Homework 4

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1. (Boyd & Vandenberghe, Exercise 4.3) The feasibility set for this problem is $\mathcal{F} = [-1,1]^3$. Since the objective function

$$f_0(x) = \frac{1}{2}x^T P x + q^T x + r$$

is differentiable and $x^\star = (1, 1/2, -1) \in \mathcal{F}$, $x^\star$ is optimal if and only if

$$\nabla f_0(x^\star)^T(x - x^\star) \geq 0 \quad \text{for every } x \in \mathcal{F}.$$

Since P is symmetric, we have

$$\nabla f_0(x^\star) = \frac{1}{2}(P + P^T)x^\star + q = Px^\star + q,$$

and so the optimality condition becomes

$$(Px^\star + q)^T (x - x^\star) \geq 0 \quad \text{for every } x \in \mathcal{F},$$

or

$$(Px^\star + q)^T x - (Px^\star + q)^T x^\star \geq 0 \quad \text{for every } x \in \mathcal{F}.$$

Using the given values for P and q, we get

$$\begin{aligned} (Px^\star + q)^T &= \begin{pmatrix} -1 & 0 & 2 \end{pmatrix} \\ (Px^\star + q)^T x^\star &= -3, \end{aligned}$$

Substituting these values into the optimality condition, we see that $x^\star$ is optimal if and only if

$$-x_1 + 2x_3 + 3 \geq 0 \quad \text{for every } x = (x_1, x_2, x_3) \in \mathcal{F}.$$

But since $\mathcal{F} = [-1,1]^3$, $x_i \in [-1,1]$ $(i = 1,2,3)$, and it follows easily that this condition is satisfied. Thus $x^\star$ is optimal.

2. (Boyd & Vandenberghe, Exercise 4.8)

(b) The feasible set for this problem is

$$\mathcal{F} = \{x \in \mathbb{R}^n : a^T x \leq b\},$$

and the objective function is $f_0(x) = c^T x$, which is differentiable. If $c = 0$, then any feasible value is optimal, so we will assume that $c \neq 0$. We have $\nabla f_0(x) = c$. Note that since $a \neq 0$, we can write $\mathbb{R}^n = V \oplus V^\perp$, where $V = \operatorname{span}\{a\}$. Thus we can write $c = \alpha_1 c_1 + \alpha_2 c_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $c_1 \in V$ and $c_2 \in V^\perp$ satisfy $\|c_1\|_2 = 1$ and $\|c_2\|_2 = 1$. In particular, we can take $c_1 = a/\|a\|_2$.

To determine an explicit solution, there are several cases to consider:

i. *If $\alpha_1 \geq 0$, then the problem is unbounded below.* To see this, let $x \in \mathcal{F}$ and $t \geq 0$. Then

$$\begin{aligned}
a^T(x - t\nabla f_0(x)) &= a^T(x - tc) \\
&= a^T x - t a^T c \\
&= a^T x - t a^T(\alpha_1 c_1 + \alpha_2 c_2) \\
&= a^T x - t\alpha_1 a^T c_1 \\
&= a^T x - t\alpha_1 \|a\|_2 \\
&\leq b
\end{aligned}$$

since $\alpha_1 \geq 0$, and so $x - t\nabla f_0(x) \in \mathcal{F}$. Moreover,

$$c^T(x - t\nabla f_0(x)) = c^T(x - tc) = c^T x - t c^T c \to -\infty \quad \text{as} \quad t \to \infty$$

Thus the problem is unbounded below in this case.

ii. *If $\alpha_2 \neq 0$, then the problem is unbounded below.* To see this, let $x \in \mathcal{F}$ and $t \geq 0$. Then

$$a^T(x - t\alpha_2 c_2) = a^T x \leq b,$$

so that $x - t\alpha_2 c_2 \in \mathcal{F}$. Moreover,

$$\begin{aligned}
c^T(x - t\alpha_2 c_2) &= \alpha_1 c_1^T(x - t\alpha_2 c_2) + \alpha_2 c_2^T(x - t\alpha_2 c_2) \\
&= \alpha_1 c_1^T x + \alpha_2 c_2^T x - t\alpha_2^2 c_2^T c_2 \\
&= c^T x - t\alpha_2^2 c_2^T c_2,
\end{aligned}$$

so that $c^T(x - t\alpha_2 c_2) \to -\infty$ as $t \to \infty$ since $\alpha_2 \neq 0$.

iii. *If $\alpha_1 < 0$ and $\alpha_2 = 0$, then the problem has a unique (finite) solution.* In this case $c = \alpha_1 c_1 = \alpha_1 a/\|a\|_2$. Note that for any $x \in \mathcal{F}$,

$$c^T x = \frac{a^T x}{\|a\|_2} \leq \frac{b}{\|a\|_2}.$$

Let $x^\star = ba/\|a\|_2$. Then $a^T x^\star = b$, so that $x^\star \in \mathcal{F}$, and

$$c^T x^\star = \frac{a^T x^\star}{\|a\|_2} = \frac{b}{\|a\|_2}.$$

Thus $c^T x^\star \leq c^T x$ for every $x \in \mathcal{F}$, so that $x^\star$ is optimal.

To summarize, the optimal value is given by

$$p^\star = \begin{cases} b/\|a\|_2 & \text{if } \alpha_1 < 0,\ \alpha_2 = 0 \\ -\infty & \text{otherwise} \end{cases}.$$

(c) The feasible set for this problem is

$$\mathcal{F} = \{x \in \mathbb{R}^n : l \preceq x \preceq u\} = \{x \in \mathbb{R}^n : l_i \leq x_i \leq u_i \text{ for every } 1 \leq i \leq n\}.$$

The objective function is $f_0(x) = c^T x$. We can optimize f_0 in each component x_i of the optimization variable x individually. Specifically, for every $1 \leq i \leq n$, we want to minimize $c_i x_i$ in x_i subject to the constraint $l_i \leq x_i \leq u_i$. We can therefore construct an optimal value $x^\star \in \mathcal{F}$ as follows: for every $1 \leq i \leq n$,

- if $c_i = 0$, any value $x_i^\star \in [l_i, u_i]$ is optimal;
- if $c_i < 0$, let $x_i^\star = u_i$;
- if $c_i > 0$, let $x_i^\star = l_i$.

The optimal value is therefore given by

$$p^\star = c^T x^\star = l^T c^+ + u^T c^-,$$

where $c^+ \in \mathbb{R}^n$ and $c^- \in \mathbb{R}^n$ are defined by $c_i^+ = \max\{c_i, 0\}$ and $c_i^- = \max\{-c_i, 0\}$, respectively, for every $1 \leq i \leq n$.

3. (Boyd & Vandenberghe, Exercise 4.9) The feasibility set for this problem is

$$\mathcal{F} = \{x \in \mathbb{R}^n : Ax \preceq b\}.$$

Let $y = Ax$, so that $x = A^{-1}y$. Then the constraint $Ax \preceq b$ becomes $y \preceq b$, and the objective function becomes

$$f_0(x) = c^T x = c^T A^{-1} y = (A^{-T}c)^T y = d^T y,$$

where $d = A^{-T}c$.

We can optimize f_0 in each component y_i of y individually. Specifically, for every $1 \leq i \leq n$, we want to minimize $d_i y_i$ subject to the constraint $y_i \leq b_i$.

First, note that if $d_i > 0$ for some $1 \leq i \leq n$, i.e., $d = A^{-T}c \not\preceq 0$, then the problem is unbounded below. To see this, let $y = (b_1, \ldots, b_i - t, \ldots, b_n)$, where $t \geq 0$. Then $y \in \mathcal{F}$, but

$$f_0(y) = d^T y = d^T b - d_i t \to -\infty \quad \text{as} \quad t \to \infty.$$

On the other hand, if $d_i \geq 0$ for every $1 \leq i \leq n$, i.e., $d = A^{-T}c \preceq 0$, we can construct an optimal value $y^\star \in \mathcal{F}$ as follows: for every $1 \leq i \leq n$,

- if $d_i = 0$, any value $y_i^\star \le b_i$ is optimal;
- if $d_i < 0$, let $y_i^\star = b_i$.

In this case, the optimal value is therefore

$$p^\star = d^T b = (A^{-T}c)^T b = c^T A^{-1} b$$

Thus, in general, the optimal value is given by

$$p^\star = \begin{cases} c^T A^{-1} b & \text{if } A^{-T}c \preceq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

4. (Boyd & Vandenberghe, Exercise 4.11)

   (a) An equivalent LP is

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & -(Ax-b) \preceq t\mathbf{1} \end{array},$$

   where $t \in \mathbb{R}$.

   Let $x \in \mathbb{R}^n$ be optimal for the original problem. Then $(x,t) \in \mathbb{R}^{n+1}$ is optimal for the LP above if $t = \|Ax - b\|_\infty$. Conversely, if we fix $x \in \mathbb{R}^n$ and optimize with respect to t in the LP, then we get $t = \|Ax - b\|_\infty$. If we then optimize in x, the resulting value of x is optimal for the original problem. The two problems are therefore equivalent.

   (b) An equivalent LP is

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \preceq t \\ & -(Ax-b) \preceq t \end{array},$$

   where $t \in \mathbb{R}^m$.

   Let $x \in \mathbb{R}^n$ be optimal for the original problem. Then $(x,t) \in \mathbb{R}^{n+m}$ is optimal for the LP above if t satisfies $t_i = |(Ax-b)_i|$ for every $1 \le i \le m$. Conversely, if we fix $x \in \mathbb{R}^n$ and optimize with respect to t in the LP, then t will satisfy $t_i = |(Ax-b)_i|$ for every $1 \le i \le m$. If we then optimize in x, the resulting value of x is optimal for the original problem. The two problems are therefore equivalent.

   (c) An equivalent LP is

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \preceq t \\ & -(Ax-b) \preceq t \\ & x \preceq \mathbf{1} \\ & -x \preceq \mathbf{1} \end{array},$$

   where $t \in \mathbb{R}^m$.

Let $x \in \mathbb{R}^n$ be optimal for the original problem. Then $(x, t) \in \mathbb{R}^{n+m}$ is optimal for the LP above if t satisfies $t_i = |(Ax - b)_i|$ for every $1 \le i \le m$. Conversely, if we fix $x \in \mathbb{R}^n$ and optimize with respect to t in the LP, then t will satisfy $t_i = |(Ax - b)_i|$ for every $1 \le i \le m$. If we then optimize in x, the resulting value of x is optimal for the original problem. The two problems are therefore equivalent.

(d) An equivalent LP is

$$\begin{array}{ll} \text{minimize} & 1^T t \\ \text{subject to} & Ax - b \preceq 1 \\ & -(Ax - b) \preceq 1 \\ & x \preceq t \\ & -x \preceq t \end{array},$$

where $t \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ be optimal for the original problem. Then $(x, t) \in \mathbb{R}^{2n}$ is optimal for the LP above if t satisfies $t_i = |x_i|$ for every $1 \le i \le n$. Conversely, if we fix $x \in \mathbb{R}^n$ and optimize with respect to t in the LP, then t will satisfy $t_i = |x_i|$ for every $1 \le i \le n$. If we then optimize in x, the resulting value of x is optimal for the original problem. The two problems are therefore equivalent.

(e) An equivalent LP is

$$\begin{array}{ll} \text{minimize} & 1^T t + s \\ \text{subject to} & Ax - b \preceq t \\ & -(Ax - b) \preceq t \\ & x \preceq s1 \\ & -x \preceq s1 \end{array},$$

where $t \in \mathbb{R}^m$ and $s \in \mathbb{R}$.

Let $x \in \mathbb{R}^n$ be optimal for the original problem. Then $(x, t, s) \in \mathbb{R}^{n+m+1}$ is optimal for the LP above if $s = \|x\|_\infty$ and t satisfies $t = |(Ax - b)_i|$ for every $1 \le i \le m$. Conversely, if we fix x and optimize with respect to t and s in the LP, then t will satisfy $t = |(Ax - b)_i|$ for every $1 \le i \le m$ and $s = \|x\|_\infty$. If we then optimize in x, the resulting value of x is optimal for the original problem. The two problems are therefore equivalent.

5. (Boyd & Vandenberghe, Exercise 4.25) First, note that

$$\begin{array}{ll} a^T x + b > 0 & \text{for every } x \in \mathcal{E}_i \ (1 \le i \le K) \\ a^T x + b < 0 & \text{for every } x \in \mathcal{E}_j \ (K + 1 \le j \le K + L) \end{array}$$

if and only if there exists $\epsilon > 0$ such that

$$\begin{array}{ll} a^T x + b \ge \epsilon & \text{for every } x \in \mathcal{E}_i \ (1 \le i \le K) \\ a^T x + b \le -\epsilon & \text{for every } x \in \mathcal{E}_j \ (K + 1 \le j \le K + L) \end{array}$$

If we write $\tilde{a} = a/\epsilon$ and $\tilde{b} = b/\epsilon$, then we obtain the equivalent conditions

$$\begin{aligned} &\tilde{a}^T x + \tilde{b} \geq 1 \quad \text{for every } x \in \mathcal{E}_i \ (1 \leq i \leq K) \\ &\tilde{a}^T x + \tilde{b} \leq -1 \quad \text{for every } x \in \mathcal{E}_j \ (K+1 \leq j \leq K+L) \end{aligned}$$

The feasibility problem in question can therefore be written as

$$\begin{array}{ll} \text{find} & (a,b) \in \mathbb{R}^{n+1} \\ \text{subject to} & a^T x + b \geq 1 \quad \text{for every } x \in \mathcal{E}_i \ (1 \leq i \leq K) \\ & a^T x + b \leq -1 \quad \text{for every } x \in \mathcal{E}_j \ (K+1 \leq j \leq K+L) \end{array}.$$

We want to express this problem as an SOCP feasibility problem. Let $1 \leq i \leq K$. We compute

$$\begin{aligned} \inf\{a^T x + b : x \in \mathcal{E}_i\} &= \inf\{a^T(P_i u + q_i) + b : \|u\|_2 \leq 1\} \\ &= b + a^T q_i + \inf\{a^T P_i u : \|u\|_2 \leq 1\} \\ &= b + a^T q_i + a^T P_i \left(-\frac{P_i^T a}{\|P_i^T a\|_2} \right) \\ &= b + q_i^T a - \|P_i a\|_2. \end{aligned}$$

Similarly, for $K+1 \leq j \leq K+L$, we compute

$$\sup\{a^T x + b : x \in \mathcal{E}_j\} = b + q_j^T a + \|P_j a\|_2.$$

The constraints in the original optimization problem given above are therefore equivalent to the conditions

$$\begin{aligned} &b + q_i^T a - \|P_i a\|_2 \geq 1 \quad (1 \leq i \leq K) \\ &b + q_j^T a + \|P_j a\|_2 \leq -1 \quad (K \leq j \leq K+L). \end{aligned}$$

The problem can therefore be written as the following SOCP feasibility problem:

$$\begin{array}{ll} \text{find} & (a,b) \in \mathbb{R}^{n+1} \\ \text{subject to} & b + q_i^T a - \|P_i a\|_2 \geq 1 \quad (1 \leq i \leq K) \\ & b + q_j^T a + \|P_j a\|_2 \leq -1 \quad (K \leq j \leq K+L). \end{array}.$$

6. (CVX problem) We solve the following LP using CVX:

$$\begin{array}{ll} \text{minimize} & 1^T u \\ \text{subject to} & x \succeq 0 \\ & Ax \preceq c^{\max} \\ & p_j x_j \geq u_j \quad (1 \leq j \leq n) \\ & p_j q_j + p_j^{\text{disc}}(x_j - q_j) \geq u_j \quad (1 \leq j \leq n) \end{array}.$$

The code to solve this LP is contained in the file `optimal_activity_levels.m`. Running this script, we obtain the following output:

```
Status: Solved
Optimal value (cvx_optval): +192.5

Optimal activity levels:
    4.0000
   22.5000
   31.0000
    1.5000

Associated revenues:
   12.0000
   32.5000
  139.0000
    9.0000

Total revenue: 192.5

Average price per unit:
    3.0000
    1.4444
    4.4839
    6.0000
```

From this output we see that among all activities, activities 2 and 3 have the highest optimal activity levels, along with the highest associated revenues. In particular, among all activities, activity 3 contributes by far the most to the total revenue.