Homework 9

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Section 6.1

11. Let $x, y \in V$. We have the following:

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

In $\mathbb{R}^2$ this equation has the following interpretation: the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.

27. (b) Let $x, u, y \in V$. Then by the parallelogram law (exercise 11),

$$\|x+u+2y\|^2 + \|x-u\|^2 = 2\|x+y\|^2 + 2\|u+y\|^2 \tag{1}$$

and

$$\|x+u-2y\|^2 + \|x-u\|^2 = 2\|x-y\|^2 + 2\|u-y\|^2 \tag{2}$$

By subtracting (2) from (1), we get

$$\|x+u+2y\|^2 - \|x+u-2y\|^2 = 2(\|x+y\|^2 - \|x-y\|^2) + 2(\|u+y\|^2 - \|u-y\|^2),$$

or equivalently,

$$4\langle x+u, 2y\rangle = 8\langle x, y\rangle + 8\langle u, y\rangle. \tag{3}$$

Thus since $4\langle x+u, 2y\rangle = 8\langle x+u, y\rangle$, multiplying both sides of (3) by 1/8 yields

$$\langle x+u, y\rangle = \langle x, y\rangle + \langle u, y\rangle.$$

(c) Let $x, y \in V$. I will proceed by induction on n. When $n = 1$ the result holds trivially. Suppose that $\langle nx, y\rangle = n\langle x, y\rangle$ for some $n > 1$. Then by part (b),

$$\langle (n+1)x, y\rangle = \langle nx + x, y\rangle = \langle nx, y\rangle + \langle x, y\rangle = n\langle x, y\rangle + \langle x, y\rangle = (n+1)\langle x, y\rangle.$$

So $\langle nx, y\rangle = n\langle x, y\rangle$ for every positive integer n by induction.

28. Let $x, y, z \in V$ and $c \in \mathbb{R}$. Using basic properties of complex numbers, we get the following:

(a) $[x+z, y] = \mathrm{Re}\langle x+z, y\rangle = \mathrm{Re}[\langle x, y\rangle + \langle z, y\rangle] = \mathrm{Re}\langle x, y\rangle + \mathrm{Re}\langle z, y\rangle = [x, y] + [z, y]$

(b) $[cx, y] = \mathrm{Re}\langle cx, y\rangle = \mathrm{Re}[c\langle x, y\rangle] = c\,\mathrm{Re}\langle x, y\rangle = c[x, y]$ (since $c \in \mathbb{R}$)

(c) $[x, y] = \mathrm{Re}\langle x, y\rangle = \mathrm{Re}\,\overline{\langle y, x\rangle} = \mathrm{Re}\langle y, x\rangle = [y, x]$

(d) If $x \neq 0$, then $\langle x, x\rangle > 0$ (note that $\langle x, x\rangle \in \mathbb{R}$). Thus $[x, x] = \mathrm{Re}\langle x, x\rangle = \langle x, x\rangle > 0$.

Thus $[\cdot, \cdot]$ is an inner product for V when V is regarded as a vector space over $\mathbb{R}$. Finally, for any $x \in V$,

$$[x, ix] = \mathrm{Re}\langle x, ix\rangle = \mathrm{Re}[\bar{i}\langle x, x\rangle] = \mathrm{Re}[-i\langle x, x\rangle] = 0.$$

This is due to the fact that since $\langle x, x\rangle \in \mathbb{R}$, the complex number $-i\langle x, x\rangle$ is pure imaginary and hence has no real part.

Section 6.2

6. By Thm. 6.6 there exist unique vectors $y \in W^\perp$ and $z \in W$ such that $x = y + z$. Since $x \notin W$, we must have $y \neq 0$. Therefore,

$$\langle x, y\rangle = \langle y + z, y\rangle = \langle y, y\rangle + \langle z, y\rangle = \langle y, y\rangle \neq 0.$$

10. By Thm. 6.6 and Thm. 5.10, $V = W \oplus W^\perp$, and thus the projection $T : V \to V$ on W along $W^\perp$ exists. Let $x \in V$. Then there exist unique vectors $x_1 \in W$ and $x_2 \in W^\perp$ with $x = x_1 + x_2$. Notice that $x \in \ker T$ if and only if $0 = T(x) = x_1$, or equivalently, $x = x_2 \in W^\perp$. Therefore $\ker T = W^\perp$. Moreover, since x_1 and x_2 are orthogonal by definition,

$$\|T(x)\|^2 = \|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2 = \|x_1 + x_2\|^2 = \|x\|^2$$

by exercise 10 of section 6.1. Thus upon taking square roots,

$$\|T(x)\| \leq \|x\|.$$

Section 6.3

17. Let $v \in \mathrm{range}(T^*)^\perp$. Then $\langle v, u\rangle_1 = 0$ for every $u \in \mathrm{range}(T^*)$; that is, for every $w \in W$, $0 = \langle v, T^*(w)\rangle_1 = \langle T(v), w\rangle_2$. In particular, since $T(v) \in W$, $\langle T(v), T(v)\rangle_2 = 0$, so that $T(v) = 0$. Thus $v \in \ker T$, so that $\mathrm{range}(T^*)^\perp \subseteq \ker T$. Conversely, let $v \in \ker T$. Then $T(v) = 0$, so that $0 = \langle T(v), w\rangle_2 = \langle v, T^*(w)\rangle_1$ for every $w \in W$; that is, $\langle v, u\rangle_1 = 0$ for every $u \in \mathrm{range}(T^*)$. Thus $v \in \mathrm{range}(T^*)^\perp$, so that $\ker T \subseteq \mathrm{range}(T^*)^\perp$. Therefore $\mathrm{range}(T^*)^\perp = \ker T$, as desired.

22. (c) Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

We want to find a solution u to the system $AA^*u = b$. We have

$$A^* = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

and therefore

$$AA^* = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 6 & 4 \\ -1 & 4 & 3 \end{pmatrix}.$$

Thus we can represent the system $AA^*u = b$ in matrix form as follows:

$$\left(\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ 0 & 6 & 4 & 3 \\ -1 & 4 & 3 & 2 \end{array}\right) \xrightarrow{\text{Row reduces to}} \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 2/3 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

One solution is therefore $u = (1, -3/2, 3)^t$. Now let

$$s = A^*u = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3/2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}.$$

Then since $As = AA^*u = b$, the original system is consistent (i.e., it has a solution, namely s). So by Thm. 6.13, s is the unique minimal solution to the original system.

Section 6.5

6. Note that for any $f \in V$,

$$\begin{aligned} \|T(f)\|^2 &= \langle T(f), T(f)\rangle = \langle hf, hf\rangle \\ &= \int_0^1 h(t)f(t)\overline{h(t)f(t)}\,dt = \int_0^1 f(t)\overline{f(t)}h(t)\overline{h(t)}\,dt \\ &= \int_0^1 |f(t)|^2|h(t)|^2\,dt. \end{aligned} \tag{1}$$

Thus if $|h(t)| = 1$ for every $0 \le t \le 1$, (1) implies that for any $f \in V$,

$$\|T(f)\|^2 = \int_0^1 |f(t)|^2\,dt = \int_0^1 f(t)\overline{f(t)}\,dt = \langle f, f\rangle = \|f\|^2,$$

so that $\|T(f)\| = \|f\|$. Hence T is unitary. Conversely, suppose that T is unitary, so that $\|T(f)\| = \|f\|$ for any $f \in V$. Thus (1) implies that

$$0 = \|T(f)\|^2 - \|f\|^2 = \int_0^1 |f(t)|^2|h(t)|^2\,dt - \int_0^1 |f(t)|^2\,dt = \int_0^1 |f(t)|^2(|h(t)|^2 - 1)\,dt. \tag{2}$$

But in order for (2) to hold for every choice of $f \in V$, we must have $|h(t)|^2 - 1 = 0$ for every $0 \le t \le 1$, so that $|h(t)| = 1$ for every $0 \le t \le 1$. (The fact that $|h(t)| = 1$ must hold for *every* $0 \le t \le 1$ rather than for *almost every* $0 \le t \le 1$ follows from the continuity of h).

7. Since T is unitary, Cor. 2 to Thm. 6.18 implies that there exists an ordered orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ for V where, for each $1 \le i \le n$, the eigenvalue λ_i corresponding to v_i satisfies

$|\lambda_i| = 1$. Thus we can write

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}.$$

Define a linear operator $U : V \to V$ by

$$[U]_\beta = \begin{pmatrix} \lambda_1^{1/2} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{1/2} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^{1/2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n^{1/2} \end{pmatrix}.$$

Then since $[U]_\beta^2 = [T]_\beta$, $U^2 = T$. Furthermore, for any $1 \leq i \leq n$, $U(v_i) = \lambda_i^{1/2} v_i$ by construction, and since $|\lambda_i^{1/2}| = 1$, we see that U is unitary by Cor. 2 to Thm. 6.18.