Homework 3

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1. Here the model is of the form

$$e_c(i,j) = r(i,j)e(i,j),$$

where $e_c(i,j)$ is the corrupted image, $e(i,j)$ is the true (uncorrupted) image, and $r(i,j) = ai + bj + c$ is an affine function of the pixel coordinates (i,j), satisfying $|r(i,j)| \leq 1$. The goal is to estimate the parameters a, b, and c. In addition, we know that the true (uncorrupted) image $e(i,j)$ satisfies

$$e(i,j) = 255 \quad \text{for every } 1 \leq i < 50 \text{ and } 1 \leq j < 250.$$

We can use this information to find suitable values for a, b, and c. Specifically, we solve the (convex) optimization problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{49}\sum_{j=1}^{249}(e_c(i,j) - 255r(i,j))^2 \\ \text{subject to} & |r(i,j)| \leq 1. \end{array}$$

Finally, we compute an estimate $\hat{e}(i,j)$ of the true (uncorrupted) image by computing $\hat{e}(i,j) = e_c(i,j)/r(i,j)$ for all pixel coordinates (i,j).

2. We have raw data points X and Y, which consist of ordered collections $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$ of corresponding x- and y-coordinates, respectively.

First, we divide these collections into M disjoint segments of K coordinates each: $X_1, \ldots, X_M$ of x-coordinates and $Y_1, \ldots, Y_M$ of y-coordinates, where $Y_i = (y_{i1}, \ldots, y_{iK})$ is the collection of y-coordinates associated with the x-coordinates $X_i = (x_{i1}, \ldots, x_{iK})$. The reason for segments of equal length K is to simplify the implementation.

For each of the segments X_i and Y_i we want to fit cubic polynomials

$$\begin{aligned} x_i(t) &= a_{i3}t^3 + a_{i2}t^2 + a_{i1}t + a_{i0} \\ y_i(t) &= b_{i3}t^3 + b_{i2}t^2 + b_{i1}t + b_{i0}, \end{aligned}$$

respectively, each of which is parameterized by a variable $t \in [0,1]$. For each segment we fit the polynomial via least squares, subject to second-order smoothness constraints.

Specifically, we require that, for every $1 \leq i < M$,

$$\begin{aligned} x_i(1) &= x_{i+1}(0) & & & y_i(1) &= y_{i+1}(0) \\ x_i'(1) &= x_{i+1}'(0) & &\text{and} & y_i'(1) &= y_{i+1}'(0) \\ x_i''(1) &= x_{i+1}''(0) & & & y_i''(1) &= y_{i+1}''(0). \end{aligned}$$

Written in terms of the polynomial coefficients a_{ij} and b_{ij}, these constraints become

$$\begin{aligned} a_{i3} + a_{i2} + a_{i1} + a_{i0} &= a_{i+1,0} \\ 3a_{i3} + 2a_{i2} + a_{i1} &= a_{i+1,1} \\ 6a_{i3} + 2a_{i2} &= 2a_{i+1,2} \end{aligned} \tag{1}$$

$$\begin{aligned} b_{i3} + b_{i2} + b_{i1} + b_{i0} &= b_{i+1,0} \\ 3b_{i3} + 2b_{i2} + b_{i1} &= b_{i+1,1} \\ 6b_{i3} + 2b_{i2} &= 2b_{i+1,2} \end{aligned} \tag{2}$$

for every $1 \leq i < M$.

We therefore want to solve two (convex) optimization problems: one to compute the coefficients a_{ij} and one to compute the coefficients b_{ij}, subject to the constraints (1) and (2), respectively. Let $0 = t_1 < t_2 < \cdots < t_K = 1$ be equally spaced elements of $[0, 1]$. To compute the coefficients a_{ij}, we solve the problem

$$\begin{aligned} &\text{minimize} \ \sum_{i=1}^{M} \sum_{j=1}^{K} (x_{ij} - x_i(t_j))^2 \\ &\text{subject to (1).} \end{aligned}$$

To compute the coefficients b_{ij}, we solve the problem

$$\begin{aligned} &\text{minimize} \ \sum_{i=1}^{M} \sum_{j=1}^{K} (y_{ij} - y_i(t_j))^2 \\ &\text{subject to (2).} \end{aligned}$$

Thus we obtain a collection of curves $(x_i(t), y_i(t))$, $1 \leq i \leq M$, in the plane that together form a cubic spline approximating the raw data points X and Y.