Homework 2

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1. (Boyd & Vandenberghe, Exercise 2.12)

   (e) This set is not necessarily convex. As a counterexample, define the sets S and T by $S = (-\infty, -2] \cup [2, \infty)$ and $T = [-1, 1]$. Then the set

   $$\{x \in \mathbb{R} : \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\} = (-\infty, -3/2] \cup [3/2, \infty)$$

   is not convex.

   (f) For every $s \in S_2$, define the set

   $$C_s = \{x \in \mathbb{R}^n : x + s \in S_1\}.$$

   To see that these sets are convex, let $s \in S_2$, let $x_1, x_2 \in C_s$, and let $0 \leq \theta \leq 1$. Then since $x_1 + s, x_2 + s \in S_1$, convexity of S_1 implies that

   $$\theta x_1 + (1-\theta)x_2 + s = \theta(x_1 + s) + (1-\theta)(x_2 + s) \in S_1.$$

   Thus $\theta x_1 + (1-\theta)x_2 \in C_s$, so that C_s is convex. Therefore

   $$\{x \in \mathbb{R}^n : x + S_2 \subseteq S_1\} = \bigcap_{s \in S_2} C_s$$

   is convex, since set intersection preserves convexity.

   (g) Let

   $$C = \{x \in \mathbb{R}^n : \|x - a\|_2 \leq \theta \|x - b\|_2\}.$$

   Then $x \in C$ if and only if

   $$(x-a)^T(x-a) \leq \theta^2(x-b)^T(x-b),$$

   or equivalently (after rearrangement),

   $$(1-\theta^2)x^T x + 2(\theta^2 b - a)^T x + a^T a - \theta^2 b^T b \leq 0.$$

   Thus, letting $A = (1-\theta^2)I$, $\beta = 2(\theta^2 b - a)$, and $\gamma = a^T a - \theta^2 b^T b$, the set in question becomes

   $$C = \{x \in \mathbb{R}^n : x^T A x + \beta^T x + \gamma \leq 0\}.$$

   Thus, since $A \in S_+^n$, Exercise 2.10 (a) implies that C is convex.

2. (Boyd & Vandenberghe, Exercise 3.20)

   (c) Recall from Exercise 3.18 (a) that the map $g(X) = \mathrm{Tr}(X^{-1})$ on S^m_{++} is convex. In addition, the map $h(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ on $\mathbb{R}^n$ is affine. The map f in question is the composition $f = g \circ h$, with domain $\mathrm{dom}\, f = \{x \in \mathbb{R}^n : h(x) \in \mathrm{dom}\, g\}$. A composition of this form is convex, and therefore f is convex.

3. (Boyd & Vandenberghe, Exercise 3.21)

   (a) For every $1 \leq i \leq k$, define the map $f_i(x) = \|A^{(i)}x - b^{(i)}\|$ on $\mathbb{R}^m$. Since norms are convex, and each of the maps f_i is the composition of a norm with an affine map, we see that each f_i is convex. Since the map f in question is the pointwise maximum of $f_1, \ldots, f_k$, we conclude that f itself is convex.