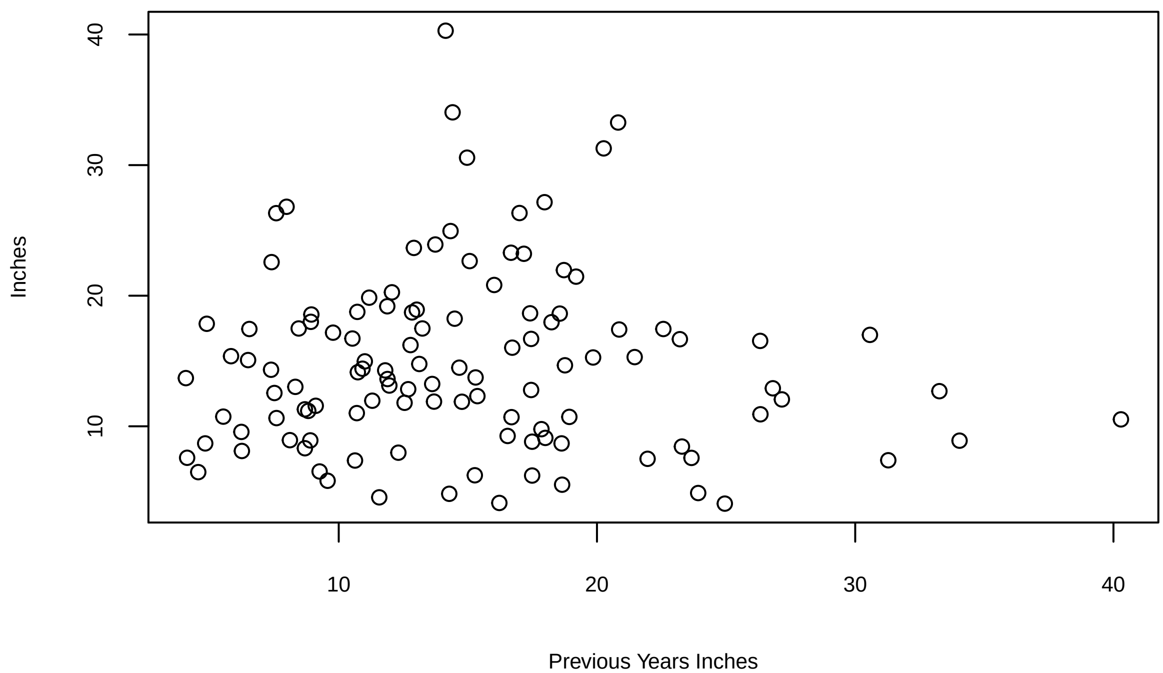
Homework 1 Solution

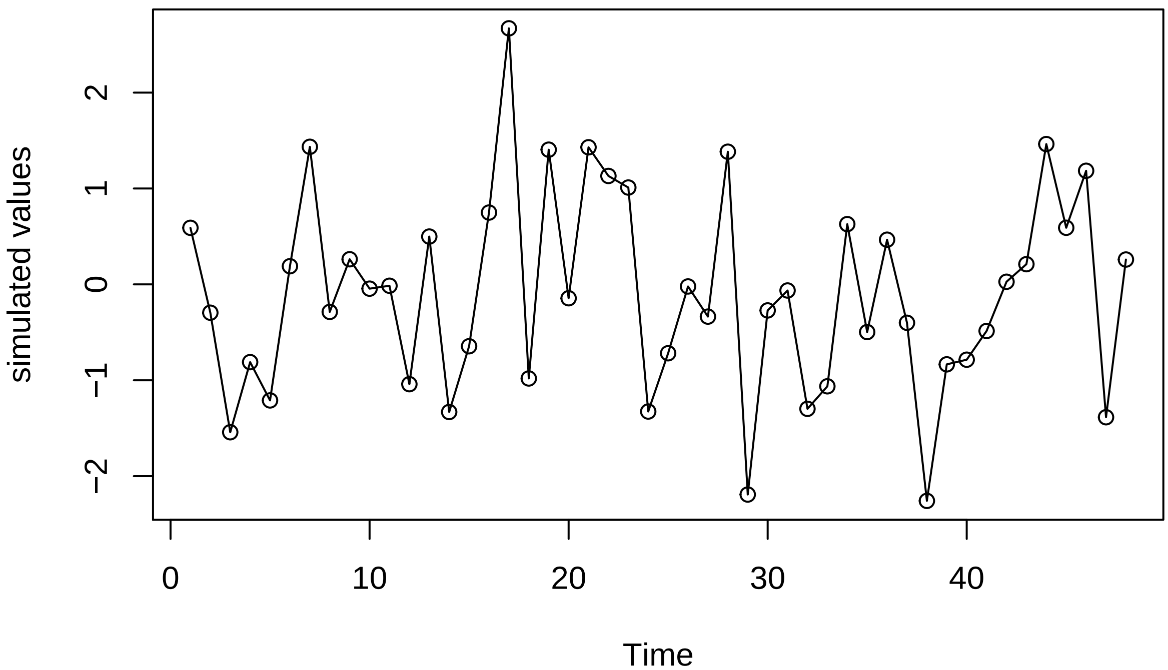
1.1

```r
library(TSA)
data(larain)
win.graph(width=3,height=3,pointsize = 8)
plot(y=larain,x=zlag(larain),ylab='Inches',xlab='Previous Years Inches')
```



1.3

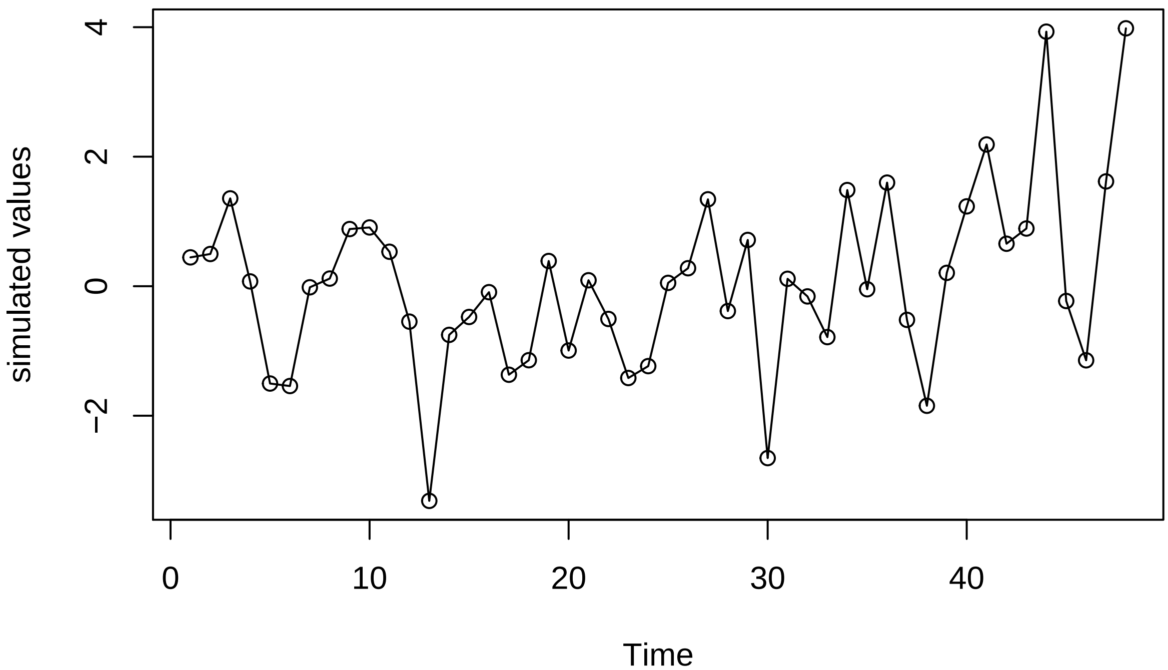
```r
plot(ts(rnorm(n=48)),type='o',ylab='simulated values')
```



Note:If you repeat this command R will use a new “random numbers” each time. If you want to reproduce the same simulation first use the command set.seed(######) where ###### is an integer of your choice

1.5

```
plot(ts(rt(n=48,df=5)),type='o',ylab='simulated values')
```



simulated values
4
2
0
−2
0
10
20
30
40
Time

Problem 2.1.

Part (a):

$$\begin{aligned}
&\text{Cov}X, Y = \text{Corr}(X, Y) \cdot \sqrt{\text{Var}X\text{Var}X} = 0.25 \cdot 3 \cdot 2 = 1.5. \\
&\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = 9 + 4 + 2(3 \cdot 2 \cdot 0.25) = 16.
\end{aligned}$$

Part (b):

$$\text{Cov}(X, X+Y) = \text{Cov}(X, X) + \text{Cov}(X, Y) = 9 + 1.5 = 10.5.$$

Part (c):

$$\begin{aligned}
\text{Var}(X-Y) &= 9 + 4 - 2 \cdot 1.5 = 10. \\
\text{Cov}(X+Y, X-Y) &= \text{Cov}(X, X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Cov}(Y, Y), \\
&= \text{Var}(X) - \text{Var}(Y) = 9 - 4 = 5. \\
\text{Corr}(X+Y, X-Y) &= \frac{\text{Cov}(X+Y, X-Y)}{\sqrt{\text{Var}(X+Y)\text{Var}(X-Y)}} = \frac{5}{\sqrt{16 \cdot 10}} = 0.3953.
\end{aligned}$$

Problem 2.2.

$$\begin{aligned}
\text{Cov}(X+Y, X-Y) &= \text{Cov}(X, X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Cov}(Y, Y) \\
&= \text{Var}(X) - \text{Var}(Y) = 0.
\end{aligned}$$

Problem 2.5.

Part a:

$$\mathbb{E}(Y_t) = \mathbb{E}(5 + 2t + X_t) = 5 + 2t.$$

Part (b):

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}), \\
&= \text{Cov}(X_t, X_{t-k}), \\
&= \gamma_k.
\end{aligned}$$

Part (c): from part(a) we have that the mean of $\{Y_t\}$ varies over time, so it is not stationary.

Problem 2.6.

Part a:

$$Y_t = \begin{cases} X_t & \text{for } t \text{ odd} \\ X_{t+3} & \text{for } t \text{ even} \end{cases}$$

$$\text{Cov}(Y_t, Y_{t-k}) = \text{Cov}(X_t, X_{t-k}) \ , \quad \text{No matter } t, t-k \text{ even or odd.}$$

Since X_t is stationary, $\text{Cov}(X_t, X_{t-k}) = \gamma_k$ is free of t for all lags k.

Part b:

$$\mathbb{E}(Y_t) = \begin{cases} \mathbb{E}(X_t) & \text{for } t \text{ odd} \\ \mathbb{E}(X_t) + 3 & \text{for } t \text{ even} \end{cases}$$

So Y_t is not stationary because the mean is varying.

Problem 2.10.

Part (a):

$$\begin{aligned} \mathbb{E}(Y_t) &= \mathbb{E}(\mu_t + \sigma_t X_t) = \mu_t + \sigma_t \mathbb{E}(X_t) = \mu_t. \text{ since } \{X_t\} \text{ is zero-mean.} \\ \mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k}) = \sigma_t \sigma_{t-k} \mathrm{Cov}(X_t, X_{t-k}), \end{aligned}$$

We notice that X_t is uni-variance, which means that:

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t-k}) &= \mathrm{Corr}(X_t, X_{t-k}) \cdot \sqrt{\mathrm{Var}(X_t)\mathrm{Var}(X_{t-k})} = \rho_k \\ \mathrm{Cov}(Y_t, Y_{t-k}) &= \sigma_t \sigma_{t-k} \rho_k. \end{aligned}$$

Part (b):

$$\begin{aligned} \mathrm{Var}(Y_t) &= \mathrm{Var}(\mu_t + \sigma_t X_t) = \sigma_t^2, \\ \mathrm{Corr}(Y_t, Y_{t-k}) &= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\sigma_t^2 \sigma_{t-k}^2}} = \rho_k. \end{aligned}$$

So we have the autocorrelation function for Y_t only depends on the lag k, it is **not** stationary since the mean $\mathbb{E}(Y_t) = \mu_t$ is not a constant.

Part (c):
Based on what we learn form part(a) and (b), we can let $Y_t = \mu + \sigma_t X_t$, where μ is a now constant. We can check that:

$$\mathbb{E}(Y_t) = \mu \text{ and } \mathrm{Corr}(Y_t, Y_{t-k}) = \rho_k.$$

However,Y_t is **not** stationary because $\mathrm{Cov}(Y_t, Y_{t-k}) = \sigma_t \sigma_{t-k} \rho_k$, which depends on t.

Problem 2.15.

Part (a):

$$\mathbb{E}(Y_t) = (-1)^t \mathbb{E}(X) = 0.$$

Part (b):

$$\mathrm{Cov}(Y_t, Y_{t-k}) = \mathrm{Cov}((-1)^t X, (-1)^{t-k} X) = (-1)^{2t-k} \mathrm{Var}(X) = (-1)^k \cdot \mathrm{Var}(X).$$

Part (c):
It is stationary since the mean is a constant and autocovariance only depends on the lag k.

Problem 2.26.

Part (a):
Suppose $\{Y_t\}$ is a stationary process with mean μ and autocovariance γ_k. Then we have:

$$\begin{aligned}
\mathbb{E}(Y_t - Y_s) = \mu - \mu = 0,\ \text{thus}\ \mathbb{E}[(Y_t - Y_s)^2] = \text{Var}(Y_t - Y_s).\\
\Gamma_{t,s} &= \frac{1}{2}\mathbb{E}[(Y_t - Y_s)^2] = \frac{1}{2}\text{Var}(Y_t - Y_s),\\
&= \frac{1}{2}[\text{Var}(Y_t) + \text{Var}(Y_s) - 2\text{Cov}(Y_t, Y_s],\\
&= \gamma_0 - \gamma_{|t-s|}.
\end{aligned}$$

Part (b):
Without loss of generality, we can assume t¿s, so that:

$$\begin{aligned}
Y_t - Y_s &= e_{s+1} + e_{s+2} + \cdots + e_{t-1} + e_t.\\
\Gamma_{t,s} &= \frac{1}{2}\mathbb{E}[(Y_t - Y_s)^2] = \frac{1}{2}\text{Var}(Y_t - Y_s),\\
&= \frac{1}{2}\text{Var}(e_{s+1} + e_{s+2} + \cdots + e_{t-1} + e_t),\\
&= \frac{1}{2}(t-s)\sigma_e^2.\ \text{which only depends on the time difference.}
\end{aligned}$$