Homework 5

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Problems from the book

Exercise 2.1

Note: For this exercise I opted to solve the recurrence directly rather than appeal to Exercise 1.12.

For every $1 \leq k \leq n$, let $\Delta_k = f_k - f_{k-1}$. Then for every $1 \leq k \leq n-1$,

$$\begin{aligned}\Delta_k = f_k - f_{k-1} &= (f_{k+1} - f_k) + (f_k - f_{k+1}) + (f_k - f_{k-1}) \\ &= \Delta_{k+1} + (2f_k - f_{k+1} - f_{k-1}) = \Delta_{k+1} + 2.\end{aligned}$$

In addition,

$$\sum_{k=1}^{n} \Delta_k = \sum_{k=1}^{n} (f_k - f_{k-1}) = f_n - f_0 = 0.$$

First, I claim that $\Delta_k = \Delta_1 - 2(k-1)$ for every $1 \leq k \leq n$. I will proceed by induction on k. When $k = 1$, the statement reduces to $\Delta_1 = \Delta_1 - 2(1-1)$, which holds trivially. Now let $1 < k \leq n-1$, and assume that $\Delta_k = \Delta_1 - 2(k-1)$. Then,

$$\Delta_{k+1} = \Delta_k - 2 = [\Delta_1 - 2(k-1)] - 2 = \Delta_1 - 2[(k+1) - 1],$$

and so the result holds by induction.

Next, note that we have the relation

$$\begin{aligned}0 = \sum_{k=1}^{n} \Delta_k = \sum_{k=1}^{n} [\Delta_1 - 2(k-1)] &= n\Delta_1 - 2\sum_{k=1}^{n} (k-1) \\ &= n\Delta_1 - 2\sum_{k=1}^{n-1} k = n\Delta_1 - 2\left[\frac{(n-1)n}{2}\right] = n\Delta_1 - 2(n-1)n,\end{aligned}$$

and rearrangement yields $\Delta_1 = n - 1$. Therefore, for every $1 \leq k \leq n$,

$$\Delta_k = (n-1) + 2(k-1),$$

so that

$$f_k = f_{k-1} + (n-1) + 2(k-1).$$

I claim that $f_k = k(n-k)$ for every $1 \leq k \leq n$. When $k = 1$, this statement reduces to $f_1 = n - 1$, which holds trivially since $f_1 = f_1 - f_0 = \Delta_1 = n - 1$. Now let $1 < k \leq n - 1$, and assume that $f_k = k(n-k)$. Then by the above,

$$\begin{aligned} f_{k+1} &= f_k + (n-1) - 2k = k(n-k) + (n-1) + 2k \\ &= (kn - k^2 - k) + (n - k - 1) = (k+1)(n-k-1) = (k+1)[n-(k+1)], \end{aligned}$$

and so the result holds by induction. Moreover, $f_0 = 0(n-0)$, and so in fact $f_k = k(n-k)$ for every $0 \leq k \leq n$, and the above argument shows that this is the unique solution to the recurrence relation in question. Therefore $\mathrm{E}_k(\tau) = k(n-k)$ for every $0 \leq k \leq n$.

Exercise 2.2

Let τ denote the number of rounds the gambler plays before reaching a fortune of either 0 or n dollars. I claim that $\mathrm{E}_k(\tau) = k(n-k)/p$ for every $0 \leq k \leq n$. To show this, it suffices to verify the conditions given in Exercise 1.12 ($A = \{0, n\}$ in this case). Specifically, if we define $f_k = k(n-k)/p$ for every $0 \leq k \leq n$, then we need to show that

(1) $f_0 = f_n = 0$

(2) $f_k = 1 + \sum_{m=0}^{n} P(k,m) f_m$ for every $1 \leq k \leq n-1$.

Here P denotes the transition matrix of the chain, and is defined by

$$P(k,m) = \begin{cases} 1-p & \text{if } m = k \\ p/2 & \text{if } m = k \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $0 \leq m \leq n$ and $0 < k \leq n$. Thus condition (2) can be written

$$f_k = 1 + \sum_{m=0}^{n} P(k,m) f_m = 1 + (1-p) f_k + \frac{p}{2} f_{k-1} + \frac{p}{2} f_{k+1}.$$

Some tedious algebra shows that $f_k = k(n-k)/p$ satisfies the above for every $0 \leq k \leq n$. Moreover, $f_0 = 0(n-0)/p = 0$ and $f_n = n(n-n)/p = 0$, so that condition (1) is satisfied as well. Therefore $\mathrm{E}_k(\tau) = f_k = k(n-k)/p$ for every $0 \leq k \leq n$ by Exercise 1.12.

Exercise 2.3

(Ran out of time).

Exercise 2.4

Using the inequality $\int_k^{k+1} \frac{dx}{x} \leq 1/k$, we get the following:

$$\sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{dx}{x} = \int_1^{n+1} \frac{dx}{x} = \log(n+1),$$

thus establishing the lower bound. Next, using the inequality $1/(k+1) \leq \int_k^{k+1} \frac{dx}{x}$, we get:

$$\sum_{k=1}^{n} \frac{1}{k} - 1 = \sum_{k=2}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{1}{k+1} \leq \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{x} = \int_1^{n} \frac{dx}{x} = \log(n),$$

and so rearrangement yields

$$\sum_{k=1}^{n} \frac{1}{k} \leq \log(n) + 1.$$

Therefore, putting the two bounds together, we get

$$\log(n+1) \leq \sum_{k=1}^{n} \frac{1}{k} \leq \log(n) + 1.$$

Exercise 2.9

Define a relation $\sim$ on $\mathbb{Z}$ as follows: for any $x, y \in \mathbb{Z}$, $x \sim y$ if and only if $x - y = kn$ for some $k \in \mathbb{Z}$ (i.e., $x \equiv y \pmod{n}$). This is easily seen to be an equivalence relation. Let $x, y, z \in \mathbb{Z}$. Then:

- $x \sim x$ since $x - x = 0n$.
- If $x \sim y$, then $x - y = kn$ for some $k \in \mathbb{Z}$, and so $y - x = (-k)n$. Hence $y \sim x$.
- If $x \sim y$ and $y \sim z$, then $x - y = kn$ and $y - z = mn$ for some $k, m \in \mathbb{Z}$. Thus, $x - z = (x - y) + (y - z) = kn + km = (k + m)n$, so that $x \sim z$ as well.

Therefore $\sim$ is an equivalence relation on $\mathbb{Z}$, as claimed. Let $\mathcal{X}^\sharp = \{[x] : x \in \mathbb{Z}\}$ denote the set of equivalence classes under $\sim$. Now let $x, x' \in \mathbb{Z}$ satisfy $x \sim x'$. Let $[y] \in \mathcal{X}^\sharp$. Then if P denotes the transition matrix for the simple random walk on $\mathbb{Z}$, we have the following:

$$\begin{aligned} P(x, [y]) \equiv \mathrm{P}\{X_{t+1} \in [y] | X_t = x\} &= \sum_{z \in [y]} \mathrm{P}\{X_{t+1} = z | X_t = x\} \\ &= \sum_{z \in [y]} P(x, z) = \sum_{z \in \mathbb{Z}} P(x, y + kn). \end{aligned}$$

Note, however, that for any $k \in \mathbb{Z}$,

$$P(x, y+kn) = \begin{cases} 1/2 & \text{if } y+kn = x+1 \\ 1/2 & \text{if } y+kn = x-1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & \text{if } y \sim (x+1) \\ 1/2 & \text{if } y \sim (x-1) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1/2 & \text{if } y \sim (x'+1) \\ 1/2 & \text{if } y \sim (x'-1) \\ 0 & \text{otherwise} \end{cases}$$
$$= P(x', y+kn).$$

Using this, we get the following:

$$P(x, [y]) = \sum_{z \in \mathbb{Z}} P(x, y+kn) = \sum_{z \in \mathbb{Z}} P(x', y+kn) = P(x', [y]).$$

Hence Lemma 2.5 applies, so that if (X_t) denotes the simple random walk on $\mathbb{Z}$, then $([X_t])$ is a Markov chain with state space $\mathcal{X}^\sharp$ and transition matrix $P^\sharp$ given by

$$P^\sharp([x],[y]) \equiv P(x,[y]) = \sum_{z \in \mathbb{Z}} P(x, y+kn),$$

for any $[x],[y] \in \mathcal{X}^\sharp$. If $n > 1$, then $P(x, y+k'n) = 1/2$ for precisely one value of $k' \in \mathbb{Z}$, and so

$$P^\sharp([x],[y]) = \sum_{z \in \mathbb{Z}} P(x, y+kn) = P(x, y+k'n) = \begin{cases} 1/2 & \text{if } y \sim (x+1) \\ 1/2 & \text{if } y \sim (x-1) \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, if $n = 1$ (the degenerate case), then $P(x, y+k'n) = P(x, y+k''n) = 1/2$ for precisely two values $k', k'' \in \mathbb{Z}$, and so

$$P^\sharp([x],[y]) = \sum_{z \in \mathbb{Z}} P(x, y+kn) = P(x, y+k'n) + P(x, y+k'n) = 1.$$

Finally, note that $\mathcal{X}^\sharp$ can be identified with $\mathbb{Z}_n$, since for any $x \in \mathbb{Z}$, we can write $x = kn+y$ for some unique $y \in \mathbb{Z}_n$ by the division algorithm, and so we can write $\mathcal{X}^\sharp = \{[y] : y \in \mathbb{Z}_n\}$. Thus $P^\sharp$ is simply the transition matrix for the simple random walk on $\mathbb{Z}_n$. The simple random walk on $\mathbb{Z}_n$ is therefore a projection of the simple random walk on $\mathbb{Z}$.

Exercise 2.10

(Ran out of time).

Metropolis chain on self-avoiding paths

Exercise 1

See the file `pivot_chain.ipynb`.

Exercise 2

Let x and y be self-avoiding paths of length n, and let P denote the transition matrix for the chain. Let ν denote the stationary distribution on the set of self-avoiding paths of length n. If

$$\nu(x)P(x,y) = \nu(y)P(y,x), \tag{1}$$

then ν is stationary for the chain. If y cannote be obtained from x by through of the possible transformations (rotations clockwise by $\pi/2$, π, $3\pi/2$, reflection across the x-axis, and reflection across the y-axis), then $P(x,y) = P(y,x) = 0$, and so the ν satisfies (1) trivially. On the other hand, suppose y can be obtained from x through one of these transformations. Then x can be obtained from y through the corresponding inverse transformation (for example, if y is obtained from x by through a rotation clockwise by $\pi/2$, then x can be obtained from y through a rotation by $3\pi/2$). Thus, since at each step the next state of the chain is determined by choosing a node in the path and a transformation, both uniformly at random, it follows from the above that $P(x,y) = P(y,x)$ in this case as well, and therefore ν satisfies (1). Thus ν is stationary for this chain.

Exercise 3

Let $\mathcal{X}$ denote the set of self-avoiding paths of length n. Note that $\mathcal{X}$ is finite. Let $f : \mathcal{X} \to \mathbb{R}$ be a bounded function, and let ν denote the uniform distribution for the chain. Then ν is stationary for the chain by Exercise 2 of this problem. Moreover,

$$E_\nu(f) = \sum_{x \in \mathcal{X}} \nu(x) f(x) = \sum_{x \in \mathcal{X}} \frac{1}{|\mathcal{X}|} f(x),$$

the average value of f over all self-avoiding paths. The Ergodic Theorem states that

$$\mathrm{P}_{x_0} \left\{ \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} f(X_t) = E_\nu(f) \right\} = 1,$$

where x_0 denotes the starting configuration of the chain. Thus if we use the code to compute an observed value of the average $\frac{1}{T} \sum_{t=0}^{T} f(X_t)$ for a large value of T, then we expect this value to be close to the true average $E_\nu(f)$.

Thus we can estimate each of the quantities in question by choosing an appropriate bounded function $f : \mathcal{X} \to \mathbb{R}$ and appealing to the discussion above:

(a) Define $f : \mathcal{X} \to \mathbb{R}$ as follows: for any $x \in \mathcal{X}$, if $x = (0, v_1, \ldots, v_n)$, then $f(x) = \|v_n - 0\|$, where $\|\cdot\|$ is the Euclidean norm. Then for any $x \in \mathcal{X}$, $0 \leq f(x) \leq n$, so that f is bounded. Thus the above discussion is applicable to f.

(b) Define $f : \mathcal{X} \to \mathbb{R}$ as follows: for any $x \in \mathcal{X}$, if $x = (0, v_1, \ldots, v_n)$, then

$$f(x) = \max_{a,b \in \{0, v_1, \ldots, v_n\}} \|a - b\|,$$

where $\|\cdot\|$ is the Euclidean norm. Then for any $x \in \mathcal{X}$, $0 \leq f(x) \leq n$. so that f is bounded. Thus the above discussion is applicable to f.

(c) Define $f : \mathcal{X} \to \mathbb{R}$ as follows: for any $x \in \mathcal{X}$, let $f(x)$ be the number of nodes that are between two parallel edges of x (in the sense given in the problem statement). Then $0 \leq f(x) \leq n$, so that f is bounded. Thus the above discussion is applicable to f.

Exercise 4

See the file `pivot_chain.ipynb`.