Homework 4 Solution

4.11 (a)Without loss of generality we assume the mean of the series is zero and we then have:

$$
\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathbb{E}[(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-k}], \\
&= 0.8\mathbb{E}(Y_{t-1}Y_{t-k}) + \mathbb{E}(e_tY_{t-k}) + 0.7\mathbb{E}(e_{t-1}Y_{t-k}) + 0.6\mathbb{E}(e_{t-2}Y_{t-k}), \\
&= 0.8\mathrm{Cov}(Y_{t-1}, Y_{t-k}) + 0 + 0 + 0. \quad \text{(since k>2 and we assume zero mean)}
\end{aligned}
$$

Since $\gamma_k = 0.8\gamma_{k-1}$, it is easily seen that:

$$
\rho_k = \frac{\gamma_k}{\sqrt{\mathrm{Var}(Y_t)\mathrm{Var}(Y_{t-k})}} = \frac{0.8\gamma_{k-1}}{\sqrt{\mathrm{Var}(Y_{t-1})\mathrm{Var}(Y_{t-k})}} = 0.8\rho_{k-1}.
$$

(b)

$$
\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathbb{E}[(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-2}], \\
&= \mathbb{E}[(0.8Y_{t-1} + e_t)Y_{t-2}], \\
&= 0.8\mathrm{Cov}(Y_{t-1}, Y_{t-2}) + 0.6\mathbb{E}(e_{t-2}Y_{t-2}).
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(e_{t-2}Y_{t-2}) = \mathbb{E}(e_tY_t) = \mathbb{E}(e_t(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})) = \sigma_e^2. \\
\gamma_2 = 0.8\gamma_1 + 0.6\sigma_e^2.
\end{aligned}
$$

Then using the same logic in (a), we have the required result.

4.12 (a) plug the values into Equation(4.2.3) at page 63 or alternatively,use the following R code.

```
ARMAacf(ma=c(-1/6,-1/6))
```

```
##          0          1          2          3
##  1.0000000 -0.1315789 -0.1578947  0.0000000
```

```
ARMAacf(ma=c(1,-6))
```

```
##          0          1          2          3
##  1.0000000 -0.1315789 -0.1578947  0.0000000
```

(b) The roots of the two polynomials are reciprocals of one another. And only the model with $\theta_1 = \theta_2 = 1/6$ is invertible.

4.21 (a)start from $k = 0$ we get the following results:

$$
\begin{aligned}
\mathrm{Var}(Y_t) &= [1+(-1)^2+(0.5)^2]\sigma_e^2 = 2.25\sigma_e^2. \\
\mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}(e_{t-1}-e_{t-2}+0.5e_{t-3}, e_{t-2}-e_{t-3}+0.5e_{t-4}) \\
&= \mathrm{Cov}(-e_{t-2}, e_{t-2}) + \mathrm{Cov}(0.5e_{t-3}, -e_{t-3}) \\
&= -1.5\sigma_e^2. \\
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathrm{Cov}(e_{t-1}-e_{t-2}+0.5e_{t-3}, e_{t-3}-e_{t-4}+0.5e_{t-5}) \\
&= \mathrm{Cov}(0.5e_{t-3}, e_{t-3}) \\
&= 0.5\sigma_e^2.
\end{aligned}
$$

And it is easily seen that the autocovariance with lags larger than 2 are all zero.

(b)We could define a new series $e_t^* = e_{t-1}$, and now we have $Y_t = e_t^* - e_{t-1}^* + 0.5e_{t-2}^*$ and since the error term has the same propertise as e_t, we found out that the series is actually a MA(2) process in disguise.

Problem 4:

(a)

(i)This is an AR(2) model with AR characteristic polynomial $\phi(x) = 1 + 0.2x - 0.48x^2$,the two roots are 1.67 and -1.25, both larger than 1 in modulus, so it is **stationary**.And the process is **invertible** since its an AR process.

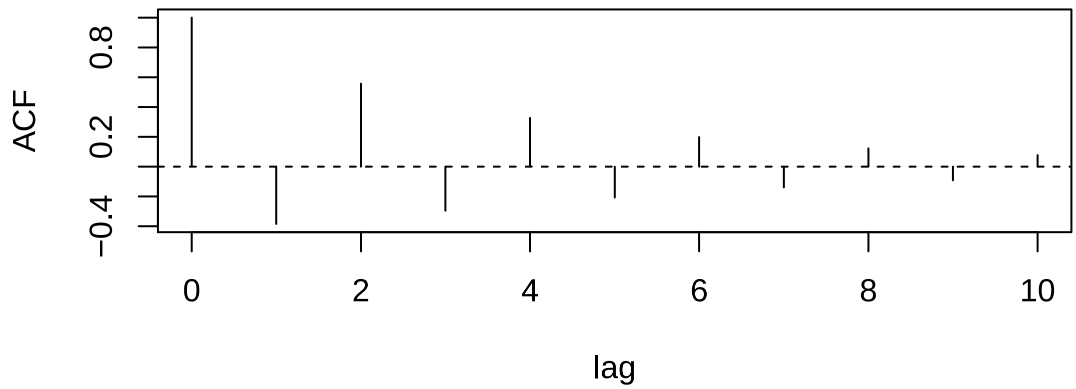
(ii)This is an ARMA(1,1) model with AR characteristic polynomial $\phi(x) = 1 + 0.6x$,the root is -1.67, larger than 1 in modulus, so it is **stationary**. The MA characteristic polynomial is $\theta(x) = 1 + 1.2x$ with root 0.83 , so the process is **not invertible**.

(iii)This is an AR2(2) model with AR characteristic polynomial $\phi(x) = 1 + 1.8x + 0.81x^2$,the root is -1.11, larger than 1 in modulus, so it is **stationary**. Again it is obviously **invertible**.

(iv)This is an ARMA(1,1) model with AR characteristic polynomial $\phi(x) = 1 + 1.6x$,the root is -0.625, smaller than 1 in modulus, so it is **not stationary**. The MA characteristic polynomial is $\theta(x) = 1 - 0.4x$ with root 2.5 , so the process is **invertible**.

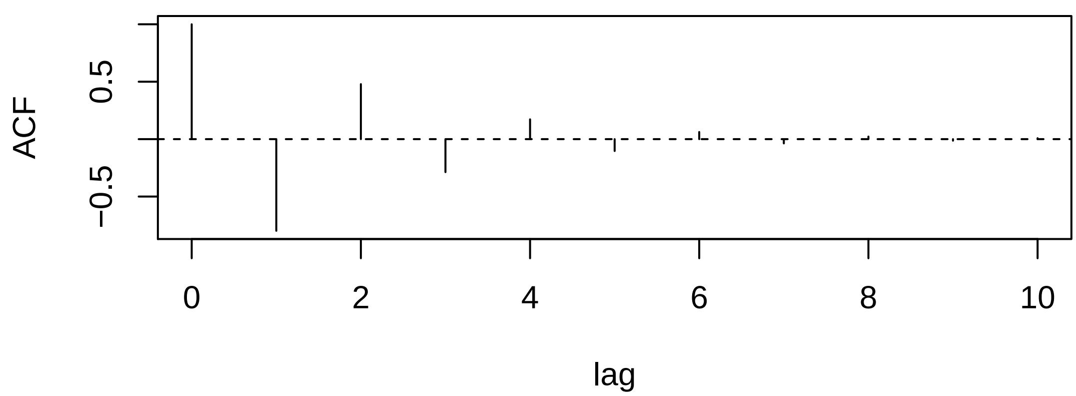
(b)The following code will give you the acf plot for the three stationary process.

```
plot(c(0:10),ARMAacf(ar=c(-0.2,0.48),lag.max = 10),xlab="lag",ylab="ACF",type="h")
abline(h=0,lty=2)
```



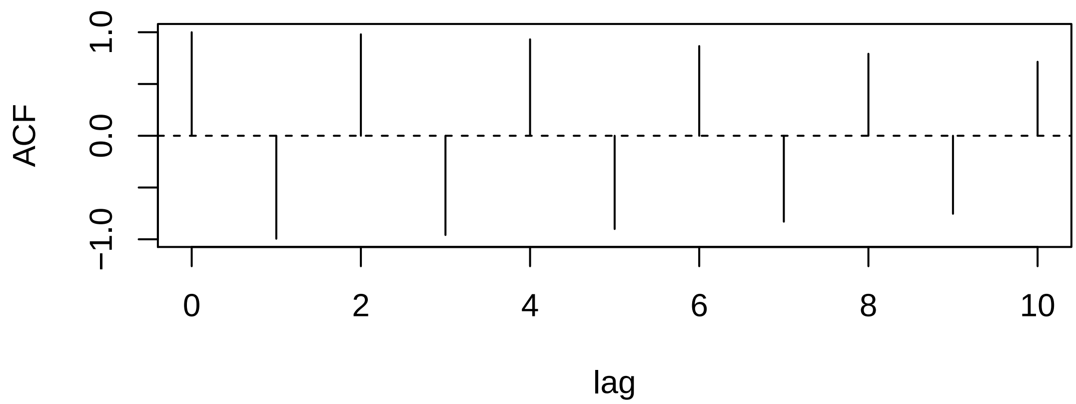
(ii)

```
plot(c(0:10),ARMAacf(ar=c(-0.6),ma=c(-1.2),lag.max = 10),xlab="lag",ylab="ACF",type="h")
abline(h=0,lty=2)
```



(iii)

```
plot(c(0:10),ARMAacf(ar=c(-1.8,-0.81),lag.max = 10),xlab="lag",ylab="ACF",type="h")
abline(h=0,lty=2)
```



(c)Using formula(4.4.7) in the text book, we can calculate the coefficients recursively, with the result follow below:

$$Y_t = e_t - 0.2e_{t-1} + 0.52e_{t-2} - 0.20e_{t-3} \tag{1}$$
$$Y_t = e_t + 0.6e_{t-1} - 0.36e_{t-2} + 0.216e_{t-3} \tag{2}$$
$$Y_t = e_t - 1.8e_{t-1} + 2.43e_{t-2} - 2.916e_{t-3} \tag{3}$$

Problem 5.1 (a)This looks like an ARMA(2,1) model with $\phi_1 = 1$ and $\phi_2 = -0.25$. We need to check the stationary conditions of Equation(4.3.11) at page 72. Here $\phi_1 + \phi_2 = 0.75 < 1$, $\phi_2 - \phi_1 = -1.25 < 1$, $|\phi_2| = 0.25 < 1$ so the process is a stationary and invertible ARMA(2,1) model with $\phi_1 = 1$, $\phi_2 = -0.25$ and $\theta_1 = 0.1$.

(b)Initially it looks like an AR(2) model but 2+(-1)=1 which is not strictly less than 1. Rewrite as $Y_t - Y_{t-1} = (Y_{t-1} - Y_{t-2}) + e_t$ suggests an AR(1) model after the differences but the AR coefficients is 1 now. Actually, the second difference $Y_t - 2Y_{t-1} + Y_{t-2} = e_t$ is a white noise, so that the original series is an IMA(2,0) model.

(c)The AR part is stationary since the inequalities of Equation(4.3.11) are satisfied. Applying the same equation to the MA part of the model, we see that the MA part is invertible. Therefore, the model is a stationary, and invertible ARMA(2,2) model with $\phi_1 = 0.5$, $\phi_2 = -0.5$, $\theta_1 = 0.5$ and $\theta_2 = -0.25$.

Problem 5.4 (a)We have that $E[Y_t] = A + Bt$, so the mean varies with t(suppose B is not zero), so the process is **not stationary**.

(b)$\triangledown Y_t = (A+Bt+X_t) - [A+B(t-1)+X_{t-1}] = B+X_t - X_{t-1} = B + \triangledown X_t$. Since X_t is a random walk which means that $\triangledown X_t$ is zero-mean. So we have $\mathbb{E}(\triangledown Y_t) = B$. $\text{Cov}(\triangledown Y_t, \triangle Y_{t-k}) = \text{Cov}(B + \triangledown X_t, B + \triangledown X_{t-k}) = 0$, since $\triangledown X_t$ is white noise. So $\triangledown Y_t$ is **stationary**.

(c)It is still not stationary since the mean still depends on t(suppose $\mathbb{E}(B) \neq 0$).

(d)Now $\mathbb{E}(\triangledown Y_t = \mathbb{E}(B)$ is still constant. $\text{Cov}(\triangledown Y_t, \triangle Y_{t-k}) = \text{Cov}(B + \triangledown X_t, B + \triangledown X_{t-k})) = \text{Var}(B)$ for all k. Wo we do have **stationarity**