High Dimensional Fruits

Ben Lucas - October 26, 2016

The curse of dimensionality is a well known but surprisingly counterintuitive phenomena. Loosely formulated, the curse of dimensionality describes the fact that most of a high dimensional space is found toward the extremities of that space. We all have some experience of this being the case in low dimensions. For example, if we take a slice of pizza, the tip is eaten in only a single bite, but the crust takes six. There is more area on the edge of the slice than in the center.

Interestingly enough, as the number of dimensions the magnitude of this high dimensional skew toward the edges increases increases. If we made a spherical pizza (a terrible idea), we would find that this 3d pizza has a higher percentage of crust than the classic two dimensional pizza. To explore this trend in more detail, let us consider the how it would effect the composition of high dimensional fruits.

For a radially symmetric fruit[1], the hypervolume is purely a function of the radius R and the dimension n. More specifically

$$V_n(R) = k \cdot R^n \qquad \text{(Eq 1)}$$

Here k represents a proportionality constant related the dimension. This is given by the equation:

$$k = \frac{\pi^{(n/2)}}{\Gamma(n/2+1)} \qquad \text{(Eq 2)}$$

where Γ is Euler's Gamma. This gets complicated, but since we're only going to be looking at the ratios between the hyper-volumes of hyper-fruit with the same dimension, this term is a constant and will cancel when we look at the ratio of sections of a hyper-fruit.

From equation 1 we can easily calculate the hyper volume of a spherical hypershell of inner radius a and outer radius b as V_n shell(a,b) = V_nsphere(b) - V_nsphere(a) = k(b^n - a^n)

$$V_{n-shell}(a,b) = V_{n-sphere}(b) - V_{n-sphere}(a) = k \cdot (b^n - a^n) \qquad \text{(Eq 3)}$$

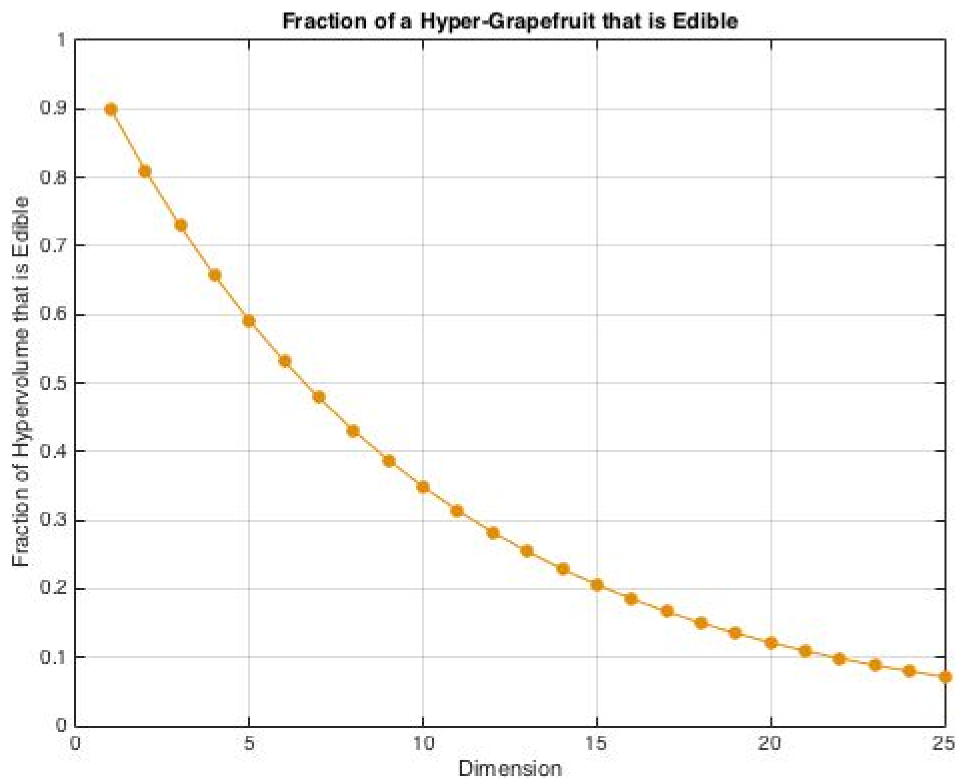
The hyper-grapefruit:

For starters let's consider the hyper-peel of a hyper-grapefruit. If we look at 3d grapefruit, we can see that it is composed of a thin shell (the peel) that surrounds an inner sphere of nutritious fruit.

Thus by equation 1, the ratio of edible hyper-volume V_{edible} to the total hyper-volume V_{total} for a hyper-grapefruit of fruit radius a and total (fruit + peel) radius b, can be represented as by the following equation:

$$\frac{V_{edible}}{V_{total}} = \frac{k(a^n)}{k(b^n)} = (a/b)^n \quad \text{(Eq 4)}$$

Based on a highly scientific scientific measurement[2], we can determine that the inner radius of a grapefruit is exactly 4.5 cm, while the outer radius is exactly 5.0 cm.
From this, we can plot the ratio of edible fruit vs dimension given by equation 4:



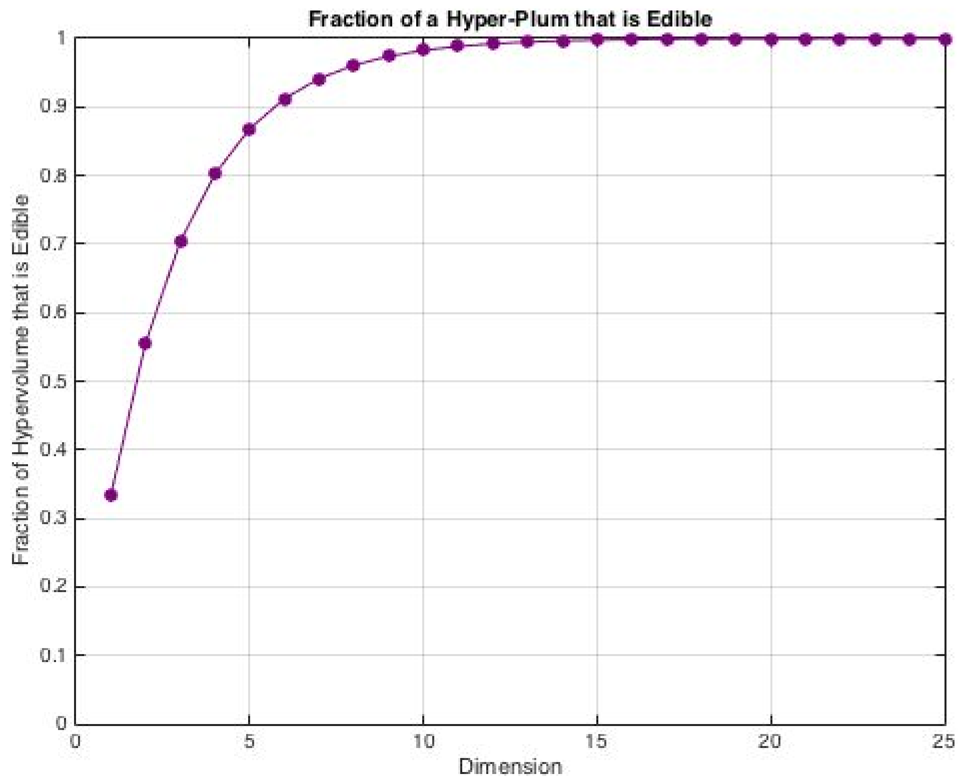
Since a/b <1, so as n-> \infty, the total fraction will decrease to zero very quickly. At 25 dimensions, our grape fruit is more than 90% hyper-peel by hyper-volume.

The hyper-plum:
The story in high dimension isn't all bad, for the sake of perspective let's consider a hyper-plum. This presents the exact opposite configuration as the grapefruit, where the fruit is a shell on that surrounds a spherical pit. By equations 3 and 1, the fraction of edible fruit can be given as:

$$\frac{V_{edible}}{V_{total}} = \frac{k(b^n - a^n)}{k(b^n)} = 1 - (a/b)^n \quad \text{(Eq 5)}$$

Since a/b < 1, we see the opposite trend, where the pit becomes negligible with increasing dimensions, and the fruit dominates the hypervolume.

Below, we plot the relationship given in equation 5 for a hyper-plum of pit radius = 1.5cm, and total radius = 3cm.

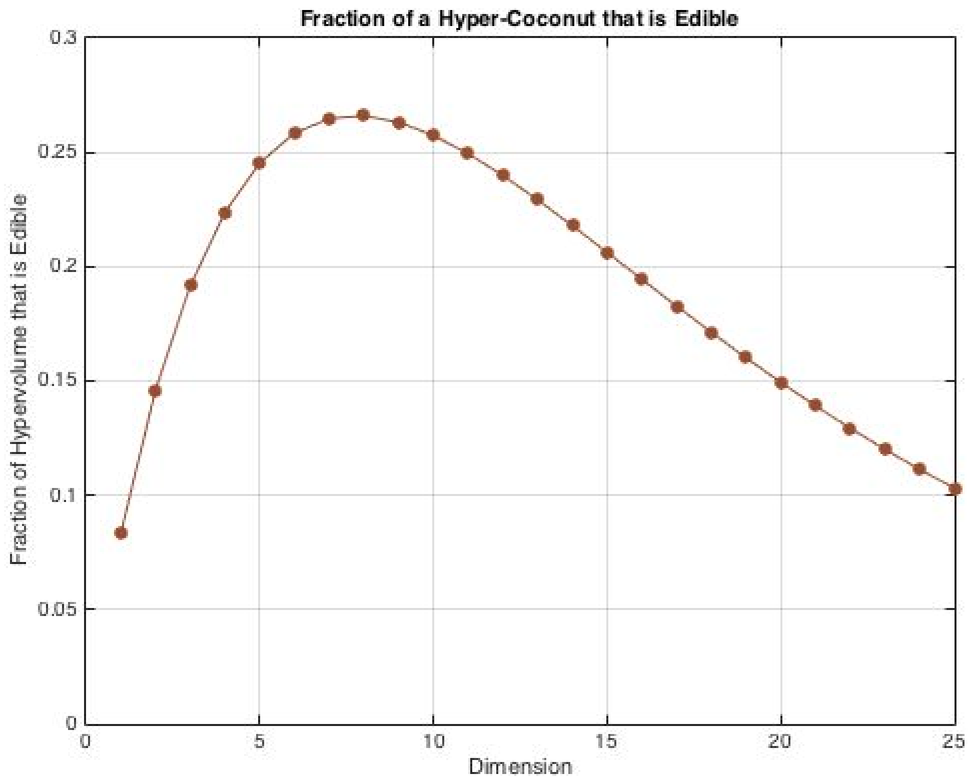


The hyper-coconut

Finally, let us turn to the case of the coconut. This consists of a central sphere of empty matter, a shell of fruit, with inner radius a, and outer radius b, as well as a hard shell of inedible husk with inner radius b and outer radius c. By equations 1 and 3, we can derive the following equation:

$$\frac{V_{edible}}{V_{total}} = \frac{k(b^n - a^n)}{k(c^n)} = (b/c)^n - (a/c)^n \qquad \text{(Eq 6)}$$

The behavior of this relationship is plotted below for a hyper-coconut with inner fruit radius a = 5.0cm, outer fruit radius b = 5.5cm, and outer husk radius c = 6.0cm.



As you can see, the behavior is more interesting in the case of a coconut: the curve increases for low dimensions and then decreases for high dimensions. The edible shell of coconut takes up an increasing fraction of the hyper-volume before diminishing to zero. This results in an optimal coconut in the 8th dimension.

Interestingly enough, this result closely follows a similar result regarding the hyper-surface area to hyper-volume ratio of a unit sphere [3], where the surface of volume ratio reaches is maximum value for a 7-dimensional sphere.

Conclusions:

In closing, we have seen that surprising patterns can appear in high dimensional spaces. As a general rule, it can be seen that the majority of a space is found away from the central point. This means that for more complicated systems, the mean value is less representative of the whole. This has deep implications for the reliability of inferences made in a high dimensional space. Finally, it is important to remember that effect of increasing dimension is not necessarily monotonic. As the hyper-coconut illustrates, the geometry of a high dimensional object can exhibit surprising patterns.

Notes:

[1] Note that we will only be considering roughly spherical fruits for the sake of simplicity. This captures the core idea with the least effort. It would be an interesting excursion to consider oblate spheroids in high dimension.

[2] All measurements are taken from fruit bought at lee's market with a sample size n = 1. Measurements were made using my hand, scaling that to the our best estimate of the hand size of the king of France circa 800 AD. This figure was then multiplied by a geometric factor related to the circumference of the earth. Finally, this was rounded to the nearest half centimeter to smooth out the high degree of imprecision involved in this method. It is worth noting that there are good reasons to suspect that the size and shape of fruits would not remain constant in higher dimensions, but this issue will be ignored in the present treatment.

[3] *Les Nombres Remarquables (Actualites Scientifiques et Industrielles)*, Le Lionnais, Francois, 1983, Hermann Books, Paris, France.

Addendum:

Computer Generated Projection of a 4 Dimensional Coconut: