Homework 1

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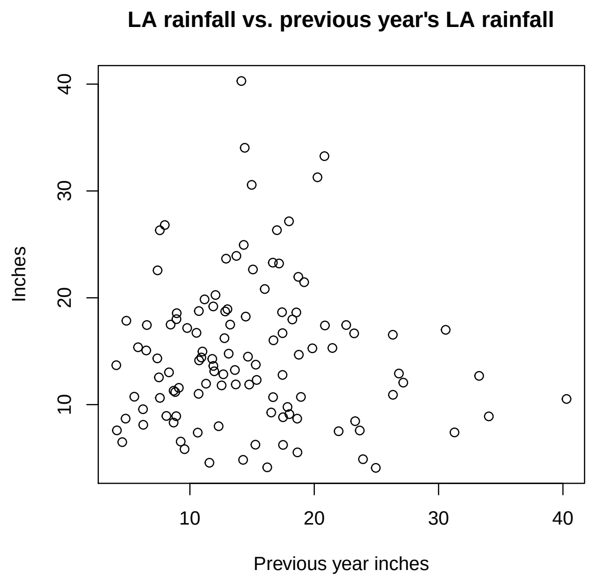
1. (Cryer & Chan, Exercise 1.1) This plot can be reproduced using the following R code:

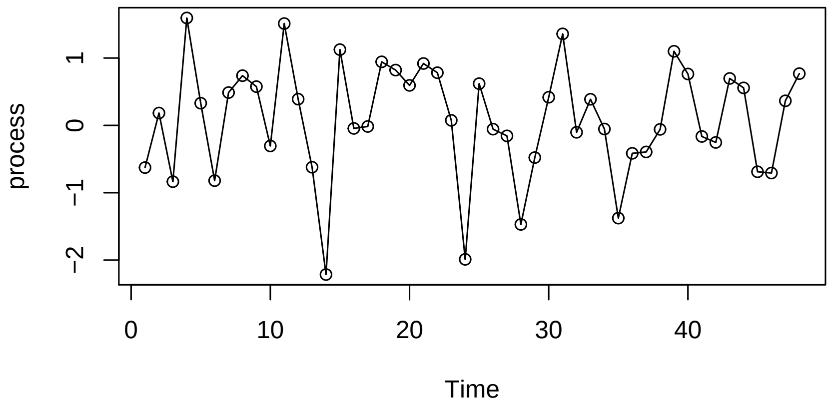
```
data(larain)
plot(
  y = larain, x = zlag(larain),
  xlab = "Previous year inches",
  ylab = "Inches",
  main = "LA rainfall vs. previous year's LA rainfall"
)
```



   The lack of linearity in the plot indicates that there is little correlation between this year's rainfall amount and the previous year's.

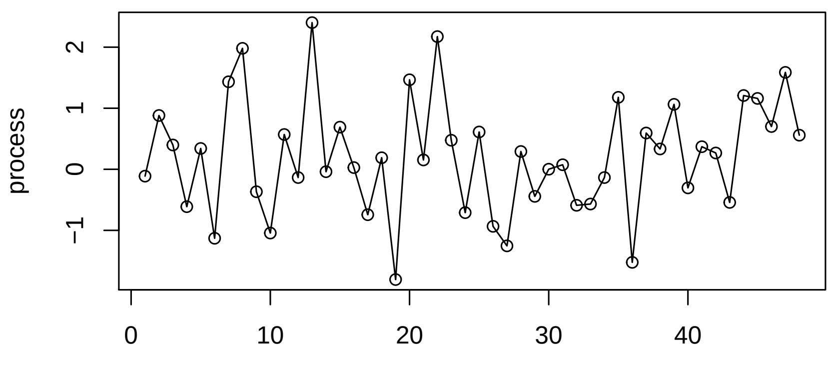
2. (Cryer & Chan, Exercise 1.3) The following function simulates a random process of length 48 with independent, normal values, and plots the associated time series plot:

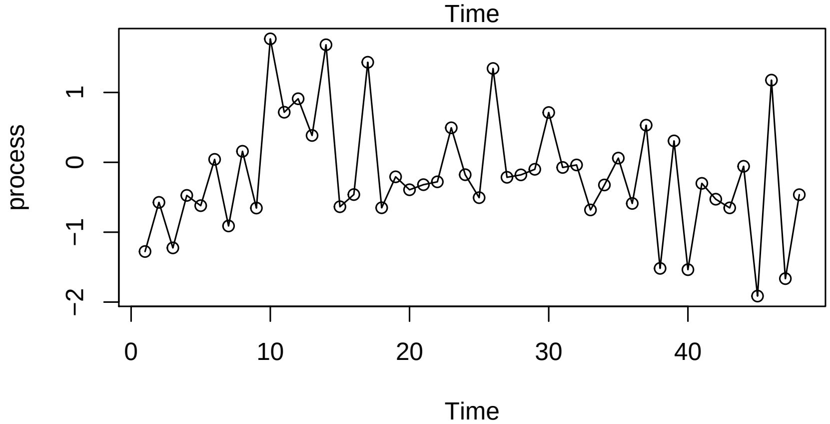
```
set.seed(1)  # Set a random seed for reproducability.

plot_process <- function() {
  process <- ts(rnorm(48))
  plot(process, type = "o")
}

plot_process()
```



The resulting output does appear to consist of random fluctuations about the mean of the process (which is zero in this case). Performing the experiment a few more times yields simular results:

```
plot_process()
plot_process()
```

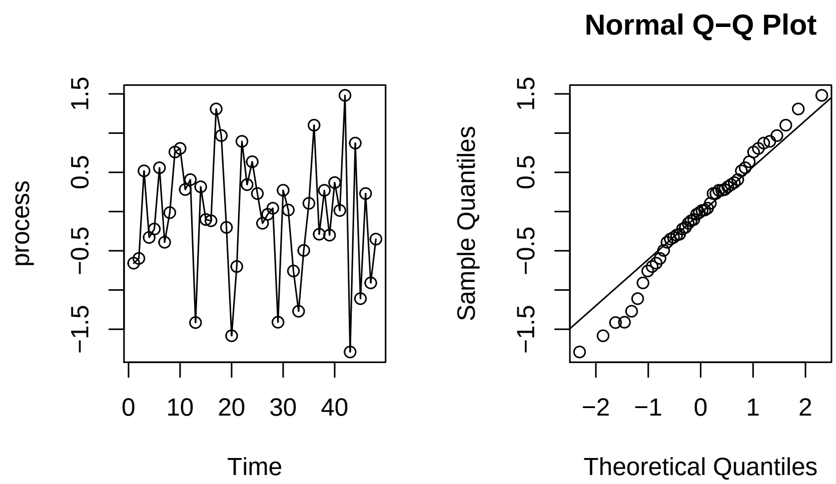




3. (Cryer & Chan, Exercise 1.5) The following function simulates a random process of length 48 with independent t-distributed values, each with 5 degrees of freedom, and plots the associated time series plot, along with a normal Q-Q plot of the generated values:

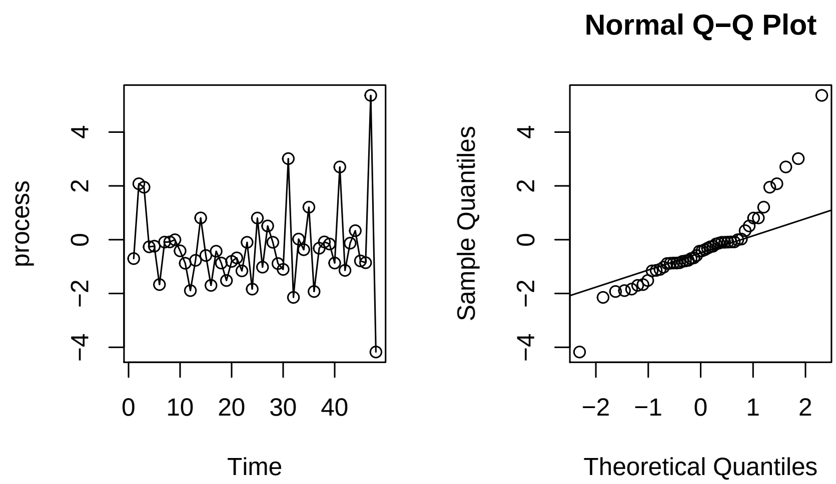
```
set.seed(1)  # Set a random seed for reproducability.
```

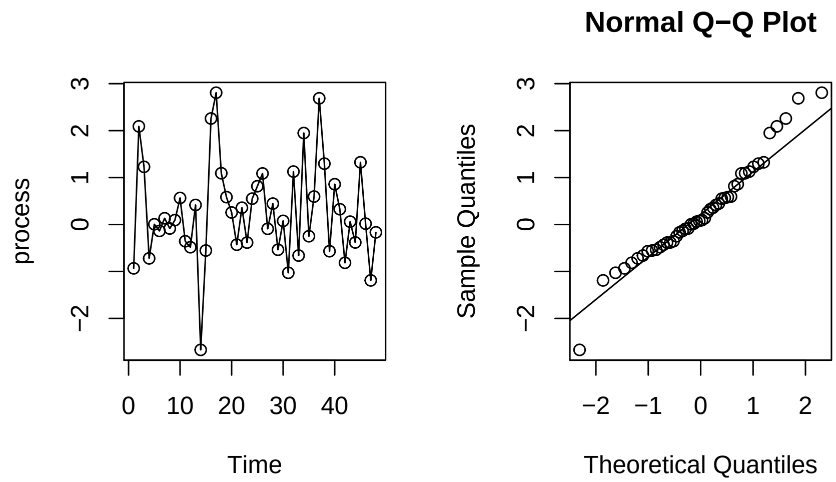
```r
plot_process <- function() {
  process <- ts(rt(48, 5))
  old_par <- par(mfrow = c(1, 2))
  plot(process, type = "o")
  qqnorm(process)
  qqline(process)
  par(old_par)
}

plot_process()
```



The resulting time series plot does appear to consist of random fluctuations about the mean of the process (which is zero in this case). Moreover, the normal Q-Q plot indicates that the data are unlikely to be from a normal distribution, since the tails of the empirical distribution are too heavy. Performing the experiment a few more times yields similar results:

```r
plot_process()
plot_process()
```





4. (Cryer & Chan, Exercise 2.1)

   a. Since $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 4$, and $\mathrm{Corr}(X, Y) = 1/4$, we have
   $$\mathrm{Cov}(X, Y) = \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}\mathrm{Corr}(X, Y) = 3/2.$$
   Thus,
   $$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y) = 16.$$

   b. Since $\mathrm{Var}(X) = 9$, and $\mathrm{Cov}(X, Y) = 3/2$ by part (a), we have
   $$\mathrm{Cov}(X, X + Y) = \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) = \mathrm{Var}(X) + \mathrm{Cov}(X, Y) = 21/2.$$

   c. First, note that
   $$\mathrm{Corr}(X + Y, X - Y) = \frac{\mathrm{Cov}(X + Y, X - Y)}{\sqrt{\mathrm{Var}(X + Y)\mathrm{Var}(X - Y)}}.$$
   Since $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 4$, and by part (a), $\mathrm{Cov}(X, Y) = 3/2$,
   $$\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y) = 10.$$
   and
   $$\begin{aligned}\mathrm{Cov}(X + Y, X - Y) &= \mathrm{Cov}(X, X) - \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) - \mathrm{Cov}(Y, Y) \\ &= \mathrm{Var}(X) - \mathrm{Var}(Y) = 5.\end{aligned}$$
   Thus, since $\mathrm{Var}(X + Y) = 16$ by part (a), we get
   $$\mathrm{Corr}(X + Y, X - Y) = 5/\sqrt{160}.$$

5. (Cryer & Chan, Exercise 2.2) Since $\mathrm{Var}(X) = \mathrm{Var}(Y)$, we have
$$\begin{aligned}\mathrm{Cov}(X + Y, X - Y) &= \mathrm{Cov}(X, X) - \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) - \mathrm{Cov}(Y, Y) \\ &= \mathrm{Var}(X) - \mathrm{Var}(Y) = 0.\end{aligned}$$

6. (Cryer & Chan, Exercise 2.5)

   a. Since $\{X_t\}$ is a mean-zero process, the mean function for $\{Y_t\}$ is given by

   $$\tilde{\mu}_t = \mathrm{E}(Y_t) = 5 + 2t + \mathrm{E}(X_t) = 5 + 2t$$

   for any time t.

   b. Since the process $\{X_t\}$ is stationary, the autocovariance function of $\{Y_t\}$ is given by

   $$\begin{aligned}\tilde{\gamma}_{t,s} &= \mathrm{Cov}(Y_t, Y_s) \\ &= \mathrm{Cov}(5 + 2t + X_t, 5 + 2s + X_s) \\ &= \mathrm{Cov}(X_t, X_s) \\ &= \mathrm{Cov}(X_0, X_{|t-s|}) = \gamma_{|t-s|}\end{aligned}$$

   for any times t and s.

   c. The process $\{Y_t\}$ is *not* stationary, since by part (a) its mean function $\tilde{\mu}_t$ is not constant with respect to time t.

7. (Cryer & Chan, Exercise 2.6)

   a. Define

   $$\delta_t = \begin{cases} 0 & \text{if } t \text{ is odd} \\ 1 & \text{if } t \text{ is even} \end{cases}.$$

   Then we can write $Y_t = 3\delta_t$ for any time t. Thus, for any time t and lag k,

   $$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(X_t + 3\delta_t, X_{t-k} + 3\delta_{t-k}) \\ &= \mathrm{Cov}(X_t, X_{t-k}) \\ &= \mathrm{Cov}(X_0, X_k),\end{aligned}$$

   where the final equality is due to the fact that $\{X_t\}$ is stationary.

   b. The mean function for $\{Y_t\}$ is given by

   $$\mu_t = \mathrm{E}(Y_t) = \mathrm{E}(X_t) + 3\delta_t$$

   for any time t. Since $\{X_t\}$ is stationary, $\mathrm{E}(X_t)$ is constant with respect to t, and so μ_t depends on t through δ_t. Thus μ_t is not constant with respect to t, and hence $\{Y_t\}$ is *not* stationary.

8. (Cryer & Chan, Exercise 2.10)

   a. Since $\{X_t\}$ is a mean-zero process, the mean function for $\{Y_t\}$ is given by

   $$\tilde{\mu}_t = \mathrm{E}(Y_t) = \mu_t + \sigma_t \mathrm{E}(X_t) = \mu_t$$

   for any time t. Since $\{X_t\}$ is a unit-variance process, we have

   $$\rho_k = \mathrm{Corr}(X_0, X_k) = \frac{\mathrm{Cov}(X_0, X_k)}{\sqrt{\mathrm{Var}(X_k)\mathrm{Var}(X_0)}} = \mathrm{Cov}(X_0, X_k)$$

for any k. Therefore, since $\{X_t\}$ is stationary, the autocovariance function for $\{Y_t\}$ is given by

$$\begin{aligned}\tilde{\gamma}_{t,s} &= \mathrm{Cov}(Y_t, Y_s)\\ &= \mathrm{Cov}(\mu_t + \sigma_t X_t, \mu_s + \sigma_s X_s)\\ &= \sigma_t\sigma_s \mathrm{Cov}(X_t, X_s)\\ &= \sigma_t\sigma_s \mathrm{Cov}(X_0, X_{|t-s|})\\ &= \sigma_t\sigma_s \rho_{|t-s|},\end{aligned}$$

for any times t and s.

b. By part (a) and the fact that $\rho_0 = \mathrm{Corr}(X_0, X_0) = 1$, the autocorrelation function for $\{Y_t\}$ is given by

$$\tilde{\rho}_{t,s} = \frac{\tilde{\gamma}_{t,s}}{\sqrt{\tilde{\gamma}_{t,t}\tilde{\gamma}_{s,s}}} = \frac{\sigma_t\sigma_s\rho_{|t-s|}}{\sqrt{\sigma_t^2\rho_0\sigma_s^2\rho_0}} = \rho_{|t-s|}$$

for any times t and s. However, the process $\{Y_t\}$ is *not* stationary, since by part (a) its mean function $\tilde{\mu}_t$ is not constant with respect to t.

c. Yes. Define a process $\{Y_t\}$ as follows: let $Z \sim \mathrm{N}(0,1)$ and let

$$Y_t = 2^t Z \quad \text{for every } t = 0, 1, 2, \ldots$$

Then $\mathrm{E}(Y_t) = 2^t\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Y_t) = 4^t\mathrm{Var}(Z) = 4^t$ for any time t. In addition,

$$\mathrm{Cov}(Y_t, Y_{t-k}) = \mathrm{Cov}(2^t Z, 2^{t-k}Z) = 4^t 2^{-k}\mathrm{Cov}(Z,Z) = 4^t 2^{-k}$$

for any time t and lag k with $t \geq k$. Thus the autocovariance function for the process $\{Y_t\}$ is not completely determined by the lag k, so that $\{Y_t\}$ is *not* stationary. However,

$$\mathrm{Corr}(Y_t, Y_{t-k}) = \frac{\mathrm{Cov}(Y_t, Y_{t-k})}{\sqrt{\mathrm{Var}(Y_t)\mathrm{Var}(Y_{t-k})}} = \frac{4^t 2^{-k}}{4^t 2^{-k}} = 1$$

for any time t and lag k with $t \geq k$.

9. (Cryer & Chan, Exercise 2.15)

a. The mean function for $\{Y_t\}$ is given by

$$\mu_t = \mathrm{E}(Y_t) = (-1)^t\mathrm{E}(X) = 0$$

for any time t.

b. The autocovariance function for $\{Y_t\}$ is given by

$$\begin{aligned}\gamma_{t,s} &= \mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}((-1)^t X, (-1)^s X)\\ &= (-1)^{t+s}\mathrm{Cov}(X, X) = (-1)^{t+s}\mathrm{Var}(X)\\ &= (-1)^{|t-s|}\mathrm{Var}(X)\end{aligned}$$

for any times t and s. In particular, note that $\gamma_{t,s} = \gamma_{0,|t-s|}$.

c. By part (a) the mean function μ_t for $\{Y_t\}$ is constant with respect to t, and its autocovariance function satisfies $\gamma_{t,s} = \gamma_{0,|t-s|}$ (i.e., depends only on time lag), as noted in part (b). Thus $\{Y_t\}$ is stationary.

10. (Cryer & Chan, Exercise 2.26)

a. Let $\{Y_t\}$ be a stationary process. Thus $\{Y_t\}$ has constant mean function $\mathrm{E}(Y_t) = \mu$ and autocovariance function γ_k that is completely determined by the time lag k. We can therefore write

$$\begin{aligned}
\Gamma_{t,s} &= \frac{1}{2}\mathrm{E}[(Y_t - Y_s)^2] \\
&= \frac{1}{2}\mathrm{E}[Y_t^2 - 2Y_tY_s + Y_s^2] \\
&= \frac{1}{2}\mathrm{E}(Y_t^2) - \mathrm{E}(Y_tY_s) + \frac{1}{2}\mathrm{E}(Y_s^2) \\
&= \frac{1}{2}(\gamma_0 + \mu^2) - (\gamma_{|t-s|} + \mu^2) + \frac{1}{2}(\gamma_0 + \mu^2) \\
&= \gamma_0 - \gamma_{|t-s|}
\end{aligned}$$

for any times t and s.

b. Let $\{e_i\}_{i=1}^{\infty}$ be a sequence of iid random variables with mean zero and finite variance. The associated random walk process $\{Y_t\}_{t=1}^{\infty}$ is defined by

$$Y_t = \sum_{i=1}^{t} e_i \quad \text{for every } t = 1, 2, \ldots$$

Consider times t and s. Since $\Gamma_{t,s} = \Gamma_{s,t}$, we can assume without loss of generality that $t \geq s$. Then

$$\Gamma_{t,s} = \frac{1}{2}\mathrm{E}[(Y_t - Y_s)^2] = \frac{1}{2}\mathrm{E}\left[\left(\sum_{i=1}^{t} e_i - \sum_{i=1}^{s} e_i\right)^2\right] = \frac{1}{2}\mathrm{E}\left[\left(\sum_{i=s+1}^{t} e_i\right)^2\right].$$

Since the e_i's are iid and $t \geq s$, we have

$$\sum_{i=s+1}^{t} e_i \sim \sum_{i=1}^{t-s} e_i = \sum_{i=1}^{|t-s|} e_i.$$

Thus,

$$\Gamma_{t,s} = \frac{1}{2}\mathrm{E}\left[\left(\sum_{i=s+1}^{t} e_i\right)^2\right] = \frac{1}{2}\mathrm{E}\left[\left(\sum_{i=1}^{|t-s|} e_i\right)^2\right],$$

so that $\Gamma_{t,s}$ depends on t and s only through the time difference $|t-s|$ (i.e., the process $\{Y_t\}$ is intrinsically stationary).