Homework 4

Benjamin Noland

Section 2.6

10. (a) First, define polynomials $p_0, \ldots, p_n \in V$ as follows:

$$p_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - c_i}{c_j - c_i} \quad (0 \leq j \leq n).$$

Then for any $0 \leq j \leq n$ and $0 \leq k \leq n$, we have:

$$p_j(c_k) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{c_k - c_i}{c_j - c_i} = \delta_{jk}.$$

Now let $a_0, \ldots, a_n \in F$ be scalars such that

$$0 = \sum_{k=0}^{n} a_k f_k$$

(here 0 denotes the zero linear functional on V). Then for any $0 \leq j \leq n$,

$$0 = \sum_{k=0}^{n} a_k f_k(p_j) = \sum_{k=0}^{n} a_k p_j(c_k) = \sum_{k=0}^{n} a_k \delta_{jk} = a_j.$$

Therefore $a_0 = \cdots = a_n = 0$, so that the set $\beta = \{f_0, \ldots, f_n\}$ is linearly independent. Since V is finite-dimensional with $\dim(V) = n + 1$, V^* is also finite-dimensional with $\dim(V^*) = n + 1$ (see the remark at the bottom of pg. 119). Thus β is a basis for V^* by part (b) of Cor. 2 to Thm. 1.10.

(b) Since $\beta^* = \{f_0, \ldots, f_n\}$ is an ordered basis for V^* by part (a), the Cor. to Thm. 2.26 implies the existence of an ordered basis $\beta = \{p_0, \ldots, p_n\}$ for V which has β^* as its dual. In particular, this means we have:

$$\delta_{ij} = f_j(p_i) = p_i(c_j) \quad (0 \leq i \leq n, 0 \leq j \leq n).$$

To see that $p_0, \ldots, p_n$ are the unique polynomials in V with the required property, suppose that for each $1 \leq i \leq n$ there exists another polynomial $q_i \in V$ satisfying

$$q_i(c_j) = \delta_{ij} \quad (0 \leq j \leq n).$$

Since β is a basis for V, there exist scalars $a_0, \ldots, a_n \in F$ such that

$$q_i = \sum_{k=0}^{n} a_k p_k. \tag{1}$$

Then for any $0 \leq j \leq n$, we have:

$$\delta_{ij} = q_i(c_j) = \sum_{k=0}^{n} a_k p_k(c_j) = \sum_{k=0}^{n} a_k \delta_{jk} = a_j.$$

Plugging these values for $a_0, \ldots, a_n$ back into (1), we get

$$q_i = \sum_{k=0}^{n} \delta_{ik} p_k = p_i.$$

Thus the polynomials $p_0, \ldots, p_n$ are the unique polynomials in V with the required property.

(c) Let $\beta = \{p_0, \ldots, p_n\}$ be the ordered basis described in part (b). Then for any $q \in V$ there exist unique scalars $b_0, \ldots b_n \in F$ such that

$$q = \sum_{i=0}^{n} b_i p_i \tag{1}$$

For any $0 \leq j \leq n$, we have:

$$q(c_j) = \sum_{i=0}^{n} b_i p_i(c_j) = \sum_{i=0}^{n} b_i \delta_{ij} = b_j.$$

Thus we may write (1) as

$$q = \sum_{i=0}^{n} q(c_i) p_i.$$

In particular, if q is the unique polynomial in V satisfying

$$q = \sum_{i=0}^{n} a_i p_i, \tag{2}$$

for the given scalars $a_0, \ldots, a_n$, then $q(c_i) = a_i$ $(0 \leq i \leq n)$. Thus there exists a unique polynomial in $q \in V$ with the required property, given by (2).

(d) I already deduced this in part (c).

(e) Let $p \in V$. By the result of part (c), we can write

$$p(x) = \sum_{i=0}^{n} p(c_i) p_i(x) \quad (x \in F)$$

In particular, if $V = P_n(\mathbb{R})$, we have:

$$\int_a^b p(t)\,dt = \int_a^b \sum_{i=0}^{n} p(c_i) p_i(t)\,dt = \sum_{i=0}^{n} p(c_i) \int_a^b p_i(t)\,dt = \sum_{i=0}^{n} p(c_i) d_i.$$

Additional problem

We need to show that $(F[t])^* \cong F[[x]]$, but we need a lemma first.

Lemma: Let V and W be vector spaces over the field F, let $\beta = \{v_i\}_{i \in I}$ be a basis (not necessarily finite) for V, and let $\{w_i\}_{i \in I}$ be a collection of vectors in W. Then there exists a unique linear transformation $T : V \to W$ satisfying $T(v_i) = w_i$ for every $i \in I$.

Proof: Let $x \in V$. Then there exist vectors $v_{i_1}, \ldots, v_{i_n} \in \beta$ and unique scalars $a_1, \ldots, a_n \in F$ such that

$$x = \sum_{k=1}^{n} a_k v_{i_k}. \tag{1}$$

(Note that if $x \neq 0$ the vectors $v_{i_1}, \ldots, v_{i_n}$ are unique as well). Define a map $T : V \to W$ by

$$T(x) = \sum_{k=1}^{n} a_k w_{i_k}.$$

(Note that if $x = 0$, then $a_1 = \cdots = a_n = 0$ regardless of the values of the vectors $v_{i_1}, \ldots, v_{i_n}$, so that $T(x) = 0$ unambiguously. Hence this map is well-defined). Thus we see that $T(v_i) = w_i$ for every $i \in I$, as desired. To see that T is a linear transformation, let $x, y \in V$ and $c \in F$. Then there exist vectors $v_{i_1}, \ldots, v_{i_n}, v_{i_{n+1}}, \ldots, v_{i_{n+m}} \in \beta$ and unique scalars $a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m} \in F$ satisfying

$$x = \sum_{k=1}^{n} a_k v_{i_k} \quad \text{and} \quad y = \sum_{k=1}^{m} a_{n+k} v_{i_{n+k}}.$$

Therefore

$$T(cx + y) = T\left(\sum_{k=1}^{n} c a_k v_{i_k} + \sum_{k=1}^{m} a_{n+k} v_{i_{n+k}}\right) = \sum_{k=1}^{n} c a_k w_{i_k} + \sum_{k=1}^{m} a_{n+k} w_{i_{n+k}} = cT(x) + T(y),$$

so that T is a linear transformation. Finally, to see that T is unique, let $U : V \to W$ be a linear transformation such that $U(v_i) = w_i$ for every $i \in I$. Let $x \in V$, with

$$x = \sum_{k=1}^{n} a_k v_{i_k},$$

as in (1). Then

$$U(x) = \sum_{k=1}^{n} a_k U(v_{i_k}) = \sum_{k=1}^{n} a_k w_{i_k} = T(x).$$

Thus T is unique, completing the proof.

Now for the main problem. Recall that the set $\beta = \{1, t, t^2, t^3, \ldots\}$ is a basis for $F[t]$. Define a map $\psi : (F[t])^* \to F[[x]]$ by

$$\psi(f) = \sum_{k=0}^{\infty} f(t^k) x^k \quad (f \in (F[t])^*).$$

Now let $f, g \in (F[t])^*$ and $c \in F$. Then

$$\psi(cf+g) = \sum_{k=0}^{\infty}(cf+g)(t^k)x^k = \sum_{k=0}^{\infty}[cf(t^k)+g(t^k)]x^k = c\sum_{k=0}^{\infty} f(t^k)x^k + \sum_{k=0}^{\infty} g(t^k)x^k = c\psi(f)+\psi(g).$$

Thus ψ is a linear transformation.

To see that ψ is injective, let $f, g \in (F[t])^*$, with $f \neq g$. Then by the lemma there exists a basis element $t^m \in \beta$ such that $f(t^m) \neq g(t^m)$. It follows that

$$\psi(f) = \sum_{k=0}^{\infty} f(t^k)x^k \neq \sum_{k=0}^{\infty} g(t^k)x^k = \psi(g).$$

Thus ψ is injective. To see that ψ is surjective, let $\sum_{k=0}^{\infty} a_k x^k \in F[[x]]$. Then by the lemma there exists $f \in (F[t])^*$ such that $f(t^k) = a_k$ for every $k \geq 0$. Therefore

$$\psi(f) = \sum_{k=0}^{\infty} f(t^k)x^k = \sum_{k=0}^{\infty} a_k x^k.$$

Thus ψ is surjective. Therefore ψ is bijective, and hence an isomorphism. Thus $(F[t])^* \cong F[[x]]$, as desired.

Section 2.7

11. Let $1 \leq i \leq k$ and $0 \leq j \leq n_i - 1$. We need to show that $t^j e^{c_i t}$ is in the solution space of the differential equation in question. Note that by way of the Lemma to Thm. 2.34,

$$(D - c_i I)^{n_i}(t^j e^{c_i t}) = 0.$$

Thus if $p(t) = (x-c_1)^{n_1}\cdots(x-c_k)^{n_k}$ (i.e., p is the auxiliary polynomial of the differential equation in question), then

$$p(D)(t^j e^{c_i t}) = (D-c_1 I)^{n_1}\cdots(D-c_i I)^{n_i}\dots(D-c_k I)^{n_k}(t^j e^{c_i t}) = 0.$$

Thus $t^j e^{c_i t}$ is in the solution space. In particular, the set

$$\beta_k = \{e^{c_1 t}, te^{c_1 t}, \dots, t^{n_1 - 1}e^{c_1 t}, \dots, e_{c_k t}, te^{c_k t}, \dots, t^{n_k - 1}e^{c_k t}\}$$

is contained in the solution space.

By the Cor. to Thm. 2.32 and part (b) of Cor. 2 to Thm. 1.10, we need only verify that β_k is linearly independent to show that it is a basis for the solution space (we are dealing with a differential equation of order $n = n_1 + \cdots + n_k$, and β_k contains precisely n elements). I will proceed by induction on k. When $k = 1$, the result is immediate from the Lemma to Thm. 2.34. Now suppose β_{k-1} is linearly independent for some $k > 1$. Let n_k be a positive integer and c_k a complex number distinct from $c_1, \dots, c_{k-1}$. We want to show that the set

$$\beta_k = \beta_{k-1} \cup \{e^{c_k t}, te^{c_k t}, \dots, t^{n_k - 1}e^{c_k t}\}$$

is linearly independent. Let a_{ij} $(1 \le i \le k, 0 \le j \le n_i - 1)$ be scalars such that

$$0 = \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t}. \tag{1}$$

First, we want to show that $a_{km} = 0$ for every $0 \le m \le n_k - 1$. I will proceed by (strong) induction. Assume that $a_{km} = 0$ for every $1 \le m < n_k - 1$. We want to show that $a_{k,(n_k-1)} = 0$ as well. Define the differential operator

$$\widetilde{D} = (D - c_k I)^{n_k - 1} (D - c_{k-1} I)^{n_{k-1}} \cdots (D - c_1 I)^{n_1}.$$

From some computations done in class, we know the following:

- $(D - c_i I)^{n_i} (t^j e^{c_i t}) = 0$ $(1 \le i \le k-1, 0 \le j \le n_i - 1)$
- $(D - c_i I)^{n_i} (e^{c_k t}) = (c_k - c_i)^{n_i} e^{c_k t}$ $(1 \le i \le k-1)$
- $(D - c_k I)^{n_k - 1} (t^{n_k - 1} e^{c_k t}) = (n_k - 1)!\, e^{c_k t}$.

It follows that $\widetilde{D}(t^j e^{c_i t}) = 0$, and

$$\widetilde{D}(t^{n_k-1} e^{c_k t}) = (c_k - c_{k-1})^{n_{k-1}} \cdots (c_k - c_1)^{n_1} (n_k - 1)!\, e^{c_k t}.$$

Since $a_{km} = 0$ for every $1 \le m < n_k - 1$ by assumption, (1) reduces to

$$0 = \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t} = \sum_{i=1}^{k-1} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t} + \sum_{j=0}^{n_k-1} a_{kj} t^j e^{c_k t} = \sum_{i=1}^{k-1} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t} + a_{k,(n_k-1)} t^{n_k-1} e^{c_k t}. \tag{2}$$

Applying the operator $\widetilde{D}$ to both sides of (2) yields the following:

$$\begin{aligned} 0 = \widetilde{D}\left(\sum_{i=1}^{k-1} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t} + a_{k,(n_k-1)} t^{n_k-1} e^{c_k t} \right) &= \sum_{i=1}^{k-1} \sum_{j=0}^{n_i-1} a_{ij} \widetilde{D}(t^j e^{c_i t}) + a_{k,(n_k-1)} \widetilde{D}(t^{n_k-1} e^{c_k t}) \\ &= a_{k,(n_k-1)} (c_k - c_{k-1})^{n_{k-1}} \cdots (c_k - c_1)^{n_1} (n_k - 1)!\, e^{c_k t}. \end{aligned}$$

Since the constants $c_1, \ldots, c_k$ are distinct, $(c_k - c_{k-1})^{n_{k-1}} \cdots (c_k - c_1)^{n_1} (n_k - 1)!\, e^{c_k t} \neq 0$ for any value of t, and therefore $a_{k,(n_k-1)} = 0$. Thus by the inductive hypothesis, it follows that $a_{km} = 0$ for every $0 \le m \le n_k - 1$.

Using this result, (1) reduces to

$$0 = \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t} = \sum_{i=1}^{k-1} \sum_{j=0}^{n_i-1} a_{ij} t^j e^{c_i t},$$

and since β_{k-1} is linearly independent by the inductive hypothesis, $a_{ij} = 0$ for every $1 \le i \le k-1$ and $0 \le j \le n_i - 1$. Thus we see that $a_{ij} = 0$ for every $1 \le i \le k$ and $0 \le j \le n_i - 1$, so that β_k is linearly independent as well. As noted above, this shows that β_k is a basis for the solution space.

14. I will proceed by induction on n. When $n = 1$, x is a solution to a differential equation of the form

$$y' + a_0 y = 0 \tag{1}$$

for some constant a_0. The condition on x reduces to $x(t_0) = 0$ for some $t_0 \in \mathbb{R}$. By Thm 2.30, the solution space for the differential equation (1) has $\{e^{-a_0 t}\}$ as a basis. Therefore $x(t) = ke^{-a_0 t}$ for some constant $k \in \mathbb{C}$. Since $e^{-a_0 t_0} \neq 0$, the fact that $x(t_0) = 0$ implies that $k = 0$. Thus $x = 0$ identically.

Now suppose the result holds for any nth order homogeneous linear differential equation with constant coefficients. Consider an $(n+1)$th order differential equation of this type with auxiliary polynomial p. Let x be a solution to this equation that satisfies

$$x(t_0) = x'(t_0) = \cdots = x^{(n)}(t_0) = 0 \tag{2}$$

for some $t_0 \in \mathbb{R}$. Since p is of order $(n+1)$, it can be factored over $\mathbb{C}$ as $p(t) = (t-c)q(t)$, where q is an nth order polynomial over $\mathbb{C}$, and $c \in \mathbb{C}$ is a constant. Say $q(t) = a_n t^n + \cdots + a_1 t + a_0$, and define z by

$$z(t) = q(D)(x(t)) = (a_n D^n + \cdots + a_1 D + a_0 I)(x(t)) = a_n x^{(n)}(t) + \cdots + a_1 x'(t) + a_0 x(t). \tag{3}$$

Thus by (2), $z(t_0) = 0$. Furthermore, note that

$$0 = p(D)(x(t)) = (D - cI)q(D)(x(t)) = (D - cI)(z(t)),$$

so that z is a solution to $y' - cy = 0$. By Thm. 2.30, the solution space to this differential equation has $\{e^{ct}\}$ as a basis, and thus $z(t) = ke^{ct}$ for some constant $k \in \mathbb{C}$. Since $e^{ct_0} \neq 0$, this implies that $k = 0$, so that $z = 0$ identically. Thus (3) reduces to

$$0 = q(D)(x(t)).$$

That is, x is a solution to an nth order homogeneous linear differential equation with constant coefficients. Since

$$x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$$

in particular by (1), $x = 0$ identically by the inductive hypothesis.

Section 3.1

6. Suppose that B can be obtained from A by an elementary row operation. Then by Thm. 3.1 there exists an $m \times m$ elementary matrix E such that $B = EA$. Then $B^t = (EA)^t = A^t E^t$. Note that E^t can be obtained from the $m \times m$ identity matrix I by performing the same operations that produced E on the columns of I rather than the rows (this is immediate from the definition of the transpose of a matrix).

   If B is obtained from A by an elementary column operation, then by Thm. 3.1 there exists an $n \times n$ elementary matrix F such that $B = AF$. Then $B^t = (AF)^t = F^t A^t$, and an analogous argument to that above yields the desired result.

12. I will proceed by induction on the number of rows m. When $m = 1$ the matrix is (trivially) upper triangular, since there are no entries below the diagonal. Now assume the result holds for any $(m-1) \times n$ matrix, where $m \geq 1$. Let B be the matrix consisting of the upper $(m-1)$ rows of A. Then B is an $(m-1) \times n$ matrix, and so by the inductive hypothesis may be transformed into an upper triangular matrix through a finite sequence of elementary row operations of types 1 and 3.

    If $m \leq n$, let $k = m - 1$, and if $m > n$, let $k = n$, and for each of $j = 1, 2, \ldots, k$ in sequence, perform the following operation:

- if $A_{jj} = 0$, swap row j with row m (a type 1 operation);
- if $A_{jj} \neq 0$, add $(-A_{mj}/A_{jj})$ times row j to row m (a type 3 operation).

In either case, we end up with $A_{mj} = 0$. Therefore, through a finite sequence of row operations of types 1 and 3, we obtain $A_{mj} = 0$ for each $1 \leq j \leq k$.

To verify that the resulting matrix is upper triangular, first let $1 \leq i \leq m-1$ and $1 \leq j \leq n$ be such that $j < i$. Then $A_{ij} = B_{ij} = 0$ by the inductive hypothesis. Now let $1 \leq j \leq n$ be such that $j < m$. If $m \leq n$, then $k = m-1$, so that $1 \leq j \leq k$, and thus $A_{mj} = 0$. If $m > n$, then $k = n$, so that $1 \leq j \leq k$, and thus $A_{mj} = 0$. We therefore see that $A_{ij} = 0$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$ with $j < i$, so that A is upper triangular. Thus by way of the inductive hypothesis, we see that A can be transformed into an upper triangular matrix through a finite sequence of row operations of types 1 and 3.

Section 3.2

9. Let A be an $m \times n$ matrix, and let B be an $m \times n$ matrix obtained from A through an elementary column operation. Then by Thm. 3.1 there exists an $n \times n$ elementary matrix E with $B = AE$. By Thm. 3.2, E is invertible, and thus by part (a) of Thm. 3.4,

$$\operatorname{rank}(B) = \operatorname{rank}(AE) = \operatorname{rank}(A).$$

Thus elementary column operations preserve rank.

Section 3.3

10. Let $Ax = b$ denote the system in question, and let $A_1, \ldots, A_n \in F^m$ denote the n column vectors of A. Note that since $\operatorname{rank}(A) = m$, we must have $m \leq n$, and by Thm. 3.5, m is the maximum number of linearly independent columns of A. Therefore the set of $n+1 > m$ elements $\{A_1, \ldots, A_n, b\}$ in F^m must be linearly dependent. Since these are simply the column vectors of the augmented matrix $(A|b)$, we therefore see that $\operatorname{rank}(A|b) = m$ as well by way of Thm. 3.5. The system therefore has a solution by Thm. 3.11.