Homework 8

Benjamin Noland

Filtrations

Exercise 1

Let τ be a strong stationary time for (X_t) and starting state x. Let $t \geq 0$ and let $y \in \mathcal{X}$. Then

$$\mathrm{P}_x\{\tau \leq t, X_t = y\} = \sum_{s=0}^{t} \mathrm{P}_x\{\tau = s, X_t = y\} = \sum_{s=0}^{t} \sum_{z \in \mathcal{X}} \mathrm{P}_x\{\tau = s, X_s = z, X_t = y\}.$$

Note that for any $0 \leq s \leq t$, since τ is a strong stationary time, it is a stopping time, and thus there exists a function p_s such that $\{\tau = s\} = \{p_s(Z_0, \ldots, Z_s) = 1\}$. In addition, there exists a function q_s such that $X_s = q_s(Z_0, \ldots, Z_s)$. Thus, for any $z \in \mathcal{X}$,

$$\begin{aligned}
&\mathrm{P}_x\{\tau = s, X_s = z, X_t = y\} \\
&\quad = \mathrm{P}_x\{X_t = y | \tau = s, X_s = z\} \, \mathrm{P}_x\{\tau = s, X_s = z\} \\
&\quad = \mathrm{P}_x\{X_t = y | p_s(Z_0, \ldots, Z_s) = 1, q_s(Z_0, \ldots, Z_s) = z\} \, \mathrm{P}_x\{\tau = s, X_s = z\} \\
&\quad = P^{t-s}(z, y) \, \mathrm{P}_x\{\tau = s, X_\tau = z\}.
\end{aligned}$$

Therefore, since τ is a strong stationary time, and π is a stationary distribution for the chain,

$$\begin{aligned}
\mathrm{P}_x\{\tau \leq t, X_t = y\} &= \sum_{s=0}^{t} \sum_{z \in \mathcal{X}} \mathrm{P}_x\{\tau = s, X_s = z, X_t = y\} \\
&= \sum_{s=0}^{t} \sum_{z \in \mathcal{X}} P^{t-s}(z, y) \, \mathrm{P}_x\{\tau = s, X_\tau = z\} \\
&= \sum_{s=0}^{t} \sum_{z \in \mathcal{X}} P^{t-s}(z, y) \, \mathrm{P}_x\{\tau = s\} \pi(z) \\
&= \sum_{s=0}^{t} \mathrm{P}_x\{\tau = s\} \sum_{z \in \mathcal{X}} P^{t-s}(z, y) \pi(z) \\
&= \mathrm{P}_x\{\tau \leq t\} \pi(y).
\end{aligned}$$

Exercises from the book

Exercise 6.1

Since τ and τ' are both stopping times, for any $t \geq 0$ there exist functions f_t and g_t such that

$$I\{\tau = t\} = f_t(Z_0, \ldots, Z_t\} \quad \text{and} \quad I\{\tau' = t\} = g_t(Z_0, \ldots, Z_t\}.$$

Thus, for any $t \geq 0$,

$$\begin{aligned}
I\{\tau + \tau' = t\} &= I\left\{\bigcup_{s=0}^{t} \{\tau = s, \tau' = t - s\}\right\} = \sum_{s=0}^{t} I\{\tau = s, \tau' = t - s\} \\
&= \sum_{s=0}^{t} I\{\tau = s\} I\{\tau' = t - s\} = \sum_{s=0}^{t} f_s(Z_0, \ldots, Z_s) g_{t-s}(Z_0, \ldots, Z_{t-s}) \\
&= h_t(Z_0, \ldots, Z_t),
\end{aligned}$$

a function of $Z_0, \ldots, Z_t$ alone. Thus $\tau + \tau'$ is also a stopping time. In particular, if r is a non-negative integer, then for any $t \geq 0$, $I\{r = t\}$ is trivially a function of $Z_0, \ldots, Z_t$, and hence r is a stopping time. Thus, taking $\tau' = r$ in the above, we see that $\tau + r$ is a stopping time.

Exercise 6.6

By Exercise 6.4, s_x is weakly decreasing for any $x \in \mathcal{X}$. Thus, for any $x \in \mathcal{X}$ and $t > 0$,

$$s_x(t) = s_x\left(t_0 \frac{t}{t_0}\right) \leq s_x\left(t_0 \left\lfloor \frac{t}{t_0} \right\rfloor\right).$$

Thus,

$$s(t) = \max_{x \in \mathcal{X}} s_x(t) \leq \max_{x \in \mathcal{X}} s_x\left(t_0 \left\lfloor \frac{t}{t_0} \right\rfloor\right) = s\left(t_0 \left\lfloor \frac{t}{t_0} \right\rfloor\right) \leq s(t_0)^{\lfloor t/t_0 \rfloor},$$

since s is sub-multiplicative by Exercise 6.4. Note that by Lemma 6.12,

$$s(t_0) \leq \max_{x \in \mathcal{X}} \mathrm{P}_x\{\tau > t_0\} \leq \epsilon.$$

Hence,

$$s(t) \leq s(t_0)^{\lfloor t/t_0 \rfloor} \leq \epsilon^{\lfloor t/t_0 \rfloor},$$

so that by Lemma 6.16,

$$d(t) \leq s(t) \leq \epsilon^{\lfloor t/t_0 \rfloor}.$$

Exercise 6.7

(a) First, assume that $\mathrm{P}\{Y_t \geq 0\} = 1$ for every $t \geq 1$. We can write

$$\mathrm{E}\left(\sum_{t=1}^{\tau} Y_t\right) = \mathrm{E}\left(\sum_{t=1}^{\infty} Y_t I\{\tau \geq t\}\right).$$

Define

$$Y_n = \sum_{t=1}^{n} Y_t I\{\tau \geq t\} \quad \text{for any } n \geq 1$$

and

$$Y = \sum_{t=1}^{\infty} Y_t I\{\tau \geq t\}.$$

Then since $\mathrm{P}\{Y_t \geq 0\} = 1$ for every $t \geq 1$, $\mathrm{P}\{Y_n \leq Y_{n+1}\} = 1$ for every $n \geq 1$. Thus by Prop. A.11(iii),

$$\lim_{n\to\infty} \mathrm{E}(Y_n) = \mathrm{E}(Y).$$

Therefore, since Y_t is independent of $\{\tau \geq t\}$ for every $t \geq 1$,

$$\begin{aligned}
\mathrm{E}\left(\sum_{t=1}^{\tau} Y_t\right) &= \mathrm{E}\left(\sum_{t=1}^{\infty} Y_t I\{\tau \geq t\}\right) = \mathrm{E}\left(\lim_{n\to\infty}\sum_{t=1}^{n} Y_t I\{\tau \geq t\}\right) \\
&= \lim_{n\to\infty} \mathrm{E}\left(\sum_{t=1}^{n} Y_t I\{\tau \geq t\}\right) = \lim_{n\to\infty}\sum_{t=1}^{n} \mathrm{E}(Y_t I\{\tau \geq t\}) \\
&= \sum_{t=1}^{\infty} \mathrm{E}(Y_t I\{\tau \geq t\}) = \sum_{t=1}^{\infty} \mathrm{E}(Y_t)\,\mathrm{E}(I\{\tau \geq t\}) \\
&= \mathrm{E}(Y_1)\sum_{t=1}^{\infty} \mathrm{P}\{\tau \geq t\} = \mathrm{E}(Y_1)\,\mathrm{E}(\tau).
\end{aligned}$$

Now remove the assumption that $\mathrm{P}\{Y_t \geq 0\} = 1$ for every $t \geq 1$. Note that for every $t \geq 0$, we can write $Y_t = Y_t^+ - Y_t^-$, where $Y_t^+ = \max\{0, Y_t\}$ and $Y_t^- = \min\{0, -Y_t\}$. We have

$$\mathrm{P}\{Y_t^+ \geq 0\} = \mathrm{P}\{Y_t^- \geq 0\} = 1.$$

In addition, each of the sequences (Y_t^+) and (Y_t^-) consists of IID random variables with finite expectation, and for every $t \geq 1$, each of Y_t^+ and Y_t^- is independent of $\{\tau \geq t\}$. So the above discussion applies to each of the sequences (Y_t^+) and (Y_t^-). Therefore,

$$\begin{aligned}
\mathrm{E}\left(\sum_{t=1}^{\tau} Y_t\right) &= \mathrm{E}\left[\sum_{t=1}^{\tau}(Y_t^+ - Y_t^-)\right] = \mathrm{E}\left(\sum_{t=1}^{\tau} Y_t^+\right) - \mathrm{E}\left(\sum_{t=1}^{\tau} Y_t^-\right) \\
&= \mathrm{E}(Y_1^+)\,\mathrm{E}(\tau) - \mathrm{E}(Y_1^-)\,\mathrm{E}(\tau) = \mathrm{E}(Y_1^+ - Y_1^-)\,\mathrm{E}(\tau) \\
&= \mathrm{E}(Y_1)\,\mathrm{E}(\tau).
\end{aligned}$$

(b) Since τ is a stopping time for (Y_t), for every $s \geq 1$ there exists a function p_s such that

$$\{\tau = s\} = \{p_s(Y_1, \ldots, Y_s) = 1\}.$$

In particular, for any $t \geq 1$,

$$\{\tau \geq t\}^c = \{\tau \leq t-1\} = \bigcup_{s=1}^{t-1} \{\tau = s\} = \bigcup_{s=1}^{t-1} \{p_s(Y_1, \ldots, Y_s) = 1\},$$

so that the event $\{\tau \geq t\}^c$, and hence the event $\{\tau \geq t\}$, depends solely upon $Y_1, \ldots, Y_{t-1}$. But since the elements of the sequence (Y_t) are IID, Y_t is independent of $Y_1, \ldots, Y_{t-1}$, and hence Y_t is independent of $\{\tau \geq t\}$.

Exercise 6.8

(Incomplete).

Shuffling by insertion

Exercise 1

By definition, at each time step the top card of the first deck is inserted uniformly at random into the second deck. At the outset, the second deck contains no cards, and card 1 is at the top of the first deck. Thus at time $k = 1$, card 1 is transferred from the first deck to the second deck, and card 1 is the only card in the second deck. Hence $\mathrm{P}\{S_1 = (1)\} = 1$.

Exercise 2

At time $k = 1$, card 2 is at the top of the first deck and card 1 is the only card in the second deck. Thus at time $k = 2$ there are two possibilities: card 2 is placed above card 1 in the second deck ($S_2 = (2,1)$), or card 2 is placed below card 1 ($S_2 = (1,2)$). Since the insertion is uniformly at random, each of these two possibilities is equally likely. In particular, since $S_2 = (2,1)$ if and only if $R_1 = 1$, we have

$$\mathrm{P}\{S_2 = (2,1)\} = \mathrm{P}\{R_1 = 1\} = \frac{1}{2}.$$

Exercise 3

At time $k = 2$, card 3 is at the top of the first deck, and the second deck contains cards 1 and 2 in some order. Thus at time $k = 3$ there are three places to insert card

3, all equally likely. The value of S_3 depends on the value of S_2 (either $S_2 = (1, 2)$ or $S_2 = (2, 1)$). In particular, we have

$$\begin{aligned} \mathrm{P}\{S_3 = (2,1,3)\} &= \mathrm{P}\{S_3 = (2,1,3) | S_2 = (1,2)\}\, \mathrm{P}\{S_2 = (1,2)\} \\ &\qquad + \mathrm{P}\{S_3 = (2,1,3) | S_2 = (2,1)\}\, \mathrm{P}\{S_2 = (2,1)\} \\ &= (0)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \left(\frac{1}{6}\right). \end{aligned}$$

Note that $S_3 = (2, 1, 3)$ if and only if $R_2 = 1$ and $R_3 = 3$. Thus

$$\mathrm{P}\{R_2 = 1, R_3 = 3\} = \mathrm{P}\{S_3 = (2,1,3)\} = \frac{1}{6}.$$

Exercise 4

I will proceed by induction on n. When $n = 1$ we must have $\sigma = (1)$, and since $S_1 = (1)$ if and only if $R_1 = 1$, the base case holds. Now let $n > 1$, and assume the result holds for S_n. Consider $S_{n+1} = \sigma = (\sigma(1), \ldots, \sigma(n+1))$. The permutation S_{n+1} is obtained from S_n by sampling a rank uniformly at random from $\{1, \ldots, n+1\}$. Say $R_{n+1} = r_{n+1}$. This implies that $S_n = \sigma'$, where σ' is the permutation on $\{1, \ldots, n\}$ defined, for every $1 \le k \le n$, by

$$\sigma'(k) = \begin{cases} \sigma(k) & \text{if } 1 \le k < r_{n+1} \\ \sigma(k+1) & \text{if } r_{n+1} \le k \le n. \end{cases}$$

By assumption there exist unique integers $r_k \in \{1, \ldots, k\}$ for every $1 \le k \le n$ such that $S_n = \sigma'$ if and only if $R_k = r_k$ for every $1 \le k \le n$. By the above discussion, if $S_{n+1} = \sigma$, then $S_n = \sigma'$ and $R_{n+1} = r_{n+1}$, and thus $R_k = r_k$ for every $1 \le k \le n+1$. Conversely, if $R_k = r_k$ for every $1 \le k \le n+1$, then $S_n = \sigma'$ and $R_{n+1} = r_{n+1}$, and thus $S_{n+1} = \sigma$. Since the integers $r_1, \ldots, r_n, r_{n+1}$ are unique, we see that the result holds for S_{n+1}. So the result holds for every $n \ge 1$ by induction.

By the above result, $S_n = \sigma$ if and only if $R_k = r_k$ for every $1 \le k \le n$, where $r_k \in \{1, \ldots, k\}$ are unique integers. Moreover, by definition, $\mathrm{P}\{R_k = r_k\} = 1/k$ for every $1 \le k \le n$. Thus, since $R_1, R_2, \ldots, R_n$ are independent,

$$\begin{aligned} \mathrm{P}\{S_n = \sigma\} &= \mathrm{P}\{R_1 = r_1, R_2 = r_2, \ldots, R_n = r_n\} \\ &= \mathrm{P}\{R_1 = r_1\}\, \mathrm{P}\{R_2 = r_2\} \cdots \mathrm{P}\{R_n = r_n\} = \frac{1}{n!}. \end{aligned}$$

Exercise 5

(In what follows, the sums are taken over all permutations of $\{1, \ldots, n\}$). Let σ be a permutation of $\{1, \ldots, n\}$. Then

$$\begin{aligned}
\mathrm{P}\{S_n \circ S = \sigma\} &= \sum_{\sigma'} \mathrm{P}\{S_n \circ S = \sigma, S_n = \sigma'\} = \sum_{\sigma'} \mathrm{P}\{\sigma' \circ S = \sigma\}\, \mathrm{P}\{S_n = \sigma'\} \\
&= \frac{1}{n!} \sum_{\sigma'} \mathrm{P}\{S = (\sigma')^{-1}\sigma\} = \frac{1}{n!} \sum_{\sigma''} \mathrm{P}\{S = \sigma''\} = \frac{1}{n!}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{P}\{S \circ S_n = \sigma\} &= \sum_{\sigma'} \mathrm{P}\{S \circ S_n = \sigma, S_n = \sigma'\} = \sum_{\sigma'} \mathrm{P}\{S \circ \sigma' = \sigma\}\, \mathrm{P}\{S_n = \sigma'\} \\
&= \frac{1}{n!} \sum_{\sigma'} \mathrm{P}\{S = \sigma(\sigma')^{-1}\} = \frac{1}{n!} \sum_{\sigma''} \mathrm{P}\{S = \sigma''\} = \frac{1}{n!}.
\end{aligned}$$

Thus each of $S_n \circ S$ and $S \circ S_n$ has the same distribution as S_n (i.e., uniform over all permutations of $\{1, \ldots, n\}$).

Exercise 6

The second deck described in this problem can be regarded as consisting of the cards under the original bottom card (in the context of Prop. 6.11). Exercise 4 shows that when there are k cards under the original bottom card, each of the $k!$ possible permutations of these cards are equally likely. In particular, the time τ_{top} corresponds to the case when all n cards are in the second deck, and so the distribution of the chain is uniform over all possible permutations of the n cards.