Homework 4

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1. (Cryer & Chan, Exercise 4.11) The process $\{Y_t\}$ is of the form

$$Y_t = \phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2},$$

with $\phi = 0.8$, $\theta_1 = -0.7$, and $\theta_2 = -0.6$. In particular, note that the process has AR characteristic polynomial $\phi(x) = 1 - 0.8x$, which has the single root $x = 1.25 > 1$. Thus the process is stationary.

First, we compute

$$\mathrm{E}(e_t Y_t) = \mathrm{E}[e_t(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] = \mathrm{E}(e_t^2) = \sigma_e^2.$$

Next, we have

$$\begin{aligned}
\mathrm{E}(e_{t-1} Y_t) &= \mathrm{E}[e_{t-1}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(e_{t-1} Y_{t-1}) - \theta_1 \mathrm{E}(e_{t-1}^2) \\
&= \phi \sigma_e^2 - \theta_1 \sigma_e^2 \\
&= (\phi - \theta_1)\sigma_e^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{E}(e_{t-2} Y_t) &= \mathrm{E}[e_{t-2}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(e_{t-2} Y_{t-1}) - \theta_2 \mathrm{E}(e_{t-2}^2) \\
&= \phi(\phi - \theta_1)\sigma_e^2 - \theta_2 \sigma_e^2 \\
&= [\phi(\phi - \theta_1) - \theta_2]\sigma_e^2.
\end{aligned}$$

Thus we can write the autocovariance function as

$$\begin{aligned}
\gamma_k &= \mathrm{Cov}(Y_t, Y_{t-k}) \\
&= \mathrm{E}[Y_{t-k}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(Y_{t-k} Y_{t-1}) + \mathrm{E}(Y_{t-k} e_t) - \theta_1 \mathrm{E}(Y_{t-k} e_{t-1}) - \theta_2 \mathrm{E}(Y_{t-k} e_{t-2}) \\
&= \phi \gamma_{k-1} + \mathrm{E}(Y_{t-k} e_t) - \theta_1 \mathrm{E}(Y_{t-k} e_{t-1}) - \theta_2 \mathrm{E}(Y_{t-k} e_{t-2})
\end{aligned}$$

for any time t and lag k.

a. Let $k > 2$. Then the expression for γ_k above immediately gives us

$$\gamma_k = \phi \gamma_{k-1},$$

so that

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi \rho_{k-1} = 0.8 \rho_{k-1}.$$

b. Again, using the expressions computed above, we get

$$\gamma_2 = \phi \gamma_1 - \theta_2 \mathrm{E}(Y_{t-2} e_{t-2}) = \phi \gamma_1 - \theta_2 \sigma_e^2,$$

so that

$$\phi_2 = \frac{\gamma_2}{\gamma_0} = \phi \rho_1 - \frac{\theta_2 \sigma_e^2}{\gamma_0} = 0.8\rho_1 + \frac{0.6\sigma_e^2}{\gamma_0}.$$

2. (Cryer & Chan, Exercise 4.12)

   a. Note that in general, an MA(2) process with parameters θ_1 and θ_2 has autocorrelation function ρ_k given by

$$\begin{aligned}
\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_k &= 0 \quad \text{for every } k > 2.
\end{aligned}$$

   Thus we find that for both of the processes in question, $\rho_1 = -5/38$, $\rho_2 = -3/19$, and $\rho_k = 0$ for every $k > 2$. The two processes therefore have the same autocorrelation structure.

   b. The process with $\theta_1 = \theta_2 = 1/6$ has MA characteristic polynomial $\theta(x) = 1 - (1/6)x - (1/6)x^2$, which has roots $x = -3, 2$. So this process is invertible. On the other hand, the process with $\theta_1 = -1$ and $\theta_2 = 6$ has MA characteristic polynomial $\theta(x) = 1 + x - 6x^2$, which has roots $x = -1/3, 1/2$, and so this second process is *not* invertible.

   This is an example of the following result: there is only one set of parameter values $\theta_1, \ldots, \theta_q$ that yield an invertible MA(q) process with a given autocorrelation function.

3. (Cryer & Chan, Exercise 4.21)

4.

5. (Cryer & Chan, Exercise 5.1)

6. (Cryer & Chan, Exercise 5.4)