On density of smooth functions in weighted Sobolev spaces with variable exponent.

Mikhail D. Surnachev
Computational Aeroacoustics Laboratory,
Keldysh Institute of Applied Mathematics RAS
Moscow 125047, Russia
peitsche@yandex.ru

In this work we are concerned with the question of density of smooth functions in weighted Sobolev-Orlicz spaces. In a bounded Lipshitz domain $\Omega \subset \mathbb{R}^n$ for a weight (nonnegative measurable function) $\rho \in L^1(\Omega)$ we define the weighted Sobolev-Orlicz space

$$W := \left\{ u \in W_0^{1,1}(\Omega) \ : \ \int_\Omega |\nabla u|^{p(x)} \rho \, \mathrm{d}x < \infty \right\},$$

with a measurable exponent $p = p(x) > 1$, equipped with the norm

$$\|u\|_W = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|\nabla u|}{\lambda} \right)^p \rho \, \mathrm{d}x \le 1 \right\}.$$

We assume that the weight additionally satifsfies $\rho^{-1/p} \in L^{p'}(\Omega; \ \mathrm{d}x)$, which guarantees completeness of W. Next, we can define another space H as the closure of C_0^∞ in W.

Recently V.V. Zhikov proved the following interesting theorem.

Theorem (V.V. Zhikov). *Let $\rho = \omega\omega_0$, $p \equiv 2$ and $w_0 \in A_2$. Assume that*

$$\liminf_{t\to\infty} \frac{\left(\int_\Omega \omega^t \omega_0 \, \mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \omega^{-t} \omega_0 \, \mathrm{d}x\right)^{1/t}}{t^2} < \infty.$$

Then $H = W$.

First, we obtain a generalization of this result for all constant $p > 1$.

Theorem 1. *Let $\rho = \omega\omega_0$, $p = const > 1$ and $w_0 \in A_p$. Assume that*

$$\liminf_{t\to\infty} \frac{\left(\int_\Omega \omega^t \omega_0 \, \mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \omega^{-t} \omega_0 \, \mathrm{d}x\right)^{1/t}}{t^p} < \infty.$$

Then $H = W$.

Next, we extend this result to the case of Sobolev spaces with variable exponent. We assume that the exponent $p = p(x) : \Omega \to (1, \infty)$ satisfies

$$1 < \alpha < p(x) < \beta < \infty, \tag{1}$$

$$|p(x) - p(y)| \le \frac{k_0}{\ln \frac{1}{|x-y|}}, \quad |x - y| < 1. \tag{2}$$

This is the widely known logarithmic condition introduced originally by V.V. Zhikov and X.L. Fan in the first half of 1990s. To state the analogue of Theorem 1 we need to introduce a generalization of the Muckenhoupt classes for variable exponents.

Definition. We say that a nonnegative locally integrable function (weight) $\omega \in A^{loc}_{p(\cdot)}(\Omega)$, if

$$\sup_{Q}\sup_{x\in Q}\left(\int_Q \omega\,\mathrm{d}y\right)^{1/p(x)}\frac{\|\omega^{-1/p}\|_{L^{p'}(Q)}}{|Q|}<\infty, \tag{3}$$

where the supremum is taken over all cubes $Q\subset\Omega$ with faces parallel to the coordinate hyperplanes.

A very important feature of this classes is that they inherit the self-improvement property of the classical Muckenhoupt classes.

Theorem 2. *Let the exponent $p(\cdot)$ satisfy the logarithmic condition* (1)-(2) *and $\omega \in A^{loc}_{p(\cdot)}(\Omega)$. Then $\omega\in A^{loc}_{p(\cdot)-\varepsilon}(\Omega)$ for some $\varepsilon>0$.*

The next theorem is the main result of this work.

Theorem 3. *Let $\rho=\omega\omega_0$, where $\omega_0\in A^{loc}_{p(\cdot)}(\Omega)$. Let the exponent $p=p(x)$ satisfy the logarithmic condition* (1)-(2). *Let*

$$\liminf_{t\to\infty}\left(\int_\Omega \omega^{-t}\omega_0\,\mathrm{d}x\right)^{\frac{t-1}{t(t+1)}}\left(\int_\Omega \left(t^{-p(x)}\omega\right)^t\omega_0\,\mathrm{d}x\right)^{1/t}<\infty,$$

Then $H=W$.

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References

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