Homework 7

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Section 5.3

6. In any given month, a patient is in precisely one of the following states: ambulatory, bedridden, recovered, or dead. The given data yield the transition matrix

$$A = \begin{pmatrix} 0.20 & 0.20 & 0 & 0 \\ 0.20 & 0.50 & 0 & 0 \\ 0.60 & 0.10 & 1 & 0 \\ 0 & 0.20 & 0 & 1 \end{pmatrix}$$

and initial probability vector

$$P = \begin{pmatrix} 0.30 \\ 0.70 \\ 0 \\ 0 \end{pmatrix}.$$

The first entry of P is the initial proportion of ambulatory patients, the second the proportion who are bedridden, the third the proportion who have recovered, and the fourth the proportion who have died.

We compute

$$AP = \begin{pmatrix} 0.20 & 0.20 & 0 & 0 \\ 0.20 & 0.50 & 0 & 0 \\ 0.60 & 0.10 & 1 & 0 \\ 0 & 0.20 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.30 \\ 0.70 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.20 \\ 0.41 \\ 0.25 \\ 0.14 \end{pmatrix}.$$

Thus after one month of arrival, 20% of the patients are ambulatory, 41% are bedridden, 25% have recovered, and 14% have died.

Next, we compute

$$\begin{aligned} \det(tI_4 - A) &= \det \begin{pmatrix} t-0.20 & -0.20 & 0 & 0 \\ -0.20 & t-0.50 & 0 & 0 \\ -0.60 & -0.10 & t-1 & 0 \\ 0 & -0.20 & 0 & t-1 \end{pmatrix} \\ &= (t-1)\det \begin{pmatrix} t-0.20 & -0.20 & 0 \\ -0.20 & t-0.50 & 0 \\ -0.60 & -0.10 & t-1 \end{pmatrix} \\ &= (t-1)^2 \det \begin{pmatrix} t-0.20 & -0.20 \\ -0.20 & t-0.50 \end{pmatrix} = (t-1)^2(t-0.1)(t-0.6). \end{aligned}$$

Setting this expression equal to zero and solving for t, we find the eigenvalues of A to be $\lambda_1 = 1$ (with multiplicity 2), $\lambda_2 = 0.1$, and $\lambda_3 = 0.6$. We have:

$$B_1 = A - \lambda_1 I_4 = \begin{pmatrix} -0.80 & 0.20 & 0 & 0 \\ 0.20 & -0.50 & 0 & 0 \\ 0.60 & 0.10 & 0 & 0 \\ 0 & 0.20 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Row reduces to}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = A - \lambda_2 I_4 = \begin{pmatrix} 0.10 & 0.20 & 0 & 0 \\ 0.20 & 0.40 & 0 & 0 \\ 0.60 & 0.10 & 0.90 & 0 \\ 0 & 0.20 & 0 & 0.90 \end{pmatrix} \xrightarrow{\text{Row reduces to}} \begin{pmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4.5 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_3 = A - \lambda_3 I_4 = \begin{pmatrix} -0.40 & 0.20 & 0 & 0 \\ 0.20 & -0.10 & 0 & 0 \\ 0.60 & 0.10 & 0.40 & 0 \\ 0 & 0.20 & 0 & 0.40 \end{pmatrix} \xrightarrow{\text{Row reduces to}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we find:

$$\ker(B_1) = \left\{ s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$\ker(B_2) = \left\{ s \begin{pmatrix} 18 \\ -9 \\ -11 \\ 2 \end{pmatrix} : s \in \mathbb{R} \right\}$$

$$\ker(B_3) = \left\{ s \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\}.$$

For each $1 \leq i \leq 3$, the eigenvalues of A corresponding to λ_i are precisely the non-zero elements of $\ker(B_i)$. By part (a) of Thm. 5.9, A is diagonalizable. Thus by part (b) of Thm. 5.9,

$$\beta' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 18 \\ -9 \\ -11 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is an ordered basis for $\mathbb{R}^4$. Consider the linear operator $L_A : \mathbb{R}^4 \to \mathbb{R}^4$. Then

$$[L_A]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.6 \end{pmatrix},$$

and by Thm. 2.23, $[L_A]_{\beta'} = Q^{-1}[L_A]_\beta Q$, where Q is the matrix that changes β'-coordinates into β-coordinates. Through a simple computation we find that

$$Q = \begin{pmatrix} 0 & 0 & 18 & -1 \\ 0 & 0 & -9 & -2 \\ 1 & 0 & -11 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad Q^{-1} = \frac{1}{45}\begin{pmatrix} 40 & 25 & 45 & 0 \\ 5 & 20 & 0 & 45 \\ 2 & -1 & 0 & 0 \\ -9 & -18 & 0 & 0 \end{pmatrix}.$$

Letting $D = [L_A]_{\beta'}$, we have the identity $A = QDQ^{-1}$. Thus, using the Cor. to Thm. 5.12, we get:

$$\begin{aligned} L &= \lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = Q(\lim_{m\to\infty} D^m)Q^{-1} \\ &= \begin{pmatrix} 0 & 0 & 18 & -1 \\ 0 & 0 & -9 & -2 \\ 1 & 0 & -11 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \left[\lim_{m\to\infty} \begin{pmatrix} 1^m & 0 & 0 & 0 \\ 0 & 1^m & 0 & 0 \\ 0 & 0 & 0.1^m & 0 \\ 0 & 0 & 0 & 0.6^m \end{pmatrix}\right] \frac{1}{45}\begin{pmatrix} 40 & 25 & 45 & 0 \\ 5 & 20 & 0 & 45 \\ 2 & -1 & 0 & 0 \\ -9 & -18 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 18 & -1 \\ 0 & 0 & -9 & -2 \\ 1 & 0 & -11 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{45}\begin{pmatrix} 40 & 25 & 45 & 0 \\ 5 & 20 & 0 & 45 \\ 2 & -1 & 0 & 0 \\ -9 & -18 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8/9 & 5/9 & 1 & 0 \\ 1/9 & 4/9 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore:

$$LP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8/9 & 5/9 & 1 & 0 \\ 1/9 & 4/9 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.30 \\ 0.70 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 59/90 \\ 31/90 \end{pmatrix}.$$

So in the long run 59/90 ($\approx 65.6\%$) of the patients recover, 31/90 ($\approx 34.4\%$) die, and none remain either ambulatory or bedridden.

Section 5.4

13. Note that $W = \text{span}\{T^k(v) : k \geq 0\}$ by definition. Thus if $w \in W$, then there exist scalars $a_0, \dots, a_n$ satisfying

$$w = \sum_{i=0}^{n} a_i T^i(v) = \left(\sum_{i=0}^{n} a_i T^i\right)(v) = g(T)(v),$$

where g is the polynomial $g(t) = \sum_{i=0}^{n} a_i t^i$. Conversely, suppose $w \in V$ satisfies $w = g(T)(v)$ for some polynomial g. Say $g(t) = \sum_{i=0}^{n} a_i t^i$. Then

$$w = g(T)(v) = \left(\sum_{i=0}^{n} a_i T^i\right)(v) = \sum_{i=0}^{n} a_i T^i(v),$$

and so we see that $w \in W$.

17. By part (a) of Thm. 5.3, the characteristic polynomial f of A is of the form

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

By the Cor. to Thm. 5.23,

$$O_n = f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n, \tag{1}$$

where O_n and I_n denote the $n \times n$ zero and identity matrices, respectively. In particular, from (1), we see that A^n can be expressed as a linear combination of $I_n, A, \ldots, A^{n-1}$; that is, $A^n \in W$, where $W = \operatorname{span}\{I_n, A, \ldots, A^{n-1}\}$. Now let $m > n$, and assume that $A^m \in W$, so that $A^m = b_{n-1}A^{n-1} + \cdots + b_1 A + b_0 I_n$ for some scalars $b_0, \ldots, b_{n-1}$. Then

$$A^{m+1} = AA^m = A(b_{n-1}A^{n-1} + \cdots + b_1 A + b_0 I_n) = b_{n-1}A^n + \cdots + b_1 A^2 + b_0 A.$$

Thus since $A^n \in W$, we see that $A^{m+1} \in W$ as well. Therefore $A^m \in W$ for every $m \geq n$ by induction. Thus if $V = \operatorname{span}\{I_n, A, A^2, \ldots\}$, then $V = W$. Since W is finite-dimensional with $\dim(W) \leq n$, we therefore see that $\dim(V) \leq n$, as desired.

21. Suppose that V is not a T-cyclic subspace of itself. Let $\{v_1, v_2\}$ be a basis for V. Then by assumption, neither $\{v_1, T(v_1)\}$ nor $\{v_2, T(v_2)\}$ spans V. Thus $\{v_i, T(v_i)\}$ $(i = 1, 2)$ either contains a single element (if $T(v_i) = v_i$), or it contains $2 = \dim(V)$ elements (if $T(v_i) \neq v_i$), in which case it is linearly dependent (by part (b) of Cor. 2 to Thm. 1.10). In either case there exist scalars c_1 and c_2 with $T(v_1) = c_1 v_1$ and $T(v_2) = c_2 v_2$. By the same sort of argument, there exists a scalar c with $T(v_1 - v_2) = c(v_1 - v_2)$, so that

$$c_1 v_1 - c_2 v_2 = T(v_1) - T(v_2) = T(v_1 - v_2) = c(v_1 - v_2) = cv_1 - cv_2. \tag{1}$$

Upon rearrangement, (1) becomes

$$0 = (c - c_1)v_1 + (c_2 - c)v_2.$$

Thus since v_1 and v_2 are linearly independent, $c - c_1 = 0$ and $c_2 - c = 0$, so that $c_1 = c_2 = c$. Now let $v \in V$. Then $v = a_1 v_1 + a_2 v_2$ for some scalars a_1 and a_2. Thus

$$T(v) = T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2) = a_1(cv_1) + a_2(cv_2) = c(a_1 v_1 + a_2 v_2) = cv.$$

From this we see that $T = cI_V$, where $I_V : V \to V$ denotes the identity linear operator on V.

27. (a) Let $v, v' \in V$ be such that $v + W = v' + W$. This means that $v - v' \in W$. Since W is T-invariant, $T(v - v') = T(v) - T(v') \in W$ as well, so that $\overline{T}(v + W) = T(v) + W = T(v') + W = \overline{T}(v' + W)$. Thus $\overline{T}$ is well-defined.

    (b) Let $v_1, v_2 \in V$ and let c be a scalar. Then

$$\begin{aligned}
\overline{T}(c(v_1 + W) + (v_2 + W)) &= \overline{T}((cv_1 + v_2) + W) \\
&= T(cv_1 + v_2) + W \\
&= (cT(v_1) + T(v_2)) + W \\
&= c(T(v_1) + W) + (T(v_2) + W) \\
&= c\overline{T}(v_1 + W) + \overline{T}(v_2 + W).
\end{aligned}$$

    Thus $\overline{T}$ is a linear operator on V/W.

(c) Let $v \in V$. Then

$$(\eta \circ T)(v) = \eta(T(v)) = T(v) + W = \overline{T}(v + W) = \overline{T}(\eta(v)) = (\overline{T} \circ \eta)(v)$$

Therefore $\eta \circ T = \overline{T} \circ \eta$, as desired.

36. Suppose that T is diagonalizable. Then by Thm. 5.1 there exists an ordered basis $\{w_1, \ldots, w_n\}$ for V consisting of eigenvectors of T. For each $1 \leq i \leq n$, let $W_i = \text{span}\{w_i\}$. Then since $w_i \neq 0$, $\{w_i\}$ is a basis for W_i. Thus $\dim(W_i) = 1$. Furthermore, note that $T(w_i) = \lambda_i w_i$, where λ_i is the eigenvalue of T corresponding to w_i. Thus for any $w \in W_i$, since $w = cw_i$ for some scalar c,

$$T(w) = T(cw_i) = cT(w_i) = c(\lambda_i w_i) = \lambda_i(cw_i) = \lambda_i w,$$

so that $T(w) \in W$. Thus W_i is T-invariant. Since for each $1 \leq i \leq n$, $\{w_i\}$ is an ordered basis for W_i, and $\{w_1\} \cup \cdots \cup \{w_n\} = \{w_1, \ldots, w_n\}$ is an ordered basis for V, Thm. 5.10 implies that $V = W_1 \oplus \cdots \oplus W_n$.

Conversely, suppose that $V = W_1 \oplus \cdots \oplus W_k$, where for each $1 \leq i \leq k$, W_i is a one-dimensional T-invariant subspace of V. Note that since $\dim(W_i) = 1$ for each $1 \leq i \leq k$, we must have $k = n$, where $n = \dim(V)$ (see the result of exercise 20 of section 5.2). For each $1 \leq i \leq n$, let $\{w_i\}$ be an ordered basis for W_i. Then since W_i is T-invariant, $T(w_i) \in W_i$, so that $T(w_i) = \lambda_i w_i$ for some scalar λ_i. Thus since $w_i \neq 0$, w_i is an eigenvector of T. By Thm. 5.10, $\beta = \{w_1\} \cup \cdots \cup \{w_n\} = \{w_1, \ldots, w_n\}$ is an ordered basis for V. Since β consists of eigenvectors of T, Thm. 5.1 implies that T is diagonalizable.