Drift-diffusion Model of a Feed-forward Loop

May 30, 2017

1 The Indirect Path

We'll refer to the three nodes of the FFL as the Transmitter (T), Middle (M), and Receiver (R). The probability of two particles arriving at R at times separated by an interval Δt, given that they were released from T at times separated by $\Delta\tau$ is given in the previous notes for the direct path as

$$p_{TR}(\Delta t|\Delta\tau) = \int_0^\infty dt\, IG(t+\Delta t|\Delta\tau;\mu,\lambda) IG(t|0;\mu,\lambda). \tag{1}$$

We now wish to determine the equivalent expression for the indirect path (T$\rightarrow$ M$\rightarrow$ R). The general expression will be the following:

$$p_{TMR}(\Delta t|\Delta\tau) = \int_{-\infty}^\infty d\Delta T\, p_{MR}(\Delta t|\Delta T) p_{TM}(\Delta T|\Delta\tau), \tag{2}$$

where ΔT is the time interval between the release of the two particles from M. The quantities p_{MR} and p_{TM} are both of the same form as p_{TR} above (Eq. (1)). Note that the order in which the particles arrive at M may be inverted, so ΔT can be negative. It can't actually be $-\infty$, of course, but you can prevent any unphysical combinations of the time intervals from contributing to the probability distributions by extending the inverse Gaussian distribution to negative values of its arguments as follows:

$$\begin{aligned} IG(x;\mu,\lambda) &= \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\left\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \text{ for } x \geq 0 \\ &= 0, \text{ for } x < 0. \end{aligned} \tag{3}$$

2 The Feed-forward Loop

For the full feed-forward loop, we once again want to compute $p(\Delta t|\Delta\tau)$, except now it can have contributions from both p_{TR} and p_{TMR}. Assuming that a particle is equally likely to be sent along the direct path as the indirect path, there are four separate possibilities to consider. If two particles leave T separated in time by $\Delta\tau$ and arrive at R separated by Δt, they either both followed the direct path, OR they both followed the indirect path, OR the first particle to be released followed the direct path and the second followed the indirect path, OR the first particle to be released followed the indirect path and the second followed the direct path. As clear from my use of "OR," these four possibilities are mutually exclusive and statistically independent, so the total probability is given by summing their individual probabilities:

$$p(\Delta t|\Delta\tau) = \frac{1}{4}\left[p_{TR}(\Delta t|\Delta\tau) + p_{TMR}(\Delta t|\Delta\tau) + p_{DI}(\Delta t|\Delta\tau) + p_{ID}(\Delta t|\Delta\tau)\right]. \tag{4}$$

In the above, the first two probabilities on the right hand side can be found from numerical evaluations of Eq. (1) and (2). The second two probabilities (for the Direct/Indirect and Indirect/Direct combinations) have yet to be computed.

To determine p_{DI}, we must construct the probability distribution for a particle leaving T at time zero and arriving at R at time t AND the probability distribution for a second particle leaving T at time $\Delta\tau$, arriving at M at time t', and finally arriving at R at time $t+\Delta t$. This first probability is given simply by $IG(t|0;\mu,\lambda)$ and the second by the more complicated

$$\int_{\Delta\tau}^{\infty} dt'\, IG(t+\Delta t|t';\mu,\lambda) IG(t'|\Delta\tau;\mu,\lambda).$$

The use of "AND" above makes it clear that these two probabilities should be multiplied together, and, to get $p_{DI}(\Delta t|\Delta\tau)$, we must further integrate over all values of t:

$$p_{DI}(\Delta t|\Delta\tau) = \int_0^{\infty} dt\, IG(t|0;\mu,\lambda) \int_{\Delta\tau}^{\infty} dt'\, IG(t+\Delta t|t';\mu,\lambda) IG(t'|\Delta\tau;\mu,\lambda). \tag{5}$$

The probability $p_{ID}(\Delta t|\Delta\tau)$ can be constructed using very similar arguments to those above, and I leave that as an exercise.

3 Additional Notes