Homework 6

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Section 5.1

3. (b) (i) We compute

$$\begin{aligned}
\det(tI_3 - A) &= \det \begin{pmatrix} t & 2 & 3 \\ 1 & t-1 & 1 \\ -2 & -2 & t-5 \end{pmatrix} \\
&= t\det \begin{pmatrix} t-1 & 1 \\ -2 & t-5 \end{pmatrix} - 2\det \begin{pmatrix} 1 & 1 \\ -2 & t-5 \end{pmatrix} + 3\det \begin{pmatrix} 1 & t-1 \\ -2 & -2 \end{pmatrix} \\
&= t^3 - 6t^2 + 11t - 6 = (t-3)(t-2)(t-1).
\end{aligned}$$

By setting this expression equal to zero and solving for t we find the eigenvalues of A to be $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

(ii) *Eigenvectors corresponding to* λ_1:

Let

$$B_1 = A - \lambda_1 I_3 = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}.$$

Through a series of elementary row operations, we find that the homogeneous system of linear equations defined by B_1 is equivalent to the homogeneous system defined by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solution set of this system is simply $\ker(L_{B_1})$. Thus we find that

$$\ker(L_{B_1}) = \left\{ \begin{pmatrix} -x \\ -x \\ x \end{pmatrix} : x \in \mathbb{R} \right\}$$

In particular, $v \in \mathbb{R}^3$ is an eigenvector of A corresponding to λ_1 if and only if

$$v = \begin{pmatrix} -x \\ -x \\ x \end{pmatrix} \text{ for some } x \in \mathbb{R}^3, x \neq 0.$$

Eigenvectors corresponding to λ_2:

Let

$$B_2 = A - \lambda_2 I_3 = \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}.$$

Through the same sort of computations as performed above, we find that

$$\ker(L_{B_2}) = \left\{ \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

In particular, $v \in \mathbb{R}^3$ is an eigenvector of A corresponding to λ_2 if and only if

$$v = \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} \text{ for some } x \in \mathbb{R}^3, x \neq 0.$$

Eigenvectors corresponding to λ_3:

Let

$$B_3 = A - \lambda_3 I_3 = \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Through the same sort of computations as performed above, we find that

$$\ker(L_{B_3}) = \left\{ \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

In particular, $v \in \mathbb{R}^3$ is an eigenvector of A corresponding to λ_3 if and only if

$$v = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} \text{ for some } x \in \mathbb{R}^3, x \neq 0.$$

(iii) Let

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{and } v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Then for each $1 \leq i \leq 3$, v_i is an eigenvector of A corresponding to the eigenvalue λ_i. Let $\beta' = \{v_1, v_2, v_3\}$. It is a simple matter to check that β' is linearly independent, and since β' contains precisely $\dim(\mathbb{R}^3) = 3$ elements, it is therefore an ordered basis for $\mathbb{R}^3$ (by part (b) of Cor. 2 to Thm. 1.10).

(iv) Let $\beta = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ be the standard ordered basis for $\mathbb{R}^3$. Consider the linear operator $L_A : \mathbb{R}^3 \to \mathbb{R}^3$. By definition, $[L_A]_\beta = A$. Furthermore, since by definition the eigenvectors of A are precisely those of L_A (see pg. 246), we have

$$[L_A]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

By Thm. 2.23, $[L_A]_{\beta'} = Q^{-1}[L_A]_\beta Q$, where Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Since

$$\begin{aligned} v_1 &= (-1)\epsilon_1 + (-1)\epsilon_2 + 1\epsilon_3 \\ v_2 &= 1\epsilon_1 + (-1)\epsilon_2 + 0\epsilon_3 \\ v_3 &= 1\epsilon_1 + 0\epsilon_2 + (-1)\epsilon_3, \end{aligned}$$

we find that

$$Q = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Finally, if we let $D = [L_A]_{\beta'}$, we have the relation $D = Q^{-1}AQ$, as desired.

20. By definition the characteristic polynomial f of A is given by $f(t) = \det(A - tI_n)$. We therefore have

$$f(t) = \det(A - tI_n) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

Substituting $t = 0$ into this expression we see that $f(0) = \det(A) = a_0$, as desired. Since A is invertible if and only $\det(A) \neq 0$ by the Cor. to Thm. 4.7, we therefore see that A is invertible if and only if $a_0 \neq 0$.

22. (a) Write

$$g(t) = \sum_{k=0}^{n} a_k t^k.$$

Then

$$g(T)(x) = \left(\sum_{k=0}^{n} a_k T^k\right)(x) = \sum_{k=0}^{n} a_k T^k(x), \tag{1}$$

where for any $k \geq 0$, T^k denotes T composed with itself k times (T^0 is defined to be the identity map $I_V : V \to V$). Note that $T^0(x) = I_V(x) = x = \lambda^0 x$. If we assume that $T_k(x) = \lambda^k x$ for some $k > 0$, then

$$T^{k+1}(x) = T(T^k(x)) = T(\lambda^k x) = \lambda^k T(x) = \lambda^k(\lambda x) = \lambda^{k+1}x.$$

Thus $T^k(x) = \lambda^k x$ for every $k \geq 0$ by induction. In particular (1) becomes

$$g(T)(x) = \sum_{k=0}^{n} a_k T^k(x) = \sum_{k=0}^{n} a_k \lambda^k x = \left(\sum_{k=0}^{n} a_k \lambda^k\right) x = g(\lambda)x.$$

Section 5.2

4. Let $A \in M_{n\times n}(F)$, and suppose that A has n distinct eigenvalues. Note that the eigenvalues and eigenvectors of A are defined to be those of the linear operator $L_A : F^n \to F^n$, and that A is said to be diagonalizable if and only if L_A is diagonalizable (see pgs. 245-246). Thus by assumption, L_A has n distinct eigenvalues, and since $\dim(F^n) = n$, L_A is diagonalizable by the Cor. to Thm. 5.5. This means that A is diagonalizable.

9. (a) Let $n = \dim(V)$ and let $A = [T]_\beta$. By definition, the characteristic polynomial f of T is given by $f(t) = \det(A - tI_n)$. By assumption, A is an upper triangular matrix, and thus so is $(A - tI_n)$. The determinant of $(A - tI_n)$ is therefore simply the product of its diagonal entries $(A_{ii} - t)$, $1 \leq i \leq n$. Thus we have the following:

$$f(t) = \det(A - tI_n) = \prod_{i=1}^{n}(A_{ii} - t) = \prod_{i=1}^{n}(-1)(t - A_{ii}) = (-1)^n \prod_{i=1}^{n}(t - A_{ii}).$$

We therefore see that the characteristic polynomial f of T splits.

12. (a) Note that since T is invertible, T does not have zero as an eigenvalue, and that $\lambda \neq 0$ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1} (by exercise 8 of section 5.1 – cited in the statement of this problem). Let λ be an eigenvalue of T. Then $\lambda \neq 0$, so that λ^{-1} exists, and λ^{-1} is an eigenvalue of T^{-1}. Let E_λ denote the eigenspace of T corresponding to λ, and let $E^{-1}_{\lambda^{-1}}$ denote the eigenspace of T^{-1} corresponding to λ^{-1}. Then since for any $x \in V$, $T(x) = \lambda x$ if and only if $T^{-1}(x) = \lambda^{-1}x$, we have the following:

$$E_\lambda = \{x \in V : T(x) = \lambda x\} = \{x \in V : T^{-1}(x) = \lambda^{-1}x\} = E^{-1}_{\lambda^{-1}}.$$

(b) Suppose T is diagonalizable, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. For each $1 \leq i \leq k$, let β_i be an ordered basis for the eigenspace E_{λ_i}. Then by Thm. 5.9, $\beta = \beta_1 \cup \cdots \cup \beta_k$ is an ordered basis for V (with the natural ordering – see the footnote on pg. 268) consisting of eigenvectors of T. But since $E_{\lambda_i} = E^{-1}_{\lambda_i^{-1}}$ for every $1 \leq i \leq k$, β is also an ordered basis for V consisting of eigenvectors of T^{-1}. Thus $[T^{-1}]_\beta$ is a diagonal matrix, so that T^{-1} is diagonalizable.