V' \cap Y$ so that they are open in $(Y, \mathcal{T}_Y)$. We have that $U \cap V \subset U' \cap V' = \emptyset$. Since $p \in U'$ and $p \in Y$, $p \in U$, and similarly, $D \subset V'$ and $D \subset Y$, so $D \subset V$. Since p, A, $(Y, \mathcal{T}_Y)$, and $(X, \mathcal{T})$ were arbitrary, all subspaces of regular spaces are regular, meaning that regular spaces are hereditarily regular. $\square$

Theorem 4.23. If Y is closed in a normal space $(X, \mathcal{T})$, then $(Y, \mathcal{T}_Y)$ is normal.

Proof. Let $(X, \mathcal{T})$ be normal and let Y be closed in $(X, \mathcal{T})$. Let A and B be disjoint closed sets in $(Y, \mathcal{T}_Y)$. Then we have that there exist sets C and D that are closed in $(X, \mathcal{T})$ such that $A = C \cap Y$ and $B = D \cap Y$. Since Y is closed in $(X, \mathcal{T})$, A and B are each the intersections of closed sets and are therefore themselves closed in $(X, \mathcal{T})$. By the normality of $(X, \mathcal{T})$, there exist disjoint open sets U' and V' such that $A \subset U'$ and $B \subset V'$. Then since $A, B \subset Y$, we have that $A \subset U := U' \cap Y$ and $B \subset V := V' \cap Y$. By the definition of $\mathcal{T}_Y$, U and V are open in $(Y, \mathcal{T}_Y)$ and we have that $U \cap V \subset U' \cap V' = \emptyset$, so we have found disjoint open sets separating A and B, meaning that $(Y, \mathcal{T}_Y)$ is normal. $\square$

Lemma 4.25 (Corollary 3.29). Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $\overline{C} \cap Y = C$ where the closure is taken in $(X, \mathcal{T})$.

Proof. ($\Longrightarrow$) Let C be closed in $(Y, \mathcal{T}_Y)$. Then by Theorem 3.28, there exists a D closed in $(X, \mathcal{T})$ such that $C = D \cap Y$. Note that $C \subset Y$, so $C \subset \overline{C} \cap Y$, so to show equality, it suffices to show that $\overline{C} \cap Y \subset C$. We have:

$$\overline{C} \cap Y = \overline{D \cap Y} \cap Y \subset \overline{D} \cap \overline{Y} \cap Y = \overline{D} \cap Y = D \cap Y = C$$

where $\overline{D} = D$ since D is closed in $(X, \mathcal{T})$.

($\Longleftarrow$) Let C be a subset of Y such that $\overline{C} \cap Y = C$, so there is a closed set $D = \overline{C}$ such that $C = D \cap Y$, and therefore C is closed in $(Y, \mathcal{T}_Y)$ by Theorem 3.28. $\square$

Theorem 4.25. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$ and let A and B be disjoint closed sets in $(Y, \mathcal{T}_Y)$. Then we have that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ where the closures are taken in $(X, \mathcal{T})$.

Proof. Let A and B be disjoint closed sets in $(Y, \mathcal{T})$. Then by the Lemma, $\overline{A} \cap B \subset \overline{A} \cap Y = A$ and therefore $\overline{A} \cap B = (\overline{A} \cap B) \cap B \subset A \cap B = \emptyset$. Similarly, $A \cap \overline{B} \subset Y \cap \overline{B} = B$ since B is closed in $(Y, \mathcal{T}_Y)$, and $A \cap \overline{B} = A \cap (A \cap \overline{B}) \subset A \cap B = \emptyset$. Hence $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. $\square$

Theorem 4.26. A topological space $(X, \mathcal{T})$ is completely normal if and only if it is hereditarily normal.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a completely normal space, let $(Y, \mathcal{T}_Y)$ be a subspace, and let A and B be closed sets in $(Y, \mathcal{T}_Y)$. By Theorem 4.25, A and B are separated in $(X, \mathcal{T}_Y)$, and so there exist disjoint open sets U' and V' such that $A \subset U'$ and $B \subset V'$. Then we also have that $A \subset U := U' \cap Y$, $B \subset V := V' \cap Y$, and $U \cap V \subset U' \cap V' = \emptyset$. By definition the definition of $\mathcal{T}_Y$, U and V are open in $(Y, \mathcal{T}_Y)$, and therefore $(X, \mathcal{T})$ is hereditarily normal since $(Y, \mathcal{T}_Y)$ was an arbitrary subspace.

($\Longleftarrow$) Let $(X, \mathcal{T})$ be a hereditarily normal space and let A and B be separated sets. Define the set $Y = X - (\overline{A} \cap \overline{B})$, an open set in $(X, \mathcal{T})$. Note that

$$\overline{A} \cap Y = \left(\bigcup_{D \supset A, D \in \mathcal{C}} D \right) \cap Y = \bigcup_{D \supset A, D \in \mathcal{C}} (D \cap Y) = \mathrm{Cl}_Y(A)$$

since $\bigcup_{D \supset A, D \in \mathcal{C}} (D \cap Y)$ is the intersection of all closed sets in $(Y, \mathcal{T}_Y)$ containing A ($\mathcal{C}$ here is the set of all closed sets in $(X, \mathcal{T})$). Similarly, we have that $\mathrm{Cl}_Y(B) = \overline{B} \cap Y$. Therefore

$$\mathrm{Cl}_Y(A) \cap \mathrm{Cl}_Y(B) = (\overline{A} \cap \overline{B}) \cap Y = (\overline{A} \cap \overline{B}) \cap (X - (\overline{A} \cap \overline{B})) = \emptyset.$$

Since $(X, \mathcal{T})$ is hereditarily normal and $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_Y(B)$ are disjoint closed sets in the subspace $(Y, \mathcal{T}_Y)$, there exist disjoint open sets U and V such that $\mathrm{Cl}_Y(A) \subset U$ and $\mathrm{Cl}_Y(B) \subset V$. These sets are open in $(Y, \mathcal{T}_Y)$, so there exist open sets in U' and V' in $(X, \mathcal{T})$ such that $U = U' \cap Y$ and $V = V' \cap Y$. Then since Y is open in $(X, \mathcal{T})$, U and V are the (finite) intersections of open sets and are therefore open. We have that $A \subset Y$, $B \subset Y$, $A \subset \overline{A}$, and $B \subset \overline{B}$, so therefore $A \subset \overline{A} \cap Y = \mathrm{Cl}_Y(A) \subset U$ and $B \subset \overline{B} \cap Y = \mathrm{Cl}_Y(B) \subset V$. We have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, and since A and B were arbitrary separated sets, $(X, \mathcal{T})$ is completely normal. $\square$

Exercise 4.27. (2) Consider the set $Y = \left\{\left(x, \frac{1}{2}\right) : x \in [0,1]\right\}$ as a subspace of the lexicographically ordered square $(X, \mathcal{T})$. Let $p = (x_0, y_0) \in Y$ be arbitrary. Note that in the lexicographically ordered square, the set $U_p = \{(x_0, y) : 0 < y < 1\}$ is open since the lexicographically ordered square is an order topology and $(x_0, 0) < (x_0, 1)$ Therefore U_p is open and contains the point p. Therefore $U_p \cap Y = \{p\}$ is open in the subspace $(Y, \mathcal{T}_Y)$. But this means that $\mathcal{T}_Y$ is not the order topology on Y, since by Theorem 4.15, order topologies are T_1, meaning that points are closed. This differs from $\mathcal{T}_Y$ since in $(Y, \mathcal{T}_Y)$, points are open, making this is the discrete topology on Y.

(3) Consider the set $Y = \{(x, 1) : x \in [0, 1)\}$ as a subspace of the lexicographically ordered square $(X, \mathcal{T})$. $\mathcal{T}$ has a basis consisting of sets of the form $\{x \in X : x < (x_0, y_0)\}$, $\{x \in X : (x_0, y_0) < x\}$, $\{x \in X : (x_1, y_1) < x < (x_2, y_2)\}$, so by Theorem 3.30, the subspace $(Y, \mathcal{T}_Y)$ has a basis of sets of the form $\{(x, 1) : 0 \leq x < x_0\}$, $\{(x, 1) : x_0 \leq x < 1\}$ (if $y_0 < 1$), $\{(x, 1) : x_0 < x < 1\}$ (if $y_0 = 1$), $\{(x, 1) : x_1 \leq x < x_2\}$ (if $y_0 < 1$), and $\{(x, 1) : x_1 < x < x_2\}$ (if $y_0 = 1$). Sets of forms 3 and 5 form a basis for the standard topology on Y, and sets of forms 1, 2, and 4 form a basis for the lower limit topology. Therefore $\mathcal{T}_Y$ is the union of the standard topology and the lower limit topology, which is just the lower limit topology since it is strictly finer than the standard topology. But the lower limit topology is not an order topology, so we have another example of a non-order topology subspace of an order topology.

4.4 The Normality Lemma

Theorem 4.29 (The Normality Lemma). Let A and B be sets in a topological space X and let $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ be collections of open sets such that

$$1) \quad A \subset \bigcup_{i \in \mathbb{N}} U_i$$

$$2) \quad B \subset \bigcup_{i \in \mathbb{N}} V_i$$

$$3) \quad \overline{U_i} \cap B = \emptyset \text{ and } \overline{V_i} \cap A = \emptyset \text{ for all } i \in \mathbb{N}.$$

Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Proof. To construct sets U and V, we build them up out of the parts of the $\{U_i\}$ and $\{V_i\}$ that don't overlap. Begin with the first set U_1 as the first building block of U and the first set V_1 excluding anything overlapping with U_1 as the first building block of V. Then add all the elements in U_2 that aren't in what we have so far of V into U, and add all the elements in V_2 that aren't in what we have so far of U into V. Continue doing this to construct the sets U and V.

More formally, we define two collections of sets $\{U_i'\}_{i\in\mathbb{N}}$ and $\{V_i'\}_{i\in\mathbb{N}}$ recursively. Let $U_1' = U_1$ and let $V_1' = V_1 - \overline{U_1'}$. Then for $n \in \mathbb{N}$, define $U_{n+1}' = (U_n' \cup U_{n+1}) - \overline{V_n'}$, and define $V_{n+1}' = (V_n' \cup V_{n+1}) - \overline{U_{n+1}'}$. We claim that the sets

$$U = \bigcup_{i\in\mathbb{N}} U_i' \quad \text{and} \quad V = \bigcup_{i\in\mathbb{N}} V_i'$$

are open such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

We first argue by induction on n that U_n' and V_n' are open for all $n \in \mathbb{N}$. For the base case $n = 1$, $U_1' = U_1$ is open by assumption, and $V_1' = V_1 - \overline{U_1'}$ is an open set minus a closed set, so it is open by Theorem 2.15. Now suppose there is a $k \in \mathbb{N}$ such that U_k' and V_k' are open. Then $U_{k+1}' = (U_k' \cup U_{k+1}) - \overline{V_k'}$, which is the union of two open sets minus a closed set, so it is open. Similarly, $V_{k+1}' = (V_k' \cup V_{k+1}) - \overline{U_{k+1}'}$ is open. This completes the induction, and so all U_i' and V_i' are open, and since unions of open sets are open, U and V are also open.

Now we claim that if $a \in A$, $a \notin \overline{V_n'}$, and if $b \in B$, $b \notin \overline{U_n'}$ for all $n \in \mathbb{N}$ again by induction on n. For the base case $n = 1$, $a \notin \overline{V_1}$ by the hypotheses of the Theorem, so since $V_1' = V_1 - \overline{U_1'} \subset V_1$ implies that $\overline{V_1'} \subset \overline{V_1}$, we have that $a \notin \overline{V_1'}$. We also have that $b \notin \overline{U_1} = \overline{U_1'}$. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $a \notin \overline{V_k'}$ and $b \notin \overline{U_k'}$. Then since $a \notin \overline{V_{k+1}}$, $a \notin \overline{V_k'} \cup \overline{V_{k+1}} = \overline{V_k' \cup V_{k+1}}$, and since $V_{k+1}' = (V_k' \cup V_{k+1}) - \overline{U_{k+1}'} \subset V_k' \cup V_{k+1}$, we have that $\overline{V_{k+1}'} \subset \overline{V_k' \cup V_{k+1}}$, and therefore $a \notin \overline{V_{k+1}'}$. Similarly, $b \notin \overline{U_{k+1}'}$. Now since $a \in A \subset \bigcup_{i\in\mathbb{N}} U_i$, there exists a $j \in \mathbb{N}$ such that $a \in U_j$. If $j = 1$, then $a \in U_1 = U_1'$. If $j \neq 1$, then $a \in U_j \subset U_{j-1}' \cup U_j$, and since $a \notin \overline{V_n'}$ for all $n \in \mathbb{N}$, we have that $a \in (U_{j-1}' \cup U_j) - \overline{V_{j-1}} = U_j' \subset U$, so $A \subset U$. A similar argument shows that $B \subset V$.

Now we show that $U \cap V = \emptyset$. Note that $V_1' = V_1 - U_1'$ so $V_1' \cap U_1' = \emptyset$. Also if $n > 1$, then $V_n' = (V_{n-1}' \cup V_n) - \overline{U_n'}$, so $U_n' \cap V_n' = \emptyset$ for all $n \in \mathbb{N}$. Now let $m \in \mathbb{N}$ be fixed. We argue by induction on n that $U_m' \subset U_n'$ for all $n \geq m$. For the base case $n = m$, $U_m' \subset U_m' = U_n'$. Suppose there exists a $k \geq m$ such that $U_m' \subset U_k'$. Then $U_{k+1}' = (U_k' \cup U_{k+1}) - \overline{V_k'}$. If $y \in U_m' \subset U_k'$, then $y \notin V_k'$ since $U_k' \cap V_k' = \emptyset$. But since U_k' is an open set containing

y, y cannot be a limit point of V'_k by Theorem 2.9, so $y \notin \overline{V'_k}$, and therefore $y \in U'_{k+1}$. We have shown that $U'_m \subset U'_{k+1}$ and this completes the induction. We also have that $V'_m \subset V'_n$ for all $n \geq m$. For the base case $n = m$, $V'_m \subset V'_m = V'_n$. Suppose there exists a $k \geq m$ such that $V'_m \subset U'_k$. Then $V'_{k+1} = (V'_k \cup V_{k+1}) - \overline{U'_{k+1}}$. If $y \in V'_m \subset V'_k \subset \overline{V'_k}$, then $y \notin (U'_k \cup U_{k+1}) - \overline{V'_k} = U'_{k+1}$. Since V'_k is an open set containing y and $V'_k \cap U'_{k+1} = \emptyset$, y cannot be a limit point of U'_{k+1}, again by Theorem 2.9. Therefore $y \in (V'_k \cup V_{k+1}) - \overline{U'_{k+1}} = V'_{k+1}$, so $V'_m \subset V'_{k+1}$. Now if there exists an $x \in U \cap V$, then there exist $j, k \in \mathbb{N}$ such that $x \in U'_j$ and $x \in V'_k$. If $j \leq k$, then $x \in U'_j \cap V'_k \subset U'_k \cap V'_k = \emptyset$, a contradiction. On the other hand, if $j \geq k$, then $x \in U'_j \cap V'_k \subset U'_j \cap V'_j = \emptyset$, again a contradiction. Therefore there is no x such that $x \in U \cap V$, so $U \cap V = \emptyset$ and we have shown that there exist disjoint open sets U and V such that $A \in U$ and $B \in V$. $\square$

Theorem 4.31. Let X be regular and countable. Then X is normal.

Proof. Let A and B be disjoint closed subsets of X. Since X is countable, A and B are also countable. Then there exist surjective sequences $(a_i)_{i \in \mathbb{N}}$ with $a_i \in A$ and $(b_i)_{i \in \mathbb{N}}$ with $b_i \in B$. Since X is regular, for each $a_i \in A \subset X - B$, there exists an open U_i such that $a_i \in U_i$ and $\overline{U_i} \subset X - B$ by Theorem 4.8. That is, $\overline{U_i} \cap B = \emptyset$. Similarly, for each $b_i \in B$, there exists an open V_i such that $b_i \in V_i$ and $\overline{V_i} \cap A = \emptyset$. This means condition (3) of the Normality Lemma is satisfied, and since there is a U_i for each $a_i \in A$ such that $a_i \in U_i$, we have that $A \subset \bigcup_{i \in \mathbb{N}} U_i$. Similarly, $B \subset \bigcup_{i \in \mathbb{N}} V_i$, so conditions (1) and (2) are satisfied as well, meaning there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since A and B were arbitrary disjoint closed sets, X is normal. $\square$

Theorem 4.32. Let X be regular with a countable basis $\mathcal{B}$. Then X is normal.

Proof. Let A and B be disjoint closed subsets of X. Since X is regular, for all $a \in A$, there exists an open set U_a such that $a \in U_a$ and $\overline{U_a} \subset X - B$. Since X has a basis $\mathcal{B}$, there exists a $U'_a \in \mathcal{B}$ such that $a \in U'_a \subset U_a$. Therefore $\overline{U'_a} \subset \overline{U_a} \subset X - B$, meaning $\overline{U'_a} \cap B = \emptyset$. Since there exists such a U'_a for all $a \in A$, $A \subset \bigcup_{i \in \mathbb{N}} U'_a$. Then because $\mathcal{B}$ is countable, there are only a countable number of distinct U'_a, although A may be uncountable. Therefore there is a surjection $f : \mathbb{N} \to \{U'_a \mid a \in A\}$, so we have that

$$A \subset \bigcup_{a \in A} U'_a = \bigcup_{i \in \mathbb{N}} f(i)$$

where $\overline{f(i)} \cap B = \emptyset$. Similarly, for each $b \in B$ there exists a $V'_b \in \mathcal{B}$ defined similarly such that $\overline{V'_b} \cap B$, and there exists a surjective $g : \mathbb{N} \to \{V'_b \mid b \in B\}$ such that $B \subset \bigcup_{i \in \mathbb{N}} g(i)$ and

$\overline{g(i)} \cap B = \emptyset$. Hence the conditions of the Normality Lemma are satisfied and so there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$, so X is normal. $\square$

5 Countable Features of Spaces: Size Restrictions

5.1 Separable Spaces, An Unfortunate Name

Exercise 5.1. A subset $A \subset X$ is dense in X if and only if every non-empty open set contains a point of A.

Proof. ($\Longrightarrow$) Suppose A is dense in X and let U be an open set. Then since $U \subset X = \overline{A}$, if $p \in U$, then either $p \in A$ and we are done or p is limit point of A. If this is the case, then $p \in U$ so $(U - \{p\}) \cap A \neq \emptyset$, which means U contains a point in A, as required.

($\Longleftarrow$) Suppose all open sets U contain a point in A. Then certainly $\overline{A} \subset X$, so to A is dense in X it suffices to show that $X \subset \overline{A}$. Let $x \in X$. Then either $x \in A \subset \overline{A}$ and we are done, or $x \notin A$. If $x \notin A$, then we claim that x is a limit point of A. Let U be an open set containing x. Since $x \notin A$, $x \notin U \cap A$, which means $(U - \{x\}) \cap A = U \cap A \neq \emptyset$. Therefore x is a limit point of A, and so in all cases, $x \in \overline{A}$, meaning $X \subset \overline{A}$ and therefore A is dense in X. $\square$

Exercise 5.2. $\mathbb{R}_{\text{std}}$ is separable, but $\mathbb{R}$ with the discrete topology is not.

Proof. The set $\mathbb{Q}$ of rational numbers is countable, and we claim it is dense in $\mathbb{R}_{\text{std}}$. Let U be a nonempty basic open set $\mathcal{B}$, the basis of open intervals in $\mathbb{R}$. Then there exist $a, b \in \mathbb{R}$ such that $U = (a, b)$, and since U is nonempty, $|a - b| > 0$. Since $\mathbb{N}$ is unbounded, there exists an $n \in \mathbb{N}$ such that $n > \frac{1}{|a-b|} > 0$. Then we have that

$$|na - nb| = n|a - b| > 1,$$

so there exists an integer $m \in \mathbb{Z}$ such that $na < m < nb$, and therefore $a < m/n < b$ $(n > 0)$. Therefore since $m/n \in \mathbb{Q}$ and the basic open set U was arbitrary, there exists an element of $\mathbb{Q}$ in all nonempty basic open sets. Because all open sets in $\mathbb{R}_{\text{std}}$ are unions of basic open sets, there exists an element of $\mathbb{Q}$ in all nonempty open sets, meaning $\mathbb{Q}$ is dense in $\mathbb{R}_{\text{std}}$ by Exercise 5.1, and therefore $\mathbb{R}_{\text{std}}$ is separable.

However, $\mathbb{R}$ is not separable with the discrete topology. Let C be a countable subset of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable, there exists an $\alpha \in \mathbb{R} - C$, and since $\{\alpha\}$ is open in the discrete topology, $\{\alpha\}$ is a nonempty open set that does not contain a point of C, and since such a set exists, C is not dense in $\mathbb{R}$ with the discrete topology, so $\mathbb{R}$ is not separable in this case. □

Exercise 5.4. Consider the x-axis X as a subset of $\mathbb{H}_{\text{bub}}$. Then the set $\{(\alpha,0)\} = X \cap (\{(\alpha,0)\} \cup B((\alpha,1),1)$ is open in the relative topology because $(\{(\alpha,0)\} \cup B((\alpha,1),1)$ is a basic open set in $\mathbb{H}_{\text{bub}}$. But $\{(\alpha,0)\}$ was an arbitrary singleton in X and is open, so the relative topology on X is the discrete topology, which means X is not separable even though it is a subspace of the separable space $\mathbb{H}_{\text{bub}}$.

Theorem 5.5. If X and Y are separable spaces, then $X \times Y$ is separable.

Proof. Since X and Y are separable, there exist countable sets A and B such that A is dense in X and B is dense in Y. Then $A \times B \subset X \times Y$ and is countable, so it remains to show that $A \times B$ is dense in $X \times Y$. Let W be an open set in $X \times Y$. Then $W = U \times V$ for some open U in X and open V in Y. Since A is dense in X, there exists an $a \in A \cap U$ and since B is dense in Y, there exists a $b \in B \cap V$. Therefore $(a,b) \in (A \times B) \cap W$, and so every open set in $X \times Y$ contains an element of the countable subset $A \times B$, which means $A \times B$ is dense in $X \times Y$ and therefore $X \times Y$ is separable. □

5.2 2nd Countable Spaces

Theorem 5.9. Let X be a 2^{nd} countable space. Then X is separable.

Proof. Let X be 2^{nd} countable. Then there exists a countable basis $\mathcal{B}$ of nonempty sets V_i, $i \in \mathbb{N}$. Since V_i is nonempty, there exists an $a_i \in V_i$ for all $i \in \mathbb{N}$. Define the set A to be the union $A = \bigcup_{i \in \mathbb{N}} \{a_i\}$. Then A is a countable subset of X. Suppose U is a nonempty open set in X. Then there exists a point $p \in U$, and since $\mathcal{B}$ is a basis, there exists a basic open set V_j for some $j \in \mathbb{N}$ such that $p \in V_j \subset U$. By construction of A, there exists a point a_j such that $a_j \in V_j \subset U$. Because U was an arbitrary nonempty open set in X, all nonempty open sets in X contain a point in A, so A is dense in X by Theorem 5.1, and since A is countable, X is separable. □

Exercise 5.10. (1) The standard topology $\mathbb{R}^n_{\text{std}}$ has a basis of open balls with rational radii centered at rational points in $\mathbb{R}^n$, and this basis is countable.

(2) Let $\mathcal{B}$ be a basis for $\mathbb{R}_{\text{LL}}$. We compare cardinalities by constructing a map from $\mathbb{R}$ to $\mathcal{B}$. Since for each $x \in \mathbb{R}$, the set $[x, x+1)$, is open, there exists a basic open set V_x such

that $x \in V_x \subset [x, x+1)$ by Theorem 3.1. This defines a map from $\mathbb{R}$ to $\mathcal{B}$ that takes a real number x to this basic open set V_x. Suppose there exist $\alpha, \beta \in \mathbb{R}$ such that $V_\alpha = V_\beta$. Then $\alpha, \beta \in V_\alpha \subset [\alpha, \alpha+1)$, and $\alpha, \beta \in V_\alpha = V_\beta \subset [\beta, \beta+1)$. Since $\alpha = \min[\alpha, \alpha+1) \supset V_\alpha$, $\alpha = \min V_\alpha$. But similarly, $\beta = \min V_\beta = \min V_\alpha = \alpha$. Therefore this map is injective, and so $|\mathbb{R}| \leq |\mathcal{B}|$, and we see that our arbitrary basis $\mathcal{B}$ is not countable. Hence $\mathbb{R}_{\text{LL}}$ is not 2^{nd} countable.

(3) The argument that $\mathbb{H}_{\text{bub}}$ is not 2^{nd} countable is very similar. This time, define a map from $\mathbb{R}$ to a potential basis $\mathcal{B}$ by associating to each $x \in \mathbb{R}$ a basic open set $V_x \in \mathcal{B}$ such that $(x, 0) \in V_x \subset B_x := B((x, 1), 1) \cup \{(x, 0)\}$. Since B_x is open in $\mathbb{H}_{\text{bub}}$, Theorem 3.1 again guarantees such a V_x exists. Then as earlier if there exist $\alpha, \beta \in \mathbb{R}$ such that $V_\alpha = V_\beta$, then we have that $(\beta, 0) \in V_\alpha \subset B_\alpha$. But $(\beta, 0) \notin B((\alpha, 1), 1)$ since everything in this ball has y-coordinate strictly greater than 0, so $\beta \in B_\alpha$ implies that $(\beta, 0) \in \{(\alpha, 0)\}$, which is only the case if $\alpha = \beta$ and our map is injective. Then this shows that $|\mathbb{R}| \leq |\mathcal{B}|$, and so $\mathbb{H}_{\text{bub}}$ is not 2^{nd} countable.

Theorem 5.11. Every uncountable set in a 2^{nd} countable space has a limit point.

Proof. Let A be an uncountable set in a space X, and suppose it has no limit points. Then every point of A is an isolated point, so by Exercise 2.10, for each $x \in A$, there exists an open U_x such that $U_x \cap A = \{x\}$. Suppose $\mathcal{B}$ is a basis for the topology on X. Then there exists a $V_\alpha \in \mathcal{B}$ such that $x \in V_x \subset U_x$. Then $V_x \cap A = \{x\}$, so this defines a map from A to $\mathcal{B}$ associating a point of A with a basic open set containing it. Then if there exist $\alpha, \beta \in A$ such that $V_\alpha = V_\beta$, then there exists $\beta \in V_\beta = V_\alpha$ and so $\beta \in V_\alpha \cap A = \{\alpha\}$. Therefore $\beta = \alpha$ and this map is injective, meaning $|A| \leq |\mathcal{B}|$. This means the basis $\mathcal{B}$ is uncountable and so our space X is not 2^{nd} countable. We have shown the contrapositive of the claim. $\square$

Exercise 5.12. All 2^{nd} countable spaces are hereditarily 2^{nd} countable.

Proof. This is a corollary of Theorem 3.30, which says that if $\mathcal{B}$ is a basis for a space $(X, \mathcal{T})$, and $(Y, \mathcal{T}_Y)$ is a subspace, then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$. If $\mathcal{B}$ is countable, then so is $\mathcal{B}_Y$: the map $f : \mathcal{B} \to \mathcal{B}_Y$ given by $f(B) = B \cap Y$ is a surjection. $\square$

Exercise 5.13. If X and Y are 2^{nd} countable, then so is $X \times Y$.

Proof. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be countable bases for X and Y. Then the product $\mathcal{B} = \mathcal{B}_X \times \mathcal{B}_Y$ is countable, and since for just two sets X and Y the product topology on $X \times Y$ contains all products $U \times V$ where U is open in X and V is open in Y, all elements of $\mathcal{B}$ are in the topology, meaning condition (1) of Theorem 3.1 is satisfied. Let W be an open set in $X \times Y$ and let $p = (x, y) \in W$. Then by the definition of the product topology, it can be written as the union of sets of the for $U_\alpha \times V_\alpha$ for all $\alpha \in \lambda$ for some index set λ. Therefore there exists a particular α such that $p \in U_\alpha \times V_\alpha$ where U_α is open in X and V_α is open in Y. Since $\mathcal{B}_X$ is a basis for the topology on X, there exists a basic open set $B_{x,\alpha} \in \mathcal{B}_X$ such that $x \in B_{x,\alpha} \subset U_\alpha$. Similarly, there is a $B_{y,\alpha} \in \mathcal{B}_Y$ such that $y \in B_{y,\alpha} \subset V_\alpha$. Therefore $p = (x, y) \in B_{x,\alpha} \times B_{y,\alpha} \subset U_\alpha \times V_\alpha \subset W$. Therefore condition (2) is satisfied, and so $\mathcal{B}$ is a basis for the product topology on $X \times Y$. Since $\mathcal{B}$ is countable, $X \times Y$ is 2^{nd} countable. □

5.3 1st Countable Spaces

Theorem 5.14. If X is a 2^{nd} countable space, then it is 1^{st} countable.

Proof. Let X be a 2^{nd} countable space. Then it has a countable basis $\mathcal{B}$. Let $p \in X$ and define the set $\mathcal{B}_p$ to be $\mathcal{B}_p = \{V \in \mathcal{B} \mid p \in V\}$. By definition, every set in B_p is an open set containing p, and if W is an open set containing p, then since $\mathcal{B}$ is a basis, there exists a $V_p \in \mathcal{B}$ such that $p \in V_p \subset W$, and since $p \in V_p$, $V_p \in \mathcal{B}_p$, which means both conditions for being a neighborhood basis are satisfied by $\mathcal{B}_p$. Since $\mathcal{B}_p \subset \mathcal{B}$ and $\mathcal{B}$ is countable, $\mathcal{B}_p$ is countable. Since p was an arbitrary point of X and has a countable neighborhood basis, the space X is 1^{st} countable. □

Theorem 5.15. If p is a point in a space X with a countable neighborhood basis, then there exists a nested countable neighborhood basis for p.

Proof. Let $\mathcal{B} = \{U_n\}_{n \in \mathbb{N}}$ be a countable neighborhood basis for p. Define the set $V_n = \bigcap_{i=1}^{n} U_i$. Then the set $\mathcal{B}_{\text{nested}} = \{V_n \mid n \in \mathbb{N}\}$ is countable and nested, since

$$V_n = U_n \cap \bigcap_{i=1}^{n-1} = U_n \cap V_{n-1} \subset V_{n-1}$$

for all $n \in \mathbb{N}$. Since every set in $\mathcal{B}_{\text{nested}}$ is in $\mathcal{B}$ which is a neighborhood basis for p, all V_n contain p, and if W is an open set containing p, then there exists a $U_j \in \mathcal{B}$ such that $p \in U_j \subset W$. Then by the above, we have that $V_n \subset U_n$ for all $n \in \mathbb{N}$, so $p \in V_j \subset U_j \subset W$ and $\mathcal{B}_{\text{nested}}$ is also a countable neighborhood basis for p. □

Theorem 5.18. Let X be a 1^{st} countable space with a subset $A \subset X$. If $x \in X$ is a limit point of A, then there exists a sequence $\{a_i\}_{i \in \mathbb{N}}$ contained in A that converges to x.

Proof. Since X is 1^{st} countable, there exists a countable neighborhood basis for x that is nested (by Theorem 5.15, $\mathcal{B} = \{V_i \mid i \in \mathbb{N}\}$. Since each V_i is open and x is a limit point of A, the sets $(V_i - \{x\}) \cap A$ are all nonempty, so for each there exists a point $a_i \in (V_i - \{x\}) \cap A$. Then the sequence $(a_i)_{i\in\mathbb{N}}$ is contained in A and converges to x, because for any open set U containing x, there exists an $N \in \mathbb{N}$ such that $x \in V_N \subset U$, and so for all $i > N$, $a_i \in V_i \subset V_N \subset U$ since the neighborhood basis we chose for x is nested. $\square$

Exercise 5.19. All 1^{st} countable spaces are hereditarily 1^{st} countable.

Proof. Let $(X, \mathcal{T})$ be a 1^{st} countable space with subspace $(Y, \mathcal{T}_Y)$, and let $p \in Y$ be a point with countable neighborhood basis $\mathcal{B}$. Then the set $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a countable collection of subsets of Y. It is also a neighborhood basis for p because $p \in Y$ and $p \in B$ for all $B \in \mathcal{B}$ implies p is in all elements of $\mathcal{B}_Y$. We also have that if U is an open set in $(Y, \mathcal{T}_Y)$ containing p, then there exists an open set V in $(X, \mathcal{T})$ such that $U = V \cap Y$. Therefore $p \in V$ and so there exist a $B_0 \in \mathcal{B}$ such that $p \in B_0 \subset V$, which means $p \in B_0 \cap Y \subset V \cap Y = U$. Since $B_0 \cap Y \in \mathcal{B}_Y$, $\mathcal{B}_Y$ is a countable neighborhood basis for p in $(Y, \mathcal{T}_Y)$, and because p was an arbitrary point, we have shown that $(Y, \mathcal{T}_Y)$ is 1^{st} countable, as required. $\square$

6 Compactness: The Next Best Thing to Being Finite

6.1 Compact Sets

Theorem 6.1. Let X be a finite topological space. Then X is compact.

Proof. Let $(X, \mathcal{T})$ be a finite topological space and let $\mathscr{C}$ and open cover. Then $\mathscr{C} \subset \mathcal{T}$ because elements of $\mathscr{C}$ are open sets and are therefore in $\mathcal{T}$, and $\mathcal{T} \subset 2^X$ because elements of $\mathcal{T}$ are subsets of X. The space X is finite, so its power set 2^X is also finite, which means $\mathcal{T}$ and $\mathscr{C}$ are also finite and $\mathscr{C}$ is therefore a finite subcover of itself. $\square$

Theorem 6.2. If C is a compact subset of $\mathbb{R}_{\text{std}}$, then C has a maximum point.

Proof. We will show the contrapositive. Let C be a nonempty subset of $\mathbb{R}_{\text{std}}$ without a maximum point. Define the function $f : C \to \mathcal{T}_{\text{std}}$ by $f(x) = (-\infty, x)$ for $x \in C$. Then the image of f is an open cover for C because if $a \in C$, then there exists a $b \in C$ such that $a < b$ (since C has no maximum), and therefore $a \in f(b)$. Let F be a finite, nonempty subset of $f(C)$. Then $f^{-1}(F)$ is a finite, nonempty subset of C (f is an injection) and hence has a maximum, α. However, there is no maximum of C, so there exists some $\beta \in C$ such that $\alpha < \beta$. Let $F_0 \in F$. Then $F_0 = (-\infty, x_0)$ for some $x_0 \in f^{-1}(F)$. Since α is the maximum of $f^{-1}(F)$, $x_0 \leq \alpha < \beta$, which means $\beta \notin F_0$. We have shown that β is not in any element of F, which means F is not a cover of C. Since F was an arbitrary finite subset of $f(C)$, we have shown that there exists an open cover of C without a finite subcover, meaning C is not compact. $\square$

Theorem 6.3. Let X be compact. Then every infinite subset of X has a limit point.

Proof. We again show the contrapositive. Let A be an infinite subset of a topological space X such that A has no limit points. For each point $p \in A$, p is not a limit point, so there exists at least one open set U_p such that $(U_p - \{p\}) \cap A = \emptyset$. Since A has no limit points, it vacuously contains all its limit points and is therefore closed, meaning $X - A$ is open. The set $\mathscr{C} = \{U_p \mid p \in A\} \cup \{(X - A)\}$ is an open cover for X because each U_p is open and $p \in A$ means $p \in U_p$ and $p \notin A$ means $p \in X - A$, so every point in X is in some element of $\mathscr{C}$ and therefore $X \subset \bigcup_{V \in \mathscr{C}} V$. Let F be a finite subset of $\mathscr{C}$. Then there exists an $x \in A$ such that $U_x \notin F$, otherwise $F = \mathscr{C} - \{(X - A)\}$ which is infinite. Let $U_y \in F$ be arbitrary. Then $(U_y - \{y\}) \cap A = \emptyset$, and $x \in A$ therefore implies x must not be in $U_y - \{y\}$. We have that $x \neq y$, because otherwise we would have $U_x = U_y \in F$, so in order for $(U_y - \{y\}) \cap A$ to be empty, we must have $x \notin U_y$. The set U_y was arbitrary, so x does not belong to any element of F of this form, and $x \in A$ implies $x \notin (X - A)$, so there is no element of F that x belongs to, meaning F is not a cover of X. Since F was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ has no finite subcover and therefore X is not compact. $\square$

Corollary 6.4. If X is compact and $E \subset X$ has no limit points, then E is finite.

Proof. By Theorem 6.3, every infinite subset of X has a limit point, which means in order to have no limit points, E must be finite. $\square$

Theorem 6.5. A space X is compact if and only if every collection of closed sets with the finite intersection property has a nonempty intersection.

Proof. ($\Longrightarrow$) Let X be a compact space and let $\{D_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets. To show that the finite intersection property implies a nonempty intersection, we will show that $\bigcap_{\alpha \in \lambda} D_\alpha = \emptyset$ implies that this collection does not have the finite intersection property. Suppose this intersection is empty. Then

$$\bigcap_{\alpha \in \lambda} D_\alpha = \emptyset \implies X = X - \bigcap_{\alpha \in \lambda} D_\alpha = \bigcup_{\alpha \in \lambda} (X - D_\alpha),$$

and since all $X - D_\alpha$ are open, the collection $\{X - D_\alpha\}_{\alpha\in\lambda}$ is an open cover for X. Because X is compact, this open cover has a finite subcover, call it $\{X - D_i\}_{i\in N}$ where N is a finite subset of the index set λ. Therefore we have that

$$X = \bigcup_{i\in N}(X - D_i) = X - \bigcap_{i\in N} D_i \implies \bigcap_{i\in N} D_i = \emptyset,$$

and so we have found a finite subcollection of $\{D_\alpha\}_{\alpha\in\lambda}$ that does not have a nonempty intersection, meaning this collection does not have the finite intersection property.

($\Longleftarrow$) Now suppose that in a space X, every collection of closed sets with the finite intersection property has a nonempty intersection. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be an open cover of X. Then

$$X = \bigcup_{\alpha\in\lambda} C_\alpha \implies \bigcap_{\alpha\in\lambda}(X - C_\alpha) = \emptyset.$$

Therefore $\{X - C_\alpha\}_{\alpha\in\lambda}$ is a collection of closed sets with empty intersection, which means it does not have the finite intersection property by our hypotheses. Therefore there exists a finite subcollection $\{X-C_i\}_{i\in N}$ for some finite $N \subset \lambda$ such that $\bigcap_{i\in N}(X-C_i) = \emptyset$. Therefore $\bigcup_{i\in N} C_i = X$, which means $\{C_i\}_{i\in N}$ is a finite subcover of $\{C_\alpha\}_{\alpha\in\lambda}$. Since $\{C_\alpha\}_{\alpha\in\lambda}$ was an arbitrary open cover, all open covers have finite subcovers, and therefore X is compact. □

Theorem 6.6. A space X is compact if and only if for any open set U in X and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$, there exists a finite subcollection of K_αs whose intersection is a subset of U.

Proof. ($\Longrightarrow$) Let X be a compact space, let U be an open set, and let $\{K_\alpha\}_{\alpha\in\lambda}$ be a collection of closed sets with $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$. Then we have that

$$X - U \subset X - \bigcap_{\alpha\in\lambda} K_\alpha = \bigcup_{\alpha\in\lambda}(X - K_\alpha) \implies X = U \cup \bigcup_{\alpha\in\lambda}(X - K_\alpha),$$

so the collection $\{U\} \cup \{X - K_\alpha\}_{\alpha\in\lambda}$ is an open cover for X, and since X is compact, there exists a finite subcover which either looks like $\{U\} \cup \{X - K_i\}_{i\in N}$ or $\{X - K_i\}_{i\in N}$ for some finite $N \subset \lambda$. Then either

$$X = U \cup \bigcup_{i\in N}(X - K_i) \quad \text{or} \quad X = \bigcup_{i\in N}(X - K_i),$$

and in both cases we have

$$X - U \subset \bigcup_{i \in N} (X - K_i) = X - \bigcap_{i \in N} K_i \implies \bigcap_{i \in N} K_i \subset U,$$

as required.

($\Longleftarrow$) Let X be a space in which every open set U and collection of closed sets with intersection in U has a finite subcollection with intersection in U. Let $\{K_\alpha\}_{\alpha\in\lambda}$ be a collection of closed sets with empty intersection. Then $\bigcap_{\alpha\in\lambda} K_\alpha \subset \emptyset = U$, an open set, so there exists a finite subcollection of the K_αs with empty intersection, meaning $\{K_\alpha\}_{\alpha\in\lambda}$ does not have the finite intersection property. Therefore all collections of closed sets with the finite intersection property have nonempty intersections, meaning X is compact by Theorem 6.5. $\square$

Exercise 6.7. The union of finitely many compact subsets of X is compact.

Proof. Let $\{A_i\}_{i=1}^n$ be compact subsets of X, and let $\mathscr{C}$ be an open cover of $\bigcup_{i=1}^n A_i$. Then $\mathscr{C}$ is an open cover of A_i for all $i = 1, \ldots, n$, so there exist finite subcovers C_i such that C_i is a cover of A_i. The collection $\mathscr{C}' = \bigcup_{i=1}^n C_i$ is a subset of $\mathscr{C}$ because if $U \in \mathscr{C}'$, then $U \in C_j \subset \mathscr{C}$ for some $j = 1, \ldots, n$, and it is a finite collection because it is the finite union of finite sets. If $a \in \bigcup_{i=1}^n A_i$, then $a \in A_j$ for some j, and therefore $a \in U^*$ for some $U^* \in C_j \subset \mathscr{C}'$ since C_j is a cover of A_j. Therefore all elements of $\bigcup_{i=1}^n A_i$ are in some element of $\mathscr{C}'$, so $\mathscr{C}'$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, $\bigcup_{i=1}^n A_i$ is compact. $\square$

Theorem 6.8. Let A be a closed subspace of a compact space X. Then A is compact.

Proof. Let $\mathscr{C}$ be an open cover of A. Then since A is closed, $X - A$ is open and so the collection $\mathscr{C} \cup \{X - A\}$ is an open cover for X. X is compact, so this open cover has a finite subcover $\mathscr{C}'$. Then the collection $\mathscr{C}^* = \mathscr{C}' - \{X - A\}$ is a finite subcover of $\mathscr{C}$ since every element of $\mathscr{C}^*$ is an element of $\mathscr{C}$, $|\mathscr{C}^*| \leq |\mathscr{C}'|$ which is finite, and every element in A is in some element of $\mathscr{C}'$ (because it covers X) other than $X - A$. $\mathscr{C}$ was an arbitrary open cover of A, so A is compact. $\square$

Theorem 6.9. Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Proof. Suppose X is Hausdorff and that $A \subset X$ is not closed. We will show that A is not compact. A not being closed means there is a limit point p of A such that $p \notin A$. For all $a \in A$, $a \neq p$, so since X is Hausdorff, there exist disjoint open sets U_a and V_a such that

$a \in U_a$ and $p \in V_a$. Since all U_a are open, the collection $\mathscr{C} = \{U_a\}_{a \in A}$ is an open cover of A. Let $F = \{U_i\}_{i \in N}$ be a finite subcollection for some finite $N \subset A$. The set $V = \bigcap_{i \in N} V_i$ is the finite intersection of open sets and is therefore open. Additionally, $p \in V_a$ for all $a \in A$, so $p \in V$. Since p is a limit point of A, $(V - \{p\}) \cap A$ is nonempty, so there exists an $x \in V \cap A$. Because $x \in V$ means $x \in V_i$ for all $i \in N$, $x \notin U_i$ for all $i \in N$, and so we have an element $x \in A$ that is not covered by the collection $F = \{U_i\}_{i \in N}$. Therefore A is not compact because it has an open cover with no finite subcover. We have shown that A not closed implies that A is not compact or not Hausdorff, so this is the contrapositive of the theorem statement. $\square$

Exercise 6.10. Consider the interval $(0, 1)$ as a subset of $\mathbb{R}$ with the finite complement topology. There are infinitely many points in $(0, 1)$, so it is not closed. Let $\mathscr{C}$ be an open cover of $(0, 1)$. $(0, 1)$ is nonempty, so there must be at least one set in $\mathscr{C}$. Call this set C_0. Because C_0 is open, its complement $\mathbb{R} - C_0$ is finite, which means the set $(0, 1) - C_0 \subset \mathbb{R} - C_0$ is also finite, so $(0, 1) - C_0 = \{a_1, \ldots, a_n\}$ for some $a_i \in (0, 1)$. Since $\mathscr{C}$ is an open cover of $(0, 1)$ and $a_i \in (0, 1)$, there exists a $C_i \in \mathscr{C}$ such that $a_i \in C_i$ for all $i = 1, \ldots, n$. Hence the collection $\{C_i\}_{i=0}^{n}$ is a cover of $(0, 1)$, and since each $C_i \in \mathscr{C}$ for $i = 0, \ldots, n$, this collection is a finite subcover of the arbitrary open cover $\mathscr{C}$. Therefore $(0, 1) \subset \mathbb{R}$ with the finite complement topology is an example of a subset the is compact but not closed.

Exercise 6.11. Intersections of compact sets need not be compact (I think the Double Headed Snake should have a counterexample?). However if we add the assumption that we are working with subsets of a Hausdorff space, then we can say that arbitrary intersections of compact sets are compact. If $\{K_\alpha\}_{\alpha \in \lambda}$ is a collection of compact sets, then Theorem 6.9 implies that K_α is closed for all $\alpha \in \lambda$, and therefore the intersection $K = \bigcap_{\alpha \in \lambda} K_\alpha$ is also closed. Since K is a closed subset of the compact set K_β for any $\beta \in \lambda$, Theorem 6.8 implies that K is itself compact.

Theorem 6.12. Every compact, Hausdorff space is normal.

Proof. Let X be a compact, Hausdorff space with closed set $A \subset X$ and $p \in X - A$. We will first show that X is regular. For each $\alpha \in A$, X being Hausdorff means that there exist disjoint open sets U_α and V_α such that $p \in U_\alpha$ and $\alpha \in V_\alpha$. Therefore $\{V_\alpha\}_{\alpha \in A}$ is an open cover of A. Since A is a subset of the compact space X, by Theorem 6.8, A is compact, and so the open cover $\{V_\alpha\}_{\alpha \in A}$ has a finite subcover $\{V_i\}_{i \in N}$ for some finite subset $N \subset A$. Define the sets $U = \bigcap_{i \in N} U_i$ and $V = \bigcup_{i \in N} V_i$. U is open because it is the *finite* intersection of open sets, and V is open because it is the union of open sets. We also have that $p \in U$ because $p \in U_i$ for all $i \in N$, and $A \subset V$ because $\{V_i\}_{i \in N}$ is a cover of A. It remains to show that U and V are disjoint. Suppose for contradiction that there exists an element $x \in U \cap V$. Then $x \in V$ means there exists a $j \in N$ such that $x \in V_j$, and $x \in U$ means $x \in U_i$ for all $i \in N$, in particular, $x \in U_j$. Therefore $x \in U_j \cap V_j$, but this is a contradiction since U_j and V_j are disjoint. Therefore $U \cap V = \emptyset$, and so X is regular.

Now let A and B be closed subsets of X. Since X is regular, for each $\alpha \in A$, there exist disjoint open sets U_α and V_α such that $\alpha \in U_\alpha$ and $B \subset V_\alpha$. Therefore $\{U_\alpha\}_{\alpha \in \lambda}$ is an open cover of A, and since A is a closed subset of the compact space X, A is compact, so there exists a finite subcover $\{U_i\}_{i \in N}$ for some finite subset $N \subset A$. Define $U = \bigcup_{i \in N} U_i$ and $V = \bigcap_{i \in N} V_i$. U is open because it is the union of open sets, and V is open because it is the *finite* intersection of open sets. The collection $\{U_i\}_{i \in N}$ is a cover of A, so $A \subset U$, and $B \subset V_i$ for all $i \in N$ by construction, so $B \subset V$ as well. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means there exists a $j \in N$ such that $x \in U_j$, and $x \in V$ means $x \in V_i$ for all $i \in N$. Therefore $x \in U_j \cap V_j$, but this is a contradiction since U_j and V_j are disjoint. Therefore $U \cap V = \emptyset$, and so X is normal. □

Theorem 6.13. Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets has a finite subcover.

Proof. ($\Longrightarrow$) Let $\mathcal{B}$ be a basis for a compact space X, and let $\mathscr{C}$ be a basic open cover. A cover by basic open sets is an open cover, and since X is compact, $\mathscr{C}$ has a finite subcover.

($\Longleftarrow$) Suppose every cover by basic open sets has a finite subcover and let $\mathscr{C}$ be an open cover. Then for all $x \in X$, there exists a $C_x \in \mathscr{C}$ such that $x \in C_x$. Since C_x is open, there exists a basic open V_x such that $x \in V_x \subset C_x$. Since such a V_x exists for all $x \in X$, the collection $\mathscr{C}' = \{V_x\}_{x \in X}$ is a cover of X by basic open sets and therefore has a finite subcover $\{V_i\}_{i \in N}$ for some finite subset $N \subset X$. Define $\mathscr{C}^* = \{C_i\}_{i \in N}$. Then since $C_i \in \mathscr{C}$ for all $i \in N$, $\mathscr{C}^*$ is a finite subcollection of $\mathscr{C}$. We also have that

$$X = \bigcup_{i \in N} V_i \subset \bigcup_{i \in N} C_i$$

since $V_x \subset C_x$ for all $x \in X$. Therefore $\mathscr{C}^*$ covers X and so is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitary open cover, all open covers have finite subcovers and X is compact. □

6.2 The Heine-Borel Theorem

Theorem 6.14. For all $a \leq b$, the subspace $[a, b]$ is compact.

Proof. If $a = b$, then any open cover $\mathscr{C}$ of $[a, b]$ contains an open set C with $a \in C$, so $\{C\}$ is a finite subcover of $[a, b]$. Assume $a < b$ and let $\mathscr{C}$ be an open cover of $[a, b]$. Then the set

$$A = \{x \in [a, b] \mid [a, x] \text{ is covered by a finite subcover of } \mathscr{C}\}$$

is nonempty ($a \in A$) and is bounded above (for example by b). Therefore $s = \sup A$ exists. Since b is an upper bound of A by construction, $s \leq b$. Suppose for contradiction that $s < b$. Then since $s \in [a, b]$ and $\mathscr{C}$ is an open cover of $[a, b]$, there exists a $C_0 \in \mathscr{C}$ such that $s \in C_0$. Therefore there exists a basic open set $B_0 = (s - \varepsilon_0, s + \varepsilon_0)$ such that $s \in B_0 \subset C_0$. Set $\varepsilon = \min\{s - a, b - s, \varepsilon_0\}$. Then $a \leq s - \varepsilon < s$ so $s - \varepsilon$ is not an upper bound of A, which means there exists a $y \in A$ such that $s - \varepsilon < y < s$. Since $y \in A$, the set $[a, y]$ is covered by a finite subcover of $\mathscr{C}$, call it $\mathscr{C}'$. Define $\mathscr{C}^* = \mathscr{C}' \cup C_0$. Then $\mathscr{C}^*$ is a finite subcollection of $\mathscr{C}$, and since $[a, y] \subset \bigcup_{C \in \mathscr{C}'} C$, we have that

$$\left[a, s + \frac{\varepsilon}{2}\right] \subset [a, y] \cup (s - \varepsilon, s + \varepsilon) \subset [a, y] \cup B_0 \subset \left(\bigcup_{C \in \mathscr{C}'} C\right) \cup C_0 = \bigcup_{C \in \mathscr{C}^*} C.$$

Therefore $[a, s + \varepsilon/2]$ is covered by a finite subcover of $\mathscr{C}$, and so $s + \varepsilon/2 \in A$, contradicting $s = \sup A$. Since $s \not< b$, $s = b$. Since $b \in [a, b]$, there exists a $C_1 \in C$ such that $b \in C_1$, and so there also exists a basic open set $B_1 = (b - \varepsilon_1, b + \varepsilon_1)$ such that $b \in B_1 \subset C_1$. If $b - \varepsilon_1 < a$, then $[a, b] \subset B_1 \subset C_1$ and $\{C_1\}$ is a finite subcover of $\mathscr{C}$ for $[a, b]$. If $a \leq b - \varepsilon_1$, then $b - \varepsilon_1$ is not an upper bound of A since $b = \sup A$, so there exists a $z \in (b - \varepsilon_1, b)$ such that $z \in A$. Then $z \in A$ means $[a, z]$ is finitely covered by some subcollection of $\mathscr{C}$, call it $\mathscr{C}'$, and so we have that

$$[a, b] \subset [a, z] \cup (b - \varepsilon_1, b + \varepsilon_1) \subset \left(\bigcup_{C \in \mathscr{C}'} C\right) \cup C_1.$$

Therefore $[a, b]$ is covered by a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover of $[a, b]$, all open covers have finite subcovers and $[a, b]$ is compact. □

Theorem 6.15. Let A be a subset of $\mathbb{R}_{\text{std}}$. Then A is compact if and only if A is closed and bounded.

Proof. ($\Longrightarrow$) Let A be a compact subset of $\mathbb{R}_{\text{std}}$. Then since $\mathbb{R}_{\text{std}}$ is Hausdorff, by Theorem 6.9 we have that A is closed. Suppose for contradiction that A is unbounded, and without loss of generality, assume it is unbounded above. Then the set $\mathscr{C} = \{(-\infty, a) \mid a \in A\}$ is an open cover for A, and if $\mathscr{C}'$ is a finite subset of $\mathscr{C}$, then the set $\{a \in A \mid (-\infty, a) \in \mathscr{C}'\}$ has a maximum. Since A is unbounded above, there exists an $\alpha \in A$ such that $\alpha > \max\{a \in A \mid (-\infty, a) \in \mathscr{C}'\}$, and therefore $\alpha \notin \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. Since $\mathscr{C}'$ was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ is an open cover of A without a finite subcover, and therefore A is not compact, a contradiction. Therefore A being compact means A is closed and bounded.

($\Longleftarrow$) Let A be a closed and bounded subset of $\mathbb{R}_{\text{std}}$. Then there exists an $M \in \mathbb{R}$ such that $A \subset [-M, M]$, and since $[-M, M]$ is compact by Theorem 6.14, the set A is also compact by Theorem 6.8. $\square$

Exercise 6.16. Consider the set $A = [0, \sqrt{2}] \cap \mathbb{Q}$. Since $[0, \sqrt{2}]$ is closed in $\mathbb{R}_{\text{std}}$, A is closed in $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$ by Theorem 3.28, and note that A is bounded below by 0 and above by 2. Define $\mathscr{C} = \{(-\infty, q) \cap \mathbb{Q} \mid q \in A\}$ and let $\mathscr{C}'$ be a finite subset of $\mathscr{C}$. Then the set $\{a \in A \mid (-\infty, a) \cap \mathbb{Q} \in \mathscr{C}'\}$ is finite and therefore has a maximum. Note that A has no maximum since $\sqrt{2} \notin A \subset \mathbb{Q}$, so there exists an $\alpha \in A$ such that $\alpha > \max\{a \in A \mid (-\infty, a) \cap \mathbb{Q} \in \mathscr{C}'\}$. Therefore $\alpha \notin \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. Since $\mathscr{C}$ is therefore an open cover of A without a finite subcover, A is not compact in $(\mathbb{Q}, \mathcal{T}_{\mathbb{Q}})$.

Theorem 6.17. Every compact subset $C \subset \mathbb{R}_{\text{std}}$ has a maximum.

Proof. Assume C is nonempty By the Heine-Borel Theorem, C is closed and bounded. Since C is bounded and nonempty, $\alpha = \sup C$ exists. We claim that $\alpha \in C$. Let U be an open set containing α. Then there exists a basic open set $V = (\alpha - \varepsilon, \alpha + \varepsilon)$ for some $\varepsilon > 0$ such that $\alpha \in V \subset U$. Since $\alpha = \sup C$, the point $\alpha - \varepsilon$ is not an upper bound for C, which means there exists a point $c \in C$ such that $\alpha - \varepsilon < c \leq \alpha = \sup C$. Therefore $c \in V \subset U$. If $c = \alpha$, then $\alpha \in C$ and we are done. If $c \neq \alpha$, then $c \in U - \{\alpha\}$, so $(U - \{\alpha\}) \cap C$ is nonempty. For an arbitrary open set U containing α, we have that shown that either $\alpha \in C$ or $(U - \{\alpha\}) \cap C \neq \emptyset$. Therefore if there exists a choice of U such that $\alpha \in C$, we are done, and if not, then $(U - \{\alpha\}) \cap C \neq \emptyset$ for all open sets U containing α, which means α is a limit point of C. Becuase C is closed, it contains all of its limit points, so $\alpha \in C$ in this case as well. Therefore $\alpha \in C$ and $\alpha = \sup C \geq c$ for all $c \in C$, which means α is a maximum of C. $\square$

6.3 Compactness and Products

Lemma 6.18. Let Y be compact, let $x_0 \in X$, and let U be an open set containing the slice $x_0 \times Y$. Let $y \in Y$ be arbitrary. Then $(x_0, y) \in x_0 \times Y \subset U$, so there exists a basic open set $W_y \times V_y$ (where W_y is open in X and V_y is open in Y) such that $(x_0, y) \in W_y \times V_y \in U$. Then since y was arbitary, there exists such an open V_y for all $y \in Y$, and therefore $\mathscr{C} = \{V_y\}_{y \in Y}$ is an open cover of Y. Since Y is compact, there exists a finite subcover $\mathscr{C}' = \{V_i\}_{i \in N}$ for some finite $N \subset Y$. Define the set $W = \bigcap_{i \in N} W_i$. Since N is finite, W is the intersection of a finite number of sets that are open in X and is therefore also open in X. W also contains x_0 because $x_0 \in W_y$ for all $y \in Y$ and $N \subset Y$. Therefore the set $W \times Y$ is an open tube in $X \times Y$ containing the slice $x_0 \times Y$. Let (α, β) be a point in this tube. Then $\alpha \in W$ and $\beta \in Y$. Recall that $\mathscr{C}' = \{V_i\}_{i \in N}$ is a cover of Y. Therefore $\beta \in Y$ implies that there exists a $j \in N$ such that $\beta \in V_j$. Then because $\alpha \in W$, $\alpha \in W_i$ for all $i \in N$, and in particular, $\alpha \in W_j$. Therefore $(\alpha, \beta) \in W_j \times V_j$, which is a basic open subset of U, meaning we also have $(\alpha, \beta) \in U$. Therefore W is open in X, contains x_0, and $W \times Y \subset U$, as required.

Theorem 6.19. Let X and Y be compact spaces. Then $X \times Y$ is compact.

Proof. Let X and Y be compact spaces and let $\mathscr{C} = \{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of $X \times Y$ by basic open sets. By Theorem 6.13, it is sufficient to show that this cover has a finite subcover in order to show that $X \times Y$ is compact. Since U_α is a basic open set for all $\alpha \in \lambda$, we can rewrite $\mathscr{C}$ as $\mathscr{C} = \{S_\alpha \times T_\alpha\}_{\alpha \in \lambda}$ such that S_α is open in X for all $\alpha \in \lambda$ and T_α is open in Y for all $\alpha \in \lambda$. For $x \in X$, define $\gamma_x = \{\alpha \in \lambda \mid x \in S_\alpha\}$. Then $\{S_\alpha \times T_\alpha\}_{\alpha \in \gamma_x}$ covers the slice $x \times Y$ for all $x \in X$. Therefore $\{T_\alpha\}_{\alpha \in \gamma_x}$ is an open cover for Y, and since Y is compact, $\{T_\alpha\}_{\alpha \in \gamma_x}$ has a finite subcover, call it $\{T_i\}_{i \in N_x}$ for some finite subset $N_x \subset \gamma_x$. This implies that $x \times Y$ is covered by the finite collection $\{S_i \times T_i\}_{i \in N_x}$, meaning $x \times Y$ subset $\bigcup_{i \in N_x} (S_i \times T_i)$. Since Y is compact, there exists an open set $W_x \subset X$ containing x such that $x \times Y \subset W_x \times Y \subset \bigcup_{i \in N_x} (S_i \times T_i)$ by the tube lemma. Since there exists such a W_x for all $x \in X$, the collection $\{W_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{W_x\}_{x \in X}$, call it $\{W_j\}_{j \in M}$ for some finite subset $M \subset X$. Define $N = \bigcup_{j \in M} N_j$. We claim that $\{S_i \times T_i\}_{i \in N}$ is a finite subcover of $\mathscr{C}$. Since N_j is finite for all $j \in M$ and M is also finite, the union N is finite. Since $N_j \subset \gamma_j \subset \lambda$ for all $j \in M$, the union N is also a subset of λ. Therefore $\{S_i \times T_i\}_{i \in N}$ is a finite subcollection of $\mathscr{C}$ and it only remains to show that this subcollection covers $X \times Y$. To show this, let $(x_0, y_0) \in X \times Y$ be arbitrary. Then $x_0 \in X$ and X is covered by $\{W_j\}_{j \in M}$, so there exists a $j_0 \in M$ such that $x_0 \in W_{j_0}$. Therefore the point (x_0, y_0) is in the tube $W_{j_0} \times Y$ since $y_0 \in Y$, and this tube is covered by $\{S_i \times T_i\}_{i \in N_{j_0}}$. Therefore there exists an $i_0 \in N_{j_0} \subset N$ such that $(x_0, y_0) \in S_{i_0} \times T_{i_0} \in \{S_i \times T_i\}_{i \in N}$. Therefore $X \times Y \subset \bigcup_{i \in N}(S_i \times T_i)$, so $\{S_i \times T_i\}_{i \in N}$ is a finite subcover of $\mathscr{C}$. $\square$

Theorem 6.20. A subset $A \subset \mathbb{R}^n_{\text{std}}$ is compact if and only if A is closed and bounded.

Proof. ($\Longrightarrow$) Suppose $A \subset \mathbb{R}^n_{\text{std}}$ is compact. Then since $\mathbb{R}_{\text{std}}$ is Hausdorff, A is closed by Theorem 6.9. Suppose for contradiction that A is unbounded. Then for every open ball $B(\mathbf{0}, M)$, we have that $A \not\subset B(\mathbf{0}, M)$, otherwise A would be bounded. However, the set $\mathscr{C} = \{B(\mathbf{0}, M) \mid M \in \mathbb{R}\}$ is an open cover for $\mathbb{R}^n$, so it is also an open cover for A. Let $\mathscr{C}'$ be a finite subcollection of $\mathscr{C}$. Then the set $\{M \in \mathbb{R} \mid B(\mathbf{0}, M) \in \mathscr{C}'\}$ is finite and so it has a maximum, call it M'. Then $\bigcup_{C \in \mathscr{C}'} C = B(\mathbf{0}, M')$, and so since A is unbounded, $A \not\subset \bigcup_{C \in \mathscr{C}'} C$, so $\mathscr{C}'$ does not cover A. But this means A is not compact, so we have a contradiction and therefore have that A is bounded in addition to being closed.

($\Longleftarrow$) Suppose $A \subset \mathbb{R}^n_{\text{std}}$ is closed and bounded. Then there exists some ball $B(\mathbf{0}, M)$ such that $A \subset B(\mathbf{0}, M)$, and this ball is in turn a subset of the product of intervals $\prod_{i=1}^n [-M, M]$. By Theorem 6.14, $[-M, M] \subset \mathbb{R}_{\text{std}}$ is compact, which means that by Theorem 6.19, $\prod_{i=1}^n$ is also compact. Since A is a subset of a compact set, A is compact by Theorem 6.8. $\square$

Theorem 6.21. Let $\mathscr{S}$ be a subbasis for a space X. Then X is compact if and only if every subbasic open cover has a finite subcover.

Proof. ($\Longrightarrow$) Suppose X is compact, let $\mathscr{S}$ be a subbasis for X, and let $\mathscr{C}$ be an arbitrary subbasic open cover of X. Then since subbasic open sets are open sets, $\mathscr{C}$ is an open cover and therefore has a finite subcover since X is compact.

($\Longleftarrow$) Let X be a space with a subbasis $\mathscr{S}$ satisfing the property that every subbasic open cover has a finite subcover, and suppose for contradiction that X is not compact. Define the set $\mathscr{C}$ to be the set of all open covers of X that do not have finite subcovers. Because X is not compact, $\mathscr{C}$ is not empty since there is at least one open cover without a finite subcover. The elements of $\mathscr{C}$ are collections of open covers, so we may partially order them by set inclusion: if $C \in \mathscr{C}$, then $C \subset C$; if $A, B \in \mathscr{C}$ with $A \subset B$ and $B \subset A$, then $A = B$; and if $A, B, C \in \mathscr{C}$ with $A \subset B$ and $B \subset C$, then $A \subset C$, so this does define a partial order. Let $\mathfrak{T} \subset \mathscr{C}$ be an arbitary totally ordered subset. Define $C_{\mathfrak{T}} = \bigcup_{T \in \mathfrak{T}} T$ and note that for all $T \in \mathfrak{T}$, $T \subset C_{\mathfrak{T}}$, so $C_{\mathfrak{T}}$ is an upper bound for $\mathfrak{T}$. In order to apply Zorn's Lemma to $\mathscr{C}$, we claim that $C_{\mathfrak{T}} \in \mathscr{C}$, that is, we claim $C_{\mathfrak{T}}$ is an open cover of X with no finite subcover. Since $\mathfrak{T}$ is a subset of $\mathscr{C}$, each $T \in \mathfrak{T} \subset \mathscr{C}$ is an open cover of X, so $C_{\mathfrak{T}}$ is also an open cover of

X. Suppose for contradiction that $C_{\mathfrak{T}}$ had a finite subcover, call it $\{C_i\}_{i\in F}$ for some finite index set F. Since $C_{\mathfrak{T}} = \bigcup_{T\in\mathfrak{T}} T$, for all $i \in F$, there exists some $T_i \in \mathfrak{T}$ such that $C_i \in T_i$. Then since the set $\{T_i \mid i \in F\}$ is a subset of $\mathfrak{T}$, it is finite and totally ordered by inclusion (so all elements can be compared), and therefore this set has a maximum, call it T_j. Then $C_i \in T_j$ for all $i \in F$, and since $\{C_i\}_{i\in F}$ covers X, it is a finite subcover of $T_j \in \mathfrak{T} \subset \mathscr{C}$, a contradiction since elements of $\mathscr{C}$ are exactly those open covers that do not have finite subcovers. Therefore the upper bound $C_{\mathfrak{T}}$ of $\mathfrak{T}$ is an open cover without a finite subcover, and so $C_{\mathfrak{T}} \in \mathscr{C}$, as claimed. Since $\mathfrak{T}$ as an arbitrary totally ordere subset of $\mathscr{C}$, all such subsets have upper bounds in $\mathscr{C}$, and so we may apply Zorn's Lemma to $\mathscr{C}$ to say that $\mathscr{C}$ itself has a maximum, call it $M \in \mathscr{C}$.

We claim that $M \cap \mathscr{S}$ is not an open cover of X. If it were, $M \cap \mathscr{S} \subset \mathscr{S}$ would imply that it has a finite subcover since all subbasic open covers of X have finite subcovers. However, this cannot be the case because also $M \cap \mathscr{S} \subset M$, and so any finite subcover of $M \cap \mathscr{S}$ is a finite subcover of M, an open cover with no finite subcover. Therefore $M \cap \mathscr{S}$ is not an open cover of X, so there exists an $x \in X$ such that for all $V \in M \cap \mathscr{S}$, $x \notin N$. Since M itself is a cover of X, there exists a $U \in M$ such that $x \in U$. Because $\mathscr{S}$ is a subbasis for the topology on X, there exists a finite collection of subbasic open sets $S_1, \ldots, S_n$ such that $x \in \bigcap_{i=1}^n S_i \subset U$. For all $i = 1, \ldots, n$, we have that $S_i \notin M$. This is because $x \in \bigcap_{i=1}^n S_i$ implies that $x \in S_i$ for all $i = 1, \ldots n$, and therefore that $x \in U \cap S_i$. If S_i were an element of M, $U \cap S_i$ would be an element of $M \cap \mathscr{S}$ containing x, which would contradict our choice of x as an element not covered by $M \cap \mathscr{S}$.

We now have a finite number of subbasic open sets $S_1, \ldots, S_n$ such that $S_i \notin M$. Therefore for all $i = 1, \ldots, n$, M is a subset of $M \cup \{S_i\}$. Recall that M was defined to be the maximum of the set $\mathscr{C}$ with respect to the ordering by set inclusion, so M being a proper subset of $M \cup \{S_i\}$ means that for all $i = 1, \ldots, n$, $M \cup \{S_i\} \notin \mathscr{C}$. Since M on its own is an open cover of X, so is $M \cup \{S_i\}$. Therefore $M \cup \{S_i\}$ is an open cover of X not in $\mathscr{C}$, meaning it has a finite subcover, and this finite subcover must be of the form $M_i \cup \{S_i\}$ for some $M_i \subset M$. Otherwise, the finite subcover of $M \cup \{S_i\}$ would also be a finite subcover of $M \in \mathscr{C}$, which is known to have no finite subcover.

Define the set $\overline{M} = \bigcup_{i=1}^n M_i$. Since $M_i \cup \{S_i\}$ is finite for all $i = 1, \ldots, n$, each M_i is also finite, and therefore so is $\overline{M} \subset M$. Suppose now for contradiction that there exists some $x_0 \in X$ that is not covered by the finite collection $\overline{M} \cup \{U\}$. Then for all $M' \in \overline{M}$, we have that $x_0 \notin M'$, and that $x_0 \notin U$, otherwise x_0 would be covered. Since $x_0 \notin U$, also $x_0 \notin \bigcap_{i=1}^n S_i \subset U$. This means there exists a $j = 1, \ldots, n$ such that $x_0 \notin S_j$. However, the collection $M_j \cup \{S_j\}$ covers X, so $x_0 \notin S_j$ means there exists an $M_0 \in M_j$ such that $x_0 \in M_0$. But $M_0 \in M_j$ and $M_j \subset \overline{M}$, so $x_0 \in M_0 \in \overline{M} \cup \{U\}$, which was assumed to not cover x_0. Therefore there does not exists such an x_0, and so $\overline{M} \cup \{U\}$ covers X. But $\overline{M} \subset M$ and $U \in M$, so $\overline{M} \cup \{U\}$ is a finite subcover of M, a contradiction. Therefore X is compact since in assuming otherwise, we have reached a contradiction. □

Theorem 6.23. Products of compact spaces are compact.

Proof. Let $\{X_\alpha\}_{\alpha \in \lambda}$ be a collection of compact spaces and suppose for contradiction that the product space $\prod_{\alpha \in \lambda} X_\alpha$ is not compact. Recall that the product topology has a subbasis $\mathscr{S}$ consisting of sets of the form $\pi_\alpha^{-1}(U_\alpha)$ for sets U_α open in X_α. By Theorem 6.21, $\prod_{\alpha \in \lambda} X_\alpha$ not being compact means it has a subbasic open cover $\mathscr{C} = \{S_\beta\}_{\beta \in \gamma}$ without a finite subcover. Since for all $\beta \in \gamma$, S_β is a subbasic open set, each S_β is identified with an open set U_β in some factor space X_{α_β} for an index $\alpha_\beta \in \lambda$, and so it can be written as $\pi_{\alpha_\beta}^{-1}(U_\beta)$. Define C_α to be the collection of open sets in X_α given by $C_\alpha = \{U_\beta \mid \alpha_\beta = \alpha\}$.

Let $\delta \in \lambda$ be an arbitrary index. We claim that C_δ does not cover X_δ. X_δ is compact, so C_δ being an open cover would mean it would have to have a finite subcover, call it $C'_\delta = \{U_i\}_{i \in F}$ for some finite index set $F \subset \gamma$. Then define $\mathscr{C}' = \{S_i\}_{i \in F}$ and let $(x_\alpha)_{\alpha \in \lambda}$ be an arbitrary point in $\prod_{\alpha \in \lambda} X_\alpha$. Then $x_\delta \in X_\delta$ which is covered by C'_δ, and so there exists a $j \in F$ such that $x_\delta \in U_j$, and therefore that $(x_\alpha)_{\alpha \in \lambda} \in \pi_\delta^{-1}(U_j) = S_j \in \mathscr{C}'$. This implies that $\mathscr{C}'$ is a finite subcover of $\mathscr{C}$, which is a contradiction meaning that C_δ does not cover X_δ.

Since $\delta \in \lambda$ was arbitrary, we have that for all $\alpha \in \lambda$, C_α does not cover X_α, so for all $\alpha \in \lambda$, there exists an $x'_\alpha \in X_\alpha$ such that x'_α is not covered by C_α. Consider the point $(x'_\alpha)_{\alpha \in \lambda} \in \prod_{\alpha \in \lambda} X_\alpha$. Because we have assumed that $\mathscr{C}$ is a cover for the product space that contains this point, there exists some $S_{\beta'} \in \mathscr{C}$ such that $(x'_\alpha)_{\alpha \in \lambda} \in S_{\beta'} = \pi_{\alpha_{\beta'}}^{-1}(U_{\beta'})$. But this then implies that $x'_{\alpha_{\beta'}} \in X_{\alpha_{\beta'}}$ is an element of $U_{\beta'} \in C_{\alpha_{\beta'}}$, and therefore that $x'_{\alpha_{\beta'}}$ is covered by $C_{\alpha_{\beta'}}$, a contradiction since $(x'_\alpha)_{\alpha \in \lambda}$ was constructed so that each coordinate was not covered by its corresponding C_α. Therefore the original $\mathscr{C}$ does not actually cover $\prod_{\alpha \in \lambda} X_\alpha$, so our assumption that such a subbasic open cover without a finite subcover existed was false, meaning $\prod_{\alpha \in \lambda} X_\alpha$ is compact. □

Exercise 6.24. Consider the space $\{0,1\}^\omega$ as a subspace of $[0,1]^\omega$. Let $(a_i)_{i \in \mathbb{N}}$ be a sequence not in $\{0,1\}^\omega$. Then there exists an $m \in \mathbb{N}$ such that $a_m \in (0,1)$. Since the sets $[0,1]$ and $(0,1)$ are both open in $[0,1]$, the set $U = \underbrace{[0,1] \times \cdots \times [0,1]}_{(m-1) \text{ times}} \times (0,1) \times [0,1] \times [0,1] \times \dots$ is open in $[0,1]^\omega$ and contains the sequence $(a_i)_{i \in \mathbb{N}}$. If $(x_i)_{i \in \mathbb{N}}$ is a sequence in $\{0,1\}^\omega$, then

$(x_i)_{i\in\mathbb{N}} \notin U$, because $x_m = \{0,1\}$, meaning $x_m \notin (0,1)$. Therefore U is an open set containing $(a_i)_{i\in\mathbb{N}}$ such that $(U-(a_i)_{i\in\mathbb{N}})\cap\{0,1\}^\omega = \emptyset$, and so $(a_i)_{i\in\mathbb{N}}$ is not a limit point of $\{0,1\}^\omega$. We have shown that all elements of $[0,1]^\omega - \{0,1\}^\omega$ are not limit points, and so if an element of $[0,1]^\omega$ is a limit point of $\{0,1\}^\omega$, it must also be am element of $\{0,1\}^\omega$. Therefore $\{0,1\}^\omega$ is a closed subset of $[0,1]^\omega$. As a subspace of $[0,1]^\omega$ with the box topology, $\{0,1\}^\omega$ also has the box topology, so by Exercise 3.42, $\{0,1\}^\omega$ has the discrete topology, meaning singletons are open. Therefore the set $\mathscr{C} = \{\{p\}\}_{p\in\{0,1\}^\omega}$ is an open cover of $\{0,1\}^\omega$. If $\mathscr{C}'$ is a finite subset of $\mathscr{C}$, then there exists a $p_0 \in \{0,1\}^\omega$ such that $\{p_0\} \notin \mathscr{C}'$, and therefore $p_0 \notin \bigcup_{C\in\mathscr{C}'} C$, so $\mathscr{C}'$ does not cover $\{0,1\}^\omega$. Therefore the subspace $\{0,1\}^\omega$ is not compact, so applying the contrapositive of Theorem 6.8, we have that $[0,1]^\omega$ is not compact or that $\{0,1\}^\omega$ is not closed. Since we have shown that $\{0,1\}^\omega$ is a closed subset of $[0,1]^\omega$, we have that $[0,1]^\omega$ is not compact with the box topology.

6.4 Countably Compact, Lindelöf Spaces

Theorem 6.25. Every countably compact and Lindelöf space is compact.

Proof. Let X be countably compact and Lindelöf, and let $\mathscr{C}$ be an open cover of X. Then because X is Lindelöf, $\mathscr{C}$ has a countable subcover, call it $\mathscr{C}^*$. Since $\mathscr{C}^*$ is a countable open cover of X and X is countably compact, $\mathscr{C}^*$ has a finite subcover, call it $\mathscr{C}'$. Therefore $\mathscr{C}' \subset \mathscr{C}^* \subset \mathscr{C}$, $\mathscr{C}'$ is finite, and and it covers X. Since $\mathscr{C}$ was an arbitrary open cover and it has a finite subcover, X is compact. □

Theorem 6.26. Let X be a T_1 space. Then X is countably compact if and only if every infinite subset has a limit point.

Proof. ($\Longrightarrow$) We will show the contrapositive. Suppose $A^* \subset X$ is an infinite subset with no limit point. Then by Theorem 1.9, A^* has a countably infinite subset, call it A. We claim that A also has no limit points. If p is a limit point of A, then for all open U containing p, we have $(U-\{p\})\cap A \neq \emptyset$, so also $(U-\{p\})\cap A^* \neq \emptyset$, since $A \subset A^*$. Therefore if p is a limit point of A, p is also a limit point of A^* which has no limit points. Hence A has no limit points, and we proceed as in Theorem 6.3: for each point $p \in A$, there is an open set U_p containing p such that $(U_p - \{p\})\cap A = \emptyset$. Since A vacuously contains all of its limit points, A is closed and $X - A$ is open. Therefore $\mathscr{C} = \{U_p\}_{p\in A} \cup \{X-A\}$ is a countable open cover of X. Let F be a finite subset of $\mathscr{C}$. Then there exists an $x \in A$ such that $U_x \notin F$, otherwise $F = \mathscr{C} - \{(X-A)\}$ which is infinite. Let $U_y \in F$ be arbitrary. Then

$(U_y - \{y\}) \cap A = \emptyset$, and $x \in A$ therefore implies x must not be in $U_y - \{y\}$. We have that $x \neq y$, because otherwise we would have $U_x = U_y \in F$, so in order for $(U_y - \{y\}) \cap A$ to be empty, we must have $x \notin U_y$. The set U_y was arbitrary, so x does not belong to any element of F of this form, and $x \in A$ implies $x \notin (X - A)$, so there is no element of F that x belongs to, meaning F is not a cover of X. Since F was an arbitrary finite subset of $\mathscr{C}$, we see that $\mathscr{C}$ has no finite subcover and therefore X is not countably compact.

($\Longleftarrow$) We will again show the contrapositive. Suppose X is not countably compact. Then there exists a countable open cover $\mathscr{C} = \{C_n\}_{n \in \mathbb{N}}$ of X with no finite subcover. This means that for each $n \in \mathbb{N}$, the collection $\{C_i\}_{i=1,\ldots,n}$ is a finite subcollection of $\mathscr{C}$ and therefore does not cover X, so there exists an $a_n \notin \bigcup_{i=1}^{n} C_i$. Define the set $A = \{a_n \mid n \in \mathbb{N}\}$. Now suppose for contradiction that A is finite. Then $A = \{a_n \mid n \in N\}$ for some finite $N \subset \mathbb{N}$. Since $\mathscr{C}$ covers X, for each $n \in N$, there exists an $m_n \in \mathbb{N}$ such that $a_n \in C_{m_n}$. Define $M = \max\{m_n \mid n \in N\}$, which exists because $\{m_n \mid n \in N\}$ is a finite set. Then we have that

$$A = \{a_n \mid n \in N\} \subset \bigcup_{n \in N} C_{m_n} \subset \bigcup_{i=1}^{M} C_i.$$

However, $a_M \in A$, and by definition, $a_M \notin \bigcup_{i=1}^{M} C_i$, contradiction the assumption that A is finite. Therefore A is an infinite subset of X. Now let $p \in X$ be arbitrary. Then because $\mathscr{C}$ covers X, there exists an $m \in \mathbb{N}$ such that $p \in C_m$. If $(C_m - \{p\}) \cap A = \emptyset$, we are done. Otherwise, set $A' = (C_m - \{p\}) \cap A$. If $a_n \in A'$, then $a_n \neq p$ and $a_n \in C_m$, so $n < m$ since $a_n \notin \bigcup_{i=1}^{n} C_i$ means for all $n \geq m$, $C_m \subset \bigcup_{i=1}^{n} C_i$. Therefore $a_n \in \bigcup_{i=1}^{m-1} \{a_i\}$, so $A' \subset \bigcup_{i=1}^{m-1} \{a_i\}$ and is therefore finite. A' being finite means it is the union of singletons, all of which are closed because X is a T_1 space. Therefore A' is closed, so the set $C_m - A'$ is open by Theorem 2.15. By definition, $p \notin A'$, so $C_m - A'$ is an open set containing p and

$$((C_m - A') - \{p\}) \cap A = ((C_m - \{p\}) \cap A) - A' = A' - A' = \emptyset.$$

Since we have found such an open set $C_m - A'$, the point $p \in X$ is not a limit point of A, and since p was arbitrary, A has no limit points despite being an infinite set. □

Theorem 6.27. Let X be a Lindelöf space. Then every uncountable set has a limit point.

Proof. Let X be a space with uncountable subset A such that A does not have a limit point. We will show that X is not Lindelöf. For all $p \in A$, p is not a limit point of A, so there exists an open set U_p containing p such that $(U_p - \{p\}) \cap A = \emptyset$. Additionally, A vacuously contains all of its limit points, so A is closed and $X - A$ is open. Consider the collection $\mathscr{C} = \{U_p\}_{p\in A} \cup \{X - A\}$. Note that if $p \neq p'$, for $p, p' \in A$ then $p' \notin U_p$, and therefore $U_p \neq U_p'$, meaning the map $f : A \to \mathscr{C}$ defined by $f(p) = U_p$ is an injection and $\mathscr{C}$ is therefore an uncountable open cover of X. Suppose $N \subset \mathscr{C}$ is countable. Then there exists an $x \in A$ such that $U_x \notin N$, otherwise $\{U_p\}_{p\in A} \subset N$ and N is uncountable. Let $U_y \in N$ be arbitrary. Then $(U_y - \{y\}) \cap A = \emptyset$, and since $x \in A$, we have that $x \notin (U_y - \{y\})$. We also have that $x \neq y$ since otherwise $U_x = U_y \in N$. Therefore $(U_y - \{y\}) \cap A$ implies that $x \notin U_y$. However, U_y was an arbitrary element of $N - \{X - A\}$, and because $x \in A$, also $x \notin X - A$, meaning there is no element of N that contains x. Therefore N does not cover X, and since N was an arbitrary countable subcollection of $\mathscr{C}$, no such subcollection covers X, so X is not Lindelöf. □

Theorem 6.29. If A is a closed subspace of a countably compact (Lindelöf) space, then A is countably compact (Lindelöf).

Proof. Let X be countably compact (Lindelöf) with closed subspace $A \subset X$ and let $\mathscr{C}$ be a countable open cover (an open cover) of A. Then $X - A$ is open, and so $\mathscr{C} \cup \{X - A\}$ a countable open cover (an open cover) of X, meaning it has a finite (countable) subcover, call it $\mathscr{C}'$. Then since $\mathscr{C}'$ covers X, it also covers A, and every element of A is in some $C \in \mathscr{C}'$ other than $X - A$, meaning $\mathscr{C}^* = \mathscr{C}' - \{X - A\}$ is a finite (countable) subcover of $\mathscr{C}$. Therefore A is countably compact (Lindelöf). □

Theorem 6.30. Every regular, Lindelöf space is normal.

Proof. Let X be regular and Lindelöf and let A and B be disjoint closed subsets of X. Since X is regular, for each $\alpha \in A$, there exists an open set U_α such that $\alpha \in U_\alpha \subset \overline{U_\alpha} \subset X - B$, meaning U_α contains α and $\overline{U_\alpha} \cap B = \emptyset$. Then since $\alpha \in U_\alpha$ for all $\alpha \in A$, we have that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of A. Since A is a closed subspace of the Lindelöf space X, A is Lindelöf by Theorem 6.29, and so there exists a countable subcover $\{U_{\alpha_i}\}_{i\in\mathbb{N}} \subset \{U_\alpha\}_{\alpha\in A}$. Similarly, for all $\beta \in B$, there exists an open V_β containing β with $\overline{V_\beta} \cap A = \emptyset$ such that there exists a countable collection $\{V_{\beta_j}\}_{j\in\mathbb{N}}$ that covers B. Note that these two collections $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ and $\{V_{\beta_j}\}_{j\in\mathbb{N}}$ satisfy all three conditions of the Normality Lemma (4.29), and so there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Therefore X is normal. □

Theorem 6.31. Let $\mathcal{B}$ be a basis for a space X. Then X is Lindelöf if and only if every cover of X by basic open sets in $\mathcal{B}$ has a countable subcover.

Proof. ($\Longrightarrow$) Let $\mathcal{B}$ be a basis for a Lindelöf space X and let $\mathscr{C}$ be a cover by basic open sets in $\mathcal{B}$. Then we have that $\mathscr{C}$ is an open cover, and so there exists a countable subcover because X is Lindelöf.

($\Longleftarrow$) Let $\mathcal{B}$ be a basis for a space X and suppose every cover of X by basic open sets has a countable subcover and let $\mathscr{C}$ be an open cover of X. Then for all $x \in X$, there exists some open $U_x \in \mathscr{C}$ such that $x \in U_x$. Then because $\mathcal{B}$ is a basis, there exists a basic open V_x such that $x \in V_x \subset U_x$. Then we have that $\{V_x\}_{x\in X}$ is an open cover of X. Because it is a cover of X by basic open sets, there exists a countable subcover $\{V_{x_i}\}_{i\in\mathbb{N}}$. Then we have that

$$X \subset \bigcup_{i\in\mathbb{N}} V_{x_i} \subset \bigcup_{i\in\mathbb{N}} C_{x_i},$$

so the collection $\{C_{x_i}\}_{i\in\mathbb{N}}$ is a countable subcover of $\mathscr{C}$, and therefore X is Lindelöf. $\square$

Corollary 6.32. Every 2^{nd} countable space is Lindelöf.

Proof. Let X be 2^{nd} countable. Then there exists a countable basis $\mathcal{B}$ for the topology on X. Let $\mathscr{C}$ be a cover of X by basic open sets. Then $\mathscr{C} \subset \mathcal{B}$ and so $\mathscr{C}$ itself is countable and since it is a subcover of itself, we have that every cover of X by basic open sets has a countable subcover. By Theorem 6.31, X is Lindelöf. $\square$

7 Continuity: When Nearby Points Stay Together

7.1 Continuous Functions

Theorem 7.1. Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:

(1) The function f is continuous.

(2) For every closed set K in Y, the inverse image $f^{-1}(K)$ is closed in X.

(3) For every limit point p of a set A in X, the image $f(p) \in \overline{f(A)}$.

(4) For every $x \in X$ and open set V with $f(x) \in V$, there exists an open set U containing x such that $f(U) \subset V$.

Proof. (1) $\Longrightarrow$ (2): Let $f : X \to Y$ be continuous and let K be closed in Y. Then $Y - K$ is open in Y, so $f^{-1}(Y - K)$ is open in X. We also have that

$$f^{-1}(K) = f^{-1}(Y - (Y - K)) = f^{-1}(Y) - f^{-1}(Y - K) = X - f^{-1}(Y - K),$$

so $f^{-1}(K)$ is a closed set minus an open set and is therefore closed by Theorem 2.15.

(2) $\implies$ (3) Let p be a limit point of A and let $f : X \to Y$ be a function satisfying property (2). Since $f(A) \subset \overline{f(A)}$, we have that

$$A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)}).$$

Since $\overline{f(A)}$ is closed in Y, $f^{-1}(f(\overline{A}))$ is closed in X, and since it contains A, it contains $\overline{A}$ by Theorem 2.20. Therefore if p is a limit point of A, $p \in \overline{A} \subset f^{-1}(\overline{f(A)})$, so

$$f(p) \in f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$$

as required.

(3) $\implies$ (4) Let $f : X \to Y$ satisfy (3), let $x \in X$ be a point with $f(x) \in V$ for some set V open in Y, and define $U = X - \overline{f^{-1}(Y - V)}$. Then $x \in U$, since otherwise, we would have $x \in \overline{f^{-1}(Y - V)}$, which means either $x \in f^{-1}(Y - V)$, implying that $f(x) \in Y - V$, a contradiction, or that x is a limit point of $f^{-1}(Y - V)$, implying by (3) that

$$f(x) \in \overline{f(f^{-1}(Y - V))} \subset \overline{Y - V} = Y - V,$$

giving the same contradiction (the last equality follows from V being open in Y and $Y - V$ therefore already being closed). Therefore $x \in U$, and since U is the complement of the closed set $\overline{f^{-1}(Y - V)}$, U is open in X. To show $f(U) \subset V$, let $y_0 \in f(U)$ be arbitrary. Then there exists some $x_0 \in U$ such that $y_0 = f(x_0)$. Since $x_0 \in U$, $x_0 \notin \overline{f^{-1}(Y - V)}$ and therefore also $x_0 \notin f^{-1}(Y - V) \subset \overline{f^{-1}(Y - V)}$. This means that $y_0 = f(x_0) \notin Y - V$, and therefore we have that $y_0 \in V$, so indeed $f(U) \subset V$. Therefore for all $x \in X$ and open sets V containing $f(x)$, there exists an open U containing x such that $f(U) \subset X$.

(4) $\implies$ (1) Let $f : X \to Y$ be a function satisfying (4) and let V be open in Y. Then for all $x \in f^{-1}(V)$, there exists a U_x open in X and containing x such that $f(U_x) \subset V$. Define the set U as $U = \bigcup_{x \in f^{-1}(V)} U_x$. Then U is the union of open sets in X and is therefore itself open in X. We claim that $U = f^{-1}(V)$. To show $U \subset f^{-1}(V)$, let $x_0 \in U$. Then $x_0 \in \bigcup_{x \in f^{-1}(V)} U_x$, so there exists an $x' \in f^{-1}(V)$ such that $x_0 \in U_{x'}$, meaning $f(x_0) \in f(U_{x'}) \subset V$. Therefore $x \in f^{-1}(V)$. To show $f^{-1}(V) \subset U$, let $x_0 \in f^{-1}(V)$. Then $x_0 \in U_{x_0} \subset U$. Therefore $f^{-1}(V) = U$ is open in X and so f is continuous. $\square$

Theorem 7.2. Let $y_0 \in Y$ and define the map $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$. Then f is continuous.

Proof. Let $x \in X$ be arbitrary and let V be an open set containing $f(x)$. Then X is open and contains x, and we have that $f(X) = \{y_0\} = \{f(x)\}$, so $f(X) \subset V$. Therefore f is continuous by (4) in Theorem 7.1. □

Theorem 7.3. Let $X \subset Y$ be topological spaces and define the inclusion map $i : X \to Y$ by $i(x) = x$. Then i is continuous.

Proof. Let $x \in X$ be arbitrary and let V be an open set containing $i(x) = x$. Define $U = X \cap V \subset V$. Then U is open in X as a subspace of Y and contains x. Then since $i(U) = U$, $i(U) \subset V$ and so i is continuous by (4) in Theorem 7.1. □

Theorem 7.4. Let $f : X \to Y$ be a continuous map and let A be a subset of X. Then the restriction map $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.

Proof. Let A' be an arbitrary subset of $A \subset X$. Since $A' \subset X$, if $p \in A$ is a limit point of A', then $f(p) \in \overline{f(A')}$. Now since $A' \subset A$, $a \in A'$ means that $f|_A(a) = f(a)$, so $f|_A(A') = f(A')$. Therefore we have that $f|_A(p) = f(p) \in \overline{f(A')} = \overline{f|_A(A')}$, meaning $f|_A$ is continuous by (3) in Theorem 7.1. □

Theorem 7.5. A function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous if and only if for every point $x \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Proof. ($\Longrightarrow$) Suppose $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Then the interval $V = (f(x) - \varepsilon, f(x) + \varepsilon)$ is open in $\mathbb{R}_{\text{std}}$ and contains $f(x)$, so by (4) of Theorem 7.1, there exists an open set U in $\mathbb{R}_{\text{std}}$ such that U contains x and $f(U) \subset V$. Since U contains x, there exists a basic open set in $\mathbb{R}_{\text{std}}$ that contains x and is a subset of U, that is, there exists a $\delta > 0$ such that $x \in (x - \delta, x + \delta) \subset U$. Let $y \in \mathbb{R}$ be a point such that $|x - y| < \delta$. Then we have that $y \in (x - \delta, x + \delta) \subset U$, so $f(y) \in f(U) \subset V = (f(x) - \varepsilon, f(x) + \varepsilon)$. Therefore $|f(x) - f(y)| < \varepsilon$.

($\Longleftarrow$) Suppose $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is a function such that for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $x \in X$ and let V be an open set in $\mathbb{R}_{\text{std}}$ containing $f(x)$. We will show that there exists an open U containing X such that $f(U) \subset V$. Since $f(x) \in V$, there exists a basic open set in $\mathbb{R}_{\text{std}}$ containing $f(x)$ that is a subset of V. That is, there exists an $\varepsilon > 0$ such that $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$. Since $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Define $U = (x - \delta, x + \delta)$. Then U is open in $\mathbb{R}_{\text{std}}$ and contains x. Now let $z_0 \in f(U)$. Then there exists some $y_0 \in U$ such that $z_0 = f(y_0)$. Since $y_0 \in U$, we have that $|x - y_0| < \delta$, meaning $|f(x) - f(y_0)| < \varepsilon$. Therefore $z_0 = f(y_0) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$, so we have $f(U) \subset V$. By (4) of Theorem 7.1, f is continuous. $\square$

Theorem 7.6. Let X be 1^{st} countable. Then a function $f : X \to Y$ for some topological space Y is continuous if and only if for all convergent sequences $x_n \to x$ in X, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be a continuous map. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence converging to $x \in X$ and consider the sequence $(f(x_n))_{n\in\mathbb{N}}$. We claim that $f(x_n) \to f(x)$. Let V be an open set containing $f(x)$. By (4) of Theorem 7.1, there exists an open set U in X containing x such that $f(U) \subset V$. Since $x_n \to x$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$. Therefore for all $n > N$, $f(x_n) \in f(U) \subset V$, so since such an N exists, $f(x_n) \to f(x)$, as required.

($\Longleftarrow$) Let X be 1^{st} countable and let $f : X \to Y$ be a function such that for all convergent sequences $x_n \to x$ in X, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in Y. Let p be a limit point of a set A in X. Then by Theorem 5.18, X being 1^{st} countable implies that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A that converges to p. By assumption, this means $f(a_n) \to f(p)$. Since $a_n \in A$ for all $n \in \mathbb{N}$, we have that $f(a_n) \in f(A)$. By Theorem 2.30, this means that the limit of the sequence $(f(a_n))_{n\in\mathbb{N}}$ is in the closure $\overline{f(A)}$. That is, $f(p) \in \overline{f(A)}$, and since A was an arbitrary subset of X and p an arbitrary limit point of A, we have that f is continuous by (3) of Theorem 7.1. $\square$

Theorem 7.7. Let X be a space with a dense set D and let Y be Hausdorff. Then if $f, g : X \to Y$ are continuous functions such that $f(d) = g(d)$ for all $d \in D$, then $f(x) = g(x)$ for all $x \in X$.

Proof. Assume the hypotheses of the claim and suppose for contradiction that there exists some $x \in X$ such that $f(x) \neq g(x)$. Then because Y is Hausdorff, there exist two disjoint open sets V_f and V_g such that $f(x) \in V_f$ and $g(x) \in V_g$. Because both f and g are continuous, the preimages $U_f = f^{-1}(V_f)$ and $U_g = g^{-1}(V_g)$ are open in X and contain x. Therefore $U_f \cap U_g$ is a nonempty open set, and since D is dense in X, there exists a $d \in D$ such that $d \in U_f \cap U_g = f^{-1}(V_f) \cap g^{-1}(V_g)$. Therefore $f(d) \in V_f$ and $f(d) = g(d) \in V_g$, so $f(d) \in V_f \cap V_g$, but this is a contradiction since $V_f \cap V_g = \emptyset$. Therefore $f = g$. $\square$

Theorem 7.9. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions, and let V be an open set in Z. Then $g^{-1}(V)$ is open in Y, so $f^{-1}(g^{-1}(V))$ is open in X. We claim that $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Let $x \in f^{-1}(g^{-1}(V))$. Then $f(x) \in g^{-1}(V)$, meaning $(g \circ f)(x) = g(f(x)) \in V$, so $x \in (g \circ f)^{-1}(V)$. Now let $x \in (g \circ f)^{-1}(V)$. Then $g(f(x)) \in V$, meaning $f(x) \in g^{-1}(V)$ and therefore $x \in f^{-1}(g^{-1}(V))$. Since both inclusions hold, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X, and since V was an arbitrary open set in Z, $g \circ f$ is continuous. $\square$

Theorem 7.10. Let $X = A \cup B$ for closed subsets $A, B \subset X$. Then if $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined to be f on A and g on B is continuous.

Proof. Assume the hypotheses of the claim and let V be an arbitrary closed set in Y. Then

$$h^{-1}(V) \cap A = \{x \in X \mid h(x) \in V\} \cap A = \{x \in A \mid f(x) = h(x) \in V\} = f^{-1}(V).$$

Similarly, $h^{-1}(V) \cap B = g^{-1}(V)$. Since V is closed in Y, $f^{-1}(V)$ is closed in A and $g^{-1}(V)$ is closed in B. Then by Corollary 3.29, there exist sets C and D closed in X such that $f^{-1}(V) = A \cap C$ and $g^{-1}(V) = B \cap D$. The sets A and B are also closed in X, so $f^{-1}(V)$ and $g^{-1}(V)$ are the intersections of closed sets and are therefore closed. Therefore

$$h^{-1}(V) = (h^{-1}(V) \cap A) \cup (h^{-1}(V) \cap B) = f^{-1}(V) \cup g^{-1}(V)$$

is the union of sets that are closed in X, so also $h^{-1}(V)$ is closed in X. By (2) in Theorem 7.1, h is continuous. $\square$

Theorem 7.11. Let $X = A \cup B$ for open subsets $A, B \subset X$. Then if $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined to be f on A and g on B is continuous.

Proof. Assume the hypotheses of the claim and let V be an arbitrary open set in Y. Then

$$h^{-1}(V) \cap A = \{x \in X \mid h(x) \in V\} \cap A = \{x \in A \mid f(x) = h(x) \in V\} = f^{-1}(V).$$

Similarly, $h^{-1}(V) \cap B = g^{-1}(V)$. Since V is open in Y, $f^{-1}(V)$ is open in A and $g^{-1}(V)$ is open in B. Then by the definition of being open in the relative topology on a subspace, there exist sets C and D open in X such that $f^{-1}(V) = A \cap C$ and $g^{-1}(V) = B \cap D$. The sets A and B are also open in X, so $f^{-1}(V)$ and $g^{-1}(V)$ are the intersections of open sets and are therefore open. Therefore

$$h^{-1}(V) = (h^{-1}(V) \cap A) \cup (h^{-1}(V) \cap B) = f^{-1}(V) \cup g^{-1}(V)$$

is the union of sets that are open in X, so also $h^{-1}(V)$ is open in X. Therefore h is continuous. $\square$

Exercise 7.12. The pasting lemma does not work on arbitrary sets A and B. For instance, let $A = (0,1)$, let $B = [1,2]$, let $f : A \to \mathbb{R}_{\text{std}}$ be given by $f(x) = 0$, and let $g : B \to \mathbb{R}_{\text{std}}$ be given by $g(x) = 1$. Since f and g are constant functions, they are continuous by Theorem 7.2, and they agree on $A \cap B = \emptyset$. However, the function $h : (0,2] \to \mathbb{R}_{\text{std}}$ such that $h = f$ on A and $h = g$ on B is not continuous, since 1 is a limit point of the set $A = (0,1) \subset (0,2]$ but $h(1) = 1 \notin \{0\} = \overline{h(A)}$, violating (3) of Theorem 7.1.

Theorem 7.13. Let $f : X \to Y$ be a function and let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if for every open set $V \in \mathcal{B}$, the preimage $f^{-1}(V)$ is open in X.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be continuous and let $\mathcal{B}$ be a basis for Y. If $V \in \mathcal{B}$, then V is open in Y and so $f^{-1}(V)$ is open in X since f is continuous.

($\Longleftarrow$) Let $f : X \to Y$ be a function such that for all $V \in \mathcal{B}$, a basis for Y, $f^{-1}(V)$ is open in X. Let U be an arbitrary open set in Y. Then for all $y \in U$, there exists a $V_y \in \mathcal{B}$ such that $y \in V_y \subset U$. Then we have that $f^{-1}(V_y)$ is open in X. We claim that $f^{-1}(U) = \bigcup_{y \in U} f^{-1}(V_y)$ is the union of open sets and is therefore open in X. Let $x \in f^{-1}(U)$. Then $f(x) \in U$, so there is a corresponding $V_{f(x)}$ containing $f(x)$, and we have that $x \in f^{-1}(V_{f(x)}) \subset \bigcup_{y \in U} f^{-1}(V_y)$. Now let $x \in \bigcup_{y \in U} f^{-1}(V_y)$. Then there exists a $y_0 \in U$ such that $x \in f^{-1}(V_{y_0})$, so $f(x) \in V_{y_0} \subset U$, meaning $x \in f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in X, and since U was an arbitrary open set in Y, the function $f : X \to Y$ is continuous. $\square$

Theorem 7.14. Let $f : X \to Y$ be a function and let $\mathscr{S}$ be a subbasis for Y. Then f is continuous if and only if for every open set $V \in \mathscr{S}$, the preimage $f^{-1}(V)$ is open in X.

Proof. ($\Longrightarrow$) Let $f : X \to Y$ be continuous and let $\mathscr{S}$ be a subbasis for Y. If $V \in \mathscr{S}$, then V is open in Y and so $f^{-1}(V)$ is open in X since f is continuous.

($\Longleftarrow$) Let $f : X \to Y$ be a function such that for all $V \in \mathscr{S}$, a subbasis for Y, $f^{-1}(V)$ is open in X. Recall that $\mathcal{B}$, the set of all finite intersections of elements in $\mathscr{S}$, is a basis for the topology on Y, and let $V \in \mathcal{B}$. Then we have that $V = \bigcap_{i=1}^{n} V_i$ for some $V_i \in \mathscr{S}$, so

$$f^{-1}(V) = f^{-1}\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} f^{-1}(V_i).$$

Since $V_i \in \mathscr{S}$, $f^{-1}(V_i)$ is open in X. Since $f^{-1}(V)$ is therefore the finite intersection of open sets in X, it is also open in X. Since V was an arbitrary basic open set, by Theorem 7.13, f is continuous. □

7.2 Properties Preserved by Continuous Functions

Theorem 7.15. If X is compact and $f : X \to Y$ is continuous and surjective, then Y is compact.

Proof. Let X be compact with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be an open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_x \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x \in X}$ is an open cover of X and therefore has a finite subcover, call it $\{f^{-1}(C_i)\}_{i \in F}$ for some finite subset $F \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i \in F}$ covers X, there exists an $i_0 \in F$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in F$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i \in F}$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, Y is compact. □

Theorem 7.16. If X is Lindelöf and $f : X \to Y$ is continuous and surjective, then Y is Lindelöf.

Proof. Let X be Lindelöf with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be an open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_X \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x \in X}$ is an open cover of X and therefore has a countable subcover, call it $\{f^{-1}(C_i)\}_{i \in N}$ for some countable subset $N \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i \in N}$ covers X, there exists an $i_0 \in N$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in N$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i \in N}$ is a countable subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary open cover, Y is Lindelöf. $\square$

Theorem 7.17. If X is countably compact and $f : X \to Y$ is continuous and surjective, then Y is countably compact.

Proof. Let X be countably compact with continuous surjection $f : X \to Y$ and let $\mathscr{C}$ be a countable open cover of Y. Then for all $x \in X$, $f(x) \in Y$, so there exists an open set $C_x \in \mathscr{C}$ containing $f(x)$. Therefore $f^{-1}(C_x)$ is open in X and contains x. Since all $x \in X$ are contained in a corresponding $f^{-1}(C_x)$, the collection $\{f^{-1}(C_x)\}_{x \in X}$ is a countable open cover of X and therefore has a finite subcover, call it $\{f^{-1}(C_i)\}_{i \in F}$ for some finite subset $F \subset X$. Now let $y_0 \in Y$ be arbitrary. Since f is surjective, there exists an $x_0 \in X$ such that $y_0 = f(x_0)$. Since $\{f^{-1}(C_i)\}_{i \in F}$ covers X, there exists an $i_0 \in F$ such that $x_0 \in f^{-1}(C_{i_0})$. Therefore we have $y_0 = f(x_0) \in C_{i_0}$. So since for every $y \in Y$, there exists an $i \in F$ such that $y \in C_i \in \mathscr{C}$, the collection $\{C_i\}_{i \in F}$ is a finite subcover of $\mathscr{C}$. Since $\mathscr{C}$ was an arbitrary countable open cover, Y is countably compact. $\square$

Theorem 7.18. If $f : X \to Y$ is continuous and surjective and X has a dense subset D, then $f(D)$ is dense in Y.

Proof. Let $f : X \to Y$ be continuous and surjective and let V be a nonempty open set in Y. Then $f^{-1}(V)$ is open in X, and because f is surjective and V is nonempty, there exists a $y \in V$ and therefore also an $x \in f^{-1}(V)$. Because D is dense in X and $f^{-1}(V)$ is open and nonempty, there exists a $d \in D \cap f^{-1}(V)$. Then $d \in f^{-1}(V)$ means $f(d) \in V$. Since V was an arbitrary open set in Y and we have found an element $f(d) \in f(D) \cap V$, the set $f(D)$ is dense in Y by Theorem 5.1. $\square$

Corollary 7.19. If X is separable and $f : X \to Y$ is continuous and surjective, then Y is separable.

Proof. Suppose X is separable. Then it has a countable dense subset D, and by Theorem 7.18, $f(D)$ is dense in Y. Since $f : X \to Y$ is surjective, the cardinality of D is greater than or equal to the cardinality of $f(D)$, meaning $f(D)$ is also countable. This means Y has a countable dense subset and is therefore separable. $\square$

Exercise 7.20. (1) Let f be the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology. Then U is open in $\mathbb{R}_{\text{std}}$, $f(U) \subset \mathbb{R}$ and because $\mathbb{R}$ here has the discrete topology, $f(U)$ is open. Therefore f is an open function. However, f is not continuous, because $\{0\} \subset \mathbb{R}$ is open in the discrete topology but $f^{-1}(\{0\}) = \{0\} \subset \mathbb{R}_{\text{std}}$. If $\varepsilon > 0$, then $(-\varepsilon, \varepsilon) \not\subset \{0\}$, so $\{0\}$ is not open in $\mathbb{R}_{\text{std}}$.

(2) The same function as in (1) is also closed: Let $A \subset \mathbb{R}_{\text{std}}$ be a closed subset. Then $X - f(A) \subset \mathbb{R}$ is open in the discrete topology, so $f(A)$ is closed in $\mathbb{R}$ with the discrete topology. However, f is not continuous.

(3) Now let f be the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the indiscrete topology. Then if $(0, 1)$ is open in $\mathbb{R}_{\text{std}}$ but not in $\mathbb{R}$ with the indiscrete topology, and similarly, $[0, 1]$ is closed in $\mathbb{R}_{\text{std}}$ but not in $\mathbb{R}$ with the indiscrete topology, so f is neither closed nor open. However, f is continuous. The open sets in $\mathbb{R}$ with the indiscrete topology are $\mathbb{R}$ and $\emptyset$ with preimages $f^{-1}(\mathbb{R}) = \mathbb{R}$ and $f^{-1}(\emptyset) = \emptyset$, both of which are open in $\mathbb{R}_{\text{std}}$.

(4) Consider $(0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$. Then by Theorem 7.3, the inclusion map $i : (0, 1) \to \mathbb{R}_{\text{std}}$ is continuous. If $U \subset (0, 1)$ is open in the relative topology on $(0, 1)$, then there exists a V open in $\mathbb{R}_{\text{std}}$ such that $U = (0, 1) \cap V$. The set $(0, 1)$ is open in $\mathbb{R}_{\text{std}}$, so $i(U) = U$ is the intersection of two open sets and is therefore open, meaning i is an open function. However, i is not closed. Note that $(0, 1)$ is closed in $(0, 1)$ (it is the whole space), but $i\left((0, 1)\right) = (0, 1)$ is not closed in $\mathbb{R}_{\text{std}}$ because it does not contain the limit points at 0 and 1.

(5) Consider $[0, 1]$ as a subspace of $\mathbb{R}_{\text{std}}$. As in (4), $i : [0, 1] \to \mathbb{R}_{\text{std}}$ is continuous, and if $A \subset [0, 1]$ is closed, by Theorem 3.28 there exists a closed D in $\mathbb{R}_{\text{std}}$ such that $A = [0, 1] \cap D$. Then $i(A) = A$ is the intersection of two closed sets and is therefore closed, so i is a closed map. However, i is not open. Note that $[0, 1]$ is open in $[0, 1]$ (it is the whole space), but $i([0, 1]) = [0, 1]$ is not open in $\mathbb{R}_{\text{std}}$ because for all $\varepsilon > 0$, the interval $(-\varepsilon, \varepsilon)$ contains 0 but is not a subset of $[0, 1]$.

Theorem 7.21. If X is normal and $f : X \to Y$ is continuous, surjective, and closed, then Y is normal.

Proof. Assume the hypotheses of the claim and let A and B be disjoint closed subsets of Y. Then because f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are closed in X, and we have that $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$. Since $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X and X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset f^{-1}(B)$. Since U is open, $X - U$ is closed in X, and therefore $f(X - U)$ is closed in Y since f is a closed function. We claim that $f(X - U) \subset Y - A$. Let $y \in f(X - U)$. Then there exists an $x \in X - U$ such that $y = f(x)$. Then $x \notin U$, and therefore $x \notin f^{-1}(A) \subset U$. Since

$x \notin f^{-1}(A) = \{x \in X \mid f(x) \in A\}$, we have that $y = f(x) \notin A$, so $y \in Y - A$. Therefore $f(X-U) \subset Y - A$, meaning also $A \subset Y - f(X-U)$. Since $f(X-U)$ is closed, $Y - f(X-U)$ is an open set containing A. Similarly, $Y - f(X-V)$ is an open set containing B. To show Y is normal, we show that these two open sets are disjoint. Suppose for contradiction that there exists some $y \in (Y - f(X-U)) \cap (Y - f(X-V))$. Then $y \notin f(X-U)$ and $y \notin f(X-V)$. Since f is surjective, there exists an $x \in X$ such that $y = f(x)$, and therefore $x \notin X - U$ and $x \notin X - V$. This means $x \in U \cap V = \emptyset$, a contradiction. Therefore A and B can be put in disjoint open sets, so Y is normal. □

Theorem 7.22. If $\{B_\alpha\}_{\alpha\in\lambda}$ is a basis for X and $f : X \to Y$ is continuous, surjective, and open, then $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.

Proof. Assume the hypotheses of the claim. Since B_α is open in X for all α and f is open, $f(B_\alpha)$ is open in Y, satisfying (1) of Theorem 3.1. Now let $y \in Y$ be contained in some open set V. Then because f is surjective, there exists an $x \in X$ such that $y = f(x)$. Since f is continuous, there exists an open U in X containing x such that $f(U) \subset V$. Because $\{B_\alpha\}_{\alpha\in\lambda}$ is a basis for X, there exists a B_β such that $x \in B_\beta \subset U$. Therefore $y = f(x) \in f(B_\beta) \subset f(U) \subset V$. Because $y \in Y$ was arbitrary with V an arbitrary open set containing y and we have shown there exists an $f(B_\beta)$ such that $y \in f(B_\beta) \subset V$, the collection $\{f(B_\alpha)\}_{\alpha\in\lambda}$ satisfies condition (2) of Theorem 3.1 and is therefore a basis for the topology on Y. □

Corollary 7.23. If X is 2^{nd} countable and $f : X \to Y$ is continuous, surjective, and open, then Y is 2^{nd} countable.

Proof. Assume the hypotheses of the claim. Since X is 2^{nd} countable, X has a countable basis $\{B_i\}_{i\in\mathbb{N}}$. Then by Theorem 7.22, $\{f(B_i)\}_{i\in\mathbb{N}}$ is a countable basis for Y and Y is therefore also 2^{nd} countable. □

Theorem 7.24. Let X be compact and Y be Hausdorff. If $f : X \to Y$ is continuous, then it is closed.

Proof. Let X be compact, let Y be Hausdorff, let $f : X \to Y$ be continuous, and let $A \subset X$ be closed. Then by Theorem 6.8, A is compact, and by Theorem 7.4, $f|_A : A \to Y$ is continuous. Therefore $f|'_A : A \to f(A)$ is continuous as well and is also surjective, so by Theorem 7.15, $f(A) \subset Y$ is compact. Since Y is Hausdorff, Theorem 6.9 implies $f(A)$ is closed. Since A was an arbitrary closed subset in X, $f(A)$ being closed in Y implies the function f is closed. □

Theorem 7.25. Let X be compact and 2^{nd} countable, and let Y be Hausdorff. Then if $f : X \to Y$ is continuous and surjective, Y is 2^{nd} countable.

Proof. Assume the hypotheses of the claim. Then X has a countable basis, call it $\mathcal{B}$. Denote by $\mathcal{B}^*$ the set of all finite subsets of $\mathcal{B}$, which is countable by Theorem 1.14, and denote by $\mathcal{B}'$ the set of all finite unions of elements of $\mathcal{B}$, that is

$$\mathcal{B}' = \left\{ \bigcup_{B \in B^*} B \mid B^* \in \mathcal{B}^* \right\}.$$

Then the cardinality of $\mathcal{B}'$ is less than or equal to that of $\mathcal{B}^*$ and so is countable. Therefore we write it as $\mathcal{B}' = \{B_i\}_{i \in \mathbb{N}}$. We claim that the collection $\mathcal{B}_Y = \{Y - f(X - B_i)\}_{i \in \mathbb{N}}$ is a countable basis for Y and that Y is therefore 2^{nd} countable. That this proposed basis is countable follows from $\mathcal{B}'$ being countable. Now note that every element of $B_i \in \mathcal{B}'$ is the union of basic open sets and is therefore open in X, meaning the sets $X - B_i$ are all closed in X. Because X is compact, Y is Hausdorff, and f is continuous, Theorem 7.24 implies that f is also closed. Therefore for all $i \in \mathbb{N}$, $f(X - B_i)$ is closed in Y, making $Y - f(X - B_i)$ open in Y. Since all elements of $\mathcal{B}_Y$ are open in Y, condition (1) of Theorem 3.1 is satisfied.

To show condition (2), let U be open in Y containing a point y. Since $\{y\} \subset U$, we have that $f^{-1}(\{y\}) \subset f^{-1}(U)$, which is open because f is continuous. Because f is surjective, $f^{-1}(\{y\})$ is nonempty. Then for all $x \in f^{-1}(\{y\})$, x is contained in the open set $f^{-1}(U)$, and since $\mathcal{B}$ is a basis for X, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset f^{-1}(U)$. Then we have that the collection $\{B_x\}_{x \in f^{-1}(\{y\})}$ is an open cover for $f^{-1}(\{y\})$. Since Y is Hausdorff, the singleton $\{y\}$ is closed, and therefore $f^{-1}(\{y\})$ is closed in X because f is continuous and nonempty because f is surjective. Since $f^{-1}(\{y\})$ is closed in X and X is compact, by Theorem 6.8, $f^{-1}(\{y\})$ is also compact. This means the open cover $\{B_x\}_{x \in f^{-1}(\{y\})}$ has a finite subcover, call it $\mathcal{B}_y$. Since it is a cover, we have

$$f^{-1}(\{y\}) \subset \bigcup_{B \in \mathcal{B}_y} B,$$

and since $\mathcal{B}_y$ is a finite subset of $\mathcal{B}$, $\mathcal{B}_y \in \mathcal{B}^*$, and therefore $\bigcup_{B \in \mathcal{B}_y} B = B_k \in \mathcal{B}'$ for some $k \in \mathbb{N}$. We now wish to show that $y \in Y - f(X - B_k) \subset U$.

Suppose $x \in X$ such that $f(x) \notin Y - f(X - B_k)$. Then $f(x) \in f(X - B_k)$, which means there exists an $x' \in X - B_k$ (not necessarily equal to x) such that $f(x') = f(x)$. Then $x' \in X - B_k$ means $x' \notin B_k$, and since $f^{-1}(\{y\}) \subset B_k$, we have that $x' \notin f^{-1}(\{y\})$.

Therefore $f(x) = f(x') \neq y$. Since $f(x) \notin Y - f(X - B_k)$ implies that $f(x) \neq y$, we have that if $y = f(x)$, then $y = f(x) \in Y - f(X - B_k)$. Since y can indeed be written as $f(x)$ for some $x \in X$ by the surjectivity of f, we have $y \in Y - f(X - B_k)$ as desired. Now recall that for all $x \in f^{-1}(\{y\})$, $B_x \subset f^{-1}(U)$. Therefore all elements of the finite subcover $\mathcal{B}_y$ are subsets of $f^{-1}(U)$, and so the union B_k of the sets in $\mathcal{B}_y$ is also a subset of $f^{-1}(U)$. Then $B_k \subset f^{-1}(U)$ implies that $X - f^{-1}(U) \subset X - B_k$, so also $f(X - f^{-1}(U)) \subset f(X - B_k)$. Now because for all $C, D \subset X$, $f(C) - f(D) \subset f(C - D)$, we have that

$$f(X) - f(f^{-1}(U)) \subset f(X - f^{-1}(U)) \subset f(X - B_k).$$

Since f is surjective, $f(X) = Y$ and $f(f^{-1}(U)) = U$. This means we have $Y - U \subset f(X - B_k)$, and therefore that $Y - f(X - B_k) \subset U$. We have taken an arbitrary y contained in an arbitrary open set U in Y, and have shown that there exists a $k \in \mathbb{N}$ such that $y \in Y - f(X - B_k) \subset U$, so (2) of Theorem 3.1 is satisfied in addition to (1), and therefore the set $\mathcal{B}_Y = \{Y - f(X - B_i)\}_{i \in \mathbb{N}}$ is a basis for Y. Because it is a countable basis, Y is 2^{nd} countable. □

7.3 Homeomorphisms

Theorem 7.26. Being homeomorphic is an equivalence relation on topological spaces.

Proof. Let X be a topological space and define $i : X \to X$ to be the inclusion map $i(x) = x$. Then i is a bijection and $i^{-1} = i$. Theorem 7.3 guarantees i and i^{-1} are continuous, so i is a homeomorphism, meaning X is homeomorphic to itself, so the relation is reflexive.

Let X and Y be topological spaces such that X is homeomorphic to Y. Then there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Then the map $f^{-1} : Y \to X$ is also bijective, and $(f^{-1})^{-1} = f$, so both directions are continuous, meaning Y is also homeomorphic to X and the relation is therefore symmetric.

Let X, Y, and Z be topological spaces such that X is homeomorphic to Y and Y is homeomorphic to Z. Then there exist homeomorphisms $f : X \to Y$ and $g : Y \to Z$. Then $f \circ g$ is a bijection since both f and g are bijections, and $f \circ g$ is continuous by Theorem 7.9. Sine $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and g^{-1} and f^{-1} are both continuous (since f and g are homeomorphisms), $(f \circ g)^{-1}$ is continuous as well, again by Theorem 7.9. Therefore $f \circ g : X \to Z$ is a homeomorphism, and so the relation is transitive, meaning it is an equivalence relation on the set of topological spaces. □

Exercise 7.27. If $a < b$, then $\mathbb{R}_{\text{std}}$ is homeomorphic to (a, b) as a subspace of $\mathbb{R}_{\text{std}}$.

Proof. The function $f : \mathbb{R} \to (a, b)$ given by

$$f(x) = \frac{b-a}{1+e^{-x}} + a$$

is a homeomorphism. □

Theorem 7.28. If $f : X \to Y$ is continuous, then the following are equivalent:

(a) f is a homeomorphism.

(b) f is a closed bijection.

(c) f is an open bijection.

Proof. (a) $\implies$ (b): Suppose $f : X \to Y$ is a homeomorphism and let $K \subset X$ be closed. Then f is a bijection so f^{-1} exists, and since it is a homeomorphism, $f^{-1} : Y \to X$ is continuous. Therefore K closed in X means $(f^{-1})^{-1}(K)$ is closed in Y, but $(f^{-1})^{-1} = f$ since f is bijective, and therefore $f(K)$ is closed in Y, meaning f is a closed bijection.

(b) $\implies$ (c): Suppose $f : X \to Y$ is a closed bijection and let $U \subset X$ be open. Then $X - U$ is closed in X, which means $f(X - U)$ is closed in Y. Since f is a bijection, $f(X - U) = f(X) - f(U) = Y - f(U)$, and since this is closed in Y, $f(U)$ must be open in Y. Therefore f is an open bijection.

(c) $\implies$ (a): Suppose $f : X \to Y$ is an open bijection. Then $f^{-1} : Y \to X$ exists, and if U is an open set in X, $(f^{-1})^{-1}(U) = f(U)$ is open in Y since f is open. Therefore f^{-1} is continuous, and so f is a continuous bijection such that f^{-1} is also continuous. Therefore f is a homeomorphism. □

Theorem 7.29. Let X be compact and Y be Hausdorff. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof. If X is compact and Y is Hausdorff, $f : X \to Y$ being continuous means it is closed by Theorem 7.24. Since f is also a bijection, it is a homeomorphism by Theorem 7.28. □

Exercise 7.30. Recall that by (3) of Exercise 7.20, the identity function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}$ where the codomain has the indiscrete topology is continuous but neither closed nor open. This remains the case if we use the identity function $f : A \to A$ where the domain $A \subset \mathbb{R}_{\text{std}}$ has the relative topology from $\mathbb{R}_{\text{std}}$ and the codomain has the indiscrete topology. Here, f is a continuous bijection and the domain is compact by Theorem 6.15, but f is neither closed nor open, so f is not a homeomorphism, which shows that the codomain being Hausdorff is necessary.

Recall that by (1) of Exercise 7.20, the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology is not continuous, and consider the identity function $f : \mathbb{R} \to \mathbb{R}_{\text{std}}$ where the domain has the discrete topology. This is a bijection, and it is continuous because if U is open in $\mathbb{R}_{\text{std}}$, then $f^{-1}(U) = U$ is open in $\mathbb{R}$ with the discrete topology. However, f^{-1} is the identity function from $\mathbb{R}_{\text{std}}$ to $\mathbb{R}$ with the discrete topology and is therefore not continuous, meaning f is not a homeomorphism. $\mathbb{R}_{\text{std}}$ is Hausdorff, but $\mathbb{R}$ with the discrete topology is not compact, so this shows the necessity of the domain being compact.

Corollary 7.31. Let X be compact and Y be Hausdorff. Then if $f : X \to Y$ is continuous and injective, f is an embedding.

Proof. Since $f : X \to Y$ is injective, $f : X \to f(X)$ is a continuous bijection. Since Y is Hausdorff and all Hausdorff spaces are hereditarily Hausdorff, $f(X)$ is Hausdorff and so by Theorem 7.29, $f : X \to f(X)$ is a homeomorphism and therefore f is an embedding. □

7.4 Product Spaces and Continuity

Theorem 7.32. Let X and Y be topological spaces. Then the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous, surjective, and open.

Proof. Let X and Y be nonempty topological spaces. Then there exists a $y_0 \in Y$, so if $x \in X$, then $(x, y_0) \in X \times Y$ and $\pi_X(x, y_0) = x$, so π_X is surjective (and similarly so is π_Y). To check continuity, let U be open in X. Then $\pi^{-1}(U)$ is a subbasic open set in $X \times Y$, so it is open. Therefore π_X is continuous (and similarly, π_Y is as well). Now let V be open in $X \times Y$. Then V is the union $V = \bigcup_{\alpha\in\lambda} U_\alpha \times W_\alpha$ of products of nonempty open sets U_α open in X and W_α open in Y. We claim that $\pi_X(V) = \bigcup_{\alpha\in\lambda} U_\alpha$. Let $x \in \pi_X(V)$. Then there exists a $y \in Y$ such that $(x, y) \in V$, so there exists a $\beta \in \lambda$ such that $(x, y) \in U_\beta \times W_\beta$. Therefore $x \in U_\beta \subset \bigcup_{\alpha\in\lambda} U_\alpha$. Now let $x \in \bigcup_{\alpha\in\lambda} U_\alpha$. Then there exists a $\beta \in \lambda$ such that $x \in U_\beta$, and since the U_α and W_α were assumed to be nonempty, there exists a $y \in W_\beta$, and therefore $(x, y) \in U_\beta \times W_\beta \subset V$. Then $(x, y) \subset V$ means $x \in \pi_X(V)$, so we have shown both inclusions, and therefore $\pi_X(V)$ is the union of open sets in X and is itself open in X. Similarly, $\pi_Y(V)$ is open in Y, so the projection functions are continuous, surjective, and open, as required. □

Theorem 7.33. The product topology on $X \times Y$ is the coarsest topology such that the projection maps are continuous.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$ and let $\mathcal{T}'$ denote a strictly coarser topology on $X \times Y$. Since $\mathcal{T}'$ is strictly coarser, there exists a set $U \in \mathcal{T} - \mathcal{T}'$, that is, a set open in the product topology but not in the coarser topology. Then since $U \in \mathcal{T}$, U is the union of sets $\{U_\alpha\}_{\alpha \in \lambda}$ such that the U_α are finite intersections of subbasic open sets in the subbasis for the product topology. Since $\mathcal{T}'$ is a topology, if all U_α were in $\mathcal{T}'$, their union U would also be in $\mathcal{T}'$, meaning there exists some $\beta \in \lambda$ such that $U_\beta \notin \mathcal{T}'$. Then there exists a collection of subbasic open sets $\{S_i\}_{i=1}^n$ in the product topology's subbasis such that $U_\beta = \bigcap_{i=1}^n S_i$. Again because $\mathcal{T}'$ is a topology, if $S_i \in \mathcal{T}'$ for all $i = 1, \ldots, n$, we would have that their intersection $U_\beta \in \mathcal{T}'$, but since this is not the case, there must exist some $1 \leq j \leq n$ such that $S_j \notin \mathcal{T}'$. Since S_j is a subbasic open set in the product topology, it is the inverse image of some open set in either X or Y. Without loss of generality, assume S_j is of the form $\pi_X^{-1}(V)$ for some set V open in X. Then $\pi_X : X \times Y \to X$ is not continuous, because there exists V open in X such that $\pi_X^{-1}(V) = S_j$ is not open in $X \times Y$ with the coarser topology. Since $\mathcal{T}'$ was an arbitrary topology strictly coarser than the product topology and the projection maps are not necessarily continuous under it, we have that the product topology is the coarsest topology on $X \times Y$ that guarantees the projection maps are continuous. $\square$

Exercise 7.34. The set

$$S = \left\{ \left(x, \frac{1}{1+e^{-x}} \right) \,\middle|\, x \in \mathbb{R} \right\}$$

as a subset of $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is closed, but its projection $\pi_Y(S)$ is the interval $(0,1)$, which is not closed in $\mathbb{R}_{\text{std}}$.

Theorem 7.35. Let X and Y be topological spaces. Then for every $y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is homeomorphic to X.

Proof. Let X and Y be topological spaces and let $y_0 \in Y$. Then the projection function $\pi_X : X \times Y \to X$ is continuous, surjective, and open. By Theorem 7.4, the restriction map $f = \pi_X|_{X \times \{y_0\}}$ is also continuous. It is injective because if $x_1 = f(x_1, y_0) = f(x_2, y_0) = x_2$ in X, then $(x_1, y_0) = (x_2, y_0)$ in $X \times \{y_0\}$, and it is surjective because if $x \in X$, then $(x, y_0) \in X \times \{y_0\}$ and $f(x, y_0) = x$. Now consider the inverse map $f^{-1} : X \to X \times \{y_0\}$. To check continuity, by Theorem 7.13 it suffices to check that $f = (f^{-1})^{-1}$ of a basic open set in $X \times \{y_0\}$ is open in X. By Theorem 3.30, all basic open sets in $X \times \{y_0\}$ are of the form $(U \times V) \cap (X \times \{y_0\})$ where U is open in X and V is open in Y. If such a basic open set is nonempty, then we have that $(U \times V) \cap (X \times \{y_0\}) = U \times \{y_0\}$ because $U \subset X$ and $y_0 \in V$. Then

$$(f^{-1})^{-1}(U \times \{y_0\}) = f(U \times \{y_0\}) = \pi_X|_{X\times\{y_0\}}(U \times \{y_0\}) = \pi_X(U \times \{y_0\}) = U$$

where the second to last equality follows from $U \times \{y_0\} \subset X \times \{y_0\}$. Since U is open in X, we have that the preimages of basic open sets in $X \times \{y_0\}$ under f^{-1} are open in X, so f^{-1} is continuous. Therefore f is a homeomorphism since it is also a continuous bijection. Since $y_0 \in Y$ was arbitrary, all such slices are homeomorphic to X. $\square$

Theorem 7.36. Let X, Y, and Z be topological spaces. A map $g : Z \to X \times Y$ is continuous if and only if $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Proof. ($\Longrightarrow$) Let X, Y, and Z be topological spaces and let $g : Z \to X \times Y$ be continuous. Since π_X and π_Y are continuous by Theorem 7.32, Theorem 7.9 implies that $\pi_X \circ g$ and $\pi_Y \circ g$ are continuous as well.

($\Longleftarrow$) By Theorem 7.14 is suffices to check that the preimages of subbasic open sets in $X \times Y$ are open in Z. Let V be a subbbasic open set in $X \times Y$. Then V is of the form $\pi_X^{-1}(U)$ for U open in X or $\pi_Y^{-1}(W)$ for W open in Y. Without loss of generality assume the former. Then

$$g^{-1}(V) = g^{-1}(\pi_X^{-1}(U)) = (\pi_X \circ g)^{-1}(U)$$

is open in Z because U is open in X and $\pi_X \circ g : Z \to X$ is continuous. Therefore g is continuous. $\square$

Theorem 7.38. Let $\prod_{\alpha\in\lambda} X_\alpha$ be the product of spaces $\{X_\alpha\}_{\alpha\in\lambda}$. The projection function $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ is continuous, surjective, and open.

Proof. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be nonempty spaces. If one were empty, we would have $\prod_{\alpha\in\lambda} X_\alpha = \emptyset$, and so $\pi_\beta(\emptyset) = \emptyset$ is open in X_β making π_β and open map, π_β is surjective vacuously, and $\pi_\beta^{-1}(U)$ is still a subbasic open set and therefore is open, making π_β a continuous map. Therefore we assume the X_α are nonempty. Then if U is open in X_β, $\pi_\beta^{-1}(U)$ is a subbasic open set in the product, so π_β is continuous. Let $x \in X_\beta$ be an arbitrary point. Then since X_α are all nonempty, there exists an $a_\alpha \in X_\alpha$, so the point $(x_\alpha)_{\alpha\in\lambda}$ with $x_\alpha = a_\alpha$ for $\alpha \neq \beta$ and $x_\beta = x$ is in the product space and $\pi_\beta((x_\alpha)_{\alpha\in\lambda}) = x$, so π_β is surjective. Now let V be an open set in $\prod_{\alpha\in\lambda} X_\alpha$. Then V is the union of nonempty finite intersections of subbasic open sets, that is,

$$V = \bigcup_{\gamma\in\mu} V_\gamma \quad \text{where} \quad V_\gamma = \bigcap_{i=1}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i)$$

for μ an index set, $n_\gamma \in \mathbb{N}$ and U_i open in X_{α_i} for all $i = 1, \ldots, n_\gamma$. Then

$$\pi_\beta(V) = \pi_\beta\left(\bigcup_{\gamma\in\mu} V_\gamma\right) = \bigcup_{\gamma\in\mu} \pi_\beta(V_\gamma) = \bigcup_{\gamma\in\mu} \pi_\beta\left(\bigcap_{i=1}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i)\right).$$

If $\alpha_i \neq \beta$ for all i, then we claim that $\pi_\beta(V_\gamma) = X_\beta$. Because the codomain of π_β is X_β, we have that $\pi_\beta(V_\gamma) \subset X_\beta$. To show the other inclusion, let $x_0 \in X_\beta$. Since V_γ is nonempty, $U_i \neq \emptyset$ for $i = 1, \ldots, n_\gamma$, so for each i there exists an $x_i \in U_i$. Then for all X_α such that $\alpha \neq \alpha_i$, there exists a point $x'_\alpha \in X_\alpha$. Then the point $(x_\alpha)_{\alpha\in\lambda}$ given by

$$x_\alpha = \begin{cases} x_0 & \alpha = \beta \\ x_i & \alpha = \alpha_i,\ i = 1, \ldots, n_\gamma \\ x'_\alpha & \alpha \notin \{\beta\} \cup \{\alpha_i \mid i = 1, \ldots, \gamma\} \end{cases}$$

is an element of $\pi_{\alpha_i}^{-1}(U_i)$ for all $i = 1, \ldots, \gamma$, and so $(x_\alpha)_{\alpha\in\lambda} \in \bigcap_{i=1}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i) = V_\gamma$. Then x_0 is a coordinate of a point in V_γ, so $x_0 \in \pi_\beta(V_\gamma)$. Therefore $\pi_\beta(V_\gamma) = X_\beta$ is open in X_β. Now if there exists some $N \subset \{1, \ldots, n_\gamma\}$ such that $i \in N$ implies $\alpha_i = \beta$, we claim that $\pi_\beta(V_\gamma) = \bigcap_{i\in N} U_i$. Using a similar argument to the above, define x_0 now to be a point in $\bigcap_{i\in N} U_i$, define the x_i the same way as earlier but now only for $i \notin N$, and define the x'_α exactly as earlier. Then the point $(x_\alpha)_{\alpha\in\lambda}$ given by

$$x_\alpha = \begin{cases} x_0 & \alpha = \beta \\ x_i & \alpha = \alpha_i,\ i = 1, \ldots, n_\gamma, \text{ and } i \notin N \\ x'_\alpha & \alpha \notin \{\beta\} \cup \{\alpha_i \mid i = 1, \ldots, \gamma\} \end{cases}$$

is again an element of $\pi_{\alpha_i}^{-1}(U_i)$ for all $i = 1, \ldots, \gamma$, and so $(x_\alpha)_{\alpha\in\lambda} \in \bigcap_{i=1}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i) = V_\gamma$. Since $x_0 \in X_\beta$ is a coordinate of a point in V_γ, $x_0 \in \pi_\beta(V_\gamma)$, and therefore $\bigcap_{i\in N} U_i \subset \pi_\beta(V_\gamma)$. To show the other direction, recall again that $\pi_\beta(A) \subset X_\beta$ for any $A \subset \prod_{\alpha\in\lambda} X_\alpha$, so we have that following:

$$
\begin{aligned}
\pi_\beta(V_\gamma) &= \pi_\beta\left(\left(\bigcap_{\substack{i=1\\ i\notin N}}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i)\right)\cap\left(\bigcap_{i\in N}\pi_\beta^{-1}(U_i)\right)\right)\\
&\subset \pi_\beta\left(\bigcap_{\substack{i=1\\ i\notin N}}^{n_\gamma} \pi_{\alpha_i}^{-1}(U_i)\right)\cap \pi_\beta\left(\bigcap_{i\in N}\pi_\beta^{-1}(U_i)\right)\\
&\subset X_\beta \cap \pi_\beta\left(\bigcap_{i\in N}\pi_\beta^{-1}(U_i)\right) = \pi_\beta\left(\pi_\beta^{-1}\left(\bigcap_{i\in N} U_i\right)\right) = \bigcap_{i\in N} U_i
\end{aligned}
$$

where the last equality follows by the surjectivity of π_β. Therefore $\pi_\beta(V_\gamma)$ is the intersection of finitely many sets that are all open in X_β, meaning $\pi_\beta(V_\gamma)$ is open in X_β in this case as well. Since $\pi_\beta(V_\gamma)$ is open in X_β for all $\gamma \in \mu$,

$$
\pi_\beta(V) = \pi_\beta\left(\bigcup_{\gamma\in\mu} V_\gamma\right) = \bigcup_{\gamma\in\mu}\pi_\beta(V_\gamma)
$$

is the union of open sets in X_β and is therefore also open in X_β. Since V was an arbitrary open set in $\prod_{\alpha\in\lambda} X_\alpha$ and $\pi_\beta(V)$ is open in the codomain X_β, the map π_β is open. $\square$

Theorem 7.39. The product topology on $\prod_{\alpha\in\lambda} X_\alpha$ is the coarsest topology that makes the projection maps continuous.

Proof. Let $\mathcal{T}$ denote the product topology on $\prod_{\alpha\in\lambda} X_\alpha$ and let $\mathcal{T}'$ denote a strictly coarser topology on $\prod_{\alpha\in\lambda} X_\alpha$. Since $\mathcal{T}'$ is strictly coarser, there exists a set $U \in \mathcal{T} - \mathcal{T}'$, that is, a set open in the product topology but not in the coarser topology. Then since $U \in \mathcal{T}$, U is the union of sets $\{U_\gamma\}_{\gamma\in\mu}$ such that the U_γ are finite intersections of subbasic open sets in the subbasis for the product topology. Since $\mathcal{T}'$ is a topology, if all U_γ were in $\mathcal{T}'$, their union U would also be in $\mathcal{T}'$, meaning there exists some $\delta \in \mu$ such that $U_\delta \notin \mathcal{T}'$. Then there exists a collection of subbasic open sets $\{S_i\}_{i=1}^n$ in the product topology's subbasis such that $U_\delta = \bigcap_{i=1}^n S_i$. Again because $\mathcal{T}'$ is a topology, if $S_i \in \mathcal{T}'$ for all $i = 1,\dots,n$, we would have that their intersection $U_\delta \in \mathcal{T}'$, but since this is not the case, there must exist some $1 \le j \le n$ such that $S_j \notin \mathcal{T}'$. Since S_j is a subbasic open set in the product topology, it is the inverse image of some open set in one of the factors X_α. Without loss of generality, assume S_j is of the form $\pi_\beta^{-1}(V)$ for some set V open in X_β. Then $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$

is not continuous, because there exists V open in X_β such that $\pi_\beta^{-1}(V) = S_j$ is not open in $\prod_{\alpha\in\lambda} X_\alpha$ with the coarser topology. Since $\mathcal{T}'$ was an arbitrary topology strictly coarser than the product topology and the projection maps are not necessarily continuous under it, we have that the product topology is the coarsest topology on $\prod_{\alpha\in\lambda} X_\alpha$ that guarantees the projection maps are continuous. $\square$

Theorem 7.40. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of spaces. A map $g : Z \to \prod_{\alpha\in\lambda} X_\alpha$ for a space Z is continuous if and only if $\pi_\beta \circ g$ is continuous for all $\beta \in \lambda$.

Proof. ($\Longrightarrow$) Assume the hypotheses of the claim and suppose g is continuous. Since π_β is continuous for all $\beta \in \lambda$ by Theorem 7.38, Theorem 7.9 implies that $\pi_\beta \circ g$ is continuous for all $\beta \in \lambda$.

($\Longleftarrow$) By Theorem 7.14 is suffices to check that the preimages of subbasic open sets in $\prod_{\alpha\in\lambda} X_\alpha$ are open in Z. Let V be a subbasic open set in $\prod_{\alpha\in\lambda} X_\alpha$. Then V is of the form $\pi_\beta^{-1}(U)$ for some $\beta \in \lambda$ and U open in X_β. Then

$$g^{-1}(V) = g^{-1}(\pi_\beta^{-1}(U)) = (\pi_\beta \circ g)^{-1}(U)$$

is open in Z because U is open in X_β and $\pi_\beta \circ g : Z \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous. Therefore g is continuous. $\square$

Exercise 7.41. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be defined by $f(x) = (x, x, \dots)$. With the product topology, we check that $\pi_n \circ f : \mathbb{R} \to \mathbb{R}$ is continuous for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Then

$$(\pi_n \circ f)(x) = \pi_n(f(x)) = \pi_n(x, x, \dots) = x,$$

so $\pi_n \circ f$ is the inclusion map $i : \mathbb{R} \to \mathbb{R}$ and is therefore continuous by Theorem 7.3. Since $\pi_n \circ f$ is continuous for all $n \in \mathbb{N}$, Theorem 7.40 implies that f is continuous. Now consider the box topology. Since the interval $(-1/n, 1/n)$ is open in $\mathbb{R}$ for all $n \in \mathbb{N}$, the set

$$U = \prod_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is open in the box topology on $\mathbb{R}^\omega$. Let $x \neq 0$ be an arbitrary point in $\mathbb{R}$. Then $f(x) = (x, x, \dots)$, and since $\mathbb{N}$ is unbounded above, there exists an $N \in \mathbb{N}$ such that $1/N < |x|$, and therefore that $x \notin (-1/n, 1/n)$ for all $n \geq N$. Therefore $f(x) \notin U$, so $x \notin f^{-1}(U)$ for all $x \neq 0$. Since for all $n \in \mathbb{N}$, $0 \in (-1/n, 1/n)$, we have that $f(0) = (0, 0, \dots) \in U$, and therefore that $f^{-1}(U) = \{0\}$, which is not open in $\mathbb{R}$. Therefore f is not continuous with the box topology.

Exercise 7.42. Recall (MATH 200) that the Cantor set is the set of all $x \in [0,1]$ with a ternary expansion consisting only of 0s and 2s. Then define $f : C \to \prod_{n\in\mathbb{N}}\{0,1\}$ by

$$f\left(\sum_{n\in\mathbb{N}} a_n \left(\frac{1}{3}\right)^n\right) = b_n = \begin{cases} 0 & a_n = 0 \\ 1 & a_n = 2 \end{cases}$$

so that (b_n) is a sequence in the product space $\prod_{n\in\mathbb{N}}\{0,1\}$. Since each $x \in C$ can be uniquely represented by a sequence (a_n) of zeros and twos as shown above, each $x \in C$ can be uniquely represented by a sequence (b_n). That is, f is a bijection.

Note also that in the definition of the Cantor set, the set C_k is the disjoint union of 2^k closed intervals. Each of these can be labeled by the common first k terms of the ternary expansions of the elements in each closed set. For example, $C_1 = [0,1/3] \cup [2/3,1]$. All elements of $[0,1/3]$ have ternary expansions beginning with $.0\ldots$ and all elements of $[2/3,1]$ have expansions beginning with $.2\ldots$, so write C_1 as $C_1 = C_{.0} \cup C_{.2}$. Similarly, $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$ which we will write as $C_2 = C_{.00} \cup C_{.02} \cup C_{.20} \cup C_{.22}$. Therefore the set $C_n = \bigcup C_{.(a'_n)}$ where the union is taken over all 2^n truncated sequences of length n consisting of only 0s and 2s.

To show continuity, recall that by Theorem 7.14, it suffices to check that preimages of subbasic open sets are open. Let V be a subbasic open set in $\prod_{n\in\mathbb{N}}\{0,1\}$. Then V is of the form $\pi_n^{-1}(U)$ for U open in $\{0,1\}$. If U is empty, then V is empty and $f^{-1}(V)$ is empty and therefore open in C. If $U = \{0,1\}$, then V is the entire product space, the preimage of which is the Cantor set and so $f^{-1}(V)$ is open in C in this case as well. The remaining possibility is that V is a singleton. Assume without loss of generality that $V = \{0\}$. Then V is the set of all sequences in the product space whose nth coordinate is 0, meaning $f^{-1}(V)$ is the subset of the Cantor set containing all points whose nth digit is a 0 in their ternary expansions. That is, all points in $f^{-1}(V)$ are points of the Cantor set and also in the one of the intervals $C_{.(a'_n)}$ such that (a'_n) ends in a 0. Call the collection of such intervals $\{C_{.(a'_n)0}\}$. Then $f^{-1}(V) = C \cap \bigcup C_{.(a'_n)0}$. To show that this is open in C, we construct a set U open in $\mathbb{R}_{\text{std}}$ such that $f^{-1}(V) = U \cap C$. Because $\mathbb{R}_{\text{std}}$ is normal and C_n is the union of finitely many pairwise disjoint closed sets, we can find 2^n pairwise disjoint open sets $U_{.00\ldots00}, U_{.00\ldots02}, \ldots, U_{.22\ldots22}$ such that $C_{(a'_i)} \subset U_{(a'_i)}$. As above, denote by $\{U_{.(a'_n)0}\}$ the collection of these open sets with index list of length n ending in 0. Then $\bigcup C_{.(a'_n)0} \subset \bigcup U_{.(a'_n)0}$ and this union is open in $\mathbb{R}_{\text{std}}$.

Then since under our bijection f, $f^{-1}(V) = C \cap \bigcup C_{.(a'_n)_0}$, $f^{-1}(V) \subset C \cap \bigcup U_{.(a'_n)_0}$, so it remains to show the reverse inclusion. Suppose $x \in C \cap \bigcup U_{.(a'_n)_0}$. Then since all $U_{.(a'_i)}$ are disjoint, $x \notin U_{.(a'_n)_2}$, meaning $x \notin C_{.(a'_n)_2}$. This means the nth coordinate of the ternary expansion of x is not equal to 2, and because it is in the Cantor set, it must therefore be equal to 0, meaning it is mapped to a sequence in $\prod_{n\in\mathbb{N}}\{0,1\}$ with nth coordinate equal to 0, showing that $f(x) \in V$. Therefore $x \in f^{-1}(V)$ and so we have that $f^{-1}(V) = C \cap \bigcup U_{.(a'_n)_0}$ is open in C.

Because each C_i is the union of finitely many closed sets, C_i is closed in $\mathbb{R}_{\text{std}}$ for all $i \in \mathbb{N}$. Therefore the Cantor set, which is the intersection of the C_i is itself closed in $\mathbb{R}_{\text{std}}$. Since it is also bounded, it is compact by Theorem 6.15. Since $\{0,1\}$ with the discrete topology is Hausdorff, the space $\prod_{n\in\mathbb{N}}\{0,1\}$ is the product of Hausdorff spaces and is therefore Hausdorff by Theorem 4.16. Hence $f : C \to \prod_{n\in\mathbb{N}}\{0,1\}$ is a continuous bijection with compact domain and Hausdorff codomain, so by Theorem 7.29, f is a homeomorphism.

7.5 Quotient Maps and Quotient Spaces

Exercise 7.43. The identification map is the map taking each point in $[0,2] \times [0,1]$ to the set in C^* containing it. Then a basis for the topology on C^* is the collection

$$\mathcal{B} = \left\{ \bigcup_{n\in\mathbb{Z}} B((x, y+n), r) \right\}_{(x,y)\in[0,2]\times[0,1]}$$

where $B((x,y),r)$ is an open ball of radius r centered at (x,y).

Exercise 7.44. The identification space is

$$X^* = \{\{(x,y)\} \mid x \in (0,8), y \in [0,1]\} \cup \{\{(0,y),(8,1-y)\} \mid y \in [0,1]\}$$

so that each point $(0,y)$ in the slice $\{0\} \times [0,1]$ is glued to the point $(8,1-y)$ in the slice $\{8\} \times [0,1]$.

Exercise 7.45. (1) Recall that C was constructed as points $(x, \sin\theta, \cos\theta) \in \mathbb{R}^3$ for $x \in [0,2]$ and $\theta \in [0,2\pi)$. We create the torus as an identification space C^* of C as

$$\begin{aligned} C^* = \{&\{(x,\sin\theta,\cos\theta)\} \mid x \in (0,2), \theta \in [0,2\pi)\} \\ &\cup \{\{(0,\sin\theta,\cos\theta),(1,\sin\theta,\cos\theta)\} \mid \theta \in [0,2\pi)\} \end{aligned}$$

so that the left $x = 0$ circle of the cylinder is identified with the right $x = 1$ circle of the cylinder.

(2) An identification space is

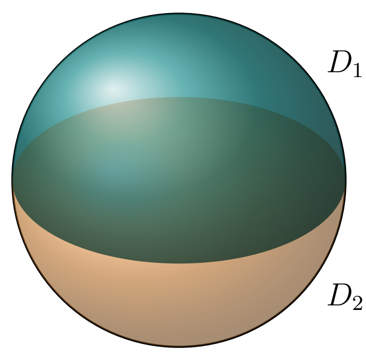
$$\begin{aligned} X^* = \{&\{(x,y)\} \mid x,y \in (0,1)\} \\ &\cup \{\{(x,0),(x,1)\} \mid x \in [0,1]\} \\ &\cup \{\{(0,y),(1,y)\} \mid y \in (0,1)\} \end{aligned}$$

so that all components are disjoint.

(3) An identification space T^* is

$$\begin{aligned} T^* = \{&\{(x+m, y+n) \mid m,n \in \mathbb{Z}\} \mid x,y \in (0,1)\} \\ &\cup \{\{(m, y+n) \mid m,n \in \mathbb{Z}\} \mid y \in (0,1)\} \\ &\cup \{\{(x+m, y) \mid m,n \in \mathbb{Z}\} \mid x \in [0,1)\}. \end{aligned}$$

Exercise 7.46. Describe the first disk as $D_1 = \{(x,y,0) \mid x^2 + y^2 \leq 1\}$ and the second as $D_2 = \{(x,y,1) \mid x^2 + y^2 \leq 1\}$. Then the identification shown here can be described as



$$D^* = \big\{\{(x,y,z)\} \mid x^2 + y^2 < 1, z \in \{0,1\}\big\} \cup \big\{\{(x,y,0),(x,y,1)\} \mid x^2 + y^2 = 1\big\}.$$

where the boundaries of the two disks are glued together.

Theorem 7.47. The quotient topology is a topology.

Proof. Let X and Y be topological spaces with a surjective map $f : X \to Y$. Consider the quotient topology on Y. We have that Y is open since $f^{-1}(Y) = X$ is open in X; $\emptyset$ is open in Y because $f^{-1}(\emptyset) = \emptyset$ is open in X; if U and V are open in Y, then $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ is the intersection of open sets in X and is therefore open in X, meaning

$U \cap V$ is open in Y; and lastly if $\{U_\alpha\}_{\alpha \in \lambda}$ is a collection of open sets in Y, then their union is also open in Y since

$$f^{-1}\left(\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha)\right)$$

is the union of open sets in X and is therefore itself open in X. $\square$

Theorem 7.48. Let X be a topological space, let Y be a set, and let $f : X \to Y$ be a surjection. The quotient topology on Y is the finest topology that make f continuous.

Proof. Let $\mathcal{Q}$ denote the quotient topology on Y and let $\mathcal{T}$ denote a strictly finer topology on Y. Since $\mathcal{T}$ is strictly finer, there exists some $U \in \mathcal{T} - \mathcal{Q}$. Consider $f^{-1}(U)$. Since U is not in $\mathcal{Q}$ and $\mathcal{Q}$ is defined to contain exactly those sets V for which $f^{-1}(V)$ is open in X, we have that $f^{-1}(U)$ is not open in X. Since U is open in Y under the topology $\mathcal{T}$ but $f^{-1}(U)$ is not open in X, the map f is not continuous when Y is given the topology $\mathcal{T}$. Since $\mathcal{T}$ was an arbitrary strictly finer topology on Y than $\mathcal{Q}$, we have that $\mathbb{Q}$ is the finest topology on Y such that f is continuous. $\square$

Theorem 7.49. Let X and Y be topological spaces. If $f : X \to Y$ is continuous, surjective, and open, then f is a quotient map.

Proof. To show that f is a quotient map, we show that the topology $\mathcal{T}$ on Y is the quotient topology $\mathcal{Q}$ with respect to f, that is, that the topology on Y contains exactly the sets $U \subset Y$ such that $f^{-1}(U)$ is open in X. If $U \in \mathcal{T}$, then by continuity of f, $f^{-1}(U)$ is open in X, meaning also $U \in \mathcal{Q}$, so we have that $\mathcal{T} \subset \mathcal{Q}$. Now let $V \in \mathcal{Q}$. Then by definition, $f^{-1}(V)$ is open in X, so since f is open, $f(f^{-1}(V)) \in \mathcal{T}$. Since f is surjective, $f(f^{-1}(V)) = V$, and so also $\mathcal{Q} = \mathcal{T}$. Therefore $\mathcal{T}$ is the quotient topology on Y, and so f is a quotient map. $\square$

Theorem 7.50. Let X and Y be topological spaces. If $f : X \to Y$ is continuous, surjective, and closed, then f is a quotient map.

Proof. To show that f is a quotient map, we show that the topology $\mathcal{T}$ on Y is the quotient topology $\mathcal{Q}$ with respect to f, that is, that the topology on Y contains exactly the sets $U \subset Y$ such that $f^{-1}(U)$ is open in X. If $U \in \mathcal{T}$, then by continuity of f, $f^{-1}(U)$ is open in X, meaning also $U \in \mathcal{Q}$, so we have that $\mathcal{T} \subset \mathcal{Q}$. Now let $V \in \mathcal{Q}$. Then by definition, $f^{-1}(V)$ is open in X, meaning also $X - f^{-1}(V)$ is closed. Since we have that

$$X - f^{-1}(V) = f^{-1}(Y) - f^{-1}(V) = f^{-1}(Y - V),$$

the set $f^{-1}(Y - V)$ is closed in X. Then because f is closed, $Y - V = f(f^{-1}(Y - V))$ is closed in Y with the topology $\mathcal{T}$. But this means that $V \in \mathcal{T}$, and so we have $\mathcal{Q} \subset \mathcal{T}$, meaning f is a quotient map. $\square$

Exercise 7.51. For a quotient map that isn't open, take the identification map f in Exercise 7.43. As an identification map, f is a quotient map, however, the set $U = B((1,1),1)$ is the open ball of radius 1 centered at $(1,1)$ and is open in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$, meaning $U \cap [0,2] \times [0,1]$ is open in $[0,2] \times [0,1]$. The image $f(U)$ is not open on the cylinder, however, because there is no open V in the topology on the cylinder such that $(1,1) \in V \subset f(U)$.

For a quotient map that isn't closed, take the projection map in Exercise 7.34 from $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$. Since projection maps are continuous, open, and surjective, they are quotient maps by Theorem 7.49. However, the set

$$S = \left\{ \left(x, \frac{1}{1 + e^{-x}} \right) \middle| \; x \in \mathbb{R} \right\}$$

as a subset of $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is closed, but its projection $\pi_Y(S)$ is the interval $(0,1)$, which is not closed in $\mathbb{R}_{\text{std}}$.

Exercise 7.52. Yes—by Theorem 7.38, projection maps are continuous, surjective, and open, so by Theorem 7.49, they are quotient maps.

Theorem 7.53. Let $f : X \to Y$ be a quotient map. Then a map $g : Y \to Z$ out of the quotient space Y is continuous if and only if $g \circ f : X \to Z$ is continuous.

Proof. ($\Longrightarrow$) Since f is a quotient map, it is continuous, meaning if g is continuous, the composition $g \circ f$ is also continuous by Theorem 7.9.

($\Longleftarrow$) Suppose $g \circ f : X \to Z$ is continuous and let U be an arbitrary open set in Z. Then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. Since $f^{-1}(g^{-1}(U))$ is open in X, $g^{-1}(U)$ is open in Y by the definition of the quotient topology on Y with respect to f. Since U was an arbitrary open set in Z and $g^{-1}(U)$ is open in Y, the map g is continuous. $\square$

Exercise 7.54. Recall that C was defined as the set of points $(x, \sin\theta, \cos\theta) \in \mathbb{R}^3$ such that $x \in [0,2]$ and $\theta \in [0, 2\pi)$ with topology given by $U \subset C$ being open in C if and only if the preimage of the gluing function is open in $[0,2] \times [0,1]$, where the gluing function is $g : [0,2] \times [0,1] \to C$ defined by $g(x,y) = (x, \sin 2\pi y, \cos 2\pi y)$. We define the map $h : C^* \to C$ as $h(S) = g(x,y)$ for some $(x,y) \in S$, where

$$S \in C^* = \{\{(x,y)\} \mid x \in [0,2], y \in (0,1)\} \cup \{\{(x,0),(x,1)\} \mid x \in [0,2]\}.$$

We first confirm that this map is well-defined. If S is a singleton, there is only one point (x, y) to choose, and if S contains two elements, then they are of the form $(x, 0)$ and $(x, 1)$. Since $g(x, 0) = g(x, 1)$, the map is well-defined. Note also that if $f : [0, 2] \times [0, 1] \to C^*$ is the identification map defining the topology on C^*, then $h \circ f = g$, that is, the gluing function at a point in $[0, 2] \times [0, 1]$ is the same as h applied to the set containing the point in C^*.

We now show that h is an injection. Let $S_1, S_2 \in C^*$ such that $h(S_1) = h(S_2)$. Then there exist $(x_1, y_1) \in S_1$ and $(x_2, y_2) \in S_2$ such that

$$(x_1, \sin 2\pi y_1, \cos 2\pi y_1) = g(x_1, y_1) = g(x_2, y_2) = (x_2, \sin 2\pi y_2, \cos 2\pi y_2).$$

Therefore $x_1 = x_2$. Since $\sin 2\pi y_1 = \sin 2\pi y_2$, either $y_1 = y_2$ or $y_1 = 1/2 - y_2$, and since $\cos 2\pi y_1 = \cos 2\pi y_2$, either $y_1 = y_2$ or $y_1 = 1 - y_2$. Therefore $(x_1, y_1) = (x_2, y_2)$, and since sets $S \in C^*$ are pairwise disjoint, $(x_1, y_1) = (x_2, y_2) \in S_1 \cap S_2$ implies that $S_1 = S_2$, meaning h is injective. For surjectivity, let $p = (x, \sin 2\pi y, \cos 2\pi y)$ be a point on C. If $y \in (0, 1)$, then $\{(x, y)\} \in C^*$ with $h(\{(x, y)\}) = p$, and if $y = 0$ or $y = 1$, then $\{(x, 0), (x, 1)\} \in C^*$ with $h(\{(x, 0), (x, 1)\}) = p$, so h is surjective. Therefore h is a bijective map.

For continuity, note that h is a map out of the quotient space. Then since the identification map $f : [0, 2] \times [0, 1]$ is a quotient map, and we have that $h \circ f = g$. Since the topology on C is defined to make g a continuous map, $h \circ f$ is continuous. Then by Theorem 7.53, h is also continuous.

We now wish to apply Theorem 7.29 to show that h is a homeomorphism. By the Heine-Borel Theorem, $[0, 2] \times [0, 1]$ is compact, and since f is continuous and surjective (it is a quotient map), Theorem 7.15 implies that its image C^* is compact as well. It remains to show that the cylinder C is Hausdorff. Let p and p' be points on the cylinder such that $p \neq p'$. Since $C \subset \mathbb{R}^3$ and $\mathbb{R}^3$ is Hausdorff, C is also Hausdorff when given the relative topology from $\mathbb{R}^3$ by Theorem 4.19. Therefore there exist two disjoint sets U and V with $p \in U$ and $p' \in V$ such that $U, V \in \mathcal{T}_{\mathbb{R}^3}$ where $\mathcal{T}_{\mathbb{R}^3}$ is the relative topology inherited from $\mathbb{R}^3$. Since the gluing map $g : [0, 2] \times [0, 1] \to C$ is defined by $g(x, y) = (x, \sin 2\pi y, \cos 2\pi y)$, we have that the map $\pi_1 \circ g : [0, 2] \times [0, 1] \to \mathbb{R}$ is the map $(\pi_1 \circ g)(x, y) = x$ and is continuous; similarly, $(\pi_2 \circ g)(x, y) = \sin 2\pi y$ is continuous; and $(\pi_3 \circ g)(x, y) = \cos 2\pi y$ is continuous. Since g is a map into a product space and its composition with each projection function is continuous, Theorem 7.40 implies that g is continuous. Since the topology on C that we are interested in is the quotient topology $\mathcal{Q}$ on C with respect to the map g, Theorem 7.48 means this topology is the finest topology on C that makes g continuous. Since g is also continuous when C is given the topology $\mathcal{T}_{\mathbb{R}^3}$, we have that $\mathcal{Q}$ is finer than $\mathcal{T}_{\mathbb{R}^3}$. Therefore

$U, V \in \mathcal{T}_{\mathbb{R}^3} \subset \mathcal{Q}$, so the sets U and V separating the points p and p' are also open in the topology of interest on the cylinder C, meaning C with this topology is Hausdorff. Since h is a continuous bijection from a compact space to a Hausdorff space, h is a homeomorphism by Theorem 7.29.

Exercise 7.55. If C is the union of all line segments from x_0 to points in X, then we can define the surjective map $f : X \times [0,1] \to C$ by $f(x,t) = (1-t)x_0 + tx$, and since the topology on C is the one inherited from $\mathbb{R}^{n+1}$, f is continuous and C is Hausdorff by Theorem 4.19. The cone over X is the space

$$X^* = \{\{(x,t)\} \mid x \in X, t \in (0,1]\} \cup \{\{(x,0) \mid x \in X\}\}$$

with the quotient topology from the identification map $f^* : X \times [0,1] \to X^*$ taking each point to the set containing it. Because X is compact by hypothesis and $[0,1]$ is compact as a closed interval in $\mathbb{R}$, the space $X \times [0,1]$ is compact and therefore so is X^* by Theorem 7.15. Now define the map $g : X^* \to C$ by $g(S) = f(x,t)$ where (x,t) is any point in S. If S is a singleton, there is only one point to choose, and if S is not a singleton, then $S = \{(x,0) \mid x \in X\}$ and each $(x,0) \to x_0$, so this function is indeed well-defined. We also have that g is continuous by Theorem 7.53 because it is a map out of a quotient space and $g \circ f^* = f$ and is continuous.

We have now that $g : X^* \to C$ is a continuous map from a compact space to a Hausdorff space, so it remains to show that g is a bijection. For injectivity, let $S_1, S_2 \in X^*$ such that $g(S_1) = g(S_2)$. Then there exist $(x_1, t_1), (x_2, t_2) \in X \times [0,1]$ such that

$$(1-t_1)x_0 + t_1 x_1 = f(x_1,t_1) = g(S_1) = g(S_2) = f(x_2,t_2) = (1-t_2)x_0 + t_2 x_2.$$

Then $(1-t_1)x_0 = (1-t_2)x_0$, so $t_1 = t_2$, and then $t_1 x_1 = t_2 x_2$ implies that $x_1 = x_2$ as well. Therefore the point (x_1, t_1) is in both S_1 and S_2, which means $S_1 = S_2$, since otherwise we would have that $S_1 \cap S_2 = \emptyset$. Therefore g is injective. For surjectivity, let $p \in C$ be an arbitrary point. Because f is surjective, there exists a point $(x,t) \in X \times [0,1]$ such that $f(x,t) = p$. Then because there exists a set $S \in X^*$ that contains this (x,t), $g(S) = p$ and so g is surjective. Because g is a continuous bijection from a compact space to a Hausdorff space, g is a homeomorphism.

7.6 Urysohn's Lemma and the Tietze Extension Theorem

Lemma 7.56. Let A and B be disjoint and closed in a normal space X. Then for every $r \in \mathbb{Q} \cap [0,1]$, there exists an open U_r such that $A \subset U_0$, $B \subset (X - U_1)$, and for $r < s$, $\bar(U_r) \subset U_s$.

Proof. Because $\mathbb{Q} \cap [0,1]$ is countable, we can list the elements as $(r_1, r_2, \dots)$. Define U_{r_1} to be the open set above separating A and B. Recall that by Theorem 4.9, normality means that A can be separated from the open set $X - B$ by an open set U_{r_1} such that $A \subset U$ and $\overline{U_{r_1}} \subset X - B$. Now that U_{r_1} has been defined, consider the rational r_n. Define q_n to be the maximum of the set $\{r_i \mid r_i < r_n, i < n\}$ if it is nonempty, and q_n' to be the minimum of the set $\{r_i \mid r_i > r_n, i < n\}$ if it is nonempty. We then define U_{r_n} to be the open set separating the closed set $\overline{U_{q_n}}$ (or using A if q_n does not exist) from the open set $U_{q_n'}$ containing it (or using $X - B$ if q_n' does not exist). Then we have that $\overline{U_{q_n}} \subset U_{r_n} \subset \overline{U_{r_n}} \subset U_{q_n'}$.

Note that $0 \in \mathbb{Q} \cap [0,1]$, so there exists $k \in \mathbb{N}$ such that $r_k = 0$. Then q_k does not exist because the set $\{r_i \mid r_i < r_k, i < k\}$ is empty, so $U_0 = U_{r_k}$ contains A by definition. Similarly, $1 \in \mathbb{Q} \cap [0,1]$, so there exists $m \in \mathbb{N}$ such that $r_m = 1$. Then q_m' does not exist because the set $\{r_i \mid r_i > r_m, i < m\}$ is empty, so $U_1 = U_{r_m}$ is defined so that $\overline{U_1} \subset X - B$. Therefore we have that $U_1 \subset X - B$, which means that $X - U_1 \supset B$, as required. Now let $r < s$. Then there exist $j, l \in \mathbb{N}$ such that $r = r_j$ and $s = r_l$. Suppose $j < l$. Then recall that q_l was defined to be $\max\{r_i \mid r_i < r_l, i < l\}$ which exists because it is finite and nonempty (it contains r_j). Then this maximum is some r_{t_1} where $j \leq t_1 < l$ such that $\overline{U_{r_{t_1}}} = \overline{U_{q_l}} \subset U_{r_l} = U_s$. If $j = t_1$, then we have that $\overline{U_r} \subset U_s$ and we are done. Otherwise, we continue this process to define a t_2 such that $j \leq t_2 < t_1 < l$ such that $\overline{U_{r_{t_2}}} = \overline{U_{q_{t_1}}} \subset U_{r_{t_1}} \subset U_s$. If $j \neq t_2$, we continue on until this process eventually terminates. It must terminate at some $j = t_p$ for some $p \in \mathbb{N}$ because each time we iterate, we are removing at least one element from the starting finite set $\{r_i \mid r_i < r_l, i < l\}$, meaning we cannot continue removing infinitely many elements. Therefore we eventually reach a point at which

$$\overline{U_r} = \overline{U_{r_j}} = \overline{U_{r_{t_p}}} \subset U_{r_{t_{p-1}}} \subset \cdots \subset \overline{U_{r_{t_2}}} \subset U_{r_{t_1}} \subset \overline{U}_{r_{t_1}} \subset U_{r_l} = U_s.$$

If instead we have $j > l$, we follow a similar procedure using q_j', defined as the minimum of $\{r_i \mid r_i > r_j, i < j\}$, which must be some r_{t_1} for some $t_1 \in \mathbb{N}$ since this set is finite and nonempty (it contains r_l). Then we have that $\overline{U_r} = \overline{U_{r_j}} \subset U_{r_{t_1}}$, and as before we repeat this process that must eventually terminate, the next step of which is to consider q_{t_1}', which is some rational r_{t_2}, and continuing on we eventually reach an r_{t_p} such that $t_p = l$. Then at this point we have the chain

$$\overline{U_r} = \overline{U_{r_1}} \subset U_{r_{t_1}} \subset \overline{U_{r_{t_1}}} \subset U_{r_{t_2}} \subset \cdots \subset \overline{U_{r_{t_{p-1}}}} \subset U_{r_{t_p}} = U_{r_l} = U_s.$$

Therefore in both cases $j < l$ and $j > l$, we have that $\overline{U_r} \subset U_s$, so we are done because $j \neq l$ (otherwise we would have $r = s$). $\square$

Theorem 7.57. A space X is normal if and only if for every pair of disjoint closed sets A and B in X, there exists a continuous $f : X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

Proof. ($\Longrightarrow$) Let X be a normal space and let A and B be disjoint closed sets. Then by Lemma 7.56, for every $r \in \mathbb{Q} \cap [0,1]$, there exists an open U_r such that $A \subset U_0$, $B \subset X - U_1$, and $r < s$ implies $\overline{U_r} \subset U_s$. For $x \in X$, define R_x to be the set $R_x = \{r \mid x \in U_r\}$. Then define the function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf R_x & R_x \neq \emptyset \\ 1 & R_x = \emptyset \end{cases}$$

where the condition $R_x \neq \emptyset$ is necessary and sufficient to ensure that $\inf R_x$ exists since R_x is bounded below by 0 for all $x \in X$. Let $x \in A$ be arbitrary. Then since $A \subset U_0$, we have that $x \in U_0$, which means $0 \in R_x$. Since R_x is bounded below by 0 and $0 \in R_x$, the set is nonempty and we have that $f(x) = \inf R_x = 0$. Therefore $x \in f^{-1}(0)$, so $A \subset f^{-1}(0)$. Now let $x \in B$ be arbitrary. Then if there existed an $r \in [0,1]$ such that $x \in U_r$, then either $r < 1$, in which case $x \in U_r \subset \overline{U_r} \subset U_1$, or $r = 1$, in which case $x \in U_r = U_1$. However, $x \in B$ and $B \subset X - U_1$, so we have that $x \in U_1$ and $x \in X - U_1$, which is a contradiction. Therefore there is no $r \in [0,1]$ such that $x \in U_r$, which means the set R_x is empty and $f(x) = 1$. Therefore $x \in f^{-1}(1)$, so $B \subset f^{-1}(1)$.

It remains now to show that f is continuous. Let $V \subset [0,1]$ be an open set containing $f(x)$ for some $x \in X$. If $f(x) \neq 0, 1$, then V being open in $[0,1]$ with the standard topology means that there exists a basic open set of the form $(f(x) - \varepsilon, f(x) + \varepsilon) \subset V$ which contains $f(x)$. By the density of the rationals in the reals, there exist rationals $r, s \in \mathbb{Q} \cap [0,1]$ such that

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$$

so that $(r,s) \subset V$. Therefore $f^{-1}(r,s) \subset f^{-1}(V)$. Consider the set set U open in X defined by $X = U_s - \overline{U_r}$. This set is open, and it is nonempty because $r < s$ implies that $\overline{U_r} \subset U_s$.

Since $r < f(x)$, there exists a rational q such that $r < q < f(x) = \inf R_x$, which means q is a lower bound for R_x not contained in R_x, so we have that $x \notin U_q$. Since $r < q$, $\overline{U_r} \subset U_q$, and therefore we also have that $x \notin \overline{U_r}$. Since $\inf R_x = f(x) < s$, s is not a lower bound of R_x, which means there exists $t \in R_x$ such that $t < s$. Because $t \in R_x$, we have that $x \in U_t \subset \overline{U_t} \subset U_s$ by the definition of R_x and that $t < s$ implies $\overline{U_t} \subset U_s$. Therefore $x \in U = U_s - \overline{U_r}$. Now let $y \in U$ be arbitrary. Then it cannot be the case that $\inf R_y < r$, as this would mean there exists a rational $q \in R_y$ with $q < r$ and we would have that $y \in U_q \subset U_r$, meaning $y \notin U = U_s - \overline{U_r}$. Similarly, it cannot be the case that $\inf R_y > s$, as this would mean s is a lower bound for R_y with $s \notin R_y$, meaning $y \notin U_s$. Therefore we have that $\inf R_y = f(y)$ satisfies $r \leq f(y) \leq s$. This means that $f(U) \subset [r, s] \subset (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$, and so we have shown that if $f(x) \neq 0, 1$, then there exists an open set U in X containing x such that $f(U) \subset V$. If instead we have that $f(x) = 0$, then there exists a basic open set $[0, \varepsilon)$ containing 0. Since $\inf R_x = 0 < \varepsilon$, we have that ε is not a lower bound of R_x, which means there exists an $s \in R_x$ such that $s < \varepsilon$. Then $s \in R_x$ implies that $x \in U_s$, so we have found an open set in X containing x. A similar argument to the one given above shows that $f(U_s) \subset [0, s] \subset [0, \varepsilon) \subset V$. Finally, if $f(x) = 1$, then we have that V contains a basic open set of the form $(1-\varepsilon, 1]$. If $f(x) = \inf R_x$, we have that $1 - \varepsilon < \inf R_x = 1$ and a similar argument to the above shows there exists a rational r such that $X - \overline{U_r}$ is an open set containing x and that $f(X - \overline{U_r}) \subset (1 - \varepsilon, 1] \subset V$. If instead R_x is empty and $f(x) = 1$ for that reason, the same set $X - \overline{U_r}$ chosen above still works, because $r < 1$ and therefore $\overline{U_r} \subset U_1$, meaning $x \notin \overline{U_r}$ so $x \in X - \overline{U_r}$. Therefore in all cases, given a point $f(x) \in [0, 1]$ and an open set V containing it, we have that there exists an open U in X containing x and satisfying $f(U) \subset V$, which means that f is a continuous function.

($\Longleftarrow$) Suppose that for every pair of disjoint open sets A and B, there exists a continuous function $f : X \to [0, 1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. Then since $\{0\} \subset [0, 1/2)$ and $\{1\} \subset (1/2, 1]$, we have that $A \subset f^{-1}(0) \subset f^{-1}([0, 1/2))$ and $B \subset f^{-1}(1) \subset ((1/2, 1])$. Since $[0, 1/2)$ and $(1/2, 1]$ are disjoint, so are their preimages, and since they are both open in $[0, 1]$ with the relative topology inherited from the standard topology on $\mathbb{R}$, their preimages are open in X since f is a continuous function. Therefore there do exist disjoint open sets separating A and B, which means X is a normal space since A and B were arbitrary closed sets. □

Lemma 7.58. Let X be normal and let A be a closed subset of X. Let $f : A \to [0, 1]$ be a continuous function, and let $r \in (0, 1)$. Then there exist disjoint open sets U_r and V_r such that $f^{-1}([0,r)) \subset U_r$ and $f^{-1}((r,1])$. Equivalently, there exists an open set U_r such that $f^{-1}([0,r)) \subset U_r$ and $\overline{U_r} \cap f^{-1}((r,1]) = \emptyset$.

Proof. First we note that given $r \in (0,1)$, the interval $[0,r)$ is the union $U = \bigcup_{s\in S}[0,s]$ where $S = \mathbb{Q} \cap [0,r)$. Then since each $s \in [0,r)$, the union U is a subset of $[0,r)$, so to show equality, let $x \in [0,r)$. Then $x < r$, so by the density of the rationals in the reals, there exists an $s_x \in (x,r)$, and we have that $x \in [0,s_x] \subset U$, so we are done. What this means then is that

$$f^{-1}([0,r)) = f^{-1}\left(\bigcup_{s\in S}[0,s]\right) = \bigcup_{s\in S} f^{-1}([0,s]),$$

and by a similar argument, we have that

$$f^{-1}((r,1]) = \bigcup_{q\in Q} f^{-1}([q,1])$$

where $Q = \mathbb{Q} \cap (r,1]$.

Since $[0,s] \subset [0,r)$, we have that $[0,s] \cap [r,1] = \emptyset$, so also $f^{-1}([0,s]) \cap f^{-1}([r,1]) = \emptyset$. Now we note that $[0,s]$ and $[r,1]$ are closed in $[0,1]$ and so their preimages are closed in A as a subspace of X by the continuity of f. Since A is a closed subset of X, $f^{-1}([0,s])$ and $f^{-1}([r,1])$ are also closed in X. (Recall that if C is closed in a subspace Y, then $C = \overline{C} \cap Y$ where the closure is taken in X. Therefore if Y is closed, it is a closed set in X containing C and therefore contains $\overline{C}$ as well, meaning $C = \overline{C} \cap Y = \overline{C}$ and so C is also closed in X.) This means $f^{-1}([0,s])$ and $f^{-1}([r,1])$ are disjoint and closed in the normal space C, and so using the characterization of normality in Theorem 4.9, there exists a set U_s such that

$$f^{-1}([0,s]) \subset U_s \subset \overline{U_s} \subset X - f^{-1}([r,1]) \subset X - f^{-1}((r,1]).$$

Therefore for each $s \in S$, the set U_s defined in this way satisfies $\overline{U_s} \cap f^{-1}((r,1]) = \emptyset$, and by the argument above, we also have that

$$f^{-1}([0,r)) = \bigcup_{s\in S} f^{-1}([0,s]) \subset \bigcup_{s\in S} U_s.$$

A similar construction shows that for each $q \in Q$, there exists a V_q such that $\overline{V_q} \cap f^{-1}([0,r)) = \emptyset$ and that $f^{-1}((r,1]) \subset \bigcup_{q\in Q} V_q$. Then since S and Q are the rationals in the intervals $[0,r)$ and $(r,1]$ respectively, they are both countable. This means that by the Normality Lemma, there exist two sets U_r and V_r that are disjoint and open in X such that

$f^{-1}([0,r)) \subset U_r$ and $f^{-1}((r,1]) \subset V_r$, as required. We also have that $\overline{U_r} \cap f^{-1}((r,1]) = \emptyset$ because if x were a point in the intersection, we would have that $x \in f^{-1}((r,1]) \subset V_r$ and therefore $x \notin U_r$, meaning $x \in \overline{U_r}$ would imply x is a limit point of U_r. But since $x \in V_r$ and $V_r \cap U_r = \emptyset$, we have a neighborhood of x that doesn't intersect U_r and therefore x cannot be a limit point. □

Theorem 7.59. A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to [0,1]$, there exists a continuous function $F : X \to [0,1]$ such that $F(x) = f(x)$ for every $x \in A$.

Proof. ($\Longrightarrow$) Let X be a normal space with closed subset $A \subset X$ and let $f : A \to [0,1]$ be continuous. We begin by constructing a collection of open sets $\{V_r\}_{r\in\mathbb{Q}\cap(0,1)}$ such that

(1) given $r, s \in \mathbb{Q} \cap (0,1)$, $r < s$ implies that $\overline{V_r} \subset V_s$,

(2) for all $r \in \mathbb{Q} \cap (0,1)$, we have that $f^{-1}([0,r)) \subset V_r$, and

(3) for all $r \in \mathbb{Q} \cap (0,1)$, we have that $\overline{V_r} \cap f^{-1}((r,1]) = \emptyset$.

Since the set $\mathbb{Q} \cap (0,1)$ is countable, we may write it as $\mathbb{Q} \cap (0,1) = \{r_1, r_2, r_3, \dots\}$. We will define the sets $\{V_r\}_{r\in\mathbb{Q}\cap(0,1)}$ inductively using this order. Beginning with r_1, define V_{r_1} to be the open set satisfying (2) and (3) that is guaranteed to exist by Lemma 7.58. Note that V_{r_1} vacuously satisfies (1) for all V_{r_i} that we have defined. Now suppose we have defined V_{r_i} for all $1 \le i < n$ with $n \ge 2$ such that (2) and (3) hold and such that (1) holds for all $r_j \ne r_i$ we have defined. We define V_{r_n} by first defining r_k and r_m as follows:

$$r_k = \max\{r_i \mid r_i < r_n \text{ and } i < n\}$$
$$r_m = \min\{r_i \mid r_i > r_n \text{ and } i < n\}.$$

Note that the sets used above are finite, and so if they are nonempty, maxima and minima must exist. Therefore in defining V_{r_n}, there are three cases to consider. In case 1, r_k and r_m both exist; in case 2, r_k exists but r_m does not; and in case 3, r_m exists but r_k does not. Note that because we have already defined V_{r_1} independently, we are only considering $n \ge 2$ and therefore one of the sets above must be nonempty (either $r_1 < r_n$ or $r_1 > r_n$). This means that the three cases above are the only possible cases. Now to define V_{r_n}, note that as in the proof of Lemma 7.58, the set $f^{-1}([0,r_n))$ is the union $\bigcup_{s\in S} f^{-1}([0,s])$ where $S = \mathbb{Q} \cap [0,r_n)$ and similarly, $f^{-1}((r_n,1]) = \bigcup_{q\in Q} f^{-1}([q,1])$ where $Q = \mathbb{Q} \cap (r_n,1]$. Again by the proof of Lemma 7.58, we have that for all $s \in S$, $f^{-1}([0,s])$ and $f^{-1}([r_n,1])$ are closed and disjoint in X. Note also that in case 1, $r_k < r_m$, and so by (1) we have that $\overline{V_{r_k}} \subset V_{r_m}$, and therefore $\overline{V_{r_k}}$ and $X - V_{r_m}$ are disjoint closed sets. By (3), $\overline{V_{r_k}} \cap f^{-1}((r_k,1]) = \emptyset$, and since $r_k < r_n$, we have that $f^{-1}([r_n, 1]) \subset f^{-1}((r_k, 1])$, which means also $\overline{V_{r_k}} \cap f^{-1}([r_n, 1]) = \emptyset$. This means that $\overline{V_{r_k}}$ and $f^{-1}([r_n, 1]) \cup (X - V_{r_m})$ are disjoint closed sets in X. Now because $s < r_n < r_m$, we have that $f^{-1}([0, s]) \subset f^{-1}([0, r_m)) \subset V_{r_m}$, which means that $f^{-1}([0, s]) \cap (X - V_{r_m}) = \emptyset$. Therefore $f^{-1}([0, s]) \cup \overline{V_{r_k}}$ and $f^{-1}([r_n, 1]) \cup (X - V_{r_m})$ are disjoint closed sets in X. Since X is a normal space, there exists an open set U_s such that

$$\begin{aligned} f^{-1}([0, s]) \cup \overline{V_{r_k}} \subset U_s \subset \overline{U_s} &\subset X - \big(f^{-1}([r_n, 1]) \cup (X - V_{r_m})\big) \\ &\subset X - (f^{-1}((r_n, 1]) \cup (X - V_{r_m})). \end{aligned}$$

Therefore we have that $\overline{U_s} \cap (f^{-1}((r_n, 1]) \cup (X - V_{r_m})) = \emptyset$ and also that

$$\overline{V_{r_k}} \cup f^{-1}([0, r_n)) = \overline{V_{r_k}} \cup \bigcup_{s \in S} f^{-1}([0, s]) = \bigcup_{s \in S} (\overline{V_{r_k}} \cup f^{-1}([0, s])) \subset \bigcup_{s \in S} U_s.$$

A similar argument now shows that for each $q \in Q$, there exists an open set U_q such that $\overline{U_q} \cap (f^{-1}([0, r_n)) \cup V_{r_k}) = \emptyset$ and that $f^{-1}((r_n, 1]) \cup (X - V_{r_m}) \subset \bigcup_{q \in Q} U_q$. Then using the Normality Lemma, there exist two disjoint open sets V^* and V' such that

$$\overline{V_{r_k}} \cup f^{-1}([0, r_n)) \subset V^* \quad \text{and} \quad (X - V_{r_m}) \cup f^{-1}((r_n, 1]) \subset V'.$$

Now if we are instead in case 2 in which V_{r_m} does not exist, replace it in the argument above by $X - f^{-1}(1)$ (which is possibly the entire space), and if we are in case 3 in which V_{r_k} does not exist, replace it in the argument above by $f^{-1}(0)$ (which is possibly empty). This still works because $[0, 1]$ is Hausdorff and therefore T_1, meaning $\{0\}$ and $\{1\}$ are closed. Because f is continuous, $f^{-1}(0)$ and $f^{-1}(1)$ are closed in A, which means they are also closed in X since $A \subset X$ is closed. Therefore in all cases, it is possible to define such a V^*, and in the remainder of the proof, we will use V_{r_k} to mean $f^{-1}(0)$ and V_{r_m} to mean $f^{-1}(1)$ where appropriate.

This set V^* satisfies (2) since $f^{-1}([0, r_n)) \subset V^*$, and since $f^{-1}((r_n, 1]) \subset V'$, if it were the case that there were some $x \in \overline{V^*} \cap f^{-1}((r_n, 1])$, then we would have that $x \in f^{-1}((r_n, 1]) \subset V'$ which is disjoint from V^*. Therefore x would have to be a limit point of V^*, but this is impossible because as we have just seen, V' is a neighborhood of x disjoint from V^*. Therefore $\overline{V^*} \cap f^{-1}((r_n, 1]) = \emptyset$, and so (3) is also satisfied. However, V^* does not satisfy (1) yet, since while we have that $V^* \subset V_{r_m}$, there may be limit points of V^* outside V_{r_m}. We do have, however, that $\overline{V_{r_k}} \subset \overline{V_{r_k}} \cup f^{-1}([0, r_n)) \subset V^*$, so we are part of the way toward (1). To define V_{r_n}, note that $\overline{V_{r_k}} \cup \overline{f^{-1}([0, r_n))}$ is closed in X and contained in V_{r_m} since

$$\overline{f^{-1}([0, r_n))} \subset f^{-1}(\overline{[0, r_n)}) = f^{-1}([0, r_n]) \subset f^{-1}([0, r_m)) \subset V_{r_m}$$

by (2). Then by normality, there exists a W' open in X such that

$$\overline{V_{r_k}} \cup \overline{f^{-1}([0, r_n))} \subset W' \subset \overline{W'} \subset V_{r_m}.$$

Similarly, note that by the above definition, $\overline{V_{r_k}}$ is contained in the open set W', and that we also then have that $\overline{V_{r_k}} \subset W' - \overline{f^{-1}((r_n, 1])}$. This is because

$$\overline{f^{-1}((r_n, 1])} \subset f^{-1}(\overline{(r_n, 1]}) \subset f^{-1}([r_n, 1]) \subset f^{-1}((r_k, 1])$$

and we have by (3) that $\overline{V_{r_k}} \cap f^{-1}((r_k, 1]) = \emptyset$, meaning that indeed

$$\overline{V_{r_k}} \subset W' - f^{-1}((r_k, 1]) \subset W' - \overline{f^{-1}((r_n, 1])}$$

as claimed. Since $W' - \overline{f^{-1}((r_n, 1])}$ is open in X and contains the closed set $\overline{V_{r_k}}$, normality implies that there exists a W^* open in X such that

$$\overline{V_{r_k}} \subset W^* \subset \overline{W^*} \subset W' - \overline{f^{-1}((r_n, 1])}.$$

We now finally define V_{r_n} to be the set $V_{r_n} = (V^* \cup W^*) \cap W'$. We begin by checking that V_{r_n} satisfies (1) for all r_j previously defined. If $r_j < r_n$, then recalling that r_k was defined to be $\max\{r_i \mid r_i < r_n \text{ and } i < n\}$ and that $\overline{V_{r_k}} \subset V^*, W^*, W'$, we have that

$$\overline{V_{r_j}} \subset V_{r_k} \subset \overline{V_{r_k}} \subset (V^* \cup W^*) \cap W' = V_{r_n}.$$

If instead we have that $r_j > r_n$, then recalling that W' was defined so that $\overline{W'} \subset V_{r_m}$ and that r_m was defined as $r_m = \min\{r_i \mid r_i > r_n \text{ and } i < n\}$, we have that

$$\overline{V_{r_n}} = \overline{(V^* \cup W^*) \cap W'} \subset \overline{V^* \cup W^*} \cap \overline{W'} \subset \overline{W'} \subset V_{r_m} \subset \overline{V_{r_m}} \subset V_{r_j},$$

so indeed V_{r_n} satisfies (1). Recall that V^* satisfied both (2) and (3). Using this, we have that $f^{-1}([0, r_n) \in V^*$ and $f^{-1}([0, r_n)) \subset \overline{f^{-1}([0, r_n))} \subset W'$, and therefore that

$$f^{-1}([0, r_n)) \subset V^* \cap W' \subset (V^* \cup W^*) \cap W' = V_{r_n},$$

so V_{r_n} satisfies (2). For (3), we already have that $\overline{V^*} \cap f^{-1}((r_n, 1]) = \emptyset$. Now if $x \in f^{-1}((r_n, 1]) \subset \overline{f^{-1}((r_n, 1])}$, then $x \notin W' - \overline{f^{-1}((r_n, 1)}$, so also $x \notin \overline{W^*}$, meaning $\overline{W^*} \cap f^{-1}((r_n, 1]) = \emptyset$ as well. Therefore because $\overline{V_{r_n}} \subset \overline{V^* \cup W^*} \cap \overline{W'} \subset \overline{V^* \cup W^*}$, we have that

$$\overline{V_{r_n}} \cap f^{-1}((r_n, 1]) \subset \overline{V^* \cup W^*} \cap f^{-1}((r_n, 1]) = (\overline{V^*} \cap f^{-1}((r_n, 1])) \cup (\overline{W^*} \cap f^{-1}((r_n, 1])) = \emptyset$$

and so V_{r_n} satisfies (3) in addition to (1) and (2) now for r_n all r_j previously defined.

This inductive procedure therefore defines a V_r for all $r \in \mathbb{Q} \cap (0, 1)$, and so we quickly check that this collection of sets satisfies (1), (2), and (3) for all possible values. For (2) and (3), this is immediate because given an $r \in \mathbb{Q} \cap (0, 1)$, there exists an $l \in \mathbb{N}$ such that $r = r_l$, and because we have defined V_{r_l} to satisfy (2) and (3) we are done. For (1), however, we have only defined V_r so that (1) is satisfied for values of r_j appearing earlier in our list $(r_1, r_2, \dots)$. To check that this is not an issue, let $r, s \in \mathbb{Q} \cap (0, 1)$ with $r < s$. Then there exist $p, q \in \mathbb{N}$ such that $r = r_p$ and $s = r_q$. If $p < q$, then V_r is already defined when V_s is being defined, and so $V_s = V_{r_q}$ is constructed so that $\overline{V_{r_p}} \subset V_{r_q}$. If instead $p > q$, then V_s is already defined when V_r is being defined, and again we have that $V_r = V_{r_p}$ is constructed so that $\overline{V_{r_p}} \subset V_{r_q}$. In both cases we have that $\overline{V_r} \subset V_s$.

We now have our collection $\{V_r\}_{r \in \mathbb{Q} \cap (0,1)}$ and so we can now define R_x for each $x \in X$ to be the set $R_x = \{r \in \mathbb{Q} \cap (0, 1) \mid x \in V_r\}$. We then define the function $F : X \to [0, 1]$ by

$$F(x) = \begin{cases} \inf R_x & R_x \neq \emptyset \\ 1 & R_x = \emptyset \end{cases}$$

where the condition $R_x \neq \emptyset$ is necessary and sufficient to ensure that $\inf R_x$ exists. This is because R_x is bounded below by 0 for all $x \in X$. That this function F is continuous follows from the proof of Theorem 7.57, since there the function was defined in exactly the same way using only that the collection of open sets under consideration satisfied condition (1).

It remains to show that $F(x) = f(x)$ for all $x \in A$. If $x \in A$, then there are two cases two check, one in which R_x is empty and one in which it is nonempty. In the first case, $F(x) = 1$, so we need to show that it must be the case that $f(x) = 1$. Suppose $f(x) \in [0, 1)$. Then $f(x) < 1$ and by the density of the rationals in the reals, there exists a rational $r \in (f(x), 1)$. However, this means that $f(x) \in [0, r)$, and therefore that $x \in f^{-1}([0, r)) \subset V_r$, meaning $r \in R_x \neq \emptyset$, a contradiction. Therefore $f(x) \notin [0, 1)$, so we have that $f(x) = F(x) = 1$. Now for the second case, suppose $R_x \neq \emptyset$. Note that in this case we cannot have that $f(x) = 1$, because that for all rationals $r \in (0, 1)$, we would have that $x \notin V_r$ meaning

$R_x = \emptyset$. Therefore for this second case, we have that $f(x) \in [0,1)$. Then if $r \in R_x$, we have that $x \in V_r$, so $x \notin f^{-1}((r,1])$ and therefore $f(x) \notin (r,1]$. This means that $f(x) \leq r$, and since r was an arbitrary element of R_x, we have that $f(x)$ is a lower bound for R_x. We will now show that in fact, $f(x)$ is the greatest lower bound of R_x. Let $\varepsilon > 0$. Then there exists a rational number $s \in (f(x), f(x)+\varepsilon)$ again by the density of $\mathbb{Q}$. Then $f(x) < s$ implies that $x \in f^{-1}([0,s)) \subset U_s$, meaning $s \in R_x$. Since for every $\varepsilon > 0$, there exists an element $s \in R_x$ such that $f(x) < s < f(x)+\varepsilon$, we have that $f(x) = \inf R_x = F(x)$. Therefore in both cases, we have that $x \in A$ implies that $F(x) = f(x)$.

($\Longleftarrow$) Let X be a space such that for every closed $A \subset X$ and continuous function $f : A \to X$, there exists a continuous function $F : X \to [0,1]$ such that $F(x) = f(x)$ for all $x \in A$. Let A and B be disjoint closed sets and define the map $f : A \cup B \to [0,1]$ by

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}.$$

Let C be closed in X. Then $0, 1 \in C$ implies that $f^{-1}(C) = A \cup B$ which is closed in $A \cup B$; $0 \in C$ but $1 \notin C$ implies that $f^{-1}(C) = A$ which is a closed subset of $A \cup B$; $0 \notin C$ and $1 \in C$ implies that $f^{-1}(C) = B$ which is a closed subset of $A \cup B$; and lastly $0, 1 \notin C$ implies that $f^{-1}(C) = \emptyset$, which is also a closed subset of $A \cup B$. Therefore this map f is continuous since the preimages of closed subsets of $[0,1]$ are closed. Since A and B are closed in X, $A \cup B$ is also closed in X, and so there exists a continuous extension $F : X \to [0,1]$ such that $F(x) = f(x)$ for all $x \in A \cup B$. Since $[0, 1/2)$ and $(1/2, 1]$ are open and disjoint in $[0,1]$, the sets $F^{-1}([0,1/2))$ and $F^{-1}((1/2,1])$ are open and disjoint in X. Then since $x \in A$ implies $F(x) = f(x) = 0$, we have that $A \subset F^{-1}(0) \subset F^{-1}([0,1/2))$, and similarly, we have that $B \subset F^{-1}((1/2,1])$. Since A and B were arbitrary disjoint closed sets in X and we have found disjoint open sets separating them, X is normal. $\square$

Theorem 7.64. Let X be a normal space, and let A be a closed subspace of X homeomorphic to $[0,1]$ with the relative topology from $\mathbb{R}_{\text{std}}$. Then there exists a function $f : X \to A$ such that for every $x \in A$, $r(x) = x$.

Proof. Let X be normal, let A be a closed subspace homeomorphic to $[0,1]$, and let $f : A \to [0,1]$ be the homeomorphism. Since X is normal and f is continuous, the Tietze Extension Theorem implies that there exists a continuous $F : X \to [0,1]$ such that for all $x \in A$, $F(x) = f(x)$. Define $r : X \to A$ to be $r = f^{-1} \circ F$. Then F is continuous, and so is f^{-1} since

f is a homeomorphism. Therefore r is continuous as well as the composition of continuous functions. Let $x \in A$. Then we have that

$$r(x) = f^{-1}(F(x)) = f^{-1}(f(x)) = x$$

as required. $\square$

Theorem 7.69. Let X be a normal, T_1 space. Then X is homeomorphic to a subspace of $[0,1]^\lambda = \prod_{\alpha\in\lambda}[0,1]_\alpha$ for some λ.

Proof. We let $\lambda = \{\alpha : X \to [0,1] \mid \alpha \text{ is continuous}\}$ be the set of all continuous functions from X to the interval $[0,1]$. We then define $f : X \to [0,1]^\lambda$ by $f(x) = (\alpha(x))_{\alpha\in\lambda}$ (that is, the αth coordinate of $f(x)$ is α evaluated at x) and claim that $g : X \to f(X)$ given by $g(x) = f(x)$ is a homeomorphism between X and $f(X) \subset [0,1]^\lambda$. Since functions are surjective onto their images, g is a bijection onto $f(X)$ if and only if it is injective. Suppose $p \neq q$ for $p, q \in X$. Then because X is T_1, $\{p\}$ and $\{q\}$ are disjoint closed sets. Then by Urysohn's Lemma, there exists a $\beta \in \lambda$ such that $\{p\} \subset \beta^{-1}(0)$ and $\{q\} \subset \beta^{-1}(1)$. Therefore $g(p) \neq g(q)$ because they differ at least at the βth coordinate:

$$\pi_\beta(g(p)) = \beta(p) = 0 \neq 1 = \beta(q) = \pi_\beta(g(q)).$$

We now have that g is a bijection, so it remains to check that g and g^{-1} are continuous. For continuity, note that f is a map into a product space and therefore Theorem 7.40 means that to show f is continuous, we need only show that $\pi_\alpha \circ f : X \to [0,1]$ is continuous for all $\alpha \in \lambda$. Since $f(x) = (\alpha(x))_{\alpha\in\lambda}$, we have that $\pi_\alpha(f(x)) = \alpha(x)$, so $\pi_\alpha \circ f = \alpha \in \lambda$, which is continuous since λ is the set of all continuous functions from X to $[0,1]$. Now that we have that f is continuous, we also have that g is continuous since g can be thought of as the composition of the identity map restricted to $f(X)$ and f, both of which are continuous.

Checking now that $g^{-1} : f(X) \to X$ is continuous, let $y \in f(X)$ and let U be an open set in X containing $g^{-1}(y)$. We will show continuity by constructing a set V open in $f(X)$ such that V contains y and $g^{-1}(V) \subset U$. First note that because $y \in f(X)$ and g is a bijection, there exists a unique $x \in X$ such that $y = g(x) = (\alpha(x))_{\alpha\in\lambda}$ and that $x = g^{-1}(y) \in U$. Again using the fact that X is T_1, we have that $\{x\}$ and $X - U$ are disjoint closed sets in X. By Urysohn's Lemma, there exists a $\beta \in \lambda$ such that $\{x\} \subset \beta^{-1}(0)$ and $X - U \subset \beta^{-1}(1)$. Then because $[0,1/2)$ is open in $[0,1]$, we have that $W = \pi_\beta^{-1}([0,1/2))$ is a subbasic open set in $[0,1]^\lambda$, and therefore that $V = f(X) \cap W$ is open in $f(X)$. Because $\{x\} \subset \beta^{-1}(0)$, we have that $\beta(x) = 0$. Therefore the βth coordinate of $y = g(x) = (\alpha(x))_{\alpha\in\lambda}$ is $\beta(x) = 0 \in [0, 1/2)$, which means that in addition to $y \in f(X)$, we also have that $y \in \pi_\beta^{-1}([0, 1/2))$. Therefore V contains y. It remains to show that $g^{-1}(V) \subset U$, so let $q \in g^{-1}(V)$ be arbitrary. Then $g(q) \in V$ means that $g(q) \in \pi_\beta^{-1}([0, 1/2))$, so we have that

$$\beta(q) = \pi_\beta((\alpha(q))_{\alpha\in\lambda}) = \pi_\beta(g(q)) \in [0, 1/2).$$

In particular then, $\beta(q) \neq 1$, which means that $q \notin \beta^{-1}(1)$. Since $X - U \subset \beta^{-1}(1)$, we have that $q \notin X - U$, so $q \in U$. Since q was an arbitrary element of $g^{-1}(V)$, we have that $g^{-1}(V) \subset U$ as required. Therefore g^{-1} is continuous, and so we have that g is a homeomorphism from X to $f(X) \subset [0, 1]^\lambda$. □

Scholium 7.70. A space X is completely regular and T_1 if and only if X can be embedded in $\prod_{\alpha\in\lambda}[0, 1]_\alpha$ for some λ.

Proof. ($\Longrightarrow$) This follows directly from the proof of Theorem 7.69.

($\Longleftarrow$) Suppose X can be embedded in $[0, 1]^\lambda$. Therefore there exists a homemorphism $f : X \to I$ for some subset $I \subset [0, 1]^\lambda$. Let $p \in X$ be a point not contained in some closed subset $A \subset X$. Then $f(p) \cap f(A) = \emptyset$ and $f(A)$ is closed in I. Therefore there exists a set $B \subset [0, 1]^\lambda$ such that $B \cap I = f(A)$. Since $f(p) \in I$, $f(p) \notin B$, otherwise we would have $f(p) \in f(A)$. Therefore $f(p) \cap B = \emptyset$ and $f(A) \subset B$. Since products of compact spaces are compact and products of Hausdorff spaces are Hausdorff, $[0, 1]^\lambda$ is compact Hausdorff and therefore normal. Since $[0, 1]^\lambda$ is normal and $\{f(p)\}$ and B are disjoint closed sets, Urysohn's Lemma implies the existence of a continuous function $g : [0, 1]^\lambda \to [0, 1]$ such that $\{f(p)\} \subset g^{-1}(0)$ and $f(A) \subset B \subset g^{-1}(1)$. Then if we define $h = g \circ f$, we have that h is a continuous function satisfying $p \in h^{-1}(0)$ and $A \subset h^{-1}(1)$, so X is completely regular. To see that X is T_1, note that X is homeomorphic to a subspace of a T_1 space and being T_1 is hereditary. □

8 Connectedness: When Things Don't Fall into Pieces

8.1 Connectedness

Theorem 8.1. The following are equivalent:

(1) X is connected.

(2) There is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$.

(3) X is not the union of two disjoint nonempty separated sets.

(4) X is not the union of two disjoint nonempty closed sets.

(5) The only subsets of X that are both closed and open in X are $\emptyset$ and X itself.

(6) For every pair of points p and q and every open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X, there exist a finite number of U_α's, $\{U_{\alpha_1},\dots,U_{\alpha_n}\}$ such that $p\in U_{\alpha_1}$, $q\in U_{\alpha_2}$, and for each $i<n$, $U_{\alpha_i}\cap U_{\alpha_{i+1}}\neq\emptyset$.

Proof. (1) $\Longrightarrow$ (2): We show the contrapositive. Let $f:X\to\mathbb{R}_{\text{std}}$ be a continuous function such that $f(X)=\{0,1\}$. Since $\mathbb{R}_{\text{std}}$ is Hausdorff, there exist disjoint open sets U and V such that $0\in U$ and $1\in V$. Then we have that $f(X)\subset U\cup V$, so $X\subset f^{-1}(U\cup V)=f^{-1}(U)\cup f^{-1}(V)$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty open sets in X whose union is all of X. Therefore X is not connected.

(2) $\Longrightarrow$ (4): We again show the contrapositive. Let X be a space that is the union of two disjoint nonempty open sets, A and B. Then define $f:X\to\mathbb{R}_{\text{std}}$ by

$$f(x)=\begin{cases}0 & x\in A\\ 1 & x\in B\end{cases}.$$

Then we have that $f(X)=\{0,1\}$ as required, and we claim that f is continuous. Let C be closed in $\mathbb{R}_{\text{std}}$. If $0,1\in C$, then $f^{-1}(C)=X$ which is closed; if $0\in C$ and $1\notin C$, then $f^{-1}(C)=A$ which is closed; and if $1\in C$ and $0\notin C$, then $f^{-1}(C)=B$ which is closed. Therefore f is continuous since the preimages of closed sets are closed.

(4) $\Longrightarrow$ (3): Again working with the contrapositive, suppose X is the union of two disjoint nonempty separated sets A and B. Then we have that

$$\overline{A}=\overline{A}\cap X=\overline{A}\cap(A\cup B)=(\overline{A}\cap A)\cup(\overline{A}\cap B)=A\cup\emptyset=A$$

because $X=A\cup B$ and $\overline{A}\cap B=\emptyset$ since A and B are separated. Similarly, $\overline{B}=B$, so actually A and B are both closed sets and therefore $X=A\cup B$ is the union of disjoint nonempty closed sets.

(3) $\Longrightarrow$ (4): Showing the contrapositive, suppose X is the union of two disjoint nonempty closed sets A and B. Then A and B are also separated and so X is the union of nonempty separated sets.

(4) $\Longrightarrow$ (5): We will prove the contrapositive. Suppose $A\neq X$ is nonempty and is both open and closed in X. Then A and $X-A$ are closed and disjoint nonempty sets such that $X=A\cup(X-A)$.

(5) $\Longrightarrow$ (6): Again we show the contrapositive. Suppose that there exist points $p, q \in X$ and an open cover $\{U_\alpha\}_{\alpha\in\lambda}$ such that for every finite subcollection $\{U_{\alpha_i}\}_{i=1}^n$ with $p \in U_{\alpha_1}$ and $q \in U_{\alpha_n}$, there exists an $i < n$ such that $U_{\alpha_i} \cap U_{\alpha_{i+1}} = \emptyset$. Denote the collection of all ordered finite subcollections of this open cover that have $p \in U_{\alpha_1}$ and $q \in U_{\alpha_n}$ by $\mathfrak{C} = \{C_\beta\}_{\beta\in\gamma}$ (where by ordered we mean that two different assignments of indices to the same collection of sets would count as distinct elements of $\mathfrak{C}$). Since for each C_β, there exists an $i < n$ such that $U_{\alpha_i} \cap U_{\alpha_{i+1}} = \emptyset$, define i_β to be the minimum i for which this occurs (a minimum exists because the index set is $i \in \{1, \ldots, n\}$ which is finite). Then define the set V_β as the union of all U_{α_j} such that $j \le i_\beta$. Then V_β is open, and therefore so is the set $V = \bigcup_{\beta\in\gamma} V_\beta$. We claim that V is also closed. Suppose for contradiction that V has a limit point $x \notin V$. Then there exists a U_μ in the open cover that contains x, and since x is a limit point of V, we have that $V \cap U_\mu \neq \emptyset$, meaning there exists a $y \in V \cap U_\mu$. Since $y \in V$, there exists a $\beta \in \gamma$ such that $y \in V_\beta$. Since $y \in V_\beta$, there exists a finite subcollection C_β such that $y \in U_{\alpha_j}$ for some $\alpha \le i_\beta$. Since $y \in U_\mu$ as well, the finite collection containing all open sets in C_β along with the set U_μ such that $U_\mu = U_{j+1}$ is an element $C'_\beta \in \mathfrak{C}$ satisfying $j + 1 \le i'_\beta$. Therefore $x \in U_\mu \subset V'_\beta \subset V$, but this is a contradiction since we assumed $x \notin V$. Therefore V has no limit points it doesn't contain and so is closed in addition to being open.

(6) $\Longrightarrow$ (1): For the contrapositive, suppose there exist disjoint nonempty open sets U and V such that $X = U \cup V$. Since they are nonempty, there exist $p \in U$ and $q \in V$. Then $\{U, V\}$ is an open cover for X with $\{U_1, U_2\}$ as a finite subcollection such that $p \in U_1 = U$ and $q \in U_2 = V$, however, $U_1 \cap U_2 = \emptyset$, so we have shown the contrapositive. $\square$

Exercise 8.2. (1) $\mathbb{R}$ with the discrete topology is not connected because it is the union of the nonempty open sets $(-\infty, 0)$ and $[0, \infty)$.

(2) $\mathbb{R}$ with the indiscrete topology is connected because the only nonempty open set is $\mathbb{R}$ itself, so the whole space can't be made out of two disjoint open sets.

(3) $\mathbb{R}$ with the finite complement topology is connected because if $U \neq \mathbb{R}$ is a nonempty open set, its complement is finite and is therefore not open because $\mathbb{R} - (\mathbb{R} - U) = U$ must be uncountable in order to have a finite complement. Therefore U is not closed. Since the only sets that are simultaneously open and closed are $\mathbb{R}$ and $\emptyset$, $\mathbb{R}$ with the finite complement topology is connected.

(4) $\mathbb{R}_{\mathrm{LL}}$ is not connected because it is the union of the nonempty open sets $(-\infty, 0)$ and $[0, \infty)$.

(5) $\mathbb{Q}$ is not connected because the sets $(-\infty, \sqrt{2})$ and $(\sqrt{2}, \infty)$ are open in $\mathbb{R}_{\mathrm{std}}$ and therefore these sets intersected with $\mathbb{Q}$ are open in $\mathbb{Q}$ with the relative topology. Since

$\sqrt{2} \notin \mathbb{Q}$, these nonempty disjoint sets have union equal to $\mathbb{Q}$.

(6) $\mathbb{R} - \mathbb{Q}$ is not connected for similar reasons, this time using the sets $(-\infty, 0)$ and $(0, \infty)$ intersected with $(\mathbb{R} - \mathbb{Q})$.

Theorem 8.3. The space $\mathbb{R}_{\text{std}}$ is connected.

Proof. Suppose not. Then there exist disjoint nonempty open sets U and V such that $U \cup V = \mathbb{R}$. Since these sets are nonempty, there exist $x \in U$ and $y \in V$. Without loss of generality, assume $x < y$. Then consider the set $A = \{a \in U \mid a < y\}$. Since $x \in U$ and $x < y$, this set is nonempty, and it is bounded above by y, so it has a supremum, call it $\alpha < y$. Then if $\alpha \in U$ (or if $\alpha \in V$, there exists a $0 < \varepsilon < y - \alpha$ such that $(\alpha - \varepsilon, \alpha + \varepsilon)$ is a subset of U (or of V) since U and V are open sets. Then if $\alpha \in U$, $\alpha + \varepsilon < y$ and is in U, meaning $\alpha + \varepsilon \in A$ and $\alpha \neq \sup A$. If instead $\alpha \in V$, we still have $\alpha = \sup A$, so there exists an element $a \in A$ such that $\alpha - \varepsilon < a < \alpha$, and since $a \in A$, also $a \in U$. However, $a \in (\alpha - \varepsilon, \alpha + \varepsilon) \subset V$, so $a \in U \cap V = \emptyset$, a contradiction. Therefore $\alpha \notin U \cup V = \mathbb{R}$, which contradicts the assumption that such U and V exist, and so we have that $\mathbb{R}_{\text{std}}$ is connected. □

Theorem 8.4. Let A and B be separated subsets of a space X. Then if C is a connected subset of $A \cup B$, either $C \subset A$ or $C \subset B$.

Proof. If C is empty or one of A or B is empty, we have that $C \subset A \cup B$ immediately implies $C \subset A$ or $C \subset B$. In the case that $C \neq \emptyset$ and A and B are nonempty and separated, we will show the contrapositive. Suppose $C \subset A \cup B$ but $C \not\subset A$ and $C \not\subset B$. Because $C \subset A \cup B$, $C = C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$. We have that

$$\overline{C \cap A} \cap (C \cap B) \subset (\overline{C} \cap \overline{A}) \cap (\overline{C} \cap B) = \overline{C} \cap (\overline{A} \cap B) = \emptyset,$$

so $\overline{C \cap A}$ and $C \cap B$ are disjoint. Similarly, $C \cap A$ and $\overline{C \cap B}$ are disjoint, so C is the union of nonempty separated sets and is therefore not connected. □

Theorem 8.5. Let $\{C_\alpha\}_{\alpha \in \lambda}$ be a collection of connected subsets of X, and let $E \subset X$ such that E is connected and for every $\alpha \in \lambda$, $E \cap C_\alpha \neq \emptyset$. Then $E \cup \left(\bigcup_{\alpha \in \lambda} C_\alpha\right)$ is connected.

Proof. Assume the hypotheses of the claim and let $p, q \in C := E \cup \left(\bigcup_{\alpha \in \lambda} C_\alpha\right)$. If $p \in E$, define $p' \in E$ to be $p' = p$. If $p \notin E$, then there exists a $\alpha_p \in \lambda$ such that $p \in C_{\alpha_p}$. Then by hypothesis, $E \cap C_{\alpha_p} \neq \emptyset$, so define p' to be a point in $E \cap C_{\alpha_p}$. Similarly, if $q \in E$, define

$q' \in E$ to be $q' = q$, and if $q \notin E$, there exists a C_{α_q} containing q and we may define q' to be a point in $E \cap C_{\alpha_q}$. Let $\mathscr{C} = \{U_\beta\}_{\beta \in \gamma}$ be an open cover of C. Then it also covers C_{α_p}, and since C_{α_p} is connected, there exists a finite subcollection $U_p := \{U_{\beta_1}, \ldots, U_{\beta_k}\}$ such that $p \in U_{\beta_1}$, $p' \in U_{\beta_k}$, and for all $i < k$, $U_{\beta_i} \cap U_{\beta_{i+1}} \neq \emptyset$ by (6) of Theorem 8.1. Similar finite subcollections U_E and U_q exist where U_E is a collection of m open sets connecting p' to q' and U_q is a collection of n open sets connecting q to q'. Relabel the sets in U_E and U_q so that $U_E = \{U_{\beta_{k+1}}, \ldots, U_{\beta_{k+m}}\}$ and $U_q = \{U_{\beta_{k+m+1}}, \ldots, U_{\beta_{k+m+n}}\}$. Then the finite subcollection $U_p \cup U_E \cup U_q$ satisfies $p \in U_{\alpha_1}$ and $q \in U_{\beta_{k+m+n}}$, and if $i < k+m+n$ and $i \neq k, k+m$, we have that $U_{\beta_i} \cap U_{\beta_{i+1}} \neq \emptyset$ since both U_{β_i} and $U_{\beta_{i+1}}$ are in one of U_p, U_E, or U_q. Then for $i = k$, we have that $p' \in U_{\beta_k} \cap U_{\beta_{k+1}}$ since U_{β_k} is the last set in U_p and $U_{\beta_{k+1}}$ is the first set in U_E. Similarly, we have that $q' \in U_{\beta_{k+m}} \cap U_{\beta_{k+m+1}}$, so indeed for all $i < k+m+n$, we have that $U_{\beta_i} \cap U_{\beta_{i+1}} \neq \emptyset$. Since p and q were arbitrary points in C and $\mathscr{C}$ was an arbitrary open cover of C, we have that C is connected by (6) of Theorem 8.1. □

Theorem 8.6. Suppose C is connected and $C \subset D \subset \overline{C}$. Now let $\mathscr{C} = \{U_\alpha\}_{\alpha \in \lambda}$ be an open cover for D, and let $p, q \in D$ be arbitrary. If $p \in C$, define $p' = p$, and if $p \notin C$, then $D \subset \overline{C}$ means that p is a limit point of C, so for a neighborhood $U_{\alpha_p} \in \mathscr{C}$ of p (which exists because $\mathscr{C}$ covers $p \in D$), we have that $U_{\alpha_p} \cap C \neq \emptyset$ (since $p \notin C$). Define p' to be a point in $U_{\alpha_p} \cap C$. Similarly for $q \in D$, if $q \in C$ then $q' = q$, and if not, define q' to be a point in $U_{\alpha_q} \cap C$ for a neighborhood $U_{\alpha_q} \in \mathscr{C}$ of q. Then since $C \subset D$, $\mathscr{C}$ is also an open cover for C, and so by Theorem 8.1(6), there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ such that $p' \in U_{\alpha_1}$, $q' \in U_{\alpha_n}$, and for all $i < n$, $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$. Then also $U_{\alpha_p} \cap U_{\alpha_1} \neq \emptyset \neq U_{\alpha_n} \cap U_{\alpha_q}$, so relabeling this finite subcollection as $\{U_{\alpha_2}, \ldots, U_{\alpha_{n+1}}\}$ and setting $U_{\alpha_1} = U_{\alpha_p}$ and $U_{\alpha_{n+2}} = U_{\alpha_q}$ gives us a new finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_{n+2}}\}$ that connects p to q in the way specified in Theorem 8.1(6). Then since p and q were arbitrary points in D and $\mathscr{C}$ was an arbitrary open cover of D, we have that D is connected.

Exercise 8.7. Note that the topologist's sine curve

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \middle| \, x \in (0,1) \right\}$$

is itself connected by Theorem 8.9 (since the map $f : (0,1) \to S$ defined by $f(x) = (x, \sin(1/x))$ is continuous and surjective and the interval $(0,1)$ is connected for the same reason $\mathbb{R}_{\text{std}}$ is connected). Then since S is connected, $S \subset \overline{S} \subset \overline{S}$ implies that its closure is also connected by Theorem 8.6.

Theorem 8.8. Let X be connected with connected subset C. If $X - C = A \,|\, B$, then $A \cup C$ and $B \cup C$ are connected.

Proof. If $A = \emptyset$, then $X - C = A \,|\, B$ means $X - C = B$, so $B \cup C = X$ is connected, and $A \cup C = C$ is connected. The claim holds similarly if $B = \emptyset$. Now assume A and B are nonempty, and suppose for contradiction that $A \cup C$ is not connected. Then there exist separated nonempty sets U and V such that $A \cup C = U \cup V$. Therefore C is a connected subset of $U \cup V$, and so must either be contained in U or contained in V. Without loss of generality, assume $C \subset V$ Then since $U \cap C \subset U \cap V = \emptyset$ but $U \subset A \cup C$ we have that $U \subset A$. Since $X - C = A \cup B$, we have that

$$X = A \cup B \cup C = (A \cup C) \cup B = (U \cup V) \cup B = U \cup (V \cup B),$$

and we claim that U and $V \cup B$ are separated sets. We have that

$$\overline{U} \cap (V \cup B) = (\overline{U} \cap V) \cup (\overline{U} \cap B) = \overline{U} \cap B \subset \overline{A} \cap B = \emptyset$$

and that

$$U \cap (\overline{V \cup B}) = U \cap (\overline{V} \cup \overline{B}) = (U \cap \overline{V}) \cup (U \cap \overline{B}) = U \cap \overline{B} \subset A \cap \overline{B} = \emptyset$$

where in both cases we have used the fact that A and B are separated. Therefore $X = U \,|\, (V \cup B)$, and so X is not connected, a contradiction. Therefore it must be the case that $A \cup C$ is connected, and a similar argument shows that $B \cup C$ is connected. $\square$

Theorem 8.9. Let $f : X \to Y$ be continuous and surjective. If X is connected, so is Y.

Proof. Let $f : X \to Y$ be a continuous surjection and suppose Y is not connected. Then there exist disjoint nonempty open sets U and V in Y such that $Y = U \cup V$. Then we have that

$$X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V).$$

Since f is surjective and U and V are both nonempty, $f^{-1}(U)$ and $f^{-1}(V)$ are also nonempty. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are both open in X. Lastly, since $U \cap V = \emptyset$, we have that

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset.$$

Therefore X is the union of disjoint nonempty open sets and is therefore not connected. We have shown the contrapositive of the claim. $\square$

Theorem 8.10. Let $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ be a continuous function and let $a < b$ be real numbers. If $r \in \mathbb{R}$ such that $f(a) < r < f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = r$.

Proof. The proof that $\mathbb{R}_{\text{std}}$ is connected also applies to show that the interval $[a, b]$ is connected. Then the function $f|_{[a,b]}$ is continuous by Theorem 7.4, and it is a surjection onto its image, $f([a, b])$. Therefore by Theorem 8.9, $f([a, b])$ is connected. Suppose for contradiction that $r \notin f([a, b])$. Then since $(-\infty, r)$ and (r, ∞) are both open in $\mathbb{R}_{\text{std}}$ and disjoint, $(-\infty, r) \cap f([a, b])$ and $(-\infty, r) \cap f([a, b])$ are open in $f([a, b])$ with the relative topology inherited from $\mathbb{R}_{\text{std}}$ and are disjoint. Then since $f(a) < r < f(b)$, we have that

$$f(a) \in (-\infty, r) \cap f([a, b]) \quad \text{and} \quad f(b) \in (r, \infty) \cap f([a, b]).$$

Since this means that $f([a, b])$ is the union of disjoint nonempty open sets and so is not connected, we have a contradiction. Therefore it must be the case that $r \in f([a, b])$, so indeed there exists $c \in [a, b]$ such that $f(c) = r$. □

Theorem 8.11. For spaces X and Y, the product $X \times Y$ is connected if and only if X and Y are both connected.

Proof. ($\Longrightarrow$) Suppose $X \times Y$ is connected. Then $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous and surjective functions by Theorem 7.32, so by Theorem 8.9, X and Y are both connected.

($\Longleftarrow$) Suppose X and Y are connected. If one of X or Y is empty, then $X \times Y$ is empty as well and is therefore connected. Therefore suppose X and Y are connected and nonempty. Then there exists a $y_0 \in Y$, and by Theorem 7.35, the subspace $X \times \{y_0\}$ is homeomorphic to X and is therefore connected by Theorem 8.9, and for all $x \in X$, $\{x\} \times Y$ is homeomorphic to Y and is connected. Then since we have that

$$X \times Y = \bigcup_{x \in X} (\{x\} \times Y) = (X \times \{y_0\}) \cup \left(\bigcup_{x \in X} (\{x\} \times Y) \right),$$

Theorem 8.5 implies that $X \times Y$ is connected since for each $x \in X$,

$$(X \times \{y_0\}) \cap (\{x\} \times Y) = \{(x, y_0)\} \neq \emptyset$$

where $X \times \{y_0\}$ plays the role the set E played in Theorem 8.5. □

Lemma 8.12. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of topological spaces and let $\gamma \subset \lambda$ be arbitrary. Then for every $(p_\alpha)_{\alpha\in\lambda-\gamma} \in \prod_{\alpha\in\lambda-\gamma} X_\alpha$, we have that $\prod_{\alpha\in\gamma} X_\alpha$ is homeomorphic to $\prod_{\alpha\in\gamma} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-\gamma}\}$.

Proof. The homeomorphism is $f : \prod_{\alpha\in\gamma} X_\alpha \to \prod_{\alpha\in\gamma} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-\gamma}\}$ defined by $f((x_\alpha)_{\alpha\in\gamma}) = (q_\alpha)_{\alpha\in\lambda}$ where

$$q_\alpha = \begin{cases} x_\alpha & \alpha \in \gamma \\ p_\alpha & \alpha \in \lambda - \gamma. \end{cases}$$

It is injective because if $(x_\alpha)_{\alpha\in\gamma}, (x'_\alpha)_{\alpha\in\gamma} \in \prod_{\alpha\in\gamma} X_\alpha$ satisfy $(q_\alpha)_{\alpha\in\lambda} = (q'_\alpha)_{\alpha\in\lambda}$, then we have that $q_\alpha = q'_\alpha$ for all $\alpha \in \lambda$, and in particular, $q_\alpha = q'_\alpha$ for all $\alpha \in \gamma$, meaning $(x_\alpha)_{\alpha\in\gamma} = (x'_\alpha)_{\alpha\in\lambda}$. The map is surjective because if $(y_\alpha)_{\alpha\in\lambda} \in \prod_{\alpha\in\gamma} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-\gamma}\}$, then $(y_\alpha)_{\alpha\in\gamma} \in \prod_{\alpha\in\gamma} X_\alpha$ and f maps it to $(y_\alpha)_{\alpha\in\lambda}$. Then since f is a map into a product space, we check that $\pi_\beta \circ f : \prod_{\alpha\in\gamma} X_\alpha \to X_\beta$ is continuous for all $\beta \in \lambda$. For $\beta \in \gamma$, $(\pi_\beta \circ f)((x_\alpha)_{\alpha\in\gamma}) = x_\beta$ and so is a projection function, and for $\beta \in \lambda - \gamma$, $(\pi_\beta \circ f)((x_\alpha)_{\alpha\in\gamma}) = p_\beta$ and so is a constant function. In both cases, $\pi_\beta \circ f$ is continuous, and since $\beta \in \lambda$ was arbitrary, f is continuous. Since f^{-1} is also a function into a product space, we check that $\pi_\beta \circ f^{-1} : \prod_{\alpha\in\gamma} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-\gamma}\} \to X_\beta$ is continuous for all $\beta \in \gamma$. Since $(\pi_\beta \circ f^{-1})((q_\alpha)_{\alpha\in\lambda}) = q_\beta = x_\beta \in X_\beta$, this map is just a projection function and is therefore continuous. Since f is a continuous bijection with continuous inverse, f is a homeomorphism. $\square$

Theorem 8.12. For spaces $\{X_\alpha\}_{\alpha\in\lambda}$, the product $\prod_{\alpha\in\lambda} X_\alpha$ is connected if and only if X_α is connected for all $\alpha \in \lambda$.

Proof. ($\Longrightarrow$) Suppose $\prod_{\alpha\in\lambda} X_\alpha$ is connected and let $\beta \in \lambda$ be arbitrary. Then $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ is continuous and surjective by Theorem 7.38, so by Theorem 8.9, X_β is connected. Since $\beta \in \lambda$ was arbitrary, we have that X_α is connected for all $\alpha \in \lambda$.

($\Longleftarrow$) Suppose X_α is connected for all $\alpha \in \lambda$ and fix a point $(p_\alpha)_{\alpha\in\lambda}$. Then for any finite subset $F \subset \lambda$, induction and the result of Theorem 8.11 implies that the product space $\prod_{\alpha\in F} X_\alpha$ is connected. By Lemma 8.12, we have that the "slice" $\prod_{\alpha\in F} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-F}\}$ is homeomorphic to $\prod_{\alpha\in F} X_\alpha$ and is therefore connected. Since for each $\alpha \in F$, we have that $p_\alpha \in X_\alpha$, we also have that $(p_\alpha)_{\alpha\in\lambda} \in \prod_{\alpha\in F} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-F}\}$. Therefore if $E = \{(p_\alpha)_{\alpha\in\lambda}\}$, we have that $E \cap (\prod_{\alpha\in F} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-F}\}) \neq \emptyset$ for all finite subsets $F \subset \lambda$. Then if we define

$$\mathfrak{F} = \bigcup_{F\subset\lambda} \left(\prod_{\alpha\in F} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-F}\} \right),$$

we have that $\mathfrak{F} = E \cap \mathfrak{F}$ is connected by Theorem 8.5. We claim that $\overline{\mathfrak{F}} = \prod_{\alpha\in\lambda} X_\alpha$. By Exercise 5.1, it suffices to show that every nonempty open set intersects $\mathfrak{F}$. Let $U \neq \emptyset$ be open. Then there exists a point $(x_\alpha)_{\alpha\in\lambda} \in U$ and therefore a basic open set V such that $(x_\alpha)_{\alpha\in\lambda} \in V \subset U$. Since V is a basic open set in the product topology, V is of the form $V = \prod_{\alpha\in\lambda} V_\alpha$ such that $V_\alpha = X_\alpha$ for all but finitely many α, say, $\alpha \in F'$ for some finite $F' \subset \lambda$. Then for all $\alpha \in F'$, $x_\alpha \in V_\alpha \subset X_\alpha$ since $(x_\alpha)_{\alpha\in\lambda} \in V$, and for all $\alpha \in \lambda - F'$, $p_\alpha \in X_\alpha = V_\alpha$. Therefore the point $(y_\alpha)_{\alpha\in\lambda}$ defined by

$$y_\alpha = \begin{cases} x_\alpha & \alpha \in F' \\ p_\alpha & \alpha \in \lambda - F' \end{cases}$$

is an element of V, and therefore an element of U as well since $V \subset U$. This point is also an element of $\prod_{\alpha\in F'} X_\alpha \times \{(p_\alpha)_{\alpha\in\lambda-F'}\} \subset \mathfrak{F}$ since $y_\alpha = p_\alpha$ for all $\alpha \in \lambda - F'$. Therefore $(y_\alpha)_{\alpha\in\lambda} \in U \cap \mathfrak{F}$, so we have that $\overline{\mathfrak{F}} = \prod_{\alpha\in\lambda} X_\alpha$ as claimed. Then by Theorem 8.6, $\mathfrak{F}$ being connected with $\mathfrak{F} \subset \prod_{\alpha\in\lambda} X_\alpha \subset \overline{\mathfrak{F}}$ means that $\prod_{\alpha\in\lambda} X_\alpha$ is connected. □

Exercise 8.13. Recall that $\mathbb{R}^\omega$ is the set of all real-valued sequences, and let A denote the set of all sequences that converge to 0. To show that $\mathbb{R}^\omega$ with the box topology is not connected, we will show that A is both open and closed. To see that A is open, let $(a_n)_{n\in\mathbb{N}} \in A$ be arbitrary. Then the set $U = \prod_{n\in\mathbb{N}}(a_n - 1/n, a_n + 1/n)$ contains the sequence $(a_n)_{n\in\mathbb{N}}$ and is open in the box topology since it is the product sets that are all open in $\mathbb{R}_{\text{std}}$. We claim now that $U \subset A$. Let $(b_n)_{n\in\mathbb{N}} \in U$. To show this sequence is in A, we must show that $b_n \to 0$. Note that $(b_n)_{n\in\mathbb{N}} \in U$ means that for all $n \in \mathbb{N}$,

$$a_n - \frac{1}{n} \leq b_n \leq a_n + \frac{1}{n}.$$

Then since $(a_n)_{n\in\mathbb{N}}$ converges to 0, so do $(a_n \pm 1/n)_{n\in\mathbb{N}}$, and so we have that $b_n \to 0$ as well. Since $(a_n)_{n\in\mathbb{N}}$ was an arbitrary point of A and we were able to find a neighborhood of $(a_n)_{n\in\mathbb{N}}$ contained in A, A is an open set.

To show that A is closed, take an arbitrary $(x_n)_{n\in\mathbb{N}} \notin A$. We will show this is not a limit point of A. Define V similar to U before: $V = \prod_{n\in\mathbb{N}}(x_n - 1/n, x_n + 1/n)$. Then V contains $(x_n)_{n\in\mathbb{N}}$ and is open, but we will show that $V \cap A = \emptyset$. Since this sequence is not in A, it does not converge to 0, which means there exists an ε_0 such that for all $N \in \mathbb{N}$, there exists an $n > N$ such that $x_n \notin (-\varepsilon_0, \varepsilon_0)$. Now suppose $(y_n)_{n\in\mathbb{N}} \in A$. To show that $V \cap A = \emptyset$, we will show that $(y_n)_{n\in\mathbb{N}} \notin V$. Since this new sequence is in A, it converges to 0, which means there exists an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that $y_n \in (-\varepsilon_0/2, \varepsilon_0/2)$. Because $\mathbb{N}$ is unbounded, there also exists an $N_2 \in \mathbb{N}$ such that $1/N_2 < \varepsilon_0/2$. Take N to be $N = \max\{N_1, N_2\}$. Recall that there exists an $n_0 > N$ such that $x_{n_0} \notin (-\varepsilon_0, \varepsilon_0)$, and that $n_0 > N \geq N_2$ implies $1/n_0 < \varepsilon/2$. This means that $(x_{n_0} - 1/n_0, x_{n_0} + 1/n_0) \cap (-\varepsilon_0/2, \varepsilon_0/2) = \emptyset$, and since $n_0 > N_1$, $y_{n_0} \in (-\varepsilon_0/2, \varepsilon_0/2)$ and therefore $y_{n_0} \notin (x_{n_0} - 1/n_0, x_{n_0} + 1/n_0)$, meaning $(y_n)_{n\in\mathbb{N}} \notin V$. Therefore $A \cap V = \emptyset$, and so we have that for all sequences $(x_n)_{n\in\mathbb{N}} \notin A$, $(x_n)_{n\in\mathbb{N}}$ is not a limit point of A. This means that if there are limit points, they are contained in A, and so A is closed in addition to being open. Since $(0, 0, 0, \dots) \in A$, A is nonempty, and since $(1, 1, 1, \dots) \notin A$, $A \neq \mathbb{R}^\omega$. Therefore $\mathbb{R}^\omega$ with the box topology is not connected by Theorem 8.1(5).

8.3 Components and Continua

Theorem 8.18. Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.

Proof. Let A be an arbitrary component of X. Then there exists a $p \in A$ such that A is the component of p in X. We will first prove that A is connected. By the definition of a component, we may write $A = \bigcup_{p\in U\subset X} U$ where the union is taken over connected sets U that contain the point p. Then since $p \in U$ for all of these U, the set $E = \{p\}$ intersects all of these U and is itself connected because it contains a single point. Therefore by Theorem 8.5, $A = \bigcup_{p\in U\subset X} U = E \cup \left(\bigcup_{p\in U\subset X} U\right)$ is connected. We now prove that A is not a proper subset of any connected subset $B \subset X$. Suppose $A \subset B$. Then $p \in B$ and since B is connected, $A = \bigcup_{p\in U\subset X} U$ contains B because the union is taken over all connected subsets containing p. Therefore $A \subset B$ for a connected B implies that $B \subset A$ as well, so A is not contained in any strictly larger connected subset of X. From this, it follows that A is closed. This is because $A \subset \overline{A}$, and by Theorem 8.6, $\overline{A}$ is connected. Therefore $\overline{A} \subset A$ as well, and so we have that $\overline{A} = A$, meaning A is closed. $\square$

Theorem 8.19. The set of components of a space X is a partition of X.

Proof. Let $C = \{A \subset X \mid A \text{ is a component of } X\}$ be the set of components. We will show that $X = \bigsqcup_{A\in C} A$. First note that for all $p \in X$, $p \in A_p$ where A_p is the component of p in X. Since A_p is a component of X, $A_p \in C$, and so we have that $X = \bigcup_{A\in C} A$. It remains to show that for all $A, B \in C$ with $A \neq B$, $A \cap B = \emptyset$. Suppose there existed a $q \in A \cap B$. Then because A and B are components, they are connected, and since $E = \{q\}$ is also connected as a set containing a single point, Theorem 8.5 implies that $A \cup B = E \cup (A \cup B)$ is also connected. Because A and B are components and $A, B \subset A \cup B$, we have that $A = A \cup B = B$ since components are not contained in any strictly larger connected subsets of X. Therefore $A \cap B \neq \emptyset$ implies that $A = B$, so we have that $A \neq B$ implies that $A \cap B = \emptyset$ as required. Therefore $X = \bigsqcup_{A \in C} A$, and so the set C of components of X is indeed a partition of X. $\square$

Lemma 8.20. Let X be a topological space and let $\{H_\alpha\}_{\alpha \in \lambda}$ be the set of subsets of X that are both open and closed. Then the following are equivalent:

(1) For every two components A and B, there exists a separation of X into two disjoint closed sets such that A is in one and B is in the other.

(2) For every component A, $\bigcap_{A \subset H_\alpha} H_\alpha = A$.

Proof. (1) $\implies$ (2) Let A be a component and note that $A \subset \bigcap_{A \subset H_\alpha} H_\alpha$ since we are taking the union of all H_α that contain A. To show that they are equal, we will prove that the intersection is a subset of A. Let $p \in \bigcap_{A \subset H_\alpha}$ and suppose for contradiction that $p \notin A$. Then p is in some other component by Theorem 8.19, call it B. Then since we are assuming (1), there exist disjoint closed sets C and D such that $X = C \cup D$, $A \subset C$, and $B \subset D$. Since $X - C = D$ is closed, C is open in addition to being closed, and therefore $C = H_\beta$ for some $\beta \in \lambda$. Since $p \in B \subset D$ but also $p \in \bigcap_{A \subset H_\alpha} H_\alpha \subset H_\beta = C$ since $A \subset H_\beta = C$. But this is a contradiction since $C \cap D = \emptyset$, and so we have that $p \in A$. Therefore $A = \bigcap_{A \subset H_\alpha} H_\alpha$.

(2) $\implies$ (1) We will show the contrapositive. Suppose then that there exists a pair of components A and B such that for every separation of X into disjoint closed sets C and D, we have either that $A, B \subset C$ or $A, B \subset D$. Now note that for every set H_α that contains A and is both closed and open, there exists a separation $X = H_\alpha \cup (X - H_\alpha)$. Because H_α is closed and open, we have that H_α and $X - H_\alpha$ are both closed. Then since $A \subset H_\alpha$, it is not the case that $A, B \subset X - H_\alpha$, so by assumption, we have that $A, B \subset H_\alpha$. But H_α was an arbitrary closed and open set containing A, so all such sets also contain B. Since B is a component, $B \neq \emptyset$ and $A \cap B = \emptyset$, and so we have that A is a proper subset of $A \cup B$. Therefore

$$A \subset A \cup B \subset \bigcap_{A \subset H_\alpha} H_\alpha$$

means that $\bigcap_{A \subset H_\alpha} H_\alpha \neq A$, so we have that (2) does not hold. $\square$

Lemma 8.21. Let X be compact and let U be an open set in X. Let $\{H_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets such that $\bigcap_{\alpha \in \lambda} H_\alpha \subset U$. Then there exists a finite subset $N \subset \lambda$ such that $\bigcap_{\alpha \in N} H_\alpha \subset U$.

Proof. This is Theorem 6.6. $\square$

Lemma 8.22. Let A and B be components of a compact Hausdorff space X. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.

Proof. We will show that (2) of Lemma 8.20 holds. Let A be a component and $\{H_\alpha\}_{\alpha\in\lambda}$ the collection of sets that are both closed and open, and let $H_A = \{H_\alpha \mid A \subset H_\alpha\}$. Now suppose for contradiction that $H = \bigcap_{H_\alpha \in H_A} H_\alpha \neq A$. Then it must be the case that A is a proper subset of H. Since A is a component, H is not connected, which means there exist nonempty disjoint sets C and D that are closed in H such that $H = C \cup D$. By Theorem 8.4, $A \subset C$ or $A \subset D$. Assume without loss of generality that $A \subset C$ Since C and D are closed in H and H is closed in X (since it is the intersection of closed sets), we have that C and D are disjoint closed sets in X. Then since X is compact and Hausdorff, it is normal by Theorem 6.12, and since it is normal, by Theorem 4.10 there exist sets U and V open in X such that $A \subset C \subset U$, $D \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$. Then we have that $H = \bigcap_{H_\alpha \in H_A} H_\alpha \subset U \cup V$. By Lemma 8.21, there exists a finite subcollection $N \subset H_A$ such that $\bigcap_{H_\alpha \in N} H_\alpha \subset U \cup V$. Because this is a finite intersection of sets that are all closed and open, the intersection itself is closed and open, and because $A \subset H_\alpha$ for all $H_\alpha \in N \subset H_A$, we have that $\bigcap_{H_\alpha \in N} = H_\beta$ for some $H_\beta \in H_\alpha$. Therefore $H_\beta \subset U \cup V$, $H = \bigcap_{H_\alpha \in H_A} H_\alpha \subset H_\beta$, and also H_β is a closed and open set containing A. We will now show that $H_\beta \cap U$ is also a closed and open set containing A. Since both H_β and U are open, we have that $H_\beta \cap U$ is open, and then we have that

$$\overline{H_\beta \cap U} \subset \overline{H_\beta} \cap \overline{U} = H_\beta \cap \overline{U} = (H_\beta \cap (U \cup V)) \cap \overline{U}$$

where we have used that H_β is closed and a subset of $U \cup V$. But this means that

$$\overline{H_\beta \cap U} \subset H_\beta \cap ((U \cup V) \cap \overline{U}) = H_\beta \cap ((U \cap \overline{U}) \cup (V \cap \overline{U})) = H_\beta \cap (U \cup \emptyset) = H_\beta \cap U.$$

Since also $H_\beta \cap U \subset \overline{H_\beta \cap U}$, we have that $\overline{H_\beta \cap U} = H_\beta \cap U$. So indeed $H_\beta \cap U$ is closed and open, and it contains A because both H_β and U contain A. Therefore $H_\beta \cap U = H_\gamma$ for some $H_\gamma \in H_A$, so we have that $H = \bigcap_{H_\alpha \in H_A} H_\alpha \subset H_\gamma$. But this then means that

$$D \subset C \cup D = H \subset H_\gamma = H_\beta \cap U \subset U,$$

which is a contradiction because also $D \subset V$, meaning $D \subset U \cap V = \emptyset$. However, D is nonempty, so we have that (2) must hold. By Lemma 8.20, we have that for components A and B of X, there exists a separation $X = H \cup K$ for disjoint closed sets H and K with

$A \subset H$ and $B \subset K$. Since H and K are closed and disjoint, we have that they are separated sets, as required. □

Theorem 8.23. Let X be a compact Hausdorff space and let X^* be the partition of X into its components. Then the identification space X^* is a compact Hausdorff space.

Proof. Recall that the identification map $f : X \to X^*$ is continuous and surjective. Then X being compact means that X^* is compact as well by Theorem 6.15. To see that X^* is Hausdorff, let $p, q \in X^*$ be distinct points. Then $f^{-1}(p)$ and $f^{-1}(q)$ are components of the space X, and so by the proof of Lemma 8.22, X can be written as $X = H \cup K$ for disjoint closed sets H and K such that $f^{-1}(p) \subset H$ and $f^{-1}(q) \subset K$. Then we have that $H = X - K$ and $K = X - H$, so actually H and K are also open sets. Now for each component C of X, C is connected, and since H and K are separated subsets of X, Theorem 8.4 implies that for each component C, either $C \subset H$ or $C \subset K$. We then define the sets U and V as follows:

$$U = \{x \in X^* \mid f^{-1}(x) \subset H\} \quad \text{and} \quad V = \{x \in X^* \mid f^{-1}(x) \subset K\}.$$

Because the preimage of a point in X^* is a component of X and therefore $x \in X^*$ cannot be in both U and V, we have that U and V are disjoint sets such that $p \in U$ and $q \in V$. To show that X^* is Hausdorff, it remains to show that U and V are open sets. Consider $f^{-1}(U)$. Let $a \in f^{-1}(U)$. Then $f(a) \in U$, which means that $f^{-1}(f(a))$ is a component in X such that $a \in f^{-1}(f(a)) \subset H$. Therefore $f^{-1}(U) \subset H$. Now let $b \in H$. Then b is in some component C_b of X, and since $b \in H$, we have that $C_b \subset H$ because C_b is connected (again by Theorem 8.4). Because $C_b \subset H$, we have that $f(C_b) = \{y\}$ is a singleton in X^*, and we have that $f^{-1}(y) = f^{-1}(f(C_b))$, which is a set in X containing C_b. Since the identification map takes each component to a single point in X^* and different components are mapped to different points, we have that $f^{-1}(y)$ is a single component. Since $C_b \subset f^{-1}(y)$, we have that $C_b = f^{-1}(y)$. Then since $b \in C_b \subset H$, we have that $f^{-1}(y) \subset H$, meaning $y \in U$. This means that $f(b) \in f(C_b)$, so $f(b) = y \in U$. Therefore $b \in f^{-1}(U)$ and we have that $H \subset f^{-1}(U)$. We have shown that $f^{-1}(U) = H$, and since H is open in X, we have that U is open in X^* by the definition of the identification map f. A similar argument shows that V is also open in X^* because $f^{-1}(V) = K$, and so we have that X^* is Hausdorff because for arbitrary distinct points $p, q \in X^*$, we have found disjoint open sets U and V such that $p \in U$ and $q \in V$. □

Theorem 8.24. Let A and B be closed subsets of a compact Hausdorff space X such that no component intersects both A and B. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.

Proof. If one of A or B is empty, then $X = X \cup \emptyset$ is the desired separation. Assume now that A and B are nonempty. As in the lemmas, let $\{H_\alpha\}_{\alpha\in\lambda}$ denote the collection of closed and open sets in X. Define H_A to be the collection $H_A = \{H_\alpha \mid A \subset H_\alpha\}$ and define C_A to be the collection $C_A = \{C \mid A \cap C \neq \emptyset \text{ and } C \text{ is a component}\}$ of components that intersect A. Let $\mathfrak{C} = \bigcup_{C\in C_A} C$, let $\mathfrak{H} = \bigcap_{H_\alpha \in H_A} H_\alpha$, and let $C \in C_A$ be arbitrary. Then for each $H_\alpha \in H_A$, we have that $X = H_\alpha \cup (X - H_\alpha)$, and since H_α is closed and open, H_α and $X - H_\alpha$ are both closed and therefore separated. Then by Theorem 8.4, we have that either $C \subset H_\alpha$ or $C \subset X - H_\alpha$. Since $A \cap C \neq \emptyset$ and $A \subset H_\alpha$, we must have that $C \subset H_\alpha$ since it cannot be the case that $C \subset X - H_\alpha$. The component C was an arbitrary element of C_A, so we have shown that all components intersecting A are contained in all closed and open H_α that contain A. Therefore $\mathfrak{C} \subset \mathfrak{H}$.

We now show that also $\mathfrak{H} \subset \mathfrak{C}$ by supposing for contradiction there exists some point $q \in \mathfrak{H} - \mathfrak{C}$. Then $q \notin \mathfrak{C}$ means that the component of q in X does not intersect A, and therefore neither does the component C_q of q in $\mathfrak{H}$. Now for each $C \in C_A$, $C \neq C_q$, and since $\mathfrak{H}$ contains all of these components entirely, $C \in C_A$ is also a component in $\mathfrak{H}$. Then by the proof of Lemma 8.22, there exist disjoint closed sets $S_C, T_C \subset \mathfrak{H}$ such that $\mathfrak{H} = S_C \cup T_C$ with $C \subset S_C$ and $C_q \subset T_C$. Since S_C and T_C are closed in $\mathfrak{H}$ and $\mathfrak{H}$ is closed in X, S_C and T_C are closed in X. Since X is compact and Hausdorff, it is normal, and so there exist open sets U_C and V_C such that $S_C \subset U_C$, $T_C \subset V_C$, and $\overline{U_C} \cap \overline{V_C} = \emptyset$. Since $\mathfrak{H} \subset X = S_C \cup T_C$, we have that $\bigcap_{H_\alpha \in H_A} H_\alpha = \mathfrak{H} \subset U_C \cup V_C$. By Lemma 8.21, there exists a finite subset $N \subset H_A$ such that $\bigcap_{H_\alpha\in N} H_\alpha \subset U_C \cup V_C$. Since this is the finite intersection of sets that are all closed and open, the intersection is also closed and open, and since each set contains A, the intersection also contains A, and so we have that $\bigcap_{H_\alpha \in N} H_\alpha = H_{\alpha_C}$ for some $H_{\alpha_C} \in H_A$. Then H_{α_C} contains C as shown previously, and since $C \subset S_C \subset U_C$, we have that $C \subset H_{\alpha_C} \cap U_C$. Then as in Lemma 8.22, we have that

$$\begin{aligned}\overline{H_{\alpha_C} \cap U_C} \subset \overline{H_{\alpha_C}} \cap \overline{U_C} = H_{\alpha_C} \cap \overline{U_C} &= H_{\alpha_C} \cap ((U_C \cup V) \cap \overline{U_C}) \\ &= H_{\alpha_C} \cap ((U_C \cap \overline{U_C}) \cup (V \cap \overline{U_C})) = H_{\alpha_C} \cap (U_C \cup \emptyset) = H_{\alpha_C} \cap U_C,\end{aligned}$$

so $H_{\alpha_C} \cap U_C$ is a closed set containing C, and because both of these sets are open, $H_{\alpha_C} \cap U_C$ is also open. Therefore $\{H_{\alpha_C} \cap U_C\}_{C\in C_A}$ is an open cover for $\mathfrak{C}$.

We claim that $\mathfrak{C}$ is compact. Since X is compact and $\mathfrak{C} \subset X$, it suffices to show that $\mathfrak{C}$ is closed. Let X^* be the partition of X into its components and let $f : X \to X^*$ be the identification map taking each point to its component in X^*. We have that f is continuous, that X is compact, and by Theorem 8.23 that X^* is Hausdorff. Therefore f is a closed map by Theorem 7.24, and so $f(A)$ is closed in X^*. Then because f is continuous, we have that $f^{-1}(f(A))$ is closed in X. We claim now that $f^{-1}(f(A)) = \mathfrak{C}$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, which means there exists an $a \in A$ such that $f(x) = f(a)$, which in turn means x and a have the same component in X. Therefore the component of x in X intersects A, and so we have that $x \in \mathfrak{C}$. Now let $x \in \mathfrak{C}$. Then the component of x in X intersects A, and so we have that for some $a \in A$, $f(x) = f(a)$, meaning $f(x) \in f(A)$ and therefore that $x \in f^{-1}(f(A))$. So indeed, $\mathfrak{C} = f^{-1}(f(A))$ is closed.

Since $\mathfrak{C}$ is compact, the open cover $\{H_{\alpha_C} \cap U_C\}_{C \in C_A}$ has a finite subcover, so there exists some finite $M \subset C_A$ such that $\mathfrak{C} \subset \bigcup_{C \in M}(H_{\alpha_C} \cap U_C)$. This union is finite, so since $H_{\alpha_C} \cap U_C$ is open and closed for all $C \in M$, their union is closed and open as well. Since the union contains $\mathfrak{C}$, it also contains A. Therefore $\bigcup_{C \in M}(H_{\alpha_C} \cap U_C) = H_\beta$ for some $H_\beta \in H_A$. Since $\mathfrak{H} = \bigcap_{H_\alpha \in H_A} H_\alpha$, we have that $\mathfrak{H} \subset H_\beta$, and so also

$$\bigcup_{C \in M} T_C \subset \bigcup_{C \in M} (S_C \cup T_C) = \mathfrak{H} \subset H_\beta = \bigcup_{C \in M} (H_{\alpha_C} \cap U_C) \subset \bigcup_{C \in M} U_C$$

since $S_C \cup T_C = \mathfrak{H}$ for all $C \in M \subset C_A$. Then we have that $q \in C_q \subset T_C \subset V_C$ for all $C \in M$, which implies that $q \notin U_C$ for any $C \in M$, which is a contradiction because it would imply that $q \notin \bigcup_{C \in M} U_C$. Therefore we have that $\bigcup_{C \in C_A} C = \mathfrak{C} = \mathfrak{H} = \bigcap_{H_\alpha \in H_A} H_\alpha$.

Now suppose for contradiction that for every separation of X into disjoint closed sets $X = H \cup K$, we have that either $A, B \subset H$ or $A, B \subset K$. Again we note that if $H_\alpha \in H_A$, then $X = H_\alpha \cap (X - H_\alpha)$, so either $A, B \subset H_\alpha$ or $A, B \subset X - H_\alpha$. Since $H_\alpha \in H_A$ contains A, we must have the former, meaning $B \subset H_\alpha$ for all $H_\alpha \in H_A$. Therefore for all $H_\alpha \in H_A$, we have that

$$B \subset \bigcap_{H_\alpha \in H_A} H_\alpha = \mathfrak{H} = \mathfrak{C} = \bigcup_{C \in C_A} C.$$

Since $B \neq \emptyset$, there exists some $p \in B$. Then we have that $p \in \mathfrak{C}$, so $p \in C'$ for some $C' \in C_A$. That is, the component of p in X is $C' \in C_A$, which means the component of p in X intersects A. However, it also intersects B because it contains $p \in B$. Therefore C' is a component intersecting both A and B, but this is a contradiction since we assumed that no such component exists. This means it cannot be the case that either $A, B \subset H$ or $A, B \subset K$ for every pair of disjoint closed sets H and K satisfying $X = H \cup K$, and so we have that there must exist some pair of H and K such that $A \subset H$ and $B \subset K$, as required. $\square$

Theorem 8.24. Let A and B be closed subsets of a compact Hausdorff space X such that no component intersects both A and B. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.

Proof. If one of A or B is empty, then $X = X \cup \emptyset$ is the desired separation. Assume now that A and B are nonempty. As in the lemmas, let $\{H_\alpha\}_{\alpha\in\lambda}$ denote the collection of closed and open sets in X. Define H_A to be the collection $H_A = \{H_\alpha \mid A \subset H_\alpha\}$ and define C_A to be the collection $C_A = \{C \mid A \cap C \neq \emptyset \text{ and } C \text{ is a component}\}$ of components that intersect A. Let $\mathfrak{C} = \bigcup_{C\in C_A} C$, let $\mathfrak{H} = \bigcap_{H_\alpha\in H_A} H_\alpha$, and let $C \in C_A$ be arbitrary. Then for each $H_\alpha \in H_A$, we have that $X = H_\alpha \cup (X - H_\alpha)$, and since H_α is closed and open, H_α and $X - H_\alpha$ are both closed and therefore separated. Then by Theorem 8.4, we have that either $C \subset H_\alpha$ or $C \subset X - H_\alpha$. Since $A \cap C \neq \emptyset$ and $A \subset H_\alpha$, we must have that $C \subset H_\alpha$ since it cannot be the case that $C \subset X - H_\alpha$. The component C was an arbitrary element of C_A, so we have shown that all components intersecting A are contained in all closed and open H_α that contain A. Therefore $\mathfrak{C} \subset \mathfrak{H}$.

We now show that also $\mathfrak{H} \subset \mathfrak{C}$ by supposing for contradiction there exists some point $q \in \mathfrak{H} - \mathfrak{C}$. Then $q \notin \mathfrak{C}$ means that the component of q in X does not intersect A, and therefore neither does the component C_q of q in $\mathfrak{H}$. Now for each $C \in C_A$, $C \neq C_q$, and since $\mathfrak{H}$ contains all of these components entirely, $C \in C_A$ is also a component in $\mathfrak{H}$. Then by the proof of Lemma 8.22, there exist disjoint closed sets $S_C, T_C \subset \mathfrak{H}$ such that $\mathfrak{H} = S_C \cup T_C$ with $C \subset S_C$ and $C_q \subset T_C$. Since S_C and T_C are closed in $\mathfrak{H}$ and $\mathfrak{H}$ is closed in X, S_C and T_C are closed in X. Since X is compact and Hausdorff, it is normal, and so there exist open sets U_C and V_C such that $S_C \subset U_C$, $T_C \subset V_C$, and $\overline{U_C} \cap \overline{V_C} = \emptyset$. Since $\mathfrak{H} \subset X = S_C \cup T_C$, we have that $\bigcap_{H_\alpha\in H_A} H_\alpha = \mathfrak{H} \subset U_C \cup V_C$. By Lemma 8.21, there exists a finite subset $N \subset H_A$ such that $\bigcap_{H_\alpha\in N} H_\alpha \subset U_C \cup V_C$. Since this is the finite intersection of sets that are all closed and open, the intersection is also closed and open, and since each set contains A, the intersection also contains A, and so we have that $\bigcap_{H_\alpha\in N} H_\alpha = H_{\alpha_C}$ for some $H_{\alpha_C} \in H_A$. Then H_{α_C} contains C as shown previously, and since $C \subset S_C \subset U_C$, we have that $C \subset H_{\alpha_C} \cap U_C$. Then as in Lemma 8.22, we have that

$$\begin{aligned}\overline{H_{\alpha_C} \cap U_C} \subset \overline{H_{\alpha_C}} \cap \overline{U_C} = H_{\alpha_C} \cap \overline{U_C} &= H_{\alpha_C} \cap ((U_C \cup V) \cap \overline{U_C}) \\ &= H_{\alpha_C} \cap ((U_C \cap \overline{U_C}) \cup (V \cap \overline{U_C})) = H_{\alpha_C} \cap (U_C \cup \emptyset) = H_{\alpha_C} \cap U_C,\end{aligned}$$

so $H_{\alpha_C} \cap U_C$ is a closed set containing C, and because both of these sets are open, $H_{\alpha_C} \cap U_C$ is also open. Therefore $\{H_{\alpha_C} \cap U_C\}_{C\in C_A}$ is an open cover for $\mathfrak{C}$.

We claim that $\mathfrak{C}$ is compact. Since X is compact and $\mathfrak{C} \subset X$, it suffices to show that $\mathfrak{C}$ is closed. Let X^* be the partition of X into its components and let $f : X \to X^*$ be the identification map taking each point to its component in X^*. We have that f is continuous, that X is compact, and by Theorem 8.23 that X^* is Hausdorff. Therefore f is a closed map by Theorem 7.24, and so $f(A)$ is closed in X^*. Then because f is continuous, we have that $f^{-1}(f(A))$ is closed in X. We claim now that $f^{-1}(f(A)) = \mathfrak{C}$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, which means there exists an $a \in A$ such that $f(x) = f(a)$, which in turn means x and a have the same component in X. Therefore the component of x in X intersects A, and so we have that $x \in \mathfrak{C}$. Now let $x \in \mathfrak{C}$. Then the component of x in X intersects A, and so we have that for some $a \in A$, $f(x) = f(a)$, meaning $f(x) \in f(A)$ and therefore that $x \in f^{-1}(f(A))$. So indeed, $\mathfrak{C} = f^{-1}(f(A))$ is closed.

Since $\mathfrak{C}$ is compact, the open cover $\{H_{\alpha_C} \cap U_C\}_{C \in C_A}$ has a finite subcover, so there exists some finite $M \subset C_A$ such that $\mathfrak{C} \subset \bigcup_{C \in M}(H_{\alpha_C} \cap U_C)$. This union is finite, so since $H_{\alpha_C} \cap U_C$ is open and closed for all $C \in M$, their union is closed and open as well. Since the union contains $\mathfrak{C}$, it also contains A. Therefore $\bigcup_{C \in M}(H_{\alpha_C} \cap U_C) = H_\beta$ for some $H_\beta \in H_A$. Since $\mathfrak{H} = \bigcap_{H_\alpha \in H_A} H_\alpha$, we have that $\mathfrak{H} \subset H_\beta$, and so also

$$\bigcup_{C \in M} T_C \subset \bigcup_{C \in M} (S_C \cup T_C) = \mathfrak{H} \subset H_\beta = \bigcup_{C \in M} (H_{\alpha_C} \cap U_C) \subset \bigcup_{C \in M} U_C$$

since $S_C \cup T_C = \mathfrak{H}$ for all $C \in M \subset C_A$. Then we have that $q \in C_q \subset T_C \subset V_C$ for all $C \in M$, which implies that $q \notin U_C$ for any $C \in M$, which is a contradiction because it would imply that $q \notin \bigcup_{C \in M} U_C$. Therefore we have that $\bigcup_{C \in C_A} C = \mathfrak{C} = \mathfrak{H} = \bigcap_{H_\alpha \in H_A} H_\alpha$.

Now suppose for contradiction that for every separation of X into disjoint closed sets $X = H \cup K$, we have that either $A, B \subset H$ or $A, B \subset K$. Again we note that if $H_\alpha \in H_A$, then $X = H_\alpha \cap (X - H_\alpha)$, so either $A, B \subset H_\alpha$ or $A, B \subset X - H_\alpha$. Since $H_\alpha \in H_A$ contains A, we must have the former, meaning $B \subset H_\alpha$ for all $H_\alpha \in H_A$. Therefore for all $H_\alpha \in H_A$, we have that

$$B \subset \bigcap_{H_\alpha \in H_A} H_\alpha = \mathfrak{H} = \mathfrak{C} = \bigcup_{C \in C_A} C.$$

Since $B \neq \emptyset$, there exists some $p \in B$. Then we have that $p \in \mathfrak{C}$, so $p \in C'$ for some $C' \in C_A$. That is, the component of p in X is $C' \in C_A$, which means the component of p in X intersects A. However, it also intersects B because it contains $p \in B$. Therefore C' is a component intersecting both A and B, but this is a contradiction since we assumed that no such component exists. This means it cannot be the case that either $A, B \subset H$ or $A, B \subset K$ for every pair of disjoint closed sets H and K satisfying $X = H \cup K$, and so we have that there must exist some pair of H and K such that $A \subset H$ and $B \subset K$, as required. $\square$

8.4 Path or Arcwise Connectedness

Theorem 8.35. A path connected space is connected.

Proof. Let X be a path connected space and suppose for contradiction that X has more than one distinct component. Then X has at least two distinct components, call them A and B. Since they are components, they are nonempty, and so there exists an $a \in A$ and $b \in B$. Because X is path connected, there exists a path $f : [0,1] \to X$ with $f(0) = a$ and $f(1) = b$. Note that $[0,1]$ is connected in $\mathbb{R}_{\text{std}}$, $f([0,1])$ is connected in X by Theorem 8.9 because f is continuous and surjective onto its image. Therefore $E = f([0,1])$ is connected, and we have that $a \in E \cap A \neq \emptyset$ and $b \in E \cap B \neq \emptyset$. Then by Theorem 8.5, we have that $E \cup (A \cup B)$ is connected. But then because A is a proper subset of $A \cup B$ (since A and B are disjoint and nonempty), we have that

$$A \subset A \cup B \subset E \cup (A \cup B),$$

meaning A is a proper subset of a connected set in X. But this contradicts A being a component of X, and so we have that X has only one component, call it C. By Theorem 8.19, the set of components of X is a partition of X, so $X = \bigcup_{C \in \{C\}} C = C$ is connected. $\square$

Exercise 8.36. Before investigating the flea and comb space, we begin by showing that the topologist's comb is path connected. Let (a,b) and (c,d) be point in C, the topologist's comb. Then there is a path $f : [0,1] \to C$ given by

$$\begin{cases} (a, (1-3t)b) & t \in [0, 1/3] \\ (a(2-3t) + (3t-1)c, 0) & t \in [1/3, 2/3] \\ (c, (3t-2)d) & t \in [2/3, 1] \end{cases}.$$

Note that $f(0) = (a,b)$, $f(1) = (c,d)$, and that $f(t) = (x,y)$ where either $y > 0$ and $x = a = 1/n$ or $x = c = 1/m$ for some integers m and n, or $y = 0$, meaning in all cases $f(t) \in C$. Then since C is path connected, it is connected, so showing that the flea and comb space F is connected means showing that $(0,1)$ is connected to C but not path connected to C. To show connectedness, note that $(0,1)$ is a limit point of C and so we have that $C \subset F \subset \overline{C}$, meaning F is connected. To show that F is not path connected, suppose for contradiction that there exists a path f connecting $f(0) = (0,1)$ to $f(1) = (1,0)$ and consider the set $A = f^{-1}(\{(0,1)\})$. Since f is a path, it is continuous, and so we have that A is a closed set as the preimage of a closed set in F (F is a subspace of the T_1 space $\mathbb{R}^2_{\text{std}}$). We also note that $A \subset [0,1]$, meaning $A \neq \emptyset$ (since $0 \in A$) and that A is bounded above (for example by 1), so $\alpha = \sup A$ exists. Becuase A is closed, $\alpha \in A$. Now define the set $B = \{t \in (\alpha, 1] \mid \pi_Y(f(t)) = 0\}$ where π_Y is the projection onto the y axis. Since $f(1) = (1,0)$, we have that $1 \in B$ and therefore B is nonempty, and we also have that α is a lower bound for B. Therefore $\beta = \inf B$ exists. We now note that we do not have $\alpha = \beta$. This is because $\pi_Y \circ f : [0,1] \to \mathbb{R}$ is continuous, and so if $\alpha = \beta$, we would have that α is a limit point of B and Theorem 7.1(3) would imply that $\pi_Y(f(\alpha)) \in \pi_Y(f(B))$. However, for all $t \in B$, we have that $\pi_Y(f(t)) = 0$ by definition, so this would imply that $1 = \pi_Y(f(\alpha)) \in \pi_Y(f(B)) = \{0\}$, a contradiction. Therefore $\alpha < \beta$. Then for all $t \in (\alpha, \beta)$, we have that $\alpha < t$ implies that $f(t) \neq (0,1)$ (so $f(t) \in C$) and $t < \beta$ implies $\pi_Y(f(t)) \neq 0$. Therefore $f(t)$ is of the form $(1/n, y)$ for some $n \in \mathbb{N}$ and $y \in (0,1]$ since $f(t) \in C$. Suppose now for contradiction that there existed $t_1 < t_2$ in the interval (α, β) such that $1/n_1 = \pi_X(f(t_1)) \neq \pi_X(f(t_2)) = 1/n_2$. Assume without loss of generality that $1/n_1 < 1/n_2$. Then by the density of the irrationals in the reals, there exists a $\gamma \in (1/n_1, 1/n_2)$. Because $\pi_X \circ f|_{[t_1,t_2]} : [t_1, t_2] \to \mathbb{R}$ is continuous, $1/n_1 < \gamma < 1/n_2$ implies that there exists a $t_3 \in [0,1]$ such that $\pi_X(f(t_3)) = \gamma$ by Theorem 8.10. However, $f(t_3) \in C$, so the irrational x coordinate implies $f(t_3) = (\gamma, 0)$, contradicting $t_3 < t_2 < \beta = \inf B$. Therefore we have that there exists some $n \in \mathbb{N}$ such that for all $t \in (\alpha, \beta)$, $f(t)$ is of the form $(1/n, y)$ for $y \in (0,1]$. This means that $(\alpha, \beta) \subset D = (\pi_X \circ f)^{-1}(\{1/n\})$. Since $\pi_X \circ f : [0,1] \to \mathbb{R}$ is continuous and $\mathbb{R}$ is T_1, $\{1/n\}$ being closed in $\mathbb{R}$ implies that D is closed in $[0,1]$. Therefore

$$[\alpha, \beta] = \overline{(\alpha, \beta)} \subset \overline{D} = D,$$

so in particular we have that $\alpha \in D$. But this implies that $\pi_X(f(\alpha)) = 1/n$, which is a contradiction since $f(\alpha) = (0,1)$. Therefore no path exists connecting $(0,1)$ to $(0,1)$. Both of these points are in F, so F is not path connected.

Exercise 8.37. We already have that $\overline{S}$ is connected (Exercise 8.7), so we will show that $\overline{S}$ is not path connected by showing there is no path connecting $(0,0)$ to $(s, \sin(1/s))$ for some $s \in (0,1)$. Suppose for contradiction that there is such a path $f : [0,1] \to \overline{S}$ such that f is continuous with $f(0) = (0,0)$ and $f(1) = (s, \sin(1/s))$. Let $A = \{t \in [0,1] \mid \pi_X(f(t)) = 0\}$ where π_X is the projection function onto the x axis. Then we have that $0 \in A$, and A is bounded above by 1, so $\alpha = \sup A$ exists. Note that $[0,1]$ is compact and $\overline{S}$ is Hausdorff since it is a subspace of the Hausdorff space $\mathbb{R}^2_{\text{std}}$ and Hausdorffness is hereditary. Therefore f is a function from a compact space to a Hausdorff space, and so is a closed map by Theorem 7.24. Therefore, $f([\alpha,1])$ is closed in $\overline{S}$. We claim that $(x, \sin(1/x)) \in f([\alpha,1])$ for all $x \in (0,s)$. Note that $\pi_X \circ f : [0,1] \to \mathbb{R}_{\text{std}}$ is a continuous map, so by the Intermediate Value Theorem, $0 = \pi_X(f(0)) < y < \pi_X(f(s)) = s$ implies that there exists an $x \in (0,s)$ such that $\pi_X(f(x)) = y$, which then means that $f(x) = (y, \sin(1/y))$. So indeed, $f[\alpha,1]$ contains the set $S' = \{(x, \sin(1/x)) \mid x \in (0,s)\}$. But this set has the same closure as S, and since $f([\alpha,1])$ is a closed set containing S', we have that $\overline{S'} \subset f([\alpha,1])$. In particular, we have that $(0,0),(0,1) \in f([\alpha,1])$ since both are limit points of S'. This means there exist $x_1, x_2 \in [\alpha,1]$ such that $f(x_1) = (0,0)$ and $f(x_2) = (0,1)$. Since we have that $\pi_X(f(x_1)) = \pi_X(f(x_2)) = 0$, we have that $x_1, x_2 \in A \cap [\alpha,1]$. Being in A means $x_1, x_2 \leq \sup A = \alpha$, so we have that $\alpha \leq x_1, x_2 \leq \alpha$, meaning $x_1 = x_2 = \alpha$. But this means $(0,0) = f(\alpha) = (0,1)$, a contradiction. Therefore there is no path connecting $(0,0)$ to $(s, \sin(1/s))$, and so $\overline{S}$ is not path connected.

Theorem 8.38. The product of path connected spaces is path connected.

Proof. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of path connected topological spaces, and let p and q be points in the product space $\prod_{\alpha\in\lambda} X_\alpha$. Then $p = (x_\alpha)_{\alpha\in\lambda}$ and $q = (y_\alpha)_{\alpha\in\lambda}$ for $x_\alpha, y_\alpha \in X_\alpha$. Because X_α is path connected, for each $\alpha \in \lambda$ there exists a path $f_\alpha : [0,1] \to X_\alpha$ such that $f_\alpha(0) = x_\alpha$ and $f_\alpha(1) = y_\alpha$. We now define $f : [0,1] \to \prod_{\alpha\in\lambda} X_\alpha$ by $f(z) = (f_\alpha(z))_{\alpha\in\lambda}$ for $z \in [0,1]$. Then we have that $f(0) = (f_\alpha(0))_{\alpha\in\lambda} = (x_\alpha)_{\alpha\in\lambda} = p$, and similarly, $f(1) = q$. To check continuity, we note that by Theorem 7.40, f mapping into a product space means it suffices to check that $\pi_\alpha \circ f$ is continuous for all $\alpha \in \lambda$. Therefore let $\beta \in \lambda$ be arbitrary, and note that $(\pi_\beta \circ f)(z) = \pi_\beta((f_\alpha(z))_{\alpha\in\lambda}) = f_\beta(z)$. This means that $\pi_\beta \circ f = f_\beta$, which is continuous by construction. Therefore f is a path from p to q in the product space $\prod_{\alpha\in\lambda} X_\alpha$, and since p and q were arbitrary points, the product space is path connected. □

Exercise 8.39. (1) The same argument used in Exercise 8.37 shows that no point in $Y = \{(0,y) \mid y \in [-1,1]\}$ has a path connecting it to S. We claim that there are two path components of $\overline{S} = S \cup Y$, namely S and Y. To see that S is path connected, note that it is homeomorphic to $(0,1)$ as a subspace of $\mathbb{R}_{\text{std}}$ (the projection function onto the x axis is a homeomorphism). To see that Y is path connected, let $(0,p)$ and $(0,q)$ be points in Y. Then $f : [0,1] \to Y$ given by $f(x) = (0, (1-t)p + tq)$ is a path from $(0,p)$ to $(0,q)$.

(2) The path given in Exercise 8.36 to show that the topologist's comb is path connected also works to show that its closure is path connected if we allow (a,b) and (c,d) to be points in $\overline{C}$ (that is, if we allow $a = 0$ and $c = 0$).

Exercise 8.40. We claim that indeed every nonempty open and connected subset of $\mathbb{R}^n$ is path connected. Let p and q be points in U, and note that for every $x \in U$, there exists a basic open $B(x, \varepsilon_x)$ such that $x \subset B(x, \varepsilon_x) \subset U$. Therefore the collection $\{B(x, \varepsilon_x)\}_{x \in U}$ is an open cover for U, and so by (6) of Theorem 8.1, there exists a finite subcollection $\{B(x_i, \varepsilon_i)\}_{i=1}^n$ such that $p \in B(x_1, \varepsilon_1)$, $q \in B(x_n, \varepsilon_n)$, and for all $i < n$, we have that there exists some $r_i \in B(x_i, \varepsilon_i) \cap B(x_{i+1}, \varepsilon_{i+1}) \neq \emptyset$. Define the function $f : [0, 1] \to U$ by

$$f(t) = \begin{cases} p(1 - nt) + r_1(nt) & t \in [0, 1/n] \\ r_1(2 - nt) + r_2(nt - 1) & t \in [1/n, 2/n] \\ r_2(3 - nt) + r_3(nt - 2) & t \in [2/n, 3/n] \\ \vdots & \vdots \\ r_{n-1}(n - nt) + q(nt - (n - 1)) & t \in [(n - 1)/n, 1] \end{cases}$$

so that $f([(i-1)/n, i/n]) \subset B(x_i, \varepsilon_i) \subset U$. Then we have that f is continuous with $f(0) = p$, $f(1) = q$, and $f(t) \in U$, so f is a path from p to q.

9 Metric Spaces: Getting Some Distance

9.1 Metric Spaces

Exercise 9.1. Consider $d((0, \ldots, 0), (1, \ldots, 1))$. With the Euclidean metric, we have that

$$d\left((0, \ldots, 0), (1, \ldots, 1)\right) = \sqrt{(0 - 1)^2 + \cdots + (0 - 1)^2} = \sqrt{n};$$

with the box metric, we have that

$$d\left((0, \ldots, 0), (1, \ldots, 1)\right) = \max\{|0 - 1|, \ldots, |0, 1|\} = 1;$$

and with the taxi-cab metric, we have that

$$d\left((0, \ldots, 0), (1, \ldots, 1)\right) = |0 - 1| + \cdots + |0 - 1| = n.$$

Therefore all of these metrics on $\mathbb{R}^n$ are different for $n \geq 2$ since we have that $1 < \sqrt{n} < n$. These metrics are all positive definite and symmetric, so we check that they all satisfy the triangle equality as follows:

(1) For the Euclidean metric, recall that from calculus that

$$d(\mathbf{a},\mathbf{c}) = \|\mathbf{a}-\mathbf{c}\| = \|\mathbf{a}-\mathbf{b}+\mathbf{b}-\mathbf{c}\| \leq \|\mathbf{a}-\mathbf{b}\| + \|\mathbf{b}-\mathbf{c}\| = d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c}).$$

(2) For the box metric, we have that $d(\mathbf{a},\mathbf{c}) = |a_i - c_i|$ for some $i = 1,\ldots,n$, and then $|a_i - c_i| = |a_i - b_i + b_i - c_i| \leq |a_i - b_i| + |b_i - c_i|$. Then because $|a_i - b_i| \in \{|a_j - b_j| \mid j = 1,\ldots,n\}$, we have that $|a_i - b_i| \leq \max\{|a_j - b_j| \mid j = 1,\ldots n\} = d(\mathbf{a},\mathbf{b})$. Similarly, $|b_i - c_i| \leq d(\mathbf{b},\mathbf{c})$, so we have that $d(\mathbf{a},\mathbf{c}) \leq |a_i - b_i| + |b_i - c_i| \leq d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c})$.

(3) For the taxi-cab metric, each $i = 1,\ldots,n$ satisfies $|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i|$, so also

$$d(\mathbf{a},\mathbf{c}) = \sum_{i=1}^{n} |a_i - c_i| \leq \sum_{i=1}^{n} |a_i - b_i| + \sum_{i=1}^{n} |b_i - c_i| = d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c}).$$

Exercise 9.2. Since $|f(x) - g(x)| \geq 0$ for all $x \in X$, we have that $\sup_{x\in X} |f(x) - g(x)| \geq 0$ as well. If $\sup_{x\in X} |f(x) - g(x)| = 0$, then we have that $|f(x) - g(x)| \leq 0$ for all $x \in X$, meaning $|f(x) - g(x)| = 0$. Since this is for all $x \in X$, we have $f = g$, and if $f = g$, then also $|f(x) - g(x)| = 0$ for all $x \in X$ and $d(f,g) = 0$. Also since $|f(x) - g(x)| = |g(x) - f(x)|$, we have that $d(f,g) = d(g,f)$. Lastly, we note that

$$D_1 = \{|f(x) - h(x)| \mid x \in X\} = \{|f(x) - g(x) + g(x) - h(x)| \mid x \in X\},$$

and therefore for every point $a \in D_1$, there exists a point

$$b \in D_2 = \{|f(x) - g(x)| + |g(x) - h(x)| \mid x \in X\}$$

such that $a \leq b$. Therefore

$$d(f,h) = \sup D_1 \leq \sup D_2 = \sup_{x\in X} |f(x) - g(x)| + \sup_{x\in X} |g(x) - h(x)| = d(f,g) + d(g,h),$$

so indeed this is a metric on the space of continuous functions from X to $\mathbb{R}$.

Theorem 9.3. Let d be a metric on a set X. Then the collection of all open balls

$$\mathcal{B} = \{B(p,\varepsilon) = \{y \in X \mid d(p,y) < \varepsilon\} \text{ for every } p \in X \text{ and every } \varepsilon > 0\}$$

forms a basis for a topology on X.

Proof. We begin by noting that for all $p \in X$, $p \in B(p,1)$, so (1) of Theorem 3.3 is satisfied. Now let $U, V \in \mathcal{B}$ be such that there exists a point $p \in U \cap V$. Then $U = B(p_1, \varepsilon_1)$ and

$V = B(p_2, \varepsilon_2)$ for some $p_1, p_2 \in X$ and $\varepsilon_1, \varepsilon_2 > 0$. Then define $W = B(p, \varepsilon)$ where

$$\varepsilon = \min\{\varepsilon_1 - d(p, p_1), \varepsilon_2 - d(p, p_2)\}.$$

Then $p \in W$ and $W \in \mathcal{B}$, so let $q \in W$ be arbitrary. Then

$$d(q, p_1) \leq d(q, p) + d(p, p_1) < \varepsilon + d(p, p_1) \leq \varepsilon_1 - d(p, p_1) + d(p, p_1) = \varepsilon_1,$$

so $q \in B(p_1, \varepsilon_1) = U$, and similarly, $q \in V$. Therefore $W \subset U \cap V$, so (2) of Theorem 3.3 is satisfied and therefore $\mathcal{B}$ is the basis for a topology on X. $\square$

Exercise 9.4. The Euclidean metric generates the same topology as the product topology on $\mathbb{R}^n$ because the product topology on $\mathbb{R}^n$ is the standard topology on $\mathbb{R}^n$, a basis for which consist of open balls with the Euclidean metric. For the box metric, let U be open in $\mathbb{R}^n_{\text{std}}$. Then there exists a basic open $B_E(p, \varepsilon) \subset U$ with the Euclidean metric. Set $\varepsilon' = \varepsilon/\sqrt{n}$. Then $p \in B_B(p, \varepsilon') \subset B_E(p, \varepsilon) \subset U$ where $B_B(p, \varepsilon')$ has the box metric. Therefore U is open in $\mathbb{R}^n$ with the box metric. Now let V be open in the $\mathbb{R}^n$ with the box metric. Then there exists a basic open $B_B(p, \varepsilon)$, and we have that if $\varepsilon' = \varepsilon$, then $p \in B_E(p, \varepsilon') \subset B_B(p, \varepsilon) \subset V$ where $B_E(p, \varepsilon')$ has the Euclidean metric. Therefore V is open in $\mathbb{R}^n_{\text{std}}$, so we have that the box metric also generates the standard / product topology on $\mathbb{R}^n$. Lastly, if U is open in $\mathbb{R}^n_{\text{std}}$ containing p, then there exists a basic open $B_E(p, \varepsilon) \subset U$, and we have that $p \in B_T(p, \varepsilon') \subset B_E(p, \varepsilon) \subset U$ for $\varepsilon' = \varepsilon$. If V is open in the topology generated by the taxi-cab metric containing p, then there exists a basic open $B_T(p, \varepsilon) \subset V$, and for $\varepsilon' = \varepsilon/\sqrt{(n)}$ we have that $p \in B_E(p, \varepsilon') \subset B_T(p, \varepsilon) \subset V$, so the taxi-cab metric also generates the product topology on $\mathbb{R}^n$.

Exercise 9.5. The discrete metric generates the discrete topology on $\mathbb{R}^n$, which is not the product topology on $\mathbb{R}^n$.

Theorem 9.6. Let (X, d) be a metric space and define $\bar{d}(x, y) = d(x, y)/(1 + d(x, y))$. Since $d(x, y) \geq 0$ for all $x, y \in X$, we have that $0 \leq \bar{d}(x, y) < 1$ for all $x, y \in X$. Then to check that this is a metric we need only check it satisfies the triangle inequality. Note that $d(x, y) + d(y, z) - d(x, z) \geq 0$. Then we have that

$$\bar{d}(x, y) + \bar{d}(y, z) - \bar{d}(x, z) = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} - \frac{d(x, z)}{1 + d(x, z)}.$$

Since the common denominator will be positive, the whole expression is nonnegative if and only if the numerator is nonnegative. The numerator then is

$$\begin{gathered}
d(x,y)(1+d(y,z))(1+d(x,z)) \\
+d(y,z)(1+d(x,y))(1+d(x,z)) \\
-d(x,z)(1+d(x,y))(1+d(y,z))
\end{gathered}$$

which can be rewritten as

$$\begin{aligned}
&d(x,y)(1+d(y,z)+d(x,z)+d(y,z)d(x,z)) \\
&+d(y,z)(1+d(x,y)+d(x,z)+d(x,y)d(x,z)) \\
&-d(x,z)(1+d(x,y)+d(y,z)+d(x,y)d(y,z)) \\
&= d(x,y)(1+d(y,z))+d(y,z)(1+d(x,y)+d(x,y)d(x,z))-d(x,z) \\
&= d(x,y)+d(y,z)-d(x,z)+2d(x,y)d(y,z)+d(x,y)d(y,z)d(x,z) \\
&\geq 2d(x,y)d(y,z)+d(x,y)d(y,z)d(x,z)\geq 0
\end{aligned}$$

where we have use the fact that the metric d is positive definite. Therefore $\bar{d}$ does satisfy the triangle inequality and so is a metric because it is also positive definite and symmetric given that d is. To show it generates the same topology as d, we note that the bases they generate are the same. For any basic open $B_d(p,\varepsilon)$ in (X,d), $B_d(p,\varepsilon)=B_{\bar{d}}(p,\bar{\varepsilon})$ for $\bar{\varepsilon}=\varepsilon/(1+\varepsilon)$ and is therefore also a basic open set in $(X,\bar{d})$. Similarly, if $B_{\bar{d}}(p,\bar{\varepsilon})$ is a basic open set in $(X,\bar{d})$, then $B_{\bar{d}}(p,\bar{\varepsilon})=B_d(p,\varepsilon)$ for $\varepsilon=\bar{\varepsilon}/(1-\bar{\varepsilon})$ and so is open in (X,d) as well. Since the bases they generate are the same, the two metrics generate the same topology on X.

Theorem 9.7. If X is a metric space with $Y\subset X$, then the subspace Y is a metric space.

Proof. We will show that the relative topology Y inherits from (X,d) is the topology generated by the metric d. Because X is a metric space, it has a basis of open balls $B(p,\varepsilon)$, and so by Theorem 3.30, the collection of open balls intersected with Y is a basis for the relative topology on Y. That is, the relative topology on Y has a basis

$$\mathcal{B}_Y=\{B(p,\varepsilon)\cap Y=\{x\in X\mid d(p,x)<\varepsilon\}\cap Y \text{ for every } p\in X \text{ and } \varepsilon>0\}$$

which is the same as

$$\mathcal{B}_Y = \{B(p,\varepsilon) = \{y \in Y \mid d(p,y) < \varepsilon\} \text{ for every } p \in Y \text{ and } \varepsilon > 0\},$$

the basis generated by the metric d. Therefore $(Y, \mathcal{T}_Y) = (Y, d)$, so Y is a metric space. □

Theorem 9.8. Every metric space is Hausdorff, regular, and normal.

Proof. Let (X, d) be a metric space. We will first show that it is Hausdorff. Let $p, q \in X$ be distinct points and set $\varepsilon = d(p,q)/2$. Then $U = B(p,\varepsilon)$ and $V = B(q,\varepsilon)$ are open sets such that $p \in U$ and $q \in V$, and if we suppose that there exists an $s \in U \cap V$, we then have that

$$2\varepsilon = d(p,q) \leq d(p,s) + d(s,q) < \varepsilon + \varepsilon = 2\varepsilon,$$

a contradiction. Therefore U and V are also disjoint, meaning (X,d) is Hausdorff. Now we also have that (X,d) is T_1, so if it is also normal, then it is T_4 which implies regularity by Theorem 4.7. Let A and B be disjoint closed sets in X. If one of A or B is empty, take $U = X$ and $V = \emptyset$. Then U and V are disjoint open sets where U contains the nonempty one (if it exists) and V contains (one of) the empty one(s). Now assume A and B are both nonempty. For $a \in A$ and $b \in B$ define $\varepsilon_a = \inf\{d(a,b) \mid b \in B\}/2$ and $\varepsilon_b = \inf\{d(a,b) \mid a \in A\}/2$. These infima exist because A and B are nonempty and each set contains values of $d(x,y)$ for points $x, y \in X$, meaning the sets are both bounded below by 0. Now define

$$U = \bigcup_{a \in A} B(a, \varepsilon_a) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \varepsilon_b).$$

Then we have that U and V are the unions of basic open sets and therefore are open, and we have that $a \in A$ implies $a \in B(a, \varepsilon_a) \subset U$, so $A \subset U$ and similarly, $B \subset V$. It remains to show that U and V are disjoint. Suppose for contradiction that there exists an $s \in U \cap V$. Then $s \in U$ means there exists an $\alpha \in A$ such that $s \in B(\alpha, \varepsilon_\alpha)$, which means $d(\alpha, s) < \varepsilon_\alpha$, and similarly, $s \in V$ means there exists a $\beta \in B$ such that $s \in B(\beta, \varepsilon_\beta)$, so $d(s,\beta) < \varepsilon_\beta$. But then we also have that $\alpha \in A$, meaning $d(\alpha,\beta) \in \{d(\alpha,b) \mid b \in B\}$, which implies that $d(\alpha,\beta) \geq \inf\{d(\alpha,b) \mid b \in B\} = 2\varepsilon_\alpha$. Similarly, $d(\alpha,\beta) \geq 2\varepsilon_\beta$. Therefore $2d(\alpha,\beta) \geq 2\alpha + 2\beta$, and so we have that $d(\alpha,\beta) \geq \varepsilon_\alpha + \varepsilon_\beta$. This means that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha,\beta) \leq d(\alpha,s) + d(s,\beta) < \varepsilon_\alpha + \varepsilon_\beta,$$

a contradiction meaning that $U \cap V = \emptyset$. Therefore for our arbitrary closed sets $A, B \subset X$, we have found disjoint open sets U and V such that $A \subset U$ and $B \subset V$, meaning (X, d) is normal, and therefore also regular because (X, d) is T_1. $\square$

Theorem 9.10. Every metric space is 1^{st} countable.

Proof. Let (X, d) be a metric space and let $p \in X$ be arbitrary. We claim that the countable collection $\{B(p, 1/n)\}_{n \in \mathbb{N}}$ is a neighborhood basis for p. We have that $p \in B(p, 1/n)$ for all $n \in \mathbb{N}$, so we only need to check that every open set containing p contains some $B(p, 1/n)$. Let U be an open set containing p. Then because the set of open balls of radius $\varepsilon > 0$ is a basis for (X, d), there exists an $\varepsilon > 0$ such that $p \in B(p, \varepsilon) \subset U$. Then because the naturals are unbounded above, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$, and so we have that

$$p \in B(p, 1/n) \subset B(p, \varepsilon) \subset U.$$

Therefore (X, d) is 1^{st} countable. $\square$

11 Classification of 2-Manifolds: Organizing Surfaces

11.1 Examples of 2-Manifolds

Exercise 11.1. We will think of $\mathbb{T}^2$ as the identification space of $[0, 1] \times [0, 1]$ given by

$$\begin{aligned}\mathbb{T}^2 = \{\{(x, y) \mid x, y \in (0, 1)\}\} &\cup \{\{(x, 0), (x, 1)\} \mid x \in (0, 1)\} \\ \cup \{\{(0, y), (1, y)\} \mid y \in (0, 1)\} &\cup \{\{(0, 0), (1, 0), (0, 1), (1, 1)\}\}\end{aligned}$$

and we will think of $\mathbb{S}^1$ as the identification space of $[0, 1]$ given by

$$\mathbb{S}^1 = \{\{x\} \mid x \in (0, 1)\} \cup \{\{0, 1\}\}.$$

Let $f : [0, 1] \times [0, 1] \to \mathbb{T}^2$ and $g : [0, 1] \to \mathbb{S}^1$ be the respective identification maps. Then the map $h : \mathbb{T}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $h(T) = (A, B)$ is a homeomorphism where A and B are sets in $\mathbb{S}^1$ such $x \in A$ and $y \in B$ for some $(x, y) \in T$.

Exercise 11.2. Consider the n-holed torus with holes arranged in a circle lying flat on the plane with center at the origin, viewed from above as in figure 11.6. Stretch the edges of the torus between each hole out so that the boundary of the projection onto the plane it lies on looks like a disk. This transformation shows a homeomorphism between the starting torus and a pancake shaped 2-manifold with n holes in it, still arranged in a circle. Suppose the plane the torus lies on is the xy plane, and squish the edges of the torus inward toward the x axis so that the n holes are gradually forced into a region roughly along the x axis. To avoid holes colliding for even n, add a slight angle to the squishing. This gives a homeomorphism to a more rectangular blob with n holes arranged in a line. Finally, squish in the edges of the torus between each hole, undoing the first step of the homeomorphism and making the boundary of the projection of the torus onto the plane it lies on look more like a sine function.

Exercise 11.3. (1) We note that for each pair of antipodal points p and $-p$ on $\mathbb{S}^2$, there is a unique line passing through p, $-p$ and the origin in $\mathbb{R}^3$. We define the homeomorphism $f : \mathbb{S}^2/\langle x \sim -x\rangle \to \mathbb{R}\mathrm{P}^2$ by defining $f(p)$ to be this line passing through p, $-p$, and the origin. Because this line is unique, f is injective, and because each line in $\mathbb{R}\mathrm{P}^2$ passes through two antipodal points of $\mathbb{S}^2$, f is surjective. To show that f is indeed a homeomorphism then, we check that f is continuous from a compact space to a Hausdorff space. To see that f is continuous, we let g denote the quotient map that gives $\mathbb{S}^2/\langle x \sim -x\rangle$ its topology. Then let U be a double open cone in $\mathbb{R}\mathrm{P}^2$. Then $(g \circ f)^{-1}(U)$ is the intersection of this double open cone with $\mathbb{S}^2$. Consider the double open cone (thought of now as a collection of points in $\mathbb{R}^3$) minus a small enough closed ball centered at the origin such that its intersection with $\mathbb{S}^2$ is unchanged. Then this modified cone is open in $\mathbb{R}^3$, and viewing $\mathbb{S}^2$ as a subspace of $\mathbb{R}^3$, we see that this intersection is by definition open in $\mathbb{S}^2$. Therefore $g \circ f$ is continuous, and so Theorem 7.53 implies that f is continuous as well. Note that the domain of f is an identification of a compact space and so is compact, and the codomain is Hausdorff. To see this last claim, let l_1 and l_2 be distinct lines and let θ be the angle between them. Then the double open cones C_1 and C_2 are disjoint open sets containing l_1 and l_2 where C_1 is the set of all lines that make an angle less than $\theta/2$ with l_1 and C_2 is the set of all lines that make an angle less than $\theta/2$ with l_2. Therefore $\mathbb{R}\mathrm{P}^2$ is Hausdorff and f is a homeomorphism.

(2) We think of the identified bigon as the identification space

$$B^* = \{\{(x, y)\} \mid x^2 + y^2 < 1\} \cup \{\{(x, y), (-x, -y)\} \mid x^2 + y^2 = 1\}$$

and of the identified 2-sphere as the identification space

$$\mathbb{S}^2/\langle x \sim -x\rangle = \{\{(x, y, z), (-x, -y, -z)\} \mid x^2 + y^2 + z^2 = 1\}.$$

Then the map $f : B^* \to \mathbb{S}^2/\langle x \sim -x\rangle$ given by $f(B) = S$ is a homeomorphism where S is the set in $\mathbb{S}^2/\langle x \sim -x\rangle$ containing the point $(x, y, \sqrt{1-x^2-y^2})$ for a point $(x, y) \in B$.

(3) The Klein bottle is homeomorphic to the identification space of $[0,1]\times[0,1]$ given by

$$\begin{aligned} X^* = &\{\{(x,y)\} \mid x, y \in (0,1)\} \cup \{\{(x,0),(x,1)\} \mid x \in (0,1)\} \\ &\cup \{\{(0,y),(1,1-y)\} \mid y \in (0,1)\} \cup \{\{(0,0),(0,1),(1,0),(1,1)\}\}, \end{aligned}$$

that is, the Klein bottle can be thought of as the unit square with the left and top and bottom edges glued in the same direction and the left and right edges glued in the opposite direction.

11.6 Polygonal Presentations of 2-Manifolds

Exercise 11.24. (a) This is $\mathbb{S}^2$ since identifying the boundary in this way can be thought of as taking the lower hemisphere and bringing the edges up from the equator to form the upper hemisphere and then gluing it closed.

(b) This is a Klein bottle—cutting along the diagonal from upper left to lower right and identifying the new edges in the same direction creates two triangles that can be moved in order to glue the two edges labeled 'b'. This results in a parallelogram with edges identified as in Exercise 11.3 (3).

(c) This is $\mathbb{RP}^2$—cutting along the diagonal from upper left to lower right and identifying the new edges in the same direction creates two triangles that can be moved in order to glue the two edges labeled 'b'. This results in a triangle subdivided along what used to be the edge 'b', where the left edge is a straight line, the upper half of which is identified with the lower half such that the top point is glued to the bottom point. This gluing can be performed, which results in the bigon identified as in Exercise 11.3 (2).

Theorem 11.25. Let P^* be a polygonal presentation. Then P^* is a 2-manifold.

Proof. Let P^* be a polygonal presentation. Then there exists a polygon $P \subset \mathbb{R}^2_{\text{std}}$ and an identification map $f : P \to P^*$ that gives P^* its topology. Without loss of generality, we will assume that P is a regular polygon with $2n$ sides, all of which have length equal to 1 (since any other polygon with $2n$ sides is homeomorphic). Then to create the polygonal presentation P^* as an identification space of P, we note that every edge $a \subset \partial P$ is identified with another edge $a' \subset \partial P$. Since a and a' are both line segments of length 1 in the plane, to identify them, we may pick an endpoint of each to be glued together, call them $a_0 \in a$ and $a'_0 \in a'$, and construct our gluing function as the homeomorphism $g_a : a \to a'$ where $g_a(x)$

is defined as follows: for $x = a_0$, we define $g(x) = a_0'$, and for $x \neq a_0$, we let $b = d(a_0, x)$ and define $g(x)$ to be the unique point in a' satisfying $d(a_0', g(x)) = b$. This point is unique because it can be thought of as the intersection of the line segment a' and a circle of radius b centered at a_0'. Recalling that all edges of P are of length 1, we will also denote by a_1 the point at which $d(a_0, a_1) = 1$ (the second endpoint of edge a). This means that for points p and q on an edge a that get glued to points p' and q' on an edge a', we have that $d(p, q) = d(p', q') \in [0, 1]$. Now we note that to create the polygonal presentation, denote the set of corner points by C, and for every $p \in \partial P - C$, there is exactly one other point to which it is glued ($g_a(p)$ where a is the edge containing p) since all edges are glued pairwise. Therefore we can define $E = \{(a_i, a_i') \mid i = 1, \dots, n\}$ to be the set of edge pairs, and we can describe the polygonal presentation as the identification space

$$P^* = \{\{x\} \mid x \in P^\circ\} \cup \bigcup_{i=1}^{n} \{\{x, g_{a_i}(x)\} \mid x \in a_i - C\} \cup \bigcup_{x \in C} \{\{\tilde{g}_1(x), \dots, \tilde{g}_l(x) \mid \tilde{g}_j \in CG_x\}\}$$

where CG_x is the set of all compositions of gluing maps and their inverses defined at x. There are infinitely many compositions of gluing maps, but corners must be mapped to corners, so there are only finitely many elements $\tilde{g}_1, \dots, \tilde{g}_l$ because there are only finitely many corners. This means the last kind of set in the identification space is the set of all corners a given corner x is glued to in the presentation. We note that the identification map $f : P \to P^*$ is continuous by definition, and we now claim that P^* is Hausdorff. For every set in $T \in P^*$ we define $B_\varepsilon(T) \subset P$ to be $B_\varepsilon(T) = \bigcup_{x \in T} B(x, \varepsilon)$ for $\varepsilon > 0$ small enough such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ for all $x \neq y$, and additionally that $B(x, \varepsilon) \cap a = \emptyset$ for all edges a such that $x \notin a$ (that is, we allow ε to be arbitrarily small but not so large that one of these conditions would be violated). Finding such an ε is possible because the set T is finite, as is the set of edges in P. We claim that $f(B_\varepsilon(T))$ is an open set in P^* containing T. Recall that a set U is open in P^* if and only if $f^{-1}(U)$ is open in P. Therefore we will show that $f^{-1}(f(B_\varepsilon(T)))$ is open in P. To this end, let $q \in f^{-1}(f(B_\varepsilon(T)))$. Then there exists a $p \in B_\varepsilon(T)$ such that $f(p) = f(q)$. If $p = q$, then $q \in B_\varepsilon(T)$. If $p \neq q$, then $f(p) = f(q)$ implies that there is an edge pair (a, a') such that $p \in a$, $q \in a'$, and $g_a(p) = q$. We claim that there exists an $x_0 \in T \cap a$ such that $p \in B(x_0, \varepsilon)$. If not, then we would have $p \in B_\varepsilon(T)$, so $p \in B(x, \varepsilon)$ for some $x \notin a$, but by the choice of ε, $B(x, \varepsilon) \cap a = \emptyset$ and therefore $p \in \emptyset$, a contradiction. Therefore such an x_0 exists, and we have that $d(p, x_0) < \varepsilon$. Now because $x_0 \in a$, there exists an $x_0' \in a'$ such that $g_a(x_0) = x_0'$. Then by the way we have defined the gluing map $g_a : a \to a'$, we have that $d(p, x_0) < \varepsilon$ implies that

$$d(q, x_0') = d(g_a(p), g_a(x_0)) < \varepsilon,$$

which in turn implies that $q \in B(x_0', \varepsilon)$. Then $x_0 \in T$ means $x_0' = g_a(x_0) \in T$ because either $T = \{x_0, g_a(x_0)\}$ or $T \subset C$ and $g_a(x_0) \in T$ because g_a is a composition of gluing maps defined on x_0. Therefore $q \in B(x_0', \varepsilon) \subset B_\varepsilon(T)$, so we have shown that $f^{-1}(f(B_\varepsilon(T))) \subset B_\varepsilon(T)$. For all functions we have $A \subset f^{-1}(f(A))$, so we now have that $f^{-1}(f(B_\varepsilon(T))) = B_\varepsilon(T)$. Since $B_\varepsilon(T)$ is the union of open balls, it is open in P, which by definition means $f(B_\varepsilon(T))$ is open in P^*. Since for all $x \in T$, $x \in B(x, \varepsilon)$, we have that $x \in B_\varepsilon(T)$ and so $\{x\} \subset B_\varepsilon(T)$ implies that $T = f(\{x\}) \subset f(B_\varepsilon(T))$. Therefore $f(B_\varepsilon(T))$ is an open set containing T, as claimed. Recall that our goal was to show P^* is Hausdorff. Let $S, T \in P^*$ be distinct. Then for all $\varepsilon > 0$ such that $B_\varepsilon(S)$ and $B_\varepsilon(T)$ are defined, $f(B_\varepsilon(S))$ and $f(B_\varepsilon(T))$ are open sets in P^* containing S and T respectively. Because there are only finitely many points in S and T and finitely many edges in P, we may choose an ε small enough that both $B_\varepsilon(S)$ and $B_\varepsilon(T)$ are defined, and such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ for all $x \in S$ and $y \in T$ (because S and T being distinct implies that $S \cap T = \emptyset$). Then since $B_\varepsilon(S) = \bigcup_{x \in S} B(x, \varepsilon)$ and $B_\varepsilon(T) = \bigcup_{y \in T} B(y, \varepsilon)$, we have that $B_\varepsilon(S) \cap B_\varepsilon(T) = \emptyset$. Therefore

$$f(B_\varepsilon(S)) \cap f(B_\varepsilon(T)) \subset f(B_\varepsilon(S) \cap B_\varepsilon(T)) = \emptyset,$$

so $f(B_\varepsilon(S)$ and $f(B_\varepsilon(T))$ are disjoint open sets containing S and T respectively. Since S and T were arbitrary distinct sets in P^*, we have that P^* is Hausdorff.

Since the polygon P is a subspace of the plane $\mathbb{R}^2_{\text{std}}$ and $\mathbb{R}^2_{\text{std}}$ is 2^{nd} countable, Theorem 5.12 implies that P is 2^{nd} countable as well, and so Theorem 5.9 implies that P is separable as well. Since the polygon P is closed and bounded in $\mathbb{R}^2_{\text{std}}$, Theorem 6.20 implies that P is compact. Since the identification map $f : P \to P^*$ is continuous, Corollary 7.19 implies that P^* is separable and Theorem 7.15 implies that P^* is compact. Since P is compact and P^* is Hausdorff, Theorem 7.24 implies that f is closed, and Theorem 7.25 implies that P^* is 2^{nd} countable. Since P^* is a compact, 2^{nd} countable, and Hausdorff, Theorem 9.49 implies that P^* is metrizable.

Therefore to show that the polygonal presentation P^* is a 2-manifold, it only remains to show that every point $T \in P^*$ has a neighborhood homeomorphic to an open ball in $\mathbb{R}^2$. We know that for all $T \in P^*$, the set $f(B_\varepsilon(T))$ is a neighborhood of T for all $\varepsilon > 0$ for which it is defined. We now claim that it is also homeomorphic to an open ball in the plane. To see this, we work in cases. If T is a singleton, then $B_\varepsilon(T) = B(x, \varepsilon) \subset P^\circ$, and $f|_{P^\circ}$ is a homeomorphism, so $f(B_\varepsilon(T))$ is homeomorphic to an open ball. If $T = \{x, g_a(x)\}$ for some edge a with $x \in a$ and $g_a(x) \in a'$, then $B_\varepsilon(T) = B(x, \varepsilon) \cup B(g_a(x), \varepsilon)$ is the union of two half disks in the plane (the edges $a \cap B(x, \varepsilon)$ and $a' \cap B(g_a(x), \varepsilon)$ are the diameters of these half disks). The edges a and a' are identified by the gluing function g_a, and we have that for all $y \in a \cap B(x, \varepsilon)$, $d(x, y) < \varepsilon$, so also $d(g_a(x), g_a(y)) < \varepsilon$, which means $g_a(y) \in B(g_a(x), \varepsilon)$. Therefore for all y on the diameter of the first half disk, the point $g_a(y)$ with which it is identified is on the diameter of the second half disk. Then since $B_\varepsilon(T)$ is the union of these two half disks, $f(B_\varepsilon(T))$ is the result of gluing the two half disks together, meaning indeed $f(B_\varepsilon(T))$ is homeomorphic to an open ball. Now if $T = \{\tilde{g}_1(x), \ldots, \tilde{g}_l(x) \mid \tilde{g}_j \in CG_x\}$ is a subset of C containing l corners, then $B_\varepsilon(T)$ is homeomorphic to the union of l half disks in the plane, each of which is centered at $\tilde{g}_j(x)$ for some corner $\tilde{g}_j(x)$. To determine $f(B_\varepsilon(T))$, we note that each corner $\tilde{g}_j(x)$ is an endpoint of two (not necessarily distinct) edges a_m and a_k, and for the edge a_m, the corner could be either the initial endpoint a_{m0} or the final endpoint a_{m1} to use our notation from earlier. We now relabel the corners so that $\tilde{g}_j(x) = C_{m\alpha,k\beta}$ where $\alpha, \beta \in \{0, 1\}$ and $\tilde{g}_j(x)$ is the corner that is simultaneously the endpoint α (either 0 or 1—initial or final) of the edge a_m and the endpoint β of the edge a_k such that a path moving clockwise around the boundary of the polygon passes along edge a_m, then through the corner $C_{i\alpha,j\beta}$, and then passes along edge a_j (so that the corners $C_{i\alpha,j\beta}$ and $C_{j\beta,i\alpha}$ are distinct). Now note that

$$f(B_\varepsilon(T)) = f\left(\bigcup_{C_{i\alpha,j\beta} \in T} B(C_{i\alpha,j\beta}, \varepsilon)\right) = \bigcup_{C_{i\alpha,j\beta} \in T} f(B(C_{i\alpha,j\beta}, \varepsilon))$$

and that each $f(B(C_{i\alpha,j\beta}))$ is homeomorphic to an half disk of radius ε with boundary along the diameter but open along the other edge such that the diameter is divided into two radii coming from edges a_i and a_j such that the center of the half disk is the corner $C_{i\alpha,j\beta}$. We will therefore denote $f(B(C_{i\alpha,j\beta}))$ by $H_{i\alpha,j\beta}$, the half disk around the corner $C_{i\alpha,j\beta}$. We also note that although the corners $C_{i\alpha,j\beta}$ and $C_{j\beta,i\alpha}$ correspond to distinct points on the polygon P (if both exist), their half disk neighborhoods $H_{i\alpha,j\beta}$ and $H_{j\beta,i\alpha}$ are homeomorphic—$H_{h\beta,i\alpha}$ can be picked up and flipped over to get $H_{i\alpha,j\beta}$. To show that $f(B_\varepsilon(T))$ is homeomorphic to an open ball, we split it up into its component parts (the $H_{i\alpha,j\beta}$s) and reglue them:

$$f(B_\varepsilon(T)) = \bigcup_{C_{i\alpha,j\beta} \in T} H_{i\alpha,j\beta}.$$

We note now that $H_{i\alpha,j\beta} \cup H_{i\alpha,k\gamma}$ is homeomorphic to the half disk $H_{j\beta,k\gamma}$. This is because they share the common radius that goes out from the center along edge a_i a distance ε, which means the gluing function g_{a_i} glues each pair of points along these radii together (by the same argument as in the case when we are gluing half disks centered at points on identified edges, each point here on one radius does have a corresponding point it is identified with on the other). To perform the gluing, pick a starting half disk $H_{i^*\alpha,j^*\beta}$. Then we have that $C_{i^*\alpha,j^*\beta} = a_{i^*_\alpha} \in T$, and so the gluing function $g_{a_{i^*}}$ is defined at this corner, meaning the corner it is glued to is of the form $C_{i^*\alpha,k\gamma} \in T$ or $C_{k\gamma,i^*\alpha} \in T$ (to represent $g_{a_{i^*}(a_{i^*_\alpha})} = a_{k_\gamma}$, the endpoint γ of edge a_k). Since in either case the half disk that corresponds to this corner is homeomorphic to $H_{i^*\alpha,k\gamma}$, we have that

$$f(B_\varepsilon(T)) = \bigcup_{C_{i\alpha,j\beta}\in T} H_{i\alpha,j\beta} \cong \left(\bigcup_{C_{i\alpha,j\beta}\in T-\{C_{i^*\alpha,j^*\beta},C_{j^*\beta,k\gamma}\}} H_{i\alpha,j\beta} \right) \cup H_{j^*\beta,k\gamma}$$

since $H_{i^*\alpha,j^*\beta} \cup H_{i^*\alpha,k\gamma} \cong H_{i^*\alpha,k\gamma}$. We continue this process of gluing half disks a pair at a time, noting that because the corner $C_{j'\beta,k\gamma} = a_{k_\gamma}$ (also the corner $a_{j^*_\beta}$) is an element of T, we must have that there is another half disk $H_{k\gamma,l\mu}$ in our union of half disks, since a_{k_γ} is glued to a'_{k_γ} by the gluing function g_{a_k}, we have that either $C_{k\gamma,l\mu}$ or $C_{l\mu,k\gamma}$ is in T because these options correspond to

$$a'_{k_\gamma} = g_{a_k}(a_{k_\gamma}) = g_{a_k}(g_{a_{i^*}}(a_{i^*_\alpha})) = (g_{a_k} \circ g_{a_{i^*}})(a_{i^*_\alpha})$$

which is in T because the composition $g_{a_k} \circ g_{a_{i^*}}$ is defined at our starting corner $a_{i^*_\alpha}$. Therefore

$$H_{i^*\alpha,j^*\beta} \cup H_{i^*\alpha,k\gamma} \cup H_{k\gamma,l\mu} \cong H_{j^*\beta,k\gamma} \cup H_{k\gamma,l\mu} \cong H_{k\gamma,k^*\beta} \cup H_{k\gamma,l\mu} \cong H_{j^*\beta,l\mu}.$$

And we continue on by finding another corner involving the endpoint a'_{l_μ} and its corresponding half disk to glue. We note that this process must eventually stop because there are only finitely many corners. This means we eventually run out of corners not that are not $a'_{j^*_\beta}$, and so we eventually end up in a situation where we have glued half disks as follows:

$$H_{i^*\alpha,j^*\beta} \cup H_{i^*\alpha,k\gamma} \cup H_{k\gamma,l\mu} \cup \cdots \cup H_{s\pi,m\lambda} \cong H_{j^*\beta,m\lambda}$$

where $C_{s\pi,m\lambda} = a_{m_\lambda}$ and we must now glue on the half disk corresponding to the corner a'_{m_λ} but find that $a'_{m_\lambda} = C_{m\lambda,j^*_\beta} = a'_{j^*_\beta}$ or $a'_{m_\lambda} = C_{j^*_\beta,m\lambda} = a'_{j^*_\beta}$. We claim that we have not missed any corners in T in doing this. This is because each element of T is a composition of gluing functions evaluated at $a_{i^*_\alpha}$ and so there is a path to it beginning at $a_{i^*_\alpha} = C_{i^*\alpha,j^*\beta}$ and changing one of the edges a_{i^*} or a_{j^*} to proceed. Because our list of half disks above does exactly this and contains each edge appearing twice (including $a_{j^*_\beta}$ and a_{m_λ}), we have included all corners reachable via a series of gluing functions. Therefore we have that

$$f(B_\varepsilon(T)) = \bigcup_{C_{i\alpha,j\beta}\in T} H_{i\alpha,j\beta} \cong H_{j^*\beta,m\lambda} \cup H_{m\lambda,j^*\beta} \cong H_{m\lambda,j^*\beta} \cup H_{m\lambda,j^*\beta} \cong H_{j^*\beta,j^*\beta}.$$

Our last claim is that the half disk $H_{j^*\beta,j^*\beta}$ is homeomorphic to an open ball. To see this, note that the notation $H_{j^*\beta,j^*\beta}$ is describing a half disk in which only includes the boundary along the diameter, and that this diameter is split into two radii that are glued in the same direction. We know they are glued in the same direction because the radii are the half open line segments beginning at $a_{j^*_\beta}$ and $a'_{j^*_\beta}$ of length ε, and these endpoints are either both the initial endpoint of the identified edge a_{j^*} (for $\beta = 0$) or are both the finial end point of a_{j^*} (when $\beta = 1$). Therefore performing the gluing given by $g_{a_{j^*}}$ results in an open disk of radius ε. Therefore $f(B_\varepsilon(T))$ is a neighborhood of T homeomorphic to an open ball in $\mathbb{R}^2$. Since all $T \in P^*$ have such neighborhoods and P^* is separable and metrizable, we have that P^* is a 2-manifold. □

Theorem 11.26. Let M be a compact, connected, triangulable 2-manifold. Then M is homeomorphic to a polygonal presentation.

Proof. Let M be a compact, connected, triangulable 2-manifold with triangulation $T = \{\sigma_i\}_{i=1}$. Following Theorem 11.7, we have that each edge of a triangle σ_i is shared by exactly two triangles. Therefore M is homeomorphic to the union of all triangles $\bigcup_{i=1}^k \sigma_i$ lying flat in the plane with edges identified pairwise (seen by cutting M along all edges and using the identification to 'reglue' these cuts). Denote the set of edges by $E = \{a_i\}_{i=1}^{3k/2}$ where we are taking an 'edge' here to be a pair of identified edges, so that one edge is associated to a unique pair of triangles. To show that M is homeomorphic to a polygonal presentation, we partially reassemble M out of these triangles. We begin by taking σ_1 lying flat in the plane, and we pick an edge $a_i \in \sigma_1$ to reglue. Regluing this edge means we now have two triangles σ_1 and σ_j where σ_j is the unique second triangle containing the edge a_i. Relabel σ_j to be σ_2. Now we have $k-2$ triangles left to connect. Continue by regluing the other two edges a_j and a_k of σ_1. Since T is a triangulation, we know that each pair of triangles is either disjoint, connected by a vertex, or connected along a single edge. Since σ_1 and σ_2 already share the edge a_i, we have that $a_j, a_k \notin \sigma_2$, which means there is another unique triangle σ_3

(again relabeling if necessary) sharing the edge a_j with σ_1, and we glue them together along this edge. Then because σ_1 and σ_3 already share this edge, the same reasoning means there is another distinct triangle σ_4 sharing edge a_k with σ_1. We now have $k-4$ triangles left to connect. To connect the next new triangle, move on to the unglued edges of σ_2, of which there are two, call them a_s and a_t. Note that now, however, the triangles sharing these edges may already be accounted for. Since σ_2 shares edge a_i with σ_1 already, we know that a_s and a_t are not shared with σ_2 by σ_1, but they may be shared with σ_2 by σ_3 or σ_4. Therefore we have two cases when decided what to do with the edge a_s. In the first case, we have not already connected a triangle containing a_s, and so there is a unique σ_5 intersecting σ_2 along a_s, and so we reglue the a_s joining σ_2 and σ_5 and now have $k-5$ edges left to reglue. In the second case, $a_s \in \sigma_3$ or $a_s \in \sigma_4$, so we do nothing and move on to edge a_t. In this case, the edge a_s will be remain unglued and will be identified by a pair of arrows in the resulting polygonal presentation. This is how we proceed in general: let T' and E' be the sets of all triangles and edges (respectively) appearing in the partial reassembly of M (appearing twice for edges) and suppose that for $\sigma_i \in T'$, all three edges of σ_i are in E' for all $i \leq m$. If $m = k$, we are done. Otherwise, move to triangle σ_{m+1} where there is at least one edge only appearing once (but possibly two edges). Call this edge a_l (or call them a_l and $a_{l'}$ if there are two). There is a unique σ_n in $T - T'$ containing a_l, so we glue σ_m and σ_n back together along this shared edge, and do the same with $\sigma_{n'}$ along $a_{l'}$ for the unique $\sigma_{n'}$ containing $a_{l'}$ if necessary. We have shown that if $\sigma_i \in T'$ for all $i < m$, we can expand T' to include the triangle σ_{m+1}. Since the number of triangles k is finite, we can eventually add all k triangles to the collection T' using this procedure. Since cutting M into these k pieces and identifying the edges along which the cuts were made left us with a collection of identified triangles homeomorphic to M and our procedure to partially reassemble M involved regluing edges that were already identified, the resulting object is still homeomorphic to M. Since each triangle added to T' was connected along an edge, the resulting object is connected, lies flat in the plane, and has some number of edges strictly less than $3k/2$. In other words, the resulting object is a polygon, call it P^*. Let a^* be some unglued edge of P^*. The edge a^* being unglued means it is identified with another unglued edge $a^{*\prime}$, and so we see that P^* is a polygon with all of its unglued edges identified pairwise, that is, P^* is a polygonal presentation homeomorphic to M. Since M was an arbitrary compact, connected, and triangulated 2-manifold and we have found a homeomorphic polygonal presentation, all such 2-manifolds are homeomorphic to polygonal presentations. □

Theorem 11.27. Let P^* be a polygonal presentation. Then P^* is a compact, connected, triangulable 2-manifold.

Proof. Let P^* be a polygonal presentation. Then by Theorem 11.25, P^* is a 2-manifold. Let P be the original polygon in the plane and let $f : P \to P^*$ be the quotient map giving P its topology. Then since P is closed and bounded in $\mathbb{R}^2_{\text{std}}$, it is compact, and it is also connected. Then the continuity of f implies the compactness of P^* by Theorem 7.15 and the connectedness of P^* by Theorem 8.9. To show triangulability, we first note that the original polygon P is homeomorphic to a regular polygon with $2n$ sides (since the edges have to be identified in pairs, there must be an even number of edges). In the case where $n = 1$ and P is a bigon, there are two possibilities. Either the edges are identified in the same direction, in which case the corresponding word is aa^{-1}, or the edges are identified in opposite directions, in which case the corresponding word is aa. In both cases, we can cut along a line segment from the top vertex to the center of the bigon and then 'reglue' by identifying the resulting new edges, which we will label b. Therefore in the first case, the polygonal presentation with word aa^{-1} is homeomorphic to the one with word $baa^{-1}b^{-1}$ and the polygonal presentation with word aa is homeomorphic to the one with word $baab^{-1}$ (where we label edges from the top vertex around clockwise). This means that every polygonal presentation can be represented by a polygon with at least 4 sides. To triangulate a polygonal presentation then, represent it as a regular polygon with $2n$ sides with edges identified in pairs and $n \geq 2$. Draw a line segment from each corner to the center. Now we have a polygon in the plane made up of at least four triangles such that each edge is shared by exactly two triangles. Call this collection of triangles C. This isn't yet a triangulation since each of the triangles shares both an edge and a vertex with the triangle opposite it, whereas in a triangulation, any pair of triangles either is disjoint, meets at a single vertex, or shares a common edge. To fix this, we take the triangulation to be the first barycentric subdivision T of C. Then we still have that each edge is shared by a unique pair of triangles, but triangles that share an identified edge along one of the edges of the polygon P no longer share a vertex in the center of the polygon—this identified edge is now the only edge they share, and so T is a triangulation. Therefore P^* is a compact, connected, triangulable 2-manifold. □

11.7 Another Classification of Compact 2-Manifolds

Theorem 11.28. If A and C are (possibly empty) words and $Abb^{-1}C$ is a string of $2n$ letters where each letter occurs twice, neglecting superscripts (and there is at least one pair other than b and b^{-1}), then the 2-manifold obtained from $Abb^{-1}C$ is homeomorphic to that obtained from the word AC.

Proof. Assume the hypotheses of the claim and consider the polygon P corresponding to $Abb^{-1}C$ lying flat in the plane. Without loss of generality, assume P is regular. Then at the corner where the edges b and b^{-1} meet, we note that the identification arrows both point toward the corner, since b means the arrow is in the direction of clockwise movement (toward the corner and along a clockwise trajectory) and b^{-1} means the arrow is in the direction going backwards, again toward the corner. Continuously pushing the corner where b and b^{-1} meet in toward the center of the polygon gives a homeomorphism from the regular polygon P to a polygon P' which is the same as P but with a dent inward at this corner. We now perform the gluing indicated by the identification of the two edges labeled b by closing the dent in P'. We can do this because both arrows along edges labeled b point inward toward the corner, so closing up the dent by gluing edge b means pushing the corner where A meets b toward the corner where b^{-1} meets C (since these corners are identified). We are left with a polygon P^* still homeomorphic to the original presentation (since we have only continuously deformed P and glued edges that were already identified) such that P^* has $2n-2$ edges identified pairwise with word AC. $\square$

Theorem 11.29. Let P be a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then P is homeomorphic to a polygonal presentation where all the vertices are identified (in the same equivalence class).

Proof. Let P be a polygonal presentation not homeomorphic to $\mathbb{S}^2$, and apply Theorem 11.28 to remove all letter pairs of the form aa^{-1} were a is adjacent to a^{-1} in the word corresponding to P. We note that after doing this we must still be left with a nonempty word, because the only way to generate an empty word would be if we started with something homeomorphic to the polygonal presentation obtained from the word bb^{-1}, that is, if we began with a polygonal presentation homeomorphic to the 2-sphere. Since P is not homeomorphic to $\mathbb{S}^2$, we have some number $2n$ of remaining letters in the word corresponding to P.

If we already have that all vertices are identified, we are done. Suppose now instead that there are $m \geq 2$ distinct equivalence classes of vertices, call them $C_1, \ldots, C_m$. Then there exist vertices $V_i \in C_i$ and $V_j \in C_j$ such that V_j is immediately clockwise of V_i for some $i \neq j = 1, \ldots, m$. Denote by V_k the vertex immediately clockwise of V_j. Note that such a pair V_i and V_j must exist, as otherwise we would have that for every vertex V, its immediately clockwise neighbor V' is in the same equivalence class, and so by induction we would have that all vertices are in this one equivalence class, contradicting our assumption that there are multiple equivalence classes. If the edge from V_i to V_j were identified with the edge from V_j to V_k, we would either have the letter pair aa in our word corresponding to P, which would mean the vertex V_i identified with V_j, contradicting our assumption that $V_i \in C_i$ and $V_j \in C_j$, or we would have the letter pair aa^{-1}, which would also be impossible because we have removed all such letter pairs using Theorem 11.28. Therefore the edge a from V_i to V_j is not identified with the edge b from V_j to V_k (where we are assuming without loss of generality that the arrows along a and b point in the clockwise direction). Therefore the word corresponding to P is of the form abX (where we begin the word at the vertex V_i). Because we know also that b or b^{-1} appears in the word later on, we can say that the word is of the form $abXbY$ or $abXb^{-1}Y$. We now make a cut from vertex V_k to V_i and identify the new edges by labeling them c so that the corresponding word is now $abcc^{-1}Xb^{(-1)}Y$. The triangle abc has edges all going clockwise labeled a, b, and c, and we may choose to start the labeling instead at vertex V_j so that abc is homeomorphic to bca. We could also pick this triangle up and flip it over to get $b^{-1}a^{-1}c^{-1}$ where we begin the labeling at the vertex V_k. If our word is $(abc)(c^{-1}XbY)$, we may glue the edges b together by detaching the triangle $abc \cong b^{-1}a^{-1}c^{-1}$ and reattaching it to be next to the second appearance of b in the word we are working with to obtain $c^{-1}Xb(b^{-1}a^{-1}c^{-1})Y$. Applying Theorem 11.28, this is homeomorphic to the polygonal presentation $c^{-1}Xa^{-1}c^{-1}Y$. If instead the word is $(abc)(c^{-1}Xb^{-1}Y)$, we glue the edges together by detaching the triangle $abc \cong bca$ and reattaching it to the appearance of b^{-1} to obtain $c^{-1}Xb^{-1}(bca)Y$, which is homeomorphic to $c^{-1}XcaY$ again by Theorem 11.28.

At the beginning of this process, we had vertices $V_i \in C_i$, $V_j \in C_j$, and $V_k \in C_k$, as well as the vertices V_j' and V_k' on either side of the second appearance of the letter b in the word $abXbY$ (or $abXb^{-1}Y$) where V_j is glued to V_j' and V_k is glued to V_k'. Therefore $V_j' \in C_j$ and $V_k' \in C_k$, so two vertices each from equivalence classes C_j and C_k, and one vertex from the class C_i. At the end of this process, we have that P is homeomorphic to one of $c^{-1}Xa^{-1}c^{-1}Y$ or $c^{-1}XcaY$. In both cases, we have that c^{-1} and c connect $V_k \in C_k$ to $V_i \in C_i$ (or the other way around), and that a^{-1} and a connect V_i to V_j, so we now have two vertices each from equivalence classes C_i and C_k, and only one vertex (an endpoint of either a or a^{-1}) belonging to the equivalence class C_j. This means that if to begin with there were l vertices in equivalence class V_j in the polygonal presentation given by $abXbY$, then this cutting and gluing process is a homeomorphism to a polygonal presentation with only $l-1$ vertices in equivalence class V_j. If necessary, we may repeat this process a finite number of times to produce a homeomorphism from P to a polygonal presentation with no vertices in equivalence class V_j. Therefore a polygonal presentation P not homeomorphic to $\mathbb{S}^2$ whose vertices lie in m equivalence classes (for $m \geq 2$) is homeomorphic to a polygonal presentation whose vertices lie in $m-1$ equivalence classes. Repeating this procedure a finite number of times if necessary therefore produces a homeomorphism from P to a polygonal presentation whose vertices are all in the same equivalence class. $\square$

Theorem 11.30. Let P be a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then P is homeomorphic to a polygonal presentation such that all the vertices are identified and for every pair of edges with the same orientation, the edges are consecutive.

Proof. By Theorem 11.29, we know that P is homeomorphic to a polygonal presentation with all vertices identified, so this part is done. Now suppose there is a pair of edges a identified in the same direction. Without loss of generality, assume they are identified clockwise so that the word obtained from this presentation is $aXaY$ for some nonempty words X and Y (if X or Y were empty, we would be done since the edges a would be consecutive or consecutive after a relabeling). Make a cut joining the initial endpoints of a, beginning at the vertex between X and a and ending at the vertex between Y and a (where by "vertex between α and β" we mean the vertex a path going around ∂P clockwise encounters moving from edge α to edge β). Then P is homeomorphic to the presentation obtained from $aXbb^{-1}aY$, and the cut splits P into aXb and $b^{-1}aY$. We then pick up the piece $b^{-1}aY$ and flip it over to obtain $Za^{-1}b$ (where Z is the word that results from flipping Y like this). Since the edges a are still identified, we can glue them, showing that P is homeomorphic to the presentation obtained from the word $Za^{-1}(aXb)b$, which is homeomorphic to the presentation from $ZXbb$. Note that flipping Y into Z changes the superscript of each letter and reads in the opposite direction but does not change whether edges with the same orientation are adjacent or not. We also have that the words $aXaY$ and $ZXbb$ contain the same number of pairs of edges identified in the same direction. Therefore this cutting and gluing procedure can be applied to all pairs identified in the same direction a finite number of times to produce a polygonal presentation homeomorphic to the original presentation P such that all such edge pairs are adjacent. $\square$

Theorem 11.31. Let P be a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then P is homeomorphic to a polygonal presentation such that all the vertices are identified, for every pair of edges with the same orientation, the edges are consecutive, and all other edges are grouped in disjoint sets following the pattern $aba^{-1}b^{-1}$.

Proof. By Theorems 11.28 through 11.30, we know that P is homeomorphic to a polygonal presentation in which the pattern aa^{-1} does not appear, in which all vertices are identified, and in which all pairs of edges identified with the same orientation are consecutive. Now if there are no pairs of edges one identified clockwise and one counter clockwise, we are done, so assume there exists at least one such pair so that the word corresponding to P can be written as $aXa^{-1}Y$. We claim that there is at least one other pair b and b^{-1} identified in opposite directions. We can assume that this word satisfies the conclusions of the previous three theorems, so if there is no other pair (b, b^{-1}) appearing in the word, X and Y are of the form $X = x_1^{s_1}x_1^{s_1}x_2^{s_2}x_2^{s_2}\ldots x_i^{s_i}x_i^{s_i}$ and $Y = y_1^{q_1}y_1^{q_1}\ldots y_j^{q_j}y_j^{q_j}$ for letters $x_1,\ldots,x_i,y_1,\ldots,y_j$ and powers $s_1,\ldots,s_i,q_1,\ldots,q_j$. This means that the initial endpoint of a is also one of the endpoints of y_j, and since y_j is glued to its neighbor in the same direction, we see that the initial endpoint of a is identified with both endpoints of y_j. The same reasoning implies that both endpoints of y_j are identified with both endpoints of y_{j-1}, and so on, stopping again at the initial endpoint of a that is also one of the endpoints of y_1. Therefore the initial endpoint of a is identified with all vertices in Y. This also means that the initial endpoint of a is not identified with any vertices in X. Since we cannot have an edge pair glued in opposite directions in which edges are adjacent (we have applied Theorem 11.28), the words X and Y are nonempty, so if one endpoint of a is not in the equivalence class of any vertex in X, there is more than one equivalence class, which contradicts the assumption that we have applied the conclusion of Theorem 11.29. Therefore there exists another pair of edges b identified in opposite directions.

If we begin with edge the labeling with edge a, there are $3! = 6$ possible forms of words:

$$\begin{array}{ll}
aX_1a^{-1}X_2bX_3b^{-1}X_4 & aX_1a^{-1}X_2b^{-1}X_3bX_4 \\
aX_1bX_2b^{-1}X_3a^{-1}X_4 & aX_1b^{-1}X_2bX_3a^{-1}X_4 \\
aX_1bX_2a^{-1}X_3b^{-1}X_4 & aX_1b^{-1}X_2a^{-1}X_3bX_4.
\end{array}$$

where the X_is are (possibly empty) words. However, the first four are impossible for the same reason a word of the form $aXa^{-1}Y$ was impossible since they all contain a subword of this form, meaning they also correspond to presentations with more than one equivalence class of vertices. Also note that if we began the labeling of P at edge b, the sixth word would have exactly the form of the fifth word. Therefore we may assume that P corresponds to the word $aX_1bX_2a^{-1}X_3b^{-1}X_4$.

First, we cut along a new edge c from the vertex between b^{-1} and X_4 to the vertex between X_1 and b so that P is homeomorphic also to the presentation obtained from $cbX_2a^{-1}X_3b^{-1}X_4aX_1c^{-1}$ (where we have relabeled starting at c). Then the last piece $X_4aX_1c^{-1}$ can be relabeled as $aX_1c^{-1}X_4$ and reglued along edge a to get the P is homeomorphic to

$cbX_2a^{-1}(aX_1c^{-1}X_4)X_3b^{-1}$, which using Theorem 11.28 is homeormorphic to $cbX_2X_1c^{-1}X_4X_3b^{-1}$. Note we may skip this step if X_1 and X_4 both are empty, since in this case we would start from $abX_2a^{-1}X_3b^{-1}$, which is of the same form as the resulting $cbX_2X_1c^{-1}X_4X_3b^{-1}$.

Second, we cut from the vertex between c and b to the vertex between X_1 and c^{-1} along the new edge d, so that our working word is $cdd^{-1}bX_2X_1c^{-1}X_4X_3b^{-1}$. Then detaching the piece $d^{-1}bX_2X_1$ and relabeling it to $bX_2X_1d^{-1}$, we reglue it along edge b to get $cdc^{-1}X_4X_3b^{-1}(bX_2X_1d^{-1})$, which using Theorem 11.28 is homeomorphic to the word $cdc^{-1}X_4X_3X_2X_1d^{-1}$, which we then relabel as $d^{-1}cdc^{-1}X_4X_3X_2X_1$. Note that we may skip this step if both X_1 and X_2 are empty, as in this case we would start the step with the word $b^{-1}cbc^{-1}X_4X_3$ of the same form as the result (after relabeling).

Third, we cut along a new edge e from the vertex between d and c^{-1} to the vertex between d^{-1} and c so that P is homeomorphic to the presentation corresponding to $d^{-1}cdee^{-1}c^{-1}X_4X_3X_2X_1$. We then detach cde and relabel it as ecd before regluing along edge d to get $(ecd)d^{-1}e^{-1}c^{-1}X_4X_3X_2X_1$. By Theorem 11.28, P is homeomorphic to $ece^{-1}c^{-1}X_4X_3X_2X_1$, so we have shown that if there is one pair of edges identified in opposite directions, there is always a second pair and the presentation P containing them is homeomorphic to the presentation obtained from an intertwined pair $ece^{-1}c$ followed by everything that was once between the edges labeled a and b. Since we never cut in a way that separated an X_i, these are all still intact, and so we may handle any other pairs of pairs of oppositely identified edges in the same manner without risk of disturbing the new $ece^{-1}c^{-1}$ set of edges. Therefore after a finite process of cutting and gluing, we can group all pairs of pairs of oppositely identified edges together as desired, proving the claim. □

Theorem 11.32. If A and C are (possibly empty) words, then the polygonal presentation $Aaba^{-1}b^{-1}ccC$ is homeomorphic to that represented by $AddeeffC$.

Proof. Consider the polygonal presentation $Aaba^{-1}b^{-1}ccC$. First, we cut along a new edge f from the vertex between b and a^{-1} to the vertex between c and c to obtain the word $Aabff^{-1}a^{-1}b^{-1}ccC$. Detaching the piece $f^{-1}a^{-1}b^{-1}c$, we flip it over and relabel to get $c^{-1}baf$. Then gluing along edge c, we have the word $Aabfc(c^{-1}baf)C$, which is homeomorphic to $AabfbafC$ by Theorem 11.28.

Second, we cut along a new edge d from the vertex between A and a to the vertex between b and a, resulting in the word $Add^{-1}abfbafC$. We detach $d^{-1}abfb$, flip it, and relabel it to $a^{-1}db^{-1}f^{-1}b^{-1}$. We then reglue along edge a to get $Ada(a^{-1}db^{-1}f^{-1}b^{-1})fC$, and so the corresponding polygonal presentation is homeomorphic to that obtained from $Addb^{-1}f^{-1}b^{-1}fC$.

Third, we create a new edge f by cutting from the vertex between d and b^{-1} to the vertex between f^{-1} and b^{-1}. Our new word is therefore $Addee^{-1}b^{-1}f^{-1}b^{-1}fC$. Detaching $e^{-1}b^{-1}f^{-1}$, flipping, and relabeling starting at b, we have the word bef, which we glue back now along edge b to obtain the word $Addeb^{-1}(bef)fC$, which by Theorem 11.28 is homeomorphic to $AddeeffC$ as required. □

Theorem 11.33. Any compact, connected, triangulable 2-manifold M is homeomorphic to the polygonal presentation given by one of the following words: aa^{-1}, $a_1a_1 \ldots a_na_n$ ($n \geq 1$) or $a_1a_2a_1^{-1}a_2^{-1} \ldots a_{n-1}a_na_{n-1}^{-1}a_n^{-1}$ ($n \geq 2$ an even number).

Proof. Let M be a compact, connected, and triangulable 2-manifold. By Theorem 11.26, M is homeomorphic to a polygonal presentation, call it P'. If P' is homeomorphic to a sphere, it is homeomorphic to the polygonal presentation given by the word aa^{-1} (Exercise 11.24(a)). If instead P' is not homeomorphic to a sphere, Theorems 11.28 through 11.31 imply that P' is homeomorphic to a polygonal presentation P in which subwords of the form aa^{-1} do not appear, in which all vertices are in one equivalence class, in which all pairs of edges identified in the same direction are consecutive and form subwords of the form aa, and in which all other edges are grouped in disjoint intertwined pairs of the form $aba^{-1}b^{-1}$. If no subwords of the form aa appear in the presentation, then the word corresponding to P is of the form $a_1a_2a_1^{-1}a_2^{-1} \ldots a_{n-1}a_na_{n-1}^{-1}a_n^{-1}$ for some even $n \geq 2$, and we are done. Therefore suppose the word corresponding to P contains at least one subword of the form aa. We claim that in this case, P is homeomorphic to a polygonal presentation of the form $a_1a_1 \ldots a_na_n$ for some $n \geq 1$.

If the word is already in this form, we are done, so assume the word is not yet in this form. There are two possible cases. In the first, there is a subword of the form $a^{-1}a^{-1}$ (since Theorem 11.30 only guarantees pairs of this form are consecutive, not that all pairs are identified in the same direction), and in the second, there are no subwords of the form $a^{-1}a^{-1}$, but there is at least one subword of the form $aba^{-1}b^{-1}$. In the first case, suppose the word is $Xa^{-1}a^{-1}Y$. Make a cut from the vertex between X and a^{-1} to the vertex between a^{-1} and a^{-1} along a new edge b so that the word becomes $Xbb^{-1}a^{-1}a^{-1}Y$. Detach the half disk $b^{-1}a^{-1}$ and flip it over to get ab before regluing along edge a. Then the word is $Xba^{-1}(ab)Y$, homeomorphic to $XbbY$ by Theorem 11.28. For any pairs of the form $a^{-1}a^{-1}$ we can use this procedure finitely many times to show that the presentation P is homeomorphic to a presentation that has all edge pairs aa identified clockwise, so that $M \cong P$ is homeomorphic to a presentation with no subwords of the form $a^{-1}a^{-1}$. After doing so, we are either done or there is at least one subword of the form $aba^{-1}b^{-1}$ and we are in the second case.

In this second case, there is at least one subword of the form aa and one subword of the form $aba^{-1}b^{-1}$, so there is at least one subword of the form $aba^{-1}b^{-1}cc$ since otherwise, we would have that immediately clockwise of every $aba^{-1}b^{-1}$ subword is another subword of the same form and so there are no subwords of the form aa. Therefore the word corresponding to P in this case is of the form $Aaba^{-1}b^{-1}ccC$, which means P is homeomorphic to a presentation of the form $AddeeffC$ by Theorem 11.32. Therefore if at the beginning of this case we had k subwords of the form $aba^{-1}b^{-1}$, we now have $k-1$ since A and C are unaltered by this procedure. Doing this again $k-1$ times produces a word of the form $a_1a_1 \dots a_na_n$ for some $n \geq 1$ and so we are done. $\square$

Exercise 11.34. The polygonal presentation in Figure 11.3 has word $a^{-1}b^{-1}c^{-1}abc$ (starting from the top edge and moving clockwise). The first thing to note is that there are two equivalence classes of vertices, so we start by applying Theorem 11.29. Make a cut from the vertex between b^{-1} and c^{-1} to the vertex between c and a^{-1} along a new edge d so that the word is $a^{-1}b^{-1}dd^{-1}c^{-1}abc$. Then detach $a^{-1}b^{-1}d$ and reglue it along edge a to get $d^{-1}c^{-1}a(a^{-1}b^{-1}d)bc$. Then this presentation is homeomorphic to $d^{-1}c^{-1}b^{-1}dbc$. We still have two equivalence classes, so we do this again by making a cut along new edge e from the vertex between b^{-1} and d to the vertex between d^{-1} and c^{-1}. Then the word is $d^{-1}c^{-1}b^{-1}ee^{-1}dbc$, and we detach $c^{-1}b^{-1}e$ and reglue it along edge c to get $d^{-1}e^{-1}dbc(c^{-1}b^{-1}e)$, which is homeomorphic to $d^{-1}e^{-1}dbb^{-1}e$, which in turn is homeomorphic to $d^{-1}e^{-1}de$. Relabeling this starting at d, we have $ded^{-1}e^{-1}$, a standard presentation for the torus.

Exercise 11.35. Let w_1 and w_2 represent these two compact, connected 2-manifolds, call them M_1 and M_2. Then their connected sum is formed by removing the interiors of a 2-simplex in each and identifying their boundaries, which is the same as removing open disks and identifying their boundaries. Since this can be done anywhere on the manifold, we choose to remove an open disk from M_1 and M_2 by taking the vertices at the beginning of each word and removing an open ball with small enough radius so as to not intersect any edges other than the first and last appear in w_1 and w_2 respectively. To do this, cut out the starting vertex of M_1 cutting clockwise and the starting vertex of M_2 cutting counterclockwise. The boundaries of these open balls are then identified, and we can show this by labeling the boundaries as a. Since the starting vertex is also the ending vertex, in cutting in this manner we have appended a letter to each word so that $M_1 - B$ is represented by w_1a and $M_2 - B$ is represented by $a^{-1}w_2$. In constructing the connected sum, we glue along the boundaries of the removed disks, which means gluing these spaces together along edge a, resulting in $w_1aa^{-1}w_2$. Applying Theorem 11.28, we have that the connected sum $M_1\#M_2$ is homeomorphic to the polygonal presentation given by the word w_1w_2.

Exercise 11.36. Theorem 11.32 can now be read as the statement that the connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.

Theorem 11.37. Any compact, connected, triangulated 2-manifold is homeomorphic to $\mathbb{S}^2$, a connected sum of tori, or a connected sum of projective planes.

Proof. In Theorem 11.33, we showed that such a 2-manifold is homeomorphic to a polygonal presentation represented by one of the following words: aa^{-1}, $a_1a_1\ldots a_na_n$ $(n \geq 1)$ or $a_1a_2a_1^{-1}a_2^{-1}\ldots a_{n-1}a_na_{n-1}^{-1}a_n^{-1}$ ($n \geq 2$ an even number). In the first case, we have that the manifold is homeomorphic to $\mathbb{S}^2$, in the second case, we have that the manifold is homeomorphic to the connected sum of $n/2$ tori (n is even), and in case three, we have that the manifold is homeomorphic to the connected sum of n projective planes. $\square$

11.9 The Euler Characteristic

Exercise 11.43. We consider each of these spaces as a polygonal presentation of the square in the plane, and triangulate each using the method in Theorem 11.27. Then regardless of the way the edges of the square are identified, there are 9 vertices, 32 edges, and 24 faces (2-simplices) not including those vertices, edges, and faces that come directly from the boundary of the original square. Since the boundary of the square contributes 0 faces, there are a total of 24 faces for all four of these polygonal presentations when triangulated as described in Theorem 11.27. Then number of edges also does not change. This is because we begin with the words (1) $aa^{-1}bb^{-1}$, (2) $aba^{-1}b^{-1}$, (3) $aba^{-1}b$, and (4) $aa^{-1}bb$, and constructing the triangulation means splitting each of the four boundary edges into two pieces (for example, edge a splits into ac and edge a^{-1} into edge $c^{-1}a^{-1}$). Each of these new edges then has a pair with which it is identified, so this always adds 4 edges to the triangulation, for a total of 36 edges and 24 faces. Adding vertices is where things changes: since we divide each original boundary edge into two pieces, this adds a vertex on each side, but these come in two pairs of identified vertices, so the boundary of the presentation always adds at least 2 vertices, in addition to the number of vertices that come from the corners of the presentation. The number of equivalence classes of corners is what changes the Euler characteristic. For the sphere $\mathbb{S}^2$, the word $aa^{-1}bb^{-1}$ when triangulated gives $acc^{-1}a^{-1}bdd^{-1}b^{-1}$, with the corners falling between c and c^{-1} (one equivalence class), between d and d^{-1} (a second equivalence class), between a^{-1} and b, and between b^{-1} and a (these last two make a third equivalence class). Because there are three equivalence classes, the boundary of the polygonal presentation for $\mathbb{S}^2$ adds 5 vertices, for a total of $V = 14$, $E = 36$, and $F = 24$, which means $\chi(\mathbb{S}^2) = 2$. Similarly, the torus and Klein bottle both have only one corner equivalence class in this particular presentation, and so there are only three new vertices added by the boundary, meaning $\chi(\mathbb{T}^2) = \chi(\mathbb{K}^2) = 0$, whereas the presentation for the projective plane that we are using has two corner equivalence classes, meaning 4 new vertices are added, so $\chi(\mathbb{RP}^2) = 1$.

Lemma 11.44. Suppose M_1 and M_2 are compact 2-manifolds. If $M_1 \# M_2$ is any choice for the connected sum of M_1 and M_2, then $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

Proof. By Theorem 11.7, M_1 and M_2 are triangulable, so let $T_1 = \{\sigma_i\}_{i=1}^n$ and $T_2 = \{\sigma_i'\}_{i=1}^m$ be triangulations. Then to form the connected sum, remove faces $\mathrm{Int}(\sigma_i)$ from M_1 and $\mathrm{Int}(\sigma_j')$ from M_2 for some $\sigma_i \in T_1$ and $\sigma_j \in T_2$. Then identifying $\partial\sigma_i$ with $\partial\sigma_j'$, we have that to add the Euler characteristics of M_1 and M_2 and subtract 1 from each (for the missing faces) would be to double count the newly identified vertices and edges of σ_i and σ_j', so we have that $M_1 \# M_2$ has 2 fewer faces, 3 fewer edges, and 3 fewer vertices than total of M_1 and M_2 separately, so $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2 + 3 - 3 = \chi(M_1) + \chi(M_2) - 2$ as claimed. $\square$

Exercise 11.45. (1) We argue by induction on n that $\chi(\#_{i=1}^n \mathbb{RP}^2) = 2 - n$. For the base case $n = 1$, we have that $\chi(\mathbb{RP}^2) = 1 = 2 - 1$, which holds. Suppose $\chi(\#_{i=1}^k \mathbb{RP}^2) = 2 - k$. Then

$$\chi(\#_{i=1}^{k+1} \mathbb{RP}^2) = \chi((\#_{i=1}^k \mathbb{RP}^2) \# \mathbb{RP}^2) = \chi(\#_{i=1}^k \mathbb{RP}^2) + \chi(\mathbb{RP}^2) - 2 = 2 - k + 1 - 2 = 2 - (k+1),$$

so the claim holds for all $n \in \mathbb{N}$.

(2) A similar argument shows that $\chi(\#_{i=1}^n \mathbb{T}^2) = 2 - 2n$.

Exercise 11.47.

$$
\begin{aligned}
\text{(a) } \mathbb{T}\#\mathbb{RP} &= \mathbb{RP}\#\mathbb{RP}\#\mathbb{RP} \\
\text{(b) } \mathbb{K}\#\mathbb{RP} &= \mathbb{RP}\#\mathbb{RP}\#\mathbb{RP} \\
\text{(c) } \mathbb{RP}\#\mathbb{T}\#\mathbb{K}\#\mathbb{RP} &= \mathbb{RP}\#\mathbb{T}\#(\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP}) = \#_{i=1}^{6}\mathbb{RP} \\
\text{(d) } \mathbb{K}\#\mathbb{T}\#\mathbb{T}\#\mathbb{RP}\#\mathbb{K}\#\mathbb{T} &= \mathbb{K}\#\mathbb{T}\#\mathbb{T}\#(\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP})\#\mathbb{T} \\
&= \mathbb{K}\#\mathbb{T}\#(\mathbb{T}\#\mathbb{RP})\#\mathbb{RP}\#(\mathbb{RP}\#\mathbb{T}) \\
&= \mathbb{K}\#\mathbb{T}\#(\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP})\#\mathbb{RP}\#(\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP}) \\
&= \mathbb{K}\#(\mathbb{T}\#\mathbb{RP})\#(\#_{i=1}^{6}\mathbb{RP}) \\
&= \mathbb{K}\#(\mathbb{RP}\#\mathbb{RP}\#\mathbb{RP})\#(\#_{i=1}^{6}\mathbb{RP}) \\
&= (\mathbb{K}\#\mathbb{RP})\#(\#_{i=1}^{8}\mathbb{RP}) = \#_{i=1}^{11}\mathbb{RP}
\end{aligned}
$$

Exercise 11.48. The word corresponding to the decagon in Figure 11.23 is $bcdbdceeaa$, which is the connected sum of the presentation $bcdbdc$ and two projective planes. Following the procedure in the proof of Theorem 11.30, we see that the presentation $bcdbdc$ is homeomorphic to $ffc^{-1}d^{-1}cd$. Cutting along a new edge g from the vertex between d and f to the vertex between d^{-1} and c, we have the word $ffc^{-1}d^{-1}cdgg^{-1}$. Detaching the word cdg and gluing it back along edge c gives the word $ffc^{-1}(cdg)d^{-1}g^{-1}$, which means the word $bcdbdc$ is homeomorphic to $ffdgd^{-1}g^{-1}$, the connected sum of a projective plane and a torus, which by Theorem 11.32 is homeomorphic to the connected sum of three projective planes. Therefore this decagon is homeomorphic to the connected sum of five projective planes.