Homework 1

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Exercise 1.1

Fix an initial state $x \in \mathbb{Z}_n$. We want to transition to an arbitrary state $y \in \mathbb{Z}_n$. Note that there are two simple paths in the state diagram leading from x to y. If one of these simple paths follows k edges, then the other follows $(n-k)$ edges. Thus if k is odd, then $(n-k)$ is even since n is odd. Thus there exists an even-lengthed simple path from x to y. This means it is possible to transition from x to y in an even number of steps, say m. Note that $(n-1)-m$ is even. Thus after following m steps to get from x to y, we can make $(n-1)-m$ transitions between y and one of its neighboring states, returning at the end to state y. Thus we can transition from x to y in $(n-1)$ steps, so that $P^{n-1}(x,y) > 0$.

Next I claim that if $0 \leq s < n-1$, then there exists a state $y \in \mathbb{Z}_n$ with $P^s(x,y) = 0$. Suppose that it is possible to transition from x to any state y in s steps. This implies that s must be even (this includes the possibility of $s = 0$), since otherwise it would be impossible to transition from x to x itself. Now suppose y is a state neighboring x in the state diagram (i.e., x and y are separated by a single edge). Since s is even, and we are trying to transition from x to y in s steps, we cannot transition along this single edge to get from x to y. Thus we must transition through the remaining $(n-1)$ edges in order to get from x to y. However, since $s < n-1$, this means it is impossible to transition from x to y in s steps. In particular, $P^s(x,y) = 0$.

Therefore the minimum value of $t \geq 0$ such that $P^t(x,y) > 0$ for every pair of states x and y is $t = n-1$.

Exercise 1.4

I will assume that the definition of ***tree*** taken here assumes that a tree contains at least two vertices, since otherwise we could consider a graph containing a single vertex to be a tree, yet this tree has no leaves.

(a) I will proceed by induction on the number n of vertices in the tree. Suppose $n = 2$. In this case, since any tree is connected and acyclic, each vertex has a single incident edge, and hence degree 1. Thus both vertices are leaves. Now let $n > 2$, and suppose that every tree with n vertices has a leaf. Suppose that T is a tree with $(n+1)$ vertices, and let T' be a subtree of T with n vertices, i.e., T' results from deleting a single vertex

v from T. Since T is connected, $\deg(v) \geq 1$. We must also have $\deg(v) \leq 1$. To see this, suppose that $\deg(v) > 1$, and let v_1 and v_2 be two neighbors of v. Then there exists a simple path $v_1 \to v \to v_2$ in T, and since trees are acyclic, every path from v_1 to v_2 must pass through v. However, since T' consists precisely of T with v removed, there is no path from v_1 to v_2 in T', and hence T' is disconnected, a contradiction. Therefore $\deg(v) = 1$, so that v is a leaf in T'. Hence every tree contains a leaf by way of induction.

(b) I will proceed by induction on the number n of vertices in the tree. Suppose $n = 2$, and let v_1 and v_2 be the tree's sole vertices. Then since any tree is connected and acyclic, there exists a unique simple path $v_1 \to v_2$ along the single edge connecting v_1 and v_2. Now let $n > 2$, and suppose the result holds for any tree with n vertices. Let T be a tree with $(n+1)$ vertices, and let T' be a subtree of T with n vertices, i.e., T' results from deleting a single vertex v from T. By the argument in part (a), v must be a leaf in T, and thus has a sole neighboring vertex w (note that w is a vertex in T' as well). By assumption, there exists a unique simple path from any vertex in T' to w, and hence to v via the single edge connecting w to v. Hence there exists a unique simple path between any two vertices in T, and so the result follows by way of induction.

(c) I will proceed by induction on the number n of vertices in the tree. When $n = 2$, there is a single edge connecting the two vertices, and so both vertices are leaves. Now let $n > 2$, and suppose that any tree with n vertices contains at least two leaves. Let T be a tree with $(n+1)$ vertices, and let T' be a subtree of T with n vertices, i.e., T' results from deleting a single vertex v from T. By the argument in part (a), v must be a leaf, and thus has a sole neighboring vertex w (note that w is a vertex in T' as well). By assumption, T' has at least two leaves, and so T' has a leaf z with $z \neq w$. Since z is a vertex in T as well, and $\deg(z)$ is unaffected by the removal of v (since v and z are not neighbors), z is a leaf in T. Thus v and z are two leaves in T, so that T contains at least two leaves. Therefore every tree contains at least two leaves by way of induction.

Exercise 1.5

I will proceed by (strong) induction on the number n of vertices in the tree T. For the sake of simplicity, in what follows I will refer to an acyclic graph containing a single vertex as a "tree". Let $n \geq 2$, and suppose that for every tree T with $1 \leq m < n$ vertices it is possible to transform a proper 3-coloring $\mathcal{C}_1(T)$ of T into any other proper 3-coloring $\mathcal{C}_2(T)$ of T by changing the color of a single vertex at a time. Now let T be a tree with n vertices, and delete a leaf v of T to yield a subtree T' of T with $(n-1)$ vertices (T has a leaf by Exercise 1.4 (a)). Let w denote the sole neighbor of v in the original tree T. Consider proper 3-colorings $\mathcal{C}_1(T)$ and $\mathcal{C}_2(T)$ of T. Since T' is simply T with a single leaf removed, $\mathcal{C}_1(T)$ and $\mathcal{C}_2(T)$ induce well-defined proper 3-colorings $\mathcal{C}_1(T')$ and $\mathcal{C}_2(T')$, respectively, on T'. By assumption it is possible to transform $\mathcal{C}_1(T')$ into $\mathcal{C}_2(T')$ by changing the color of a single vertex at a time.

Thus we can, step by step, recolor T', while at each step changing the color of v so that v and w have different colors (this yields a valid proper 3-coloring at each step, since w is the sole neighbor of v). We therefore eventually transform $\mathcal{C}_1(T')$ into $\mathcal{C}_2(T')$. At this stage we simply change the color of v in order to obtain $\mathcal{C}_2(T)$. Thus it is possible to transform $\mathcal{C}_1(T)$ into $\mathcal{C}_2(T)$ as well. It therefore follows, by way of induction, that the graph whose vertices are proper 3-colorings of a given tree is connected.

Additional problem

Throughout, $\mathbb{Z}^+$ denotes the non-negative integers.

1. The initial distribution is given by $\mu_0 = \delta_1$. That is, for any state $x \in \mathbb{Z}^+$,

$$\mu_0(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}.$$

2. Define the following independent random variables:

$$\begin{aligned} Z &\sim \text{discrete unif}\{-5, -4, \ldots, 5\} \\ U &\sim \text{unif}[0, 1] \end{aligned}$$

and form the random vector (Z, U). Let $\Lambda = \{-5, -4, \ldots, 5\} \times [0, 1]$. Define a map $f : \mathbb{Z}^+ \times \Lambda \to \mathbb{Z}^+$ by

$$f(x, z, u) = \begin{cases} x + z & \text{if } u \leq \alpha(x, x + z) \\ x & \text{if } u > \alpha(x, x + z) \end{cases}.$$

The map $\alpha : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ denotes the acceptance probability, defined by

$$\alpha(x, y) = \min\left\{1, \frac{\pi(y)}{\pi(x)}\right\},$$

where π is the pmf for a binomial$(100, 0.75)$ random variable. The random vector (Z, U), paired with the map f, defines a random mapping representation for the chain in question.

3. The transition probabilities can be obtained from the random mapping representation given in part (2). In general, for any states $x, y \in \mathbb{Z}^+$,

$$\begin{aligned} P(x, y) &= \mathrm{P}\{f(x, Z, U) = y\} \\ &= \mathrm{P}\{x + Z = y, U \leq \alpha(x, x + Z)\} + \mathrm{P}\{x = y, U > \alpha(x, x + Z)\} \\ &= \mathrm{P}\{Z = y - x, U \leq \alpha(x, y)\} + \mathrm{P}\{x = y, U > \alpha(x, x + Z)\} \\ &= \mathrm{P}\{Z = y - x\}\mathrm{P}\{U \leq \alpha(x, y)\} + \mathrm{P}\{x = y\}\mathrm{P}\{U > \alpha(x, x + Z)\}. \end{aligned} \tag{1}$$

There are several cases to consider. Let $\mathcal{T}_x = \{x-5, x-4, \ldots, x+5\}$. If $y \notin \mathcal{T}_x$, then $x \neq y$ and $\mathrm{P}\{Z = y - x\} = 0$, so that $P(x, y) = 0$ as well by (1). Now suppose $y \in \mathcal{T}_x$. If $y \neq x$, then

$$P(x,y) = \mathrm{P}\{Z = y - x\}\mathrm{P}\{U \leq \alpha(x,y)\} = \frac{1}{11}\alpha(x,y).$$

On the other hand (noting that $\alpha(x,x) = 1$), we have

$$\begin{aligned}
P(x,x) &= \mathrm{P}\{Z = 0\}\mathrm{P}\{U \leq \alpha(x,x)\} + \mathrm{P}\{U > \alpha(x, x+Z)\} \\
&= \frac{1}{11}\alpha(x,x) + \sum_{z=-5}^{5} \mathrm{P}\{U > \alpha(x,x+Z) | Z = z\}\mathrm{P}\{Z = z\} \\
&= \frac{1}{11} + \frac{1}{11}\sum_{z=-5}^{5} \mathrm{P}\{U > \alpha(x, x+z)\} \\
&= \frac{1}{11} + \frac{1}{11}\sum_{z=-5}^{5} (1 - \alpha(x, x+z)).
\end{aligned}$$

4. Consider states $x, y \in \mathbb{Z}^+$. If $y = x$, then this relation holds trivially, so assume $y \neq x$. Let

$$\mathcal{T}_x = \{x-5, x-4, \ldots, x+5\} \quad \text{and} \quad \mathcal{T}_y = \{y-5, y-4, \ldots, y+5\}.$$

If $y \notin \mathcal{T}_x$, then $x \notin \mathcal{T}_y$, so that $P(x,y) = P(y,x) = 0$ by part (3), and the relation holds trivially. On the other hand, if $y \in \mathcal{T}_x$, then $x \in \mathcal{T}_y$, and by part (3),

$$P(x,y) = \frac{1}{11}\alpha(x,y) = \frac{1}{11}\min\left\{1, \frac{\pi(y)}{\pi(x)}\right\}.$$

(Since $x \in \mathcal{T}_y$, an analogous expression holds for $P(y,x)$). Assume without loss of generality that $\pi(y)/\pi(x) \geq 1$. Then $\pi(x)/\pi(y) \leq 1$, and we have

$$P(x,y) = \frac{1}{11} \quad \text{and} \quad P(y,x) = \frac{1}{11}\frac{\pi(x)}{\pi(y)}.$$

Hence,

$$\pi(x)P(x,y) = \frac{1}{11}\pi(x) = \pi(y)\left(\frac{1}{11}\frac{\pi(x)}{\pi(y)}\right) = \pi(y)P(x,y).$$

Therefore,

$$\sum_{x \in \mathbb{Z}^+} \pi(x)P(x,y) = \sum_{x \in \mathbb{Z}^+} \pi(y)P(y,x) = \pi(y)\sum_{x \in \mathbb{Z}^+} P(y,x) = \pi(y),$$

so that the distribution $\pi \sim \text{binomial}(100, 0.75)$ is stationary for this chain.

5. The distribution described by the histogram is approximately the stationary distribution π described in part (4). In particular, the histogram has the correct shape and is approximately centered at $\mathrm{E}(\pi) = 75$.