Topology Through Inquiry Self-Study

Ben Clingenpeel

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1 Cardinality: To Infinity and Beyond

1.1 Sets and Functions

Theorem 1.2 (DeMorgan's Laws). Let X be a set and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then

$$X - \left(\bigcup_{k=1}^N A_k\right) = \bigcap_{k=1}^N (X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^N A_k\right) = \bigcup_{k=1}^N (X - A_k).$$

Proof. Let $a \in X - \left(\bigcup_{k=1}^N A_k\right)$ be arbitrary. Then $a \notin \bigcup_{k=1}^N A_k$, so for all k, $a \notin A_k$, which means that $a \in X - A_k$ for all k. Therefore $x \in \bigcap_{k=1}^N (X - A_k)$ and so $X - \left(\bigcup_{k=1}^N A_k\right) \subset \bigcap_{k=1}^N (X - A_k)$. Now let $a \in \bigcap_{k=1}^N (X - A_k)$ be arbitrary. Then we have that $a \in X - A_k$ for all k, which means that $a \notin A_k$ for all k. Therefore $a \notin \bigcup_{k=1}^N A_k$, so we have that $a \in X - \left(\bigcup_{k=1}^N A_k\right)$ and therefore $\bigcap_{k=1}^N (X - A_k) \subset X - \left(\bigcup_{k=1}^N A_k\right)$. Therefore

$$X - \left(\bigcup_{k=1}^N A_k\right) = \bigcap_{k=1}^N (X - A_k).$$

Let $a \in X - \left(\bigcap_{k=1}^N\right)$ be arbitrary. Then $a \notin \bigcap_{k=1} A_k$, so there exists some j such that $a \notin A_j$, which means that $a \in X - A_j$. Therefore $a \in \bigcup_{k=1}^N (X - A_k)$ and so $X - \left(\bigcap_{k=1}^N A_k\right) \subset$

$\bigcup_{k=1}^N (X - A_k)$. Now let $a \in \bigcup_{k=1}^N (X - A_k)$. Then there exists some j such that $a \in X - A_j$, so $a \notin A_j$ and therefore $a \notin \bigcap_{k=1}^N$. This means that $a \in X - \left(\bigcap_{k=1}^N A_k\right)$ and so $\bigcup_{k=1}^N (X - A_k) \subset X - \left(\bigcap_{k=1}^N A_k\right)$. Therefore we have that

$$X - \left(\bigcap_{k=1}^N A_k\right) = \bigcup_{k=1}^N (X - A_k).$$

□

Exercise 1.3. For a function $f : X \to Y$ and sets $A, B \subset Y$, we have that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proof. See MATH 200 final exam review sheet notes in the graph paper notebook. □

Exercise 1.4. If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y)$ contains more than one point. Then there exist $x_1, x_2 \in X$ such that $x_1, x_2 \in f^{-1}(y)$ and $x_1 \neq x_2$. By the definition of $f^{-1}(y)$, we have that $f(x_1), f(x_2) \in \{y\}$, so $f(x_1) = y = f(x_2)$ and f is therefore not injective since $x_1 \neq x_2$. We have shown the contrapositive of the claim. □

Exercise 1.5. If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.

Proof. Let $f : X \to Y$ be a function and let $y \in Y$ be arbitrary. Suppose $f^{-1}(y) = \emptyset$. Then for all $x \in X$, $f(x) \notin f^{-1}(y)$, and by the definition of $f^{-1}(y)$, this means that for all $x \in X$, $f(x) \neq y$, so f is not surjective. We have shown the contrapositive of the claim. □

1.2 Cardinality and Countable Sets

Theorem 1.8. Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Proof. Let $S \subset \mathbb{N}$ be an arbitrary subset of $\mathbb{N}$. If S is finite, then we are done. If S is not finite, then it is infinite. Let $s_0 = \min S$, let $s_1 = \min(S - \{s_0\})$, and let $s_i = \min(S - \{s_0, \ldots, s_{i-1}\})$. Define the function $f : \mathbb{N} \to S$ by the following: $f(n) = s_n$. Suppose $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. Without loss of generality, assume that $n_1 < n_2$. Then $f(n_2) = \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\})$. Since $f(n_1) = s_{n_1} \notin S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}$, we have that $f(n_1) \neq \min(S - \{s_0, \ldots, s_{n_1}, \ldots, s_{n_2-1}\}) = f(n_2)$, which means that f is injective. Let $s \in S \subset \mathbb{N}$ be arbitrary. Then set $j = |\{r \in S : r < s\}| + 1 \in \mathbb{N}$. Then we have that

$$f(j) = s_j = \min(S - \{s_0, \ldots, s_{|\{r \in S : r < s\}|}\}) = s$$

where the last equality follows from the fact that s must be the smallest element of the subset of S from which all elements smaller than s have been removed. Therefore f is surjective, and since it is also injective, f is a bijection, meaning that the cardinality of S is the same as the cardinality of $\mathbb{N}$. $\square$

Theorem 1.9. Every infinite set has a countable subset.

Proof. I think I need the axiom of choice here? I will return in the future! $\square$

Theorem 1.10. A set is infinite if and only if there is an injection from the set into a proper subset of itself.

Proof. I think I also need the axiom of choice here... $\square$

Theorem 1.11. The union of two countable sets is countable.

Proof. Let A and B be countable sets. There are two cases to consider. In the first case, the intersection of A and B is finite, and so there exists a bijection $h : A \cap B \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ where n is the size of the set $A \cap B$ If $A \cap B$ is empty, we instead use $n = 0$. Since $A \cap B$ is finite, $A - (A \cap B)$ is infinite, so there exists a bijection $f : A - (A \cap B) \to \mathbb{E}$, where $\mathbb{E}$ is the countable set containing all positive even integers greater than n. Similarly, there exists a bijection $g : B - (A \cap B) \to \mathbb{O}$, where $\mathbb{O}$ is the countable set containing all positive odd integers greater than n. Then we have that the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the second case, $A \cap B$ is infinite and there are three subcases. In the first subcase, one of $A - (A \cap B)$ or $B - (A \cap B)$ (assume without loss of generality that this is $A - (A \cap B)$) is finite. In this case, we use the same construction as earlier, since now $h : A - (A \cap B) \to \{1, \ldots, n\}$

is a bijection for some $n \in \mathbb{N}$, $f : A \cap B \to \mathbb{E}$ is a bijection, and $g : B - (A \cap B) \to \mathbb{O}$ is a bijection. Then the function $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A \cap B \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A - (A \cap B) \end{cases}.$$

In the second subcase, $A - (A \cap B)$ and $B - (A \cap B)$ are finite. Then there are bijections $f : A - (A \cap B) \to \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $g : B - (A \cap B) \to \{n+1, \ldots n+m\}$ for some $m \in \mathbb{N}$. Since $A \cap B$ is countably infinite, there is a bijection $h : A \cap B \to \{n+m+1, n+m+2, \ldots\}$, the set of positive integers greater than $n+m$. Therefore $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In the third subcase, all three sets are countably infinite. In this case, there is a bijection $f : A - (A \cap B) \to \{n \in \mathbb{N} : n = 3k, k \in \mathbb{Z}\}$, a bijection $g : B - (A \cap B) \to \{n \in \mathbb{N} : n = 3k+1, k \in \mathbb{Z}\}$, and a bijection $h : A \cap B \to \{n \in \mathbb{N} : n = 3k+2, k \in \mathbb{Z}\}$. Then we have that $\varphi : A \cup B \to \mathbb{N}$ is a bijection where

$$\varphi(a) = \begin{cases} f(a) & a \in A - (A \cap B) \\ g(a) & a \in B - (A \cap B) \\ h(a) & a \in A \cap B \end{cases}.$$

In all cases, we have that there exists a bijection $\varphi : A \cup B \to \mathbb{N}$, so $A \cup B$ is countable. $\square$

Theorem 1.12. The union of countably many sets is countable.

Vague idea. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of countably manycountable sets. Arrange the elements of $\{A_i\}$ like so:

$$\begin{array}{lllll} A_1 : & a_{1_1} & a_{1_2} & a_{1_3} & \ldots \\ A_2 : & a_{2_1} & a_{2_2} & a_{2_3} & \ldots \\ A_3 : & a_{3_1} & a_{3_2} & a_{3_3} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then define a bijection $h : \bigcup_{i=1}^{\infty} A_i \to \mathbb{N}$ where $h(a_{1_1}) = 1$, $h(a_{1_2}) = 2$ if $a_{1_2} \neq a_{1_1}$, and $h(a_{2_1}) = 3$ if $a_{2_1} \notin \{a_{1_1}, a_{1_2}\}$. The function h orders the elements of $\bigcup_{i=1}^{\infty}$ starting in the upper left and working down diagonals, skipping elements that have already been mapped. ♥

1.3 Uncountable Sets and Power Sets

Exercise 1.17. If $A = \{a, b, c\}$, then $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Theorem 1.18. If the set A is finite, then its power set has cardinality $2^{|A|}$ ($|2^A| = 2^{|A|}$).

Proof. If a set A is finite, then it has cardinality $|A| = n \in \mathbb{N}$. We argue by induction on n that its power set 2^A has cardinality 2^n. For the base case $n = 1$, let A_1 be a set with one element. Then $A_1 = \{a\}$ and $2^{A_1} = \{\emptyset, a\}$, which has two elements, so indeed $|2^{A_1}| = 2^{|A_1|}$. Now for the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that all sets with cardinality k have power sets with cardinality 2^k. Then let A_{k+1} be an arbitrary set with cardinality $k+1$ so that $A_{k+1} = \{a_1, \ldots, a_{k+1}\}$. Note that the set $A_k = \{a_1, \ldots, a_k\}$ has cardinality k, and therefore has 2^k subsets. For each subset of A_k, call them A_{k_i} for $1 \leq i \leq 2^k$, the sets A_{k_i} and $A_{k_i} \cup \{a_{k+1}\}$ are subsets of A_{k+1}. Therefore there are $2 \cdot 2^k = 2^{k+1}$ subsets of A_{k+1}, and so $|2^{A_{k+1}}| = 2^{|A_{k+1}|}$. Therefore by induction, all finite sets A have $|2^A| = 2^{|A|}$. □

Theorem 1.19. For any set A, there is an injection from A to 2^A.

Proof. Define $f : A \to 2^A$ as $f(a) = \{a\} \in 2^A$ since $\{a\} \subset A$. Let $x, y \in A$ such that $f(x) = f(y)$. Then we have that $\{x\} = \{y\}$, so $x = y$ and f is an injection, as required. □

Theorem 1.20. If P is the set of all functions from a set A to the set $\{0, 1\}$, then $|P| = |2^A|$.

Proof. Define $\varphi : P \to 2^A$ as $\varphi(h) = h^{-1}(1)$. Let $f, g \in P$ be arbitrary such that $f \neq g$. Then there exists an $a \in A$ such that $f(a) \neq g(a)$. Without loss of generality, assume that $f(a) = 0$ and $g(a) = 1$. Then we have that $a \notin f^{-1}(1) = \varphi(f)$ and $a \in g^{-1}(1) = \varphi(g)$, so $\varphi(f) \neq \varphi(g)$ and therefore φ is injective. Now let $Y \in 2^A$ be arbitrary. Then define the function $f : A \to \{0, 1\}$ such that for $a \in A$,

$$f(a) = \begin{cases} 0 & a \notin Y \\ 1 & a \in Y \end{cases}.$$

Then since $Y \in 2^A$ means that $Y \subset A$, we have that

$$\varphi(f) = f^{-1}(1) = \{a \in A : f(a) = 1\} = \{a \in A : a \in Y\} = Y,$$

so φ is also surjective, making it a bijection. Since there is a bijection from P to 2^A, we have that $|P| = |2^A|$, as required. □

Theorem 1.22. There is no surjection between a set A and its power set 2^A ($|A| \neq |2^A|$)

Proof. Let A be an arbitrary set and suppose for contradiction that $f : A \to 2^A$ is a surjection. Consider the set $X = \{a \in A : a \notin f(a)\}$. Since $X \subset A$, $X \in 2^A$, and since f is surjective, there exists an $a_0 \in A$ such that $f(a_0) = X$. There are two cases, both of which lead to a contradiction. In the first case, $a_0 \in X$, and so we have that $a_0 \notin f(a_0) = X$, a contradiction. In the second case, $a_0 \notin X$, so $a_0 \in f(a_0) = X$, also a contradiction. Since both cases lead to a contradiction, we see that the assumption that f is surjective was false, so there can be no surjection between A and 2^A, which means $|A| \neq |2^A|$. □

1.4 The Schroeder-Bernstein Theorem

Theorem 1.28. The unit square and the unit interval have the same cardinality.

Proof. Let $a \in [0, 1]$ be arbitrary. Then it can be represented as

$$a = \sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers with $a_n \in \{0, 1, \ldots, 9\}$ that does not end all in 9s (that is, if $a_i = 9$, then there exists $j > i$ such that $a_j \neq 9$). Now we claim that the function $f : [0, 1] \to [0, 1] \times [0, 1]$ is surjective, where

$$f\left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n\right) = \left(\sum_{n=1}^{\infty} a_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} a_{2n} \left(\frac{1}{10}\right)^n\right).$$

To show surjectivity, let $y \in [0, 1] \times [0, 1]$ be arbitrary. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ such that

$$y = \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n\right).$$

Let $x \in [0,1]$ be the point

$$x = \sum_{n=1}^{\infty} \left[a_n \left(\frac{1}{10}\right)^{2n-1} + b_n \left(\frac{1}{10}\right)^{2n} \right] = 0.a_1b_1a_2b_2a_3b_3 \cdots = 0.c_1c_2c_3c_4c_5c_6 \ldots .$$

Then

$$\begin{aligned} f(x) = f(0.c_1c_2c_3\ldots) &= \left(\sum_{n=1}^{\infty} c_{2n-1} \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} c_{2n} \left(\frac{1}{10}\right)^n \right) \\ &= \left(\sum_{n=1}^{\infty} a_n \left(\frac{1}{10}\right)^n, \sum_{n=1}^{\infty} b_n \left(\frac{1}{10}\right)^n \right) = y. \end{aligned}$$

Therefore f is surjective. Now consider $g : [0,1] \times [0,1] \to [0,1]$ where $g(a,b) = a$. Let $y \in [0,1]$ be arbitrary and set $x = (y,0) \in [0,1] \times [0,1]$. Then $g(x) = g(y,0) = y$, and so g is surjective. Since there is a surjection from $[0,1]$ to $[0,1] \times [0,1]$ and a surjection from $[0,1] \times [0,1]$ to $[0,1]$, by the surjective version of the Schroeder Bernstein Theorem (1.26), we have that $|[0,1] \times [0,1]| = |[0,1]|$ as required. $\square$

1.5 The Axiom of Choice

Exercise 1.32. Let X be a set and let P be the poset of all subsets of X partially ordered by inclusion. Let $p \in P$ be an element of the poset with $X \leq p$. Then we have that $X \subset p$. We also have that $p \subset X$ since $p \in P$ and P is the poset of all subsets of X. Therefore $p = X$, and so X is by definition a maximal element of P. Suppose there exists a set $Y \in P$ such that Y is also a maximal element of P. Then we have that $X \subset Y$ since Y is a maximal element, and also that $Y \subset X$ since $Y \in P$. Therefore $X = Y$, which shows that X is the unique maximal element of P. Now let $p \in P$ be an elemnet of the poset with $p \leq \emptyset$. Then we have that $p \subset \emptyset$, but also that $\emptyset \subset p$ since $\emptyset \subset x$ for all sets x. Therefore $p = \emptyset$, and so by definition we have that $\emptyset$ is a least element of the poset P. Now let $Z \in P$ be a least element. Then $Z \subset \emptyset$ since it is a least element, but as before, $\emptyset \subset Z$, which means $Z = \emptyset$ and so $\emptyset$ is the unique least element of the poset P.

Exercise 1.33. Let P be the poset ordered by cardinality with $P = \{\{0\},\{1\},\{0,1\},\{1,2\}\}$. Then $\{0\}$ and $\{1\}$ are least elements and $\{0,1\}$ and $\{1,2\}$ are maximal elements.

Exercise 1.34. Consider $\mathbb{R}$ with the $\leq$ relation. The relation is reflexive, transitive, and antisymmetric, so it is a partial order on $\mathbb{R}$. We also have that for any two elements $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$, so they are comparable. This means that $\mathbb{R}$ is totally ordered with the relation $\leq$. However, $\mathbb{R}$ is not well-ordered, because there exist nonempty subsets of $\mathbb{R}$ that do not have least elements, for example, $(0, 1) \subset \mathbb{R}$.

2 Topological Spaces: Fundamentals

2.2 Open Sets and the Definition of a Topological Space

Theorem 2.1. If $\{U_i\}_{i=1}^n$ is a finite collection of open sets in a topological space $(X, \mathcal{T})$, then $\bigcap_{i=1}^n U_i$ is open.

Proof. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. We argue by induction on n that $\bigcap_{i=1}^n U_i$ is open. For the base case $n = 1$, we have that U_1 is open. For the inductive step, assume as inductive hypothesis that there exists a $k \in \mathbb{N}$ such that $\bigcap_{i=1}^k U_i$ is open. Then since U_{k+1} is open, we have that $\bigcap_{i=1}^{k+1} U_i = \left(\bigcap_{i=1}^k U_i\right) \cap U_{k+1}$ is the intersection of two open sets and is therefore open. By induction, $\bigcap_{i=1}^n U_i$ is open for all $n \in \mathbb{N}$, in otherwords, the intersection of finitely many open sets is open. $\square$

Exercise 2.2. The above theorem does not show that the intersection of infinitely many open sets is open since the intersection of infinitely many open sets cannot be represented as $\bigcap_{i=1}^n U_i$ for any $n \in \mathbb{N}$.

Theorem 2.3. A set U is open in a topological space $(X, \mathcal{T})$ iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.

Proof. Let $(X, \mathcal{T})$ be a topological space with $U \in \mathcal{T}$. Let $x \in U$ be arbitrary and set $U_x = U$. Then U_x is open and $x \in U_x \subset U$, as required. Now let U be a set such that for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Then we have that $\bigcup_{x \in U} U_x$ is open, since the union of a collection of open sets is open. We also have that $y \in U$ implies that $y \in \bigcup_{x \in U} U_x$ since $y \in U_y$, so $U \subset \bigcup_{x \in U} U_x$. If $y \in \bigcup_{x \in U} U_x$, then $y \in U_z \subset U$ for some $z \in U$, so $\bigcup_{x \in U} U_x \subset U$, which means that the open set $\bigcup_{x \in U} U_x$ is the set U. Therefore U is open iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. $\square$

Exercise 2.4. First note that, vacuously, $\emptyset \in \mathcal{T}_{\text{std}}$, so the first property is satisfied. Next, consider the set $\mathbb{R}^n \subset \mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be an arbitrary point, and since $B(p, 1) \subset \mathbb{R}^n$ and p was arbitrary, we have that $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$, so the second property is satisfied. For the third property, let $U, V \in \mathcal{T}_{\text{std}}$ be arbitrary open sets in $\mathbb{R}^n$. Let p be an arbitrary point in $U \cap V$. Then because $p \in U$, there exists an ε_1 such that $B(p, \varepsilon_1) \subset U$, and because $p \in V$, there exists an ε_2 such that $B(p, \varepsilon_2) \subset V$. Set $\varepsilon_p = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B(p, \varepsilon_p) \subset B(p, \varepsilon_1) \subset U$ and $B(p, \varepsilon_p) \subset B(p, \varepsilon_2) \subset V$, so we have that $B(p, \varepsilon_p) \subset U \cap V$. Therefore $U \cap V \in \mathcal{T}_{\text{std}}$ and the third property is satisfied. For the fourth property, let $\{U_i\}_{i \in \lambda}$ be a collection of sets $U_i \in \mathcal{T}_{\text{std}}$. Then let $p \in \bigcup_{i \in \lambda} U_i$ be arbitrary. Since p is in the union of all the U_i, we have that $p \in U_j$ for some $j \in \lambda$. Then since $U_j \in \mathcal{T}_{\text{std}}$, there exists an ε_p such that $B(p, \varepsilon_p) \subset U_j \subset \bigcup_{i \in \lambda} U_i$, and therefore $\bigcup_{i \in \lambda} U_i \in \mathcal{T}_{\text{std}}$, so the fourth property is satisfied and we have that $\mathcal{T}_{\text{std}}$ is indeed a topology on $\mathbb{R}^n$.

Exercise 2.6. The unit interval $(0, 1) \subset \mathbb{R}$ is open in the standard topology on $\mathbb{R}$, open in the discrete topology, not open in the indiscrete topology, not open in the finite complement topology, and not open in the countable complement topology.

Exercise 2.7. In the topological space $(\mathbb{R}, \mathcal{T}_{\text{std}})$, the interval $(0, 1)$ is open and for all $n \geq 0$, the set $U_n \subset (0, 1)$ where

$$U_n = \left(\frac{2^n - 1}{2^{n+1}}, \frac{2^n + 1}{2^{n+1}} \right).$$

Then $\frac{1}{2} \in U_n$ for all n and therefore $\frac{1}{2} \in \bigcap_{n=0}^{\infty} U_n$. Let $x \in \mathbb{R}$ such that $x \neq \frac{1}{2}$. Then there exists some $m \geq 0$ such that $|x - \frac{1}{2}| > \frac{1}{2^{m+1}}$, which means that

$$x \notin \left(\frac{1}{2} - \frac{1}{2^{m+1}}, \frac{1}{2} + \frac{1}{2^{m+1}} \right) = \left(\frac{2^m - 1}{2^{m+1}}, \frac{2^m + 1}{2^{m+1}} \right) = U_m.$$

Since there exists an $m \geq 0$ such that $x \notin U_m$, we have that $x \notin \bigcap_{n=0}^{\infty} U_n$, and since x was arbitrary, we have that $\bigcap_{n=0}^{\infty} U_n = \{\frac{1}{2}\} \notin \mathcal{T}_{\text{std}}$. Therefore the infinite intersection of open sets is not necessarily open.

2.3 Limit Points and Closed Sets

Exercise 2.8. In the indiscrete topology on $\mathbb{R}$, the point 0 is in $\mathbb{R}$ but not $\emptyset$, so since $(\mathbb{R} - \{0\}) \cap (1, 2) = (1, 2) \neq \emptyset$, we have that $(U - \{0\}) \cap (1, 2) \neq \emptyset$ for all open sets U containing 0. Therefore 0 is a limit point of $(1, 2)$ in the indiscrete topology. In the finite complement topology, let $U \subset \mathbb{R}$ be an open set containing 0. Then suppose for contradiction that $(U - \{0\}) \cap (1, 2) = \emptyset$. Then for all $p \in (1, 2) \subset \mathbb{R}$, we have that $p \notin U - \{0\}$, which means that $p \in \mathbb{R} - (U - \{0\}) = (\mathbb{R} - U) \cup \{0\}$. Therefore $(1, 2) \subset (\mathbb{R} - U) \cup \{0\}$, but this is a contradiction since U is open and therefore we have that an infinite set is a subset of a finite set. Hence it must be the case that $(U - \{0\} \cap (1,2)) \neq \emptyset$ for all U containing 0, and therefore 0 is a limit point of $(1,2)$. In the standard topology and in the discrete topology, the set $(-1,1)$ is an open set containing 0, but $(-1,1) \cap (1,2) = \emptyset$, so 0 is not a limit point of $(1,2)$.

Theorem 2.9. Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A iff there exists a neighborhood U of p such that $A \cap U = \emptyset$.

Proof. Suppose $p \notin A$ is not a limit point of A. Then there exists an open set U containing p (a neighborhood U of p) such that $(U - \{p\}) \cap A = \emptyset$. Since $p \notin A$, we have that $p \notin A \cap U$, and therefore $A \cap U = \emptyset$, as required. Now suppose there exists a neighborhood U of p such that $A \cap U = \emptyset$. Then since $p \notin A$, we have that $(U - \{p\}) \cap A = \emptyset$ as well, and therefore p is not a limit point of A. □

Exercise 2.10. If p is an isolated point of A in a topological space $(X, \mathcal{T})$, then by the definition of an isolated point, we have that $p \in A$ but p is not a limit point of A. Therefore there exists an open set U such that $(U - \{p\}) \cap A = \emptyset$, and since $p \in A, U$, we have that

$$U \cap A = ((U - \{p\}) \cap A) \cup \{p\} \cap A = \emptyset \cup \{p\} = \{p\}.$$

Therefore if p is an isolated point of A, there exists an open set U such that $A \cap U = \{p\}$.

Exercise 2.11. Let $X = \{a, b, c, d\}$ be a set, let $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$ be a topology on X, and let $A = \{b, c\}$ be a set.

(1) $c \in A$ is a limit point of A since the only open set containing c is X, and we have that $(X - \{c\}) \cap A = \{b\} \neq \emptyset$.

(2) $d \notin A$ is a limit point of A also because X is the only open set containing d and $(X - \{d\}) \cap A = \{b, c\} \neq \emptyset$.

(3) $b \in A$ is an isolated point of A because it is in A but is not a limit point since $\{a, b\}$ is open but $(\{a, b\} - \{b\}) \cap A = \emptyset$.

(4) $a \notin A$ is not a limit point of A since $\{a\}$ is an open set but $(\{a\} - \{a\}) \cap A = \emptyset$.

Exercise 2.12. Let X be a set and let $\mathcal{T}$ be a topology on X. If the set X has a limit point p, then $p \in X$, so $\overline{X} = X$ and X is closed. If Now let $p \in X$ and let U be an open set containing p. Then $(U - \{p\}) \cap \emptyset = \emptyset$, so there are no limit points of the empty set, and therefore, vacuously, $\overline{\emptyset} = \emptyset$ and the empty set is closed.

(1) In the discrete topology, all sets are closed. Let A be a nonempty proper subset of X and let $p \in X$ be arbitrary. Then there exists a point $q \in X$ with $q \notin A$, and since all sets are open in the discrete topology, then set $\{p, q\}$ is a neighborhood of p that satisfies $(\{p, q\} - \{p\}) \cap A = \emptyset$. Thus we have shown that there are no limit points of A, and so vacuously, $\overline{A} = A$.

(2) In the indiscrete topology, only X and $\emptyset$ are closed. Let A be a nonempty propoer subset of X and let $p \in X$ such that $p \notin A$. Then p is a limit point of A since X is the only open set containing p and it satisfies $(X - \{p\}) \cap A \neq \emptyset$. However, $p \notin A$, so $\overline{A} \neq A$.

Theorem 2.13. For any topological space $(X, \mathcal{T})$, and $A \subset X$, the set $\overline{A}$ is closed, that is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

Proof. Let A be a set in a topological space $(X, \mathcal{T})$. Since the closure of a set contains all the points in the set, we have that $\overline{A} \subset \overline{\overline{A}}$. Now let $x \in \overline{\overline{A}}$ and let U be an arbitrary neighborhood of x. Either $x \in \overline{A}$ (in which case we're done), or x is a limit point of $\overline{A}$, in which case we have that $(U - \{x\}) \cap \overline{A} \neq \emptyset$. Since $(U - \{x\}) \cap \overline{A} \subset U \cap \overline{A}$, so $U \cap \overline{A}$ is nonempty in either case. Therefore there exists some $y \in U \cap \overline{A}$, which means we have a neighborhood U of y such that $y \in \overline{A}$. As before, either $y \in A$ or y is a limit point of A, in which case we have that $\emptyset \neq (U - \{y\}) \cap A \subset U \cap A$. Therefore for an arbitrary neighborhood U of $x \in \overline{\overline{A}}$, we have shown that $U \cap A$ is nonempty. Either $x \in A \subset \overline{A}$, or $x \notin A$ and therefore $x \notin U \cap A \neq \emptyset$. Since this intersection is nonempty, we have also that $(U - \{x\}) \cap A \neq 0$, but U was an arbitrary neighborhood of x, so we have shown that $(U - \{x\}) \cap A$ is nonempty for all neighborhoods U of x. Therefore x is a limit point of A, and so $x \in \overline{A}$. In both cases, $x \in \overline{A}$ and so $\overline{\overline{A}} \subset \overline{A}$, and since $\overline{A} \subset \overline{\overline{A}}$ also, we have that $\overline{\overline{A}} = \overline{A}$, as required. $\square$

Theorem 2.14. For any topological space $(X, \mathcal{T})$, a subset $A \subset X$ is closed if and only if $X - A$ is open.

Proof. Suppose A is closed and let $x \in X - A$. Then since A is closed, it contains all its limit points and therefore x is not a limit point. This means there exists an open set U such that $U - \{x\} \cap A = \emptyset$. Let $y \in U - \{x\}$. Then $y \notin A$, so $y \in X - A$, which means $U - \{x\} \subset X$. Since x was an arbitrary element of $X - A$, by Theorem 2.3, we have that $X - A$ is open. Now suppose that $X - A$ is open. $A \subset \overline{A}$, and we will show that $\overline{A} \subset A$. Suppose for contradiction that $\overline{A} \not\subset A$. Then there exists an $a \in \overline{A} - A$, which means that a is a limit point of A, so all open intervals U containing a satisfy $U \cap A \neq \emptyset$ (by Theorem 2.9).

In particular, since $X - A$ is open, we have that $(X - A) \cap A \neq \emptyset$, which is a contradiction. Therefore $\overline{A} = A$ and A is closed. $\square$

Corollary 2.14. If A is an open set, then $X - (X - A)$ is open, which means $X - A$ is closed.

Theorem 2.15. For any topological space $(X, \mathcal{T})$ with an open set $U \in \mathcal{T}$ and a closed set $A \in \mathcal{T}$, $U - A$ is open and $A - U$ is closed.

Proof. Since A is closed, $X - A$ is open, and so $U \cap (X - A)$ is also open since the intersection of two open sets is open. Then we have that $X - (U \cap (X - A))$ is closed by the corollary to Theorem 2.14. $X - (U \cap (X - A)) = X - (U - A)$ is closed, so $U - A$ is open, as claimed. The union of open sets is open, so we also have that $U \cup (X - A)$ is open. $U \cup (X - A) = X - (A - U)$ is open, so $A - U$ is closed, as claimed. $\square$

Theorem 2.16. Let $(X, \mathcal{T})$ be a topological space. Then:
(i) $\emptyset$ is closed.
(ii) X is closed.
(iii) The union of finitely many sets is closed.
(iv) Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

Proof. For (i) and (ii), see exercise 2.12. For (iii), let $\{A_i\}$, $1 \leq i \leq n$ for some $n \in \mathbb{N}$. Then for each A_i, $X - A_i$ is open, and we have that the intersection of finitely many open sets is open, so $\bigcap_{i=1}^{n}(X - A_i)$ is open. By the corollary to Theorem 2.14 and DeMorgan's Laws,

$$X - \left(\bigcap_{i=1}^{n} (X - A_i) \right) = X - \left(X - \bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} A_i$$

is closed. For (iv), let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed sets in $(X, \mathcal{T})$. Then for each A_α, $X - A_\alpha$ is open, and we have that the union of a collection of open sets is open, so $\bigcup_{\alpha \in \lambda}(X - A_\alpha)$ is open. Again by DeMorgan's Laws,

$$X - \left(\bigcup_{\alpha \in \lambda} (X - A_\alpha) \right) = X - \left(X - \bigcap_{\alpha \in \lambda} A_\alpha \right) = \bigcap_{\alpha \in \lambda} A_\alpha$$

is closed. $\square$

Exercise 2.19. (1) In $\mathbb{Z}$ with the finite complement topology, the set $\{0, 1, 2\}$ is not open, since $\mathbb{Z} - \{0, 1, 2\}$ is infinite. $\mathbb{Z} - \{0, 1, 2\}$, however, is open since $\{0, 1, 2\}$ is finite, and therefore $\mathbb{Z} - (\mathbb{Z} - \{0, 1, 2\}) = \{0, 1, 2\}$ is closed. The set of prime numbers has an infinite complement, so it is not open, but there are infinitely many prime numbers so it is also not closed. The set $\{n : |n| > 10\}$ has a finite complement, so it is open, but the set itself is infinite and therefore is not closed.

(2) In $\mathbb{R}$ with the standard topology, the set $(0, 1)$ is open, and its limit points are 0 and 1, neither of which are in $(0, 1)$, so it is not closed. The set $(0, 1]$ is neither closed nor open, since it contains one of its limit points but not both. The set $[0, 1]$ contains both limit points and is therefore closed, and it is not open. The set $\{0, 1\}$ has no limit points so is vacuously closed, and it is not open. The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not open since there is no $\varepsilon > 0$ such that $(1-\varepsilon, 1+\varepsilon) \subset \{\frac{1}{n} : n \in \mathbb{N}\}$. Note that 0 is a limit point of this set since for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ and therefore $\frac{1}{n_0} \in ((-\varepsilon, \varepsilon) - \{0\}) \cap \{\frac{1}{n} : n \in \mathbb{N}\} \neq \emptyset$. Since $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, the set is not closed in $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

(3) In $\mathbb{R}^2$ with the standard topology, the set $C = \{(x, y) : x^2+y^2 = 1\}$ is not open since if $p \in C$ and $\varepsilon_p > 0$, then $p - \left(\frac{\varepsilon_p}{2}, \frac{\varepsilon_p}{2}\right)$ is in $B(p, \varepsilon_p)$ but not in C. If $p = (\cos\theta_0, \sin\theta_0) \in C$, then p is a limit point of C since the point $(\cos\theta_1, \sin\theta_1) \in C$ and if $|\theta_1 - \theta_0| < \arccos\left(1 - \frac{\varepsilon_p^2}{2}\right)$, then $(\cos\theta_1, \sin\theta_1) \in (B(p, \varepsilon_p) - \{p\}) \cap C \neq \emptyset$. If $p \notin C$, then set ε to be the distance from p to the nearest point of C. We have that $(B(p, \frac{\varepsilon}{2}) - \{p\}) \cap C = \emptyset$, so all points of C are limit points and all points not in C are not limit points, which means $\overline{C} = C$ and therefore C is closed. Let $D = \{(x, y) : x^2 + y^2 < 1\}$. Then C is the set of all the limit points of D, and so $C \cup D = \overline{D}$ is closed. Therefore, $\{(x, y) : x^2 + y^2 > 1\} = \mathbb{R}^2 - \overline{D}$ is open, and its limit points are also all in the set C and therefore this set is not closed. The set D is open since $D = B(0, 1) \in \mathcal{T}_{\text{std}}$, and so the set $\{(x, y) : x^2 + y^2 \geq 1\} = \mathbb{R}^2 - D$ is closed. This set is not open since there is no ε such that $B(p, \varepsilon) \subset \{(x, y) : x^2 + y^2 \geq 1\}$ for $p \in \{(x, y) : x^2 + y^2 \geq 1\} \cap C$.

Theorem 2.20. For any set A in a topological space $(X, \mathcal{T})$, the closure of A is the intersection of all closed sets containing A, that is, $\overline{A} = \bigcap_{B \supset A, B \in \mathfrak{C}} B$, where $\mathfrak{C}$ is the set of all closed sets in $(X, \mathcal{T})$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$ and let A be a subset of X. Since its closure $\overline{A}$ is closed and $A \subset \overline{A}$, we have that $\overline{A} \in \mathfrak{C}$, and since for any sets M and N we have that $M \cap N \subset M, N$, we have that $\bigcap_{B \supset A, B \in \mathfrak{C}} B \subset \overline{A}$. To show equality, then, we need only show that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. Let $a \in \overline{A}$. There are two cases. In the first case, if

$a \in A$, then $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$, which means that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$. In the second case, we have that $a \notin A$, which means that $a \in \overline{A} - A$ is a limit point of a. This means that for all open sets U with $a \in U$, we have that $(U - \{a\}) \cap A \neq \emptyset$. Since $a \notin A$, we have that $U \cap A = (U - \{a\}) \cap A \neq \emptyset$. Let $B_0 \in \mathfrak{C}$ with $A \subset B_0$ and suppose for contradiction that $a \notin B_0$. Since B_0 is closed, $X - B_0$ is open, and since $a \notin B_0$, $a \in X - B_0$, which means that $(X - B_0) \cap A \neq \emptyset$. Therefore there exists some $y \in (X - B_0) \cap A$. We have that $y \in A \subset B_0$, which means that $y \notin X - B_0$, a contradiction. We have reached a contradiction by assuming that there exists a set $B_0 \in \mathfrak{C}$ such that $A \subset B_0$ and $a \notin B_0$, which means that $a \in B$ for all $B \in \mathfrak{C}$ such that $A \subset B$. Therefore in both cases $a \in A$ and $a \in \overline{A} - A$, we have that $a \in \bigcap_{B \supset A, B \in \mathfrak{C}} B$, which means that $\overline{A} \subset \bigcap_{B \supset A, B \in \mathfrak{C}} B$. This shows that $\overline{A}$ is indeed the intersection of all closed sets containing A, as required. $\square$

Exercise 2.21. Consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. In the discrete topology, this set is already closed and so is its own closure. In the indiscrete topology, only $\mathbb{R}$ and $\emptyset$ are closed, so the closure of H in the indiscrete topology is $\mathbb{R}$ itself. In the finite complement topology, let $p \in \mathbb{R}$ be an arbitrary point. Then let U be an arbitrary open set containing p. Since U is open, its complement is finite, which means there are only finitely many points not in $U - \{p\}$, so there must be infinitely many points in $(U - \{p\}) \cap H \neq \emptyset$. Since U was an arbitrary open set, p is a limit point, and since p was an arbitrary point in $\mathbb{R}$, we see that all points are limit points of H, which means that $\overline{H} = \mathbb{R}$. In the countable complement topology, H is closed since it contains countably many elements, so it is its own closure. In the standard topology, the only limit point of H is 0, so the closure of H is $\overline{H} = H \cup \{0\}$.

Theorem 2.22. Let A and B be subsets of a topological space X with topology $\mathcal{T}$. Then:

(1) $A \subset B$ implies $\overline{A} \subset \overline{B}$.

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) We have that $A \subset B \subset \overline{B}$, and by Theorem 2.20, we have that $\overline{A}$ is a subset of all closed sets containing A. Since $\overline{B}$ is a closed set containing A, we have that $\overline{A} \subset \overline{B}$, as required.

(2) Let $c \in \overline{A} \cup \overline{B}$. Without loss of generality, assume $c \in \overline{A}$. There are two cases. For the first case $c \in A$, we have that $c \in A \subset A \cup B \subset \overline{A \cup B}$. For the second case $c \in \overline{A} - A$, c is a limit point of A and therefore for all open sets U containing c satisfy $\emptyset \neq (U - \{c\}) \cap A \subset (U - \{c\}) \cap (A \cup B)$. Since U was an arbitrary open set, c is a limit point of $A \cup B$ and therefore $c \in \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Now let $d \in \overline{A \cup B}$ be arbitrary. Again there are two cases. In the first case, $d \in A \cup B$, so without loss of

generality assume $d \in A \subset \overline{A} \subset \overline{A} \cup \overline{B}$. In the second case, $d \in \overline{A \cup B} - (A \cup B)$, so d is a limit point of $A \cup B$. This means that for all open sets U containing d, we have that $\emptyset \neq (U - \{d\}) \cap (A \cup B) = ((U - \{d\}) \cap A) \cup ((U - \{d\}) \cap B)$, so one of the sets on the right hand side is nonempty. Without loss of generality, assume $(U - \{d\}) \cap A \neq \emptyset$. Then d is a limit point of A, so $d \in \overline{A} \subset \overline{A} \cup \overline{B}$. Therefore $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ as well, and so the two sets are equal. □

Exercise 2.25. I'm not entirely sure but I believe the Cantor Set fits this description.

2.4 Interior and Boundary

Theorem 2.26. Let A be a subset of a topological space $(X, \mathcal{T})$. Then p is an interior point if and only if there exists an open set U such that $p \in U \subset A$.

Proof. Let $A \subset X$ be arbitrary and let $p \in A$ be an interior point. Then since A° is the union of all open subsets of A, A° is open as well, and so by Theorem 2.3, there exists an open set U such that $p \in U \subset A^\circ \subset A$ where $A^\circ \subset A$ follows from the fact that if $a \in A^\circ$, then $a \in \bigcup_{U \subset A, U \in \mathcal{T}} U$ and therefore there exists an open set U_0 such that $a \in U_0 \subset A$. Now let $p \in A$ be an arbitrary point such that there is an open set U with $a \in U \subset A$. Since U is open, $U \in \mathcal{T}$ and so $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U = A^\circ$. Thus, p is an interior point, as required. □

Exercise 2.27. If U is open in a topological space, then by Theorem 2.3, for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$, which means x is an interior point of U. If x is an interior point of U, then by Theorem 2.26, there exists a set U_x such that $x \in U_x \subset U$ and so U is open. Therefore U is open in a topological space if and only if every point of U is an interior point.

Lemma 2.28. Given a set A in a topological space $(X, \mathcal{T})$, the closure of A is $\overline{A} = X - (X - A)^\circ$ and the interior of A is $A^\circ = X - \overline{X - A}$.

Proof. Let $\mathfrak{C}$ be the set of all closed sets in $(X, \mathcal{T})$. For all $B \in \mathfrak{C}$ such that $A \subset B$, we have that $X - B \subset X - A$, and since B is closed, $X - B$ is open. Given an open set U with $U \subset X - A$, $X - U \supset A$ is closed. This means that $\{B \in \mathfrak{C} : B \supset A\} = \{X - U : U \in$

$\mathcal{T}$ s.t. $U \subset X - A\}$. Therefore we have that

$$\begin{aligned}\overline{A} &= \bigcap_{B \supset A, B \in \mathfrak{C}} B = X - \left(X - \bigcap_{B \supset A, B \in \mathfrak{C}} B \right) = X - \left(\bigcup_{B \supset A, B \in \mathfrak{C}} (X - B) \right) \\ &= X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} (X - (X - U)) \right) = X - \left(\bigcup_{U \subset X - A, U \in \mathcal{T}} U \right) = X - (X - A)^\circ,\end{aligned}$$

as required. The proof that $A^\circ = X - \overline{X - A}$ is similar $\square$

Theorem 2.28. Let A be a subset of a topological space $(X, \mathcal{T})$. Then A°, ∂A, and $(X - A)^\circ$ are all disjoint and their union is X.

Proof. Let $a \in A^\circ$ and suppose for contradiction that $a \in \partial A = \overline{A} \cap \overline{X - A}$. Then either $a \in X - A$ or a is a limit point of $X - A$. In the first case, we have a clear contradiction since $a \in A^\circ \subset A$ cannot be in $X - A$. In the second case, every open set U containing a satisfies $(U - \{a\}) \cap (X - A) \neq \emptyset$. However, $a \in A^\circ$, so there exists an open set V such that $a \in V \subset A$, so we have that $\emptyset \neq (V - \{a\}) \cap (X - A) \subset A \cap (X - A) = \emptyset$, a contradiction. Therefore the sets A° and ∂A are disjoint, and a similar argument shows that ∂A and $(X - A)^\circ$ are disjoint as well. It remains to check that A° and $(X - A)^\circ$ are disjoint. Suppose not, so that there exists an $a \in A^\circ$ such that $a \in (X - A)^\circ$. By Theorem 2.26, there exists an open set U_1 containing a such that $U_1 \subset A$ and an open set U_2 containing a such that $U_2 \subset X - A$. Therefore we have that $a \in U_1 \cap U_2 \subset A \cap (X - A) = \emptyset$, a contradiction showing that A° and $(X - A)^\circ$ must be disjoint. Since $A^\circ = X - \overline{X - A}$ by Lemma 2.28, we have that

$$\begin{aligned}\overline{A} - A^\circ &= \overline{A} - (X - \overline{X - A}) \\ &= \overline{A} \cap (X - (X - \overline{X - A})) \\ &= \overline{A} \cap \overline{X - A} = \partial A,\end{aligned}$$

so we see that the closure of A is the disjoint union of the boundary of A and the interior of A. Again using Lemma 2.28 ($\overline{A} = X - (X - A)^\circ$), we also have that

$$\begin{aligned}X &= \overline{A} \cup (X - \overline{A}) = A^\circ \cup \partial A \cup (X - \overline{A}) \\ &= A^\circ \cup \partial A \cup (X - (X - (X - A)^\circ)) \\ &= A^\circ \cup \partial A \cup (X - A)^\circ.\end{aligned}$$

Therefore we have that A°, ∂A, $(X - A)^\circ$ are disjoint sets whose union is X. $\square$

Exercise 2.29. Again consider the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$. In the discrete topology, we have that $\overline{H} = H$ and $\overline{\mathbb{R} - H} = \mathbb{R} - H$, and so we have that $\partial N = \emptyset$. The interior of N is $N^\circ = \overline{H} - \partial H = H - \emptyset = H$. In the indiscrete topology, we have that $\overline{H} = \mathbb{R} = \overline{\mathbb{R} - H}$, so $\partial H = \mathbb{R}$ and $H^\circ = \overline{H} - \partial H = \mathbb{R} - \mathbb{R} = \emptyset$. In the finite complement topology, we saw in Exercise 2.21 that the closure of H was $\overline{H} = \mathbb{R}$ and by the same reasoning, $\overline{\mathbb{R} - H} = \mathbb{R}$. Therefore once again we have that $\partial H = \mathbb{R}$ and $H^\circ = \emptyset$, which makes sense since $\emptyset$ is the largest open set contained within H since H is countably infinite. In the countable complement topology, $H = \overline{H}$ is the disjoint union of H° and ∂H. If V is a nonempty open set in the countable complement topology, then $\mathbb{R} - V$ is countable, meaning that V is uncountable, so we cannot have $V \subset H$. The only open set U with $U \subset H$ is $U = \emptyset$. Therefore $H^\circ = \emptyset$ and $\partial H = H$. In the standard topology, no open set contains 1 and $\frac{1}{2}$ without also containing all points in the open set $(\frac{1}{2}, 1)$, which means that no open set is a subset of H. Therefore we have that $H^\circ = \emptyset$, and so $\partial H = \overline{H} = H \cup \{0\}$.

2.5 Convergence of Sequences

Theorem 2.30. Let A be a set in a topological space $(X, \mathcal{T})$, and let p be a point in X. If $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$.

Proof. Suppose for contradiction that $\{x_i\}_{i \in \mathbb{N}} \subset A$, $x_i \to p$, but $p \notin \overline{A}$. Then p is not a limit point of A, so there exists an open set U_0 containing p such that $(U_0 - \{p\}) \cap A = \emptyset$. Since $x_i \to p$ and $p \in U_0$, we have that there exists an $N \in \mathbb{N}$ such that $x_i \in U_0$ for all $i > N$. Since $x_i \in A$ for all $i \in \mathbb{N}$, $x_i \neq p$ for all $i \in \mathbb{N}$. Therefore, we have that for all $i > N$, $x_i \in (U_0 - \{p\}) \cap A = \emptyset$, a contradiction. Therefore if $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then $p \in \overline{A}$. In particular, p is a limit point of A. $\square$

Theorem 2.31. In the standard topology on $\mathbb{R}^n$, if p is a limit point of a set A, then there is a sequence of points in A that converges to p.

Proof. If p is a limit point of A, then $A \neq \emptyset$, so there exists a point $a \in A$. Set $\varepsilon = d(a, p) > 0$. Then for all $n \in \mathbb{N}$, $B(p, \frac{\varepsilon}{n})$ is open. Since p is a limit point and $p \in B(p, \frac{\varepsilon}{n})$, we have that $(B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A \neq \emptyset$. Since each of these sets is nonempty there exists an $a_n \in (B(p, \frac{\varepsilon}{n}) - \{p\}) \cap A$ for all n. Define a sequence $(x_n)_{n \in N}$ such that $x_n = a_n$. Therefore $\{x_n\}_{n \in \mathbb{N}} \subset A$, so it only remains to show that the sequence converges to p. Let U be an open set containing p. Since U is open in the standard topology, $U = B(p, \varepsilon_p)$ for some $\varepsilon_p > 0$, and there exists and $N \in \mathbb{N}$ such that $\frac{\varepsilon}{N} < \varepsilon_p$. Let $i > N$. We have that

$$x_i \in B\left(p, \frac{\varepsilon}{i}\right) \subset B\left(p, \frac{\varepsilon}{N}\right) \subset B\left(p, \varepsilon_p\right),$$

which means that $x_i \in U$ for all $i > N$, and therefore $x_i \to p$, as required. $\square$

Exercise 2.32. Consider $\mathbb{R}$ with the indiscrete topology. Let p be a point in $\mathbb{R}$ and set $x_n = n$. This sequence converges to p since if U is an open set containing p, then $U = \mathbb{R}$ and so we have that $x_n \in U$ for all $n \in \mathbb{N}$. Therefore $x_n \to p$, but p was arbitrary, so we see that this sequence converges to every point in $\mathbb{R}$.

3 Bases, Subspaces, Products: Creating New Spaces

3.1 Bases

Theorem 3.1. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

(1) $\mathcal{B} \subset \mathcal{T}$, and

(2) for every open set U, and point $p \in U$, there exists a set $V \in \mathcal{B}$ such that $p \in V \subset U$.

Proof. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then by definition, (1) is satisfied. Now let U be an arbitrary open set and let $p \in U$ be an arbitrary point. Since $\mathcal{B}$ is a basis, we have that $U = \bigcup_{V \in \lambda} V$ for some collection of sets $\lambda \subset \mathcal{B}$, and since $p \in U$, we have that $p \in V_0 \subset U$ for some set $V_0 \in \lambda$, so (2) is satisfied as well. Now suppose that $\mathcal{B}$ is a collection of sets satisfying (1) and (2). Let U_0 be an arbitrary open set and define λ to be the collection of sets $\lambda = \{V \in \mathcal{B} : V \subset U_0\} \subset \mathcal{B}$. Now let $p \in U_0$ be an arbitrary point in U_0. By (2), there exists a $V_0 \in \mathcal{B}$ such that $p \in V_0 \subset U_0$, which means $V_0 \in \lambda$. Therefore $p \in \bigcup_{V \in \lambda} V$, and so $U_0 \subset \bigcup_{V \in \lambda} V$. Now let p be an arbitrary point in $\bigcup_{V \in \lambda} V$. Then there exists a $V_0 \in \lambda \subset \mathcal{B}$ such that $p \in V_0$, and since $V_0 \in \lambda$, $V_0 \subset U_0$, so $p \in U_0$. Therefore $U_0 = \bigcup_{V \in \lambda} V$ since both sets are subsets of the other. Since any arbitrary open set U is the union of sets in the collection $\mathcal{B}$, $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

Exercise 3.2. $\mathcal{B}_1$ satisfies Theorem 3.1(1) since it consists only of open intervals and therefore $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Since U is open in $\mathcal{T}_{\text{std}}$, $U = (c - \varepsilon, c + \varepsilon)$ for some $c \in \mathbb{R}$ and $\varepsilon > 0$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $a, b \in \mathbb{Q}$ such that $c - \varepsilon < a < p$ and $p < b < \varepsilon$. We have that $p \in (a, b) \subset U$ and $(a, b) \in \mathcal{B}_1$, so Theorem 3.2(2) is satisfied and therefore $\mathcal{B}_1$ is a basis.

Theorem 3.3. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if

(1) each point of X is in some element of $\mathcal{B}$, and

(2) if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof. If $\mathcal{B}$ is a basis for some topology on X, then $X = \bigcup_{B \in \mathcal{B}} B$, so if $p \in X$, then $p \in B_0$ for some $B_0 \in \mathcal{B}$. Therefore (1) is satisfied. Now let U and V be sets in $\mathcal{B}$ and let p be an arbitrary point in $U \cap V$. Since U and V and in $\mathcal{B}$, they are open, and therefore $U \cap V$ is open, so by Theorem 3.1, there exists a $W \in \mathcal{B}$ such that $p \in W \subset (U \cap V)$, satisfying (2). Now suppose $\mathcal{B}$ is a collection of subsets of X satisfying (1) and (2). Then $\emptyset$ is the empty union of sets in $\mathcal{B}$, and $X = \bigcup_{B \in \mathcal{B}} B$ by (1). Suppose U and V are in $\mathcal{B}$ and set $\lambda = \{W \in \mathcal{B} : W \subset (U \cap V)\} \subset \mathcal{B}$. If p is an arbitrary point of $U \cap V$, then by (2) there exists a W_0 such that $p \in W_0 \subset (U \cap V)$, so $p \in \bigcup_{W \in \lambda} W$. If p is an arbitrary point of $\bigcup_{W \in \lambda} W$, then there exists a $W_0 \in \mathcal{B}$ such that $p \in W_0 \subset (U \cap V)$. Therefore $U \cap V = \bigcup_{W \in \lambda} W$, so the intersection of two sets that are the unions of sets in $\mathcal{B}$ is also the union of sets in $\mathcal{B}$. Now let α be a collection of sets in $\mathcal{B}$. Then for each $\beta \in \alpha$, $\beta = \bigcup_{B \in \lambda} B$ for some collection of sets $\lambda \subset \mathcal{B}$. Then we have that $\bigcup_{B \in \alpha} B$ is the union of a collection of unions of sets in $\mathcal{B}$ and is therefore itself the union of sets in $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for some topology on X since the collection of sets that are unions of sets in $\mathcal{B}$ satisfies all four properties of a topology. $\square$

Exercise 3.4. Let $\mathcal{B}_{\mathrm{LL}}$ be the set of subsets of $\mathbb{R}$ of the form $[a, b)$. Then if $x \in \mathbb{R}$, $x \in [x, x+1)$, so Theorem 3.3(1) is satisfied. Let U and V be arbitrary sets in $\mathcal{B}_{\mathrm{LL}}$. Then there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $U = [a_1, b_1)$ and $V = [a_2, b_2)$. Let $x \in U \cap V$ be arbitrary. Then $a_1, a_2 \leq x < b_1, b_2$, so $p \in W = [\max\{a_1, a_2\}, \min\{b_1, b_2\}) = U \cap V$. Therefore Theorem 3.3(2) is satisfied and so $\mathcal{B}_{\mathrm{LL}}$ is a basis for a topology on $\mathbb{R}$. ($\mathbb{R}$ together with this topology is the Sorgenfrey Line, $\mathbb{R}_{\mathrm{LL}}$.)

Exercise 3.6. As discussed in Exercise 2.21, the set $H = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed in $\mathbb{R}$ with the standard topology, which means that $\mathbb{R} - H$ is not open in the standard topology. However, H contains countably many points, so $\mathbb{R} - H$ is open in the countable complement topology. Therefore we have that the standard topology on $\mathbb{R}$ is not finer than the countable complement topology. On the other hand, the set $(0, 1)$ is open in the standard topology, but not open in the countable complement topology, so we also have that the countable complement topology is not finer than the standard topology.

Exercise 3.7. Let $\mathbb{R}_{+00}$ be the set of all positive real numbers ($\mathbb{R}_+$) together with the points $0'$ and $0''$, and let $\mathcal{B}$ be the set of all intervals of the form (a,b), $(0,b)\cup\{0'\}$, or $(0,b)\cup\{0''\}$ for $a,b \in \mathbb{R}_+$.. We claim that $\mathcal{B}$ is the basis for some topology $\mathcal{T}$. For any $x \in \mathbb{R}_+$, $x \in (\frac{x}{2}, x+1)$, $0' \in (0,1)\cup\{0'\}$, and $0'' \in (0,1)\cup\{0''\}$, so every point in $\mathbb{R}_{+00}$ is in some element of $\mathcal{B}$. Now let U and V be arbitrary elements of $\mathcal{B}$ and let $p \in U \cap V$ be arbitrary. If one of U, V is of the form $(0,b_1)\cup\{0'\}$ and the other is of the form $(0,b_2)\cup\{0''\}$, and p is an arbitrary point in $U\cap V$, then set $W = (\frac{p}{2}, \min\{b_1,b_2\})$. Then we have that $p \in W \subset (U\cap V)$ and $W \in \mathcal{B}$. Otherwise, Set $W = (U \cap V)$. Then $p \in W \subset (U\cap V)$, so to show that $\mathcal{B}$ is the basis for some topology, it remains only to show that $W \in \mathcal{B}$. If one of U, V is of the form (a,b) for some $a,b\in\mathbb{R}_+$, then $U\cap V$ is of the same form and therefore in $\mathcal{B}$. For the remaining case to check, assume without loss of generality that $U = (0,b_1)\cup\{0'\}$ and $V = (0,b_2)\cup\{0'\}$ (the case for $0''$ is the same). Then $U \cap V = (0,\min\{b_1,b_2\})\cup\{0'\} \in \mathcal{B}$. Therefore by Theorem 3.3, $\mathcal{B}$ is a basis for some topology on $\mathbb{R}_{+00}$. (This topological space is called the Double Headed Snake and will also be written as $\mathbb{R}_{+00}$.)

Exercise 3.8. In the Double Headed Snake, $\mathbb{R}_{+00}$, let p be an arbitrary point. If $p = 0'$ or $0''$, we claim that $\{p\}$ is closed. Without loss of generality, assume $p = 0'$. Then

$$\mathbb{R}_{+00} - \{p\} = (0,\infty)\cup\{0''\} = \bigcup_{n\in\mathbb{N}} \left((0,n)\cup\{0''\}\right).$$

Since $(0,n)\cup\{0''\} \in \mathcal{B}$, the basis for the Double Headed Snake given in Exercise 3.7, we have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and is therefore open in the Double Headed Snake, which means that $\{p\}$ is closed. Now if $p \neq 0', 0''$, we have that $p \in \mathbb{R}_+$. We have that

$$\begin{aligned} \mathbb{R}_{+00} - \{p\} &= (0,p)\cup(p,\infty)\cup\{0',0''\} \\ &= ((0,p)\cup\{0'\})\cup((0,p)\cup\{0''\})\cup\left(\bigcup_{p\in\mathbb{N}}(p,p+n)\right). \end{aligned}$$

Since $(p,p+n)\in\mathcal{B}$ for all $n\in\mathbb{N}$, we again have that $\mathbb{R}_{+00} - \{p\}$ is the union of elements of $\mathcal{B}$ and therefore is open in the Double Headed Snake, so $\{p\}$ is closed.

Suppose for contradiction that U and V are disjoint open sets in the Double Headed Snake such that $0' \in U$ and $0'' \in V$. Since $0' \in U$, and U is open, it is the union of sets in $\mathcal{B}$, so there exists a set of the form $(0,b_1)\cup\{0'\}\subset U$. Similarly, there exists a set of the form $(0,b_2)\cup\{0''\}\subset V$. Therefore $(0,\min\{b_1,b_2\})\subset U\cap V$, so the sets are not disjoint since

$b_1, b_2 \in \mathbb{R}_+$. This means it is impossible to have disjoint open sets each containing a different head of the snake.

Exercise 3.9. (1) In the topological space $\mathbb{R}_{\text{har}}$, the set $\mathbb{R} - H$ is open since it is the union of sets in the basis for $\mathbb{R}_{\text{har}}$:

$$\mathbb{R} - H = \bigcup_{n \in \mathbb{N}} ((-n, n) - H).$$

Since $\mathbb{R} - H$ is open in $\mathbb{R}_{\text{har}}$, H is closed and therefore $\overline{H} = H$.

(2) If $H^- = \{-\frac{1}{n} : n \in \mathbb{N}\}$, then $\overline{H^-} = H^- \cup \{0\}$.

(3) Nope!

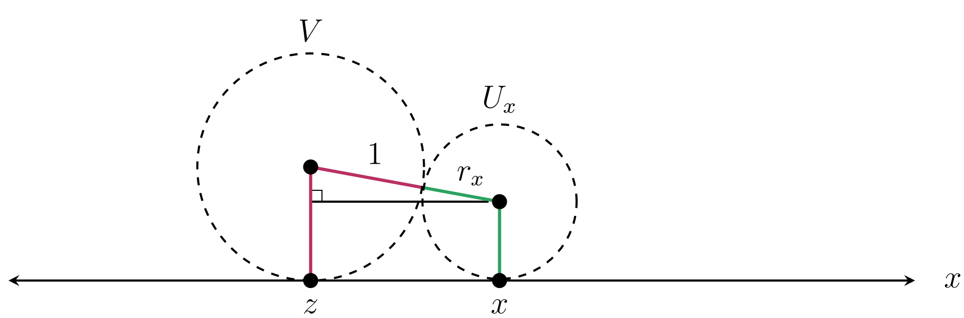
Exercise 3.10. (1) Let $\mathbb{H}_{\text{bub}}$ be the upper half plane with the Sticky Bubble Topology. Let $Q = \{(x, 0) : x \in \mathbb{Q}\}$. Then

$$\mathbb{H} - Q = \bigcup_{x \in (\mathbb{R} - \mathbb{Q})} \left(\bigcup_{n \in \mathbb{N}} (B((x, n), n) \cup \{(x, 0)\}) \right)$$

is the union of sets in the basis for the Sticky Bubble Topology, so it is open. Therefore Q is closed and $\overline{Q} = Q$.

(2) This is similar to (1) since any subset of the x-axis can be treated the way $\mathbb{Q}$ was in the previous example.

(3) If A is a countable subset of the x-axis, and z is a point on the x-axis not in A, we wish to show that there are disjoint open sets U and V such that $A \subset U$ and $z \in V$. Set $V = B((z, 1), 1) \cup \{(z, 0)\}$. Now for all $x \in A$, set $r_x = (x - z)^2/4$ and $U_x = B((x, r_x), r_x) \cup \{(x, 0)\}$. U_x is open for all x, so the set $U = \bigcup_{x \in A} U_x$ is also open. Clearly $A \subset U$, so it remains to check that $U \cap V \neq \emptyset$, which is the case provided none of the U_x bubbles overlap with the bubble V. Consider an arbitrary x and corresponding bubble U_x:



The distance from $(z, 1)$ to (x, r_x) is $((1 - r_x)^2 + (x - z)^2)^{\frac{1}{2}}$, and since U_x and V are open, we have that $U_x \cap V \neq \emptyset$ since:

$$\begin{aligned} r_x &= \frac{(x-z)^2}{4} \\ \iff (x-z)^2 &= 4r_x \\ \iff 1 + r_x^2 + (x-z)^2 &= 1 + 4r_x + r_x^2 \\ \iff 1 - 2r_x + r_x^2 + (x-z)^2 &= 1 + 2r_x + r_x^2 \\ \iff (1-r_x)^2 + (x-z)^2 &= (1+r_x)^2 \end{aligned}$$

Taking square roots shows that the distance between the centers of U_x and V is the same as the sum of the radii of U_x and V, so the bubbles overlap at a single point that is not contained in either open set. This was the case for an arbitrary $x \in A$, so it is true for all $x \in A$. Therefore we have found two open sets U and V such that $A \subset U$, $z \in V$, and $U \cap V = \emptyset$. We did not use the fact that A contains countably many points, so this holds for all subsets of the x-axis.

Exercise 3.11. Let $\mathbb{Z}_{\text{arith}}$ be the integers $\mathbb{Z}$ together with a topology generated by a basis of arithmetic progressions (the basis $\mathcal{B}$ is the collection of all sets of the form $\{az + b : z \in \mathbb{Z}\}$ for $a, b \in \mathbb{Z}$ with $a \neq 0$). To show that this is indeed a basis for some topology on $\mathbb{Z}$, note that $\mathbb{Z} = \{1 \cdot z + 0 : z \in \mathbb{Z}\}$ is itself in $\mathcal{B}$, so all points in $\mathbb{Z}$ are in some element of $\mathcal{B}$. Now let U and V be arithmetic progressions in $\mathcal{B}$ and let $z_0 \in U \cap V$ be arbitrary. Then $U = \{a_1 z + b_2 : z \in \mathbb{Z}\}$ and $V = \{a_2 z + b_2 : z \in \mathbb{Z}\}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Now $z_0 \in U \cap V$, then we have that $a_1 k_1 + b_1 = z_0 = a_2 k_2 + b_2$ for some $k_1, k_2 \in \mathbb{Z}$. Then $z_0 \in W = \{\text{lcm}(a_1, a_2) z + z_0 : z \in \mathbb{Z}\} \in \mathcal{B}$. We also have that $W \subset U \cap V$ since if $w_0 = \text{lcm}(a_1, a_2) k_0 + z_0 \in W$, then

$$\begin{aligned} a_1 \left(k_1 + \frac{a_2 k_0}{\gcd(a_1, a_2)} \right) + b_1 = a_1 k_1 + b_1 + \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 = \text{lcm}(a_1, a_2) k_0 + z_0 \\ = w_0 = \text{lcm}(a_1, a_2) k_0 + z_0 = \frac{a_1 a_2}{\gcd(a_1, a_2)} k_0 + a_2 k_2 + b_2 = a_2 \left(k_2 + \frac{a_1 k_0}{\gcd(a_1, a_2)} \right) + b_2, \end{aligned}$$

and w_0 is in both arithmetic progressions. Therefore $\mathbb{Z}_{\text{arith}}$ is a topological space generated by the basis $\mathcal{B}$.

Exercise 3.12. Consider the topological space $\mathbb{Z}_{\text{arith}}$ and note that if U is an open set, then since it is the union of infinite arithmetic progressions, it is itself infinite. Note also that for a prime p and $a = 1, 2, \ldots, p-1$, we have that $\{pz + a : z \in \mathbb{Z}\} \in \mathcal{B}$ is an arithmetic progression. Let $p\mathbb{Z}$ denote the set $\{pz : z \in \mathbb{Z}\}$. Then we have that

$$\mathbb{Z} - p\mathbb{Z} \;=\; \bigcup_{a=1,\ldots,p} \{pz + a : z \in \mathbb{Z}\}$$

is the union of open sets and is therefore open, meaning that $p\mathbb{Z}$ itself is closed. Let P denote the set $P = \bigcup_{\text{primes } p} p\mathbb{Z}$, and suppose for contradiction that there are finitely many primes. Then P is the union of finitely many closed sets and is therefore itself closed. Note that P contains all numbers that are integer multiples of a prime, so the only numbers not contained in P are -1 and 1, that is, $\mathbb{Z} - P = \{-1, 1\}$. But this set is open since P is closed, which is a contradiction since $\mathbb{Z} - P$ is finite. Therefore there are infinitely many primes.

3.2 Subbases

Exercise 3.13. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Then if U is in an open set in $(X, \mathcal{T})$, it is the union of sets in $\lambda \subset \mathcal{B}$. For each set $S \in \lambda$, S is the trivial intersection of itself, and this intersection is finite. Since $\mathcal{B}$ is a basis for a the topology $\mathcal{T}$, the finite intersections of sets in $\mathcal{B}$ is in $\mathcal{T}$. Therefore every open set in $\mathcal{T}$ can be generated by taking the union of sets that are themselves the finite intersections of sets in $\mathcal{B}$ and so $\mathcal{B}$ is a subbasis.

Exercise 3.14. Consider $\mathbb{R}$ together with the standard topology. Recall that $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis and note that $(a, b) = \{x \in \mathbb{R} : x > a\} \cap \{x \in \mathbb{R} : x < b\}$ and this is a finite intersection. Therefore every set in the basis is the finite intersection of sets in $\mathcal{S}$, the set of rays of the form $\{x \in \mathbb{R} : x < x_0\}$ and $\{x \in \mathbb{R} : x_0 > x\}$ for some $x_0 \in \mathbb{R}$. Therefore $\mathcal{S}$ is a subbasis for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

Theorem 3.16. Let X be a set and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for some topology $\mathcal{T}$ on X if and only if every point of X is contained in some element of $\mathcal{S}$.

Proof. Suppose $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X and let $x \in X$ be arbitrary. Since $\mathcal{S}$ is a subbasis, the set $\mathcal{B}$ of finite intersections of elements of $\mathcal{S}$ is a basis, and therefore $x \in B_0$ for some $B_0 \in \mathcal{B}$. Then since B_0 is the finite intersection of elements in $\mathcal{S}$, $x \in S_0$ for some $S_0 \in \mathcal{S}$. Now suppose that for all $x \in X$, there exists some $S_0 \in \mathcal{S}$ such that $x \in S_0$. Then if $\mathcal{B}$ is the set of finite intersections of elements of $\mathcal{S}$, $x \in S_0 \in \mathcal{B}$, so Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary and let $p \in U \cap V$. Then $U = \bigcap_{S\in\lambda_1} S$ and $V = \bigcap_{S\in\lambda_2} S$ where λ_1 and λ_2 are collections of sets in $\mathcal{S}$. Therefore

$$p \in \bigcap_{S\in\lambda_1\cap\lambda_2} S \subset U \cap V$$

since $\lambda_1 \cap \lambda_2 \subset \lambda_1, \lambda_2$. Therefore Theorem 3.3(2) is satisfied as well since $\bigcap_{S\in\lambda_1\cap\lambda_2} S \in \mathcal{B}$, and so $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, which also means that $\mathcal{S}$ is a subbasis for this topology. $\square$

Exercise 3.17. Let $\mathcal{S}$ be the set of all subsets of $\mathbb{R}$ of the form $\{x \in \mathbb{R} : x < a\}$ or $\{x \in \mathbb{R} : a \leq x\}$. We claim that this is a subbasis for the lower limit topology on $\mathbb{R}$, $\mathcal{T}_{\text{LL}}$. If $U \in \mathcal{S}$, then there are two cases. In the first case, $U = \{x \in \mathbb{R} : x < a\} = \bigcup_{n\in\mathbb{N}}[a-n, a) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. In the second case, $U = \{x \in \mathbb{R} : a \leq x\} = \bigcup_{n\in\mathbb{N}}[a, a+n) \in \mathcal{T}_{\text{LL}}$ for some $a \in \mathbb{R}$. Both inclusions in $\mathcal{T}_{\text{LL}}$ come from the fact that all open sets of the Sorgenfrey Line are the union of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Since any arbitrary $U \in \mathcal{S}$ has $U \in \mathcal{T}_{\text{LL}}$, we have that $S \subset \mathcal{T}_{\text{LL}}$ and therefore Theorem 3.15(1) is satisfied. Now suppose U is an open set of the Sorgenfrey Line and let $p \in U$ be arbitrary. Then $U = \bigcup_{W\in\lambda} W$ where $\lambda \subset \mathcal{B}_{\text{LL}}$ is a collection of sets of the form $[a, b)$ for some $a, b \in \mathbb{R}$. Therefore we have that $p \in W_0 = [a_0, b_0)$ for some $W_0 \in \lambda$. Since $W_0 = \{x \in \mathbb{R} : a_0 \leq x\} \cap \{x \in \mathbb{R} : x < b_0\}$ is the finite intersection of sets in $\mathcal{S}$ and $W_0 \subset U$, we have satisfied Theorem 3.15(2), meaning that $\mathcal{S}$ is indeed a subbasis for the lower limit topology on $\mathbb{R}$, as claimed.

3.3 Order Topology

Exercise 3.19. Consider $\mathbb{R}$ with the order topology from $\leq$. It has a basis $\mathcal{B}$ containing sets of the form $\{x \in \mathbb{R} : x < a\}$, $\{x \in \mathbb{R} : a < x\}$, and $\{x \in \mathbb{R} : a < x < b\}$. A set U is open in the standard topology on $\mathbb{R}$ if for each point $p \in U$, there is an $\varepsilon_p > 0$ such that $(p - \varepsilon_p, p + \varepsilon_p) \subset U$. Note that this is the case for every set of the forms contained in $\mathcal{B}$ (for the first and second forms, $\varepsilon_p = |a - p|$, and for the third form, $\varepsilon_p = \min\{p - a, b - p\}$), so $\mathcal{B} \subset \mathcal{T}_{\text{std}}$, satisfying Theorem 3.1(1). If $U \subset \mathbb{R}$ is open in $\mathcal{T}_{\text{std}}$ and $p \in U$, then since there exists $\varepsilon_p > 0$ such that $p \in (p - \varepsilon_p, p + \varepsilon_p) \subset U$ and $(p - \varepsilon_p, p + \varepsilon_p) \in \mathcal{B}$, Theorem 3.1(2) is satisfied as well and therefore $\mathcal{B}$ is a basis for the standard topology as well, so the order topology on $\mathbb{R}$ with $\leq$ is the standard topology.

Exercise 3.21. $A = \{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\}$ has closure $\overline{A} = A \cup \{(0, 1)\}$.

$B = \{\left(1 - \frac{1}{n}, \frac{1}{2}\right) : n \in \mathbb{N}\}$ has closure $\overline{B} = B \cup \{(1, 0)\}$.

$C = \{(x, 0) : 0 < x < 1\}$ has closure $\overline{C} = \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$.

$D = \{\left(x, \frac{1}{2}\right) : 0 < x < 1\}$ has closure $\overline{D} = D \cup \{(x, 0) : 0 < x \leq 1\} \cup \{(x, 1) : 0 \leq x < 1\}$.
$E = \{\left(\frac{1}{2}, y\right) : 0 < y < 1\}$ has closure $\overline{E} = E \cup \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right)\right\}$

3.4 Subspaces

Theorem 3.25. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $\mathcal{T}_Y$ is indeed a topology on Y.

Proof. Note that $\emptyset \in \mathcal{T}_Y$ since $\emptyset = \emptyset \cap Y$ and $\emptyset \in \mathcal{T}$. Similarly, $Y \in \mathcal{T}_Y$ since $Y = X \cap Y$ and $X \in \mathcal{T}$. Now let $A, B \in \mathcal{T}_Y$ be arbitrary. Then there exist sets $V_A, V_B \in \mathcal{T}$ such that $A = V_A \cap Y$ and $B = V_B \cap Y$ and we have that

$$A \cap B = (V_A \cap Y) \cap (V_B \cap Y) = (V_A \cap V_B) \cap Y \in \mathcal{T}_Y$$

since $V_A \cap V_B$ is the (finite) intersection of sets in the topology $\mathcal{T}$ and is therefore itself in the topology $\mathcal{T}$. Now let $\{U_\alpha\}_{\alpha \in \lambda}$ be a collection of sets in $\mathcal{T}_Y$. Then for each $\alpha \in \lambda$, there exists a set $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$ and we have that

$$\bigcup_{\alpha \in \lambda} U_\alpha = \bigcup_{\alpha \in \lambda} (V_\alpha \cap Y) = \left(\bigcup_{\alpha \in \lambda} V_\alpha\right) \cap Y \in \mathcal{T}_Y$$

since $\bigcup_{\alpha \in \lambda} V_\alpha$ is the union of sets in $\mathcal{T}$ and is therefore itself in $\mathcal{T}$. Thus we have that $\emptyset, Y \in \mathcal{T}_Y$ and that $\mathcal{T}_Y$ contains the finite intersections of sets in $\mathcal{T}_Y$ and the (possibly infinite) unions of sets in $\mathcal{T}_Y$ and is therefore a topology on Y. $\square$

Exercise 3.26. Taking $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$, we see that the set $\left[\frac{1}{2}, 1\right)$ is closed in Y. This is because the set $\left(-1, \frac{1}{2}\right)$ is open in $\mathbb{R}_{\text{std}}$ and therefore $\left(-1, \frac{1}{2}\right) \cap Y = \left[0, \frac{1}{2}\right)$ is open in Y, so $Y - \left[0, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right)$ is closed in Y.

Exercise 3.27. If Y is a subspace of a topological space $(X, \mathcal{T})$, it is not necessarily the case that every subset of Y that is open in Y is open in $(X, \mathcal{T})$. As in Exercise 3.26, $Y = [0, 1)$ is a subspace of $\mathbb{R}_{\text{std}}$ and $\left[0, \frac{1}{2}\right)$ is open in Y. However, it is neither open nor closed in $\mathbb{R}_{\text{std}}$.

Theorem 3.28. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

Proof. Let $C \subset Y$ be closed in $(Y, \mathcal{T}_Y)$. This is the case if and only if $Y - C$ is open in $(Y, \mathcal{T}_Y)$, which is the case if and only if there exists a set $V \subset X$, open in $(X, \mathcal{T})$, such that

$Y - C = V \cap Y$. This then is the case if and only if

$$C = Y - (V \cap Y) = Y - V = Y \cap (X - V) = Y \cap D$$

where $D = X - V$ is closed in $(X, \mathcal{T})$ since V is open in $(X, \mathcal{T})$. All implications here go in both directions, so both directions of the proof are complete. $\square$

Theorem 3.30. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$ that has basis $\mathcal{B}$. Then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Proof. Let $A_0 \in \mathcal{B}_Y$ be arbitrary. Then there exists $B_0 \in \mathcal{B}$ such that $A_0 = B_0 \cap Y$. Since $B_0 \in \mathcal{B}$, it is open in the topological space $(X, \mathcal{T})$. Since $\mathcal{T}_Y = \{U : U = V \cap Y \text{ for some } V \in \mathcal{T}\}$, we have that $A_0 \in \mathcal{T}_Y$ and so Theorem 3.1(1) is satisfied. Now let U be an open set in $(Y, \mathcal{T}_Y)$ and let $p \in U$ be an arbitrary point. Since $U \in \mathcal{T}_Y$, there exists a set $W \in \mathcal{T}$ such that $U = W \cap Y$, which means that $p \in Y$ and $p \in W$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, by Theorem 3.1(2), there exists a set $V \in \mathcal{B}$ such that $p \in V \subset W$. Then since $V \in \mathcal{B}$, $V \cap Y \in \mathcal{B}_Y$ and we have that

$$p \in V \cap Y \subset W \cap Y = U,$$

which satisfies Theorem 3.1(2), meaning that $\mathcal{B}_Y$ is indeed a basis for $\mathcal{T}_Y$. $\square$

3.5 Product Spaces

Exercise 3.32. Let X and Y be topological spaces and let $\mathcal{B}$ be the set of all cartesian products of open sets $U \subset X$ and $V \subset Y$. Then $p = (a, b)$ is a point of $X \times Y$, then p is in some element of $\mathcal{B}$ since X is open in X and Y is open in Y (that is, $X \times Y \in \mathcal{B}$). This means that Theorem 3.3(1) is satisfied. Now let $U, V \in \mathcal{B}$ be arbitrary. Then there exist sets X_1, X_2 open in X and Y_1, Y_2 such that $U = X_1 \times Y_1$ and $V = X_2 \times Y_2$. Now let $p \in U \cap V$ be arbitrary. Then we have that $p \in U \cap V \subset U \cap V$ and also that

$$U \cap V = (X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2) \in \mathcal{B}$$

since $X_1 \cap X_2$ is open in X and $Y_1 \cap Y_2$ is open in Y. Therefore Theorem 3.3(2) is also satisfied and we have that $\mathcal{B}$ is indeed the basis for some topology on $X \times Y$.

Exercise 3.34. The product of closed sets is closed in the product topology. Let X and Y be topological spaces, let A be closed in X and let B be closed in Y. Then there exists an open set $A_0 \subset X$ such that $A = X - A_0$ and there exists an open set $B_0 \subset Y$ such that $B = Y - B_0$ and we have that

$$\begin{aligned}
A \times B &= (X - A_0) \times (Y - B_0) \\
&= (X \times (Y - B_0)) - (A_0 \times (Y - B_0)) \\
&= ((X \times Y) - (X \times B_0)) - ((A_0 \times Y) - (A_0 \times B_0)) \\
&= ((X \times Y) \cup (A_0 \times B_0)) - ((X \times B_0) \cup (A_0 \times Y)) \\
&= (X \times Y) - ((X \times B_0) \cup (A_0 \times Y)).
\end{aligned}$$

The sets $X \times B_0$ and $A_0 \times Y$ are both basic open sets in $X \times Y$ since A_0 is open in X and B_0 is open in Y. Since the union of open sets is open, we have that $A \times B$ is the complement of open sets and is therefore closed. A and B were arbitrary closed sets in arbitrary topological spaces, so we have shown that the product of closed sets is closed in the product topology in general.

Theorem 3.35. The product topology on $X \times Y$ has a subbasis $\mathcal{S}$ that contains the inverse images of open sets under the projection functions, that is,

$$\mathcal{S} = \{\pi_X^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ is open in } Y\}.$$

Proof. Let S_0 be a set in $\mathcal{S}$. There are two cases. In the first, S_0 is of the form $S_0 = \{\pi_X^{-1}(U) : U \text{ is open in } X\} = U \times Y$ and is therefore a basic open set in the product space. In the second, S_0 is of the form $S_0 = \{\pi_Y^{-1}(V) : V \text{ is open in } Y\} = X \times V$ and is also a basic open set in the product space. Therefore S_0 is open in all cases and so $\mathcal{S} \subset \mathcal{T}$ where $\mathcal{T}$ is the product topology. Thus Theorem 3.15(1) is satisfied. Now let W be an open set in the product space and let $p \in W$ be an arbitrary point. Then since W is open, it is the union of sets of the form $U \times V$ where U is open in X and V is open in Y. This means there are sets $U_0 \subset X$ and $V_0 \subset Y$ such that $p \in U_0 \times V_0 \subset W$. All that remains to satisfy Theorem 3.15(2) is to show that $U_0 \times V_0$ is the finite intersection of sets in $\mathcal{S}$, which is the case since $U_0 \times V_0 = \pi_X^{-1}(U_0) \cap \pi_Y^{-1}(V_0)$. Therefore $\mathcal{S}$ is indeed a subbasis for the product topology on $X \times Y$. $\square$

Exercise 3.36. Let W be an open set in $\mathbb{R}^2_{\text{std}}$. Then $W = B(p, \varepsilon_p)$ for some $p \in \mathbb{R}^2$ and

$\varepsilon_p > 0$. Now let $x \in W$ be arbitrary. Suppose $p = (p_1, p_2)$ and $x = (x_1, x_2)$. Set

$$U = \left(x_1 - \left|x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1\right)\right|, x_1 + \left|x_1 - \left(\frac{\varepsilon_p(x_1 - p_1)}{d(x,p)} + p_1\right)\right|\right)$$

and

$$V = \left(x_2 - \left|x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2\right)\right|, x_2 + \left|x_2 - \left(\frac{\varepsilon_p(x_2 - p_2)}{d(x,p)} + p_2\right)\right|\right).$$

(See Desmos sketch lmao.) Then we have that $x \in U \times V \subset W$, and since $U \times V$ is an open set with the product topology on $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$, we have that W is an open set in this topology as well by Theorem 2.3 since x was an arbitrary point of W. Then since W was an arbitrary open set under the standard topology, it follows that the standard topology on $\mathbb{R}^2$ is a subset of the product topology on $\mathbb{R}^2$.

Now let W be an open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology and let $p \in W$ be arbitrary. Then since W is open, it is the union of sets of the form $U \times V$ where U and V are open sets in $\mathbb{R}_{\text{std}}$. This means that there exist some U_0 and V_0 open in $\mathbb{R}_{\text{std}}$ such that $p \in U_0 \times V_0 \subset W$. Then there exist intervals (a_x, b_x) and (a_y, b_y) such that $\pi_{U_0}(p) \in (a_x, b_x)$ and $\pi_{V_0}(p) \in (a_y, b_y)$. Therefore we have that $p \in B(p, \varepsilon) \subset U_0 \times V_0 \subset W$ where

$$\varepsilon = \min\{|a_x - \pi_{U_0}(p)|, |b_x - \pi_{U_0}(p)|, |a_y - \pi_{V_0}(p)|, |b_y - \pi_{V_0}(p)|\}.$$

Since this $B(p, \varepsilon)$ is an open set in $\mathbb{R}^2_{\text{std}}$, by Theorem 2.3 it follows that W is also open in $\mathbb{R}^2_{\text{std}}$, and since W was an arbitrary open set in $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ with the product topology, we have that this topology on $\mathbb{R}^2$ is a subset of the standard topology on $\mathbb{R}^2$. Since each topology is a subset of the other, they are equal, meaning that product topology on $\mathbb{R}^2$ from $\mathbb{R}_{\text{std}} \times \mathbb{R}_{\text{std}}$ is the same as the standard topology on $\mathbb{R}^2$.

Theorem 3.37. The product topology on $\prod_{\alpha \in \lambda} X_\alpha$ has a basis containing all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.

Proof. Since $\mathcal{S}$, the collection of sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in $(X_\beta, \mathcal{T}_\beta)$, is a subbasis for the product topology $\prod_{\alpha \in \lambda} X_\alpha$, the set of finite intersections of elements of $\mathcal{S}$ is therefore a basis for the product topology. Let W be a set in the collection described in the theorem statement. Then there exists a finite index set $\gamma \subset \lambda$ such that $W = \prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all $\alpha \in \lambda - \gamma$. Then since for some $\beta \in \gamma$, $\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha \in \lambda - \{\beta\}} X_\alpha$, we have that $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) = U_{\beta_1} \times U_{\beta_2} \times \prod_{\alpha \in \lambda - \{\beta_1, \beta_2\}} X_\alpha$

and therefore $W = \prod_{\alpha\in\gamma} U_\alpha \times \prod_{\alpha\in\lambda-\gamma} X_\alpha = \bigcap_{\alpha\in\gamma} \pi_\alpha^{-1}(U_\alpha)$ is the finite intersection of elements of $\mathcal{S}$. Similarly, any finite intersection of elements of $\mathcal{S}$ is a product of the form described in the theorem statement, and so by the definition of a subbasis, the collection described in the theorem statement is indeed a basis for the product topology. $\square$

Exercise 3.42. The set $2^{\mathbb{N}} = \prod_{n\in\mathbb{N}}\{0,1\}$ with the box topology has the discretet topology since all singletons are open. Let $\{a_n\}_{n\in\mathbb{N}}$ be a binary sequence (an arbitrary element of $2^{\mathbb{N}}$). Then the singleton containing $\{a_n\}_{n\in\mathbb{N}}$ is $\{\{a_n\}_{n\in\mathbb{N}}\} = \prod_{n\in\mathbb{N}}\{a_n\} = \{a_1\}\times\{a_2\}\times\cdots$, so it is a basic open set. Compare this to $2^{\mathbb{N}}$ with the product topology. Since with this topology, if $p \in 2^{\mathbb{N}}$ is a point in the space and U an open set containing p, then $(U - \{p\}) \cap 2^{\mathbb{N}} \neq \emptyset$ since any open set U contains infinitely many different sequences. Therefore $2^{\mathbb{N}}$ under the product topology has no isolated points since every point is a limit point of the set.

4 Separation Properties: Separating This From That

4.1 Hausdorff, Regular, and Normal Spaces

Theorem 4.1. A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.

Proof. Suppose $(X, \mathcal{T})$ is T_1 and let $x \in X$ be arbitrary. Let $y \in X - \{x\}$ be arbitrary. Then we have that $x \neq y$, so since the space is T_1, we have that there exist open sets U and V such that $x \in U - V$ and $y \in V - U$. Since V is open, $V \subset X$, and since $x \notin V$, $V \subset X - \{x\}$. Therefore $y \in V \subset X - \{x\}$ and by Theorem 2.3, we have that $X - \{x\}$ is open, meaning that the singleton $\{x\}$ is closed. But x was an arbitrary point in this T_1 space, so we have that points are closed in T_1 spaces. Now suppose $(X, \mathcal{T})$ is a topological space in which all points are closed and let $x, y \in X$ be arbitrary points such that $x \neq y$. Since $\{x\}$ and $\{y\}$ are closed, we have that $x \in X - \{y\}$, an open set, and $y \in X - \{x\}$, another open set. Since $x \notin X - \{x\}$ and $y \notin X - \{y\}$, and x and y were arbitrary elements of X, we have that the space $(X, \mathcal{T})$ is T_1. $\square$

Exercise 4.2. Let X be a set with the finite complement topology. Then let $x \in X$ be arbitrary. We have that $X - \{x\}$ is open since its complement, $\{x\}$, is finite. This means $\{x\}$ is closed, and since the point x was arbitrary, we have that all singletons are closed and therefore by Theorem 4.1, all sets with the finite complement topology are T_1.

Exercise 4.3. Let $x, y \in \mathbb{R}$ be points in the space $\mathbb{R}_{\text{std}}$. Then the sets $A = \left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right)$ and $B = \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$ are open and disjoint with $x \in A$ and $y \in B$. Therefore $\mathbb{R}_{\text{std}}$ is Hausdorff.

Exercise 4.5. Let A and B be disjoint, closed sets in $\mathbb{R}_{\text{LL}}$. For every $a \in A$ and $b \in B$, set

$$\delta_a = \frac{\inf\{b \in B : b > a\} - a}{2} \quad \text{and} \quad \delta_b = \frac{\inf\{a \in A : a > b\} - b}{2}.$$

Since for all $a \in A$, a is in the basic open set $U_a = [a, a+\delta_a)$, we have that $A \subset U = \bigcup_{a \in A} U_a$. Similarly, we have that $B \subset V = \bigcup_{b \in B} V_b$ where $V_b = [b, b+\delta_b)$ is a basic open set. Suppose for contradiction that there exists and $x \in U \cap V$. Then we have that there exist $\alpha \in A$ and $\beta \in B$ such that $x \in U_\alpha \cap V_\beta$. Without loss of generality, assume that $\alpha < \beta$. Then we have that $x \in U_\alpha = [\alpha, \alpha + \delta_\alpha)$ and $x \in V_\beta = [\beta, \beta + \delta_\beta)$. Then

$$\delta_\alpha = \frac{\inf\{b \in B : b > \alpha\} - \alpha}{2} \implies \alpha + \delta_\alpha = \frac{\alpha + \inf\{b \in B : b > \alpha\}}{2} \leq \inf\{b \in B : b > \alpha\}$$

since $\alpha \leq \inf\{b \in B : b > \alpha\}$. Since $\beta \in B$ and $\beta > \alpha$, $\beta \in \{b \in B : b > \alpha\}$ and therefore we have that

$$x < \alpha + \delta_\alpha \leq \inf\{b \in B : b > \alpha\} \leq \beta \leq x.$$

But $x < x$ is a contradiction and so we have that $U \cap V = \emptyset$ and therefore we have found disjoint, open sets U and V such that $A \subset U$ and $B \subset V$. This means that $\mathbb{R}_{\text{LL}}$ is normal.

Exercise 4.6. (1) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Suppose for contradiction that $\inf\{d(a,p) : a \in A\} = 0$. This means that there exists a sequence $(x_i)_{i \in \mathbb{N}} \subset A$ such that $d(x_i, p) \to 0$. This means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies that $|d(x_i,p) - 0| < \varepsilon$, which is equivalent to the statement that

$$\varepsilon > |\parallel x_i - p \parallel - 0| = \parallel x_i - p \parallel .$$

We now have that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $i > N$ implies $\parallel x_i - p \parallel < \varepsilon$, which in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ means that $x_i \to p$. By Theorem 2.30, we have that $p \in \overline{A} = A$ since A is closed. But this is a contradiction, so the assumption that $\inf\{d(a,p) : a \in A\} = 0$ was false. We have either that this is greater than 0 or less than 0, but it cannot be less than 0 since the distance between two points is always nonnegative. Therefore we have that $\inf\{d(a,p) : a \in A\} > 0$.

(2) Let $p \in (\mathbb{R}^2, \mathcal{T}_{\text{std}})$ and let $A \subset \mathbb{R}^2$ be a closed set with $p \notin A$. Then by (1), there exists an $\varepsilon > 0$ such that $\inf\{d(a,p) : a \in A\} = \varepsilon$. Set $U = B(p, \frac{\varepsilon}{2})$ and $V = \bigcup_{a\in A} B(a, \frac{\varepsilon}{2})$. Then U is a basic open set, and V is the union of basic open sets so both U and V are open. We also have that $p \in U$ and $A \subset V$ since if $a \in A$, $a \in B(a, \frac{\varepsilon}{2}) \subset V$. Suppose for contradiction that there exists an $x \in U \cap V$. Then $x \in U$ means that $d(p,x) < \varepsilon/2$ and $x \in V$ means that there exists an $a \in A$ such that $d(a,x) < \varepsilon/2$. Therefore we have that $d(a,p) \leq d(a,x) + d(x,p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction since $\inf\{d(a,p) : a \in A\} = \varepsilon$. Therefore $U \cap V = \emptyset$ and $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is regular.

(3) The sets $A = \{(x,0) : x \in \mathbb{R}\}$ and $B = \{(x,y) : x > 0, y \geq \frac{1}{x}\}$. Since for any points $(x_a, y_a) \in A$ and $(x_b, y_b) \in B$, $y_b \geq \frac{1}{x_b} > 0 = y_a$, we have that $A \cap B = \emptyset$. All limit points of A have y-coordinate equal to 0 and are therefore in A, meaning A is closed. Similarly, all limit points in B have y-coordinate equal to $\frac{1}{x_0}$ for some $x_0 > 0$ and are therefore in B, meaning B is closed. Hence A and B are disjoint, closed subsets of $\mathbb{R}^2$. However, $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$. To see this, let $\varepsilon > 0$. Then we have that $a_\varepsilon = (\frac{1}{\varepsilon} + 1, 0) \in A$ and $b_\varepsilon = (\frac{1}{\varepsilon} + 1, \frac{1}{\frac{1}{\varepsilon}+1}) \in B$. Then $d(a_\varepsilon, b_\varepsilon) = \frac{1}{\frac{1}{\varepsilon}+1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$, so $\inf\{d(a,b) : a \in A \text{ and } b \in B\} < \varepsilon$. However, $\varepsilon > 0$ was an arbitrary, so we have that $\inf\{d(a,b) : a \in A \text{ and } b \in B\} = 0$ since distance is nonnegative.

(4) Let A and B be disjoint, closed sets in $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$. For $a_0 \in A$ and $b_0 \in B$, define $\varepsilon_{a_0} = \frac{1}{2}\inf\{d(a_0, b) : b \in B\} > 0$ and $\varepsilon_{b_0} = \frac{1}{2}\inf\{d(a, b_0) : a \in A\} > 0$. Now set $U = \bigcup_{a\in A} B(a, \varepsilon_a)$ and $V = \bigcup_{b\in B} B(b, \varepsilon_b)$. Then since U and V are the unions of open balls in $\mathbb{R}^2$, they are open. Since if $a \in A$, then $a \in B(a, \varepsilon_a) \subset U$ and if $b \in B$, then $b \in B(b, \varepsilon_b) \subset V$, we have that $A \subset U$ and $B \subset V$. To show $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal, it only remains to show that U and V are disjoint. Suppose for contradiction that there exists $p \in \mathbb{R}^2$ such that $p \in U \cap V$. Then $p \in U$, so there exists an $\alpha \in A$ such that $p \in B(\alpha, \varepsilon_\alpha)$ and similarly, there exists a $\beta \in B$ such that $p \in B(\beta, \varepsilon_\beta)$. Since $\alpha \in A$, $d(\alpha, \beta) \in \{d(a, \beta) : a \in A\}$ and therefore $d(\alpha, \beta) \geq \inf\{d(a,\beta) : a \in A\} = 2\varepsilon_\alpha$. Similarly, $d(\alpha, \beta) \geq 2\varepsilon_\beta$, and so we have that $\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta)$. Since $p \in B(\alpha, \varepsilon_\alpha)$, we have that $d(\alpha, p) < \varepsilon_\alpha$, and since $p \in B(\beta, \varepsilon_\beta)$, we have that $d(p, \beta) < \varepsilon_\beta$. Putting this all together using the triangle inequality, we see that

$$\varepsilon_\alpha + \varepsilon_\beta \leq d(\alpha, \beta) \leq d(\alpha, p) + d(p, \beta) < \varepsilon_\alpha + \varepsilon_\beta.$$

This is a contradiction, so we have that $U \cap V = \emptyset$ and therefore $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$ is normal.

Theorem 4.7. (1) A T_2-space (Hausdorff) is a T_1-space.

(2) A T_3-space (regular and T_1) is a Hausdorff space, that is, a T_2-space.

(3) A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof. (1) Let $(X, \mathcal{T})$ be a Hausdorff space and let $x, y \in X$ be distinct, arbitrary points. Then there exist disjoint, open sets U and V such that $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$, we have that $x \notin V$ and $y \notin U$, so $(X, \mathcal{T})$ is a T_1-space. $\square$

Proof. (2) Let $(X, \mathcal{T})$ be a T_3-space and let $x, y \in X$ be distinct, arbitrary points. Since this space is T_1, by Theorem 4.1 we have that $\{y\}$ is closed. Since this space is regular, we have that there exist disjoint, open sets such that U and V such that $x \in U$ and $\{y\} \subset V$. But $\{y\} \subset V$ means $y \in V$, so we have found disjoint, open sets separating the arbitrary points x and y, so $(X, \mathcal{T})$ is Hausdorff. $\square$

Proof. (3) Let $(X, \mathcal{T})$ be a T_4-space, let $x \in X$ be arbitrary, and let A be a closed set with $x \notin A$. Since this space is T_1, by Theorem 4.1 we have that $\{x\}$ is closed. Since this space is normal, there exist disjoint, open sets U and V such that $\{x\} \subset U$ and $A \subset V$. But $\{x\} \subset U$ means $x \in U$, so we have found disjoint, open sets separating the arbitrary point x from the arbitrary closed set A, so $(X, \mathcal{T})$ is T_3 since it is normal and T_1. $\square$

Theorem 4.8. A topological space is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.

Proof. ($\Longrightarrow$) Let $(X, \mathcal{T})$ be a regular topological space and let U be an open set containing the point p. Then we have that $X - U$ is closed and since $p \in U$, $p \notin X - U$. Since this space is regular, there exist disjoint open sets V and W such that $p \in V$ and $X - U \subset W$. Therefore we have that $X - W \subset U$ since $X - U \subset W$, and that $X - W$ is closed since W is open. Let $x \in V$ be arbitrary. Then $x \notin W$ (since $V \cap W = \emptyset$) and therefore $x \in X - W$. Since x was arbitrary, we have that $V \subset X - W$. By Theorem 2.22, we have that $\overline{V} \subset \overline{X - W}$, and since $X - W$ is closed, we see that

$$p \in V \subset \overline{V} \subset \overline{X - W} = X - W \subset U.$$

Since U and p were arbitary, there exists an open set V containing p such that $\overline{V} \subset U$ for all $p \in X$ and open sets U containing p.

($\Longleftarrow$) Now let $(X, \mathcal{T})$ be a topological space with the property that for all $p \in X$ and $W \in \mathcal{T}$ with $p \in W$, there exists an open set U such that $p \in U$ and $\overline{U} \subset W$. Let $p \in X$ be arbitrary and let A be a closed subset of X such that $p \notin A$. Then we have that $p \in X - A$, which is open, and therefore there exists an open set U such that $p \in U$ and $\overline{U} \subset X - A$, which implies that $A \subset V$ where V is the open set $X - \overline{U}$. Let $x \in U$ be arbitary. Then $x \in \overline{U}$ since $U \subset \overline{U}$, and therefore $x \notin V = X - \overline{U}$. Since x was arbitary, we have that $U \cap V = \emptyset$. Therefore we have found disjoint open sets U and V such that $p \in U$ and $A \subset V$, so $(X, \mathcal{T})$ is regular, as required. $\square$

Theorem 4.9. A topological space is normal if and only if for each closed set A in $(X, \mathcal{T})$ and open set U containing Athere exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.