

# MOMENTUM IN OPTIMIZATION

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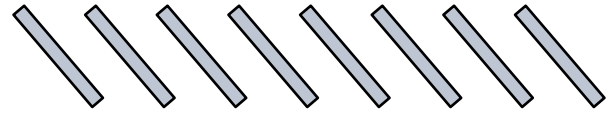
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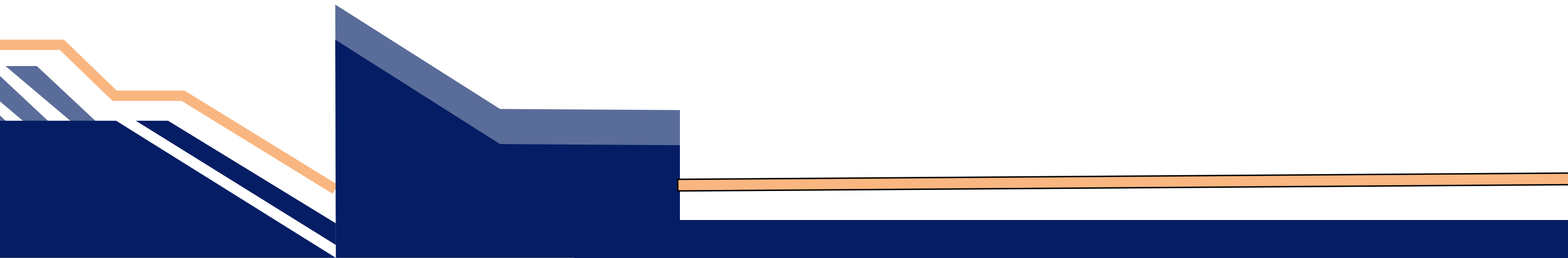
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# 01 INTRODUCTION



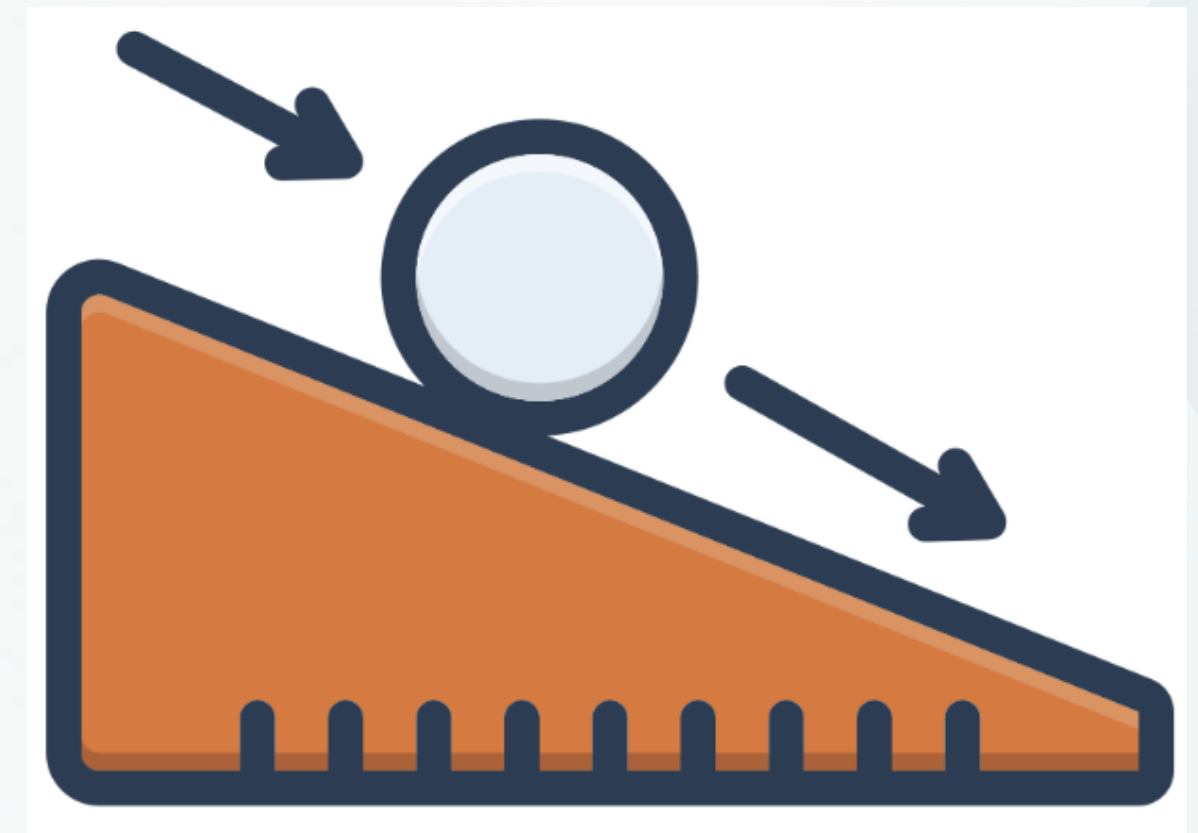
# What is momentum?

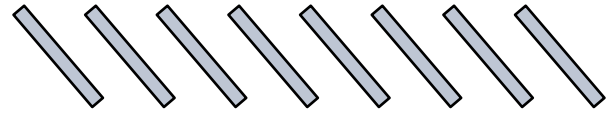
What is it? And what is the usage?

**Momentum** introduces a velocity term that accumulates past gradients to smooth the update direction and improve the convergence speed.

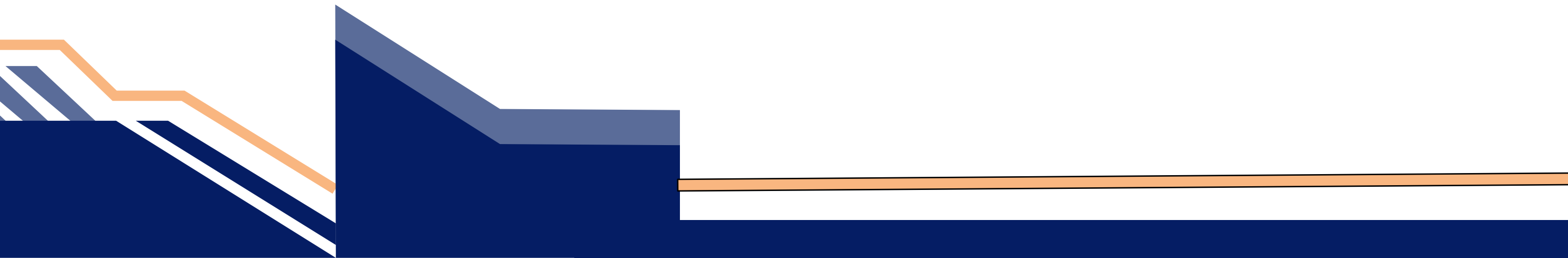
## Key benefits of Momentum:

- Smooth noisy updates
- Reduces zig-zagging in narrow valleys
- Speeds up convergence in flat regions
- More stable than plain SGD





# 02 OPTIMIZATION PROBLEM



# Optimization Problem

The core task in deep learning optimization is to find the set of parameters  $\theta$  that minimizes the objective (loss) function  $f(\theta)$

$$\min_{\theta \in \mathbb{R}^d} f(\theta)$$

The loss landscape in deep learning

- Highly non-convex
- Contains narrow valleys, plateaus, saddle points
- Gradient direction changes rapidly depending on curvature

# III-Conditioning Problem

The poor performance of plain gradient descent stems from **ill-conditioning**: the loss surface exhibits dramatically different curvature in different directions.

Consider a simple quadratic function

$$f(x, y) = 0.01x^2 + 5y^2$$

The second derivatives (curvatures) are

- The curvature in the x-direction is  $\frac{\partial^2 f}{\partial x^2} = 0.02$  (relatively flat)
- The curvature in the y-direction is  $\frac{\partial^2 f}{\partial y^2} = 10$  (much steeper)

These form the constant Hessian matrix :

$$H = \begin{bmatrix} 0.02 & 0 \\ 0 & 10 \end{bmatrix}$$

Its eigenvalues  $\lambda_1 = 0.02$  and  $\lambda_2 = 10$  are exactly the curvatures along the principal axes.

The **condition number** is

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{10}{0.02} = 500$$

Gradient descent dilemma:

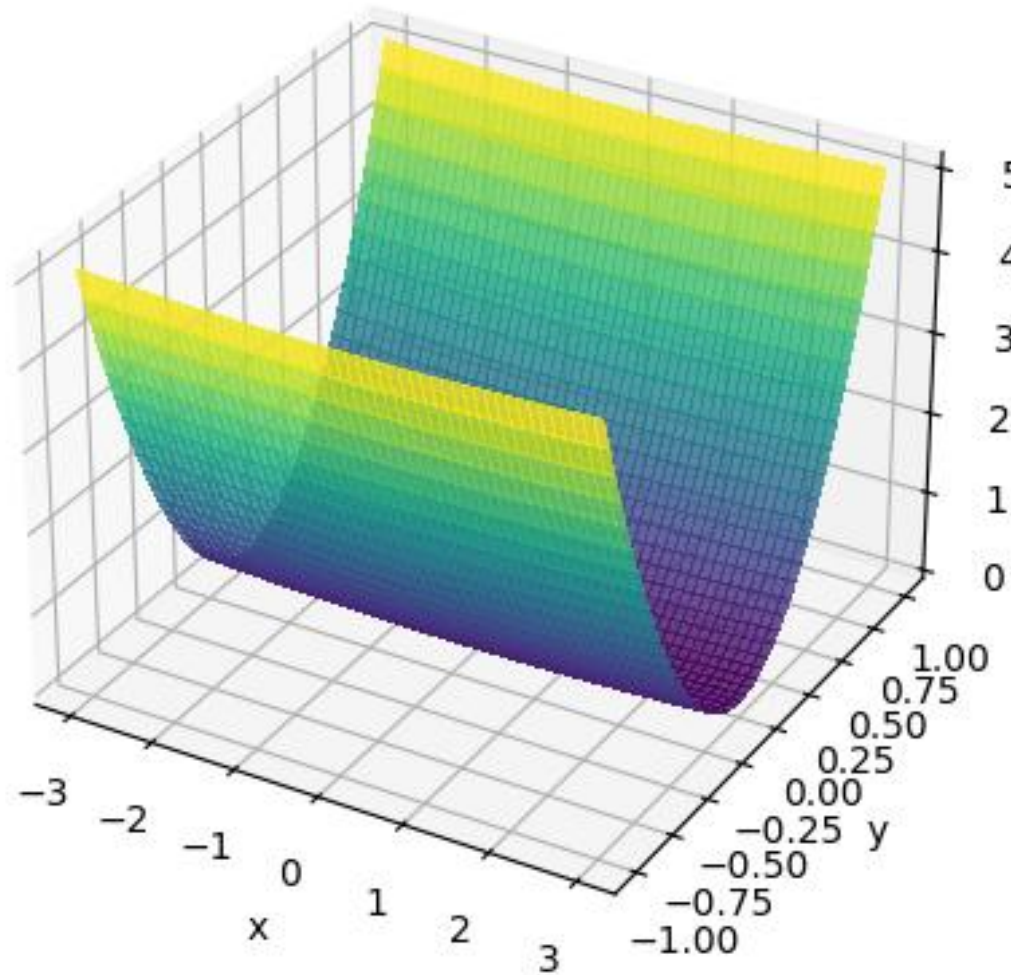
- **Small learning rate**: stable in y direction but slow in x-direction
- **Large learning rate**: good progress in x-direction but risks divergence in y-direction

# III-Conditioning Problem

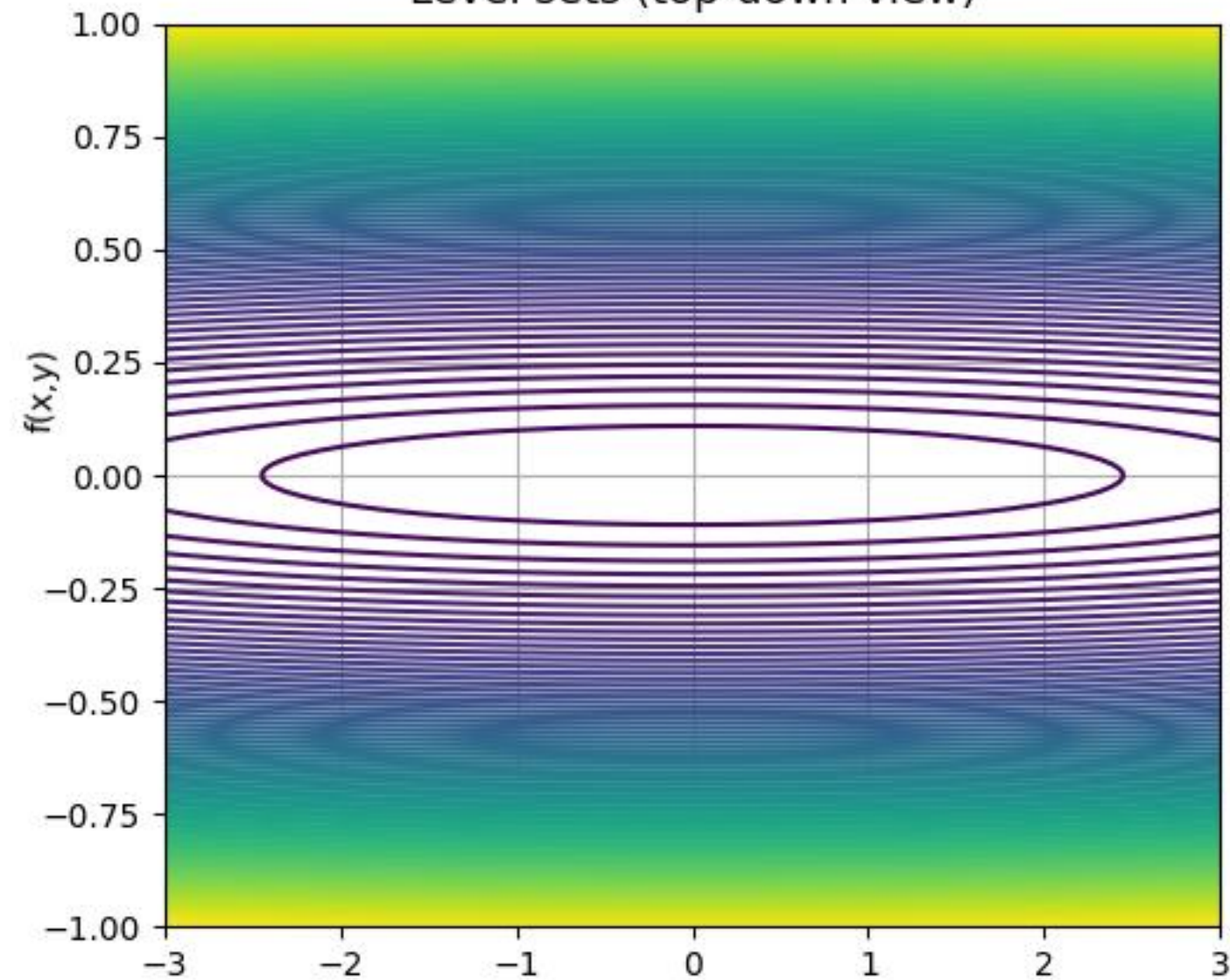
Classic III-Conditioned Loss Landscape (condition number = 500)

$$f(x,y) = 0.01x^2 + 5y^2$$

(3D view)



Level sets (top-down view)



# Geometry of Curvature

## 2.2 The Geometry of Curvature

The second-order Taylor expansion tells us how the function changes for a small step  $\Delta \mathbf{x}$  of fixed length  $\|\Delta \mathbf{x}\| = 1$ :

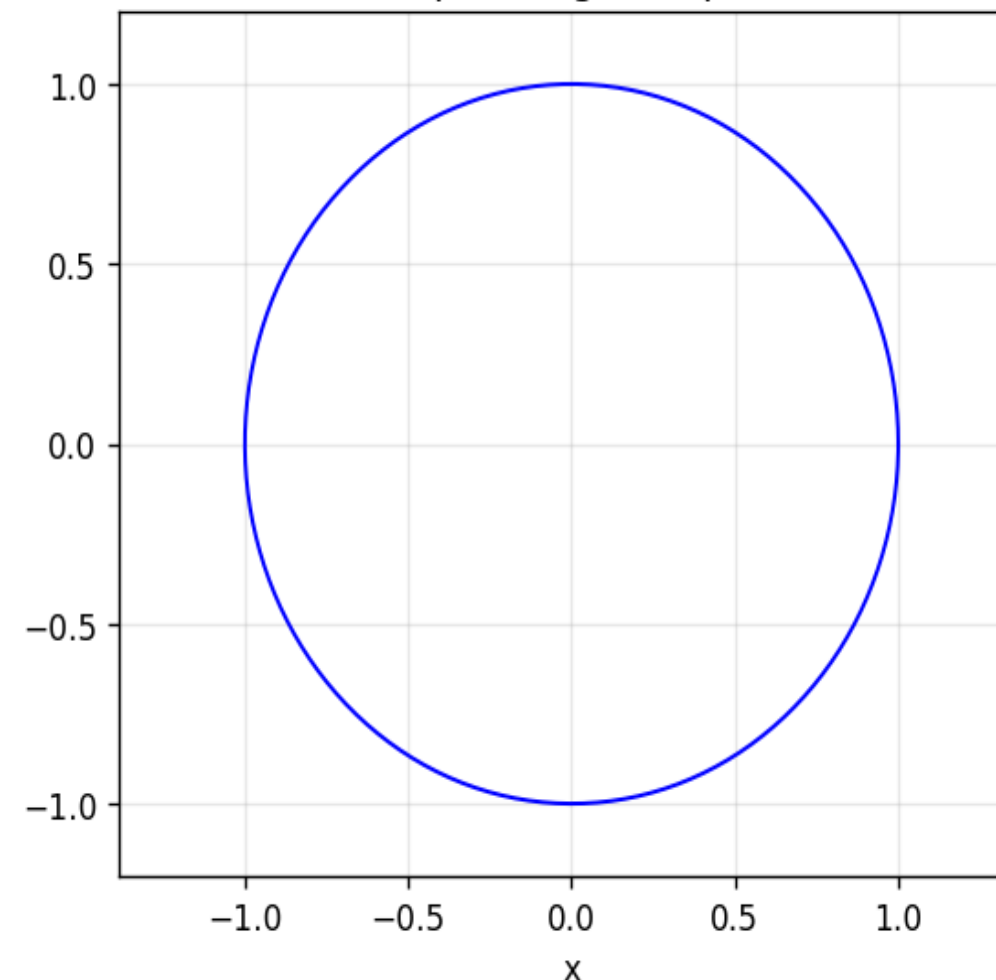
$$f(\mathbf{x} + \Delta \mathbf{x}) \approx f(\mathbf{x}) + \nabla f^\top \Delta \mathbf{x} + \frac{1}{2} \Delta \mathbf{x}^\top H \Delta \mathbf{x}.$$

The term  $\Delta \mathbf{x}^\top H \Delta \mathbf{x}$  is exactly the **curvature contribution**.

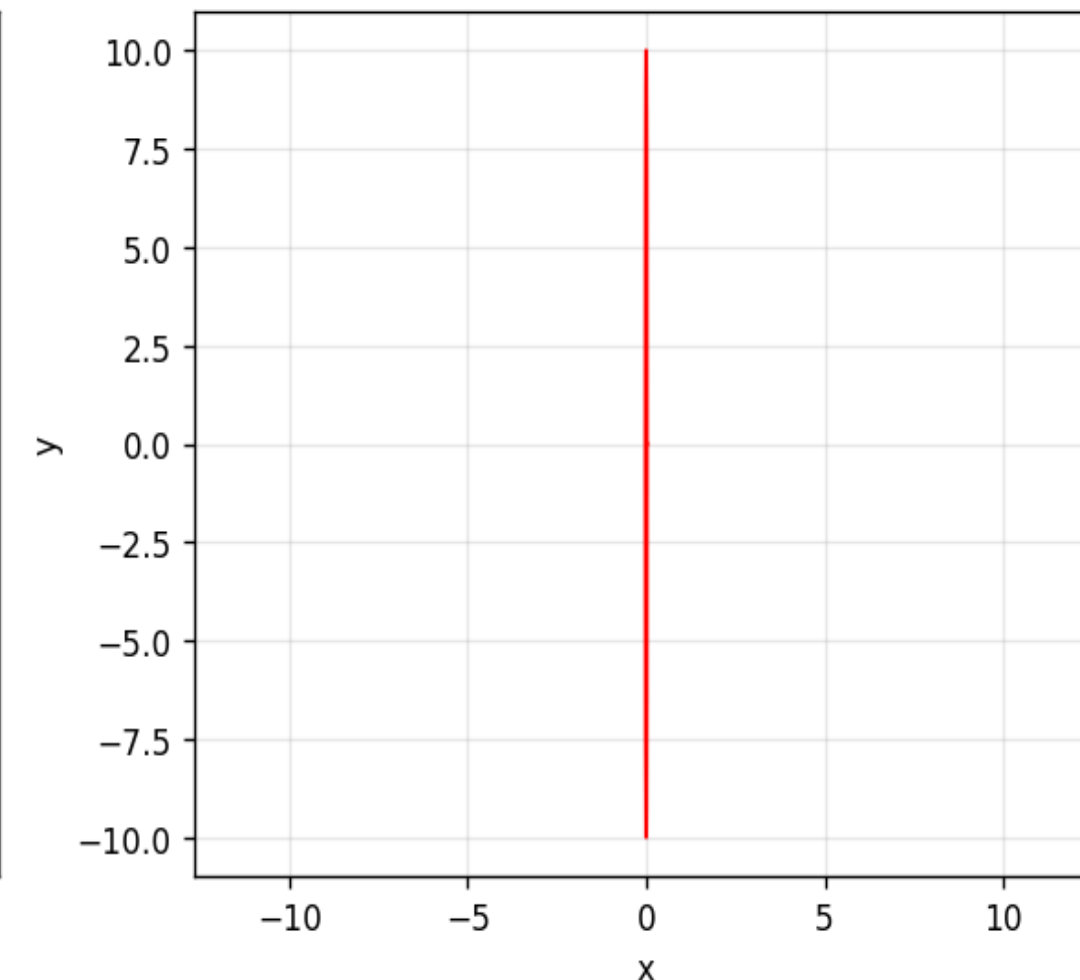
- Left: unit circle  $\rightarrow$  all directions have identical curvature (isotropic case,  $H = I$ )
- Right: after linear transformation by  $H \rightarrow$  becomes a highly elongated ellipse  $\rightarrow$  the same unit-length step now changes the function value by a factor of up to 500x depending on direction.

### Why Ill-Conditioning Causes Zig-Zagging in Gradient Descent

Unit Circle in Parameter Space  
(equal-length steps)



After Hessian Transformation  
( $\kappa = 500.0$ )



# Momentum Solution



## Damps Oscillation

By accumulating previous updates, it reduces the detrimental effect of high curvature (ill-conditioning) and the resulting zig-zag path.



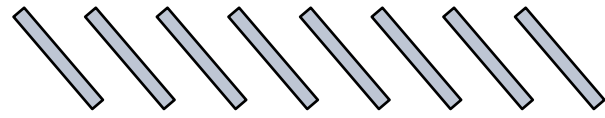
## Maintains Inertia

Maintain velocity (inertia) through flat regions or saddle points, helping to "push" past areas of small gradient.

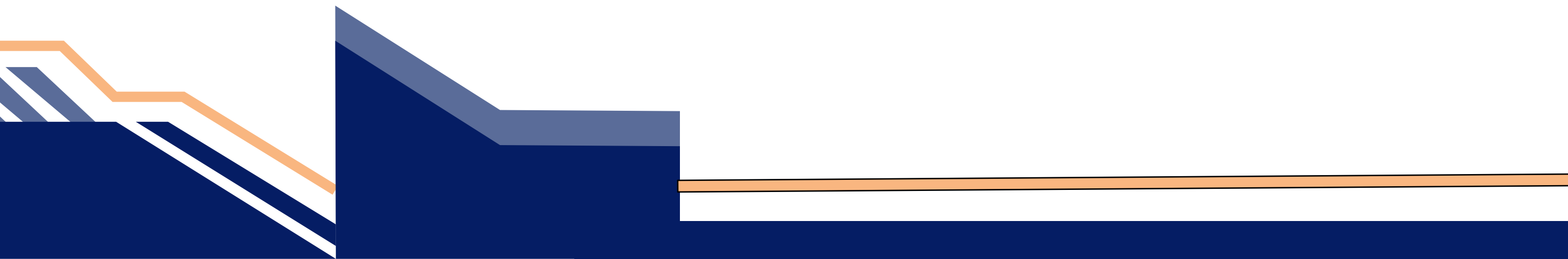


## Reduces Noise Sensitivity

By averaging the gradients over several steps, it reduces the variance caused by the SGD's minibatch sampling, leading to a more stable and reliable descent path.



# 03 MATHEMATICAL FOUNDATIONS



# When to use?

When the optimization landscape has a large disparity in curvature accross direction

Momentum modifies the update by accumulating a running average of past gradients, rather than relying solely on the current gradient. This leads to faster convergence, especially in shallow directions, by smoothing updates and reducing zig-zagging in steep directions.

$$\begin{aligned}\mathbf{v}_t &\leftarrow \beta \mathbf{v}_{t-1} + \mathbf{g}_{t,t-1}, \\ \mathbf{x}_t &\leftarrow \mathbf{x}_{t-1} - \eta_t \mathbf{v}_t.\end{aligned}$$

This motivating example sets the stage for the formal **Momentum Update Rule** and its interpretation as an **exponentially weighted (leaky) average** of past gradients

Two main benefits when adding momentum into GD:

- Accelerating convergence faster
- Smoothing the descent direction by incorporating past gradient by controlling  $\beta$  factor

# Momentum Update Rule

Momentum:

- Introduces a velocity vector that accumulates past gradients.
- Smooths noisy updates and accelerates movement in stable directions

The standard formulation used in deep learning (Sutskever et al., 2013) is:

$$\begin{aligned}\mathbf{v}_{t+1} &= \beta \mathbf{v}_t + \eta \nabla f(\mathbf{x}_t) \\ \mathbf{x}_{t+1} &= \mathbf{x}_t - \mathbf{v}_{t+1}\end{aligned}$$

Where:

- $\beta \in [0, 1)$  is the momentum coefficient (common values: 0.9 or 0.99)
- $\eta$  is the learning rate
- $v_0 = 0$

# Momentum as an Exponential Weighted Average

Momentum Update Rule:  $v_{t+1} = \beta v_t + \eta \nabla f(x_t)$

We can expand the recurrence:

$$\begin{aligned} v_{t+1} &= \beta (\beta v_{t-1} + \eta \nabla f(x_{t-1})) + \eta \nabla f(x_t) \\ &= \eta \left( \nabla f(x_t) + \beta \nabla f(x_{t-1}) + \beta^2 \nabla f(x_{t-2}) + \dots \right) \end{aligned}$$

Expanded Sum Representation:

$$\mathbf{v}_{t+1} = \eta \sum_{k=0}^t \beta^k \nabla f(\mathbf{x}_{t-k})$$

This shows that:

- Gradients from recent steps have large weight
- Gradients from older steps decay exponentially
- Momentum is literally a leaky average of past gradients

# Effective Memory Length

Total mass of the geometric weights

$$S_{\infty} = \sum_{k=0}^{\infty} \beta^k = \frac{1}{1 - \beta}$$

Momentum is often said to have an effective memory length

$$\text{Effective memory} \approx \frac{1}{1 - \beta}$$

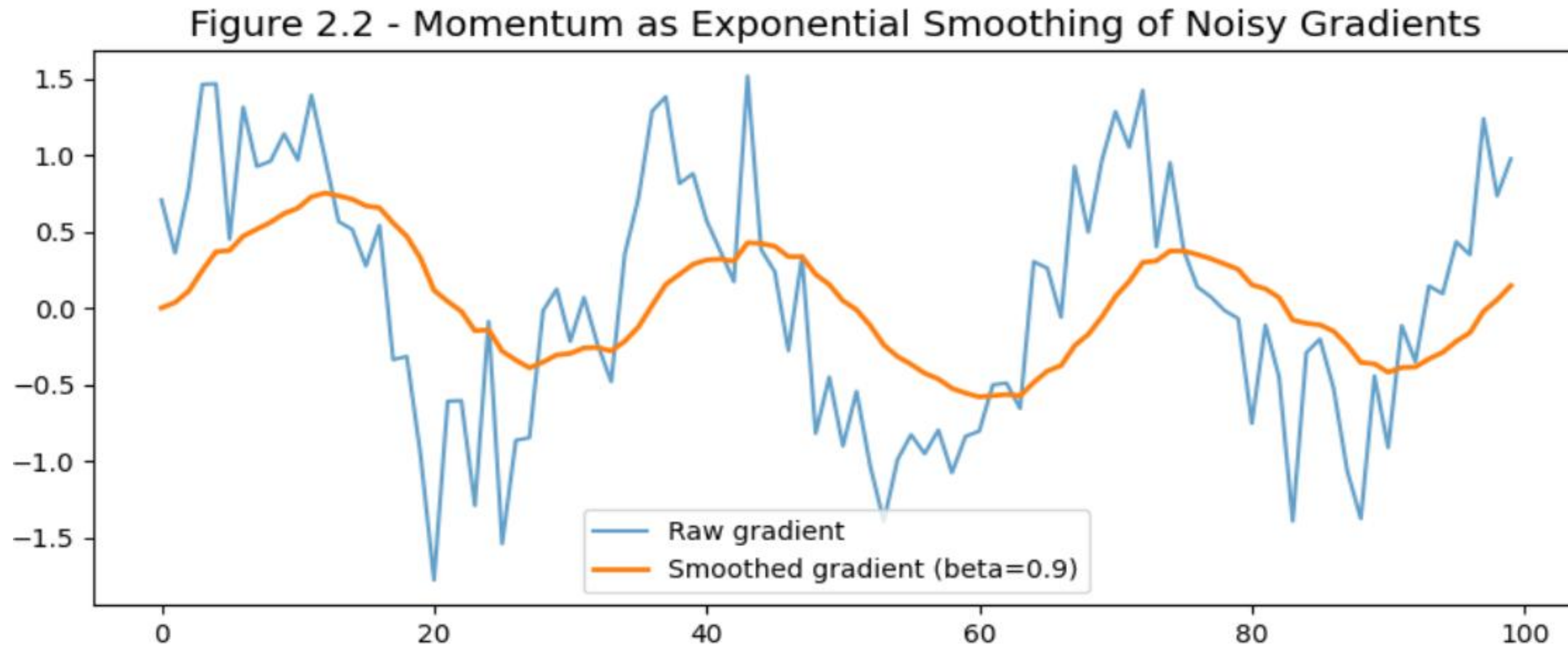
Example:

- $\beta = 0.9 \rightarrow$  remembers ~10 steps
- $\beta = 0.99 \rightarrow$  remembers ~100 steps

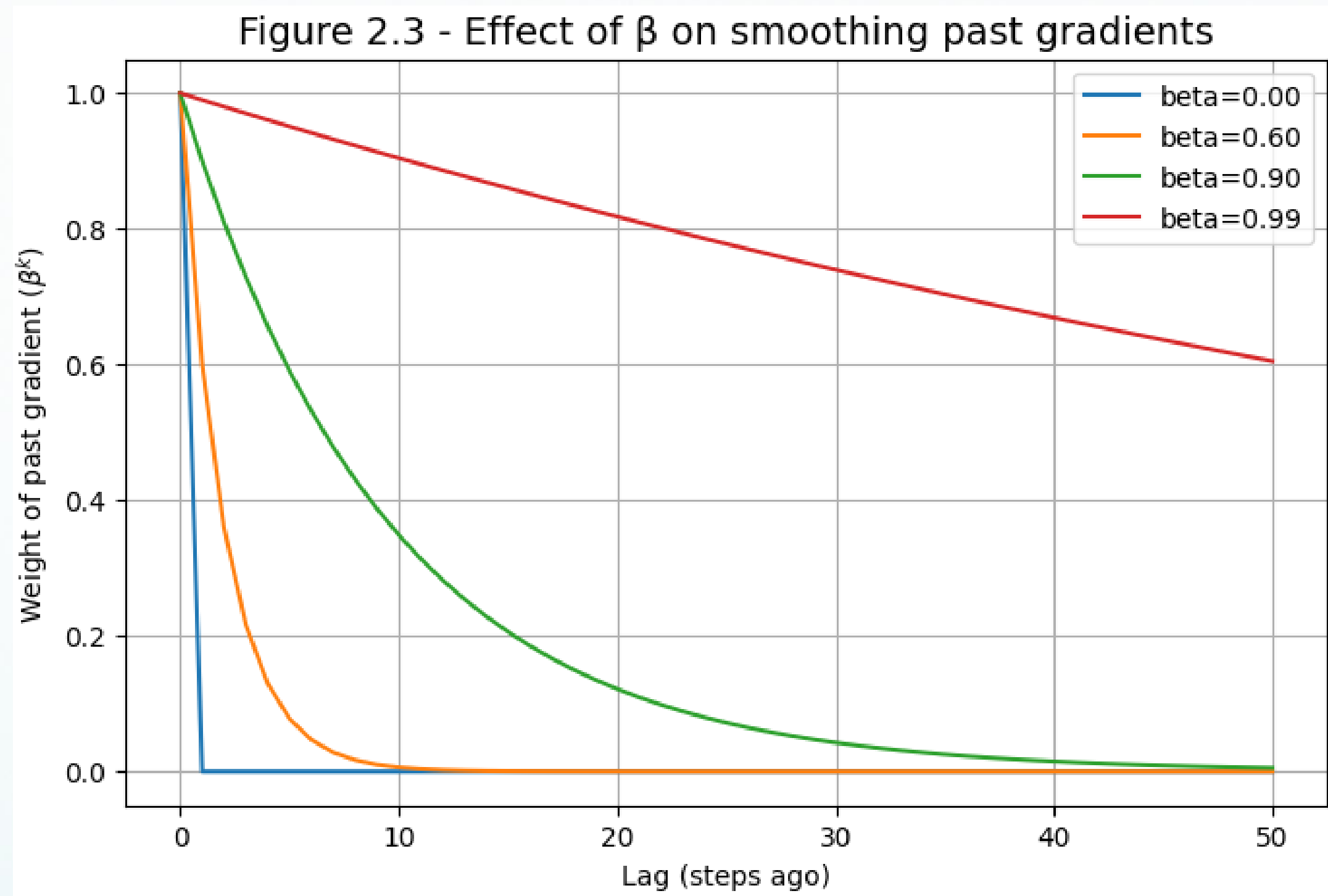
# Smoothing Effect

Momentum smooths the gradient signal in two essential ways:

- **Cancels high-frequency noise**
- **Amplifies consistent directions**



# How $\beta$ controls smoothing



# Why Momentum accelerates convergence

Recall the expanded form of the momentum velocity:

$$\mathbf{v}_t = \eta \sum_{\tau=0}^{t-1} \beta^\tau \nabla f(\mathbf{x}_{t-1-\tau})$$

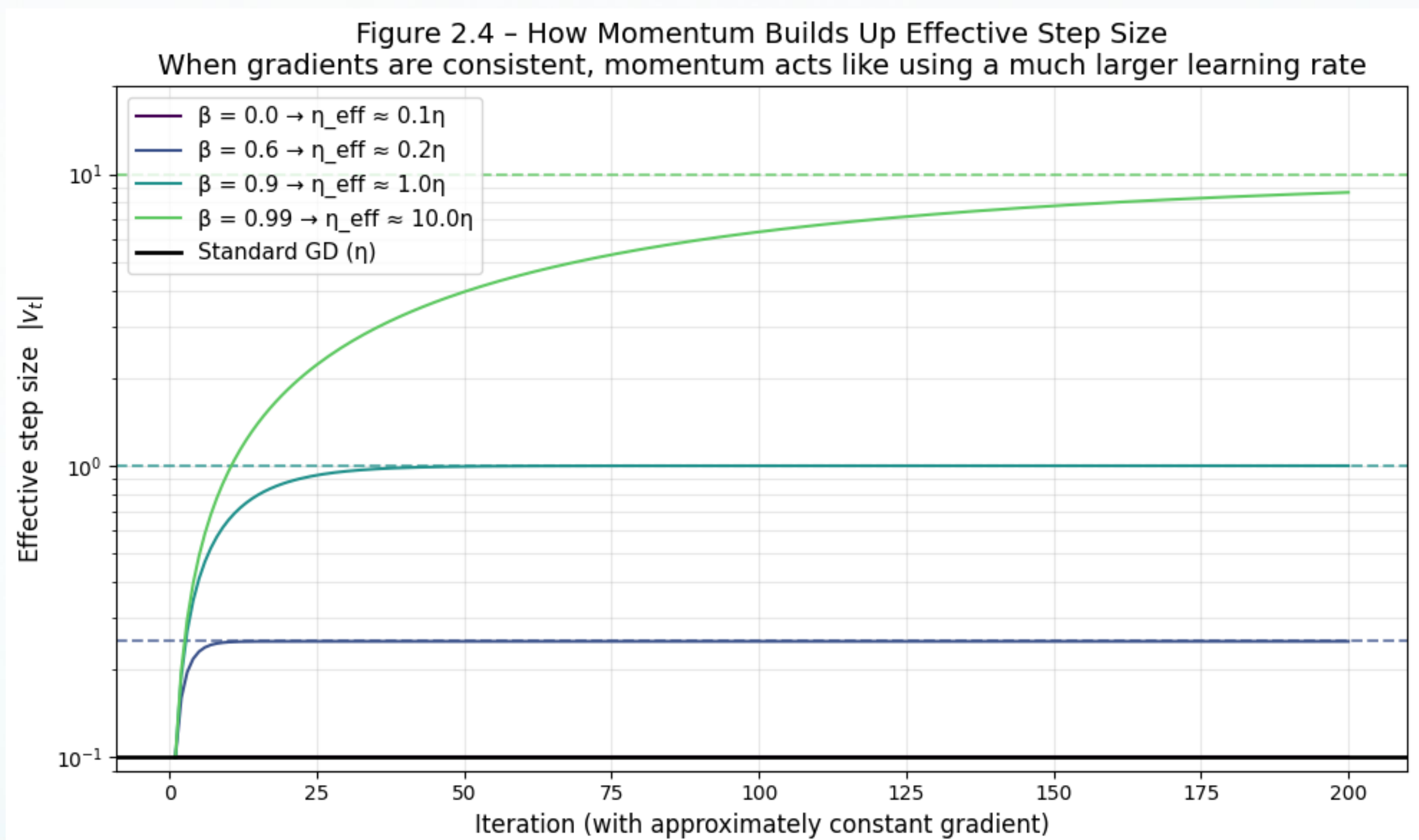
If gradients stay approximately constant,  $\nabla f(\mathbf{x}_{t-1-\tau}) \approx g$  (assume  $g=1.0$ ) then:

$$v_t \approx \sum_{\tau=0}^{t-1} \beta^\tau \approx \frac{1}{1-\beta} \text{ (for large } t \text{)}$$

Multiplying by the learning rate  $\eta$  gives an effective step size:

$$\eta_{eff} \approx \frac{\eta}{1-\beta}$$

# Why Momentum accelerates convergence





# 03 THEORETICAL ANALYSIS



# How GD & Momentum work in a mathematical way

This part will explain how and why gradient descent and momentum work in a mathematical way.

We need to prove that, in **quadratic convex function**:

- **Gradient Descent optimizes each direction independently**, scaled by how steep that direction is → That's why GD can be fast in steep direction and slow in shallow ones (Ill-conditioned problems)
- **Momentum** addresses this issue by adding velocity based on past gradient and also **acts separately in each direction**.

# The Loss is Quadratic Convex

Consider the standard supervised learning setting:

- Data matrix  $X \in \mathbb{R}^{n \times m}$  (full column rank), targets  $y \in \mathbb{R}^n$
- Parameters  $\theta \in \mathbb{R}^m$

The squared error loss is

$$\mathcal{L}(\theta) = \|X\theta - y\|^2 = (X\theta - y)^\top (X\theta - y)$$

Expanding:

$$\mathcal{L}(\theta) = \theta^\top X^\top X \theta - 2\theta^\top X^\top y + y^\top y$$

Define

$$Q = 2X^\top X, \quad \mathbf{c} = -2X^\top y, \quad b = y^\top y$$

Then

$$\mathcal{L}(\theta) = \frac{1}{2}\theta^\top Q\theta + \theta^\top \mathbf{c} + b.$$

**Why is  $Q \succ 0$  (positive definite)?**

- $Q$  is symmetric by construction.
- For any  $\theta \neq 0$ :

$$\theta^\top Q\theta = 2\|X\theta\|^2 > 0 \quad (\text{because } X \text{ has full column rank})$$

$\rightarrow Q$  is symmetric positive definite (SPD)  $\rightarrow \mathcal{L}$  is strongly convex  $\rightarrow$  unique global minimum.

# Shift Coordinates to the Minimum

Consider loss function:  $\mathcal{L}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{x}^\top \mathbf{c} + b.$

The minimizer satisfies  $\nabla \mathcal{L}(\mathbf{x}) = 0$ :  $\mathbf{Q}\mathbf{x}^* + \mathbf{c} = 0 \Rightarrow \boxed{\mathbf{x}^* = -\mathbf{Q}^{-1}\mathbf{c}}.$

Define the deviation variable  $\mathbf{v} = \mathbf{x} - \mathbf{x}^* \Rightarrow \mathbf{x} = \mathbf{x}^* + \mathbf{v}.$

Substituting  $\mathbf{x} = \mathbf{x}^* + \mathbf{v}.$  into the quadratic loss function, we have:

$$\mathcal{L}(\mathbf{x}^* + \mathbf{v}) = \frac{1}{2}(\mathbf{x}^* + \mathbf{v})^\top \mathbf{Q}(\mathbf{x}^* + \mathbf{v}) + (\mathbf{x}^* + \mathbf{v})^\top \mathbf{c} + b$$

Expand:

$$\begin{aligned}\mathcal{L}(\mathbf{x}^* + \mathbf{v}) &= \frac{1}{2}[(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{x}^* + 2(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{v} + \mathbf{v}^\top \mathbf{Q}\mathbf{v}] + (\mathbf{x}^*)^\top \mathbf{c} + \mathbf{v}^\top \mathbf{c} + b \\ &= \frac{1}{2}(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{x}^* + (\mathbf{x}^*)^\top \mathbf{Q}\mathbf{v} + \frac{1}{2}\mathbf{v}^\top \mathbf{Q}\mathbf{v} + (\mathbf{x}^*)^\top \mathbf{c} + \mathbf{v}^\top \mathbf{c} + b \\ &= \underbrace{(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{v} + \mathbf{v}^\top \mathbf{c}}_{\text{linear in } \mathbf{v}} + \underbrace{\frac{1}{2}(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{x}^* + (\mathbf{x}^*)^\top \mathbf{c} + b}_{\text{constant}} + \underbrace{\frac{1}{2}\mathbf{v}^\top \mathbf{Q}\mathbf{v}}_{\text{quadratic in } \mathbf{v}}\end{aligned}$$

# Shift Coordinates to the Minimum

$$\mathcal{L}(\mathbf{x}^* + \mathbf{v}) = \underbrace{(\mathbf{x}^*)^\top Q \mathbf{v} + \mathbf{v}^\top \mathbf{c}}_{\text{linear in } \mathbf{v}} + \underbrace{\frac{1}{2}(\mathbf{x}^*)^\top Q \mathbf{x}^* + (\mathbf{x}^*)^\top \mathbf{c} + b}_{\text{constant}} + \underbrace{\frac{1}{2} \mathbf{v}^\top Q \mathbf{v}}_{\text{quadratic in } \mathbf{v}}$$

By the linear term, we have:

$$(\mathbf{x}^*)^\top Q \mathbf{v} + \mathbf{v}^\top \mathbf{c} = (\mathbf{x}^*)^\top Q \mathbf{v} + \mathbf{v}^\top (-Q \mathbf{x}^*)$$

**Transpose property of a matrix product**  $(\mathbf{v}^\top Q \mathbf{x}^*)^\top = (\mathbf{x}^*)^\top Q^\top \mathbf{v}$

$$(\mathbf{x}^*)^\top Q \mathbf{v} + \mathbf{v}^\top \mathbf{c} = (\mathbf{x}^*)^\top Q \mathbf{v} + \mathbf{v}^\top (-Q \mathbf{x}^*) = (\mathbf{x}^*)^\top Q \mathbf{v} - (\mathbf{x}^*)^\top Q \mathbf{v} = 0$$

**Local quadratic expansion around the minimum**

$$\mathcal{L}(\mathbf{x}^* + \mathbf{v}) = \frac{1}{2} \mathbf{v}^\top Q \mathbf{v} + \mathcal{L}(\mathbf{x}^*).$$

# Eigen-Decomposition & Decoupling

Spectral Decomposition (for real symmetric and positive definite)

$$\mathbf{Q} = \mathbf{O}^\top \mathbf{\Lambda} \mathbf{O}, \quad \mathbf{O}^\top \mathbf{O} = \mathbf{I}, \quad \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m), \quad \lambda_i > 0.$$

Where:

- $\mathbf{O}$  is an orthogonal matrix of eigenvectors ( $\mathbf{O}^\top \mathbf{O} = \mathbf{I}$ )
- $\mathbf{\Lambda}$  is a diagonal matrix of eigenvalues ( $\lambda_i > 0$ ), which correspond to the curvature along each principal direction.

Rotate Coordinates:  $\mathbf{z} = \mathbf{O}\mathbf{v} \longrightarrow \boxed{\mathbf{v} = \mathbf{O}^\top \mathbf{z}}$

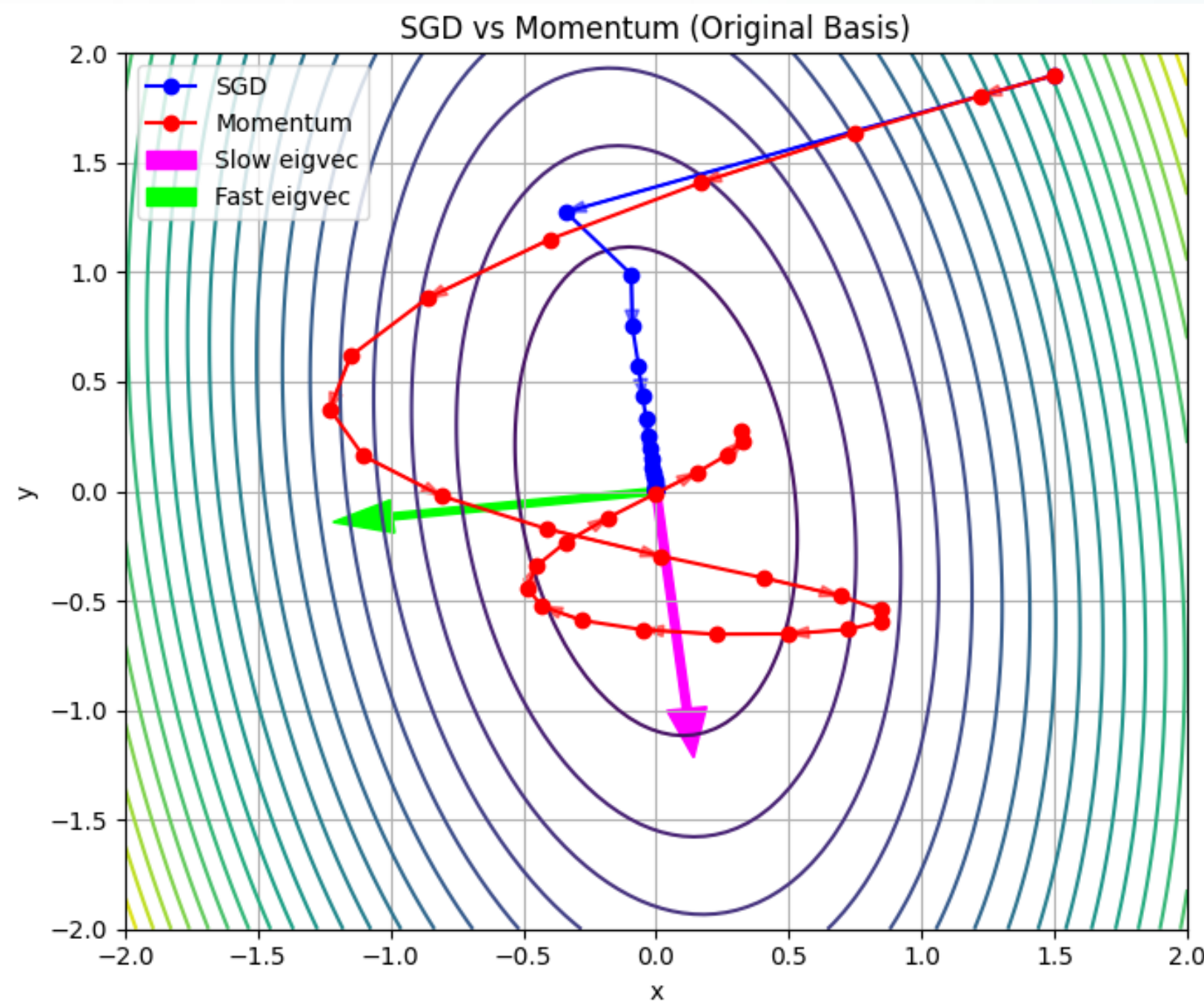
This rotation aligns the coordinate axes with eigenvectors of  $\mathbf{Q}$ .

Quadratic in Rotated basis:

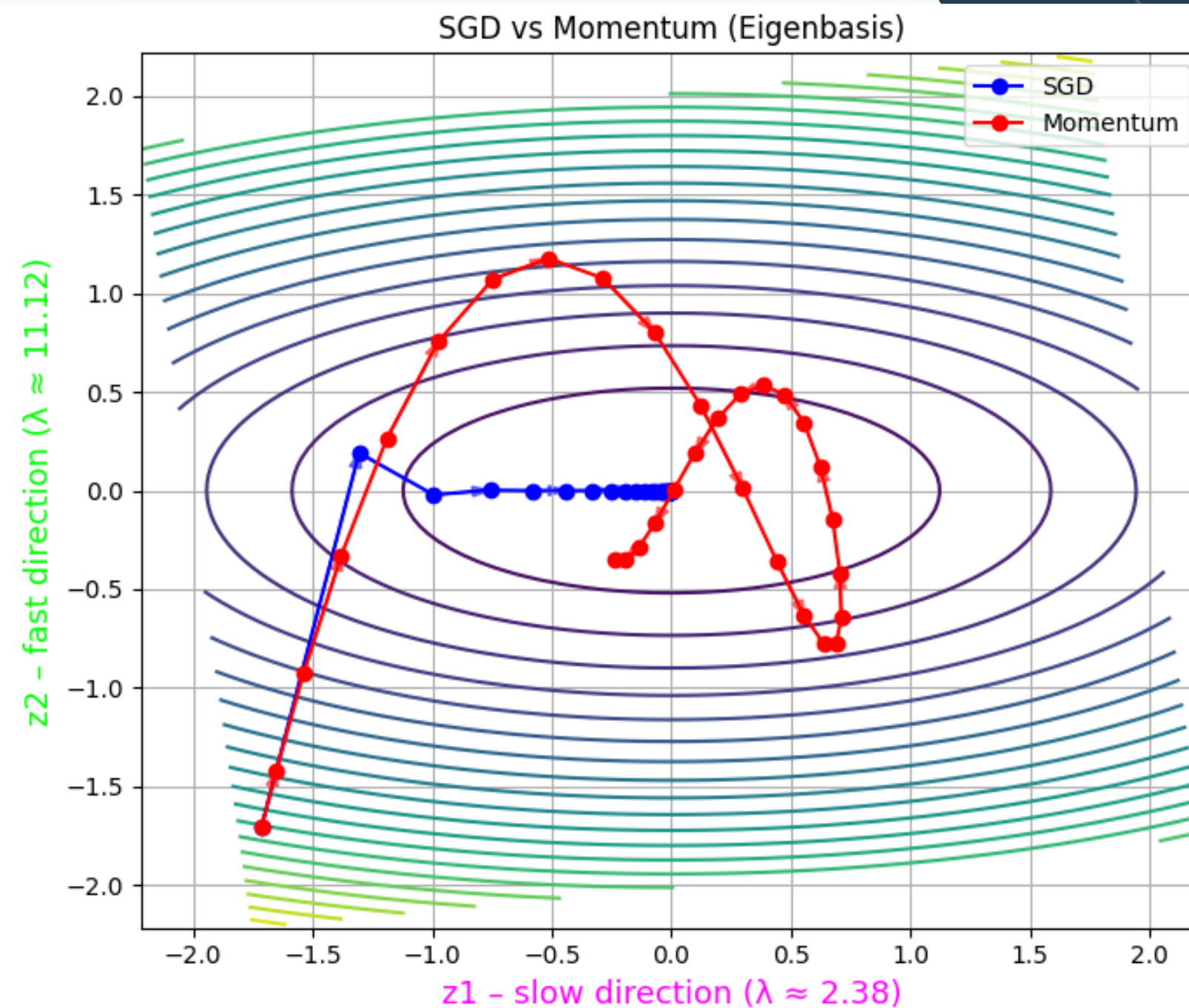
$$h(\mathbf{x}) = \frac{1}{2} \mathbf{v}^\top \mathbf{Q} \mathbf{v} + h(\mathbf{x}^*) = \frac{1}{2} \mathbf{z}^\top \mathbf{\Lambda} \mathbf{z} + b^* = \sum_{i=1}^m \lambda_i z_i^2$$

Now  $\mathbf{Q}$  becomes **diagonal**, eliminating all cross terms.

# Eigen-Decomposition & Decoupling



Eigenvalues: [ 2.3839377 11.1160623]  
Slow direction (horizontal):  $\lambda = 2.3839377008567553$   
Fast direction (vertical):  $\lambda = 11.116062299143245$



# Dynamics of GD & Momentum in the Eigen-basis

After rotating coordinates:

$$h(\mathbf{z}) = \frac{1}{2} \mathbf{z}^\top \Lambda \mathbf{z}, \quad \nabla h(\mathbf{z}) = \Lambda \mathbf{z}.$$

Since  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  each eigen-direction evolves independently.

Gradient Descent (1D per eigenvalue)

$$z_{t,i} = (1 - \eta \lambda_i) z_{t-1,i}.$$

→ Flat directions (small  $\lambda_i$ )  $\Rightarrow$  factor  $\approx 1 \Rightarrow$  extremely slow convergence.

Momentum

$$\begin{aligned} v_{t,i} &= \beta v_{t-1,i} + \eta \lambda_i z_{t-1,i}, \\ z_{t,i} &= z_{t-1,i} - v_{t,i}. \end{aligned}$$



$$\begin{bmatrix} z_{t,i} \\ v_{t,i} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 - \eta \lambda_i & -1 \\ \eta \lambda_i & \beta \end{bmatrix}}_{A_i} \begin{bmatrix} z_{t-1,i} \\ v_{t-1,i} \end{bmatrix}.$$

→ Contraction factor becomes spectral radius of  $A_i$ .

→ Massive acceleration in flat directions, unlike GD.

# 04 PRACTICAL IMPLEMENTATION





**Ill-conditioned  
quadratic function**



**Scalar function**



**Benchmark on MNIST  
dataset**

# III-Conditioned Quadratic Function

Consider the ill-conditioned quadratic function:

$$f(x_1, x_2) = 0.01 x_1^2 + 5 x_2^2.$$

The gradient of this function is:

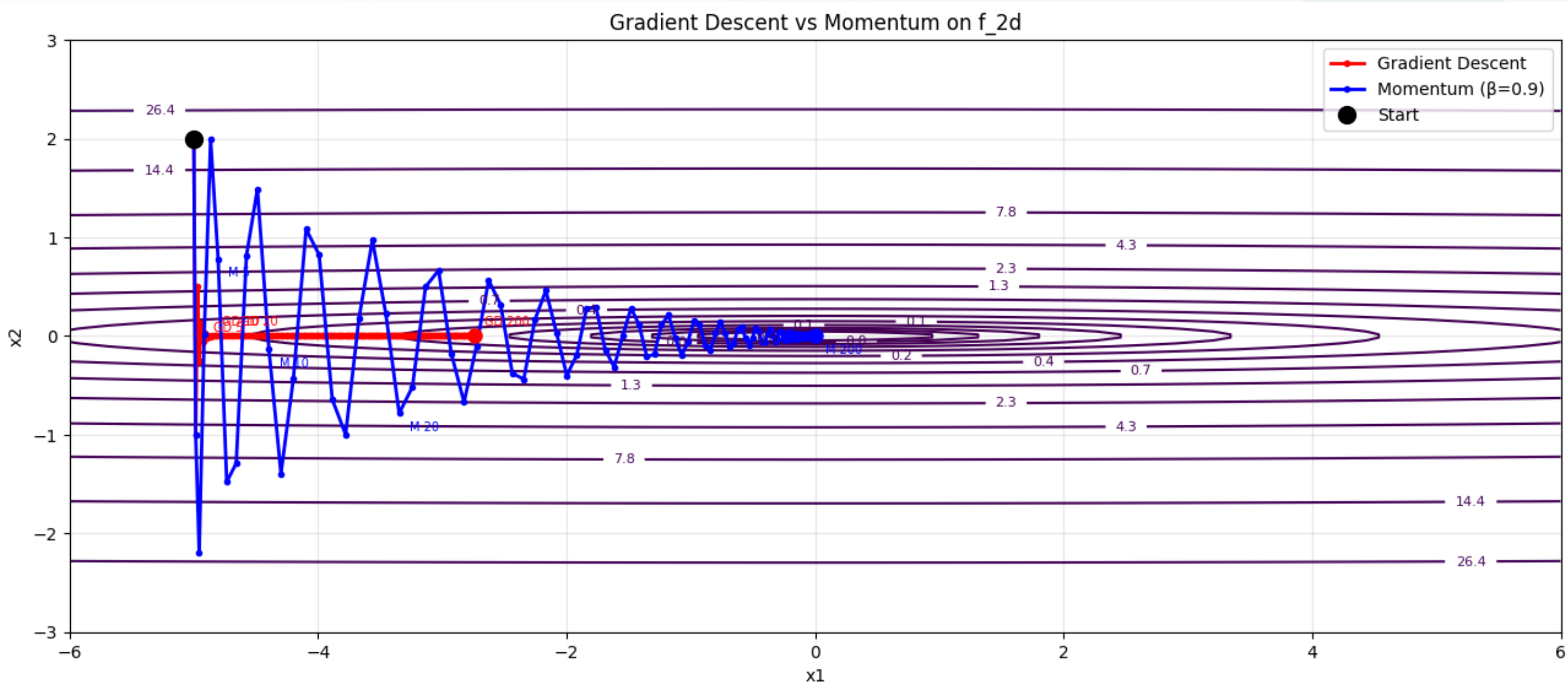
$$\frac{\partial f}{\partial x_1} = 0.02x_1, \quad \frac{\partial f}{\partial x_2} = 10x_2.$$

With learning rate  $\eta=0.15$ , we examine the behavior of standard **Gradient Descent** :

- **Along the x1 direction (small curvature):** Contraction factor is  $(1 - \eta \cdot 0.02)=0.997$   
 $\Rightarrow$  x1 decreases extremely slowly (requiring thousands of steps to reach 0).
- **Along the x2 direction (large curvature):** Contraction factor is  $(1 - \eta \cdot 10) = -0.5$   
 $\Rightarrow$  x2 changes sign at each step and exhibits strong oscillations (zig-zag)

**To address this issue, we apply Momentum with  $\beta=0.9$ .**

# III-Conditioned Quadratic Function



Iteration		GD Value	Momentum Value
Iter 0		20.250000	20.250000
Iter 5		0.262132	3.251005
Iter 10		0.235439	0.288125
Iter 15		0.228452	2.223874
Iter 20		0.221690	3.176514
Iter 25		0.215128	2.328358
Iter 200		0.075163	0.000000

Table: Function Values During Optimization on  $f_{2d}$

# Scalar Function

## Using Gradient descent

We analyze a simple scalar quadratic function:

$$f(x) = \frac{\lambda}{2}x^2$$

Using Gradient descent:

$$x_{t+1} = x_t - \eta\lambda x_t = (1 - \eta\lambda)x_t.$$

This optimization converges at exponential rate since after t steps:

$$x_t = (1 - \eta\lambda)^t x_0$$

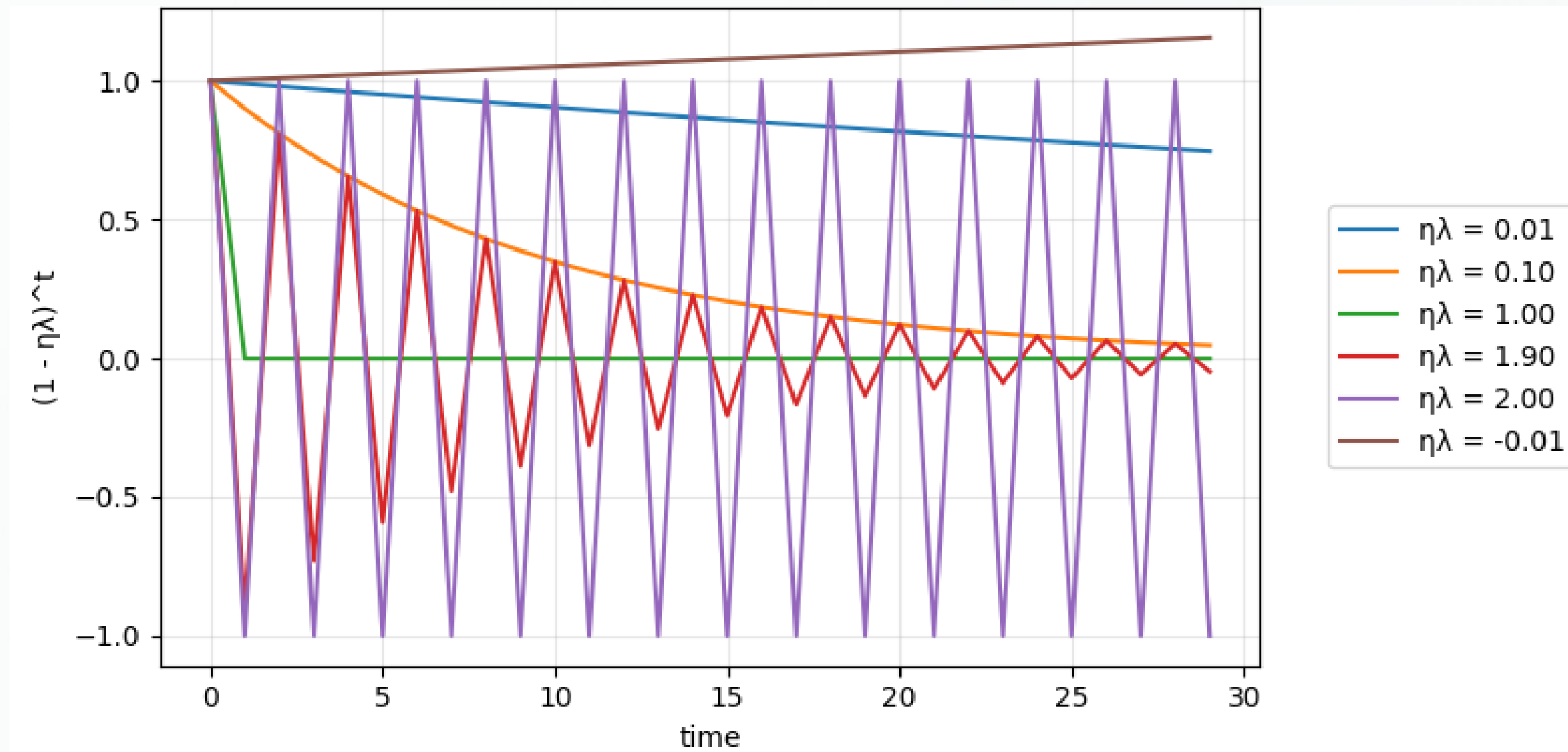
Behavior of convergence depends on the factor  $(1-\eta\lambda)$ :

- If  $|1 - \eta\lambda| < 1 \Leftrightarrow 0 < \eta\lambda < 2$  then  $x_t \rightarrow 0$ : the algorithm converges
- If  $|1 - \eta\lambda| > 1$  then  $x \rightarrow \infty$ : the algorithm diverges
- If  $|1 - \eta\lambda| = 1 \rightarrow \eta\lambda = 2/\eta\lambda = 0$ : never converge

# Scalar Function

## Using Gradient descent

The chart below showing how  $\mathbf{x}_t/\mathbf{x}_0$  change over time - how fast the variable converges to 0 under different values of  $1 - \eta\lambda$ .



# Scalar Function

## Using Momentum

$$\begin{aligned}v_{t+1} &= \beta v_t + \lambda x_t \\x_{t+1} &= x_t - \eta v_{t+1} = x_t - \eta \beta v_t + \eta \lambda x_t = -\eta \beta v_t + (1 - \eta \lambda) x_t\end{aligned}$$

Converting to matrices:

$$\begin{bmatrix} v_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} \beta & \lambda \\ -\eta \beta & (1 - \eta \lambda) \end{bmatrix} \begin{bmatrix} v_t \\ x_t \end{bmatrix} = \mathbf{R}(\beta, \eta, \lambda) \begin{bmatrix} v_t \\ x_t \end{bmatrix}.$$

After  $t$  iterations:

$$\begin{bmatrix} v_t \\ x_t \end{bmatrix} = \mathbf{R}^t \begin{bmatrix} v_0 \\ x_0 \end{bmatrix}$$

The method converges if and only if the spectral radius is strictly less than 1:

$$\rho(\mathbf{R}) = \max(|\sigma_1|, |\sigma_2|) < 1$$

Stability region for Momentum:

$$0 < \eta \lambda < 2 + 2\beta$$

# Training Neural Networks

## 1. Dataset:

- 60 000 training images, 10 000 testing images
- 28 x 28 grayscale

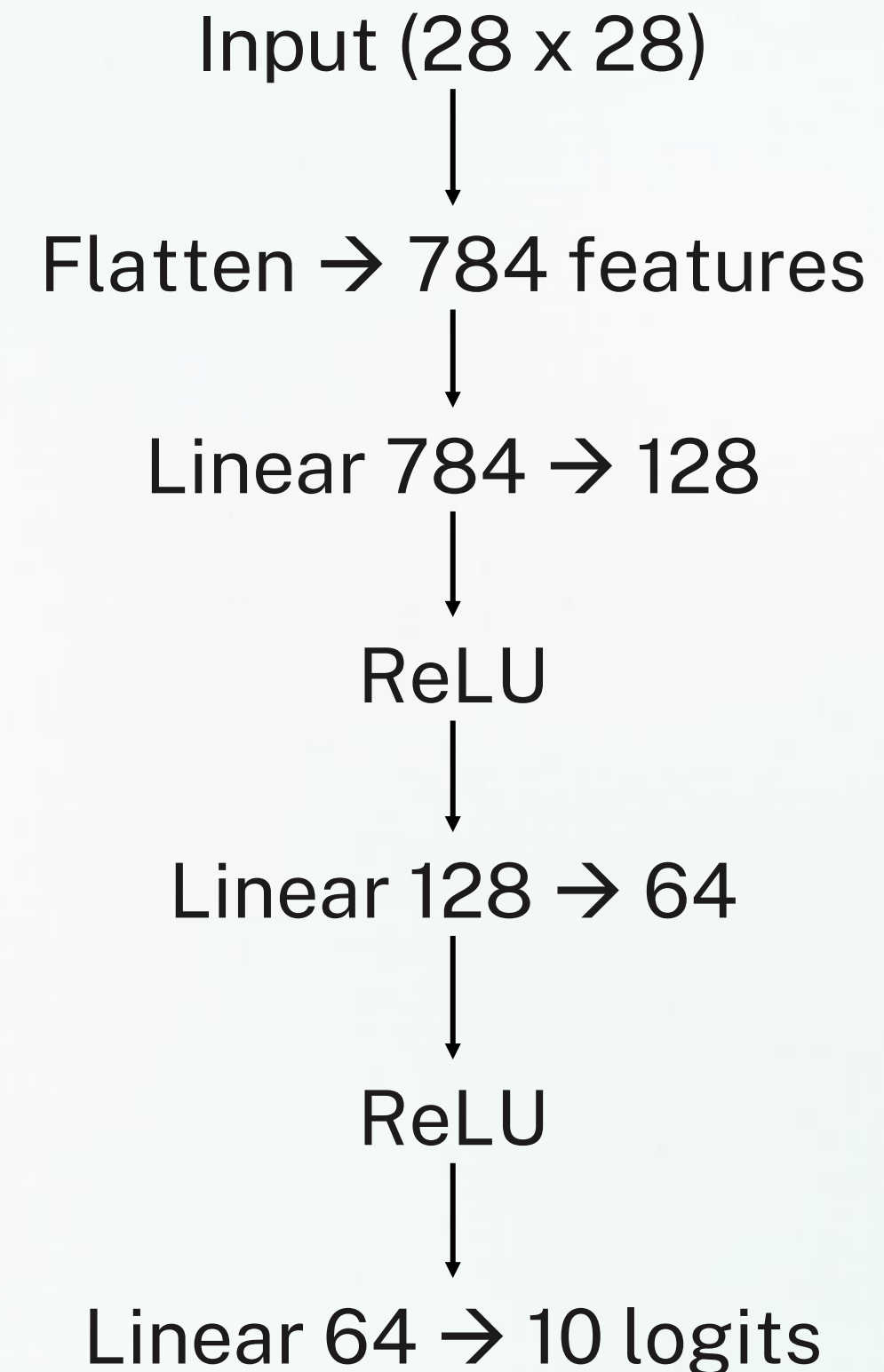
## 2. Runtime environment: Google Colab CPU

## 3. Batch size:

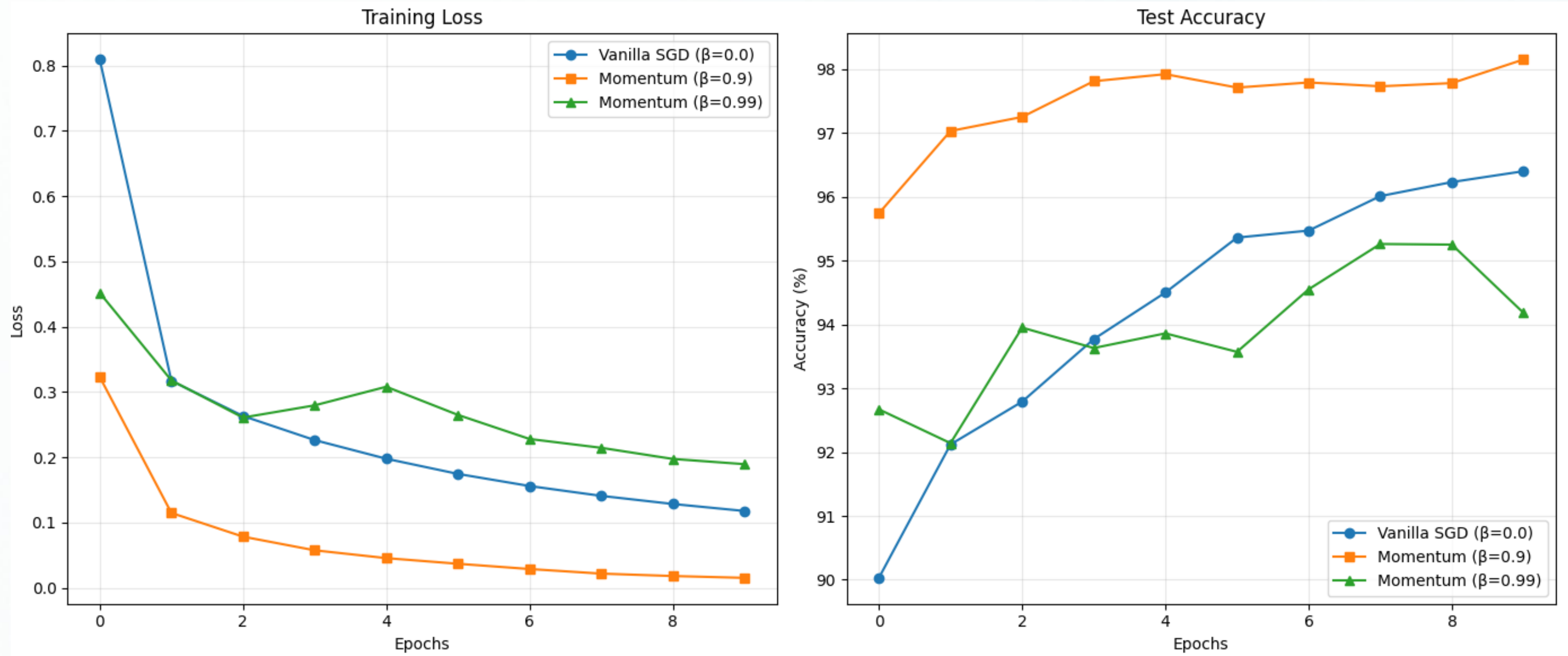
- Mini-batch = 64

## 4. Benchmark:

- Vanilla SGD:  $\beta = 0.0$
- SGD + Momentum:  $\beta = 0.9$
- SGD + High Momentum:  $\beta = 0.99$



# Momentum performs better of time and accuracy



# 05 EXERCISES



# Exercise 1:

Use other combinations of momentum hyperparameters and learning rates and observe and analyze the different experimental results.

$$f(x_1, x_2) = 0.01x_1^2 + 5x_2^2.$$

We calculate gradient descent:

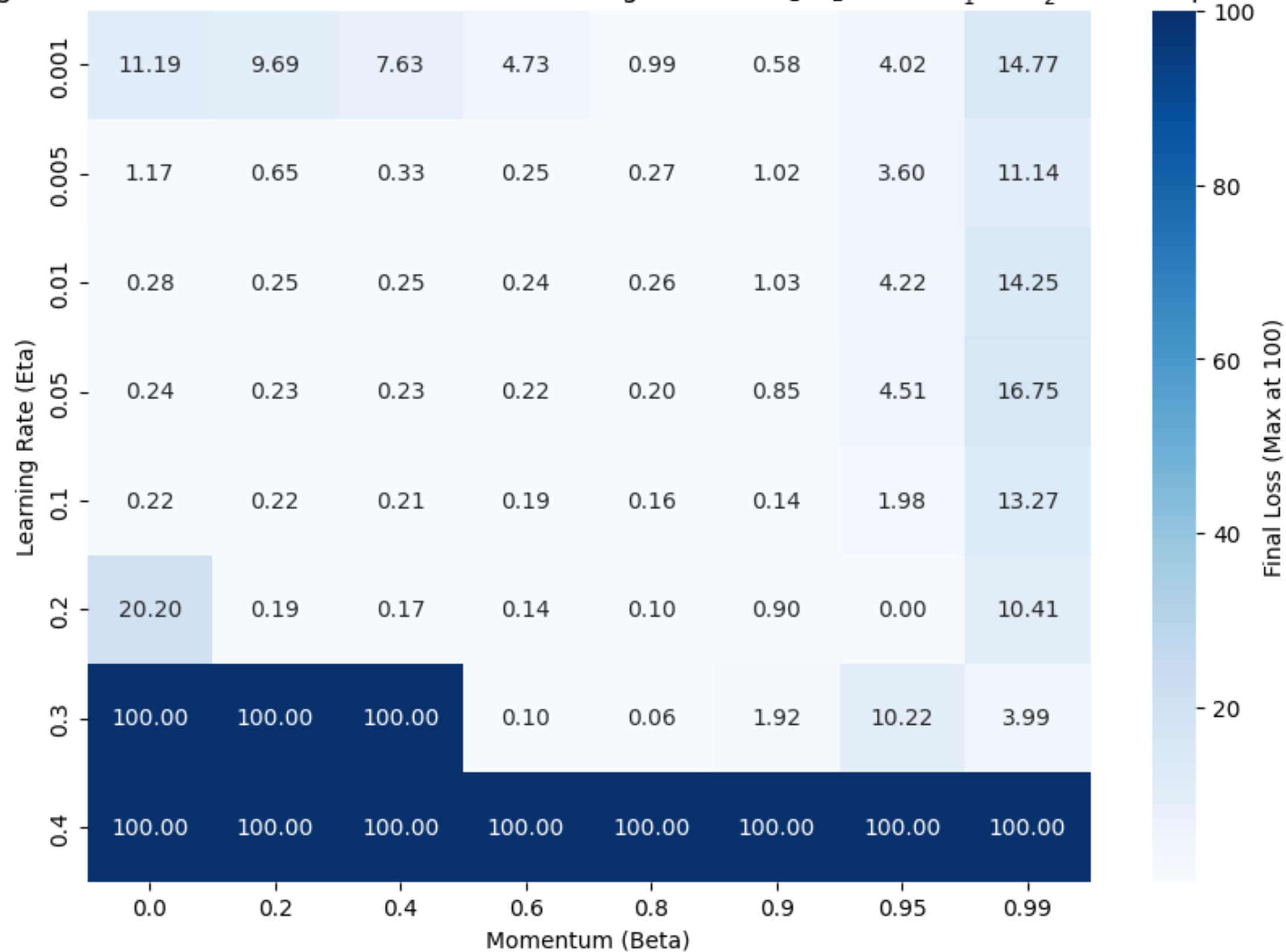
$$\frac{\partial f}{\partial x_1} = 0.02x_1, \quad \frac{\partial f}{\partial x_2} = 10x_2.$$

Momentum Update:

$$\begin{cases} v_{1,t+1} = \beta v_{1,t} + 0.02x_{1,t} \\ v_{2,t+1} = \beta v_{2,t} + 10x_{2,t} \\ x_{1,t+1} = x_{1,t} - \eta v_{1,t+1} \\ x_{2,t+1} = x_{2,t} - \eta v_{2,t+1} \end{cases}$$

[illegible]

Convergence Performance of Momentum and Learning Rate of  $f(x_1, x_2) = 0.01x_1^2 + 5x_2^2$  with 30 epochs.



# Exercise 2:

Try out gradient descent and momentum for a quadratic problem where you have multiple eigenvalues, i.e.,

$$f(x) = \frac{1}{2} \sum_i \lambda_i x_i^2, \text{ e.g. } \lambda_i = 2^{-i}$$

Plot how the values of  $x$  decrease for the initialization  $x_i = 1$ .

**We calculate gradient descent use the formula:**

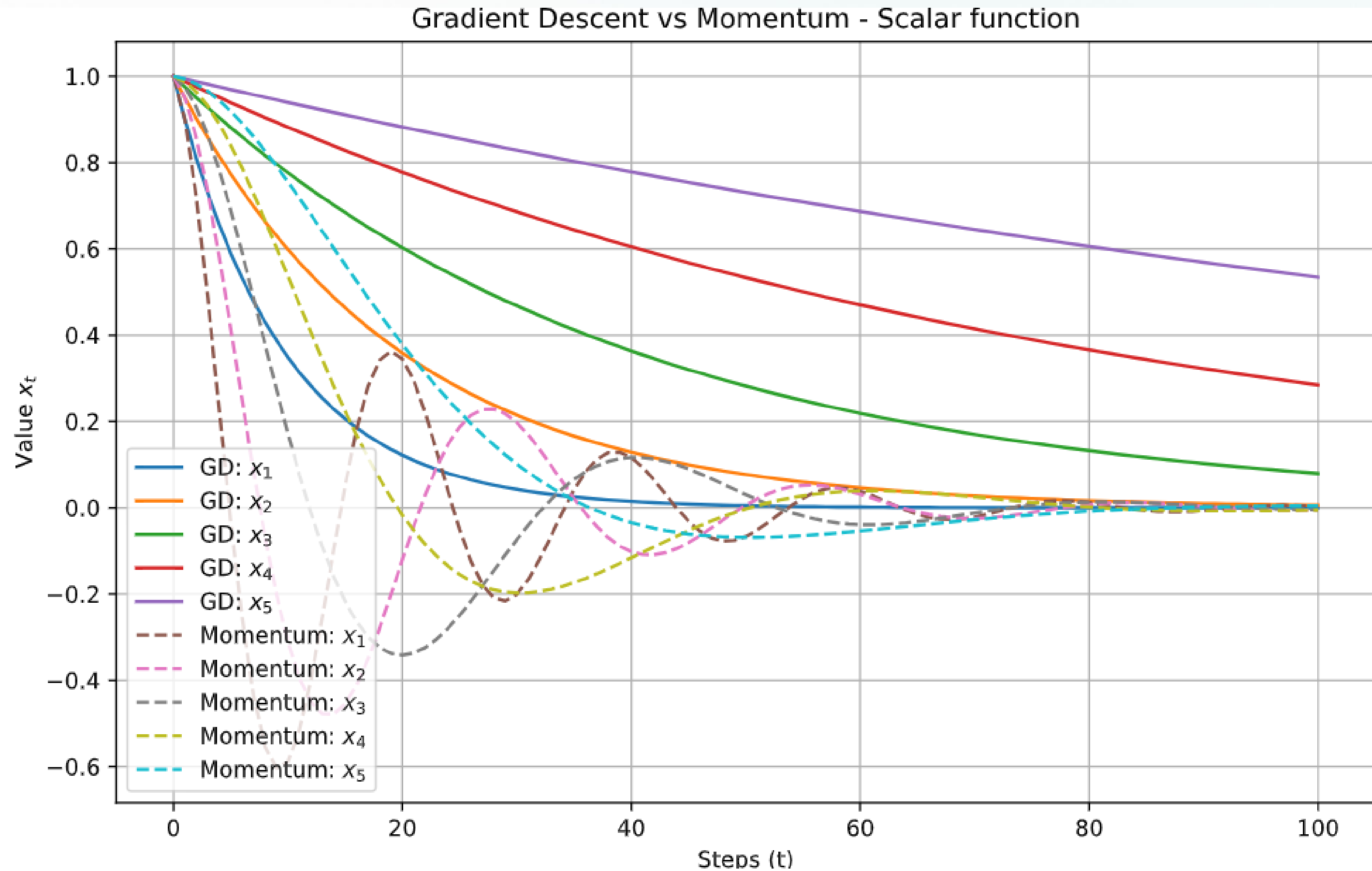
$$x_{i,t+1} = (1 - \eta \lambda_i) x_{i,t}$$

**Momentum Update:**

$$v_{i,t+1} = \beta v_{i,t} + \lambda_i x_{i,t}$$

$$x_{i,t+1} = x_{i,t} - \eta v_{i,t+1}$$

# Exercise 2:



# Exercise 3:

**Problem statement:** Derive the minimizer and minimum value of the quadratic

$$h(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{c}^\top \mathbf{x} + b, \quad \mathbf{Q} \succ 0.$$

**Step 1: Derive with respect to  $\mathbf{x}$ :**  $\nabla h(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{c}$

**Step 2: To find the minimizer, set the gradient to 0**  $\mathbf{Q}\mathbf{x} + \mathbf{c} = 0 \Rightarrow \mathbf{x}^* = -\mathbf{Q}^{-1}\mathbf{c}.$

**Step 3: Substitute  $\mathbf{x}^*$  into  $h(\mathbf{x}^*)$**

$$\begin{aligned} h(\mathbf{x}^*) &= \frac{1}{2}(\mathbf{x}^*)^\top \mathbf{Q}\mathbf{x}^* + \mathbf{c}^\top \mathbf{x}^* + b \\ &= \frac{1}{2}\mathbf{c}^\top \mathbf{Q}^{-1}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{c} - \mathbf{c}^\top \mathbf{Q}^{-1}\mathbf{c} + b \\ &= \frac{1}{2}\mathbf{c}^\top \mathbf{Q}^{-1}\mathbf{c} - \mathbf{c}^\top \mathbf{Q}^{-1}\mathbf{c} + b \\ &= b - \frac{1}{2}\mathbf{c}^\top \mathbf{Q}^{-1}\mathbf{c}. \end{aligned}$$

# Exercise 3:

Let take a numerical example to understand more:

$$h(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{c}^\top \mathbf{x} + b, \quad \mathbf{Q} \succ 0.$$

Let  $\mathbf{Q} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ ,  $\mathbf{c} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$ ,  $b = 0$ .

From previous result, we have:

First, compute  $\det \mathbf{Q}$  (for 2x2 matrix:  $ad - bc$ ):

$$\det \mathbf{Q} = 3 \cdot 2 - 1 \cdot 1 = 6 - 1 = 5.$$

The inverse is

$$\mathbf{Q}^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

The minimizer is

$$\mathbf{x}^* = -\mathbf{Q}^{-1} \mathbf{c} = -\frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 2 \\ -16 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 3.2 \end{bmatrix}.$$

# Exercise 3:

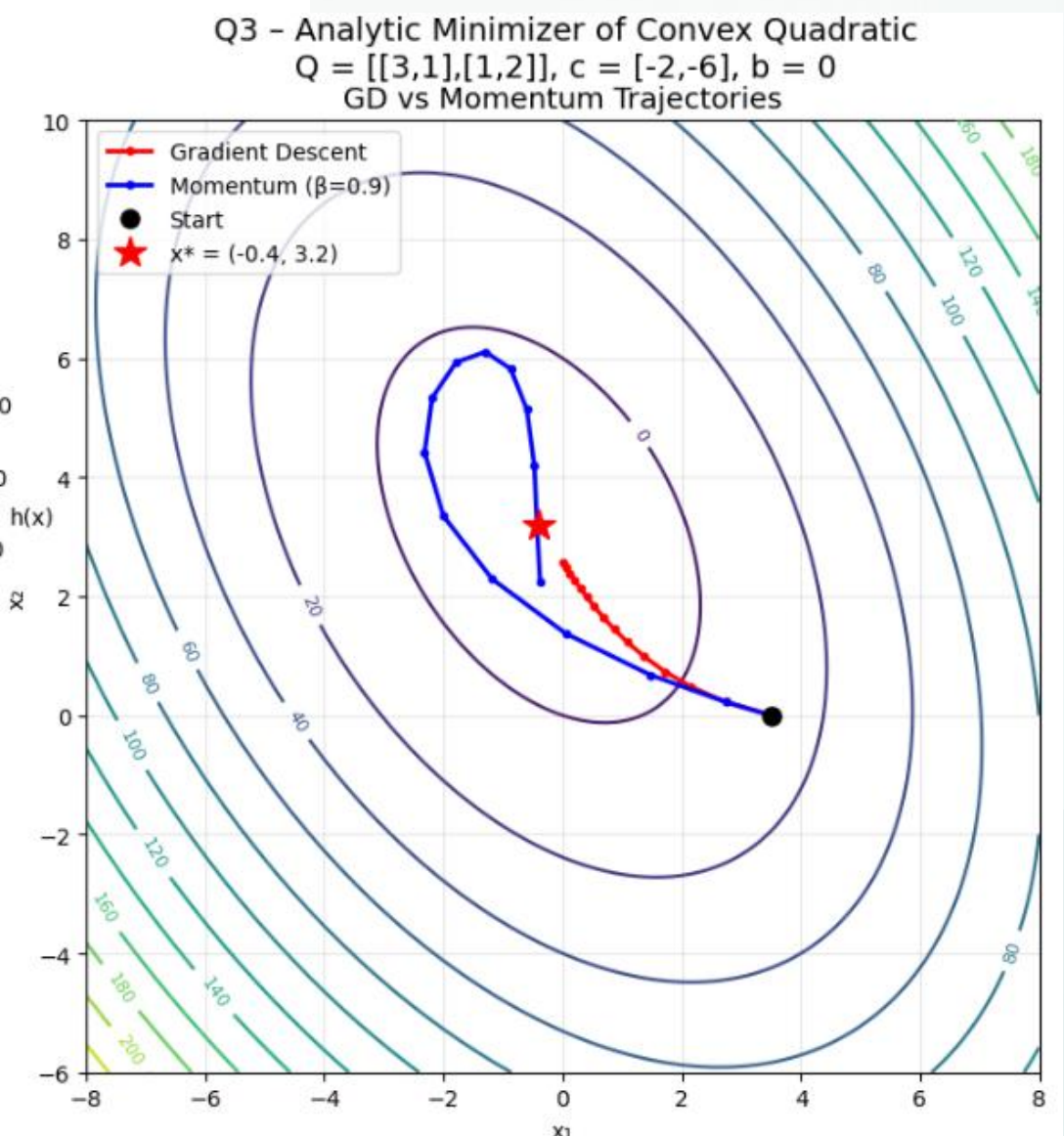
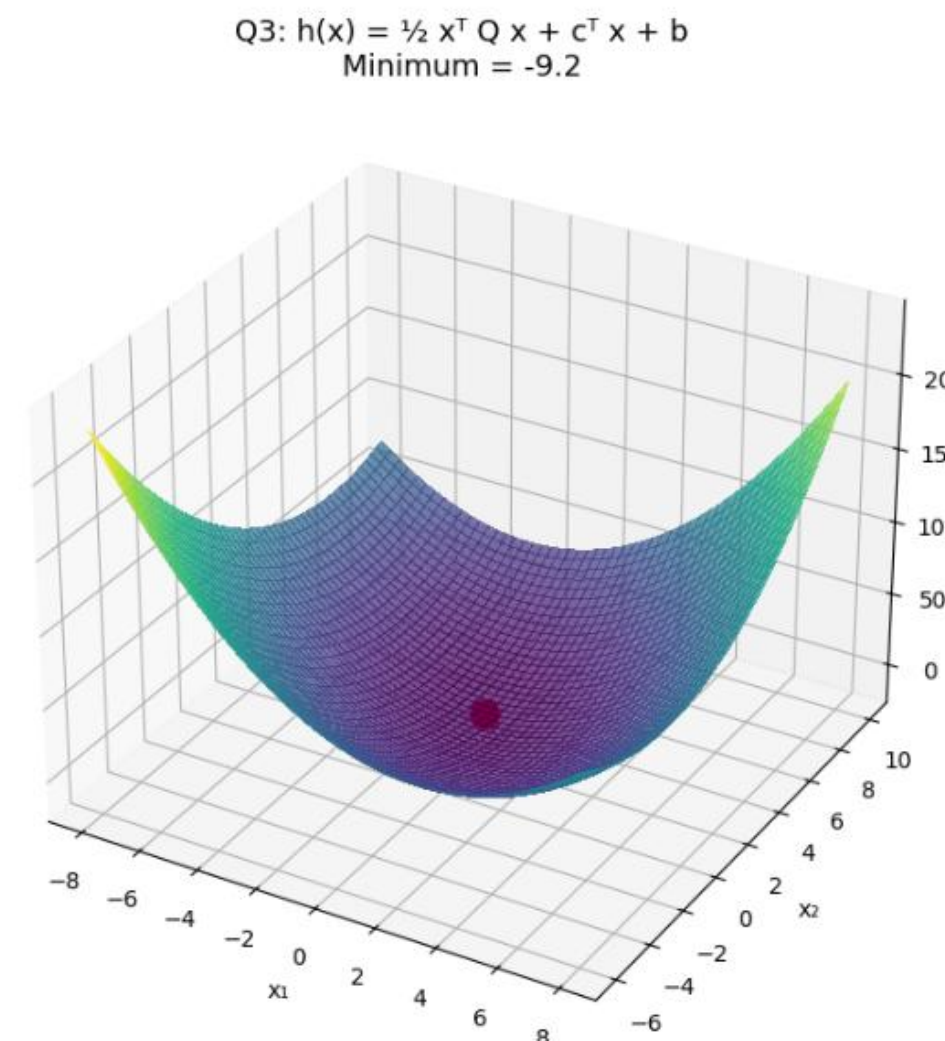
Let take a numerical example to understand more:

$$h(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{c}^\top \mathbf{x} + b, \quad \mathbf{Q} \succ 0.$$

Replace the result into the origin function to find minimum, we have:

$$\mathbf{Q}^{-1} \mathbf{c} = \frac{1}{5} \begin{bmatrix} 2 \\ -16 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -3.2 \end{bmatrix}, \quad \mathbf{c}^\top (\mathbf{Q}^{-1} \mathbf{c}) = (-2)(0.4) + (-6)(-3.2) = 18.4.$$

$$\text{Thus, } h(\mathbf{x}^*) = -\frac{1}{2} \cdot 18.4 = -9.2.$$



# Exercise 4:

**Problem statement:** What changes when we perform stochastic gradient descent with momentum?

Stochastic Gradient Descent (SGD) updates parameters using noisy gradients from one sample (or a mini-batch).

- **Without Momentum:**

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \nabla_{\mathbf{x}} f(\mathbf{x}_t)$$

- **With Momentum:**

$$\begin{aligned} v_{t+1} &= \beta v_t - \nabla_{\mathbf{x}} f(\mathbf{x}_t) \\ \mathbf{x}_{t+1} &= \mathbf{x}_t + \eta v_{t+1} \end{aligned}$$

**Key changes:**

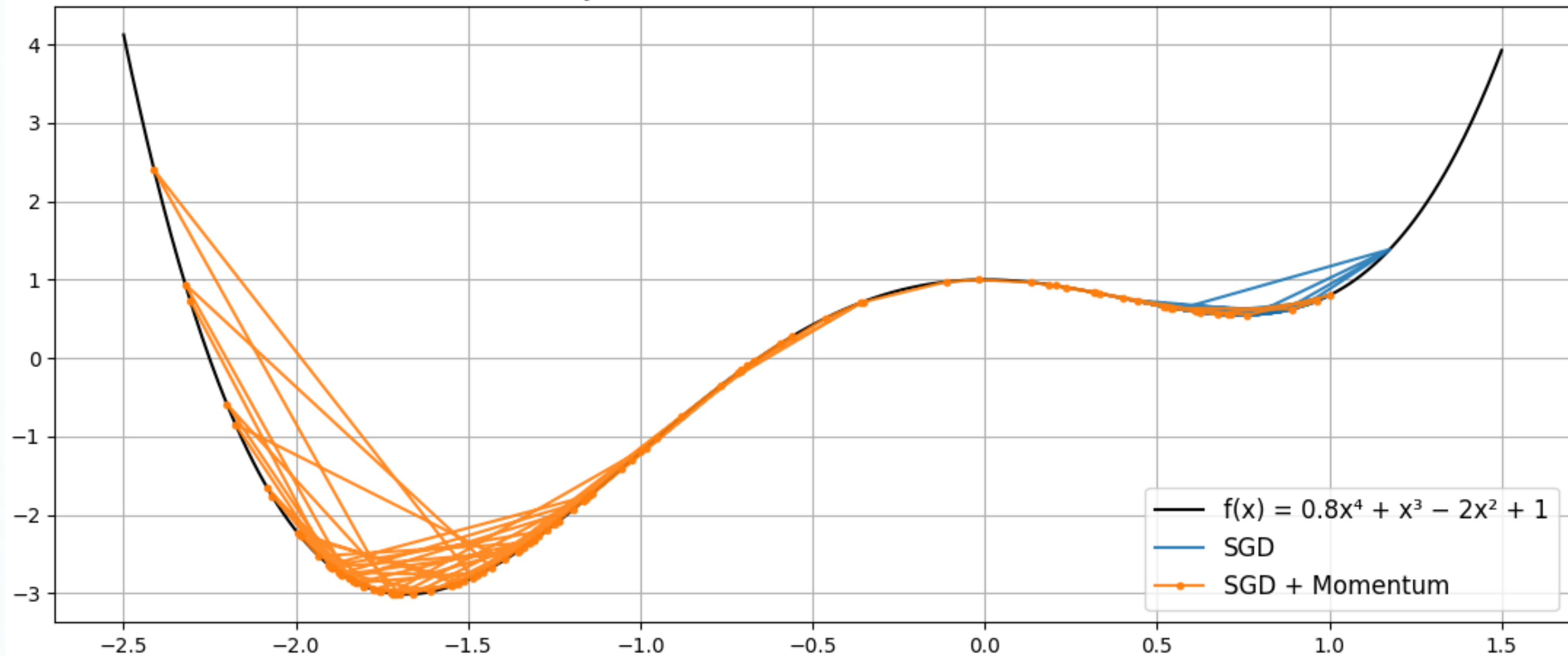
- Smoother updates - noise is filtered by the velocity term.
- Faster convergence, especially in shallow or flat regions.
- Reduced oscillation along steep directions.

# Exercise 4:

Consider function:

$$f(x) = 0.8x^4 + x^3 - 2x^2 + 1$$

SGD Trajectories with and without Momentum



# Exercise 4:

**Problem statement:** What happens when we use minibatch stochastic gradient descent with momentum?

- **Stochastic Gradient Descent (SGD):** updates parameters using one sample, so the gradient is very noisy.
  - **Mini-batch SGD:** uses a small batch (e.g., 16 or 32 samples), reducing noise while keeping updates fast.
  - **Momentum:** accumulates past gradients to smooth the trajectory and speed up convergence.
- So when we combine them: Each step uses the average gradient over a mini-batch, and Momentum accumulates these batch-gradients over time.

This produces (with momentum):

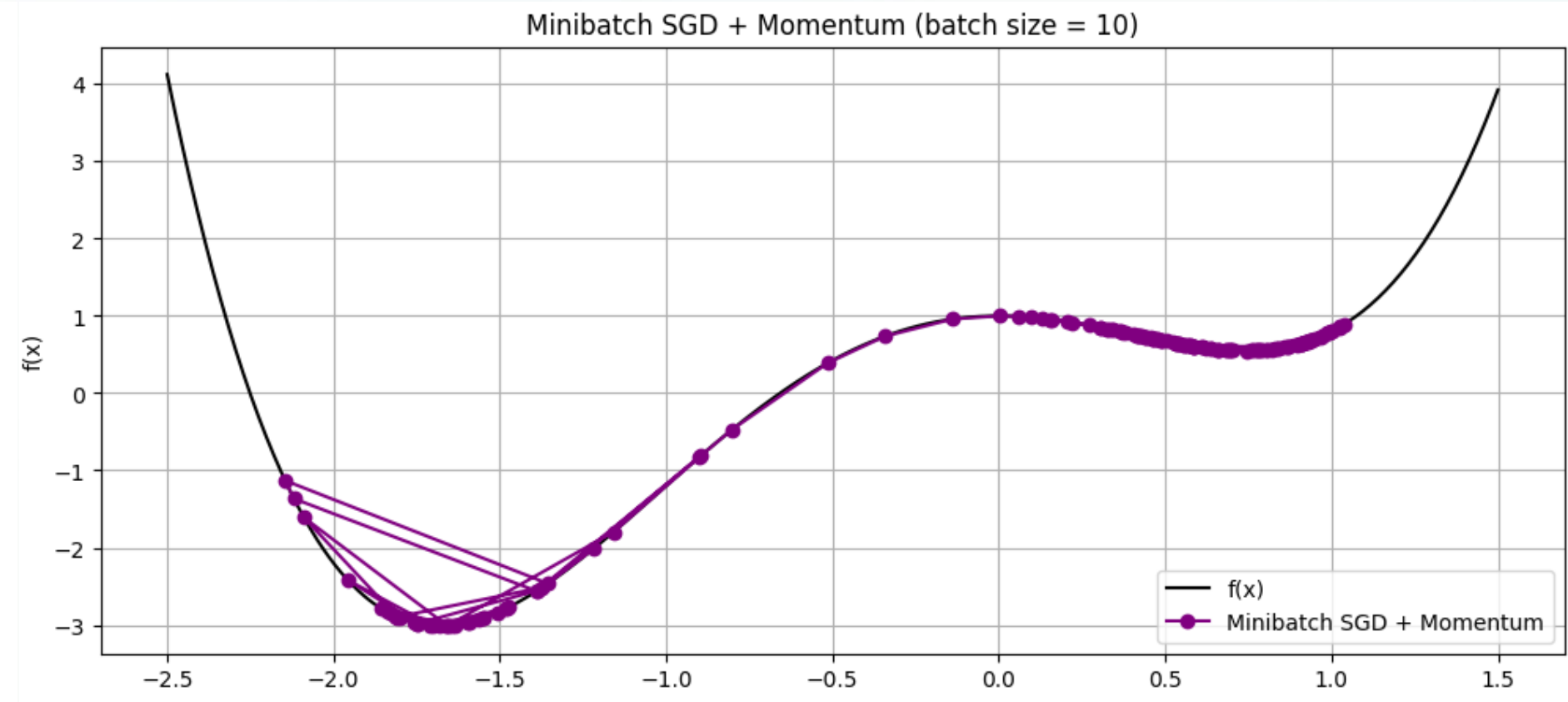
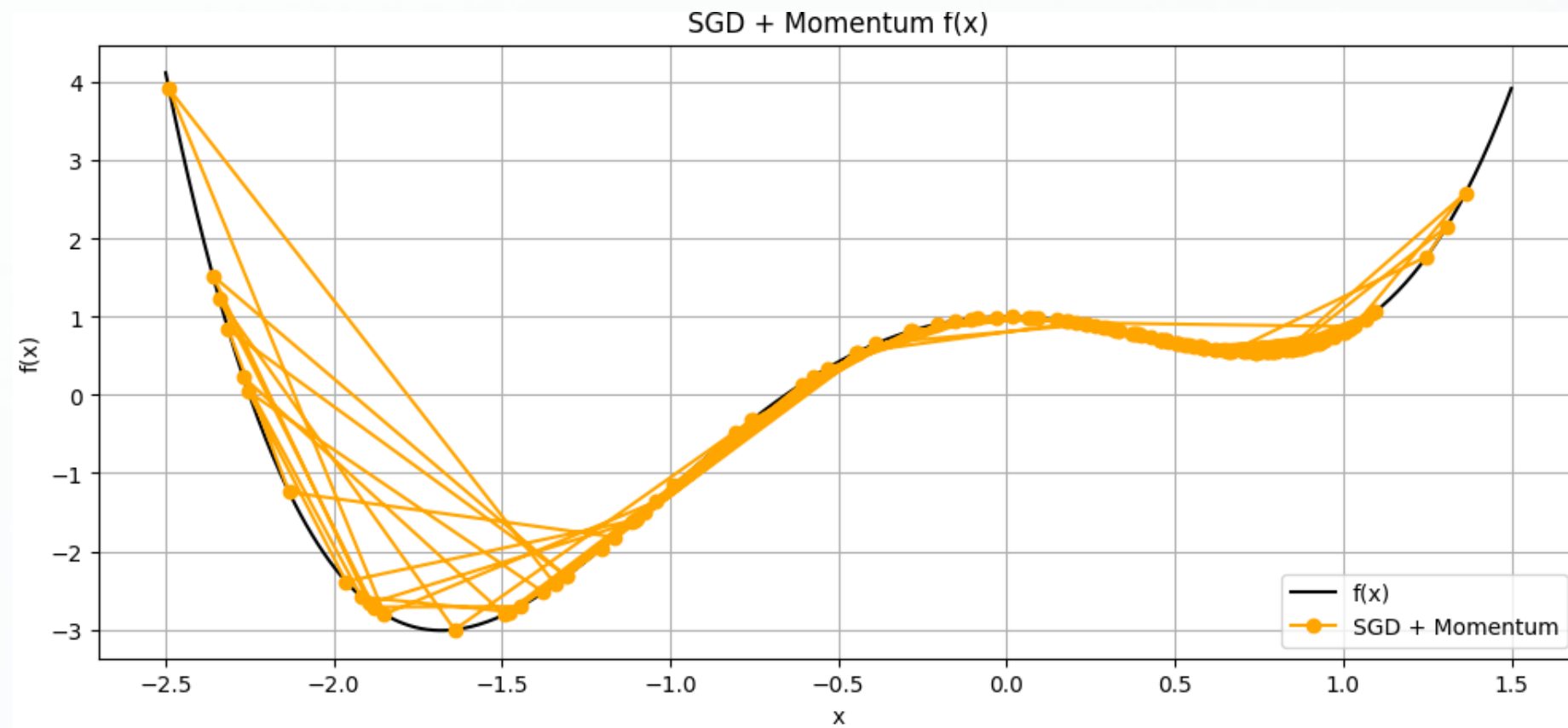
1. Faster convergence than plain mini-batch SGD
2. Lower variance in updates
3. A smoother path through the optimization landscape

$$v_{t+1} = \beta v_t + \frac{1}{|B|} \sum_{i \in B} \nabla_{\mathbf{x}} f(\mathbf{x}_t)$$
$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta v_{t+1}$$

# Exercise 4:

Consider function:

$$f(x) = 0.8x^4 + x^3 - 2x^2 + 1$$



## 06 Summary

- Momentum replaces the raw gradient with a leaky average of past gradients.
- It is desirable for both noise-free gradient descent and (noisy) stochastic gradient descent.
- The effective number of past gradients that influence each update is approximately:  $\frac{1}{1-\beta}$
- For convex quadratic functions, momentum can be analyzed exactly
- Momentum is easy to implement and widely supported. The only additional requirement is storing a velocity vector alongside the parameters.



# THANK YOU FOR LISTENING

