Homework 5

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Section 4.1

11. Let $A \in M_{2\times 2}(F)$. Then A is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Using the various properties of δ listed in the problem statement, we get the following:

$$\begin{aligned}
\delta(A) = \delta\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}\delta\begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \end{pmatrix} + a_{12}\delta\begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} \\
&= a_{11}\left[a_{21}\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] + a_{12}\left[a_{21}\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_{22}\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right] \\
&= a_{11}a_{22} + a_{12}a_{21}\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{1}$$

Furthermore, we have the following:

$$\begin{aligned}
0 = \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} &= \delta\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

and thus we see that

$$\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1.$$

Using this, (1) becomes

$$\delta(A) = a_{11}a_{22} + a_{12}a_{21}\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = \det(A),$$

as desired.

Section 4.2

24. Let $A_1, \dots, A_n \in F^{1\times n}$ denote the row vectors of A, and say $A_k = 0$ for some $1 \le k \le n$. Then the n-linearity of the determinant implies the following:

$$\begin{aligned}\det(A) &= \det(A_1, \dots, A_k, \dots, A_n)\\ &= \det(A_1, \dots, 0, \dots, A_n)\\ &= \det(A_1, \dots, 0+0, \dots, A_n)\\ &= \det(A_1, \dots, 0, \dots, A_n) + \det(A_1, \dots, 0, \dots, A_n)\\ &= 2\det(A).\end{aligned}$$

Subtracting $\det(A)$ from both ends of this equality we find that $\det(A) = 0$.

29. Note that if E is an elementary matrix obtained by performing an elementary row operation on I, then E^t is the elementary matrix obtained by performing the corresponding operation on the columns of I (this is immediate from the definition of transpose of a matrix). Thus it suffices to consider elementary matrices of the first form in the discussion below. I will show that the result holds for each type of elementary matrix individually.

Type 1: Let E be obtained from I by swapping rows i and j of I, where $i \neq j$. Note that E^t is obtained from I by performing the same operations on the columns of I, i.e., by swapping columns i and j of I. But since I is symmetric, it follows that $E^t = E$. In particular, $\det(E^t) = \det(E)$.

Type 2: Let E be obtained from I by multiplying row i by a scalar $c \neq 0$. Then E is symmetric, so that $E^t = E$. In particular, $\det(E^t) = \det(E)$.

Type 3: Let E be obtained from I by adding $c \neq 0$ times row i to row j, where $i \neq j$. Then E^t can be obtained from I by adding c times column i to column j. But note that this is the same as adding c times row j to row i. Thus by Thm. 4.6, $\det(E^t) = \det(I) = \det(E)$.

Section 4.3

10. By Thm. 4.7 we have

$$0 = \det(O) = \det(M^k) = \det(M)^k,$$

which implies that $\det(M) = 0$.

12. By Thm. 4.7 and Thm. 4.8 we have

$$1 = \det(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)^2,$$

which implies that $\det(Q) = \pm 1$.

21. As discussed in class, every square matrix A can be transformed into a diagonal matrix D through a series of elementary row operations of types 1 and 3. Since elementary row operations of type 1 flip the sign of the determinant, and those of type 3 preserve the value of the determinant (see pg. 217), we have $\det(D) = \pm\det(A)$. In particular, if $\det(D) = -\det(A)$, then multipling any row of D by -1 – an elementary row operation of type 2 – will yield another diagonal matrix D' with $\det(D') = -\det(D) = \det(A)$ (again, see pg. 217). Thus in any case, we see that A can be transformed through elementary row operations into a diagonal matrix D with $\det(D) = \det(A)$.

Say A is an $s \times s$ matrix and C is an $r \times r$ matrix. Then B is an $s \times r$ matrix. Applying the above result to the problem at hand, we see that the $n \times n$ matrix

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

can be transformed through a series of elementary row operations on the upper s rows to a matrix

$$M' = \begin{pmatrix} D & B' \\ O & C \end{pmatrix}$$

satisfying $\det(M') = \det(M)$, where D is an $s \times s$ diagonal matrix with $\det(D) = \det(A)$. Explicitly, there exist $n \times n$ elementary matrices $E_1, \dots E_k$ and $s \times s$ elementary matrices $F_1, \dots, F_k$ satisfying

$$E_1 \cdots E_k M = E_1 \cdots E_k \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} F_1 \cdots F_k A & F_1 \cdots F_k B \\ O & C \end{pmatrix} = \begin{pmatrix} D & B' \\ O & C \end{pmatrix} = M'$$

and $\det(E_1 \dots E_k) = \det(F_1 \dots F_k) = 1$. Similarly, M' can be transformed through a series of elementary row operations on the lower r rows to a matrix

$$M'' = \begin{pmatrix} D & B' \\ O & G \end{pmatrix}$$

satisfying $\det(M'') = \det(M') = \det(M)$, where G is an $r \times r$ diagonal matrix with $\det(G) = \det(C)$. Let $d_1, \dots, d_s$ denote the diagonal elements of D, and let $g_1, \dots g_r$ denote the diagonal elements of G. Then $\det(A) = \det(D) = d_1 \cdots d_s$ and $\det(C) = \det(G) = g_1 \cdots g_r$. Therefore, through the n-linearity of the determinant,

$$\det(M'') = \det \begin{pmatrix} D & B' \\ O & G \end{pmatrix} = (d_1 \cdots d_s)(g_1 \cdots g_r) \det \begin{pmatrix} I_s & B'' \\ O & I_r \end{pmatrix} = \det(A)\det(C)\det \begin{pmatrix} I_s & B'' \\ O & I_r \end{pmatrix}$$

for some $s \times r$ matrix B'', and where I_s and I_r denote the $s \times s$ and $r \times r$ identity matrices, respectively. Since the $n \times n$ matrix

$$N = \begin{pmatrix} I_s & B'' \\ O & I_r \end{pmatrix}$$

is upper triangular with 1's along the diagonal, we see that $\det(N) = 1$, and therefore

$$\det(M) = \det(M'') = \det(A)\det(C)\det(N) = \det(A)\det(C),$$

as desired.