SYMPLECTIC-MIXED FINITE ELEMENT APPROXIMATION OF LINEAR WAVE EQUATIONS

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Abstract. We apply mixed finite element approximations to the first-order form of the acoustic wave equation. Our semidiscrete method exactly conserves the system energy, and we we show that with a symplectic Euler time discretization, our method exactly conserves a pertubed energy quantity that is positive-definite and equivalent to the actual energy under a CFL condition. In addition to proving optimal-order $L^\infty(L^2)$ estimates for our methods, we also develop a bootstrap technique that allows us to derive stability and error bounds for the time derivatives and divergence of the vector variable beyond the standard under some additional regularity assumptions.

Key words. Mixed finite element, acoustic wave equation, symplectic integration

AMS subject classifications. 65M60, 65P10

1. Introduction.

We consider the linear acoustic wave equation

$$\begin{aligned} \varrho p_t + \nabla \cdot u &= f, \\ \kappa^{-1} u_t + \nabla p &= g, \end{aligned} \tag{1.1}$$

posed on some domain $\Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}$ with $d = 2, 3$. We assume that T is finite and, for simplicity, that Ω is polyhedral so that it may be tesselated exactly into simplices. We pose initial conditions $p(\cdot, 0) = p_0(\cdot)$ and $u(\cdot, 0) = u_0(\cdot)$ and the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$, where ν is the unit outward normal to Ω. We assume that the material density, ϱ, is some measurable function bounded below and above by positive ϱ_* and ϱ^*. The parameter κ is the bulk modulus of compressibility, assumed bounded between positive κ_* and κ^*. These equations are of essential interest in, among many other areas of application, seismic imaging.

We are interested in discretization of these equations using mixed finite element spaces. Let $\{\mathcal{T}_h\}_h$ be a family of quasiuniform triangulations of Ω [5]. We let $W = L^2(\Omega)$ and V the subspace of $H(\mathrm{div})$ with vanishing normal trace. We let V_h be the Raviart-Thomas-Nédélec space [17, 18] of order $r \geqslant 0$ over each triangulation $\mathcal{T}_h$ and W_h the space of discontinuous piecewise polynomials of degree r over $\mathcal{T}_h$. It is possible to extend these mixed spaces to domains with a single curved facet [15], although we do not dwell on this. Our techniques apply equally well to rectangular meshes, although the poor approximation capabilities for quadrilateral meshes discussed in [1] suggest that the techniques of Bochev and Ridzal [4] may be required to extend our analysis to that case. As for other approximating spaces on the simplex, our techniques also apply equally to the Brezzi-Douglas-Fortin-Marini spaces [6], although we suspect that the typical extra order of convergence for the velocity variable in L^2 would not be obtained for the Brezzi-Douglas-Marini elements.

Throughout, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ or $\left(L^2(\Omega)\right)^d$ inner product, as needed. We also make use of the standard Sobolev spaces $H^m(\Omega)$ and $(H^m(\Omega))^d$ with norms denoted by $\|\cdot\|_m$ and seminorms by $|\cdot|_m$ in our error estimates, where m is some nonnegative integer. For functions of space and time, we also make use of spaces of

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the form $Y(0,T;X)$, at each time level, functions are in a (semi-) normed space X, and as a function of time, the X norm is in the space Y.

Our estimates make use of coefficient-weighted norms. For some strictly positive, bounded function ω, we define the weighted L^2 norm $\|f\|_\omega$ by

$$\|f\|_\omega^2 \equiv \int_\Omega \omega |f|^2 \, dx, \tag{1.2}$$

and if $0 < \omega_* \leqslant \omega(x) \leqslant \omega^*$ throughout Ω, then we have the equivalence

$$\sqrt{\omega_*}\,\|f\| \leqslant \|f\|_\omega \leqslant \sqrt{\omega^*}\,\|f\|. \tag{1.3}$$

We will also use weighted versions of Cauchy-Schwarz. With such a weight function ω, we can bound a standard inner product as

$$(f,g) = \left(\omega^{\frac{1}{2}} f, \omega^{-\frac{1}{2}} g\right) \leqslant \|f\|_\omega \|g\|_{\omega^{-1}}. \tag{1.4}$$

We will make the standard inverse assumption about our spaces, namely that there exists a positive constant C_0 such that

$$\|\nabla \cdot v_h\| \leqslant \frac{C_0}{h} \|v_h\|. \tag{1.5}$$

for all $v_h \in V_h$.

We also use the standard projection operators $\pi : W \to W_h$ and $\Pi : V \to V_h$ for the mixed spaces. For any $q \in W$, πq is the L^2 projection satisfying

$$(\pi q, w_h) = (q, w_h) \tag{1.6}$$

for all $w_h \in W_h$, and for $v \in V$, $\Pi v \in V_h$ is defined by

$$(\nabla \cdot \Pi v, w_h) = (\nabla \cdot v, w_h) \tag{1.7}$$

for all $w_h \in W_h$.

These projections have well-known approximation properties [6]. In particular, there exists a positive constant C_1 such that

$$\|\pi q - q\| \leqslant C_1 h^m |q|_m \tag{1.8}$$

whenever $q \in H^m(\Omega)$ and $1 \leqslant m \leqslant r+1$.

There also exists a positive C_2 such that

$$\|\Pi v - v\| \leqslant C_2 h^m |v|_m \tag{1.9}$$

for any $v \in (H^m(\Omega))^d$ and for $1 \leqslant m \leqslant r+1$.

Because of the commuting relation between π, Π and the divergence (i.e., that $\nabla \cdot \Pi u = \pi(\nabla \cdot u)$, we also have the bound

$$\|\nabla \cdot (\Pi v - v)\| \leqslant C_1 h^m |\nabla \cdot v|_m, \tag{1.10}$$

provided $\nabla \cdot v \in (H^m)^d$ for $1 \leqslant m \leqslant r+1$.

We introduce the energy functional $\mathcal{E}$: $W \times V \to \mathbb{R}$ by

$$\mathcal{E}(w,v) = \frac{1}{2}\|w\|_\varrho^2 + \frac{1}{2}\|v\|_{\kappa^{-1}}^2, \tag{1.11}$$

The square root of this quantity defines a norm on the product space $W \times V$, that is

$$\|(w,v)\|_{\mathcal{E}} \equiv \sqrt{\mathcal{E}(w,v)}. \tag{1.12}$$

1.1. Weak form and discretization. The solution of (1.1) satisfies the weak form of finding $u : [0,T] \to V \equiv H_0(\mathrm{div})$ and $p : [0,T] \to W \equiv L^2$ such that

$$\begin{aligned}(\varrho p_t, w) + (\nabla \cdot u, w) &= (f, w)\,,\\ \left(\kappa^{-1} u_t, v\right) - (p, \nabla \cdot v) &= (g, v)\,,\end{aligned} \tag{1.13}$$

together with initial conditions

$$\begin{aligned}p(x,0) &= p_0(x),\\ u(x,0) &= u_0(x).\end{aligned} \tag{1.14}$$

The semidiscrete mixed formulation of (1.13) is to find $u_h : [0,T] \to V_h$ and $p_h : [0,T] \to W_h$ such that

$$\begin{aligned}(\varrho p_{h,t}, w_h) + (\nabla \cdot u_h, w_h) &= (f, w_h)\,,\\ \left(\kappa^{-1} u_{h,t}, v_h\right) - (p_h, \nabla \cdot v_h) &= (g, v_h)\end{aligned} \tag{1.15}$$

for all $w_h \in W_h$ and $v_h \in V_h$. We take as initial conditions for the semidiscrete problem

$$\begin{aligned}p_h(\cdot,0) &= \pi p_0,\\ u_h(\cdot,0) &= \Pi u_0,\end{aligned} \tag{1.16}$$

where we have chosen the divergence projection rather than the L^2 projection of the initial condition for u_h.

Let $\{\phi_i\}_{i=1}^{|W_h|}$ and $\{\psi_i\}_{i=1}^{|V_h|}$ be bases for W_h and V_h, respectively. We define the weighted mass matrix $\mathrm{M}_{ij}^{\varrho} = (\varrho\phi_j, \phi_i)$ for space W_h, with the superscript omitted when $\varrho \equiv 1$. Similarly, the weighted mass matrix for V_h is given by $\widetilde{\mathrm{M}}_{ij}^{\kappa^{-1}} = \left(\kappa^{-1}\psi_j, \psi_i\right)$ with the superscript omitted for $\kappa \equiv 1$. We also require the weak divergence operator $\mathrm{D}_{ij} = (\phi_i, \nabla \cdot \psi_j)$. We represent the discrete solutions $W_h \ni p_h(\cdot,t) \equiv \sum_{i=1}^{|W_h|} \mathrm{p}_i(t)\phi_i$ and $V_h \ni u_h(\cdot,t) \equiv \sum_{i=1}^{|V_h|} \mathrm{u}_i(t)\psi_i$. We similarly let $\mathrm{f}_i(t)$ and $\mathrm{g}_i(t)$ denote the expansion coefficients of the L^2 projections of the forcing terms f and g. That is, $\pi f(\cdot,t) = \sum_{i=1}^{|W_h|} \mathrm{f}_i(t)\phi_i$ and $\pi g(\cdot,t) = \sum_{i=1}^{|V_h|} \mathrm{g}_i(t)\psi_i$. Finally, with p^0 and u^0 be the vectors of expansion coefficients of the initial conditions (1.16), the semidiscrete form (1.15) satisfies the system of ordinary differential equations

$$\begin{aligned}\mathrm{M}^{\varrho}\mathrm{p}_t + \mathrm{D}\mathrm{u} &= \mathrm{M}\mathrm{f},\\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{u}_t - \mathrm{D}^T\mathrm{p} &= \widetilde{\mathrm{M}}\mathrm{g},\end{aligned} \tag{1.17}$$

with initial conditions

$$\begin{aligned}\mathrm{p}(0) &= \mathrm{p}^0,\\ \mathrm{u}(0) &= \mathrm{u}^0.\end{aligned} \tag{1.18}$$

We will also consider fully discrete methods. An essential feature of the wave equation is its energy conservation. The resulting fully discrete method preserves the system energy. It is well-known that the standard forward Euler method, such as used in [10], provides nondecreasing energy at each time step, while the backward Euler method, proposed by Geveci [9], generates nonincreasing energy. Here, we consider the symplectic Euler method, which is known to be a symplectic integrator for Hamilton's equations, to our mixed formulation of the wave equation.

We partition the time interval $[0, T]$ into equispaced time steps $0 \equiv t_0 < t_1 < t_2 < \cdots < t_N$, where $t_i = i\Delta t$. Then, we approximate the solution to the semidiscrete method (1.15) with $p_h(t_n) \approx p_h^n \in W_h$ and $u_h(t_n) \approx u_h^n \in V_h$ by the rule

$$\begin{aligned} \left(\varrho \frac{\Delta p_h^n}{\Delta t}, w_h\right) + (\nabla \cdot u_h^n, w_h) &= (f^n, w_h), \\ \left(\kappa^{-1} \frac{\Delta u_h^n}{\Delta t}, v_h\right) - \left(p_h^{n+1}, \nabla \cdot v_h\right) &= \left(g^{n+1}, v_h\right), \end{aligned} \tag{1.19}$$

where $f^n = f(t_n)$, $g^n = g(t_n)$, and $\Delta p_h^n = p_h^{n+1} - p_h^n$ is the standard forward difference operator. This is sometimes called the semiimplicit Euler method, since the first equation appears to be explicit and the second implicit. However, in practice, we use the first equation to solve for p_h^{n+1} only by inverting a mass matrix and then use that value of p_h^{n+1} to solve for u_h^{n+1} in the second equation, again only inverting a mass matrix.

1.2. Relevant literature. Geveci [9] first applied $H(\text{div})$ finite elements to a wave equation, proving existence and uniqueness and optimal *a priori* error estimates in $L^\infty(L^2)$ for both variables for the formulation considered here. He also formulated but did not analyze a backward Euler time-stepping scheme. Glowinski and Rossi [10] utilize this formulation in a control problem. Other anaylysis of mixed methods seems to focus on a slightly different formulation in which two time derivatives appear on the vector variable and none on the other equation. This more clearly represents the acoustic wave equation as a limiting result of elastodynamics, but less explicitly reveals Hamiltonian structure in the equations. For example, Cowsar, Dupont and Wheeler [8] prove *a priori* error estimates for this formulation and stability results for a family of time discretizations. This analysis was extended by Jenkins, Rivière, and Wheeler [14], and Jenkins provided numerical experimentation related to these results [13].

Our present work returns to the formulation of Geveci, and strengthens the existing theory in two major ways. First, we are able to control the temporal derivatives of both variables and the divergence of u_h, and such estimates are absent in related publications. Second, the first-order formulation we consider here clearly has a Hamiltonian structure. Geveci uses this fact, in the form that the time advancement operator is unitary, in his semidiscrete analysis. We carry this consideration forward in our discussion of fully discrete methods, considering the energy conservation properties of the symplectic Euler method [7].

The interaction of our spatial and temporal discretizations represent the intersection of two important trends in modern numerical analysis. On one hand, research on the finite element exterior calculus [2] and mimetic methods [3] demonstrate that the effectiveness of Raviart-Thomas-Nédeléc and related spaces derives from the discrete preservation of the de Rham complex. On the other hand, the theory of geometric integration has provided algorithms that reproduces essential qualitative structures such as energy conservation, with practical implications for long-term dynamics. In this way, our current discretization can be seen to preserve the essential structure in both the spatial and temporal aspects of the wave equation. This combination has been formulated for electromagnetics [19, 20], but ours seems to be the first theoretical analysis combining symplectic time integration with some form of mixed finite element space.

2. Stability of semidiscrete method. The existence and uniqueness of the semidiscrete method, as well as basic $L^\infty(L^2)$ stability results, are proven by Geveci [9]. We will prove similar stability results with different techniques that will more readily translate to the fully discrete methods later.

Before doing this, we prove a lemma that we will use in lieu of Young's inequality on several occasions.

LEMMA 2.1. *Suppose that a real number x satisfies the quadratic inequality*

$$x^2 \leqslant \gamma^2 + \beta x$$

for $\beta > 0$. Then

$$x \leqslant \beta + \gamma.$$

Proof. Rewrite the inequality as $x^2 - \beta x - \gamma^2 \leqslant 0$. The left-hand side is a quadratic function with positive leading term. So, for the inequality to hold, x must be to the left of the larger of the two roots of $x^2 - \beta x - \gamma^2$, which is

$$\frac{\beta + \sqrt{\beta^2 + 4\gamma^2}}{2}.$$

This is readily bounded by $\beta + \gamma$. Young's inequality instead would only give $x \leqslant \sqrt{2}\gamma + \beta$. □

We define the energy of the semidiscrete solution by

$$a^2(t) = \|(p_h(\cdot, t), u_h(\cdot, t))\|_{\mathcal{E}}. \tag{2.1}$$

THEOREM 2.2. *Let (p_h, u_h) be the solution to the semidiscrete problem (1.15). Provided that $f \in L^1\left(0, T; L^2(\Omega)\right)$ and $g \in L^1\left(0, T; \left(L^2(\Omega)\right)^d\right)$, we have the stability bound*

$$\sup_{0 \leqslant s \leqslant T} a(s) \leqslant a(0) + \sqrt{2} \int_0^T \|f(\cdot, s)\|_{\varrho^{-1}} + \|g(\cdot, s)\|_{\kappa} \, ds. \tag{2.2}$$

Proof. By selecting $w_h = p_h$ and $v_h = u_h$ at each time level in (1.15) and adding the two equations, we find that

$$(\varrho p_{h,t}, p_h) + \left(\kappa^{-1} u_{h,t}, u_h\right) = (f, p_h) + (g, u_h)$$

at each time $0 \leqslant t \leqslant T$. We rewrite the left-hand side as

$$(\varrho p_{h,t} p_h) + \left(\kappa^{-1} u_{h,t}, u_h\right) = \frac{1}{2}\frac{d}{dt}\|p_h(\cdot, t)\|_{\varrho}^2 + \frac{1}{2}\frac{d}{dt}\|u_h(\cdot, t)\|_{\kappa^{-1}}^2 = \frac{d}{dt}a^2(t).$$

Now, we pick any $0 \leqslant \tilde{t} \leqslant T$ and integrate from 0 to $\tilde{t}$ to obtain

$$a^2(\tilde{t}) = a^2(0) + \int_0^{\tilde{t}} (f, p_h) + (g, u_h) \, dt.$$

In the absence of forcing terms, this demonstrates that energy is conserved in the semidiscrete system.

To proceed, we use the weighted Cauchy-Schwarz estimate and extend the domain of integration to make the bound

$$\begin{aligned} a^2\left(\tilde{t}\right) &\leqslant a^2(0) + \int_0^{\tilde{t}} \|f(\cdot,s)\|_{\varrho^{-1}} \|p_h(\cdot,s)\|_{\varrho}\, ds + \int_0^{\tilde{t}} \|g(\cdot,s)\|_{\kappa} \|u_h(\cdot,s)\|_{\kappa^{-1}}\, ds \\ &\leqslant a^2(0) + \int_0^T \|f(\cdot,s)\|_{\varrho^{-1}} \|p_h(\cdot,s)\|_{\varrho} + \|g(\cdot,s)\|_{\kappa} \|u_h(\cdot,s)\|_{\kappa^{-1}}\, ds. \end{aligned}$$

Using the discrete Cauchy-Schwartz inequality under the integral sign, we have

$$\begin{aligned} a^2\left(\tilde{t}\right) &\leqslant a^2(0) + \int_0^T \sqrt{2} a(s) \sqrt{\|f(\cdot,s)\|_{\varrho^{-1}}^2 + \|g(\cdot,s)\|_{\kappa}^2}\, ds \\ &\leqslant a^2(0) + \sqrt{2}\left(\sup_{0\leqslant s\leqslant T} a(s)\right)\int_0^T \|f(\cdot,s)\|_{\varrho^{-1}} + \|g(\cdot,s)\|_{\kappa}\, ds. \end{aligned}$$

Now, the right-hand side is independent of t and $f(t) = t^2$ is monotonic, so we have the bound

$$\sup_{0\leqslant s\leqslant T} a^2(s) \leqslant a^2(0) + \sqrt{2}\left(\sup_{0\leqslant s\leqslant T} a(s)\right)\int_0^T \|f(\cdot,s)\|_{\varrho^{-1}} + \|g(\cdot,s)\|_{\kappa}\, ds.$$

This has the form

$$x^2 \leqslant \gamma^2 + \beta x,$$

where

$$\begin{aligned} x &= \sup_{0\leqslant t\leqslant T} a(t), \\ \beta &= \sqrt{2}\int_0^T \|f(\cdot,s)\|_{\varrho^{-1}} + \|g(\cdot,s)\|_{\kappa}\, ds, \\ \gamma &= a(0). \end{aligned}$$

Applying Lemma 2.1 finishes the proof. $\square$

Now, we will develop a bootstrap technique that will allow us to prove stronger bounds on the solution, assuming more regularity on the data. In turn, it will allow error estimates in stronger norms. The system (1.17) is a linear system of ODE, so the solution will be as differentiable as we like assuming sufficient differentiability of the forcing terms. So, assuming that f_t and g_t exist in a reasonable space (made precise in the following theorem), we may differentiate the system to obtain

$$\begin{aligned} \mathrm{M}^{\varrho}\mathrm{p}_{tt} + \mathrm{D}\mathrm{u}_t &= \mathrm{M}\mathrm{f}_t, \\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{u}_{tt} - \mathrm{D}^T\mathrm{p}_t &= \widetilde{\mathrm{M}}\mathrm{g}_t. \end{aligned}$$

From here, we define $\mathrm{q}(t) = \mathrm{p}_t$ and $\mathrm{r}(t) = \mathrm{u}_t$, and we rewrite the differentiated system as the first order system

$$\begin{aligned} \mathrm{M}^{\varrho}\mathrm{q}_t + \mathrm{D}\mathrm{r} &= \mathrm{M}\mathrm{f}_t, \\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{r}_t - \mathrm{D}^T\mathrm{q} &= \widetilde{\mathrm{M}}\mathrm{g}_t. \end{aligned} \tag{2.3}$$

Given the equivalence between the system of ODE (1.17) and the variational problem (1.15), we can take $W_h \ni q_h(\cdot,t) = \sum_{i=1}^{|W_h|} \mathrm{q}_i(t)\phi_i$ and $V_h \ni r_h(\cdot,t) = \sum_{i=1}^{|V_h|} \mathrm{r}_i(t)\psi_i$.

Then, we have that the temporal derivatives $p_{h,t} = q_h$ and $u_{h,t} = r_h$ satisfy the variational problem

$$\begin{aligned}(\varrho q_{h,t}, w_h) + (\nabla \cdot r_h, w_h) &= (f_t, w_h)\,, \\ \left(\kappa^{-1} r_{h,t}, v_h\right) - (q_h, \nabla \cdot v_h) &= (g_t, v_h)\,.\end{aligned} \tag{2.4}$$

The initial conditions for this system are specified by evaluating the system (1.15) at time 0:

$$\begin{aligned}(\varrho q_h(\cdot,0), w_h) + (\nabla \cdot u_h(\cdot,0), w_h) &= (f(\cdot,0), w_h)\,, \\ \left(\kappa^{-1} r_h(\cdot,0), v_h\right) - (p_h(\cdot,0), \nabla \cdot v_h) &= (g(\cdot,0), v_h)\,,\end{aligned}$$

so that the initial derivatives are weighted projections of the initial values of the forcing terms and derivatives of the initial conditions. With $u_h(\cdot,0) \in V_h$, $p_{h,t}(\cdot,0)$, we pick $w_h = q_h(\cdot,0)$ to give

$$\|q_h\,(\cdot,0)\|_\varrho^2 = (f(\cdot,0), q_h(\cdot,0)) - (\nabla \cdot u_h\,(\cdot,0)\,, q_h(\cdot,0))\,,$$

from which follows the bound

$$\|q_h\,(\cdot,0)\|_\varrho \leqslant \|f\,(\cdot,0)\|_{\varrho^{-1}} + \|\nabla \cdot u_h\,(\cdot,0)\|_{\varrho^{-1}}\,. \tag{2.5}$$

We may also bound $r_h(\cdot,0)$. Taking $v_h = r_h(\cdot,0)$ in the second equation in (1.15) gives

$$\|r_h\,(\cdot,0)\|_{\kappa^{-1}}^2 = (g(\cdot,0), r_h(\cdot,0)) + (p_h(\cdot,0), \nabla \cdot r_h(\cdot,0))\,.$$

Using Cauchy-Schwarz and the inverse assumption (1.5) to bound $\|\nabla \cdot r_h\,(\cdot,0)\|$ by the L^2 norm gives the estimate

$$\|r_h\,(\cdot,0)\|_{\kappa^{-1}} \leqslant \|g\,(\cdot,0)\|_\kappa + \frac{C_0\sqrt{\kappa^*}}{h}\|p_h(\cdot,0)\|. \tag{2.6}$$

This is only an $\mathcal{O}(h^{-1})$ bound, degrading as the mesh is refined. However, if the initial condition $p^0 \in H^1$, the use of the inverse assumption and hence the h^{-1} factor may be avoided. Since $\nabla \cdot r_h(\cdot,0) \in W_h$ and $p_h(\cdot,0)$ is the L^2 projection of p^0, we have

$$(p_h(\cdot,0), \nabla \cdot r_h(\cdot,0)) = \left(\pi p^0, \nabla \cdot r_h(\cdot,0)\right) = \left(p^0, \nabla \cdot r_h(\cdot,0)\right) = -\left(\nabla p^0, r_h(\cdot,0)\right),$$

giving the bound

$$\|r_h\,(\cdot,0)\|_{\kappa^{-1}} \leqslant \|g\,(\cdot,0)\|_\kappa + \left\|\nabla p^0\right\|_\kappa. \tag{2.7}$$

If $p^0 \notin H^1$, but is not merely L^2 (say, it is piecewise smooth with jump discontinuities), it may be possible to get more refined estimates using elementwise integration by parts. However, as our main application of this stability estimate will be to error estimates where the initial pressure value is zero, we do not pursue this further here.

A straightforward application of Theorem 2.2 to (2.4) gives a bound on the time derivatives. We define

$$b(t) \equiv \|(p_{h,t}\,(\cdot,t)\,, u_{h,t}\,(\cdot,t))\|_{\mathcal{E}}\,. \tag{2.8}$$

Theorem 2.3. *Let p_h and u_h be the solutions to (1.15) and. If $f_t \in L^1(0,T;L^2(\Omega))$ and $g_t \in L^1(0,T;\left(L^2(\Omega)\right)^d)$, then*

$$\sup_{0 \leqslant s \leqslant T} b(s) \leqslant b(0) + \sqrt{2} \int_0^T \left\| f_t\left(\cdot, s\right) \right\|_{\varrho^{-1}} + \left\| g_t\left(\cdot, s\right) \right\|_{\kappa} ds. \tag{2.9}$$

Bounds on the time derivatives in turn allow bounds on the divergence of the discrete solution at each time. Selecting $w_h = \nabla \cdot u_h$ in the first equation of (1.15) gives

$$\left(\varrho p_{h,t}, \nabla \cdot u_h \right) + \left\| \nabla \cdot u_h \right\|^2 = \left(f, \nabla \cdot u_h \right), \tag{2.10}$$

so that

$$\left\| \nabla \cdot u_h \right\| \leqslant \left\| f \right\| + \sqrt{\varrho^*} \left\| p_{h,t} \right\|_{\varrho}, \tag{2.11}$$

and in light of the previous theorem, we have

Theorem 2.4. *Under the assumptions of Theorem 2.3, and assuming also that $f \in L^\infty(0,T;L^2(\Omega))$, we have that*

$$\begin{aligned} \sup_{0 \leqslant s \leqslant T} \left\| \nabla \cdot u_h\left(\cdot, s\right) \right\| \leqslant & \sup_{0 \leqslant s \leqslant T} \left\| f\left(\cdot, s\right) \right\| + \sqrt{\varrho^*} b(0) \\ & + \sqrt{2\varrho^*} \int_0^T \left\| f_t\left(\cdot, s\right) \right\|_{\varrho^{-1}} + \left\| g_t\left(\cdot, s\right) \right\|_{\kappa} ds. \end{aligned} \tag{2.12}$$

3. Semidiscrete error estimates. We will bound the error in the semidiscrete method in various norms by comparing the computed solution to the projections of the true solutions. To do this, we restrict the test functions in (1.13) to the finite-dimensional spaces. Then, using the properties of the projections π and Π, we have that

$$\begin{aligned} \left(\varrho \pi p_t, w_h \right) + \left(\nabla \cdot \Pi u, w_h \right) &= \left(f, w_h \right) + \left(\varrho \left(\pi p_t - p_t \right), w_h \right), \\ \left(\kappa^{-1} \Pi u_t, v_h \right) - \left(\pi p, \nabla \cdot v_h \right) &= \left(g, v_h \right) + \left(\kappa^{-1} \left(\Pi u_t - u_t \right), v_h \right). \end{aligned} \tag{3.1}$$

We may subtract the semidiscrete form (1.15) from these to obtain the error equations

$$\begin{aligned} \left(\varrho \left(\pi p_t - p_{h,t} \right), w_h \right) + \left(\nabla \cdot \left(\Pi u - u_h \right), w_h \right) &= \left(\varrho \left(\pi p_t - p_t \right), w_h \right), \\ \left(\kappa^{-1} \left(\Pi u_t - u_{h,t} \right), v_h \right) - \left(\pi p - p_h, \nabla \cdot v_h \right) &= \left(\kappa^{-1} \left(\Pi u_t - u_t \right), v_h \right). \end{aligned} \tag{3.2}$$

We define $W_h \ni \theta_h(\cdot,t) \equiv \pi p(\cdot,t) - p_h(\cdot,t)$ and $V_h \ni \chi_h(\cdot,t) \equiv \Pi u(\cdot,t) - u_h(\cdot,t)$ to be the differences between projections and computed solutions. We also let $W \ni \xi(\cdot,t) \equiv \pi p(\cdot,t) - p(\cdot,t)$ and $V \ni \eta(\cdot,t) \equiv \Pi u(\cdot,t) - u(\cdot,t)$ be the differences between the exact solutions and their projections.

Because of the definitions of ξ and η and the properties of the projections, these differences satisfy the semidiscrete problem.

$$\begin{aligned} \left(\varrho \theta_{h,t}, w_h \right) + \left(\nabla \cdot \chi_h, w_h \right) &= \left(\varrho \xi_t, w_h \right), \\ \left(\kappa^{-1} \chi_{h,t}, v_h \right) - \left(\theta_h, \nabla \cdot v_h \right) &= \left(\kappa^{-1} \eta_t, v_h \right). \end{aligned} \tag{3.3}$$

Because of the choice of initial conditions for p_h and u_h, the initial values $\theta_h(\cdot,0)$ and $\chi_h(\cdot,0)$ are both zero.

A direct application of Theorems 2.2, 2.3 and 2.4 allows us to bound θ_h and χ_h in various norms in terms of the projection errors. We define

$$\varepsilon(t) = \|(\theta_h(\cdot,t), \chi_h(\cdot,t))\|_{\mathcal{E}}\,. \tag{3.4}$$

LEMMA 3.1. *Suppose that the true solution* (p,u) *has time derivatives* $p_t \in L^1\left(0,T; H^{r+1}(\Omega)\right)$ *and* $u_t \in L^1\left(0,T; \left(H^{r+1}(\Omega)\right)^d\right)$. *Then, for* $1 \leqslant m \leqslant r+1$, *we have the bound*

$$\varepsilon(t) \leqslant C_1 h^m \sqrt{2\varrho^*} \int_0^T |p_t(\cdot,s)|_m \, ds + C_2 h^m \sqrt{\frac{2}{\kappa_*}} \int_0^T |u_t(\cdot,s)|_m \, ds. \tag{3.5}$$

Proof. Applying Theorem 2.2 to (3.3) and using the fact that $\varepsilon^2(0)=0$, we have that

$$\sup_{0\leqslant s\leqslant T} \varepsilon(s) \leqslant \sqrt{2}\int_0^T \|\varrho\xi_t(\cdot,s)\|_{\varrho^{-1}} + \left\|\kappa^{-1}\eta_t(\cdot,s)\right\|_{\kappa} ds.$$

We rewrite the weighted norms

$$\|\varrho\xi_t\|_{\varrho^{-1}}^2 = \int_\Omega \varrho^{-1}(\varrho\xi_t)^2\,dx = \int_\Omega \varrho(\xi_t)^2\,dx = \|\xi_t\|_\varrho^2,$$

with an analagous calculation showing $\left\|\kappa^{-1}\eta_t\right\|_\kappa = \|\eta_t\|_{\kappa^{-1}}$. This gives

$$\sup_{0\leqslant s\leqslant T} \varepsilon(s) \leqslant \sqrt{2}\int_0^T \|\xi_t(\cdot,s)\|_\varrho + \|\eta_t(\cdot,s)\|_{\kappa^{-1}}\,ds,$$

and then we use norm equivalence and the approximation theoretic bounds (1.8) and (1.9) to finish the proof. □

We define the error quantity

$$\epsilon(t) = \|(p(\cdot,t)-p_h(\cdot,t), u(\cdot,t)-u_h(\cdot,t))\|_{\mathcal{E}}\,. \tag{3.6}$$

THEOREM 3.2. *Suppose that the solutions have time derivatives* $p_t \in L^1\left(0,T; H^{r+1}(\Omega)\right)$ *and* $u_t \in L^1\left(0,T; \left(H^{r+1}(\Omega)\right)^d\right)$ *and additionally that* $p \in L^\infty\left(0,T; H^{r+1}(\Omega)\right)$ *and* $u \in L^\infty\left(0,T; \left(H^{r+1}(\Omega)\right)^d\right)$. *Then, for each* $1 \leqslant m \leqslant r+1$, *we have*

$$\begin{aligned}\sup_{0\leqslant s\leqslant T} \epsilon(s) \leqslant & C_1 h^m \sqrt{\varrho^*}\left(\frac{1}{\sqrt{2}}\sup_{0\leqslant s\leqslant T}|p(\cdot,s)|_m + \sqrt{2}\int_0^T |p_t(\cdot,s)|_m\,ds\right) \\ & + \frac{C_2 h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}\sup_{0\leqslant s\leqslant T}|u(\cdot,s)|_m + \sqrt{2}\int_0^T |u_t(\cdot,s)|_m\,ds\right).\end{aligned} \tag{3.7}$$

Proof. The triangle inequality gives

$$\sup_{0\leqslant t\leqslant T} \epsilon(t) \leqslant \sup_{0\leqslant t\leqslant T} \|(\xi(\cdot,t), \eta(\cdot,t))\|_{\mathcal{E}} + \sup_{0\leqslant t\leqslant T} \|(\theta(\cdot,t), \chi(\cdot,t))\|_{\mathcal{E}}\,. \tag{3.8}$$

We use the approximation-theoretic bounds (1.8) and (1.9) on the first term:

$$\begin{aligned}\|(\xi(\cdot,t),\eta(\cdot,t))\|_{\mathcal{E}} &\leqslant \frac{1}{\sqrt{2}}\|\xi(\cdot,t)\|_{\varrho}+\frac{1}{\sqrt{2}}\|\eta(\cdot,t)\|_{\kappa^{-1}}\\ &\leqslant \sqrt{\frac{\varrho^*}{2}}\|\xi(\cdot,t)\|+\frac{1}{\sqrt{2\kappa_*}}\|\eta(\cdot,t)\|\\ &\leqslant C_1h^m\sqrt{\frac{\varrho^*}{2}}|p(\cdot,t)|_m+\frac{C_2h^m}{\sqrt{2\kappa_*}}|u(\cdot,t)|_m.\end{aligned}$$

The result follows by applying the previous theorem to the second term in (3.8) and combining terms. □

Now, we turn to the error in the time derivatives. We define

$$\beta(t)=\|(\pi p_t(\cdot,t)-p_{h,t}(\cdot,t),\Pi u_t(\cdot,t)-u_{h,t}(\cdot,t))\|_{\mathcal{E}}=\|(\theta_{h,t}(\cdot,t),\chi_{h,t}(\cdot,t))\|_{\mathcal{E}}. \quad (3.9)$$

LEMMA 3.3. *Suppose that* $p_{tt}\in L^1(0,T;H^m(\Omega))$, $u_{tt}\in L^1\left(0,T;(H^m(\Omega))^d\right)$ *and also that* $p_t(\cdot,0)\in H^m(\Omega)$ *and* $u_t(\cdot,0)\in(H^m(\Omega))^d$. *Then*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\beta(t)\leqslant &C_1\sqrt{\varrho^*}h^m\left(\frac{1}{\sqrt{2}}|p_t(\cdot,0)|_m+\sqrt{2}\int_0^T|p_{tt}(\cdot,t)|\,dt\right)\\ &+\frac{C_2h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}|u_t(\cdot,0)|_m+\sqrt{2}\int_0^T|u_{tt}(\cdot,t)|\,dt\right).\end{aligned} \quad (3.10)$$

Proof. If we apply the stability result of Theorem 2.3 to the error equations (3.3), we find that

$$\sup_{0\leqslant t\leqslant T}\beta(t)\leqslant\beta(0)+\sqrt{2}\int_0^T\|\varrho\xi_{tt}(\cdot,t)\|_{\varrho^{-1}}+\left\|\kappa^{-1}\eta_{tt}(\cdot,t)\right\|_{\kappa}dt.$$

We can bound $\beta(0)$ using the error equations (3.3) together with bounds on the initial derivatives (2.5) and (2.6):

$$\begin{aligned}\beta(0)&\leqslant\frac{1}{\sqrt{2}}\|\theta_{h,t}(\cdot,0)\|_{\varrho}+\frac{1}{\sqrt{2}}\|\chi_{h,t}(\cdot,0)\|_{\kappa^{-1}}\\ &\leqslant\frac{1}{\sqrt{2}}\|\xi_t(\cdot,0)\|_{\varrho}+\frac{1}{\sqrt{2}}\|\eta_t(\cdot,0)\|_{\kappa^{-1}}.\end{aligned}$$

The approximation results (1.8) and (1.9) give then that

$$\beta(0)\leqslant C_1h^m\sqrt{\frac{\varrho^*}{2}}|p_t(\cdot,0)|_m+\frac{C_2h^m}{\sqrt{2\kappa_*}}|u_t(\cdot,0)|_m.$$

For the remaining terms, rewriting the weighted norms and applying the approximation estimates finishes the proof. □

Now, we let

$$\gamma(t)=\|(p_t(\cdot,t)-p_{h,t}(\cdot,t),u_t(\cdot,t)-u_{h,t}(\cdot,t))\|_{\mathcal{E}}, \quad (3.11)$$

and we have an error estimate. Lemma 3.3 plus the approximation results give us a bound on $\gamma(t)$.

THEOREM 3.4. *Under the assumptions of Lemma 3.3, we have that*

$$\begin{aligned}\sup_{0\leqslant s\leqslant T}\gamma(s) \leqslant & C_1 h^m\sqrt{\varrho^*}\left(\frac{1}{\sqrt{2}}\left|p_t(\cdot,0)\right|_m+\frac{1}{\sqrt{2}}\sup_{0\leqslant s\leqslant T}\left|p_t(\cdot,s)\right|_m+\sqrt{2}\int_0^T\left|p_{tt}(\cdot,s)\right|_m ds\right)\\ &+\frac{C_2h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}\left|u_t(\cdot,0)\right|_m+\frac{1}{\sqrt{2}}\sup_{0\leqslant s\leqslant T}\left|u_t(\cdot,s)\right|_m+\sqrt{2}\int_0^T\left|u_{tt}(\cdot,s)\right|_m ds\right).\end{aligned} \tag{3.12}$$

Proof. First, we use the triangle inequality to write that

$$\sup_{0\leqslant s\leqslant T}\gamma(s)\leqslant\sup_{0\leqslant s\leqslant T}\beta(s)+\sup_{0\leqslant s\leqslant T}\left\|(\xi_t(\cdot,s),\eta_t(\cdot,s))\right\|_{\mathcal{E}}.$$

The first term is bounded by the previous lemma. The second term satisfies the bound

$$\begin{aligned}\sup_{0\leqslant s\leqslant T}\left\|(\xi_t(\cdot,s),\eta_t(\cdot,s))\right\|_{\mathcal{E}} &\leqslant \sup_{0\leqslant s\leqslant T}\sqrt{\frac{\varrho^*}{2}}\left\|\xi_t(\cdot,s)\right\|+\sup_{0\leqslant t\leqslant T}\frac{1}{\sqrt{2\kappa_*}}\left\|\eta_t(\cdot,s)\right\|\\ &\leqslant\frac{C_1h^m\sqrt{\varrho^*}}{\sqrt{2}}\sup_{0\leqslant s\leqslant T}\left|p_t(\cdot,s)\right|_m+\frac{C_2h^m}{\sqrt{2\kappa_*}}\sup_{0\leqslant s\leqslant T}\left|u_t(\cdot,s)\right|_m,\end{aligned}$$

and the final result follows by combining these bounds with Lemma 3.3. ∎

Finally, we consider the error in divergence of the vector variable.

THEOREM 3.5. *If the assumptions of Lemma 3.3 hold, plus that* $\nabla\cdot u\in L^\infty(0,T;H^m(\Omega))$, *we have*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\left\|\nabla\cdot\chi_h(\cdot,t)\right\|\leqslant & C_1h^m\varrho^*\left(\left|p_t(\cdot,0)\right|_m+\sup_{0\leqslant t\leqslant T}\left|p_t(\cdot,t)\right|_m\right.\\ &\left.+\sup_{0\leqslant t\leqslant T}\left|\nabla\cdot u(\cdot,t)\right|_m+2\int_0^T\left|p_{tt}(\cdot,t)\right|_m dt\right)\\ &+C_2h^m\sqrt{\frac{\varrho^*}{\kappa_*}}\left(\left|u_t(\cdot,0)\right|_m+2\int_0^T\left|u_{tt}(\cdot,t)\right|_m dt\right).\end{aligned} \tag{3.13}$$

Proof. First, we bound the divergence error at each time by the triangle inequality.

$$\left\|\nabla\cdot(u-u_h)(\cdot,t)\right\|\leqslant\left\|\nabla\cdot\chi_h(\cdot,t)\right\|+\left\|\nabla\cdot\eta(\cdot,t)\right\|. \tag{3.14}$$

The latter term is bounded by (1.10). For the former, we note that an estimate like (2.11) for the error equations (3.3) gives

$$\sup_{0\leqslant t\leqslant T}\left\|\nabla\cdot\chi_h(\cdot,t)\right\|\leqslant\sup_{0\leqslant t\leqslant T}\left\|\varrho\xi_t(\cdot,t)\right\|+\sqrt{\varrho^*}\sup_{0\leqslant t\leqslant T}\left\|\theta_{h,t}(\cdot,t)\right\|_{\varrho}.$$

The first term is bounded by

$$\sup_{0\leqslant t\leqslant T}\left\|\varrho\xi_t(\cdot,t)\right\|\leqslant C_1\varrho^*h^m\sup_{0\leqslant t\leqslant T}\left|p_t(\cdot,t)\right|_m.$$

For the second term, $\|\theta_{h,t}(\cdot,t)\|_{\varrho} \leqslant \sqrt{2}\beta(t)$, so applying Lemma 3.3 gives

$$\begin{aligned}
\sqrt{\varrho^*} \sup_{0\leqslant t\leqslant T} \|\theta_{h,t}(\cdot,t)\|_{\varrho} \leqslant & \sqrt{2\varrho^*} \sup_{0\leqslant t\leqslant T} \beta(t) \\
\leqslant & \sqrt{2\varrho^*}\left[C_1\sqrt{\varrho^*}h^m\left(\frac{1}{\sqrt{2}}|p_t(\cdot,0)|_m + \sqrt{2}\int_0^T |p_{tt}(\cdot,t)|_m\,dt\right)\right] \\
& + \sqrt{2\varrho^*}\left[\frac{C_2h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}|u_t(\cdot,0)|_m + \sqrt{2}\int_0^T |u_{tt}(\cdot,t)|_m\,dt\right)\right] \\
= & C_1h^m\varrho^*\left(|p_t(\cdot,0)|_m + 2\int_0^T |p_{tt}(\cdot,t)|_m\,dt\right) \\
& + C_2\sqrt{\frac{\varrho^*}{\kappa_*}}h^m\left(|u_t(\cdot,0)|_m + 2\int_0^T |u_{tt}(\cdot,t)|_m\,dt\right).
\end{aligned}$$

□

4. Fully discrete method.

4.1. Discrete energy and conservation. In analyzing this method, we will proceed in a similar fashion as for the semidiscrete method, deriving stability results en route to error estimates. First, however, we demonstrate the conservation properties of the system. In particular, we will study the system energy $a_n^2 = \mathcal{E}(p_h^n, u_h^n)$ at each time level, but we will also introduce a perturbed energy functional on $W_h \times V_h$

$$\widetilde{\mathcal{E}}_{\Delta t}(w_h, v_h) \equiv \mathcal{E}(w_h, v_h) - \frac{\Delta t}{2}\left(\nabla \cdot v_h, w_h\right). \tag{4.1}$$

This gives rise to a perturbed energy of the fully discrete solution at each time level:

$$\widetilde{a}_n^2 \equiv \widetilde{\mathcal{E}}_{\Delta t}(p_h^n, u_h^n). \tag{4.2}$$

Under a certain CFL-like restriction on Δt, the perturbed energy is an equivalent functional to the actual energy on the finite element spaces

Lemma 4.1. *Let*

$$\alpha \equiv \frac{C_0\sqrt{\kappa^*}}{2h\sqrt{\varrho_*}}, \tag{4.3}$$

and suppose that

$$\alpha\Delta t < 1. \tag{4.4}$$

Then $\widetilde{\mathcal{E}}_{\Delta t}$ is positive-definite and satisfies

$$(1-\alpha\Delta t)\,\mathcal{E}\left(w_h, v_h\right) \leqslant \widetilde{\mathcal{E}}_{\Delta t}\left(w_h, v_h\right) \leqslant 2\mathcal{E}\left(w_h, v_h\right) \tag{4.5}$$

for all $w_h \in W_h$ and $v_h \in V_h$.

Proof. Let $w_h \in W_h$ and $v_h \in V_h$. Then using weighted Cauchy-Schwarz, the equivalence of weighted norms, the inverse assumption (1.5) and Young's inequality, we have

$$
\begin{aligned}
|(\nabla \cdot v_h, w_h)| &\leqslant \|\nabla \cdot v_h\|_{\varrho^{-1}} \|w_h\|_{\varrho} \\
&\leqslant \frac{1}{\sqrt{\varrho_*}} \|\nabla \cdot v_h\| \|w_h\|_{\varrho} \\
&\leqslant \frac{C_0}{h\sqrt{\varrho_*}} \|v_h\| \|w_h\|_{\varrho} \\
&\leqslant \frac{C_0\sqrt{\kappa^*}}{h\sqrt{\varrho_*}} \|v_h\|_{\kappa^{-1}} \|w_h\|_{\varrho} \\
&\leqslant \frac{C_0\sqrt{\kappa^*}}{h\sqrt{\varrho_*}} \mathcal{E}(w_h, v_h).
\end{aligned}
$$

With this estimate, we have that

$$
\begin{aligned}
\tilde{\mathcal{E}}_{\Delta t}(w_h, v_h) &= \mathcal{E}(w_h, v_h) - \frac{\Delta t}{2} (\nabla \cdot v_h, w_h) \\
&\geqslant \mathcal{E}(w_h, v_h) - \frac{C_0 \Delta t \sqrt{\kappa^*}}{2h\sqrt{\varrho_*}} \mathcal{E}(w_h, v_h) \\
&= (1 - \alpha\Delta t)\, \mathcal{E}(w_h, v_h),
\end{aligned}
\tag{4.6}
$$

the CFL condition and the positive-definiteness of $\mathcal{E}$ rendering the quantity positive-definite. The upper bound follows in a similar fashion. □ The perturbed energy is certainly well-defined on all of $W \times V$, but our proof of equivalence required the inverse estimate, which requires the discrete spaces.

Now, we show that the perturbed energy $\tilde{a}_n^2$ is exactly conserved at each time step.

Lemma 4.2. *If the forcing functions f, g both vanish, then for each $0 \leqslant n < N$, we have that*

$$
\tilde{a}_{n+1}^2 = \tilde{a}_n^2. \tag{4.7}
$$

Proof. Select $w_h = \frac{p_h^{n+1} + p_h^n}{2}$ and $v_h = \frac{u_h^{n+1} + u_h^n}{2}$ in (1.19) to find that

$$
\begin{aligned}
\frac{1}{2\Delta t} \left\|p_h^{n+1}\right\|_{\varrho}^2 - \frac{1}{2\Delta t} \|p_h^n\|_{\varrho}^2 + \left(\nabla \cdot u_h^n, \frac{p_h^{n+1} + p_h^n}{2}\right) &= 0, \\
\frac{1}{2\Delta t} \left\|u_h^{n+1}\right\|_{\kappa^{-1}}^2 - \frac{1}{2\Delta t} \|u_h^n\|_{\kappa^{-1}}^2 - \left(p_h^{n+1}, \nabla \cdot \frac{u_h^{n+1} + u_h^n}{2}\right) &= 0.
\end{aligned}
$$

Adding the equations together, multiplying by $2\Delta t$, and some straightforward manipulations give the desired result. □

4.2. Stability. Stability results analogous to those in Section 2 hold for the fully discrete method, provided that the time-step restriction (4.4) is satisfied.

Theorem 4.3. *Supposing that the sum $\sum_{n=0}^{N-1} \left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_{\kappa}\right) \Delta t$ is bounded and the CFL condition (4.4) holds, the energy a_n of the solution to (1.19) satisfies the bound*

$$
\max_{0 \leqslant n \leqslant N} a_n \leqslant \sqrt{\frac{1}{1 - \alpha\Delta t}} \tilde{a}_0 + \frac{\sqrt{2}}{1 - \alpha\Delta t} \sum_{n=0}^{N-1} \left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_{\kappa}\right) \Delta t \tag{4.8}
$$

Proof. Selecting the test functions as in the previous theorem, the same manipulations give us

$$\tilde{a}_{n+1}^2 - \tilde{a}_n^2 = \Delta t \left(f^n, \frac{p_h^{n+1} + p_h^n}{2} \right) + \Delta t \left(g^{n+1}, \frac{u_h^{n+1} + u_h^n}{2} \right).$$

If we fix some $0 \leqslant M \leqslant N$ and sum this equation from $n = 0$ to $M - 1$, we obtain

$$\tilde{a}_M^2 = \tilde{a}_0^2 + \sum_{n=0}^{M-1} \Delta t \left(f^n, \frac{p_h^{n+1} + p_h^n}{2} \right) + \sum_{n=0}^{M-1} \Delta t \left(g^{n+1}, \frac{u_h^{n+1} + u_h^n}{2} \right).$$

Using Lemma 4.1 on the left-hand side and weighted Cauchy-Schwarz on the right and extending the interval of summation give

$$\begin{aligned}
(1 - \alpha \Delta t)\, a_M^2 &\leqslant \tilde{a}_0^2 + \sum_{n=0}^{M-1} \frac{\Delta t}{2} \left(\|f^n\|_{\varrho^{-1}} \left\|p_h^{n+1} + p_h^n\right\|_{\varrho} + \left\|g^{n+1}\right\|_{\kappa} \left\|u_h^{n+1} + u_h^n\right\|_{\kappa^{-1}} \right) \\
&\leqslant \tilde{a}_0^2 + \sum_{n=0}^{N-1} \frac{\Delta t}{2} \left(\|f^n\|_{\varrho^{-1}} \left\|p_h^{n+1} + p_h^n\right\|_{\varrho} + \left\|g^{n+1}\right\|_{\kappa} \left\|u_h^{n+1} + u_h^n\right\|_{\kappa^{-1}} \right)
\end{aligned}$$

Now, we apply the triangle inequality and the fact that $\|p_h^n\|_{\varrho}$ and $\|u_h^n\|_{\kappa^{-1}}$ are each bounded by $\sqrt{2} a_n$ to find that

$$\max_{0 \leqslant n \leqslant N} a_n^2 \leqslant \frac{1}{1 - \alpha \Delta t} \tilde{a}_0^2 + \frac{\sqrt{2}}{1 - \alpha \Delta t} \left(\max_{0 \leqslant n \leqslant N} a_n \right) \sum_{n=0}^{N-1} \left(\|f^n\|_{\varrho^{-1}} \left\|g^{n+1}\right\|_{\kappa} \right) \Delta t$$

To complete the proof, we use Lemma 2.1 with $x = \max_{0 \leqslant n \leqslant N} a_n$, $\beta = \left(\frac{\sqrt{2}}{1 - \alpha \Delta t} \right) \sum_{n=0}^{N-1} \left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_{\kappa} \right) \Delta t$, and $\gamma = \frac{1}{\sqrt{1 - \alpha \Delta t}} \tilde{a}_0$. □

As with the semidiscrete case, we can apply a boostrapping argument to obtain bounds in stronger norms. However, we will only deal with the case of discrete initial conditions $p_h^0 = 0$ and $u_h^0 = 0$. This is all that is required for subsequent error estimates. Using linearity, these results may also be adapted to nonzero initial conditions by converting them to forcing functions.

We define the difference quotients.

$$\begin{aligned}
q_h^n &= \frac{\Delta p_h^n}{\Delta t}, \\
r_h^n &= \frac{\Delta u_h^n}{\Delta t}.
\end{aligned} \tag{4.9}$$

Then, time-differencing (1.19) gives us the new system of equations

$$\begin{aligned}
\left(\varrho \frac{\Delta q_h^n}{\Delta t}, w_h \right) + \left(\nabla \cdot r_h^n, w_h \right) &= \left(\frac{\Delta f^n}{\Delta t}, w_h \right), \\
\left(\kappa^{-1} \frac{\Delta r_h^n}{\Delta t}, v_h \right) - \left(q_h^{n+1}, \nabla \cdot v_h \right) &= \left(\frac{\Delta g^{n+1}}{\Delta t}, v_h \right),
\end{aligned} \tag{4.10}$$

which holds for $0 \leqslant n \leqslant N - 1$ rather than N. The initial conditions for q_h^0 and r_h^0 are defined by evaluating (1.19) at $n = 0$

$$\begin{aligned}
\left(\varrho q_h^0, w_h \right) + \left(\nabla \cdot u_h^0, w_h \right) &= \left(f^0, w_h \right), \\
\left(\kappa^{-1} r_h^0, v_h \right) - \left(p_h^1, v_h \right) &= \left(g^1, v_h \right).
\end{aligned} \tag{4.11}$$

Selecting $w_h = q_h^0$ in the first equation, applying the fact that $u_h^0 = 0$ and using Cauchy Schwarz immediately gives that

$$\left\| q_h^0 \right\|_\varrho \leqslant \left\| f^0 \right\|_{\varrho^{-1}},$$

but the presence of p_h^1 rather than p_h^0 in the second complicates matters somewhat. We use the first equation, with $u_h^0 = 0$, in (1.19) to bound p_h^1:

$$\left(\varrho p_h^1, w_h\right) = \Delta t \left(f^0, w_h\right),$$

so that

$$\left\| p_h^1 \right\|_\varrho \leqslant \Delta t \left\| f^0 \right\|_{\varrho^{-1}}.$$

Then, the second equation in (1.19) gives that

$$\left(\kappa^{-1} r_h^0, v_h\right) - \left(p_h^1, \nabla \cdot v_h\right) = \left(g^1, v_h\right).$$

Picking $v_h = r_h^0$ and using weighted Cauchy-Schwarz gives

$$\left\| r_h^0 \right\|_{\kappa^{-1}}^2 \leqslant \left\| p_h^1 \right\|_\varrho \left\| \nabla \cdot r_h^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_\kappa \left\| r_h^0 \right\|_{\kappa^{-1}}.$$

Now, we use our bound on $\left\| p_h^1 \right\|_\varrho$, equivalence of various norms, and the inverse assumption to find that

$$\begin{aligned}
\left\| p_h^1 \right\|_\varrho \left\| \nabla \cdot r_h^0 \right\|_{\varrho^{-1}} &\leqslant \frac{\Delta t}{\sqrt{\varrho_*}} \left\| f^0 \right\|_{\varrho^{-1}} \left\| \nabla \cdot r_h^0 \right\| \\
&\leqslant \frac{C_0 \Delta t}{h\sqrt{\varrho_*}} \left\| f^0 \right\|_{\varrho^{-1}} \left\| r_h^0 \right\| \\
&\leqslant 2\alpha \Delta t \left\| f^0 \right\|_{\varrho^{-1}} \left\| r_h^0 \right\|_{\kappa^{-1}} \\
&< 2 \left\| f^0 \right\|_{\varrho^{-1}} \left\| r_h^0 \right\|_{\kappa^{-1}}.
\end{aligned}$$

Hence, we have that

$$\left\| r_h^0 \right\|_{\kappa^{-1}} < 2 \left\| f^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_\kappa. \tag{4.12}$$

The CFL condition requires $\Delta t = \mathcal{O}(h)$, so this bound does not degrade under mesh refinement.

Now, we define the sequence b_n to be the energy functional applied to q_h^n, r_h^n:

$$b_n^2 = \left\| (q_h^n, r_h^n) \right\|_{\mathcal{E}}^2 = \left\| \left(\frac{\Delta p_h^n}{\Delta t}, \frac{\Delta u_h^n}{\Delta t} \right) \right\|_{\mathcal{E}}^2 = \frac{1}{2} \left\| \frac{\Delta p_h^n}{\Delta t} \right\|_\varrho^2 + \frac{1}{2} \left\| \frac{\Delta u_h^n}{\Delta t} \right\|_{\kappa^{-1}}^2. \tag{4.13}$$

We can apply our previous stability theorem to to the equations (4.10), at least up to the penultimate time step to obtain

Theorem 4.4. *If the initial conditions p_h^0 and u_h^0 vanish and the CFL condition (4.4) holds, the time differences $\frac{\Delta p_h^n}{\Delta t}$ and $\frac{\Delta u_h^n}{\Delta t}$ satisfy the stability bound*

$$\begin{aligned}
\max_{0 \leqslant n \leqslant N-1} b_n <& \frac{1}{\sqrt{1 - \alpha \Delta t}} \left(3 \left\| f^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_\kappa \right) \\
&+ \frac{\sqrt{2}}{1 - \alpha \Delta t} \sum_{n=0}^{N-2} \left(\left\| \frac{\Delta f^n}{\Delta t} \right\|_{\varrho^{-1}} + \left\| \frac{\Delta g^{n+1}}{\Delta t} \right\|_\kappa \right) \Delta t,
\end{aligned} \tag{4.14}$$

provided that the quantities on the right-hand side are bounded.

Proof. We apply the stability result of Theorem 4.3 to the equations (4.10) to find

$$\max_{0 \leqslant n \leqslant N-1} b_n \leqslant \frac{1}{\sqrt{1-\alpha\Delta t}} \tilde{b}_0 + \frac{\sqrt{2}}{1-\alpha\Delta t} \sum_{n=0}^{N-1} \left(\left\| \frac{\Delta f^n}{\Delta t} \right\|_{\varrho^{-1}} + \left\| \frac{\Delta g^{n+1}}{\Delta t} \right\|_{\kappa} \right) \Delta t,$$

and only the bound of the initial term requires explanation. We have

$$\begin{aligned} \tilde{b}_0 \leqslant \sqrt{2} b_0 \leqslant \left\| q_h^0 \right\|_{\varrho} + \left\| r_h^0 \right\|_{\kappa^{-1}} &\leqslant \left\| f^0 \right\|_{\varrho^{-1}} + \frac{C_0 \Delta t}{h} \sqrt{\frac{\kappa^*}{\varrho_*}} \left\| f^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_{\kappa} \\ &< 3 \left\| f^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_{\kappa}, \end{aligned}$$

and collecting terms finishes the proof. ◻

Using (1.19), we can also bound the divergence at each time level:

$$\| \nabla \cdot u_h^n \| \leqslant \| f^n \| + \sqrt{\varrho^*} \left\| \frac{\Delta p_h^n}{\Delta t} \right\|_{\varrho} \leqslant \| f^n \| + \sqrt{2\varrho^*} b^n, \tag{4.15}$$

which gives the following theorem

THEOREM 4.5. *Under the hypotheses of Theorem 4.4, we have the bound*

$$\begin{aligned} \max_{0 \leqslant n \leqslant N-1} \| \nabla \cdot u_h^n \| < \max_{0 \leqslant n \leqslant N-1} \| f^n \| &+ \sqrt{\frac{2\varrho^*}{1-\alpha\Delta t}} \left(3 \left\| f^0 \right\|_{\varrho^{-1}} + \left\| g^1 \right\|_{\kappa} \right) \\ &+ \frac{2\sqrt{\varrho^*}}{1-\alpha\Delta t} \sum_{n=0}^{N-2} \left(\left\| \frac{\Delta f^n}{\Delta t} \right\|_{\varrho^{-1}} + \left\| \frac{\Delta g^{n+1}}{\Delta t} \right\|_{\kappa} \right) \Delta t. \end{aligned} \tag{4.16}$$

4.3. Error estimates. Now, we turn to the question of estimating the error. We let $p^n(\cdot) = p(\cdot, t_n)$ and $u^n(\cdot) = u(\cdot, t_n)$ be the true solution evaluated at the discrete time levels. We will also denote $\pi p^n \in W_h$ and $\Pi u^n \in V_h$ to be the projections of the true solutions at the discrete time levels.

As before, we let $\xi = \pi p - p$. Then $\xi_t = \pi p_t - p_t$, and we also define ξ^n and ξ_t^n be evaluating ξ and its time derivative at the discrete time levels. We similarly define $\eta = \Pi u - u$ and its time derivative and evaluation at each t_n. As in the semidiscrete case, we use θ_h and χ_h to denote the differences between the projections and computed solutions, but now we only have these at discrete time levels: $\theta_h^n = \pi p^n - p_h^n$ and $\chi_h^n = \Pi u^n - u_h^n$.

To handle the fully discrete estimate, differences between time derivatives and difference quotients also appear. We will need to apply difference operators to functions of time. For some $f(\cdot, t)$, we define $\Delta f(\cdot, t) = f(\cdot, t + \Delta t) - f(\cdot, t)$. This exactly agrees with differencing at discrete time levels. We define

$$\begin{aligned} \zeta(\cdot, t) &= \frac{\Delta p(\cdot, t)}{\Delta t} - p_t(\cdot, t), \\ \psi(\cdot, t) &= \frac{\Delta u(\cdot, t)}{\Delta t} - u_t(\cdot, t), \end{aligned} \tag{4.17}$$

and also $\zeta^n = \zeta(\cdot, t_n)$ and $\psi^n = \psi(\cdot, t_n)$. With these definitions, standard manipulations show that the true solution satisfies the discrete equation

$$\begin{aligned}\left(\varrho\frac{\Delta\pi p^n}{\Delta t}, w_h\right) + \left(\nabla\cdot\Pi u^n, w_h\right) &= \left(f^n, w_h\right) + \left(\varrho\frac{\Delta\xi^n}{\Delta t}, w_h\right) + \left(\varrho\zeta^n, w_h\right), \\ \left(\kappa^{-1}\frac{\Delta\Pi u^n}{\Delta t}, v_h\right) - \left(\pi p^{n+1}, \nabla\cdot v_h\right) &= \left(g^{n+1}, v_h\right) + \left(\kappa^{-1}\frac{\Delta\eta^n}{\Delta t}, v_h\right) + \left(\kappa^{-1}\psi^n, v_h\right),\end{aligned} \tag{4.18}$$

and subtracting (1.19) from this gives error equations

$$\begin{aligned}\left(\varrho\frac{\Delta\theta_h^n}{\Delta t}, w_h\right) + \left(\nabla\cdot\chi_h^n, w_h\right) &= \left(\varrho\frac{\Delta\xi^n}{\Delta t}, w_h\right) + \left(\varrho\zeta^n, w_h\right), \\ \left(\kappa^{-1}\frac{\Delta\chi_h^n}{\Delta t}, v_h\right) - \left(\theta_h^{n+1}, \nabla\cdot v_h\right) &= \left(\kappa^{-1}\frac{\Delta\eta^n}{\Delta t}, v_h\right) + \left(\kappa^{-1}\psi^n, v_h\right).\end{aligned} \tag{4.19}$$

Because the initial conditions for the discrete method coincide with the projections of the true solution, we have that θ_h^0 and χ_h^0 both vanish.

We define

$$\varepsilon^n = \|(\theta_h^n, \chi_h^n)\|_{\mathcal{E}}, \tag{4.20}$$

and make the bound

Lemma 4.6. *Suppose that $p_{tt} \in L^1\left(0, T; L^2(\Omega)\right)$, $u_{tt} \in L^1\left(0, T; \left(L^2(\Omega)\right)^d\right)$, $p_t \in L^1\left(0, T; H^m(\Omega)\right)$, $u_t \in L^1\left(0, T; \left(H^m(\Omega)\right)^d\right)$, and that (4.4) holds. Then we have the error estimate*

$$\begin{aligned}\max_{0\leqslant n\leqslant N}\varepsilon^n \leqslant &\frac{\sqrt{2\varrho^*}}{1-\alpha\Delta t}\left(C_1 h^m\int_0^T |p_t(\cdot,t)|_m\,dt + \Delta t\int_0^T \|p_{tt}(\cdot,s)\|\,ds\right) \\ &+ \frac{\sqrt{2}}{(1-\alpha\Delta t)\sqrt{\kappa_*}}\left(C_2 h^m\int_0^T |u_t(\cdot,t)|_m\,dt + \Delta t\int_0^T \|u_{tt}(\cdot,s)\|\,ds\right).\end{aligned} \tag{4.21}$$

Proof. First, we apply the stability result of Theorem 4.3 to (4.19), noting that the initial conditions vanish, to find that

$$\begin{aligned}\max_{0\leqslant n\leqslant N}\varepsilon^n \leqslant &\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\left\|\varrho\left(\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right)\right\|_{\varrho^{-1}} + \left\|\kappa^{-1}\left(\frac{\Delta\eta^n}{\Delta t}+\psi^n\right)\right\|_{\kappa}\right)\Delta t \\ = &\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right\|_{\varrho} + \left\|\frac{\Delta\eta^n}{\Delta t}+\psi^n\right\|_{\kappa^{-1}}\right)\Delta t.\end{aligned}$$

Now, we bound each of these terms separately. For the first, we have that

$$\sum_{n=0}^{N-1}\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta_n\right\|_{\varrho}\Delta t \leqslant \sqrt{\varrho^*}\sum_{n=0}^{N-1}\left[\left\|\frac{\Delta\xi^n}{\Delta t}\right\| + \|\zeta^n\|\right]\Delta t. \tag{4.22}$$

Since

$$\begin{aligned}\sum_{n=0}^{N-1}\|\Delta\xi^n\| &= \sum_{n=0}^{N-1}\left\|\int_{t_n}^{t_{n+1}}\xi_t(\cdot,s)ds\right\|\\ &\leqslant \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\|\xi_t(\cdot,s)ds\|\\ &\leqslant C_1h^m\int_0^T\left|p_t\left(\cdot,t\right)\right|_m.\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{N-1}\|\zeta^n\|\,\Delta t &= \sum_{n=0}^{N-1}\left\|p^{n+1}-p^n-\Delta t p_t^n\right\|\\ &= \sum_{n=0}^{N-1}\left\|\int_{t_n}^{t_{n+1}}\left(t_{n+1}-s\right)p_{tt}\left(\cdot,s\right)ds\right\|\\ &\leqslant \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left(t_{n+1}-s\right)\left\|p_{tt}\left(\cdot,s\right)\right\|ds\\ &\leqslant \sum_{n=0}^{N-1}\Delta t\int_{t_n}^{t_{n+1}}\left\|p_{tt}\left(\cdot,s\right)\right\|ds\\ &\leqslant \Delta t\int_0^T\left\|p_{tt}\left(\cdot,s\right)\right\|ds,\end{aligned}$$

the first term in (4.22) is bounded by

$$\frac{\sqrt{2\varrho^*}}{1-\alpha\Delta t}\left[C_1h^m\sum_{n=0}^{N-1}\left|p_t\left(\cdot,t\right)\right|_m\Delta t+\Delta t\int_0^T\left\|p_{tt}\left(\cdot,s\right)\right\|ds\right].$$

Similar techniques for the second term complete the proof. □

This lemma allows us to bound the error

$$\epsilon^n=\|(p^n-p_h^n,u^n-u_h^n)\|_{\mathcal{E}}\,. \tag{4.23}$$

THEOREM 4.7. *Under the assumptions of the previous Lemma, we have that*

$$\begin{aligned}\max_{0\leqslant n\leqslant N}\varepsilon^n\leqslant&\frac{\sqrt{2\varrho^*}}{1-\alpha\Delta t}\left(\frac{C_1h^m}{2}\sup_{0\leqslant t\leqslant T}\left|p\left(\cdot,t\right)\right|_m+C_1h^m\int_0^T\left|p_t\left(\cdot,t\right)\right|_m dt\right.\\ &\left.+\Delta t\int_0^T\left\|p_{tt}\left(\cdot,s\right)\right\|ds\right)\\ &+\frac{\sqrt{2}}{\left(1-\alpha\Delta t\right)\sqrt{\kappa_*}}\left(\frac{C_2h^m}{2}\sup_{0\leqslant t\leqslant T}\left|u\left(\cdot,t\right)\right|_m dt+C_2h^m\int_0^T\left|u_t\left(\cdot,t\right)\right|_m dt\right.\\ &\left.+\Delta t\int_0^T\left\|u_{tt}\left(\cdot,s\right)\right\|ds\right).\end{aligned} \tag{4.24}$$

Proof. From the triangle inequality, we have

$$\epsilon^n \leqslant \|(\xi^n, \eta^n)\|_{\mathcal{E}} + \varepsilon^n.$$

The first term satisfies the bound

$$\|(\xi^n, \eta^n)\|_{\mathcal{E}} \leqslant \frac{1}{\sqrt{2}} \|\xi^n\|_{\varrho} + \frac{1}{\sqrt{2}} \|\eta^n\|_{\kappa^{-1}} \leqslant \sqrt{\frac{\varrho^*}{2}} \|\xi^n\| + \frac{1}{\sqrt{2\kappa_*}} \|\eta^n\|,$$

and our approximation estimates give that

$$\max_{0\leqslant n\leqslant N} \|(\xi^n, \eta^n)\|_{\mathcal{E}} \leqslant C_1 h^m \sqrt{\frac{\varrho^*}{2}} \sup_{0\leqslant t\leqslant T} |p(\cdot,t)|_m + \frac{C_2 h^m}{\sqrt{2\kappa_*}} \sup_{0\leqslant t\leqslant T} |u(\cdot,t)|_m.$$

Combining this bound with the previous lemma and grouping terms completes the proof. ∎

Next, we estimate the error in time difference quotients. We define

$$\beta_n = \left\| \left(\frac{\Delta\theta_h^n}{\Delta t}, \frac{\Delta\chi_h^n}{\Delta t} \right) \right\|_{\mathcal{E}}. \tag{4.25}$$

LEMMA 4.8. *Suppose that $p_{tt} \in L^1(0,T;H^m(\Omega))$, $u_{tt} \in L^1\left(0,T;(H^m(\Omega))^d\right)$, and $p_{ttt} \in L^1(0,T;L^2(\Omega))$, $u_{ttt} \in L^1(0,T;(L^2(\Omega))^d)$ and that the CFL condition (4.4) holds. Then we have the estimate*

$$\begin{aligned}\max_{0\leqslant n\leqslant N-1} \beta^n < h^m \Bigg[&\frac{1}{\sqrt{1-\alpha\Delta t}} \left(3C_1\sqrt{\varrho^*} \sup_{0\leqslant t\leqslant \Delta t} |p_t(\cdot,t)|_m + \frac{C_2}{\sqrt{\kappa_*}} \sup_{0\leqslant t\leqslant \Delta t} |u_t(\cdot,t)|_m \right) \\ &+ \frac{2\sqrt{2}}{1-\alpha\Delta t} \left(C_1\sqrt{\varrho^*} \int_0^T |p_{tt}(\cdot,t)|_m \, dt + \frac{C_2}{\sqrt{\kappa_*}} \int_0^T |u_{tt}(\cdot,t)|_m \, dt \right) \Bigg] \\ + \Delta t \Bigg[&\frac{1}{\sqrt{1-\alpha\Delta t}} \left(3\sqrt{\varrho^*} \sup_{0\leqslant t\leqslant \Delta t} \|p_{tt}(\cdot,t)\| + \frac{1}{\sqrt{\kappa_*}} \sup_{0\leqslant t\leqslant \Delta t} \|u_{tt}(\cdot,t)\| \right) \\ &+ \frac{2\sqrt{2}}{1-\alpha\Delta t} \left(\sqrt{\varrho^*} \int_0^T \|p_{ttt}(\cdot,t)\| \, dt + \frac{1}{\sqrt{\kappa_*}} \int_0^T \|u_{ttt}(\cdot,t)\| \, dt \right) \Bigg].\end{aligned} \tag{4.26}$$

Proof. Since the initial conditions for the error equations (4.19) vanish, we can apply the stability estimate in Lemma 4.4 to obtain

$$\begin{aligned}\max_{0\leqslant n\leqslant N-1} \beta_n < &\frac{1}{\sqrt{1-\alpha\Delta t}} \left(3 \left\| \frac{\Delta\xi^0}{\Delta t} + \zeta^0 \right\|_{\varrho} + \left\| \frac{\Delta\eta^0}{\Delta t} + \psi^0 \right\|_{\kappa^{-1}} \right) \\ &+ \frac{\sqrt{2}}{1-\alpha\Delta t} \sum_{n=0}^{N-2} \left(\left\| \frac{\Delta^2\xi^n}{(\Delta t)^2} + \frac{\Delta\zeta^n}{\Delta t} \right\|_{\varrho} + \left\| \frac{\Delta^2\eta^n}{(\Delta t)^2} + \frac{\Delta\psi^n}{\Delta t} \right\|_{\kappa^{-1}} \right) \Delta t.\end{aligned} \tag{4.27}$$

We take each of the norms on the right in turn. The first two are evaluated at the initial condition. We start with norm equivalence and the triangle inequality:

$$\left\| \frac{\Delta\xi^0}{\Delta t} + \zeta^0 \right\|_{\varrho} \leqslant \sqrt{\varrho^*} \left\| \frac{\Delta\xi^0}{\Delta t} \right\| + \sqrt{\varrho^*} \|\zeta^0\|.$$

The first of these is estimated by

$$\left\| \frac{\Delta \xi^0}{\Delta t} \right\| = \frac{1}{\Delta t} \left\| \int_0^{\Delta t} \xi_t\left(\cdot, s\right) ds \right\| \leqslant C_1 h^m \sup_{0 \leqslant s \leqslant \Delta t} \left| p_t\left(\cdot, s\right) \right|_m,$$

and the second by

$$\left\| \zeta^0 \right\| = \left\| \frac{\Delta p^0}{\Delta t} - p_t^0 \right\| \leqslant \frac{1}{\Delta t} \left\| \int_0^{\Delta t} \left(\Delta t - s\right) p_{tt}\left(\cdot, s\right) ds \right\| \leqslant \Delta t \sup_{0 \leqslant s \leqslant \Delta t} \left\| p_{tt}\left(\cdot, s\right) \right\|.$$

In a similar fashion, we find that

$$\left\| \frac{\Delta \eta^0}{\Delta t} + \psi^0 \right\|_{\kappa^{-1}} \leqslant \frac{C_2 h^m}{\sqrt{\kappa_*}} \sup_{0 \leqslant s \leqslant \Delta t} \left| u_t\left(\cdot, s\right) \right|_m + \frac{\Delta t}{\sqrt{\kappa_*}} \sup_{0 \leqslant s \leqslant \Delta t} \left\| u_{tt}\left(\cdot, s\right) \right\|.$$

Now, we turn to the sums in (4.27). First, we have

$$\begin{aligned} \sum_{0 \leqslant n \leqslant N-2} \left\| \frac{\Delta^2 \xi^n}{\left(\Delta t\right)^2} + \frac{\Delta \zeta^n}{\Delta t} \right\|_{\varrho} \Delta t &\leqslant \sqrt{\varrho^*} \sum_{0 \leqslant n \leqslant N-2} \left(\left\| \frac{\Delta^2 \xi^n}{\Delta t} \right\| + \left\| \Delta \zeta^n \right\| \right) \\ &\equiv \sqrt{\varrho^*} \left(I + II\right), \end{aligned}$$

and we handle these in turn. To bound I, we start with the calculation

$$\begin{aligned} \frac{\Delta^2 \xi^n}{\Delta t} &= \frac{1}{\Delta t} \left(\Delta \xi^{n+1} - \Delta \xi^n\right) \\ &= \frac{1}{\Delta t} \left[\left(\Delta \xi^{n+1} - \Delta t \xi_t^{n+1}\right) - \left(\Delta \xi^n - \Delta t \xi_t^{n+1}\right) \right] \\ &= \frac{1}{\Delta t} \int_{t_{n+1}}^{t_{n+2}} (t_{n+2} - s) \xi_{tt}\left(\cdot, s\right) ds + \int_{t_n}^{t_{n+1}} (s - t_n) \xi_{tt}\left(\cdot, s\right) ds. \end{aligned}$$

In both of these integrals, we make the change variables $\sigma = s - t_{n+1}$ to find that

$$\begin{aligned} \frac{\Delta^2 \xi^n}{\Delta t} &= \frac{1}{\Delta t} \int_0^{\Delta t} \left(\Delta t - s\right) \xi_{tt}\left(\cdot, \sigma + t_{n+1}\right) ds + \int_{-\Delta t}^{0} \left(\Delta t + s\right) \xi_{tt}\left(\cdot, \sigma + t_{n+1}\right) ds \\ &= \frac{1}{\Delta t} \int_{-\Delta t}^{\Delta t} \left(\Delta t - |\sigma|\right) \xi_{tt}\left(\cdot, \sigma + t_{n+1}\right) d\sigma \\ &= \frac{1}{\Delta t} \int_{t_n}^{t_{n+2}} \left(\Delta t - |s - t_{n+1}|\right) \xi_{tt}\left(\cdot, s\right) ds. \end{aligned}$$

Using this calculation in I lets us make the bound

$$\begin{aligned} \sum_{n=0}^{N-2} \left\| \frac{\Delta^2 \xi^n}{\Delta t} \right\| &= \sum_{n=0}^{N-2} \left\| \frac{1}{\Delta t} \int_{t_n}^{t_{n+2}} \left(\Delta t - |s - t_{n+1}|\right) \xi_{tt}\left(\cdot, s\right) ds \right\| \\ &\leqslant \sum_{n=0}^{N-2} \frac{1}{\Delta t} \int_{t_n}^{t_{n+2}} \left\| \left(\Delta t - |s - t_{n+1}|\right) \xi_{tt}\left(\cdot, s\right) \right\| ds \\ &\leqslant \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+2}} \left\| \xi_{tt}\left(\cdot, s\right) \right\| ds \\ &\leqslant 2 \int_0^T \left\| \xi_{tt}\left(\cdot, s\right) \right\| ds \\ &\leqslant 2 C_1 h^m \int_0^T \left| p_{tt}\left(\cdot, s\right) \right|_m ds. \end{aligned}$$

Now, we turn to II:

$$II = \sum_{n=0}^{N-2} \|\Delta \zeta^n\| = \sum_{n=0}^{N-2} \left\| \int_{t_n}^{t_{n+1}} \zeta_t\left(\cdot, s\right) ds \right\|.$$

Differentiating (4.17), we find that

$$\zeta_t(\cdot, s) = \frac{\Delta p_t\left(\cdot, s\right)}{\Delta t} - p_{tt}\left(\cdot, s\right) = \frac{1}{\Delta t} \int_s^{s+\Delta t} \left(s + \Delta t - \tau\right) p_{ttt}\left(\cdot, \tau\right) d\tau.$$

We insert this into II and make the bounds

$$\begin{aligned}
\sum_{n=0}^{N-2} \|\Delta \zeta^n\| &= \sum_{n=0}^{N-2} \left\| \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t} \left(\int_s^{s+\Delta t} \left(s + \Delta t - \tau\right) p_{ttt}\left(\cdot, \tau\right) d\tau \right) ds \right\| \\
&\leqslant \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t} \left(\int_s^{s+\Delta t} \left(s + \Delta t - \tau\right) \|p_{ttt}\left(\cdot, \tau\right)\| d\tau \right) ds \\
&\leqslant \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} \left(\int_s^{s+\Delta t} \|p_{ttt}\left(\cdot, \tau\right)\| d\tau \right) ds.
\end{aligned}$$

To proceed, we will interchange the order of integration. However, the limits of integration on the inner integral depend on one of the variables, and this requires writing the new inner integral with two separate integrals. We have

$$\begin{aligned}
\sum_{n=0}^{N-2} \|\Delta \zeta^n\| &\leqslant \sum_{n=0}^{N-2} \left(\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \|p_{ttt}\left(\cdot, \tau\right)\| ds \right) d\tau + \int_{t_{n+1}}^{t_{n+2}} \left(\int_{s-\Delta t}^{t_{n+1}} \|p_{ttt}\left(\cdot, \tau\right)\| ds \right) d\tau \right) \\
&= \sum_{n=0}^{N-2} \left(\int_{t_n}^{t_{n+1}} \left(s - t_n\right) \|p_{ttt}\left(\cdot, \tau\right)\|_\varrho d\tau + \int_{t_{n+1}}^{t_{n+2}} \left(t_{n+2} - s\right) \|p_{ttt}\left(\cdot, \tau\right)\|_\varrho d\tau \right) \\
&\leqslant 2\Delta t \int_0^T \|p_{ttt}(\cdot, \tau)\| d\tau.
\end{aligned}$$

Combining these estimates gives that

$$\begin{aligned}
\sum_{n=0}^{N-2} \left\| \frac{\Delta^2 \xi^n}{\left(\Delta t\right)^2} + \frac{\Delta \zeta^n}{\Delta t} \right\|_\varrho \Delta t \leqslant 2C_1 h^m \sqrt{\varrho^*} \int_0^T \left|p_{tt}\left(\cdot, t\right)\right|_m dt \\
+ 2\Delta t \sqrt{\varrho^*} \int_0^T \|p_{ttt}\left(\cdot, s\right)\| ds.
\end{aligned} \tag{4.28}$$

Similar techniques allow us to write

$$\begin{aligned}
\sum_{n=0}^{N-2} \left\| \frac{\Delta^2 \eta^n}{\left(\Delta t\right)^2} + \frac{\Delta \psi^n}{\Delta t} \right\|_{\kappa^{-1}} \Delta t \leqslant \frac{2C_2 h^m}{\sqrt{\kappa_*}} \int_0^T \left|u_{tt}\left(\cdot, t\right)\right|_m dt \\
+ \frac{2\Delta t}{\sqrt{\kappa_*}} \int_0^T \|u_{ttt}(\cdot, s)\| ds.
\end{aligned} \tag{4.29}$$

□

This result allows us to give optimal-order estimates for the difference between the computed difference quotients and the true derivatives at each time step. We define

$$\Xi^n = \left\| \left(\frac{\Delta p_h^n}{\Delta t} - p_t^n, \frac{\Delta u_h^n}{\Delta t} - u_t^n \right) \right\|_{\mathcal{E}}, \tag{4.30}$$

and have the following optimal-order estimate.

THEOREM 4.9. *If the assumptions of Lemma 4.8 hold, we have the estimate*

$$\begin{aligned} \max_{0 \leqslant n \leqslant N-1} \Xi^n \leqslant & h^m \left[C_1 \sqrt{\frac{\varrho^*}{2}} \sup_{0 \leqslant t \leqslant T} \left| p_t(\cdot, t) \right|_m + \frac{C_2}{\sqrt{2\kappa_*}} \sup_{0 \leqslant t \leqslant T} \left| u_t(\cdot, t) \right|_m \right] \\ & + \Delta t \left[\sqrt{\frac{\varrho^*}{2}} \sup_{0 \leqslant t \leqslant T} \left\| p_{tt}(\cdot, t) \right\| + \frac{1}{\sqrt{2\kappa_*}} \sup_{0 \leqslant t \leqslant T} \left\| p_{tt}(\cdot, t) \right\| \right] \\ & + \max_{0 \leqslant n \leqslant N-1} \beta_n. \end{aligned} \tag{4.31}$$

Proof. With the help of the triangle inequality, we write

$$\max_{0 \leqslant n \leqslant N-1} \Xi_n \leqslant \max_{0 \leqslant n \leqslant N-1} \left\| (\zeta^n, \psi^n) \right\|_{\mathcal{E}} + \max_{0 \leqslant n \leqslant N-1} \left\| \left(\frac{\Delta \xi^n}{\Delta t}, \frac{\Delta \eta^n}{\Delta t} \right) \right\|_{\mathcal{E}} + \max_{0 \leqslant n \leqslant N-1} \beta_n.$$

The first of these terms satisfies

$$\max_{0 \leqslant n \leqslant N-1} \left\| (\zeta^n, \psi^n) \right\|_{\mathcal{E}} \leqslant \sqrt{\frac{\varrho^*}{2}} \max_{0 \leqslant n \leqslant N-1} \left\| \zeta^n \right\| + \frac{1}{\sqrt{2\kappa_*}} \max_{0 \leqslant n \leqslant N-1} \left\| \psi^n \right\|.$$

We bound the ζ^n term by

$$\begin{aligned} \max_{0 \leqslant n \leqslant N-1} \left\| \zeta^n \right\| &\leqslant \frac{1}{\Delta t} \max_{0 \leqslant n \leqslant N-1} \left\| \Delta p^n - \Delta t p_t^n \right\| \\ &= \frac{1}{\Delta t} \max_{0 \leqslant n \leqslant N-1} \left\| \int_{t_n}^{t_{n+1}} (s - t_n) \, p_{tt}(\cdot, s) \, ds \right\| \\ &\leqslant \Delta t \sup_{0 \leqslant s \leqslant T} \left\| p_{tt}(\cdot, s) \right\|. \end{aligned}$$

Similarly, we have

$$\max_{0 \leqslant n \leqslant N-1} \left\| \psi^n \right\| \leqslant \Delta t \sup_{0 \leqslant s \leqslant T} \left\| u_{tt}(\cdot, s) \right\|.$$

We also have that

$$\max_{0 \leqslant n \leqslant N-1} \left\| \left(\frac{\Delta \xi^n}{\Delta t}, \frac{\Delta \eta^n}{\Delta t} \right) \right\|_{\mathcal{E}} \leqslant \sqrt{\frac{\varrho^*}{2}} \max_{0 \leqslant n \leqslant N-1} \left\| \frac{\Delta \xi^n}{\Delta t} \right\| + \frac{1}{\sqrt{2\kappa_*}} \max_{0 \leqslant n \leqslant N-1} \left\| \frac{\Delta \eta^n}{\Delta t} \right\|.$$

The first term here satisfies

$$\max_{0 \leqslant n \leqslant N-1} \left\| \frac{\Delta \xi^n}{\Delta t} \right\| = \frac{1}{\Delta t} \max_{0 \leqslant n \leqslant N-1} \left\| \int_{t_n}^{t_{n+1}} \xi_t(\cdot, s) \, ds \right\| \leqslant C_1 h^m \sup_{0 \leqslant s \leqslant T} \left| p_t(\cdot, s) \right|_m,$$

and by the same argument,

$$\max_{0 \leqslant n \leqslant N-1} \left\| \frac{\Delta \eta^n}{\Delta t} \right\| \leqslant C_2 h^m \sup_{0 \leqslant s \leqslant T} \left| u_t(\cdot, s) \right|_m.$$

We collect these estimates to finish the proof. □

We also have optimal-order estimates for the divergence.

THEOREM 4.10. *If the assumptions of Lemma 4.8 hold and additionally* $\nabla \cdot u \in L^\infty(0,T;H^m(\Omega))$, *we have the error estimate*

$$\begin{aligned}\max_{0\leqslant n\leqslant N-1}\|\nabla\cdot(u^n-u_h^n)\| \leqslant & C_1 h^m\left[\sup_{0\leqslant t\leqslant T}|\nabla\cdot u(\cdot,t)|_m+\varrho^*\sup_{0\leqslant t\leqslant T}|p_t(\cdot,t)|_m\right]\\ &+\varrho^*\Delta t\sup_{0\leqslant t\leqslant T}\|p_{tt}(\cdot,t)\|_m+\sqrt{2\varrho^*}\max_{0\leqslant n\leqslant N-1}\beta_n.\end{aligned} \tag{4.32}$$

Proof. We start by writing

$$\max_{0\leqslant n\leqslant N-1}\|\nabla\cdot(u^n-u_h^n)\|\leqslant\max_{0\leqslant n\leqslant N-1}\|\nabla\cdot\eta^n\|+\max_{0\leqslant n\leqslant N-1}\|\nabla\cdot\chi_h^n\|.$$

The first term is purely approximation-theoretic, and (1.10) gives

$$\max_{0\leqslant n\leqslant N-1}\|\nabla\cdot\eta^n\|\leqslant C_1h^m\sup_{0\leqslant s\leqslant T}|\nabla\cdot u(\cdot,s)|_m.$$

We bound the second term by relating it back to our estimate for β_n. Much like the estimate for $\|\nabla\cdot u_h^n\|$ in (4.15), we can use the error equations (4.19) to find

$$\|\nabla\cdot\chi_h^n\|\leqslant\varrho^*\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right\|+\sqrt{2\varrho^*}\beta_n.$$

Using our standard techniques completes the proof. □

5. Other time discretizations. Here, we briefly comment on a few other possible time discretizations with interesting conservation properties. The Crank-Nicholson method, assuming no forcing terms, satisfies

$$\begin{aligned}\left(\varrho\frac{\Delta p_h^n}{\Delta t},w_h\right)+\left(\nabla\cdot\left(\frac{u_h^{n+1}+u_h^n}{2}\right),w_h\right)&=0\\ \left(\kappa^{-1}\frac{\Delta u_h^n}{\Delta t},v_h\right)-\left(\frac{p_h^{n+1}+p_h^n}{2},\nabla\cdot v_h\right)&=0.\end{aligned}\tag{5.1}$$

If we select $w_h=\frac{p_h^n+p_h^{n+1}}{2}$ and $v_h=\frac{u_h^n+u_h^{n+1}}{2}$ and add these equations the terms with spatial derivatives cancel, giving

$$\left\|(p_h^{n+1},u_h^{n+1})\right\|_{\mathcal{E}}=\|(p_h^n,u_h^n)\|_{\mathcal{E}}\tag{5.2}$$

for each time step – exact energy conservation. However, this comes at the cost of a more complicated linear system for each time step. While forward and symplectic Euler involve only inverting mass matrices, we now have a skew symmetric perturbation of the mass matrices, for the system matrix will have the form

$$\begin{bmatrix}M^\varrho & \Delta tD\\ -\Delta tD^T & \widetilde{M}^{\kappa^{-1}}\end{bmatrix}.\tag{5.3}$$

The complication of the skew pertubation could be offset by a larger allowable time step and exact conservation provided an effective preconditioner, but this is a subject of further investigation.

In fact, the Crank-Nicholson method can be seen as the lowest-order instance of a family of continuous Galerkin methods in the time variable. Logg [16] shows that this entire family of methods is exactly conservative for Hamiltonian systems and develops variable time stepping for individual components. Like Crank-Nicholson, these methods are all implicit. On the other hand, we can also consider higher-order explicit methods, such as the Störmer-Verlet method [7], which is second-order accurate and preserves a perturbation of the system energy which is quadratic rather than linear in Δt.

6. Conclusion and future directions. We have developed a method for the acoustic wave equation that preserves the essential structures of the spatial and temporal discretization. In addition to optimal estimates in the typical $L^\infty(L^2)$-based norms, our bootstrap techniques enable optimal estimates in stronger norms, as well. These results leave open many questions for further study.

Linear-algebraic questions related to handling the mass matrices for explicit methods (whether our symplectic Euler or high-order methods) and matrices like (5.3) for implicit ones will need to be addressed. On rectangular meshes, diagonalizing quadrature [21, 12, 11] would lead to explicit time-marching schemes at the cost of an additional perturbation to the conserved energy functional and restriction to highly structured geometry. In the low-order case, it may be possible to handle general quadrilaterals by the techniques in [22]. For simplicial meshes, Jenkins [13] reports that V_h mass matrices are easily handled by conjugate gradient algorithms.

Additionally, we need to study the applicability of our methodology to other kinds of equations. An application to curl-curl wave equations with edge elements should be straightforward, and current research is focused on mixed formulations of the Klein-Gordon eqatuion. Finally, extension of the formulation and analysis beyond basic reflecting boundary conditions remains an open question of interest.

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