Homework 7

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Independence

Exercise 1

Let $k \geq 1$, and let $x_1, \ldots, x_k \in \mathcal{X}$. Then

$$\begin{aligned}
&\mathrm{P}_u\{U_k = x_k | U_{k-1} = x_{k-1}, \ldots, U_1 = x_1, U_0 = u\} \\
&\qquad = \mathrm{P}_u\{f(U_{k-1}, Z_{T_k}) = x_k | f(U_{k-2}, Z_{T_{k-1}}) = x_{k-1}, \ldots, f(U_0, Z_{T_1}) = x_1, U_0 = u\} \\
&\qquad = \mathrm{P}_u\{f(x_{k-1}, Z_{T_k}) = x_k | f(x_{k-2}, Z_{T_{k-1}}) = x_{k-1}, \ldots, f(u, Z_{T_1}) = x_1, U_0 = u\} \\
&\qquad = \mathrm{P}_u\{f(x_{k-1}, Z_{T_k}) = x_k\},
\end{aligned}$$

since the event $\{f(x_{k-1}, Z_{T_k}) = x_k\}$ depends solely upon Z_{T_k}, while the event $\{f(U_{k-2}, Z_{T_{k-1}}) = x_{k-1}, \ldots, f(U_0, Z_{T_1}) = x_1, U_0 = u\}$ depends solely upon $Z_{T_1}, \ldots, Z_{T_{k-1}}$. Thus, by how the random mapping representation was defined,

$$\mathrm{P}_u\{U_k = x_k | U_{k-1} = x_{k-1}, \ldots, U_1 = x_1, U_0 = u\} = \mathrm{P}_u\{f(x_{k-1}, Z_{T_k}) = x_k\} = P(x_{k-1}, x_k).$$

Hence $(U_k)_{k=0}^{\infty}$ is a Markov chain on $\mathcal{X}$ with transition matrix P. A similar argument shows that $(V_k)_{k=0}^{\infty}$ is also a Markov chain with transition matrix P. Hence $(U_k, V_k)_{k=0}^{\infty}$ is a coupling of Markov chains with transition matrix P.

Exercise 2

Say $h : \mathcal{X}^{2(k+1)} \to \{0, 1\}$ (the map f is already defined). Then we have

$$\begin{aligned}
&\mathrm{P}_{u,v}\{h(U_0, \ldots, U_k, V_0, \ldots, V_k) = 1, T_1 = t_1, \ldots, T_k = t_k, \ldots, T_m = t_m\} \\
&\qquad = \mathrm{P}_{u,v}\{h(u, \ldots, f(U_{k-1}, Z_{T_k}), v, \ldots, g(V_{k-1}, Z_{T_k})) = 1, T_1 = t_1, \ldots, T_k = t_k, \ldots, T_m = t_m\} \\
&\qquad = \mathrm{P}_{u,v}\{h(u, \ldots, f(U_{k-1}, Z_{t_k}), v, \ldots, g(V_{k-1}, Z_{t_k})) = 1, T_1 = t_1, \ldots, T_k = t_k, \ldots, T_m = t_m\} \\
&\qquad = \mathrm{P}_{u,v}\{h(u, \ldots, f(U_{k-1}, Z_{t_k}), v, \ldots, g(V_{k-1}, Z_{t_k})) = 1\}\, \mathrm{P}_{u,v}\{T_1 = t_1, \ldots, T_k = t_k, \ldots, T_m = t_m\},
\end{aligned}$$

where the independence of the events $\{h(u, \ldots, f(U_{k-1}, Z_{t_k}), v, \ldots, g(V_{k-1}, Z_{t_k})) = 1\}$ and $\{T_1 = t_1, \ldots, T_k = t_k, \ldots, T_m = t_m\}$ is due to the fact that the T_j's are independent of the Z_t's. However, since the Z_t's are IID, we have the following equality:

$$\begin{aligned}
&\mathrm{P}_{u,v}\{h(u, \ldots, f(U_{k-1}, Z_{t_k}), v, \ldots, g(V_{k-1}, Z_{t_k})) = 1\} \\
&\qquad = \mathrm{P}_{u,v}\{h(u, \ldots, f(U_{k-1}, Z_{T_k}), v, \ldots, g(V_{k-1}, Z_{T_k})) = 1\}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathrm{P}_{u,v}\{h(U_0,\dots,U_k,V_0,\dots,V_k)=1,T_1=t_1,\dots,T_k=t_k,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{h(u,\dots,f(U_{k-1},Z_{t_k}),v,\dots,g(V_{k-1},Z_{t_k}))=1\}\,\mathrm{P}_{u,v}\{T_1=t_1,\dots,T_k=t_k,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{h(u,\dots,f(U_{k-1},Z_{T_k}),v,\dots,g(V_{k-1},Z_{T_k}))=1\}\,\mathrm{P}_{u,v}\{T_1=t_1,\dots,T_k=t_k,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{h(U_0,\dots,U_k,V_0,\dots,V_k)=1\}\,\mathrm{P}_{u,v}\{T_1=t_1,\dots,T_k=t_k,\dots,T_m=t_m\}.
\end{aligned}
$$

Hence the events $\{h(U_0,\dots,U_k,V_0,\dots,V_k)=1\}$ and $\{T_1=t_1,\dots,T_k=t_k,\dots,T_m=t_m\}$ are independent.

Exercise 3

Note that the event $\{\nu=q\}$ is determined by $U_0,\dots,U_q$ and $V_0,\dots,V_q$. Thus we can write

$$\{\nu=q\}=\{h(U_0,\dots,U_q,V_0,\dots,V_q)=1\}$$

for some map $h:\mathcal{X}^{2(q+1)}\to\{0,1\}$. Thus, for any non-negative integers $t_1,\dots,t_m$, we have

$$
\begin{aligned}
&\mathrm{P}_{u,v}\{\nu=q,T_1=t_1,\dots,T_q=t_q,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{h(U_0,\dots,U_q,V_0,\dots,V_q)=1,T_1=t_1,\dots,T_q=t_q,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{h(U_0,\dots,U_q,V_0,\dots,V_q)=1\}\,\mathrm{P}_{u,v}\{T_1=t_1,\dots,T_q=t_q,\dots,T_m=t_m\}\\
&\qquad=\mathrm{P}_{u,v}\{\nu=q\}\,\mathrm{P}_{u,v}\{T_1=t_1,\dots,T_q=t_q,\dots,T_m=t_m\}
\end{aligned}
$$

by Exercise 3. Thus the event $\{\nu=q\}$ is independent of $T_1,\dots,T_m$.

Exercise 4

Note that for any $t\ge 0$, we have

$$X_{t+1}(1)=f(X_t(1),Z_{t+1})I\{C_{t+1}=1\}+X_t(1)I\{C_{t+1}\ne 1\}.$$

For simplicitly of notation, in what follows I will simply write $X_{t+1}(1)=h(X_t(1),Z_{t+1},C_{t+1})$. Let $x_1,\dots,x_{t+1}\in\mathcal{X}$ and $x_0=x(1)$. Then

$$
\begin{aligned}
&\mathrm{P}\{X_{t+1}(1)=x_{t+1}|X_t(1)=x_t,\dots,X_1(1)=x_1,X_0=x_0\}\\
&\quad=\mathrm{P}\{h(X_t,Z_{t+1},C_{t+1})=x_{t+1}|h(X_{t-1},Z_t,C_t)=x_t,\dots,h(X_0,Z_1,C_1)=x_1,X_0=x_0\}\\
&\quad=\mathrm{P}\{h(x_t,Z_{t+1},C_{t+1})=x_{t+1}|h(x_{t-1},Z_t,C_t)=x_t,\dots,h(x_0,Z_1,C_1)=x_1,X_0=x_0\}\\
&\quad=\mathrm{P}\{h(x_t,Z_{t+1},C_{t+1})=x_{t+1}\},
\end{aligned}
$$

since the event $\{h(x_t,Z_{t+1},C_{t+1})=x_{t+1}\}$ depends solely upon Z_{t+1} and C_{t+1}, while the event $\{h(x_{t-1},Z_t,C_t)=x_t,\dots,h(x_0,Z_1,C_1)=x_1,X_0=x_0\}$ depends solely upon $Z_1,\dots,Z_t$ and

$C_1, \ldots, C_t$. If $x_{t+1} = x_t$, then this becomes

$$\begin{aligned}
\mathrm{P}\{h(x_t, Z_{t+1}, C_{t+1}) = x_t\} &= \mathrm{P}\{f(x_t, Z_{t+1}) = x_t, C_{t+1} = 1\} + \mathrm{P}\{C_{t+1} \neq 1\} \\
&= \mathrm{P}\{f(x_t, Z_{t+1}) = x_t\}\,\mathrm{P}\{C_{t+1} = 1\} + \mathrm{P}\{C_{t+1} \neq 1\} \\
&= P(x_t, x_t)\frac{1}{d} + \left(1 - \frac{1}{d}\right).
\end{aligned}$$

On the other hand, if $x_{t+1} \neq x_t$, we get

$$\begin{aligned}
\mathrm{P}\{h(x_t, Z_{t+1}, C_{t+1}) = x_{t+1}\} &= \mathrm{P}\{f(x_t, Z_{t+1}) = x_{t+1}, C_{t+1} = 1\} \\
&= \mathrm{P}\{f(x_t, Z_{t+1}) = x_{t+1}\}\,\mathrm{P}\{C_{t+1} = 1\} \\
&= P(x_t, x_{t+1})\frac{1}{d}.
\end{aligned}$$

Thus $(X_t(1))_{t=0}^{\infty}$ is a Markov chain on $\mathcal{X}$ with transition matrix Q given by

$$Q(x, y) = \begin{cases} P(x, y)/d + (1 - 1/d) & \text{if } x = y \\ P(x, y)/d & \text{if } x \neq y \end{cases} \quad \text{for any } x, y \in \mathcal{X}.$$

A similar argument shows that $(Y_t(1))_{t=0}^{\infty}$ is also a Markov chain on $\mathcal{X}$ with transition matrix Q.

Exercise 5

Note that for any $k \geq 0$, we have

$$T_{k+1} - T_k = \min\{t > 0 : C_{t+T_k} = 1\}.$$

Thus, for any $t > 0$, the independence of the C_t's implies that

$$\begin{aligned}
\mathrm{P}\{T_{k+1} - T_k = t\} &= \mathrm{P}\{C_{t+T_k} = 1, C_{t-1+T_k} \neq 1, \ldots, C_{1+T_k} \neq 1\} \\
&= \mathrm{P}\{C_{t+T_k} = 1\}\,\mathrm{P}\{C_{t-1+T_k} \neq 1\} \cdots \mathrm{P}\{C_{1+T_k} \neq 1\} \\
&= \frac{1}{d}\left(1 - \frac{1}{d}\right)^{t-1}.
\end{aligned}$$

Moreover, since $T_{k+1} > T_k$ by definition,

$$\mathrm{P}\{T_{k+1} - T_k = t\} = 0 \quad \text{for any } t \leq 0.$$

Thus $(T_{k+1} - T_k) \sim \text{geometric}(1/d)$, so that

$$\mathrm{E}(T_{k+1} - T_k) = \frac{1}{1/d} = d$$

for any $k \geq 0$.

Exercise 6

It follows from Exercise 3 that ν is independent of $T_0, T_1, \ldots$. Moreover, it follows from Exercise 5 that the random variables $(T_{k+1} - T_k)$, $k \geq 0$, are IID geometric(1/d). Thus by Exercise 5.3 of the book,

$$\mathrm{E}(T_\nu) = \mathrm{E}(T_\nu - T_0) = \mathrm{E}\left(\sum_{k=1}^{\nu}(T_k - T_{k-1})\right) = d\,\mathrm{E}(\nu),$$

since $\mathrm{E}(T_{k+1} - T_k) = d$ for every $k \geq 0$, as noted in Exercise 5.

Problems from the book

Exercise 5.1

(a) Let P denote the transition matrix of the coupling $(X_t, Y_t)_{t=0}^\infty$. Thus each of the chains $(X_t)_{t=0}^\infty$ and $(Y_t)_{t=0}^\infty$ has transition matrix P by definition. Since $X_0 \sim \mu$ and $Y_0 \sim \nu$, $X_t \sim \mu P^t$ and $Y_t \sim \nu P^t$ for any $t \geq 0$, so that (X_t, Y_t) is a coupling of the distributions μP^t and νP^t. Thus by Proposition 4.7,

$$\|\mu P^t - \nu P^t\|_{\mathrm{TV}} \leq \mathrm{P}_{\mu,\nu}\{X_t \neq Y_t\} = \mathrm{P}_{\mu,\nu}\{\tau_{\mathrm{couple}} > t\}.$$

(b) Assume that P is irreducible and aperiodic with stationary distribution π. Thus for any $t \geq 0$, $\pi P^t = \pi$. Let $x \in \mathcal{X}$. Putting $\mu = \delta_x$ and $\nu = \pi$ in the result of part (a) yields

$$\|P^t(x, \cdot) - \pi\|_{\mathrm{TV}} \leq \mathrm{P}_{x,\pi}\{\tau_{\mathrm{couple}} > t\}.$$

It remains to show that the chains $(X_t)_{t=0}^\infty$ and $(Y_t)_{t=0}^\infty$ (assumed independent of each other) coalesce almost surely. Since P is irreducible and aperiodic, Proposition 1.7 implies that there exists an integer $r_0 > 0$ such that $P^r(w, z) > 0$ for every $w, z \in \mathcal{X}$ and $r \geq r_0$. Let $\alpha = \min_{w,z\in\mathcal{X}} P^{r_0}(w, z)$. Then $\alpha > 0$. Now fix a state $x_0 \in \mathcal{X}$. Then

$$\mathrm{P}_{x,\pi}\{X_{r_0} = x_0, Y_{r_0} = x_0\} \leq \mathrm{P}_{x,\pi}\{\tau_{\mathrm{couple}} \leq r_0\}.$$

Thus since $(X_t)_{t=0}^\infty$ and $(Y_t)_{t=0}^\infty$ are independent of each other,

$$\begin{aligned}
\mathrm{P}_{x,\pi}\{\tau_{\mathrm{couple}} > r_0\} &\leq 1 - \mathrm{P}_{x,\pi}\{X_{r_0} = x_0, Y_{r_0} = x_0\} \\
&= 1 - \mathrm{P}_{x,\pi}\{X_{r_0} = x_0\}\,\mathrm{P}\{Y_{r_0} = x_0\} \\
&\leq 1 - \alpha^2.
\end{aligned}$$

Now let $k > 1$ and assume that

$$\mathrm{P}_{x,\pi}\{\tau_{\mathrm{couple}} > kr_0\} \leq (1 - \alpha^2)^k. \tag{1}$$

Then

$$\begin{aligned}
\mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > (k+1)r_0\} &= \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > (k+1)r_0, \tau_{\text{couple}} > kr_0\} \\
&= \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > (k+1)r_0 | \tau_{\text{couple}} > kr_0\}\, \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > kr_0\} \\
&= \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > r_0\}\, \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > kr_0\} \\
&= (1-\alpha^2)(1-\alpha^2)^k \\
&= (1-\alpha^2)^{k+1}.
\end{aligned}$$

Thus (1) holds for every $k \geq 1$ by induction. We therefore have the following:

$$\begin{aligned}
\mathrm{P}_{x,\pi}\{\tau_{\text{couple}} = \infty\} = \mathrm{P}_{x,\pi}\left(\bigcap_{s=1}^{\infty}\{\tau_{\text{couple}} > s\}\right) &= \mathrm{P}_{x,\pi}\left(\bigcap_{k=1}^{\infty}\{\tau_{\text{couple}} > kr_0\}\right) \\
= \mathrm{P}_{x,\pi}\{\tau_{\text{couple}} > mr_0\} &\leq (1-\alpha^2)^m
\end{aligned}$$

for any $m \geq 1$. Thus,

$$\mathrm{P}_{x,\pi}\{\tau_{\text{couple}} < \infty\} \geq 1 - (1-\alpha^2)^m \to 1 \quad \text{as } m \to \infty,$$

so that $\mathrm{P}_{x,\pi}\{\tau_{\text{couple}} < \infty\} = 1$. So the chains $(X_t)_{t=0}^{\infty}$ and $(Y_t)_{t=0}^{\infty}$ coalesce almost surely.

Exercise 5.2

Let $x, y \in \mathcal{X}$. I claim that for every $k \geq 1$,

$$\mathrm{P}_{x,y}\{\tau_{\text{couple}} > kt_0\} \leq (1-\alpha)^k. \tag{1}$$

I will proceed by induction on k. When $k = 1$, the inequality (1) reduces to

$$\mathrm{P}_{x,y}\{\tau_{\text{couple}} > t_0\} \leq 1 - \alpha,$$

which holds by assumption. Now let $k > 1$ be such that (1) holds. Then

$$\begin{aligned}
\mathrm{P}_{x,y}\{\tau_{\text{couple}} > (k+1)t_0\} &= \mathrm{P}_{x,y}\{\tau_{\text{couple}} > (k+1)t_0, \tau_{\text{couple}} > kt_0\} \\
&= \mathrm{P}_{x,y}\{\tau_{\text{couple}} > (k+1)t_0 | \tau_{\text{couple}} > kt_0\}\, \mathrm{P}_{x,y}\{\tau_{\text{couple}} > kt_0\} \\
&= \mathrm{P}_{x,y}\{\tau_{\text{couple}} > t_0\}\, \mathrm{P}_{x,y}\{\tau_{\text{couple}} > kt_0\} \\
&\leq (1-\alpha)(1-\alpha)^k \\
&= (1-\alpha)^{k+1}.
\end{aligned}$$

Therefore (1) holds for every $k \geq 1$ by induction. Finally, we have

$$\mathrm{E}_{x,y}(\tau_{\text{couple}}) = \sum_{t=0}^{\infty} \mathrm{P}_{x,y}\{\tau_{\text{couple}} > t\} \leq \sum_{k=0}^{\infty} t_0\, \mathrm{P}_{x,y}\{\tau_{\text{couple}} > kt_0\} \leq \sum_{k=0}^{\infty} t_0(1-\alpha)^k = \frac{t_0}{\alpha}.$$

Exercise 5.3

We have the following:

$$\mathrm{E}\left(\sum_{i=1}^{\tau} X_i\right) = \mathrm{E}\left[\mathrm{E}\left(\sum_{i=1}^{\tau} X_i \middle| \tau\right)\right] = \mathrm{E}(\mu\tau) = \mu\,\mathrm{E}(\tau).$$

Exercise 5.4

(a) I claim that for every $k \geq 1$,

$$\mathrm{P}_{x,y}\{\tau_1 > kdn^2\} \leq \left(\frac{1}{4}\right)^k. \tag{1}$$

We already know from the proof of Thm. 5.6 that

$$\mathrm{E}_{x,y}(\tau_1) \leq \frac{dn^2}{4},$$

and thus by Markov's Inequality,

$$\mathrm{P}_{x,y}\{\tau_1 > dn^2\} \leq \frac{\mathrm{E}_{x,y}(\tau_1)}{dn^2} \leq \frac{dn^2/4}{dn^2} = \frac{1}{4}.$$

So the base case ($k = 1$) is satisfied. Now let $k > 1$ satisfy (1). We then have the following:

$$\begin{aligned}
\mathrm{P}_{x,y}\{\tau_1 > (k+1)dn^2\} &= \mathrm{P}_{x,y}\{\tau_1 > (k+1)dn^2, \tau_1 > kdn^2\} \\
&= \mathrm{P}_{x,y}\{\tau_1 > (k+1)dn^2 | \tau_1 > kdn^2\}\,\mathrm{P}_{x,y}\{\tau_1 > kdn^2\} \\
&= \mathrm{P}_{x,y}\{\tau_1 > dn^2\}\,\mathrm{P}_{x,y}\{\tau_1 > kdn^2\} \\
&\leq \frac{1}{4}\left(\frac{1}{4}\right)^k = \left(\frac{1}{4}\right)^{k+1}.
\end{aligned}$$

Thus (1) holds for every $k \geq 1$ by induction. Now let $t \geq kdn^2$ for some $k \geq 1$. Then

$$\mathrm{P}_{x,y}\{\tau_1 > t\} \leq \mathrm{P}_{x,y}\{\tau_1 > kdn^2\} \leq \left(\frac{1}{4}\right)^k.$$

(b) The same argument as in part (a) shows that if $t \geq kdn^2$, then

$$\mathrm{P}_{x,y}\{\tau_i > kdn^2\} \leq \left(\frac{1}{4}\right)^k \quad \text{for every } 1 \leq i \leq d.$$

Note that

$$\tau_{\text{couple}} = \max_{1 \leq i \leq d} \tau_i.$$

Thus if $t \geq kdn^2$, then

$$\mathrm{P}_{x,y}\{\tau_{\text{couple}} > t\} = \mathrm{P}_{x,y}\left(\bigcup_{i=1}^{d}\{\tau_i > t\}\right) \leq \sum_{i=1}^{d} \mathrm{P}_{x,y}\{\tau_i > t\} \leq d\left(\frac{1}{4}\right)^k.$$

Let $0 < \epsilon < 1/2$. Then putting $t = \lceil \log_4(d/\epsilon) \rceil dn^2$ (i.e., $k = \lceil \log_4(d/\epsilon) \rceil$) in the above yields

$$\mathrm{P}_{x,y}\{\tau_{\text{couple}} > \lceil \log_4(d/\epsilon) \rceil dn^2\} \leq d\left(\frac{1}{4}\right)^{\lceil \log_4(d/\epsilon) \rceil} \leq d\left(\frac{1}{4}\right)^{\log_4(d/\epsilon)} = \epsilon.$$

Thus by Cor. 5.5,

$$d(\lceil \log_4(d/\epsilon) \rceil dn^2) \leq \max_{x,y \in \mathbb{Z}_n^d} \mathrm{P}_{x,y}\{\tau_{\text{couple}} > \lceil \log_4(d/\epsilon) \rceil dn^2\} \leq \epsilon,$$

and hence

$$t_{\text{mix}}(\epsilon) \leq \lceil \log_4(d/\epsilon) \rceil dn^2.$$

Harmonic functions

Exercise 1

First, let $x \in A \cup B$. Then

$$\check{P}(x,y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{for every } y \in \mathcal{X}.$$

Thus $\check{P}(x,y) \geq 0$ for every $y \in \mathcal{X}$, and

$$\sum_{y \in \mathcal{X}} \check{P}(x,y) = \check{P}(x,x) = 1.$$

Now let $x \in \mathcal{X} \setminus (A \cup B)$. Then

$$\check{P}(x,y) = \frac{P(x,y)h(y)}{h(x)} \quad \text{for every } y \in \mathcal{X}.$$

Note that $h(y) \geq 0$ for every $y \in \mathcal{X}$ by definition, and that $h(x) > 0$ since h is positive on $\mathcal{X} \setminus (A \cup B)$. Thus $\check{P}(x,y) \geq 0$ for every $y \in \mathcal{X}$, and since h is harmonic,

$$\sum_{y \in \mathcal{X}} \check{P}(x,y) = \sum_{y \in \mathcal{X}} \frac{P(x,y)h(y)}{h(x)} = \frac{h(x)}{h(x)} = 1.$$

Thus we see that $\check{P}$ is a transition matrix.

Exercise 2

(a) Let $x \in \mathcal{X} \setminus (A \cup B)$, so that $x \notin A$ and $x \notin B$. By assumption there exists $y \in A$ and a path $x = x_0, x_1, \ldots, x_k = y$ of states in $\mathcal{X}$ satisfying $P(x_i, x_{i+1}) > 0$ for every $0 \leq i < k$. Note that $x_i \notin B$ for every $0 \leq i < k$, for if $x_i \in B$ we would have $P(x_i, x_{i+1}) = 0$, since the elements of B are absorbing states. Hence there is a positive probability of reaching y from x before hitting B; that is, $\mathrm{P}_x\{\tau_y < \tau_B\} > 0$. But then, since $y \in A$,

$$0 < \mathrm{P}_x\{\tau_y < \tau_B\} \leq \mathrm{P}_x\{\tau_A < \tau_B\} = h(x).$$

Thus h is positive on $\mathcal{X} \setminus (A \cup B)$.

(b) Let $x \in \mathcal{X} \setminus (A \cup B)$. Then we have:

$$\begin{aligned}
\sum_{y \in \mathcal{X}} P(x,y)h(y) &= \sum_{y \in \mathcal{X}} \mathrm{P}_x\{X_1 = y\}\, \mathrm{P}_y\{\tau_A < \tau_B\} \\
&= \sum_{y \in \mathcal{X}} \mathrm{P}_x\{X_1 = y\}\, \mathrm{P}_x\{\tau_A < \tau_B | X_1 = y\} \\
&= \sum_{y \in \mathcal{X}} \mathrm{P}_x\{X_1 = y, \tau_A < \tau_B\} \\
&= \mathrm{P}_x\{\tau_A < \tau_B\} = h(x).
\end{aligned}$$

Note that since $x \in \mathcal{X} \setminus (A \cup B)$, $x \notin A$ and $x \notin B$, and hence the occurrence of the event $\{\tau_A < \tau_B\}$ is unaffected if, for any $y \in \mathcal{X}$, we are given that $X_0 = x$ and $X_1 = y$ rather than $X_0 = y$. This justifies the equality

$$\mathrm{P}_y\{\tau_A < \tau_B\} = \mathrm{P}_x\{\tau_A < \tau_B | X_1 = y\} \quad \text{for any } y \in \mathcal{X}.$$

Hence h is harmonic on $\mathcal{X} \setminus (A \cup B)$.

Exercise 3

Let $x \in \mathcal{X} \setminus (A \cup B)$. Then for any $y \in \mathcal{X}$,

$$\begin{aligned}
\mathrm{P}_x\{X_1 = y | \tau_A < \tau_B\} &= \frac{\mathrm{P}_x\{X_1 = y, \tau_A < \tau_B\}}{\mathrm{P}_x\{\tau_A < \tau_B\}} = \frac{\mathrm{P}_x\{\tau_A < \tau_B | X_1 = y\}\, \mathrm{P}_x\{X_1 = y\}}{h(x)} \\
&= \frac{\mathrm{P}_y\{\tau_A < \tau_B\} P(x,y)}{h(x)} = \frac{h(y)P(x,y)}{h(x)} = \check{P}(x,y).
\end{aligned}$$

Exercise 4

For any $x \in \mathcal{X} \setminus (A \cup B)$ and $y \in \mathcal{X}$, $\check{P}(x,y)$ is the probability of the chain transitioning from x to y in a single step, given that the chain hits A before it hits B.

Exercise 5

(a) If $x \in A$, then since $A \cap B = \emptyset$, $x \notin B$. Thus,

$$h(x) = \mathrm{P}_x\{\tau_A < \tau_B\} = \mathrm{P}\{\tau_A < \tau_B | X_0 = x\} = \mathrm{P}\{\tau_B > 0 | X_0 = x\} = 1.$$

Similarly, if $x \in B$, then $x \notin A$, and so

$$h(x) = \mathrm{P}_x\{\tau_A < \tau_B\} = \mathrm{P}\{\tau_A < \tau_B | X_0 = x\} = \mathrm{P}\{\tau_A < 0 | X_0 = x\} = 0.$$

(b) This was proven in Exercise 2(b). Specifically, the fact that h is harmonic on $\mathcal{X} \setminus (A \cup B)$ means that h is harmonic at every $x \in \mathcal{X} \setminus (A \cup B)$, by definition.

To see that h is the unique function satisfying (a) and (b), let $g : \mathcal{X} \to [0, \infty)$ be another function satisfying (a) and (b). That is,

(a) $g(x) = 1$ for every $x \in A$, and $g(x) = 0$ for every $x \in B$;

(b) g is harmonic at every $x \in \mathcal{X} \setminus (A \cup B)$.

Let $x \in \mathcal{X}$ be a state that maximizes $|h(x) - g(x)|$. If $x \in A \cup B$, then $h(x) = g(x)$, so that $|h(x) - g(x)| = 0$, and hence $h(y) = g(y)$ for every $y \in \mathcal{X}$. So assume $x \in \mathcal{X} \setminus (A \cup B)$. Then since h and g are both harmonic on $\mathcal{X} \setminus (A \cup B)$, we have for any $t > 0$,

$$\begin{aligned} h(x) - g(x) &= \sum_{y \in \mathcal{X}} P^t(x, y) h(y) - \sum_{y \in \mathcal{X}} P^t(x, y) g(y) \\ &= \sum_{y \in \mathcal{X}} P^t(x, y)[h(y) - g(y)] = \sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x, y)[h(y) - g(y)]. \end{aligned}$$

Hence,

$$\begin{aligned} |h(x) - g(x)| &= \left| \sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x, y)[h(y) - g(y)] \right| \le \sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x, y) |h(y) - g(y)| \\ &\le \sum_{y \in \mathcal{X}} P^t(x, y) |h(y) - g(y)| \le \sum_{y \in \mathcal{X}} P^t(x, y) |h(x) - g(x)| \\ &= |h(x) - g(x)|. \end{aligned}$$

Thus,

$$\sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x, y) |h(y) - g(y)| = |h(x) - g(x)|,$$

so that

$$\sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x, y)[|h(x) - g(x)| - |h(y) - g(y)|] = 0. \tag{1}$$

Therefore if $y \in \mathcal{X} \setminus (A \cup B)$ is such that $P^t(x,y) > 0$ for some $t > 0$, since $|h(x) - g(x)| - |h(y) - g(y)| \geq 0$, (1) implies that $|h(x) - g(x)| = |h(y) - g(y)|$. It follows that (1) can be written, for any $t > 0$,

$$\sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x,y)|h(x) - g(x)| = |h(x) - g(x)|.$$

Thus if $|h(x) - g(x)| > 0$, we have for any $t > 0$,

$$\sum_{y \in \mathcal{X} \setminus (A \cup B)} P^t(x,y) = 1. \tag{2}$$

By assumption, there exists $y \in A$ such that $P^t(x,y) > 0$ for some $t > 0$. However, (2) implies that $P^t(x,y) = 0$ for every $y \in A$ and $t > 0$, a contradiction. So we must have $|h(x) - g(x)| = 0$, so that $h(y) = g(y)$ for every $y \in \mathcal{X}$. Thus h is the unique function satisfying conditions (a) and (b).

Exercise 6

In this case the conditions (a) and (b) given in Exercise 5 reduce to

(a) $h(n) = 1$ and $h(0) = 0$;

(b) h is harmonic on $\{1, 2, \ldots, n-1\}$.

Condition (b) is equivalent to the following: for any $x \in \{1, 2, \ldots, n-1\}$,

$$\begin{aligned} h(x) &= \sum_{y \in \mathcal{X}} P(x,y)h(y) \\ &= P(x, x-1)h(x-1) + P(x, x+1)h(x+1) \\ &= \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1). \end{aligned}$$

But from the proof of Proposition 2.1 we know that the solution of this recurrence relation, with the boundary conditions specified by condition (a), is simply

$$h(x) = \frac{x}{n} \quad \text{for every } x \in \mathcal{X}.$$