Homework 4

Benjamin Noland

1. (Cryer & Chan, Exercise 4.11) The process $\{Y_t\}$ is of the form

$$Y_t = \phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2},$$

with $\phi = 0.8$, $\theta_1 = -0.7$, and $\theta_2 = -0.6$. In particular, note that the process has AR characteristic polynomial $\phi(x) = 1 - 0.8x$, which has the single root $x = 1.25 > 1$. Thus, assuming e_t is independent of $Y_{t-1}, Y_{t_2}, \ldots$ for any time t, the process is stationary.

First, we compute

$$\mathrm{E}(e_t Y_t) = \mathrm{E}[e_t(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] = \mathrm{E}(e_t^2) = \sigma_e^2.$$

Next, we have

$$\begin{aligned}
\mathrm{E}(e_{t-1}Y_t) &= \mathrm{E}[e_{t-1}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(e_{t-1}Y_{t-1}) - \theta_1 \mathrm{E}(e_{t-1}^2) \\
&= \phi\sigma_e^2 - \theta_1\sigma_e^2 \\
&= (\phi - \theta_1)\sigma_e^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{E}(e_{t-2}Y_t) &= \mathrm{E}[e_{t-2}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(e_{t-2}Y_{t-1}) - \theta_2 \mathrm{E}(e_{t-2}^2) \\
&= \phi(\phi - \theta_1)\sigma_e^2 - \theta_2\sigma_e^2 \\
&= [\phi(\phi - \theta_1) - \theta_2]\sigma_e^2.
\end{aligned}$$

Thus we can write the autocovariance function as

$$\begin{aligned}
\gamma_k &= \mathrm{Cov}(Y_t, Y_{t-k}) \\
&= \mathrm{E}[Y_{t-k}(\phi Y_{t-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})] \\
&= \phi \mathrm{E}(Y_{t-k}Y_{t-1}) + \mathrm{E}(Y_{t-k}e_t) - \theta_1 \mathrm{E}(Y_{t-k}e_{t-1}) - \theta_2 \mathrm{E}(Y_{t-k}e_{t-2}) \\
&= \phi\gamma_{k-1} + \mathrm{E}(Y_{t-k}e_t) - \theta_1 \mathrm{E}(Y_{t-k}e_{t-1}) - \theta_2 \mathrm{E}(Y_{t-k}e_{t-2})
\end{aligned}$$

for any time t and lag k.

a. Let $k > 2$. Then the expression for γ_k above immediately gives us

$$\gamma_k = \phi\gamma_{k-1},$$

so that

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi\rho_{k-1} = 0.8\rho_{k-1}.$$

b. Again, using the expressions computed above, we get

$$\gamma_2 = \phi\gamma_1 - \theta_2 \mathrm{E}(Y_{t-2}e_{t-2}) = \phi\gamma_1 - \theta_2\sigma_e^2,$$

so that

$$\phi_2 = \frac{\gamma_2}{\gamma_0} = \phi\rho_1 - \frac{\theta_2\sigma_e^2}{\gamma_0} = 0.8\rho_1 + \frac{0.6\sigma_e^2}{\gamma_0}.$$

2. (Cryer & Chan, Exercise 4.12)

   a. Note that in general, an MA(2) process with parameters θ_1 and θ_2 has autocorrelation function ρ_k given by

   $$\begin{aligned}\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\ \rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \\ \rho_k &= 0 \quad \text{for every } k > 2.\end{aligned}$$

   Thus we find that for both of the processes in question, $\rho_1 = -5/38$, $\rho_2 = -3/19$, and $\rho_k = 0$ for every $k > 2$. The two processes therefore have the same autocorrelation function.

   b. The process with $\theta_1 = \theta_2 = 1/6$ has MA characteristic polynomial $\theta(x) = 1 - (1/6)x - (1/6)x^2$, which has roots $x = -3, 2$. So this process is invertible. On the other hand, the process with $\theta_1 = -1$ and $\theta_2 = 6$ has MA characteristic polynomial $\theta(x) = 1 + x - 6x^2$, which has roots $x = -1/3, 1/2$, and so this second process is *not* invertible.

   This is an example of the following result: there is only one set of parameter values $\theta_1, \ldots, \theta_q$ that yield an invertible MA(q) process with a given autocorrelation function.

3. (Cryer & Chan, Exercise 4.21)

   a. For any time t and lag k, we have

   $$\begin{aligned}\gamma_k &= \mathrm{Cov}(Y_t, Y_{t-k}) \\ &= \mathrm{E}[(e_{t-1} - e_{t-2} + 0.5e_{t-3})(e_{t-k-1} - e_{t-k-2} + 0.5e_{t-k-3}) \\ &= \mathrm{E}(e_{t-1}e_{t-k-1}) - \mathrm{E}(e_{t-2}e_{t-k-1}) - \mathrm{E}(e_{t-2}e_{t-k-2}) \\ &\quad + 0.5\mathrm{E}(e_{t-3}e_{t-k-1}) - 0.5\mathrm{E}(e_{t-3}e_{t-k-2}) + 0.25\mathrm{E}(e_{t-3}e_{t-k-3}).\end{aligned}$$

   In particular,

   $$\begin{aligned}\gamma_0 &= \sigma_e^2 - \sigma_e^2 + 0.25\sigma_e^2 = 0.25\sigma_e^2 \\ \gamma_1 &= -\sigma_e^2 - 0.5\sigma_e^2 = -1.5\sigma_e^2 \\ \gamma_2 &= 0.5\sigma_e^2 \\ \gamma_k &= 0 \quad \text{for every } k > 2.\end{aligned}$$

   b. Since the elements of the white noise process $\{e_t\}$ are iid by definition, we have

   $$Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3} \sim e_t - e_{t-1} + 0.5e_{t-2} = W_t.$$

   Thus the process $\{W_t\}$ is MA(2), i.e., ARMA(0, 2), and satisfies $W_t \sim Y_t$ for every time t.

4. For each of the processes $\{Y_t\}$ in this problem, assume that e_t is independent of $Y_{t-1}, Y_{t-2}, \ldots$ for any time t.

a. i. This process has AR characteristic polynomial $\phi(x) = 1 + 0.2x - 0.48x^2$, which has roots $x_1 \approx -1.25, x_2 \approx 1.67$, each of which has modulus > 1, and so the process is stationary. The process has MA characteristic polynomial $\theta(x) = 1$, which has no roots, and so the process is (trivially) invertible.

   ii. This process has AR characteristic polynomial $\phi(x) = 1 + 0.6x$, which has the single root $x \approx 1.67$, which has modulus > 1, and so the process is stationary. The process has MA characteristic polynomial $\theta(x) = 1 + 1.2x$, which has the single root $x \approx -0.83$, which has modulus ≤ 1, and so the process is not invertible.

   iii. This process has AR characteristic polynomial $\phi(x) = 1 + 1.8x + 0.81x^2$, which has the single root $x \approx -1.11$, which has modulus > 1, and so the process is stationary. The process has MA characteristic polynomial $\theta(x) = 1$, which has no roots, and so the process is (trivially) invertible.

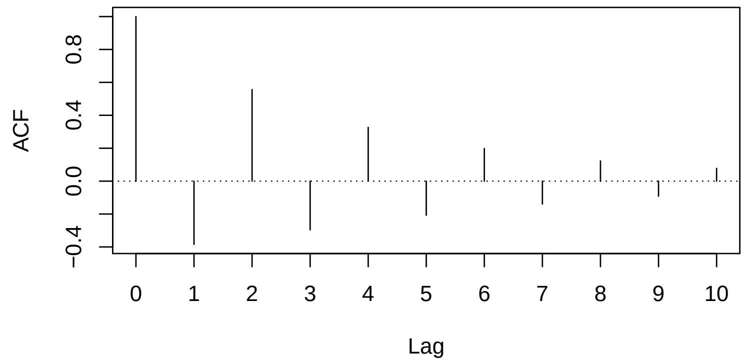
   iv. This process has AR characteristic polynomial $\phi(x) = 1 + 1.6x$, which has the single root $x \approx -0.625$, which has modulus ≤ 1, and so the process is not stationary. The process has MA characteristic polynomial $\theta(x) = 1 - 0.4x$, which has the single root $x = 2.5$, which has modulus > 1, and so the process is invertible.

b. Part (a) showed that processes (i)-(iii) are stationary, while (iv) is not. The following R function plots the autocorrelation function for a specified ARMA process:

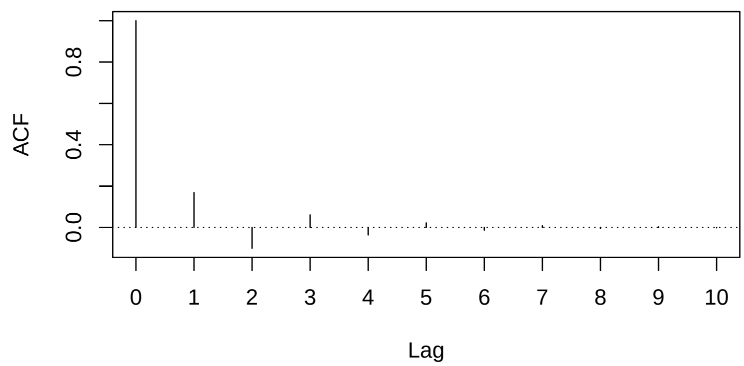
```
plot_arma_acf <- function(ar = numeric(), ma = numeric()) {
  # Negate the MA coefficients to adhere to the book's convention.
  acf <- ARMAacf(ar, -ma, lag.max = 10)
  plot(x = names(acf), y = acf, type = "h", xaxt = "n",
       xlab = "Lag", ylab = "ACF")
  abline(h = 0, lty = 3)
  axis(1, at = names(acf))
}
```

The graphs below display the autocorrelation function for each of the processes (i)-(iii):

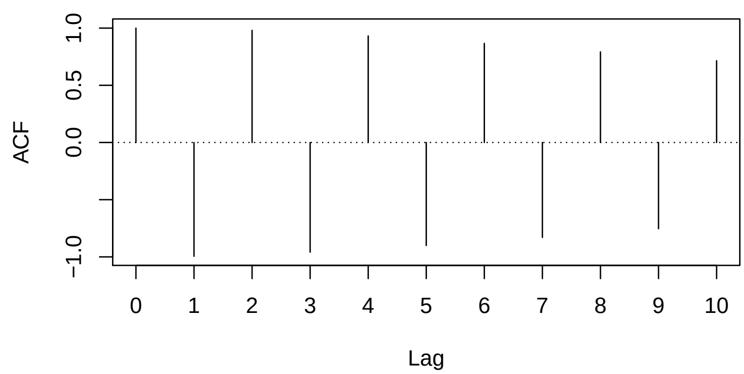
   i.
```
plot_arma_acf(ar = c(-0.2, 0.48))
```



ii.
```r
plot_arma_acf(ar = c(-0.6), ma = c(-1.2))
```



iii.
```r
plot_arma_acf(ar = c(-1.8, -0.81))
```



c. Part (a) showed that processes (i)-(iii) are stationary, while (iv) is not. The following R function computes the coefficients Ψ_j ($j = 0, 1, 2, 3$) in the general linear process representation of a given ARMA process with $p \leq 2$:

```r
psi <- function(j, phi, theta) {
  if (j == 0) {
    1
  } else if (j == 1) {
    -theta[1] + phi[1]
  } else if (j == 2 || j == 3) {
    -theta[j] + phi[2] * psi(j - 2, phi, theta)
```

```
      + phi[1] * psi(j - 1, phi, theta)
  }
}
```

We get the following values for the coefficients Ψ_j ($j = 0, 1, 2, 3$) for the processes (i)-(iii):

i.
```
sapply(0:3, psi, phi = c(-0.2, 0.48), theta = rep(0, 3))
```
```
## [1]  1.000 -0.200  0.040 -0.008
```

ii.
```
sapply(0:3, psi, phi = c(-0.6, 0), theta = c(-1.2, 0, 0))
```
```
## [1]  1.000  0.600 -0.360  0.216
```

iii.
```
sapply(0:3, psi, phi = c(-1.8, -0.81), theta = rep(0, 3))
```
```
## [1]  1.000 -1.800  3.240 -5.832
```

5. (Cryer & Chan, Exercise 5.1) For each of the processes $\{Y_t\}$ in this problem, assume that e_t is independent of $Y_{t-1}, Y_{t-2}, \ldots$ for any time t.

   a. The AR characteristic polynomial for this process is $\phi(x) = 1 - x + 0.25x^2$, which has the single root $x = 2$, which has modulus > 1, and so the process is stationary. The process has MA characteristic polynomial $\theta(x) = 1 - 0.1x$, which has the single root $x = 10$, which has modulus > 1, and so the process is invertible. So $\{Y_t\}$ is an ARMA(2, 1) process, i.e., an ARIMA(2, 0, 1) process.

   b. The AR characteristic polynomial for this process is $\phi(x) = 1 - 2x + x^2$, which has the single root $x = 1$. Thus the process is not stationary. We can rewrite the process as

   $$Y_t - Y_{t-1} = Y_{t-1} - Y_{t-2} + e_t$$

   or equivalently,

   $$\nabla Y_t = \nabla Y_{t-1} + e_t.$$

   This differenced process $\{\nabla Y_t\}$ has AR characteristic polynomial $\phi(x) = 1 - x$, which has the single root $x = 1$, and so the process is not stationary. Differencing again, we get

   $$\nabla Y_t - \nabla Y_{t-1} = e_t$$

   or equivalently,

   $$\nabla^2 Y_t = e_t.$$

   The process $\{\nabla^2 Y_t\}$ is simply white noise, i.e., an ARMA(0, 0) process, and so is stationary and (trivially) invertible. Thus $\{Y_t\}$ is an ARIMA(0, 2, 0) process.

   c. This process has AR characteristic polynomial $\phi(x) = 1 - 0.5x + 0.5x^2$, which has roots $x_1 \approx 0.5 - 1.32i, x_2 \approx 0.5 + 1.32i$, each of which has modulus $\approx 1.41 > 1$, so that the process is stationary. The process has MA characteristic polynomial

$\theta(x) = 1 - 0.5x + 0.25x^2$, which has roots $x_1 \approx 1 - 1.73i, x_2 \approx 1 + 1.73i$, each of which has modulus $\approx 2 > 1$, so that the process is invertible. Thus $\{Y_t\}$ is an ARMA(2, 2) process, i.e., an ARIMA(2, 0, 2) process.

6. (Cryer & Chan, Exercise 5.4)

   a. Assume without loss of generality that $t \geq s$. The autocovariance function of $\{Y_t\}$ is given by

   $$\gamma_{t,s} = \mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}(A + Bt + X_t, A + Bs + X_s) = \mathrm{Cov}(X_t, X_s) = s\sigma_e^2,$$

   which is not a function of the lag $(t - s)$ alone. Therefore $\{Y_t\}$ is not stationary.

   b. The random walk process $\{X_t\}$ is given by

   $$X_t = \sum_{i=1}^{t} e_t,$$

   where $\{e_t\}$ is a white noise process. The process $\{\nabla Y_t\}$ is therefore given by

   $$\nabla Y_t = Y_t - Y_{t-1} = (A + Bt + X_t) - (A + B(t-1) + X_{t-1}) = B + e_t.$$

   Its mean function is therefore given by

   $$\mu_t = \mathrm{E}(\nabla Y_t) = B,$$

   which is constant in t. Moreover, its autocovariance function is given by

   $$\gamma_{t,s} = \mathrm{Cov}(\nabla Y_t, \nabla Y_s) = \mathrm{Cov}(B + e_t, B + e_s) = \mathrm{Cov}(e_t, e_s) = 0,$$

   which is a (constant) function of the lag $(t - s)$. Thus $\{\nabla Y_t\}$ is stationary.

   c. Assume without loss of generality that $t \geq s$. Then, since A and B are independent of $\{X_t\}$,

   $$\begin{aligned}
   \gamma_{t,s} &= \mathrm{Cov}(Y_t, Y_s) \\
   &= \mathrm{Cov}(A + Bt + X_t, A + Bs + X_s) \\
   &= \mathrm{Cov}(A + Bt, A + Bs) + \mathrm{Cov}(X_t, X_s) \\
   &= \mathrm{Var}(A) + ts\mathrm{Var}(B) + 2\mathrm{Cov}(A, B) + \mathrm{Cov}(X_t, X_s) \\
   &= \mathrm{Var}(A) + ts\mathrm{Var}(B) + 2\mathrm{Cov}(A, B) + s\sigma_e^2.
   \end{aligned}$$

   Thus $\gamma_{t,s}$ is not a function of the lag $(t - s)$ alone, and so $\{Y_t\}$ is not stationary.

   d. As in part (b), the process $\{\nabla Y_t\}$ is given by

   $$\nabla Y_t = B + e_t.$$

   Its mean function is therefore given by

   $$\mu_t = \mathrm{E}(Y_t) = \mathrm{E}(B),$$

   which is a constant function of t. Moreover, its autocovariance function is given by

   $$\gamma_{t,s} = \mathrm{Cov}(\nabla Y_t, \nabla Y_s) = \mathrm{Var}(B) + \mathrm{Cov}(e_t, e_s) = \mathrm{Var}(B),$$

   which is a (constant) function of the lag $(t - s)$. Thus $\{\nabla Y_t\}$ is stationary.