Homework 8

Benjamin Noland

Section 7.1

3. (b) Firstly, we compute:

$$\begin{aligned}
T(1) &= 0 \\
T(t) &= 1 \\
T(t^2) &= 2t \\
T(e^t) &= e^t \\
T(te^t) &= e^t + te^t.
\end{aligned}$$

From this we see that the linear map $T : V \to V$ is indeed well-defined.

Note that by Thm. 2.34, the set $\gamma = \{1, t, t^2, e^t, te^t, t^2e^t\}$ is a basis for the solution space of the homogeneous linear differential equation with constant coefficients and auxiliary polynomial $t^3(t-1)^3$. In particular, γ is a linearly independent subset of the solution space. Thus since $\beta \subseteq \gamma$, β is also a linearly independent subset of the solution space by the Cor. to Thm. 1.6. This means that for any $a_1, a_2, \dots, a_5 \in \mathbb{C}$, if

$$0 = a_1 + a_2t + a_3t^2 + a_4e^t + a_5te^t$$

then $a_1 = a_2 = \cdots = a_5 = 0$. In particular, this holds if $a_1, a_2, \dots, a_5 \in \mathbb{R}$. Since V is a vector space over $\mathbb{R}$, we therefore see that β is also a linearly independent subset of V. Hence since β spans V by definition, β is an ordered basis for V.

The matrix of T in β is given by

$$A = [T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus we compute the characteristic polynomial f of T to be

$$f(t) = \det(A - tI_5) = \det \begin{pmatrix} -t & 1 & 0 & 0 & 0 \\ 0 & -t & 2 & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 1-t & 1 \\ 0 & 0 & 0 & 0 & 1-t \end{pmatrix} = -t^3(t-1)^3.$$

Note that f splits. Setting $f(t) = 0$ and solving for t, we find the eigenvalues of T to be $\lambda_1 = 0$ (with multiplicity $m_1 = 3$), and $\lambda_2 = 1$ (with multiplicity $m_2 = 2$). Thus by Thm. 7.2, the generalized eigenspaces of T are given by

$$\begin{aligned} K_{\lambda_1} &= \ker(T - 0I)^3 \\ K_{\lambda_2} &= \ker(T - 1I)^2 \end{aligned}$$

Furthermore, by part (c) of Thm. 7.4, $\dim(K_{\lambda_1}) = m_1 = 3$ and $\dim(K_{\lambda_2}) = m_2 = 2$. Thus by Thm. 7.7, K_{λ_1} has an ordered basis consisting of one cycle of length 3, or a union of one cycle of length 1 and one cycle of length 2. Similarly, K_{λ_2} must have an ordered basis consisting of one cycle of length 2.

We compute:

$$[(T - 0I)^3]_\beta = A^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{Row reduces to}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[(T - 1I)^2]_\beta = (A - I_5)^2 = \begin{pmatrix} 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Row reduces to}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore:

$$\begin{aligned} K_{\lambda_1} &= \operatorname{span}\{1, t, t^2\} \\ K_{\lambda_2} &= \operatorname{span}\{e^t, te^t\}. \end{aligned}$$

Note that $\beta_1 = \{1, t, t^2/2\}$ is a basis for K_{λ_1}, and that $T(t^2/2) = t$, $T(t) = 1$, and $T(1) = 0$. Thus β_1 consists of a single cycle of length 3. Similarly, $\beta_2 = \{e^t, te^t\}$ is a basis for K_{λ_2}, and from the discussion above we know that β_2 consists of a single cycle of length 2.

Therefore $\beta' = \beta_1 \cup \beta_2 = \{1, t, t^2/2, e^t, te^t\}$ is a Jordan canonical basis for T by part (b) of Thm. 7.5, and so a Jordan canonical form J of T is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

9. (b) Since β is linearly independent, and $\beta' = \beta \cap K_\lambda \subseteq \beta$, the Cor. to Thm. 1.6 implies that β' is linearly independent as well. Let $\lambda_1, \dots, \lambda_k$ denote the distinct eigenvalues of T. Since β is a Jordan canonical basis for T, $\beta = \beta_1 \cup \cdots \cup \beta_k$, where for each $1 \leq i \leq k$, β_i is an ordered basis for K_{λ_i} consisting of a union of disjoint cycles of generalized eigenvalues corresponding to λ_i. Thus if $\lambda = \lambda_j$,

$$\beta' = \beta \cap K_\lambda = (\beta_1 \cup \cdots \cup \beta_k) \cap K_{\lambda_j} = (\beta_1 \cap K_{\lambda_j}) \cup \cdots \cup (\beta_k \cap K_{\lambda_j}) \supseteq (\beta_j \cap K_{\lambda_j}) = \beta_j.$$

Thus $\operatorname{span} \beta' \supseteq \operatorname{span} \beta_j = K_\lambda$. But since $\beta' \subseteq K_\lambda$, $\operatorname{span} \beta' \subseteq K_\lambda$ as well. Thus $\operatorname{span} \beta' = K_\lambda$, so that β' is a basis for K_λ.

11. Let $A \in M_{n\times n}(F)$ be a matrix whose characteristic polynomial splits. Note that the characteristic polynomial of A is precisely that of the linear operator $L_A : F^n \to F^n$. Thus L_A has a Jordan canonical form by Cor. 1 to Thm. 7.7; that is, there exists an ordered basis β' for F^n such that $J = [L_A]_{\beta'}$ is a Jordan canonical form. Thus by definition, A has a Jordan canonical form as well, and by Thm. 2.23 there exists an invertible matrix Q satisfying

$$A = [L_A]_\beta = Q^{-1}[L_A]_{\beta'}Q = Q^{-1}JQ,$$

where β is the standard ordered basis for F^n. Thus A is similar to J.

Section 7.2

3. (a) Let J denote the Jordan canonical form of T (given in the problem statement). Then the characteristic polynomial f of T is given by

$$f(t) = \det(J - tI_7) = \det\begin{pmatrix} 2-t & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-t & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2-t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3-t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3-t \end{pmatrix} = (2-t)^5(3-t)^2.$$

(b) Solving $f(t) = 0$ for t, we find the eigenvalues of T to be $\lambda_1 = 2$ (with multiplicity $m_1 = 5$), and $\lambda_2 = 3$ (with multiplicity $m_2 = 2$). By observing the Jordan canonical form J, we see that K_{λ_1} has a basis consisting of the union of two disjoint cycles (one of length 3, and one of length 2). We also see that K_{λ_2} has a basis consisting of the union of two disjoint cycles (each of length 1). The dot diagram corresponding to λ_1 is therefore:

$$\begin{array}{ll} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array}$$

and the dot diagram corresponding to λ_2 is:

$$\begin{array}{ll} \bullet & \bullet \end{array}$$

(c) As discussed in part (b), K_{λ_2} has a basis consisting of two disjoint cycles of generalized eigenvectors of T corresponding to λ_2, each of length 1. Since any such cycle contains precisely one eigenvector of T corresponding to λ_2 (see the discussion at the bottom of pg. 488), we see that K_{λ_2} has a basis consisting solely of eigenvalues of T corresponding to λ_2. It follows that $K_{\lambda_2} = E_{\lambda_2}$.

(d) This is immediate from Thm. 7.9 and the dot diagrams computed in part (b). Since the dot diagram for λ_1 has 3 rows, Thm. 7.9 implies that $K_{\lambda_1} = \ker(T - \lambda_1 I)^3$, so that $p_1 = 3$. Similarly, the dot diagram for λ_2 has 1 row, and therefore $K_{\lambda_2} = \ker(T - \lambda_2 I)$, so that $p_2 = 1$.

(e) (i)

$$\operatorname{rank}(U_1) = \operatorname{rank}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 3$$
$$\operatorname{rank}(U_2) = \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

(ii)

$$\operatorname{rank}(U_1^2) = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 1$$
$$\operatorname{rank}(U_2^2) = \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

(iii) By the Dimension Thm. and part (c) of Thm. 7.4:

$$\operatorname{nullity}(U_1) = \dim(K_{\lambda_1}) - \operatorname{rank}(U_1) = m_1 - \operatorname{rank}(U_1) = 5 - 3 = 2$$
$$\operatorname{nullity}(U_2) = \dim(K_{\lambda_2}) - \operatorname{rank}(U_2) = m_2 - \operatorname{rank}(U_2) = 2 - 0 = 2$$

(iv) By the Dimension Thm. and part (c) of Thm. 7.4:

$$\operatorname{nullity}(U_1^2) = \dim(K_{\lambda_1}) - \operatorname{rank}(U_1^2) = m_1 - \operatorname{rank}(U_1^2) = 5 - 1 = 4$$
$$\operatorname{nullity}(U_2^2) = \dim(K_{\lambda_2}) - \operatorname{rank}(U_2^2) = m_2 - \operatorname{rank}(U_2^2) = 2 - 0 = 2$$

14. By the Cayley-Hamilton Thm., $(-1)^n T^n = 0$, where 0 denotes the zero linear operator on V. In particular, we see that $T^n = 0$, so that T is nilpotent.

19. (a) I will proceed by induction on r. When $r = 1$ it is clear from observation that

$$N_{ij}^r = \begin{cases} 1 & \text{if } j = i + r \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

holds for any $1 \le i, j \le m$. Now suppose that (1) holds for some $1 < r < m$. Then for any $1 \le i, j \le m$,

$$N_{ij}^r = [NN^r]_{ij} = \sum_{k=1}^{m} N_{ik} N_{kj}^r = N_{i,i+1} N_{i+1,j}^r = N_{i+1,j}^r = \begin{cases} 1 & \text{if } j = i + (r+1) \\ 0 & \text{otherwise} \end{cases}$$

Thus (1) holds for any $1 \le r < m$ by induction. In particular, it holds when $r = m - 1$. Thus for any $1 \le i, j \le m$,

$$N_{ij}^{m-1} = \begin{cases} 1 & \text{if } j = i + (m-1) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } i = 1, j = m \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

and therefore

$$N_{ij}^m = [NN^{m-1}]_{ij} = \sum_{k=1}^{m} N_{ik}N_{kj}^{m-1} = N_{i,i+1}N_{i+1,j}^{m-1} = 0,$$

since $N_{i+1,j}^{m-1} = 0$ by way of (2). Thus $N^m = O$.

Section 7.3

13. Let

$$J = \begin{pmatrix} J_1 & O & O & \cdots & O \\ O & J_2 & O & \cdots & O \\ O & O & J_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & J_k \end{pmatrix} \tag{1}$$

denote a Jordan canonical form of T, where for each $1 \le i \le k$, J_i is the collection of Jordan blocks associated with λ_i. Let $1 \le i \le k$. Then there exists an ordered basis β_i for K_{λ_i} such that $J_i = [T_{K_{\lambda_i}}]_{\beta_i}$, where β_i consists of a union of disjoint cycles $\gamma_1, \ldots, \gamma_m$. For each $1 \le j \le m$, let $W_j = \operatorname{span} \gamma_j$, so that $A_j = [T_{W_j}]_{\gamma_j}$ is a Jordan block by part (a) of Thm. 7.5. So we have the following:

$$J_i = [T_{K_{\lambda_i}}]_{\beta_i} = \begin{pmatrix} A_1 & O & O & \cdots & O \\ O & A_2 & O & \cdots & O \\ O & O & A_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & A_m \end{pmatrix}.$$

For each $1 \le j \le m$, let q_j denote the order of the block A_j. Then by part (a) of exercise 19 of section 7.2,

$$(A_j - \lambda_i I)^r \ne 0 \quad \text{if} \quad 1 \le r < q_j, \quad \text{and} \quad (A_j - \lambda_i I)^{q_j} = O.$$

Let p_i denote the order of the largest Jordan block corresponding to λ_i. Since for any integer $n \ge 0$,

$$(J_i - \lambda_i I)^n = \begin{pmatrix} (A_1 - \lambda_i I)^n & O & O & \cdots & O \\ O & (A_2 - \lambda_i I)^n & O & \cdots & O \\ O & O & (A_3 - \lambda_i I)^n & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & (A_m - \lambda_i I)^n \end{pmatrix},$$

it follows that

$$(J_i - \lambda_i I)^r \ne 0 \quad \text{if} \quad 1 \le r < p_i, \quad \text{and} \quad (J_i - \lambda_i I)^{p_i} = O. \tag{2}$$

Note that since λ_i is the sole eigenvalue of J_i, the minimal polynomial of J_i must have λ_i as its only root by Thm. 7.14. Thus (2) implies that $(t - \lambda_i)^{p_i}$ is the minimal polynomial of J_i. Moreover, by (1) the minimal polynomial of J is the least common multiple of the minimal polynomials of $J_1, \ldots, J_k$ (a result discussed in class). Thus, since the least common multiple of the polynomials $(t - \lambda_1 I)^{p_1}, \ldots, (t - \lambda_k I)^{p_k}$ is 1, the minimal polynomial of T is the product

$$(t - \lambda_1 I)^{p_1} \cdots (t - \lambda_k I)^{p_k}.$$

14. Let $V = \mathbb{R}^3$, and consider the identity linear operator $I : V \to V$. Let $\beta = \{e_1, e_2, e_3\}$ denote the standard ordered basis for V. Note that $V = W_1 \oplus W_2$, where $W_1 = \text{span}\{e_1, e_2\}$ and $W_2 = \text{span}\{e_3\}$. Since

$$[I]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the minimal polynomial of I is $p(t) = t - 1$ by Thm. 7.16. Note that $\beta_1 = \{e_1, e_2\}$ and $\beta_2 = \{e_3\}$ are bases for W_1 and W_2, respectively. Thus

$$[I_{W_1}]_{\beta_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [I_{W_2}]_{\beta_2} = \begin{pmatrix} 1 \end{pmatrix}$$

and so by Thm. 7.16 the minimal polynomials of I_{W_1} and I_{W_2} are $p_1(t) = t - 1$ and $p_2(t) = t - 1$, respectively. Thus $p(t) \neq p_1(t)p_2(t)$, and so the result does not hold in general.

Additional problem

Let $T : F^4 \to F^4$ be a linear operator satisfying $T^3 = T^2$. Then we can write

$$0 = T^3 - T^2 = T^2(T - I) = g(T),$$

where 0 denotes the zero linear operator on F^4, and g is the polynomial $g(t) = t^2(t-1)$. Thus by part (a) of Thm. 7.12, the minimal polynomial p of T divides g. In particular, any root of p is a root of g. By Thm. 7.14, the roots of p are precisely the eigenvalues of T. Thus the possible eigenvalues of T are the roots of g: $\lambda_1 = 0$ and $\lambda_2 = 1$.

Furthermore, any Jordan canonical form J of T must satisfy $J^3 = J^2$, or $g(J) = J^2(J - I) = O$. Thus the minimal polynomial q of J must divide g by way of part (a) of Thm. 7.12. Thus

$$q(t) = t^{k_1}(t-1)^{k_2}$$

for some integers $0 \leq k_1 \leq 2$ and $0 \leq k_2 \leq 1$. By exercise 13 of section 7.3, this means that every block of J associated with $\lambda_1 = 0$ must be of order 1 or 2, and that every block associated with $\lambda_2 = 1$ must be of order 1.

In enumerating the possible Jordan canonical forms of T, there are several cases to consider. Note that in each case, either the blocks associated with λ_1 are furthest to the left in the Jordan canonical form, or those associated with λ_2 are furthest to the left (see the discussion on uniqueness of Jordan canonical forms on pg. 497).

- *The only eigenvalue of T is $\lambda_1 = 0$.* In this case the Jordan canonical form of T must have either 4 blocks of order 1, 2 blocks of order 1 and 1 block of order 2, or 2 blocks of order 2 associated with λ_1. Thus in this case there are a total of 3 possible Jordan canonical forms.

- *The only eigenvalue of T is $\lambda_2 = 1$.* In this case the Jordan canonical form of T must have 4 blocks of order 1 associated with λ_2. Thus in this case there is only 1 possible Jordan canonical form.

- *The eigenvalues of T are $\lambda_1 = 0$ and $\lambda_2 = 1$.* There are several subcases to consider here.

- *There is only 1 block associated with* λ_1*.* If the block associated with λ_1 is of order 1, then there must be exactly 3 blocks of order 1 associated with λ_2. If the block associated with λ_1 is of order 2, then there must be exactly 2 blocks of order 1 associated with λ_2. Since either the blocks associated with λ_1 are furthest left, or those associated with λ_2 are furthest left, we see that there are $2 \times 2 = 4$ possible Jordan canonical forms in this case.
- *There are 2 blocks associated with* λ_1*.* If there are 2 blocks of order 1 associated with λ_1, then there must be exactly 2 blocks of order 1 associated with λ_2. If there is 1 block of order 2 and 2 blocks of order 1 associated with λ_1, then there must be exactly 1 block of order 1 associated with λ_2. Finally, if there are 2 blocks of order 2 associated with λ_1, then there cannot be any blocks associated with λ_2. Since either the blocks associated with λ_1 are furthest left, or those associated with λ_2 are furthest left, we see that there are $2 \times 2 + 1 = 5$ possible Jordan canonical forms in this case.
- *There are 3 blocks associated with* λ_1*.* If there are 3 blocks of order 1 associated with λ_1, then there must be exactly 1 block of order 1 associated with λ_2. If there is 1 block of order 2 and 2 blocks of order 1 associated with λ_1, then there then there cannot be any blocks associated with λ_2. Since either the blocks associated with λ_1 are furthest left, or those associated with λ_2 are furthest left, we see that there are $2 \times 1 + 1 = 3$ possible Jordan canonical forms in this case.
- *There are 4 blocks associated with* λ_1*.* Then there must be 4 blocks of order 1 associated with λ_1, and no blocks associated with λ_2. Thus there is only 1 possible Jordan canonical form in this case.

The total number of possible Jordan canonical forms of T is therefore $3 + 1 + 4 + 5 + 3 + 1 = 17$.