

# Topology Through Inquiry Self-Study

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## 1 Cardinality: To Infinity and Beyond

### 1.1 Sets and Functions

**Theorem 1.2 (DeMorgan's Laws).** Let  $X$  be a set and let  $\{A_k\}_{k=1}^N$  be a finite collection of sets such that  $A_k \subset X$  for each  $k = 1, 2, \dots, N$ . Then

$$X - \left( \bigcup_{k=1}^N A_k \right) = \bigcap_{k=1}^N (X - A_k)$$

and

$$X - \left( \bigcap_{k=1}^N A_k \right) = \bigcup_{k=1}^N (X - A_k).$$

*Proof.* Let  $a \in X - \left( \bigcup_{k=1}^N A_k \right)$  be arbitrary. Then  $a \notin \bigcup_{k=1}^N A_k$ , so for all  $k$ ,  $a \notin A_k$ , which means that  $a \in X - A_k$  for all  $k$ . Therefore  $a \in \bigcap_{k=1}^N (X - A_k)$  and so  $X - \left( \bigcup_{k=1}^N A_k \right) \subset \bigcap_{k=1}^N (X - A_k)$ . Now let  $a \in \bigcap_{k=1}^N (X - A_k)$  be arbitrary. Then we have that  $a \in X - A_k$  for all  $k$ , which means that  $a \notin A_k$  for all  $k$ . Therefore  $a \notin \bigcup_{k=1}^N A_k$ , so we have that  $a \in X - \left( \bigcup_{k=1}^N A_k \right)$  and therefore  $\bigcap_{k=1}^N (X - A_k) \subset X - \left( \bigcup_{k=1}^N A_k \right)$ . Therefore

$$X - \left( \bigcup_{k=1}^N A_k \right) = \bigcap_{k=1}^N (X - A_k).$$

Let  $a \in X - \left( \bigcap_{k=1}^N A_k \right)$  be arbitrary. Then  $a \notin \bigcap_{k=1}^N A_k$ , so there exists some  $j$  such that  $a \notin A_j$ , which means that  $a \in X - A_j$ . Therefore  $a \in \bigcup_{k=1}^N (X - A_k)$  and so  $X - \left( \bigcap_{k=1}^N A_k \right) \subset \bigcup_{k=1}^N (X - A_k)$ . Now let  $a \in \bigcup_{k=1}^N (X - A_k)$ . Then there exists some  $j$  such that  $a \in X - A_j$ , so  $a \notin A_j$  and therefore

$a \notin \bigcap_{k=1}^N$ . This means that  $a \in X - \left(\bigcap_{k=1}^N A_k\right)$  and so  $\bigcup_{k=1}^N (X - A_k) \subset X - \left(\bigcap_{k=1}^N A_k\right)$ . Therefore we have that

$$X - \left(\bigcap_{k=1}^N A_k\right) = \bigcup_{k=1}^N (X - A_k).$$

□

**Exercise 1.3.** For a function  $f : X \rightarrow Y$  and sets  $A, B \subset Y$ , we have that  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$  and that  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ .

*Proof.* See MATH 200 final exam review sheet notes in the graph paper notebook. □

**Exercise 1.4.** If  $f : X \rightarrow Y$  is injective and  $y \in Y$ , then  $f^{-1}(y)$  contains at most one point.

*Proof.* Let  $f : X \rightarrow Y$  be a function and let  $y \in Y$  be arbitrary. Suppose  $f^{-1}(y)$  contains more than one point. Then there exist  $x_1, x_2 \in X$  such that  $x_1, x_2 \in f^{-1}(y)$  and  $x_1 \neq x_2$ . By the definition of  $f^{-1}(y)$ , we have that  $f(x_1), f(x_2) \in \{y\}$ , so  $f(x_1) = y = f(x_2)$  and  $f$  is therefore not injective since  $x_1 \neq x_2$ . We have shown the contrapositive of the claim. □

**Exercise 1.5.** If  $f : X \rightarrow Y$  is surjective and  $y \in Y$ , then  $f^{-1}(y)$  contains at least one point.

*Proof.* Let  $f : X \rightarrow Y$  be a function and let  $y \in Y$  be arbitrary. Suppose  $f^{-1}(y) = \emptyset$ . Then for all  $x \in X$ ,  $f(x) \notin f^{-1}(y)$ , and by the definition of  $f^{-1}(y)$ , this means that for all  $x \in X$ ,  $f(x) \neq y$ , so  $f$  is not surjective. We have shown the contrapositive of the claim. □

## 1.2 Cardinality and Countable Sets

**Theorem 1.8.** Every subset of  $\mathbb{N}$  is either finite or has the same cardinality as  $\mathbb{N}$ .

*Proof.* Let  $S \subset \mathbb{N}$  be an arbitrary subset of  $\mathbb{N}$ . If  $S$  is finite, then we are done. If  $S$  is not finite, then it is infinite. Let  $s_0 = \min S$ , let  $s_1 = \min(S - \{s_0\})$ , and let  $s_i = \min(S - \{s_0, \dots, s_{i-1}\})$ . Define the function  $f : \mathbb{N} \rightarrow S$  by the following:  $f(n) = s_n$ . Suppose  $n_1, n_2 \in \mathbb{N}$  such that  $n_1 \neq n_2$ . Without loss of generality, assume that  $n_1 < n_2$ . Then  $f(n_2) = \min(S - \{s_0, \dots, s_{n_1}, \dots, s_{n_2-1}\})$ . Since  $f(n_1) = s_{n_1} \notin S - \{s_0, \dots, s_{n_1}, \dots, s_{n_2-1}\}$ , we have that  $f(n_1) \neq \min(S - \{s_0, \dots, s_{n_1}, \dots, s_{n_2-1}\}) = f(n_2)$ , which means that  $f$  is injective. Let  $s \in S \subset \mathbb{N}$  be arbitrary. Then set  $j = |\{r \in S : r < s\}| + 1 \in \mathbb{N}$ . Then we have that

$$f(j) = s_j = \min(S - \{s_0, \dots, s_{|\{r \in S : r < s\}|}\}) = s$$

where the last equality follows from the fact that  $s$  must be the smallest element of the subset of  $S$  from which all elements smaller than  $s$  have been removed. Therefore  $f$  is surjective, and since it is also injective,  $f$  is a bijection, meaning that the cardinality of  $S$  is the same as the cardinality of  $\mathbb{N}$ .  $\square$

**Theorem 1.9.** Every infinite set has a countable subset.

*Proof.* I think I need the axiom of choice here? I will return in the future!  $\square$

**Theorem 1.10.** A set is infinite if and only if there is an injection from the set into a proper subset of itself.

*Proof.* I think I also need the axiom of choice here...  $\square$

**Theorem 1.11.** The union of two countable sets is countable.

*Proof.* Let  $A$  and  $B$  be countable sets. There are two cases to consider. In the first case, the intersection of  $A$  and  $B$  is finite, and so there exists a bijection  $h : A \cap B \rightarrow \{1, \dots, n\}$  for some  $n \in \mathbb{N}$  where  $n$  is the size of the set  $A \cap B$ . If  $A \cap B$  is empty, we instead use  $n = 0$ . Since  $A \cap B$  is finite,  $A - (A \cap B)$  is infinite, so there exists a bijection  $f : A - (A \cap B) \rightarrow \mathbb{E}$ , where  $\mathbb{E}$  is the countable set containing all positive even integers greater than  $n$ . Similarly, there exists a bijection  $g : B - (A \cap B) \rightarrow \mathbb{O}$ , where  $\mathbb{O}$  is the countable set containing all positive odd integers greater than  $n$ . Then we have that the function  $\varphi : A \cup B \rightarrow \mathbb{N}$  is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the second case,  $A \cap B$  is infinite and there are three subcases. In the first subcase, one of  $A - (A \cap B)$  or  $B - (A \cap B)$  (assume without loss of generality that this is  $A - (A \cap B)$ ) is finite. In this case, we use the same construction as earlier, since now  $h : A - (A \cap B) \rightarrow \{1, \dots, n\}$  is a bijection for some  $n \in \mathbb{N}$ ,  $f : A \cap B \rightarrow \mathbb{E}$  is a bijection, and  $g : B - (A \cap B) \rightarrow \mathbb{O}$  is a bijection. Then the function  $\varphi : A \cup B \rightarrow \mathbb{N}$  is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A \cap B \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A - (A \cap B) \end{cases}.$$

In the second subcase,  $A - (A \cap B)$  and  $B - (A \cap B)$  are finite. Then there are bijections  $f : A - (A \cap B) \rightarrow \{1, \dots, n\}$  for some  $n \in \mathbb{N}$  and  $g : B - (A \cap B) \rightarrow \{n+1, \dots, n+m\}$  for some  $m \in \mathbb{N}$ . Since  $A \cap B$  is countably infinite, there is a bijection  $h : A \cap B \rightarrow \{n+m+1, n+m+2, \dots\}$ , the set of positive integers greater than  $n+m$ . Therefore  $\varphi : A \cup B \rightarrow \mathbb{N}$  is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}$$

In the third subcase, all three sets are countably infinite. In this case, there is a bijection  $f : A - (A \cap B) \rightarrow \{n \in \mathbb{N} : n = 3k, k \in \mathbb{Z}\}$ , a bijection  $g : B - (A \cap B) \rightarrow \{n \in \mathbb{N} : n = 3k+1, k \in \mathbb{Z}\}$ , and a bijection  $h : A \cap B \rightarrow \{n \in \mathbb{N} : n = 3k+2, k \in \mathbb{Z}\}$ . Then we have that  $\varphi : A \cup B \rightarrow \mathbb{N}$  is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}$$

In all cases, we have that there exists a bijection  $\varphi : A \cup B \rightarrow \mathbb{N}$ , so  $A \cup B$  is countable. □

**Theorem 1.12.** The union of countably many sets is countable.

*Vague idea.* Let  $\{A_i\}_{i=1}^{\infty}$  be a sequence of countably many countable sets. Arrange the elements of  $\{A_i\}$  like so:

$$\begin{array}{ccccccc} A_1 : & a_{1_1} & a_{1_2} & a_{1_3} & \dots & & \\ A_2 : & a_{2_1} & a_{2_2} & a_{2_3} & \dots & & \\ A_3 : & a_{3_1} & a_{3_2} & a_{3_3} & \dots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{array}$$

Then define a bijection  $h : \bigcup_{i=1}^{\infty} A_i \rightarrow \mathbb{N}$  where  $h(a_{1_1}) = 1$ ,  $h(a_{1_2}) = 2$  if  $a_{1_2} \neq a_{1_1}$ , and  $h(a_{2_1}) = 3$  if  $a_{2_1} \notin \{a_{1_1}, a_{1_2}\}$ . The function  $h$  orders the elements of  $\bigcup_{i=1}^{\infty} A_i$  starting in the upper left and working down diagonals, skipping elements that have already been mapped. ♥

### 1.3 Uncountable Sets and Power Sets

**Exercise 1.17.** If  $A = \{a, b, c\}$ , then  $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ .

**Theorem 1.18.** If the set  $A$  is finite, then its power set has cardinality  $2^{|A|}$  ( $|2^A| = 2^{|A|}$ ).

*Proof.* If a set  $A$  is finite, then it has cardinality  $|A| = n \in \mathbb{N}$ . We argue by induction on  $n$  that its power set  $2^A$  has cardinality  $2^n$ . For the base case  $n = 1$ , let  $A_1$  be a set with one element. Then  $A_1 = \{a\}$  and  $2^{A_1} = \{\emptyset, a\}$ , which has two elements, so indeed  $|2^{A_1}| = 2^{|A_1|}$ . Now for the inductive step, assume as inductive hypothesis that there exists a  $k \in \mathbb{N}$  such that all sets with cardinality  $k$  have power sets with cardinality  $2^k$ . Then let  $A_{k+1}$  be an arbitrary set with cardinality  $k+1$  so that  $A_{k+1} = \{a_1, \dots, a_{k+1}\}$ . Note that the set  $A_k = \{a_1, \dots, a_k\}$  has cardinality  $k$ , and therefore has  $2^k$  subsets. For each subset of  $A_k$ , call them  $A_{k_i}$  for  $1 \leq i \leq 2^k$ , the sets  $A_{k_i}$  and  $A_{k_i} \cup \{a_{k+1}\}$  are subsets of  $A_{k+1}$ . Therefore there are  $2 \cdot 2^k = 2^{k+1}$  subsets of  $A_{k+1}$ , and so  $|2^{A_{k+1}}| = 2^{|A_{k+1}|}$ . Therefore by induction, all finite sets  $A$  have  $|2^A| = 2^{|A|}$ .  $\square$

**Theorem 1.19.** For any set  $A$ , there is an injection from  $A$  to  $2^A$ .

*Proof.* Define  $f : A \rightarrow 2^A$  as  $f(a) = \{a\} \in 2^A$  since  $\{a\} \subset A$ . Let  $x, y \in A$  such that  $f(x) = f(y)$ . Then we have that  $\{x\} = \{y\}$ , so  $x = y$  and  $f$  is an injection, as required.  $\square$

**Theorem 1.20.** If  $P$  is the set of all functions from a set  $A$  to the set  $\{0, 1\}$ , then  $|P| = |2^A|$ .

*Proof.* Define  $\varphi : P \rightarrow 2^A$  as  $\varphi(h) = h^{-1}(1)$ . Let  $f, g \in P$  be arbitrary such that  $f \neq g$ . Then there exists an  $a \in A$  such that  $f(a) \neq g(a)$ . Without loss of generality, assume that  $f(a) = 0$  and  $g(a) = 1$ . Then we have that  $a \notin f^{-1}(1) = \varphi(f)$  and  $a \in g^{-1}(1) = \varphi(g)$ , so  $\varphi(f) \neq \varphi(g)$  and therefore  $\varphi$  is injective. Now let  $Y \in 2^A$  be arbitrary. Then define the function  $f : A \rightarrow \{0, 1\}$  such that for  $a \in A$ ,

$$f(a) = \begin{cases} 0 & a \notin Y \\ 1 & a \in Y \end{cases}.$$

Then since  $Y \in 2^A$  means that  $Y \subset A$ , we have that

$$\varphi(f) = f^{-1}(1) = \{a \in A : f(a) = 1\} = \{a \in A : a \in Y\} = Y,$$

so  $\varphi$  is also surjective, making it a bijection. Since there is a bijection from  $P$  to  $2^A$ , we have that  $|P| = |2^A|$ , as required.  $\square$

**Theorem 1.22.** There is no surjection between a set  $A$  and its power set  $2^A$  ( $|A| \neq |2^A|$ )

*Proof.* Let  $A$  be an arbitrary set and suppose for contradiction that  $f : A \rightarrow 2^A$  is a surjection. Consider the set  $X = \{a \in A : a \notin f(a)\}$ . Since  $X \subset A$ ,  $X \in 2^A$ , and since  $f$  is surjective, there exists an  $a_0 \in A$  such that  $f(a_0) = X$ . There are two cases, both of which lead to a contradiction.

In the first case,  $a_0 \in X$ , and so we have that  $a_0 \notin f(a_0) = X$ , a contradiction. In the second case,  $a_0 \notin X$ , so  $a_0 \in f(a_0) = X$ , also a contradiction. Since both cases lead to a contradiction, we see that the assumption that  $f$  is surjective was false, so there can be no surjection between  $A$  and  $2^A$ , which means  $|A| \neq |2^A|$ .  $\square$

## 1.4 The Schroeder-Bernstein Theorem

**Theorem 1.28.** The unit square and the unit interval have the same cardinality.

*Proof.* Let  $a \in [0, 1]$  be arbitrary. Then it can be represented as

$$a = \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n$$

where  $\{a_n\}_{n=1}^{\infty}$  is a sequence of numbers with  $a_n \in \{0, 1, \dots, 9\}$  that does not end all in 9s (that is, if  $a_i = 9$ , then there exists  $j > i$  such that  $a_j \neq 9$ ). Now we claim that the function  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$  is surjective, where

$$f\left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n\right) = \left(\sum_{n=1}^{\infty} a_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} a_{2n} \left(\frac{1}{10}\right)^n\right).$$

To show surjectivity, let  $y \in [0, 1] \times [0, 1]$  be arbitrary. Then there exist sequences  $\{a_n\}$  and  $\{b_n\}$  such that

$$y = \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right).$$

Let  $x \in [0, 1]$  be the point

$$x = \sum_{n=1}^{\infty} \left[ a_n \left(\frac{1}{10}\right)^{2n-1} + b_n \left(\frac{1}{10}\right)^{2n} \right] = 0.a_1 b_1 a_2 b_2 a_3 b_3 \dots = 0.c_1 c_2 c_3 c_4 c_5 c_6 \dots$$

Then

$$\begin{aligned} f(x) &= f(0.c_1 c_2 c_3 \dots) = \left(\sum_{n=1}^{\infty} c_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} c_{2n} \left(\frac{1}{10}\right)^n\right) \\ &= \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right) = y. \end{aligned}$$

Therefore  $f$  is surjective. Now consider  $g : [0, 1] \times [0, 1] \rightarrow [0, 1]$  where  $g(a, b) = a$ . Let

$y \in [0, 1]$  be arbitrary and set  $x = (y, 0) \in [0, 1] \times [0, 1]$ . Then  $g(x) = g(y, 0) = y$ , and so  $g$  is surjective. Since there is a surjection from  $[0, 1]$  to  $[0, 1] \times [0, 1]$  and a surjection from  $[0, 1] \times [0, 1]$  to  $[0, 1]$ , by the surjective version of the Schroeder Bernstein Theorem (1.26), we have that  $|[0, 1] \times [0, 1]| = |[0, 1]|$  as required.  $\square$

## 1.5 The Axiom of Choice

**Exercise 1.32.** Let  $X$  be a set and let  $P$  be the poset of all subsets of  $X$  partially ordered by inclusion. Let  $p \in P$  be an element of the poset with  $X \leq p$ . Then we have that  $X \subset p$ . We also have that  $p \subset X$  since  $p \in P$  and  $P$  is the poset of all subsets of  $X$ . Therefore  $p = X$ , and so  $X$  is by definition a maximal element of  $P$ . Suppose there exists a set  $Y \in P$  such that  $Y$  is also a maximal element of  $P$ . Then we have that  $X \subset Y$  since  $Y$  is a maximal element, and also that  $Y \subset X$  since  $Y \in P$ . Therefore  $X = Y$ , which shows that  $X$  is the unique maximal element of  $P$ . Now let  $p \in P$  be an element of the poset with  $p \leq \emptyset$ . Then we have that  $p \subset \emptyset$ , but also that  $\emptyset \subset p$  since  $\emptyset \subset x$  for all sets  $x$ . Therefore  $p = \emptyset$ , and so by definition we have that  $\emptyset$  is a least element of the poset  $P$ . Now let  $Z \in P$  be a least element. Then  $Z \subset \emptyset$  since it is a least element, but as before,  $\emptyset \subset Z$ , which means  $Z = \emptyset$  and so  $\emptyset$  is the unique least element of the poset  $P$ .

**Exercise 1.33.** Let  $P$  be the poset ordered by cardinality with  $P = \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\}$ . Then  $\{0\}$  and  $\{1\}$  are least elements and  $\{0, 1\}$  and  $\{1, 2\}$  are maximal elements.

**Exercise 1.34.** Consider  $\mathbb{R}$  with the  $\leq$  relation. The relation is reflexive, transitive, and antisymmetric, so it is a partial order on  $\mathbb{R}$ . We also have that for any two elements  $x, y \in \mathbb{R}$ , either  $x \leq y$  or  $y \leq x$ , so they are comparable. This means that  $\mathbb{R}$  is totally ordered with the relation  $\leq$ . However,  $\mathbb{R}$  is not well-ordered, because there exist nonempty subsets of  $\mathbb{R}$  that do not have least elements, for example,  $(0, 1) \subset \mathbb{R}$ .

## 2 Topological Spaces: Fundamentals

### 2.2 Open Sets and the Definition of a Topological Space

**Theorem 2.1.** If  $\{U_i\}_{i=1}^n$  is a finite collection of open sets in a topological space  $(X, \mathcal{T})$ , then  $\bigcap_{i=1}^n U_i$  is open.

*Proof.* Let  $\{U_i\}_{i=1}^n$  be a finite collection of open sets in a topological space  $(X, \mathcal{T})$ . We argue by induction on  $n$  that  $\bigcap_{i=1}^n U_i$  is open. For the base case  $n = 1$ , we have that  $U_1$  is open. For the inductive step, assume as inductive hypothesis that there exists a  $k \in \mathbb{N}$  such that  $\bigcap_{i=1}^k U_i$  is open. Then since  $U_{k+1}$  is open, we have that  $\bigcap_{i=1}^{k+1} U_i = \left(\bigcap_{i=1}^k U_i\right) \cap U_{k+1}$  is the intersection of two open sets and is therefore open. By induction,  $\bigcap_{i=1}^n U_i$  is open for all  $n \in \mathbb{N}$ , in otherwords, the intersection of finitely many open sets is open.  $\square$

**Exercise 2.2.** The above theorem does not show that the intersection of infinitely many open sets is open since the intersection of infinitely many open sets cannot be represented as  $\bigcap_{i=1}^n U_i$  for any  $n \in \mathbb{N}$ .

**Theorem 2.3.** A set  $U$  is open in a topological space  $(X, \mathcal{T})$  iff for every  $x \in U$ , there exists an open set  $U_x$  such that  $x \in U_x \subset U$ .

*Proof.* Let  $(X, \mathcal{T})$  be a topological space with  $U \in \mathcal{T}$ . Let  $x \in U$  be arbitrary and set  $U_x = U$ . Then  $U_x$  is open and  $x \in U_x \subset U$ , as required. Now let  $U$  be a set such that for every  $x \in U$ , there exists an open set  $U_x$  such that  $x \in U_x \subset U$ . Then we have that  $\bigcup_{x \in U} U_x$  is open, since the union of a collection of open sets is open. We also have that  $y \in U$  implies that  $y \in \bigcup_{x \in U} U_x$  since  $y \in U_y$ , so  $U \subset \bigcup_{x \in U} U_x$ . If  $y \in \bigcup_{x \in U} U_x$ , then  $y \in U_z \subset U$  for some  $z \in U$ , so  $\bigcup_{x \in U} U_x \subset U$ , which means that the open set  $\bigcup_{x \in U} U_x$  is the set  $U$ . Therefore  $U$  is open iff for every  $x \in U$ , there exists an open set  $U_x$  such that  $x \in U_x \subset U$ .  $\square$

**Exercise 2.4.** First note that, vacuously,  $\emptyset \in \mathcal{T}_{\text{std}}$ , so the first property is satisfied. Next, consider the set  $\mathbb{R}^n \subset \mathbb{R}^n$ . Let  $p \in \mathbb{R}^n$  be an arbitrary point, and since  $B(p, 1) \subset \mathbb{R}^n$  and  $p$  was arbitrary, we have that  $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$ , so the second property is satisfied. For the third property, let  $U, V \in \mathcal{T}_{\text{std}}$  be arbitrary open sets in  $\mathbb{R}^n$ . Let  $p$  be an arbitrary point in  $U \cap V$ . Then because  $p \in U$ , there exists an  $\varepsilon_1$  such that  $B(p, \varepsilon_1) \subset U$ , and because  $p \in V$ , there exists an  $\varepsilon_2$  such that  $B(p, \varepsilon_2) \subset V$ . Set  $\varepsilon_p = \min\{\varepsilon_1, \varepsilon_2\}$ . Then  $B(p, \varepsilon_p) \subset B(p, \varepsilon_1) \subset U$  and  $B(p, \varepsilon_p) \subset B(p, \varepsilon_2) \subset V$ , so we have that  $B(p, \varepsilon_p) \subset U \cap V$ . Therefore  $U \cap V \in \mathcal{T}_{\text{std}}$  and the third property is satisfied. For the fourth property, let  $\{U_i\}_{i \in \lambda}$  be a collection of sets  $U_i \in \mathcal{T}_{\text{std}}$ . Then let  $p \in \bigcup_{i \in \lambda} U_i$  be arbitrary. Since  $p$  is in the union of all the  $U_i$ , we have that  $p \in U_j$  for some  $j \in \lambda$ . Then since  $U_j \in \mathcal{T}_{\text{std}}$ , there exists an  $\varepsilon_p$  such that  $B(p, \varepsilon_p) \subset U_j \subset \bigcup_{i \in \lambda} U_i$ , and therefore  $\bigcup_{i \in \lambda} U_i \in \mathcal{T}_{\text{std}}$ , so the fourth property is satisfied and we have that  $\mathcal{T}_{\text{std}}$  is indeed a topology on  $\mathbb{R}^n$ .

**Exercise 2.6.** The unit interval  $(0, 1) \subset \mathbb{R}$  is open in the standard topology on  $\mathbb{R}$ , open in the discrete topology, not open in the indiscrete topology, not open in the finite complement topology, and not open in the countable complement topology.



**Exercise 2.7.** In the topological space  $(\mathbb{R}, \mathcal{T}_{\text{std}})$ , the interval  $(0, 1)$  is open and for all  $n \geq 0$ , the set  $U_n \subset (0, 1)$  where

$$U_n = \left( \frac{2^n - 1}{2^{n+1}}, \frac{2^n + 1}{2^{n+1}} \right).$$

Then  $\frac{1}{2} \in U_n$  for all  $n$  and therefore  $\frac{1}{2} \in \bigcap_{n=0}^{\infty} U_n$ . Let  $x \in \mathbb{R}$  such that  $x \neq \frac{1}{2}$ . Then there exists some  $m \geq 0$  such that  $|x - \frac{1}{2}| > \frac{1}{2^{m+1}}$ , which means that

$$x \notin \left( \frac{1}{2} - \frac{1}{2^{m+1}}, \frac{1}{2} + \frac{1}{2^{m+1}} \right) = \left( \frac{2^m - 1}{2^{m+1}}, \frac{2^m + 1}{2^{m+1}} \right) = U_m.$$

Since there exists an  $m \geq 0$  such that  $x \notin U_m$ , we have that  $x \notin \bigcap_{n=0}^{\infty} U_n$ , and since  $x$  was arbitrary, we have that  $\bigcap_{n=0}^{\infty} U_n = \{\frac{1}{2}\} \notin \mathcal{T}_{\text{std}}$ . Therefore the infinite intersection of open sets is not necessarily open.

## 2.3 Limit Points and Closed Sets

**Exercise 2.8.** In the indiscrete topology on  $\mathbb{R}$ , the point 0 is in  $\mathbb{R}$  but not  $\emptyset$ , so since  $(\mathbb{R} - \{0\}) \cap (1, 2) = (1, 2) \neq \emptyset$ , we have that  $(U - \{0\}) \cap (1, 2) \neq \emptyset$  for all open sets  $U$  containing 0. Therefore 0 is a limit point of  $(1, 2)$  in the indiscrete topology. In the finite complement topology, let  $U \subset \mathbb{R}$  be an open set containing 0. Then suppose for contradiction that  $(U - \{0\}) \cap (1, 2) = \emptyset$ . Then for all  $p \in (1, 2) \subset \mathbb{R}$ , we have that  $p \notin U - \{0\}$ , which means that  $p \in \mathbb{R} - (U - \{0\}) = (\mathbb{R} - U) \cup \{0\}$ . Therefore  $(1, 2) \subset (\mathbb{R} - U) \cup \{0\}$ , but this is a contradiction since  $U$  is open and therefore we have that an infinite set is a subset of a finite set. Hence it must be the case that  $(U - \{0\}) \cap (1, 2) \neq \emptyset$  for all  $U$  containing 0, and therefore 0 is a limit point of  $(1, 2)$ . In the standard topology and in the discrete topology, the set  $(-1, 1)$  is an open set containing 0, but  $(-1, 1) \cap (1, 2) = \emptyset$ , so 0 is not a limit point of  $(1, 2)$ .

**Theorem 2.9.** Suppose  $p \notin A$  in a topological space  $(X, \mathcal{T})$ . Then  $p$  is not a limit point of  $A$  iff there exists a neighborhood  $U$  of  $p$  such that  $A \cap U = \emptyset$ .

*Proof.* Suppose  $p \notin A$  is not a limit point of  $A$ . Then there exists an open set  $U$  containing  $p$  (a neighborhood  $U$  of  $p$ ) such that  $(U - \{p\}) \cap A = \emptyset$ . Since  $p \notin A$ , we have that  $p \notin A \cap U$ , and therefore  $A \cap U = \emptyset$ , as required. Now suppose there exists a neighborhood  $U$  of  $p$  such that  $A \cap U = \emptyset$ . Then since  $p \notin A$ , we have that  $(U - \{p\}) \cap A = \emptyset$  as well, and therefore  $p$  is not a limit point of  $A$ .  $\square$

**Exercise 2.10.** If  $p$  is an isolated point of  $A$  in a topological space  $(X, \mathcal{T})$ , then by the definition of an isolated point, we have that  $p \in A$  but  $p$  is not a limit point of  $A$ . Therefore there exists

an open set  $U$  such that  $(U - \{p\}) \cap A = \emptyset$ , and since  $p \in A, U$ , we have that

$$U \cap A = ((U - \{p\}) \cap A) \cup \{p\} \cap A = \emptyset \cup \{p\} = \{p\}.$$

Therefore if  $p$  is an isolated point of  $A$ , there exists an open set  $U$  such that  $A \cap U = \{p\}$ .

**Exercise 2.11.** Let  $X = \{a, b, c, d\}$  be a set, let  $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$  be a topology on  $X$ , and let  $A = \{b, c\}$  be a set.

(1)  $c \in A$  is a limit point of  $A$  since the only open set containing  $c$  is  $X$ , and we have that  $(X - \{c\}) \cap A = \{b\} \neq \emptyset$ .

(2)  $d \notin A$  is a limit point of  $A$  also because  $X$  is the only open set containing  $d$  and  $(X - \{d\}) \cap A = \{b, c\} \neq \emptyset$ .

(3)  $b \in A$  is an isolated point of  $A$  because it is in  $A$  but is not a limit point since  $\{a, b\}$  is open but  $(\{a, b\} - \{b\}) \cap A = \emptyset$ .

(4)  $a \notin A$  is not a limit point of  $A$  since  $\{a\}$  is an open set but  $(\{a\} - \{a\}) \cap A = \emptyset$ .

**Exercise 2.12.** Let  $X$  be a set and let  $\mathcal{T}$  be a topology on  $X$ . If the set  $X$  has a limit point  $p$ , then  $p \in X$ , so  $\bar{X} = X$  and  $X$  is closed. If Now let  $p \in X$  and let  $U$  be an open set containing  $p$ . Then  $(U - \{p\}) \cap \emptyset = \emptyset$ , so there are no limit points of the empty set, and therefore, vacuously,  $\bar{\emptyset} = \emptyset$  and the empty set is closed.

(1) In the discrete topology, all sets are closed. Let  $A$  be a nonempty proper subset of  $X$  and let  $p \in X$  be arbitrary. Then there exists a point  $q \in X$  with  $q \notin A$ , and since all sets are open in the discrete topology, then set  $\{p, q\}$  is a neighborhood of  $p$  that satisfies  $(\{p, q\} - \{p\}) \cap A = \emptyset$ . Thus we have shown that there are no limit points of  $A$ , and so vacuously,  $\bar{A} = A$ .

(2) In the indiscrete topology, only  $X$  and  $\emptyset$  are closed. Let  $A$  be a nonempty proper subset of  $X$  and let  $p \in X$  such that  $p \notin A$ . Then  $p$  is a limit point of  $A$  since  $X$  is the only open set containing  $p$  and it satisfies  $(X - \{p\}) \cap A \neq \emptyset$ . However,  $p \notin A$ , so  $\bar{A} \neq A$ .

**Theorem 2.13.** For any topological space  $(X, \mathcal{T})$ , and  $A \subset X$ , the set  $\bar{A}$  is closed, that is, for any set  $A$  in a topological space,  $\overline{\bar{A}} = \bar{A}$ .

*Proof.* Let  $A$  be a set in a topological space  $(X, \mathcal{T})$ . Since the closure of a set contains all the points in the set, we have that  $\bar{A} \subset \overline{\bar{A}}$ . Now let  $x \in \overline{\bar{A}}$  and let  $U$  be an arbitrary neighborhood of  $x$ . Either  $x \in \bar{A}$  (in which case we're done), or  $x$  is a limit point of  $\bar{A}$ , in which case we have that  $(U - \{x\}) \cap \bar{A} \neq \emptyset$ . Since  $(U - \{x\}) \cap \bar{A} \subset U \cap \bar{A}$ , so  $U \cap \bar{A}$  is nonempty in either case. Therefore there exists some  $y \in U \cap \bar{A}$ , which means we have a neighborhood  $U$  of  $y$

such that  $y \in \bar{A}$ . As before, either  $y \in A$  or  $y$  is a limit point of  $A$ , in which case we have that  $\emptyset \neq (U - \{y\}) \cap A \subset U \cap A$ . Therefore for an arbitrary neighborhood  $U$  of  $x \in \bar{\bar{A}}$ , we have shown that  $U \cap A$  is nonempty. Either  $x \in A \subset \bar{A}$ , or  $x \notin A$  and therefore  $x \notin U \cap A \neq \emptyset$ . Since this intersection is nonempty, we have also that  $(U - \{x\}) \cap A \neq \emptyset$ , but  $U$  was an arbitrary neighborhood of  $x$ , so we have shown that  $(U - \{x\}) \cap A$  is nonempty for all neighborhoods  $U$  of  $x$ . Therefore  $x$  is a limit point of  $A$ , and so  $x \in \bar{A}$ . In both cases,  $x \in \bar{A}$  and so  $\bar{\bar{A}} \subset \bar{A}$ , and since  $\bar{A} \subset \bar{\bar{A}}$  also, we have that  $\bar{\bar{A}} = \bar{A}$ , as required.  $\square$

**Theorem 2.14.** For any topological space  $(X, \mathcal{T})$ , a subset  $A \subset X$  is closed if and only if  $X - A$  is open.

*Proof.* Suppose  $A$  is closed and let  $x \in X - A$ . Then since  $A$  is closed, it contains all its limit points and therefore  $x$  is not a limit point. This means there exists an open set  $U$  such that  $U - \{x\} \cap A = \emptyset$ . Let  $y \in U - \{x\}$ . Then  $y \notin A$ , so  $y \in X - A$ , which means  $U - \{x\} \subset X - A$ . Since  $x$  was an arbitrary element of  $X - A$ , by Theorem 2.3, we have that  $X - A$  is open. Now suppose that  $X - A$  is open.  $A \subset \bar{A}$ , and we will show that  $\bar{A} \subset A$ . Suppose for contradiction that  $\bar{A} \not\subset A$ . Then there exists an  $a \in \bar{A} - A$ , which means that  $a$  is a limit point of  $A$ , so all open intervals  $U$  containing  $a$  satisfy  $U \cap A \neq \emptyset$  (by Theorem 2.9). In particular, since  $X - A$  is open, we have that  $(X - A) \cap A \neq \emptyset$ , which is a contradiction. Therefore  $\bar{A} = A$  and  $A$  is closed.  $\square$

**Corollary 2.14.** If  $A$  is an open set, then  $X - (X - A)$  is open, which means  $X - A$  is closed.

**Theorem 2.15.** For any topological space  $(X, \mathcal{T})$  with an open set  $U \in \mathcal{T}$  and a closed set  $A \in \mathcal{T}$ ,  $U - A$  is open and  $A - U$  is closed.

*Proof.* Since  $A$  is closed,  $X - A$  is open, and so  $U \cap (X - A)$  is also open since the intersection of two open sets is open. Then we have that  $X - (U \cap (X - A))$  is closed by the corollary to Theorem 2.14.  $X - (U \cap (X - A)) = X - (U - A)$  is closed, so  $U - A$  is open, as claimed. The union of open sets is open, so we also have that  $U \cup (X - A)$  is open.  $U \cup (X - A) = X - (A - U)$  is open, so  $A - U$  is closed, as claimed.  $\square$

**Theorem 2.16.** Let  $(X, \mathcal{T})$  be a topological space. Then:

- (i)  $\emptyset$  is closed.
- (ii)  $X$  is closed.
- (iii) The union of finitely many sets is closed.
- (iv) Let  $\{A_\alpha\}_{\alpha \in \lambda}$  be a collection of closed sets in  $(X, \mathcal{T})$ . Then  $\bigcap_{\alpha \in \lambda} A_\alpha$  is closed.

*Proof.* For (i) and (ii), see exercise 2.12. For (iii), let  $\{A_i\}$ ,  $1 \leq i \leq n$  for some  $n \in \mathbb{N}$ . Then for each  $A_i$ ,  $X - A_i$  is open, and we have that the intersection of finitely many open sets is open, so  $\bigcap_{i=1}^n (X - A_i)$  is open. By the corollary to Theorem 2.14 and DeMorgan's Laws,

$$X - \left( \bigcap_{i=1}^n (X - A_i) \right) = X - \left( X - \bigcup_{i=1}^n A_i \right) = \bigcup_{i=1}^n A_i$$

is closed. For (iv), let  $\{A_\alpha\}_{\alpha \in \lambda}$  be a collection of closed sets in  $(X, \mathcal{T})$ . Then for each  $A_\alpha$ ,  $X - A_\alpha$  is open, and we have that the union of a collection of open sets is open, so  $\bigcup_{\alpha \in \lambda} (X - A_\alpha)$  is open. Again by DeMorgan's Laws,

$$X - \left( \bigcup_{\alpha \in \lambda} (X - A_\alpha) \right) = X - \left( X - \bigcap_{\alpha \in \lambda} A_\alpha \right) = \bigcap_{\alpha \in \lambda} A_\alpha$$

is closed. □

**Exercise 2.19.** (1) In  $\mathbb{Z}$  with the finite complement topology, the set  $\{0, 1, 2\}$  is not open, since  $\mathbb{Z} - \{0, 1, 2\}$  is infinite.  $\mathbb{Z} - \{0, 1, 2\}$ , however, is open since  $\{0, 1, 2\}$  is finite, and therefore  $\mathbb{Z} - (\mathbb{Z} - \{0, 1, 2\}) = \{0, 1, 2\}$  is closed. The set of prime numbers has an infinite complement, so it is not open, but there are infinitely many prime numbers so it is also not closed. The set  $\{n : |n| > 10\}$  has a finite complement, so it is open, but the set itself is infinite and therefore is not closed.

(2) In  $\mathbb{R}$  with the standard topology, the set  $(0, 1)$  is open, and its limit points are 0 and 1, neither of which are in  $(0, 1)$ , so it is not closed. The set  $(0, 1]$  is neither closed nor open, since it contains one of its limit points but not both. The set  $[0, 1]$  contains both limit points and is therefore closed, and it is not open. The set  $\{0, 1\}$  has no limit points so is vacuously closed, and it is not open. The set  $\{\frac{1}{n} : n \in \mathbb{N}\}$  is not open since there is no  $\varepsilon > 0$  such that  $(1 - \varepsilon, 1 + \varepsilon) \subset \{\frac{1}{n} : n \in \mathbb{N}\}$ . Note that 0 is a limit point of this set since for any  $\varepsilon > 0$ , there exists an  $n_0 \in \mathbb{N}$  such that  $\frac{1}{n_0} < \varepsilon$  and therefore  $\frac{1}{n_0} \in ((-\varepsilon, \varepsilon) - \{0\}) \cap \{\frac{1}{n} : n \in \mathbb{N}\} \neq \emptyset$ . Since  $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$ , the set is not closed in  $(\mathbb{R}, \mathcal{T}_{\text{std}})$ .

(3) In  $\mathbb{R}^2$  with the standard topology, the set  $C = \{(x, y) : x^2 + y^2 = 1\}$  is not open since if  $p \in C$  and  $\varepsilon_p > 0$ , then  $p - (\frac{\varepsilon_p}{2}, \frac{\varepsilon_p}{2})$  is in  $B(p, \varepsilon_p)$  but not in  $C$ . If  $p = (\cos \theta_0, \sin \theta_0) \in C$ , then  $p$  is a limit point of  $C$  since the point  $(\cos \theta_1, \sin \theta_1) \in C$  and if  $|\theta_1 - \theta_0| < \arccos\left(1 - \frac{\varepsilon_p^2}{2}\right)$ , then  $(\cos \theta_1, \sin \theta_1) \in (B(p, \varepsilon_p) - \{p\}) \cap C \neq \emptyset$ . If  $p \notin C$ , then set  $\varepsilon$  to be the distance from  $p$  to the nearest point of  $C$ . We have that  $(B(p, \frac{\varepsilon}{2}) - \{p\}) \cap C = \emptyset$ , so all points of  $C$  are limit points and all points not in  $C$  are not limit points, which means  $\overline{C} = C$  and therefore  $C$  is closed. Let

$D = \{(x, y) : x^2 + y^2 < 1\}$ . Then  $C$  is the set of all the limit points of  $D$ , and so  $C \cup D = \bar{D}$  is closed. Therefore,  $\{(x, y) : x^2 + y^2 > 1\} = \mathbb{R}^2 - \bar{D}$  is open, and its limit points are also all in the set  $C$  and therefore this set is not closed. The set  $D$  is open since  $D = B(0, 1) \in \mathcal{T}_{\text{std}}$ , and so the set  $\{(x, y) : x^2 + y^2 \geq 1\} = \mathbb{R}^2 - D$  is closed. This set is not open since there is no  $\varepsilon$  such that  $B(p, \varepsilon) \subset \{(x, y) : x^2 + y^2 \geq 1\}$  for  $p \in \{(x, y) : x^2 + y^2 \geq 1\} \cap C$ .

**Theorem 2.20.** For any set  $A$  in a topological space  $(X, \mathcal{T})$ , the closure of  $A$  is the intersection of all closed sets containing  $A$ , that is,  $\bar{A} = \bigcap_{B \supset A, B \in \mathfrak{C}} B$ , where  $\mathfrak{C}$  is the set of all closed sets in  $(X, \mathcal{T})$ .

*Proof.* Let  $\mathfrak{C}$  be the set of all closed sets in  $(X, \mathcal{T})$  and let  $A$  be a subset of  $X$ . Since its closure  $\bar{A}$  is closed and  $A \subset \bar{A}$ , we have that  $\bar{A} \in \mathfrak{C}$ , and since for any sets  $M$  and  $N$  we have that  $M \cap N \subset M, N$ , we have that  $\bigcap_{B \supset A, B \in \mathfrak{C}} B \subset \bar{A}$ . To show equality, then, we need only show that  $\bar{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$ . Let  $a \in \bar{A}$ . There are two cases. In the first case, if  $a \in A$ , then  $a \in B$  for all  $B \in \mathfrak{C}$  such that  $A \subset B$ , which means that  $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$ . In the second case, we have that  $a \notin A$ , which means that  $a \in \bar{A} - A$  is a limit point of  $A$ . This means that for all open sets  $U$  with  $a \in U$ , we have that  $(U - \{a\}) \cap A \neq \emptyset$ . Since  $a \notin A$ , we have that  $U \cap A = (U - \{a\}) \cap A \neq \emptyset$ . Let  $B_0 \in \mathfrak{C}$  with  $A \subset B_0$  and suppose for contradiction that  $a \notin B_0$ . Since  $B_0$  is closed,  $X - B_0$  is open, and since  $a \notin B_0$ ,  $a \in X - B_0$ , which means that  $(X - B_0) \cap A \neq \emptyset$ . Therefore there exists some  $y \in (X - B_0) \cap A$ . We have that  $y \in A \subset B_0$ , which means that  $y \notin X - B_0$ , a contradiction. We have reached a contradiction by assuming that there exists a set  $B_0 \in \mathfrak{C}$  such that  $A \subset B_0$  and  $a \notin B_0$ , which means that  $a \in B$  for all  $B \in \mathfrak{C}$  such that  $A \subset B$ . Therefore in both cases  $a \in A$  and  $a \in \bar{A} - A$ , we have that  $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$ , which means that  $\bar{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$ . This shows that  $\bar{A}$  is indeed the intersection of all closed sets containing  $A$ , as required.  $\square$

**Exercise 2.21.** Consider the set  $H = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ . In the discrete topology, this set is already closed and so is its own closure. In the indiscrete topology, only  $\mathbb{R}$  and  $\emptyset$  are closed, so the closure of  $H$  in the indiscrete topology is  $\mathbb{R}$  itself. In the finite complement topology, let  $p \in \mathbb{R}$  be an arbitrary point. Then let  $U$  be an arbitrary open set containing  $p$ . Since  $U$  is open, its complement is finite, which means there are only finitely many points not in  $U - \{p\}$ , so there must be infinitely many points in  $(U - \{p\}) \cap H \neq \emptyset$ . Since  $U$  was an arbitrary open set,  $p$  is a limit point, and since  $p$  was an arbitrary point in  $\mathbb{R}$ , we see that all points are limit points of  $H$ , which means that  $\bar{H} = \mathbb{R}$ . In the countable complement topology,  $H$  is closed since it contains countably many elements, so it is its own closure. In the standard topology, the only limit point of  $H$  is 0, so the closure of  $H$  is  $\bar{H} = H \cup \{0\}$ .

**Theorem 2.22.** Let  $A$  and  $B$  be subsets of a topological space  $X$  with topology  $\mathcal{T}$ . Then:

- (1)  $A \subset B$  implies  $\bar{A} \subset \bar{B}$ .
- (2)  $\overline{A \cup B} = \bar{A} \cup \bar{B}$ .

*Proof.* (1) We have that  $A \subset B \subset \bar{B}$ , and by Theorem 2.20, we have that  $\bar{A}$  is a subset of all closed sets containing  $A$ . Since  $\bar{B}$  is a closed set containing  $A$ , we have that  $\bar{A} \subset \bar{B}$ , as required.

(2) Let  $c \in \bar{A \cup B}$ . Without loss of generality, assume  $c \in \bar{A}$ . There are two cases. For the first case  $c \in A$ , we have that  $c \in A \subset A \cup B \subset \overline{A \cup B}$ . For the second case  $c \in \bar{A} - A$ ,  $c$  is a limit point of  $A$  and therefore for all open sets  $U$  containing  $c$  satisfy  $\emptyset \neq (U - \{c\}) \cap A \subset (U - \{c\}) \cap (A \cup B)$ . Since  $U$  was an arbitrary open set,  $c$  is a limit point of  $A \cup B$  and therefore  $c \in \overline{A \cup B}$ , so  $\bar{A \cup B} \subset \overline{A \cup B}$ . Now let  $d \in \overline{A \cup B}$  be arbitrary. Again there are two cases. In the first case,  $d \in A \cup B$ , so without loss of generality assume  $d \in A \subset \bar{A} \subset \bar{A \cup B}$ . In the second case,  $d \in \overline{A \cup B} - (A \cup B)$ , so  $d$  is a limit point of  $A \cup B$ . This means that for all open sets  $U$  containing  $d$ , we have that  $\emptyset \neq (U - \{d\}) \cap (A \cup B) = ((U - \{d\}) \cap A) \cup ((U - \{d\}) \cap B)$ , so one of the sets on the right hand side is nonempty. Without loss of generality, assume  $(U - \{d\}) \cap A \neq \emptyset$ . Then  $d$  is a limit point of  $A$ , so  $d \in \bar{A} \subset \bar{A \cup B}$ . Therefore  $\overline{A \cup B} \subset \bar{A \cup B}$  as well, and so the two sets are equal.  $\square$

**Exercise 2.25.** I'm not entirely sure but I believe the Cantor Set fits this description.

## 2.4 Interior and Boundary

**Theorem 2.26.** Let  $A$  be a subset of a topological space  $(X, \mathcal{T})$ . Then  $p$  is an interior point if and only if there exists an open set  $U$  such that  $p \in U \subset A$ .

*Proof.* Let  $A \subset X$  be arbitrary and let  $p \in A$  be an interior point. Then since  $A^\circ$  is the union of all open subsets of  $A$ ,  $A^\circ$  is open as well, and so by Theorem 2.3, there exists an open set  $U$  such that  $p \in U \subset A^\circ \subset A$  where  $A^\circ \subset A$  follows from the fact that if  $a \in A^\circ$ , then  $a \in \bigcup_{U \subset A, U \in \mathcal{T}} U$  and therefore there exists an open set  $U_0$  such that  $a \in U_0 \subset A$ . Now let  $p \in A$  be an arbitrary point such that there is an open set  $U$  with  $a \in U \subset A$ . Since  $U$  is open,  $U \in \mathcal{T}$  and so  $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U = A^\circ$ . Thus,  $p$  is an interior point, as required.  $\square$

**Exercise 2.27.** If  $U$  is open in a topological space, then by Theorem 2.3, for every point  $x \in U$ , there exists an open set  $U_x$  such that  $x \in U_x \subset U$ , which means  $x$  is an interior point of  $U$ . If  $x$  is an interior point of  $U$ , then by Theorem 2.26, there exists a set  $U_x$  such that  $x \in U_x \subset U$  and so  $U$  is open. Therefore  $U$  is open in a topological space if and only if every point of  $U$  is an interior point.

**Lemma 2.28.** Given a set  $A$  in a topological space  $(X, \mathcal{T})$ , the closure of  $A$  is  $\bar{A} = X - (X - A)^\circ$  and the interior of  $A$  is  $A^\circ = X - \overline{X - A}$ .

*Proof.* Let  $\mathcal{C}$  be the set of all closed sets in  $(X, \mathcal{T})$ . For all  $B \in \mathcal{C}$  such that  $A \subset B$ , we have that  $X - B \subset X - A$ , and since  $B$  is closed,  $X - B$  is open. Given an open set  $U$  with  $U \subset X - A$ ,  $X - U \supset A$  is closed. This means that  $\{B \in \mathcal{C} : B \supset A\} = \{X - U : U \in \mathcal{T} \text{ s.t. } U \subset X - A\}$ . Therefore we have that

$$\begin{aligned}\bar{A} &= \bigcap_{B \supset A, B \in \mathcal{C}} B = X - \left( X - \bigcap_{B \supset A, B \in \mathcal{C}} B \right) = X - \left( \bigcup_{B \supset A, B \in \mathcal{C}} (X - B) \right) \\ &= X - \left( \bigcup_{U \subset X - A, U \in \mathcal{T}} (X - (X - U)) \right) = X - \left( \bigcup_{U \subset X - A, U \in \mathcal{T}} U \right) = X - (X - A)^\circ,\end{aligned}$$

as required. The proof that  $A^\circ = X - \overline{X - A}$  is similar □

**Theorem 2.28.** Let  $A$  be a subset of a topological space  $(X, \mathcal{T})$ . Then  $A^\circ$ ,  $\partial A$ , and  $(X - A)^\circ$  are all disjoint and their union is  $X$ .

*Proof.* Let  $a \in A^\circ$  and suppose for contradiction that  $a \in \partial A = \bar{A} \cap \overline{X - A}$ . Then either  $a \in X - A$  or  $a$  is a limit point of  $X - A$ . In the first case, we have a clear contradiction since  $a \in A^\circ \subset A$  cannot be in  $X - A$ . In the second case, every open set  $U$  containing  $a$  satisfies  $(U - \{a\}) \cap (X - A) \neq \emptyset$ . However,  $a \in A^\circ$ , so there exists an open set  $V$  such that  $a \in V \subset A$ , so we have that  $\emptyset \neq (V - \{a\}) \cap (X - A) \subset A \cap (X - A) = \emptyset$ , a contradiction. Therefore the sets  $A^\circ$  and  $\partial A$  are disjoint, and a similar argument shows that  $\partial A$  and  $(X - A)^\circ$  are disjoint as well. It remains to check that  $A^\circ$  and  $(X - A)^\circ$  are disjoint. Suppose not, so that there exists an  $a \in A^\circ$  such that  $a \in (X - A)^\circ$ . By Theorem 2.26, there exists an open set  $U_1$  containing  $a$  such that  $U_1 \subset A$  and an open set  $U_2$  containing  $a$  such that  $U_2 \subset X - A$ . Therefore we have that  $a \in U_1 \cap U_2 \subset A \cap (X - A) = \emptyset$ , a contradiction showing that  $A^\circ$  and  $(X - A)^\circ$  must be disjoint. Since  $A^\circ = X - \overline{X - A}$  by Lemma 2.28, we have that

$$\begin{aligned}\bar{A} - A^\circ &= \bar{A} - (X - \overline{X - A}) \\ &= \bar{A} \cap (X - (X - \overline{X - A})) \\ &= \bar{A} \cap \overline{X - A} = \partial A,\end{aligned}$$

so we see that the closure of  $A$  is the disjoint union of the boundary of  $A$  and the interior of  $A$ .

Again using Lemma 2.28 ( $\bar{A} = X - (X - A)^\circ$ ), we also have that

$$\begin{aligned} X &= \bar{A} \cup (X - \bar{A}) = A^\circ \cup \partial A \cup (X - \bar{A}) \\ &= A^\circ \cup \partial A \cup (X - (X - (X - A)^\circ)) \\ &= A^\circ \cup \partial A \cup (X - A)^\circ. \end{aligned}$$

Therefore we have that  $A^\circ$ ,  $\partial A$ ,  $(X - A)^\circ$  are disjoint sets whose union is  $X$ .  $\square$

**Exercise 2.29.** Again consider the set  $H = \{\frac{1}{n} : n \in \mathbb{N}\}$ . In the discrete topology, we have that  $\bar{H} = H$  and  $\overline{\mathbb{R} - H} = \mathbb{R} - H$ , and so we have that  $\partial H = \emptyset$ . The interior of  $H$  is  $H^\circ = \bar{H} - \partial H = H - \emptyset = H$ . In the indiscrete topology, we have that  $\bar{H} = \mathbb{R} = \overline{\mathbb{R} - H}$ , so  $\partial H = \mathbb{R}$  and  $H^\circ = \bar{H} - \partial H = \mathbb{R} - \mathbb{R} = \emptyset$ . In the finite complement topology, we saw in Exercise 2.21 that the closure of  $H$  was  $\bar{H} = \mathbb{R}$  and by the same reasoning,  $\overline{\mathbb{R} - H} = \mathbb{R}$ . Therefore once again we have that  $\partial H = \mathbb{R}$  and  $H^\circ = \emptyset$ , which makes sense since  $\emptyset$  is the largest open set contained within  $H$  since  $H$  is countably infinite. In the countable complement topology,  $H = \bar{H}$  is the disjoint union of  $H^\circ$  and  $\partial H$ . If  $V$  is a nonempty open set in the countable complement topology, then  $\mathbb{R} - V$  is countable, meaning that  $V$  is uncountable, so we cannot have  $V \subset H$ . The only open set  $U$  with  $U \subset H$  is  $U = \emptyset$ . Therefore  $H^\circ = \emptyset$  and  $\partial H = H$ . In the standard topology, no open set contains 1 and  $\frac{1}{2}$  without also containing all points in the open set  $(\frac{1}{2}, 1)$ , which means that no open set is a subset of  $H$ . Therefore we have that  $H^\circ = \emptyset$ , and so  $\partial H = \bar{H} = H \cup \{0\}$ .

## 2.5 Convergence of Sequences

**Theorem 2.30.** Let  $A$  be a set in a topological space  $(X, \mathcal{T})$ , and let  $p$  be a point in  $X$ . If  $\{x_i\}_{i \in \mathbb{N}} \subset A$  and  $x_i \rightarrow p$ , then  $p \in \bar{A}$ .

*Proof.* Suppose for contradiction that  $\{x_i\}_{i \in \mathbb{N}} \subset A$ ,  $x_i \rightarrow p$ , but  $p \notin \bar{A}$ . Then  $p$  is not a limit point of  $A$ , so there exists an open set  $U_0$  containing  $p$  such that  $(U_0 - \{p\}) \cap A = \emptyset$ . Since  $x_i \rightarrow p$  and  $p \in U_0$ , we have that there exists an  $N \in \mathbb{N}$  such that  $x_i \in U_0$  for all  $i > N$ . Since  $x_i \in A$  for all  $i \in \mathbb{N}$ ,  $x_i \neq p$  for all  $i \in \mathbb{N}$ . Therefore, we have that for all  $i > N$ ,  $x_i \in (U_0 - \{p\}) \cap A = \emptyset$ , a contradiction. Therefore if  $\{x_i\}_{i \in \mathbb{N}} \subset A$  and  $x_i \rightarrow p$ , then  $p \in \bar{A}$ . In particular,  $p$  is a limit point of  $A$ .  $\square$

**Theorem 2.31.** In the standard topology on  $\mathbb{R}^n$ , if  $p$  is a limit point of a set  $A$ , then there is a sequence of points in  $A$  that converges to  $p$ .



*Proof.* If  $p$  is a limit point of  $A$ , then  $A \neq \emptyset$ , so there exists a point  $a \in A$ . Set  $\varepsilon = d(a, p) > 0$ . Then for all  $n \in \mathbb{N}$ ,  $B(p, \frac{\varepsilon}{n})$  is open. Since  $p$  is a limit point and  $p \in B(p, \frac{\varepsilon}{n})$ , we have that  $(B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A \neq \emptyset$ . Since each of these sets is nonempty there exists an  $a_n \in (B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A$  for all  $n$ . Define a sequence  $(x_n)_{n \in \mathbb{N}}$  such that  $x_n = a_n$ . Therefore  $\{x_n\}_{n \in \mathbb{N}} \subset A$ , so it only remains to show that the sequence converges to  $p$ . Let  $U$  be an open set containing  $p$ . Since  $U$  is open in the standard topology,  $U = B(p, \varepsilon_p)$  for some  $\varepsilon_p > 0$ , and there exists and  $N \in \mathbb{N}$  such that  $\frac{\varepsilon}{N} < \varepsilon_p$ . Let  $i > N$ . We have that

$$x_i \in B\left(p, \frac{\varepsilon}{i}\right) \subset B\left(p, \frac{\varepsilon}{N}\right) \subset B(p, \varepsilon_p),$$

which means that  $x_i \in U$  for all  $i > N$ , and therefore  $x_i \rightarrow p$ , as required.  $\square$

**Exercise 2.32.** Consider  $\mathbb{R}$  with the indiscrete topology. Let  $p$  be a point in  $\mathbb{R}$  and set  $x_n = n$ . This sequence converges to  $p$  since if  $U$  is an open set containing  $p$ , then  $U = \mathbb{R}$  and so we have that  $x_n \in U$  for all  $n \in \mathbb{N}$ . Therefore  $x_n \rightarrow p$ , but  $p$  was arbitrary, so we see that this sequence converges to every point in  $\mathbb{R}$ .

## 3 Bases, Subspaces, Products: Creating New Spaces

### 3.1 Bases

**Theorem 3.1.** Let  $(X, \mathcal{T})$  be a topological space and let  $\mathcal{B}$  be a collection of subsets of  $X$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{T}$  if and only if

- (1)  $\mathcal{B} \subset \mathcal{T}$ , and
- (2) for every open set  $U$ , and point  $p \in U$ , there exists a set  $V \in \mathcal{B}$  such that  $p \in V \subset U$ .

*Proof.* Let  $\mathcal{B}$  be a basis for  $\mathcal{T}$ . Then by definition, (1) is satisfied. Now let  $U$  be an arbitrary open set and let  $p \in U$  be an arbitrary point. Since  $\mathcal{B}$  is a basis, we have that  $U = \bigcup_{V \in \lambda} V$  for some collection of sets  $\lambda \subset \mathcal{B}$ , and since  $p \in U$ , we have that  $p \in V_0 \subset U$  for some set  $V_0 \in \lambda$ , so (2) is satisfied as well. Now suppose that  $\mathcal{B}$  is a collection of sets satisfying (1) and (2). Let  $U_0$  be an arbitrary open set and define  $\lambda$  to be the collection of sets  $\lambda = \{V \in \mathcal{B} : V \subset U_0\} \subset \mathcal{B}$ . Now let  $p \in U_0$  be an arbitrary point in  $U_0$ . By (2), there exists a  $V_0 \in \mathcal{B}$  such that  $p \in V_0 \subset U_0$ , which means  $V_0 \in \lambda$ . Therefore  $p \in \bigcup_{V \in \lambda} V$ , and so  $U_0 \subset \bigcup_{V \in \lambda} V$ . Now let  $p$  be an arbitrary point in  $\bigcup_{V \in \lambda} V$ . Then there exists a  $V_0 \in \lambda \subset \mathcal{B}$  such that  $p \in V_0$ , and since  $V_0 \in \lambda$ ,  $V_0 \subset U_0$ , so  $p \in U_0$ . Therefore  $U_0 = \bigcup_{V \in \lambda} V$  since both sets are subsets of the other. Since any arbitrary open set  $U$  is the union of sets in the collection  $\mathcal{B}$ ,  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .  $\square$

**Exercise 3.2.**  $\mathcal{B}_1$  satisfies Theorem 3.1(1) since it consists only of open intervals and therefore  $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$ . Since  $U$  is open in  $\mathcal{T}_{\text{std}}$ ,  $U = (c - \varepsilon, c + \varepsilon)$  for some  $c \in \mathbb{R}$  and  $\varepsilon > 0$ . But since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exist  $a, b \in \mathbb{Q}$  such that  $c - \varepsilon < a < p$  and  $p < b < c + \varepsilon$ . We have that  $p \in (a, b) \subset U$  and  $(a, b) \in \mathcal{B}_1$ , so Theorem 3.2(2) is satisfied and therefore  $\mathcal{B}_1$  is a basis.

**Theorem 3.3.** Let  $X$  be a set and let  $\mathcal{B}$  be a collection of subsets of  $X$ . Then  $\mathcal{B}$  is a basis for some topology on  $X$  if and only if

- (1) each point of  $X$  is in some element of  $\mathcal{B}$ , and
- (2) if  $U$  and  $V$  are sets in  $\mathcal{B}$  and  $p$  is a point in  $U \cap V$ , there is a set  $W$  in  $\mathcal{B}$  such that  $p \in W \subset (U \cap V)$ .

*Proof.* If  $\mathcal{B}$  is a basis for some topology on  $X$ , then  $X = \bigcup_{B \in \mathcal{B}} B$ , so if  $p \in X$ , then  $p \in B_0$  for some  $B_0 \in \mathcal{B}$ . Therefore (1) is satisfied. Now let  $U$  and  $V$  be sets in  $\mathcal{B}$  and let  $p$  be an arbitrary point in  $U \cap V$ . Since  $U$  and  $V$  are in  $\mathcal{B}$ , they are open, and therefore  $U \cap V$  is open, so by Theorem 3.1, there exists a  $W \in \mathcal{B}$  such that  $p \in W \subset (U \cap V)$ , satisfying (2). Now suppose  $\mathcal{B}$  is a collection of subsets of  $X$  satisfying (1) and (2). Then  $\emptyset$  is the empty union of sets in  $\mathcal{B}$ , and  $X = \bigcup_{B \in \mathcal{B}} B$  by (1). Suppose  $U$  and  $V$  are in  $\mathcal{B}$  and set  $\lambda = \{W \in \mathcal{B} : W \subset (U \cap V)\} \subset \mathcal{B}$ . If  $p$  is an arbitrary point of  $U \cap V$ , then by (2) there exists a  $W_0$  such that  $p \in W_0 \subset (U \cap V)$ , so  $p \in \bigcup_{W \in \lambda} W$ . If  $p$  is an arbitrary point of  $\bigcup_{W \in \lambda} W$ , then there exists a  $W_0 \in \mathcal{B}$  such that  $p \in W_0 \subset (U \cap V)$ . Therefore  $U \cap V = \bigcup_{W \in \lambda} W$ , so the intersection of two sets that are the unions of sets in  $\mathcal{B}$  is also the union of sets in  $\mathcal{B}$ . Now let  $\alpha$  be a collection of sets in  $\mathcal{B}$ . Then for each  $\beta \in \alpha$ ,  $\beta = \bigcup_{B \in \lambda} B$  for some collection of sets  $\lambda \subset \mathcal{B}$ . Then we have that  $\bigcup_{\beta \in \alpha} \beta$  is the union of a collection of unions of sets in  $\mathcal{B}$  and is therefore itself the union of sets in  $\mathcal{B}$ . Therefore  $\mathcal{B}$  is a basis for some topology on  $X$  since the collection of sets that are unions of sets in  $\mathcal{B}$  satisfies all four properties of a topology.  $\square$

**Exercise 3.4.** Let  $\mathcal{B}_{\text{LL}}$  be the set of subsets of  $\mathbb{R}$  of the form  $[a, b)$ . Then if  $x \in \mathbb{R}$ ,  $x \in [x, x + 1)$ , so Theorem 3.3(1) is satisfied. Let  $U$  and  $V$  be arbitrary sets in  $\mathcal{B}_{\text{LL}}$ . Then there exist  $a_1, a_2, b_1, b_2 \in \mathbb{R}$  such that  $U = [a_1, b_1)$  and  $V = [a_2, b_2)$ . Let  $x \in U \cap V$  be arbitrary. Then  $a_1, a_2 \leq x < b_1, b_2$ , so  $p \in W = [\max\{a_1, a_2\}, \min\{b_1, b_2\}) = U \cap V$ . Therefore Theorem 3.3(2) is satisfied and so  $\mathcal{B}_{\text{LL}}$  is a basis for a topology on  $\mathbb{R}$ . ( $\mathbb{R}$  together with this topology is the Sorgenfrey Line,  $\mathbb{R}_{\text{LL}}$ .)

**Exercise 3.6.** As discussed in Exercise 2.21, the set  $H = \{\frac{1}{n} : n \in \mathbb{N}\}$  is not closed in  $\mathbb{R}$  with the standard topology, which means that  $\mathbb{R} - H$  is not open in the standard topology. However,  $H$  contains countably many points, so  $\mathbb{R} - H$  is open in the countable complement topology.

Therefore we have that the standard topology on  $\mathbb{R}$  is not finer than the countable complement topology. On the other hand, the set  $(0, 1)$  is open in the standard topology, but not open in the countable complement topology, so we also have that the countable complement topology is not finer than the standard topology.

**Exercise 3.7.** Let  $\mathbb{R}_{+00}$  be the set of all positive real numbers ( $\mathbb{R}_+$ ) together with the points  $0'$  and  $0''$ , and let  $\mathcal{B}$  be the set of all intervals of the form  $(a, b)$ ,  $(0, b) \cup \{0'\}$ , or  $(0, b) \cup \{0''\}$  for  $a, b \in \mathbb{R}_+$ . We claim that  $\mathcal{B}$  is the basis for some topology  $\mathcal{T}$ . For any  $x \in \mathbb{R}_+$ ,  $x \in (\frac{x}{2}, x + 1)$ ,  $0' \in (0, 1) \cup \{0'\}$ , and  $0'' \in (0, 1) \cup \{0''\}$ , so every point in  $\mathbb{R}_{+00}$  is in some element of  $\mathcal{B}$ . Now let  $U$  and  $V$  be arbitrary elements of  $\mathcal{B}$  and let  $p \in U \cap V$  be arbitrary. If one of  $U, V$  is of the form  $(0, b_1) \cup \{0'\}$  and the other is of the form  $(0, b_2) \cup \{0''\}$ , and  $p$  is an arbitrary point in  $U \cap V$ , then set  $W = (\frac{p}{2}, \min\{b_1, b_2\})$ . Then we have that  $p \in W \subset (U \cap V)$  and  $W \in \mathcal{B}$ . Otherwise, Set  $W = (U \cap V)$ . Then  $p \in W \subset (U \cap V)$ , so to show that  $\mathcal{B}$  is the basis for some topology, it remains only to show that  $W \in \mathcal{B}$ . If one of  $U, V$  is of the form  $(a, b)$  for some  $a, b \in \mathbb{R}_+$ , then  $U \cap V$  is of the same form and therefore in  $\mathcal{B}$ . For the remaining case to check, assume without loss of generality that  $U = (0, b_1) \cup \{0'\}$  and  $V = (0, b_2) \cup \{0'\}$  (the case for  $0''$  is the same). Then  $U \cap V = (0, \min\{b_1, b_2\}) \cup \{0'\} \in \mathcal{B}$ . Therefore by Theorem 3.3,  $\mathcal{B}$  is a basis for some topology on  $\mathbb{R}_{+00}$ . (This topological space is called the Double Headed Snake and will also be written as  $\mathbb{R}_{+00}$ .)

**Exercise 3.8.** In the Double Headed Snake,  $\mathbb{R}_{+00}$ , let  $p$  be an arbitrary point. If  $p = 0'$  or  $0''$ , we claim that  $\{p\}$  is closed. Without loss of generality, assume  $p = 0'$ . Then

$$\mathbb{R}_{+00} - \{p\} = (0, \infty) \cup \{0''\} = \bigcup_{n \in \mathbb{N}} ((0, n) \cup \{0''\}).$$

Since  $(0, n) \cup \{0''\} \in \mathcal{B}$ , the basis for the Double Headed Snake given in Exercise 3.7, we have that  $\mathbb{R}_{+00} - \{p\}$  is the union of elements of  $\mathcal{B}$  and is therefore open in the Double Headed Snake, which means that  $\{p\}$  is closed. Now if  $p \neq 0', 0''$ , we have that  $p \in \mathbb{R}_+$ . We have that

$$\begin{aligned} \mathbb{R}_{+00} - \{p\} &= (0, p) \cup (p, \infty) \cup \{0', 0''\} \\ &= ((0, p) \cup \{0'\}) \cup ((0, p) \cup \{0''\}) \cup \left( \bigcup_{n \in \mathbb{N}} (p, p + n) \right). \end{aligned}$$

Since  $(p, p + n) \in \mathcal{B}$  for all  $n \in \mathbb{N}$ , we again have that  $\mathbb{R}_{+00} - \{p\}$  is the union of elements of  $\mathcal{B}$  and therefore is open in the Double Headed Snake, so  $\{p\}$  is closed.

Suppose for contradiction that  $U$  and  $V$  are disjoint open sets in the Double Headed Snake such that  $0' \in U$  and  $0'' \in V$ . Since  $0' \in U$ , and  $U$  is open, it is the union of sets in  $\mathcal{B}$ , so there exists a set of the form  $(0, b_1) \cup \{0'\} \subset U$ . Similarly, there exists a set of the form  $(0, b_2) \cup \{0''\} \subset V$ . Therefore  $(0, \min\{b_1, b_2\}) \subset U \cap V$ , so the sets are not disjoint since  $b_1, b_2 \in \mathbb{R}_+$ . This means it is impossible to have disjoint open sets each containing a different head of the snake.

**Exercise 3.9.** (1) In the topological space  $\mathbb{R}_{\text{har}}$ , the set  $\mathbb{R} - H$  is open since it is the union of sets in the basis for  $\mathbb{R}_{\text{har}}$ :

$$\mathbb{R} - H = \bigcup_{n \in \mathbb{N}} ((-n, n) - H).$$

Since  $\mathbb{R} - H$  is open in  $\mathbb{R}_{\text{har}}$ ,  $H$  is closed and therefore  $\overline{H} = H$ .

(2) If  $H^- = \{-\frac{1}{n} : n \in \mathbb{N}\}$ , then  $\overline{H^-} = H^- \cup \{0\}$ .

(3) Nope!

**Exercise 3.10.** (1) Let  $\mathbb{H}_{\text{bub}}$  be the upper half plane with the Sticky Bubble Topology. Let  $Q = \{(x, 0) : x \in \mathbb{Q}\}$ . Then

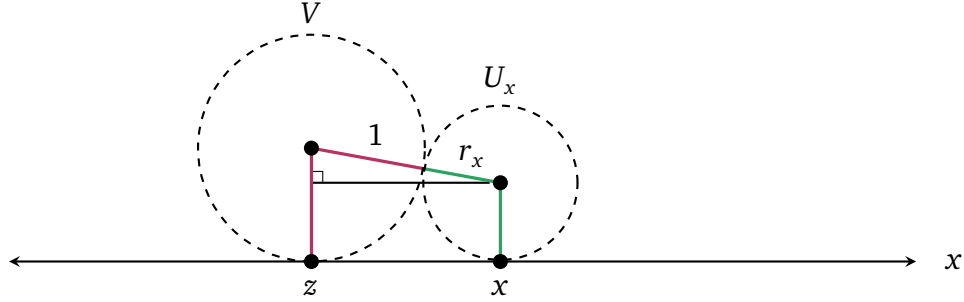
$$\mathbb{H} - Q = \bigcup_{x \in (\mathbb{R} - \mathbb{Q})} \left( \bigcup_{n \in \mathbb{N}} (B((x, n), n) \cup \{(x, 0)\}) \right)$$

is the union of sets in the basis for the Sticky Bubble Topology, so it is open. Therefore  $Q$  is closed and  $\overline{Q} = Q$ .

(2) This is similar to (1) since any subset of the  $x$ -axis can be treated the way  $\mathbb{Q}$  was in the previous example.

(3) If  $A$  is a countable subset of the  $x$ -axis, and  $z$  is a point on the  $x$ -axis not in  $A$ , we wish to show that there are disjoint open sets  $U$  and  $V$  such that  $A \subset U$  and  $z \in V$ . Set  $V = B((z, 1), 1) \cup \{(z, 0)\}$ . Now for all  $x \in A$ , set  $r_x = (x - z)^2 / 4$  and  $U_x = B((x, r_x), r_x) \cup \{(x, 0)\}$ .  $U_x$  is open for all  $x$ , so the set  $U = \bigcup_{x \in A} U_x$  is also open. Clearly  $A \subset U$ , so it remains to check that  $U \cap V \neq \emptyset$ , which is the case provided none of the  $U_x$  bubbles overlap with the bubble  $V$ . Consider an arbitrary  $x$  and corresponding bubble  $U_x$ :

The distance from  $(z, 1)$  to  $(x, r_x)$  is  $\left( (1 - r_x)^2 + (x - z)^2 \right)^{\frac{1}{2}}$ , and since  $U_x$  and  $V$  are open, we



have that  $U_x \cap V \neq \emptyset$  since:

$$\begin{aligned}
 r_x &= \frac{(x-z)^2}{4} \\
 \iff (x-z)^2 &= 4r_x \\
 \iff 1 + r_x^2 + (x-z)^2 &= 1 + 4r_x + r_x^2 \\
 \iff 1 - 2r_x + r_x^2 + (x-z)^2 &= 1 + 2r_x + r_x^2 \\
 \iff (1 - r_x)^2 + (x-z)^2 &= (1 + r_x)^2
 \end{aligned}$$

Taking square roots shows that the distance between the centers of  $U_x$  and  $V$  is the same as the sum of the radii of  $U_x$  and  $V$ , so the bubbles overlap at a single point that is not contained in either open set. This was the case for an arbitrary  $x \in A$ , so it is true for all  $x \in A$ . Therefore we have found two open sets  $U$  and  $V$  such that  $A \subset U$ ,  $z \in V$ , and  $U \cap V = \emptyset$ . We did not use the fact that  $A$  contains countably many points, so this holds for all subsets of the  $x$ -axis.

**Exercise 3.11.** Let  $\mathbb{Z}_{\text{arith}}$  be the integers  $\mathbb{Z}$  together with a topology generated by a basis of arithmetic progressions (the basis  $\mathcal{B}$  is the collection of all sets of the form  $\{az + b : z \in \mathbb{Z}\}$  for  $a, b \in \mathbb{Z}$  with  $a \neq 0$ ). To show that this is indeed a basis for some topology on  $\mathbb{Z}$ , note that  $\mathbb{Z} = \{1 \cdot z + 0 : z \in \mathbb{Z}\}$  is itself in  $\mathcal{B}$ , so all points in  $\mathbb{Z}$  are in some element of  $\mathcal{B}$ . Now let  $U$  and  $V$  be arithmetic progressions in  $\mathcal{B}$  and let  $z_0 \in U \cap V$  be arbitrary. Then  $U = \{a_1 z + b_1 : z \in \mathbb{Z}\}$  and  $V = \{a_2 z + b_2 : z \in \mathbb{Z}\}$  for some  $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ . Now  $z_0 \in U \cap V$ , then we have that  $a_1 k_1 + b_1 = z_0 = a_2 k_2 + b_2$  for some  $k_1, k_2 \in \mathbb{Z}$ . Then  $z_0 \in W = \{\text{lcm}(a_1, a_2)z + z_0 : z \in \mathbb{Z}\} \in \mathcal{B}$ . We also have that  $W \subset U \cap V$  since if  $w_0 = \text{lcm}(a_1, a_2)k_0 + z_0 \in W$ , then

$$\begin{aligned}
 a_1 \left( k_1 + \frac{a_2 k_0}{\gcd(a_1, a_2)} \right) + b_1 &= a_1 k_1 + b_1 + \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 = \text{lcm}(a_1, a_2)k_0 + z_0 \\
 = w_0 &= \text{lcm}(a_1, a_2)k_0 + z_0 = \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 + a_2 k_2 + b_2 = a_2 \left( k_2 + \frac{a_1 k_0}{\gcd(a_1, a_2)} \right) + b_2,
 \end{aligned}$$

and  $w_0$  is in both arithmetic progressions. Therefore  $\mathbb{Z}_{\text{arith}}$  is a topological space generated by the basis  $\mathcal{B}$ .

**Exercise 3.12.** Consider the topological space  $\mathbb{Z}_{\text{arith}}$  and note that if  $U$  is an open set, then since it is the union of infinite arithmetic progressions, it is itself infinite. Note also that for a prime  $p$  and  $a = 1, 2, \dots, p-1$ , we have that  $\{pz + a : z \in \mathbb{Z}\} \in \mathcal{B}$  is an arithmetic progression. Let  $p\mathbb{Z}$  denote the set  $\{pz : z \in \mathbb{Z}\}$ . Then we have that

$$\mathbb{Z} - p\mathbb{Z} = \bigcup_{a=1, \dots, p} \{pz + a : z \in \mathbb{Z}\}$$

is the union of open sets and is therefore open, meaning that  $p\mathbb{Z}$  itself is closed. Let  $P$  denote the set  $P = \bigcup_{\text{primes } p} p\mathbb{Z}$ , and suppose for contradiction that there are finitely many primes. Then  $P$  is the union of finitely many closed sets and is therefore itself closed. Note that  $P$  contains all numbers that are integer multiples of a prime, so the only numbers not contained in  $P$  are  $-1$  and  $1$ , that is,  $\mathbb{Z} - P = \{-1, 1\}$ . But this set is open since  $P$  is closed, which is a contradiction since  $\mathbb{Z} - P$  is finite. Therefore there are infinitely many primes.

## 3.2 Subbases

**Exercise 3.13.** Let  $(X, \mathcal{T})$  be a topological space with basis  $\mathcal{B}$ . Then if  $U$  is in an open set in  $(X, \mathcal{T})$ , it is the union of sets in  $\lambda \subset \mathcal{B}$ . For each set  $S \in \lambda$ ,  $S$  is the trivial intersection of itself, and this intersection is finite. Since  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ , the finite intersections of sets in  $\mathcal{B}$  is in  $\mathcal{T}$ . Therefore every open set in  $\mathcal{T}$  can be generated by taking the union of sets that are themselves the finite intersections of sets in  $\mathcal{B}$  and so  $\mathcal{B}$  is a subbasis.

**Exercise 3.14.** Consider  $\mathbb{R}$  together with the standard topology. Recall that  $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$  is a basis and note that  $(a, b) = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x < b\}$  and this is a finite intersection. Therefore every set in the basis is the finite intersection of sets in  $\mathcal{S}$ , the set of rays of the form  $\{x \in \mathbb{R} : x < x_0\}$  and  $\{x \in \mathbb{R} : x_0 > x\}$  for some  $x_0 \in \mathbb{R}$ . Therefore  $\mathcal{S}$  is a subbasis for  $(\mathbb{R}, \mathcal{T}_{\text{std}})$ .

**Theorem 3.16.** Let  $X$  be a set and let  $\mathcal{S}$  be a collection of subsets of  $X$ . Then  $\mathcal{S}$  is a subbasis for some topology  $\mathcal{T}$  on  $X$  if and only if every point of  $X$  is contained in some element of  $\mathcal{S}$ .

*Proof.* Suppose  $\mathcal{S}$  is a subbasis for a topology  $\mathcal{T}$  on  $X$  and let  $x \in X$  be arbitrary. Since  $\mathcal{S}$  is a subbasis, the set  $\mathcal{B}$  of finite intersections of elements of  $\mathcal{S}$  is a basis, and therefore  $x \in B_0$

for some  $B_0 \in \mathcal{B}$ . Then since  $B_0$  is the finite intersection of elements in  $\mathcal{S}$ ,  $x \in S_0$  for some  $S_0 \in \mathcal{S}$ . Now suppose that for all  $x \in X$ , there exists some  $S_0 \in \mathcal{S}$  such that  $x \in S_0$ . Then if  $\mathcal{B}$  is the set of finite intersections of elements of  $\mathcal{S}$ ,  $x \in S_0 \in \mathcal{B}$ , so Theorem 3.3(1) is satisfied. Now let  $U, V \in \mathcal{B}$  be arbitrary and let  $p \in U \cap V$ . Then  $U = \bigcap_{S \in \lambda_1} S$  and  $V = \bigcap_{S \in \lambda_2} S$  where  $\lambda_1$  and  $\lambda_2$  are collections of sets in  $\mathcal{S}$ . Therefore

$$p \in \bigcap_{S \in \lambda_1 \cap \lambda_2} S \subset U \cap V$$

since  $\lambda_1 \cap \lambda_2 \subset \lambda_1, \lambda_2$ . Therefore Theorem 3.3(2) is satisfied as well since  $\bigcap_{S \in \lambda_1 \cap \lambda_2} S \in \mathcal{B}$ , and so  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$ , which also means that  $\mathcal{S}$  is a subbasis for this topology.  $\square$

**Exercise 3.17.** Let  $\mathcal{S}$  be the set of all subsets of  $\mathbb{R}$  of the form  $\{x \in \mathbb{R} : x < a\}$  or  $\{x \in \mathbb{R} : a \leq x\}$ . We claim that this is a subbasis for the lower limit topology on  $\mathbb{R}$ ,  $\mathcal{T}_{\text{LL}}$ . If  $U \in \mathcal{S}$ , then there are two cases. In the first case,  $U = \{x \in \mathbb{R} : x < a\} = \bigcup_{n \in \mathbb{N}} [a - n, a) \in \mathcal{T}_{\text{LL}}$  for some  $a \in \mathbb{R}$ . In the second case,  $U = \{x \in \mathbb{R} : a \leq x\} = \bigcup_{n \in \mathbb{N}} [a, a + n) \in \mathcal{T}_{\text{LL}}$  for some  $a \in \mathbb{R}$ . Both inclusions in  $\mathcal{T}_{\text{LL}}$  come from the fact that all open sets of the Sorgenfrey Line are the union of sets of the form  $[a, b)$  for some  $a, b \in \mathbb{R}$ . Since any arbitrary  $U \in \mathcal{S}$  has  $U \in \mathcal{T}_{\text{LL}}$ , we have that  $\mathcal{S} \subset \mathcal{T}_{\text{LL}}$  and therefore Theorem 3.15(1) is satisfied. Now suppose  $U$  is an open set of the Sorgenfrey Line and let  $p \in U$  be arbitrary. Then  $U = \bigcup_{W \in \lambda} W$  where  $\lambda \subset \mathcal{B}_{\text{LL}}$  is a collection of sets of the form  $[a, b)$  for some  $a, b \in \mathbb{R}$ . Therefore we have that  $p \in W_0 = [a_0, b_0)$  for some  $W_0 \in \lambda$ . Since  $W_0 = \{x \in \mathbb{R} : a_0 \leq x\} \cap \{x \in \mathbb{R} : x < b_0\}$  is the finite intersection of sets in  $\mathcal{S}$  and  $W_0 \subset U$ , we have satisfied Theorem 3.15(2), meaning that  $\mathcal{S}$  is indeed a subbasis for the lower limit topology on  $\mathbb{R}$ , as claimed.

### 3.3 Order Topology

**Exercise 3.19.** Consider  $\mathbb{R}$  with the order topology from  $\leq$ . It has a basis  $\mathcal{B}$  containing sets of the form  $\{x \in \mathbb{R} : x < a\}$ ,  $\{x \in \mathbb{R} : a < x\}$ , and  $\{x \in \mathbb{R} : a < x < b\}$ . A set  $U$  is open in the standard topology on  $\mathbb{R}$  if for each point  $p \in U$ , there is an  $\varepsilon_p > 0$  such that  $(p - \varepsilon_p, p + \varepsilon_p) \subset U$ . Note that this is the case for every set of the forms contained in  $\mathcal{B}$  (for the first and second forms,  $\varepsilon_p = |a - p|$ , and for the third form,  $\varepsilon_p = \min\{p - a, b - p\}$ ), so  $\mathcal{B} \subset \mathcal{T}_{\text{std}}$ , satisfying Theorem 3.1(1). If  $U \subset \mathbb{R}$  is open in  $\mathcal{T}_{\text{std}}$  and  $p \in U$ , then since there exists  $\varepsilon_p > 0$  such that  $p \in (p - \varepsilon_p, p + \varepsilon_p) \subset U$  and  $(p - \varepsilon_p, p + \varepsilon_p) \in \mathcal{B}$ , Theorem 3.1(2) is satisfied as well and therefore  $\mathcal{B}$  is a basis for the standard topology as well, so the order topology on  $\mathbb{R}$  with  $\leq$  is the standard topology.

**Exercise 3.21.**  $A = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$  has closure  $\bar{A} = A \cup \{(0, 1)\}$ .

$B = \{(1 - \frac{1}{n}, \frac{1}{2}) : n \in \mathbb{N}\}$  has closure  $\bar{B} = B \cup \{(1, 0)\}$ .

$C = \{(x, 0) : 0 < x < 1\}$  has closure  $\bar{C} = \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$ .

$D = \{(x, \frac{1}{2}) : 0 < x < 1\}$  has closure  $\bar{D} = \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$ .

$E = \{(\frac{1}{2}, y) : 0 < y < 1\}$  has closure  $\bar{E} = E \cup \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$

### 3.4 Subspaces

**Theorem 3.25.** Let  $(X, \mathcal{T})$  be a topological space and let  $Y \subset X$ . Then  $\mathcal{T}_Y$  is indeed a topology on  $Y$ .

*Proof.* Note that  $\emptyset \in \mathcal{T}_Y$  since  $\emptyset = \emptyset \cap Y$  and  $\emptyset \in \mathcal{T}$ . Similarly,  $Y \in \mathcal{T}_Y$  since  $Y = X \cap Y$  and  $X \in \mathcal{T}$ . Now let  $A, B \in \mathcal{T}_Y$  be arbitrary. Then there exist sets  $V_A, V_B \in \mathcal{T}$  such that  $A = V_A \cap Y$  and  $B = V_B \cap Y$  and we have that

$$A \cap B = (V_A \cap Y) \cap (V_B \cap Y) = (V_A \cap V_B) \cap Y \in \mathcal{T}_Y$$

since  $V_A \cap V_B$  is the (finite) intersection of sets in the topology  $\mathcal{T}$  and is therefore itself in the topology  $\mathcal{T}$ . Now let  $\{U_\alpha\}_{\alpha \in \lambda}$  be a collection of sets in  $\mathcal{T}_Y$ . Then for each  $\alpha \in \lambda$ , there exists a set  $V_\alpha \in \mathcal{T}$  such that  $U_\alpha = V_\alpha \cap Y$  and we have that

$$\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (V_\alpha \cap Y) = \left( \bigcup_{\alpha \in \lambda} V_\alpha \right) \cap Y \in \mathcal{T}_Y$$

since  $\bigcup_{\alpha \in \lambda} V_\alpha$  is the union of sets in  $\mathcal{T}$  and is therefore itself in  $\mathcal{T}$ . Thus we have that  $\emptyset, Y \in \mathcal{T}_Y$  and that  $\mathcal{T}_Y$  contains the finite intersections of sets in  $\mathcal{T}_Y$  and the (possibly infinite) unions of sets in  $\mathcal{T}_Y$  and is therefore a topology on  $Y$ .  $\square$

**Exercise 3.26.** Taking  $Y = [0, 1)$  as a subspace of  $\mathbb{R}_{\text{std}}$ , we see that the set  $[\frac{1}{2}, 1)$  is closed in  $Y$ . This is because the set  $(-1, \frac{1}{2})$  is open in  $\mathbb{R}_{\text{std}}$  and therefore  $(-1, \frac{1}{2}) \cap Y = [0, \frac{1}{2})$  is open in  $Y$ , so  $Y - [0, \frac{1}{2}) = [\frac{1}{2}, 1)$  is closed in  $Y$ .

**Exercise 3.27.** If  $Y$  is a subspace of a topological space  $(X, \mathcal{T})$ , it is not necessarily the case that every subset of  $Y$  that is open in  $Y$  is open in  $(X, \mathcal{T})$ . As in Exercise 3.26,  $Y = [0, 1)$  is a subspace of  $\mathbb{R}_{\text{std}}$  and  $[0, \frac{1}{2})$  is open in  $Y$ . However, it is neither open nor closed in  $\mathbb{R}_{\text{std}}$ .

**Theorem 3.28.** Let  $(Y, \mathcal{T}_Y)$  be a subspace of a topological space  $(X, \mathcal{T})$ . A subset  $C \subset Y$  is closed in  $(Y, \mathcal{T}_Y)$  if and only if there is a set  $D \subset X$ , closed in  $(X, \mathcal{T})$ , such that  $C = D \cap Y$ .



*Proof.* Let  $C \subset Y$  be closed in  $(Y, \mathcal{T}_Y)$ . This is the case if and only if  $Y - C$  is open in  $(Y, \mathcal{T}_Y)$ , which is the case if and only if there exists a set  $V \subset X$ , open in  $(X, \mathcal{T})$ , such that  $Y - C = V \cap Y$ . This then is the case if and only if

$$C = Y - (V \cap Y) = Y - V = Y \cap (X - V) = Y \cap D$$

where  $D = X - V$  is closed in  $(X, \mathcal{T})$  since  $V$  is open in  $(X, \mathcal{T})$ . All implications here go in both directions, so both directions of the proof are complete.  $\square$

**Theorem 3.30.** Let  $(Y, \mathcal{T}_Y)$  be a subspace of a topological space  $(X, \mathcal{T})$  that has basis  $\mathcal{B}$ . Then  $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$  is a basis for  $\mathcal{T}_Y$ .

*Proof.* Let  $A_0 \in \mathcal{B}_Y$  be arbitrary. Then there exists  $B_0 \in \mathcal{B}$  such that  $A_0 = B_0 \cap Y$ . Since  $B_0 \in \mathcal{B}$ , it is open in the topological space  $(X, \mathcal{T})$ . Since  $\mathcal{T}_Y = \{U : U = V \cap Y \text{ for some } V \in \mathcal{T}\}$ , we have that  $A_0 \in \mathcal{T}_Y$  and so Theorem 3.1(1) is satisfied. Now let  $U$  be an open set in  $(Y, \mathcal{T}_Y)$  and let  $p \in U$  be an arbitrary point. Since  $U \in \mathcal{T}_Y$ , there exists a set  $W \in \mathcal{T}$  such that  $U = W \cap Y$ , which means that  $p \in Y$  and  $p \in W$ . Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ , by Theorem 3.1(2), there exists a set  $V \in \mathcal{B}$  such that  $p \in V \subset W$ . Then since  $V \in \mathcal{B}$ ,  $V \cap Y \in \mathcal{B}_Y$  and we have that

$$p \in V \cap Y \subset W \cap Y = U,$$

which satisfies Theorem 3.1(2), meaning that  $\mathcal{B}_Y$  is indeed a basis for  $\mathcal{T}_Y$ .  $\square$

### 3.5 Product Spaces

**Exercise 3.32.** Let  $X$  and  $Y$  be topological spaces and let  $\mathcal{B}$  be the set of all cartesian products of open sets  $U \subset X$  and  $V \subset Y$ . Then  $p = (a, b)$  is a point of  $X \times Y$ , then  $p$  is in some element of  $\mathcal{B}$  since  $X$  is open in  $X$  and  $Y$  is open in  $Y$  (that is,  $X \times Y \in \mathcal{B}$ ). This means that Theorem 3.3(1) is satisfied. Now let  $U, V \in \mathcal{B}$  be arbitrary. Then there exist sets  $X_1, X_2$  open in  $X$  and  $Y_1, Y_2$  such that  $U = X_1 \times Y_1$  and  $V = X_2 \times Y_2$ . Now let  $p \in U \cap V$  be arbitrary. Then we have that  $p \in U \cap V \subset U \cap V$  and also that

$$U \cap V = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2) \in \mathcal{B}$$

since  $X_1 \cap X_2$  is open in  $X$  and  $Y_1 \cap Y_2$  is open in  $Y$ . Therefore Theorem 3.3(2) is also satisfied and we have that  $\mathcal{B}$  is indeed the basis for some topology on  $X \times Y$ .

**Exercise 3.34.** The product of closed sets is closed in the product topology. Let  $X$  and  $Y$  be topological spaces, let  $A$  be closed in  $X$  and let  $B$  be closed in  $Y$ . Then there exists an open set  $A_0 \subset X$  such that  $A = X - A_0$  and there exists an open set  $B_0 \subset Y$  such that  $B = Y - B_0$  and we have that

$$\begin{aligned}
 A \times B &= (X - A_0) \times (Y - B_0) \\
 &= (X \times (Y - B_0)) - (A_0 \times (Y - B_0)) \\
 &= ((X \times Y) - (X \times B_0)) - ((A_0 \times Y) - (A_0 \times B_0)) \\
 &= ((X \times Y) \cup (A_0 \times B_0)) - ((X \times B_0) \cup (A_0 \times Y)) \\
 &= (X \times Y) - ((X \times B_0) \cup (A_0 \times Y)).
 \end{aligned}$$

The sets  $X \times B_0$  and  $A_0 \times Y$  are both basic open sets in  $X \times Y$  since  $A_0$  is open in  $X$  and  $B_0$  is open in  $Y$ . Since the union of open sets is open, we have that  $A \times B$  is the complement of open sets and is therefore closed.  $A$  and  $B$  were arbitrary closed sets in arbitrary topological spaces, so we have shown that the product of closed sets is closed in the product topology in general.

**Theorem 3.35.** The product topology on  $X \times Y$  has a subbasis  $\mathcal{S}$  that contains the inverse images of open sets under the projection functions, that is,

$$\mathcal{S} = \{\pi_X^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ is open in } Y\}.$$

*Proof.* Let  $S_0$  be a set in  $\mathcal{S}$ . There are two cases. In the first,  $S_0$  is of the form  $S_0 = \{\pi_X^{-1}(U) : U \text{ is open in } X\} = U \times Y$  and is therefore a basic open set in the product space. In the second,  $S_0$  is of the form  $S_0 = \{\pi_Y^{-1}(V) : V \text{ is open in } Y\} = X \times V$  and is also a basic open set in the product space. Therefore  $S_0$  is open in all cases and so  $\mathcal{S} \subset \mathcal{T}$  where  $\mathcal{T}$  is the product topology. Thus Theorem 3.15(1) is satisfied. Now let  $W$  be an open set in the product space and let  $p \in W$  be an arbitrary point. Then since  $W$  is open, it is the union of sets of the form  $U \times V$  where  $U$  is open in  $X$  and  $V$  is open in  $Y$ . This means there are sets  $U_0 \subset X$  and  $V_0 \subset Y$  such that  $p \in U_0 \times V_0 \subset W$ . All that remains to satisfy Theorem 3.15(2) is to show that  $U_0 \times V_0$  is the finite intersection of sets in  $\mathcal{S}$ , which is the case since  $U_0 \times V_0 = \pi_X^{-1}(U_0) \cap \pi_Y^{-1}(V_0)$ . Therefore  $\mathcal{S}$  is indeed a subbasis for the product topology on  $X \times Y$ .  $\square$

**Exercise 3.36.** Let  $W$  be an open set in  $\mathbb{R}_{\text{std}}^2$ . Then  $W = B(p, \varepsilon_p)$  for some  $p \in \mathbb{R}^2$  and  $\varepsilon_p > 0$ .

Now let  $x \in W$  be arbitrary. Suppose  $p = (p_1, p_2)$  and  $x = (x_1, x_2)$ . Set

$$U = \left( x_1 - \left| x_1 - \left( \frac{\varepsilon_p(x_1 - p_1)}{d(x, p)} + p_1 \right) \right|, x_1 + \left| x_1 - \left( \frac{\varepsilon_p(x_1 - p_1)}{d(x, p)} + p_1 \right) \right| \right)$$

and

$$V = \left( x_2 - \left| x_2 - \left( \frac{\varepsilon_p(x_2 - p_2)}{d(x, p)} + p_2 \right) \right|, x_2 + \left| x_2 - \left( \frac{\varepsilon_p(x_2 - p_2)}{d(x, p)} + p_2 \right) \right| \right).$$

(See Desmos sketch lmao.) Then we have that  $x \in U \times V \subset W$ , and since  $U \times V$  is an open set with the product topology on  $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ , we have that  $W$  is an open set in this topology as well by Theorem 2.3 since  $x$  was an arbitrary point of  $W$ . Then since  $W$  was an arbitrary open set under the standard topology, it follows that the standard topology on  $\mathbb{R}^2$  is a subset of the product topology on  $\mathbb{R}^2$ .

Now let  $W$  be an open set in  $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$  with the product topology and let  $p \in W$  be arbitrary. Then since  $W$  is open, it is the union of sets of the form  $U \times V$  where  $U$  and  $V$  are open sets in  $\mathbb{R}_{\text{std}}$ . This means that there exist some  $U_0$  and  $V_0$  open in  $\mathbb{R}_{\text{std}}$  such that  $p \in U_0 \times V_0 \subset W$ . Then there exist intervals  $(a_x, b_x)$  and  $(a_y, b_y)$  such that  $\pi_{U_0}(p) \in (a_x, b_x)$  and  $\pi_{V_0}(p) \in (a_y, b_y)$ . Therefore we have that  $p \in B(p, \varepsilon) \subset U_0 \times V_0 \subset W$  where

$$\varepsilon = \min\{|a_x - \pi_{U_0}(p)|, |b_x - \pi_{U_0}(p)|, |a_y - \pi_{V_0}(p)|, |b_y - \pi_{V_0}(p)|\}.$$

Since this  $B(p, \varepsilon)$  is an open set in  $\mathbb{R}_{\text{std}}^2$ , by Theorem 2.3 it follows that  $W$  is also open in  $\mathbb{R}_{\text{std}}^2$ , and since  $W$  was an arbitrary open set in  $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$  with the product topology, we have that this topology on  $\mathbb{R}^2$  is a subset of the standard topology on  $\mathbb{R}^2$ . Since each topology is a subset of the other, they are equal, meaning that product topology on  $\mathbb{R}^2$  from  $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$  is the same as the standard topology on  $\mathbb{R}^2$ .

**Theorem 3.37.** The product topology on  $\prod_{\alpha \in \lambda} X_\alpha$  has a basis containing all sets of the form  $\prod_{\alpha \in \lambda} U_\alpha$  where  $U_\alpha$  is open in  $X_\alpha$  for each  $\alpha$  and  $U_\alpha = X_\alpha$  for all but finitely many  $\alpha$ .

*Proof.* Since  $\mathcal{S}$ , the collection of sets of the form  $\pi_\beta^{-1}(U_\beta)$  where  $U_\beta$  is open in  $(X_\beta, \mathcal{T}_\beta)$ , is a subbasis for the product topology  $\prod_{\alpha \in \lambda} X_\alpha$ , the set of finite intersections of elements of  $\mathcal{S}$  is therefore a basis for the product topology. Let  $W$  be a set in the collection described in the theorem statement. Then there exists a finite index set  $\gamma \subset \lambda$  such that  $W = \prod_{\alpha \in \lambda} U_\alpha$  where  $U_\alpha$  is open in  $X_\alpha$  for each  $\alpha$  and  $U_\alpha = X_\alpha$  for all  $\alpha \in \lambda - \gamma$ . Then since for some  $\beta \in \gamma$ ,  $\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha \in \lambda - \{\beta\}} X_\alpha$ , we have that  $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) = U_{\beta_1} \times U_{\beta_2} \times \prod_{\alpha \in \lambda - \{\beta_1, \beta_2\}} X_\alpha$

and therefore  $W = \prod_{\alpha \in \gamma} U_\alpha \times \prod_{\alpha \in \lambda - \gamma} X_\alpha = \bigcap_{\alpha \in \gamma} \pi_\alpha^{-1}(U_\alpha)$  is the finite intersection of elements of  $\mathcal{S}$ . Similarly, any finite intersection of elements of  $\mathcal{S}$  is a product of the form described in the theorem statement, and so by the definition of a subbasis, the collection described in the theorem statement is indeed a basis for the product topology.  $\square$

**Exercise 3.42.** The set  $2^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \{0, 1\}$  with the box topology has the discrete topology since all singletons are open. Let  $\{a_n\}_{n \in \mathbb{N}}$  be a binary sequence (an arbitrary element of  $2^{\mathbb{N}}$ ). Then the singleton containing  $\{a_n\}_{n \in \mathbb{N}}$  is  $\{\{a_n\}_{n \in \mathbb{N}}\} = \prod_{n \in \mathbb{N}} \{a_n\} = \{a_1\} \times \{a_2\} \times \cdots$ , so it is a basic open set. Compare this to  $2^{\mathbb{N}}$  with the product topology. Since with this topology, if  $p \in 2^{\mathbb{N}}$  is a point in the space and  $U$  an open set containing  $p$ , then  $(U - \{p\}) \cap 2^{\mathbb{N}} \neq \emptyset$  since any open set  $U$  contains infinitely many different sequences. Therefore  $2^{\mathbb{N}}$  under the product topology has no isolated points since every point is a limit point of the set.

## 4 Separation Properties: Separating This From That

### 4.1 Hausdorff, Regular, and Normal Spaces

**Theorem 4.1.** A space  $(X, \mathcal{T})$  is  $T_1$  if and only if every point in  $X$  is a closed set.

*Proof.* Suppose  $(X, \mathcal{T})$  is  $T_1$  and let  $x \in X$  be arbitrary. Let  $y \in X - \{x\}$  be arbitrary. Then we have that  $x \neq y$ , so since the space is  $T_1$ , we have that there exist open sets  $U$  and  $V$  such that  $x \in U - V$  and  $y \in V - U$ . Since  $V$  is open,  $V \subset X$ , and since  $x \notin V$ ,  $V \subset X - \{x\}$ . Therefore  $y \in V \subset X - \{x\}$  and by Theorem 2.3, we have that  $X - \{x\}$  is open, meaning that the singleton  $\{x\}$  is closed. But  $x$  was an arbitrary point in this  $T_1$  space, so we have that points are closed in  $T_1$  spaces. Now suppose  $(X, \mathcal{T})$  is a topological space in which all points are closed and let  $x, y \in X$  be arbitrary points such that  $x \neq y$ . Since  $\{x\}$  and  $\{y\}$  are closed, we have that  $x \in X - \{y\}$ , an open set, and  $y \in X - \{x\}$ , another open set. Since  $x \notin X - \{x\}$  and  $y \notin X - \{y\}$ , and  $x$  and  $y$  were arbitrary elements of  $X$ , we have that the space  $(X, \mathcal{T})$  is  $T_1$ .  $\square$

**Exercise 4.2.** Let  $X$  be a set with the finite complement topology. Then let  $x \in X$  be arbitrary. We have that  $X - \{x\}$  is open since its complement,  $\{x\}$ , is finite. This means  $\{x\}$  is closed, and since the point  $x$  was arbitrary, we have that all singletons are closed and therefore by Theorem 4.1, all sets with the finite complement topology are  $T_1$ .

**Exercise 4.3.** Let  $x, y \in \mathbb{R}$  be points in the space  $\mathbb{R}_{\text{std}}$ . Then the sets  $A = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$  and  $B = \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$  are open and disjoint with  $x \in A$  and  $y \in B$ . Therefore  $\mathbb{R}_{\text{std}}$  is Hausdorff.

**Exercise 4.5.** Let  $A$  and  $B$  be disjoint, closed sets in  $\mathbb{R}_{\text{LL}}$ . For every  $a \in A$  and  $b \in B$ , set

$$\delta_a = \frac{\inf\{b \in B : b > a\} - a}{2} \quad \text{and} \quad \delta_b = \frac{\inf\{a \in A : a > b\} - b}{2}.$$

Since for all  $a \in A$ ,  $a$  is in the basic open set  $U_a = [a, a + \delta_a)$ , we have that  $A \subset U = \bigcup_{a \in A} U_a$ . Similarly, we have that  $B \subset V = \bigcup_{b \in B} V_b$  where  $V_b = [b, b + \delta_b)$  is a basic open set. Suppose for contradiction that there exists  $x \in U \cap V$ . Then we have that there exist  $\alpha \in A$  and  $\beta \in B$  such that  $x \in U_\alpha \cap V_\beta$ . Without loss of generality, assume that  $\alpha < \beta$ . Then we have that  $x \in U_\alpha = [\alpha, \alpha + \delta_\alpha)$  and  $x \in V_\beta = [\beta, \beta + \delta_\beta)$ . Then

$$\delta_\alpha = \frac{\inf\{b \in B : b > \alpha\} - \alpha}{2} \implies \alpha + \delta_\alpha = \frac{\alpha + \inf\{b \in B : b > \alpha\}}{2} \leq \inf\{b \in B : b > \alpha\}$$

since  $\alpha \leq \inf\{b \in B : b > \alpha\}$ . Since  $\beta \in B$  and  $\beta > \alpha$ ,  $\beta \in \{b \in B : b > \alpha\}$  and therefore we have that

$$x < \alpha + \delta_\alpha \leq \inf\{b \in B : b > \alpha\} \leq \beta \leq x.$$

But  $x < x$  is a contradiction and so we have that  $U \cap V = \emptyset$  and therefore we have found disjoint, open sets  $U$  and  $V$  such that  $A \subset U$  and  $B \subset V$ . This means that  $\mathbb{R}_{\text{LL}}$  is normal.

**Exercise 4.6.** (1) Let  $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$  and let  $A \subset \mathbb{R}^2$  be a closed set with  $p \notin A$ . Suppose for contradiction that  $\inf\{d(a, p) : a \in A\} = 0$ . This means that there exists a sequence  $(x_i)_{i \in \mathbb{N}} \subset A$  such that  $d(x_i, p) \rightarrow 0$ . This means that for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $i > N$  implies that  $|d(x_i, p) - 0| < \varepsilon$ , which is equivalent to the statement that

$$\varepsilon > | \|x_i - p\| - 0 | = \|x_i - p\|.$$

We now have that for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $i > N$  implies  $\|x_i - p\| < \varepsilon$ , which in  $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$  means that  $x_i \rightarrow p$ . By Theorem 2.30, we have that  $p \in \bar{A} = A$  since  $A$  is closed. But this is a contradiction, so the assumption that  $\inf\{d(a, p) : a \in A\} = 0$  was false. We have either that this is greater than 0 or less than 0, but it cannot be less than 0 since the distance between two points is always nonnegative. Therefore we have that  $\inf\{d(a, p) : a \in A\} > 0$ .

(2) Let  $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$  and let  $A \subset \mathbb{R}^2$  be a closed set with  $p \notin A$ . Then by (1), there exists an  $\varepsilon > 0$  such that  $\inf\{d(a, p) : a \in A\} = \varepsilon$ . Set  $U = B(p, \frac{\varepsilon}{2})$  and  $V = \bigcup_{a \in A} B(a, \frac{\varepsilon}{2})$ . Then  $U$  is a basic open set, and  $V$  is the union of basic open sets so both  $U$  and  $V$  are open. We also have that  $p \in U$  and  $A \subset V$  since if  $a \in A$ ,  $a \in B(a, \frac{\varepsilon}{2}) \subset V$ . Suppose for contradiction that there exists

an  $x \in U \cap V$ . Then  $x \in U$  means that  $d(p, x) < \varepsilon/2$  and  $x \in V$  means that there exists an  $a \in A$  such that  $d(a, x) < \varepsilon/2$ . Therefore we have that  $d(a, p) \leq d(a, x) + d(x, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ , but this is a contradiction since  $\inf\{d(a, p) : a \in A\} = \varepsilon$ . Therefore  $U \cap V = \emptyset$  and  $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$  is regular.

(3) The sets  $A = \{(x, 0) : x \in \mathbb{R}\}$  and  $B = \{(x, y) : x > 0, y \geq \frac{1}{x}\}$ . Since for any points  $(x_a, y_a) \in A$  and  $(x_b, y_b) \in B$ ,  $y_b \geq \frac{1}{x_b} > 0 = y_a$ , we have that  $A \cap B = \emptyset$ . All limit points of  $A$  have  $y$ -coordinate equal to 0 and are therefore in  $A$ , meaning  $A$  is closed. Similarly, all limit points in  $B$  have  $y$ -coordinate equal to  $\frac{1}{x_0}$  for some  $x_0 > 0$  and are therefore in  $B$ , meaning  $B$  is closed. Hence  $A$  and  $B$  are disjoint, closed subsets of  $\mathbb{R}^2$ . However,  $\inf\{d(a, b) : a \in A \text{ and } b \in B\} = 0$ . To see this, let  $\varepsilon > 0$ . Then we have that  $a_\varepsilon = (\frac{1}{\varepsilon} + 1, 0) \in A$  and  $b_\varepsilon = (\frac{1}{\varepsilon} + 1, \frac{1}{\frac{1}{\varepsilon} + 1}) \in B$ . Then  $d(a_\varepsilon, b_\varepsilon) = \frac{1}{\frac{1}{\varepsilon} + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ , so  $\inf\{d(a, b) : a \in A \text{ and } b \in B\} < \varepsilon$ . However,  $\varepsilon > 0$  was an arbitrary, so we have that  $\inf\{d(a, b) : a \in A \text{ and } b \in B\} = 0$  since distance is nonnegative.

(4) Let  $A$  and  $B$  be disjoint, closed sets in  $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ . For  $a_0 \in A$  and  $b_0 \in B$ , define  $\varepsilon_{a_0} = \frac{1}{2} \inf\{d(a_0, b) : b \in B\} > 0$  and  $\varepsilon_{b_0} = \frac{1}{2} \inf\{d(a, b_0) : a \in A\} > 0$ . Now set  $U = \bigcup_{a \in A} B(a, \varepsilon_a)$  and  $V = \bigcup_{b \in B} B(b, \varepsilon_b)$ . Then since  $U$  and  $V$  are the unions of open balls in  $\mathbb{R}^2$ , they are open. Since if  $a \in A$ , then  $a \in B(a, \varepsilon_a) \subset U$  and if  $b \in B$ , then  $b \in B(b, \varepsilon_b) \subset V$ , we have that  $A \subset U$  and  $B \subset V$ . To show  $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$  is normal, it only remains to show that  $U$  and  $V$  are disjoint. Suppose for contradiction that there exists  $p \in \mathbb{R}^2$  such that  $p \in U \cap V$ . Then  $p \in U$ , so there exists an  $\alpha \in A$  such that  $p \in B(\alpha, \varepsilon_\alpha)$  and similarly, there exists a  $\beta \in B$  such that  $p \in B(\beta, \varepsilon_\beta)$ . Since  $\alpha \in A$ ,  $d(\alpha, \beta) \in \{d(a, \beta) : a \in A\}$  and therefore  $d(\alpha, \beta) \geq \inf\{d(a, \beta) : a \in A\} = 2\varepsilon_\beta$ . Similarly,  $d(\alpha, \beta) \geq 2\varepsilon_\alpha$ , and so we have that  $\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta)$ . Since  $p \in B(\alpha, \varepsilon_\alpha)$ , we have that  $d(\alpha, p) < \varepsilon_\alpha$ , and since  $p \in B(\beta, \varepsilon_\beta)$ , we have that  $d(p, \beta) < \varepsilon_\beta$ . Putting this all together using the triangle inequality, we see that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta) \leq d(\alpha, p) + d(p, \beta) < \varepsilon_\alpha + \varepsilon_\beta.$$

This is a contradiction, so we have that  $U \cap V = \emptyset$  and therefore  $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$  is normal.

**Theorem 4.7.** (1) A  $T_2$ -space (Hausdorff) is a  $T_1$ -space.

(2) A  $T_3$ -space (regular and  $T_1$ ) is a Hausdorff space, that is, a  $T_2$ -space.

(3) A  $T_4$ -space (normal and  $T_1$ ) is regular and  $T_1$ , that is, a  $T_3$ -space.

*Proof.* (1) Let  $(X, \mathcal{T})$  be a Hausdorff space and let  $x, y \in X$  be distinct, arbitrary points. Then there exist disjoint, open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ . Since  $U \cap V = \emptyset$ , we have that  $x \notin V$  and  $y \notin U$ , so  $(X, \mathcal{T})$  is a  $T_1$ -space.  $\square$

*Proof.* (2) Let  $(X, \mathcal{T})$  be a  $T_3$ -space and let  $x, y \in X$  be distinct, arbitrary points. Since this space is  $T_1$ , by Theorem 4.1 we have that  $\{y\}$  is closed. Since this space is regular, we have that there exist disjoint, open sets  $U$  and  $V$  such that  $x \in U$  and  $\{y\} \subset V$ . But  $\{y\} \subset V$  means  $y \in V$ , so we have found disjoint, open sets separating the arbitrary points  $x$  and  $y$ , so  $(X, \mathcal{T})$  is Hausdorff.  $\square$

*Proof.* (3) Let  $(X, \mathcal{T})$  be a  $T_4$ -space, let  $x \in X$  be arbitrary, and let  $A$  be a closed set with  $x \notin A$ . Since this space is  $T_1$ , by Theorem 4.1 we have that  $\{x\}$  is closed. Since this space is normal, there exist disjoint, open sets  $U$  and  $V$  such that  $\{x\} \subset U$  and  $A \subset V$ . But  $\{x\} \subset U$  means  $x \in U$ , so we have found disjoint, open sets separating the arbitrary point  $x$  from the arbitrary closed set  $A$ , so  $(X, \mathcal{T})$  is  $T_3$  since it is normal and  $T_1$ .  $\square$

**Theorem 4.8.** A topological space is regular if and only if for each point  $p$  in  $X$  and open set  $U$  containing  $p$  there exists an open set  $V$  such that  $p \in V$  and  $\overline{V} \subset U$ .

*Proof.* ( $\implies$ ) Let  $(X, \mathcal{T})$  be a regular topological space and let  $U$  be an open set containing the point  $p$ . Then we have that  $X - U$  is closed and since  $p \in U$ ,  $p \notin X - U$ . Since this space is regular, there exist disjoint open sets  $V$  and  $W$  such that  $p \in V$  and  $X - U \subset W$ . Therefore we have that  $X - W \subset U$  since  $X - U \subset W$ , and that  $X - W$  is closed since  $W$  is open. Let  $x \in V$  be arbitrary. Then  $x \notin W$  (since  $V \cap W = \emptyset$ ) and therefore  $x \in X - W$ . Since  $x$  was arbitrary, we have that  $V \subset X - W$ . By Theorem 2.22, we have that  $\overline{V} \subset \overline{X - W}$ , and since  $X - W$  is closed, we see that

$$p \in V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since  $U$  and  $p$  were arbitrary, there exists an open set  $V$  containing  $p$  such that  $\overline{V} \subset U$  for all  $p \in X$  and open sets  $U$  containing  $p$ .

( $\impliedby$ ) Now let  $(X, \mathcal{T})$  be a topological space with the property that for all  $p \in X$  and  $W \in \mathcal{T}$  with  $p \in W$ , there exists an open set  $U$  such that  $p \in U$  and  $\overline{U} \subset W$ . Let  $p \in X$  be arbitrary and let  $A$  be a closed subset of  $X$  such that  $p \notin A$ . Then we have that  $p \in X - A$ , which is open, and therefore there exists an open set  $U$  such that  $p \in U$  and  $\overline{U} \subset X - A$ , which implies that  $A \subset V$  where  $V$  is the open set  $X - \overline{U}$ . Let  $x \in U$  be arbitrary. Then  $x \in \overline{U}$  since  $U \subset \overline{U}$ , and therefore  $x \notin V = X - \overline{U}$ . Since  $x$  was arbitrary, we have that  $U \cap V = \emptyset$ . Therefore we have found disjoint open sets  $U$  and  $V$  such that  $p \in U$  and  $A \subset V$ , so  $(X, \mathcal{T})$  is regular, as required.  $\square$

**Theorem 4.9.** A topological space is normal if and only if for each closed set  $A$  in  $(X, \mathcal{T})$  and open set  $U$  containing  $A$  there exists an open set  $V$  such that  $A \subset V$  and  $\overline{V} \subset U$ .