Model Order Reduction (MOR)

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1 MOR techniques for Linear Systems

1.1 Linear state-space Representation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1-1a}$$

$$y(t) = \mathbf{C}^T\mathbf{x}(t) \tag{1-1b}$$

From the eq. (1-1a) in Laplace domain it is obtained,

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) \;=\; \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{1-2}$$

By substituting (1-2)in (1-1b) above one can get a direct relationship between inputs and outputs signals

$$Y(s) = \mathbf{C}^T\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\mathbf{U}(\mathbf{s})\,. \tag{1-3}$$

In (1-3) the function relating inputs to outputs known as Transfer Function (TF) shown in below

$$H(s) = \mathbf{C}^T\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,. \tag{1-4}$$

1.2 Background on Arnoldi Method

Let transfer function $H(s)$ in (1-4) be a smooth (continuously derivable, C^∞) matrix function. Hence, the Taylor expansion to approximate the function in proximity of $s = 0$ is

$$H(s) \;=\; \sum_{i=0}^{\infty}\mathbf{M}_i s^i \;=\; \sum_{i=0}^{\infty}\frac{1}{i!}H^{(i)}(s)|_{s=0}s^i\,, \tag{1-5}$$

where $\frac{1}{i!}H^{(i)}(s)|_{s=0} = -\mathbf{C}^T\mathbf{A}^{-\{i+1\}}\mathbf{B}$

Proof: For the sake of simplicity of calculation let obtain a slightly different form for transfer function by factoring out $-\mathbf{A}$ from (1-4) as

$$H(s) = -\mathbf{C}^T\mathbf{A}^{-1}\left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1}\mathbf{B}\,. \tag{1-6}$$

Considering the Taylor expansion for $\left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1}$ in below

$$\left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1} \approx \mathbf{I} + \mathbf{A}^{-1}s + \mathbf{A}^{-2}s^2 + \mathbf{A}^{-3}s^3 + \dots\,, \tag{1-7}$$

Lemma 1.1 *The Taylor series approximating (1-6) will be*

$$H(s) = -\mathbf{C}^T\mathbf{A}^{-1}\left(\mathbf{I} + \mathbf{A}^{-1}s + \mathbf{A}^{-2}s^2 + \mathbf{A}^{-3}s^3 + \dots\right)\mathbf{B}\,. \tag{1-8}$$

For the proof of lemma (1.1), appendix A can be referred to. From (1-6) in lemma (1.1), we get

$$H(s) = \left(-\mathbf{C}^T\mathbf{A}^{-1}\mathbf{B}\right) + \left(-\mathbf{C}^T\mathbf{A}^{-2}\mathbf{B}\right)s + \left(-\mathbf{C}^T\mathbf{A}^{-3}\mathbf{B}\right)s^2 + \left(-\mathbf{C}^T\mathbf{A}^{-3}\mathbf{B}\right)s^3 + \dots\,. \tag{1-9}$$

$\square$

2 MOR techniques for nonlinear Systems

3 Non-Linear state-space Representation

$$\begin{cases} \frac{d}{dt}\mathbf{G}\left(\mathbf{X}(t)\right) = \mathbf{F}\left(\mathbf{X}(t)\right) + \mathbf{B}\left(\mathbf{X}(t)\right)\mathbf{U}(t) \\ \mathbf{Y}(t) = \mathbf{C}^T\mathbf{X}(t)\,. \end{cases} \tag{3-10}$$

The number of the inputs and outputs are m and n, respectively. The order of the original system is N and the order of reduced model is q. In 3-10 the vector of unknown states is

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \tag{3-11}$$

and $\mathbf{F}(\mathbf{X}(t))$ the nonlinear function of states in the most general form, is defined as

$$\begin{bmatrix} f_1(\mathbf{X}(t)) \\ f_2(\mathbf{X}(t)) \\ \vdots \\ f_N(\mathbf{X}(t)) \end{bmatrix} = \begin{bmatrix} f_1\left([x_1(t), x_2(t), \ldots, x_N(t)]^T\right) \\ f_2\left([x_1(t), x_2(t), \ldots, x_N(t)]^T\right) \\ \vdots \\ f_N\left([x_1(t), x_2(t), \ldots, x_N(t)]^T\right) \end{bmatrix}. \tag{3-12}$$

The Jacobian matrix (first order derivative) of the nonlinear function $\mathbf{F}$ with respect to $\mathbf{X}$ is

$$\mathbf{J}(\mathbf{X}) = \frac{d}{d\mathbf{X}}\mathbf{F}\left(\mathbf{X}(t)\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots, & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1}, & \frac{\partial f_2}{\partial x_2}, & \ldots, & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial x_1}, & \frac{\partial f_N}{\partial x_2}, & \ldots, & \frac{\partial f_N}{\partial x_N} \end{bmatrix}. \tag{3-13}$$

Let an arbitrarily selected initial (equilibrium) state vector be

$$\mathbf{X}_i = \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \\ \vdots \\ x_N(t_i) \end{bmatrix} = \begin{bmatrix} x_{1_i} \\ x_{2_i} \\ \vdots \\ x_{N_i} \end{bmatrix}, \tag{3-14}$$

The Jacobian matrix of the nonlinear function in (3-13) evaluated at the initial states $\mathbf{X}_i$ is represented as follows

$$\mathbf{J}(\mathbf{X}_i) \;=\; \frac{d}{d\mathbf{X}}\mathbf{F}\left(\mathbf{X}(t)\right)\Big|_{\mathbf{X}_i} \;=\; \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{\mathbf{X}_i}, & \frac{\partial f_1}{\partial x_2}\Big|_{\mathbf{X}_i}, & \ldots, & \frac{\partial f_1}{\partial x_N}\Big|_{\mathbf{X}_i} \\ \frac{\partial f_2}{\partial x_1}\Big|_{\mathbf{X}_i}, & \frac{\partial f_2}{\partial x_2}\Big|_{\mathbf{X}_i}, & \ldots, & \frac{\partial f_2}{\partial x_N}\Big|_{\mathbf{X}_i} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial x_1}\Big|_{\mathbf{X}_i}, & \frac{\partial f_N}{\partial x_2}\Big|_{\mathbf{X}_i}, & \ldots, & \frac{\partial f_N}{\partial x_N}\Big|_{\mathbf{X}_i} \end{bmatrix}. \tag{3-15}$$

The Hessian matrix (second divertive)of the above nonlinear function also is defined as

$$\mathbf{W}(\mathbf{X}) \;=\; \frac{d^2}{d\mathbf{X}^2}\mathbf{F}\left(\mathbf{X}(t)\right) \;=\; \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2}, & \frac{\partial^2 f_1}{\partial x_2 \partial x_1}, & \ldots, & \frac{\partial^2 f_1}{\partial x_N \partial x_1} \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2}, & \frac{\partial^2 f_2}{\partial x_2^2}, & \ldots, & \frac{\partial^2 f_2}{\partial x_N \partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f_N}{\partial x_1 \partial x_N}, & \frac{\partial^2 f_N}{\partial x_2 \partial x_N}, & \ldots, & \frac{\partial^2 f_N}{\partial x_N^2} \end{bmatrix}. \tag{3-16}$$

The Hessian matrix also, can be evaluated at $\mathbf{X}_i$. On the same line the Jacobian and Hessian matrices for $\mathbf{G}(\mathbf{X})$ the other nonlinear function in the left hand side of (3-10) are formed.

3.1 Linearized Approximation

Using the Jacobian matrix of $\mathbf{G}(\mathbf{X})$ and $\mathbf{F}(\mathbf{X})$ these nonlinear functions can be linearized in the neighborhood of $\mathbf{X}_i$, where $\mathbf{X} = \mathbf{X}_i + \Delta\mathbf{X}$, as follows

$$\mathbf{G}(\mathbf{X}) \;\approx\; \mathbf{G}\overset{\frown}{(\mathbf{X})} \;=\; \mathbf{G}(\mathbf{X}_i) + \mathbf{J}_{\mathbf{G}}\left(\mathbf{X}_i\right) \times \left(\mathbf{X} - \mathbf{X}_i\right) \tag{3-17}$$

and

$$\mathbf{F}(\mathbf{X}) \;\approx\; \mathbf{F}\overset{\frown}{(\mathbf{X})} \;=\; \mathbf{F}(\mathbf{X}_i) + \mathbf{J}_{\mathbf{F}}\left(\mathbf{X}_i\right) \times \left(\mathbf{X} - \mathbf{X}_i\right)\,. \tag{3-18}$$

Eq.s (3-17) and (3-18) are adequately accurate approximants for the corresponding original nonlinear functions when $\|\Delta\mathbf{X}\|$ is sufficiently small.

For the purpose of simplicity, in the rest of this context, a short-hand form for (3-17) and (3-18) are considered as follows, respectively

$$\mathbf{G}(\mathbf{X}) \;=\; \mathbf{G}_i + \mathbf{J}_{\mathbf{G}_i}\left(\mathbf{X} - \mathbf{X}_i\right) \tag{3-19}$$

and

$$\mathbf{F}(\mathbf{X}) \;=\; \mathbf{F}_i + \mathbf{J}_{\mathbf{F}_i}\left(\mathbf{X} - \mathbf{X}_i\right) \,. \tag{3-20}$$

3.2 Quadratic Form Approximation

Using the Jacobian and Hessian matrices defined above and considering $\Delta\mathbf{X}_i = (\mathbf{X} - \mathbf{X}_i)$, the nonlinear functions in (3-10) can be approximated in the quadratic form within the neighborhood of $\mathbf{X}_i$, as follows

$$G(\mathbf{X}) \;=\; \mathbf{G}_i + \mathbf{J}_{\mathbf{G}_i}\Delta\mathbf{X}_i + \Delta\mathbf{X}_i^T\mathbf{W}_{\mathbf{G}_i}\Delta\mathbf{X}_i \tag{3-21}$$

and

$$\mathbf{F}(\mathbf{X}) \;=\; \mathbf{F}_i + \mathbf{J}_{\mathbf{F}_i}\Delta\mathbf{X}_i + \Delta\mathbf{X}_i^T\mathbf{W}_{\mathbf{F}_i}\Delta\mathbf{X}_i \,. \tag{3-22}$$

The latter eq. (e.g.) can be rewritten using kronecker multiplication as

$$\mathbf{F}(\mathbf{X}) \;=\; \mathbf{F}_i + \mathbf{J}_{\mathbf{F}_i}\Delta\mathbf{X}_i + \mathbf{W}_{\mathbf{F}_i}\Delta\mathbf{X}_i^T \otimes \Delta\mathbf{X}_i \,. \tag{3-23}$$

A Appendix: proof for lemma 1.1

To make notation simpler let the core part of (1-6) be denoted as

$$F(s) = \left(\mathbf{I} - s\mathbf{A}^(- 1)\right)^{-1} . \tag{1-24}$$

Proof: To obtain Taylor expansion of $F(s)$ the subsequent derivatives can be worked out as

i . $F(s) = (\mathbf{I} - s\mathbf{A}^{-1})^{-1} \longrightarrow F(0) = F(s)|_{s=0} = \mathbf{I}$,

ii . $F^{(1)}(s) = -F(s)\frac{d}{ds}(\mathbf{I} - s\mathbf{A}^{-1})F(s) = F(s)\mathbf{A}^{-1}F(s) \longrightarrow F^{(1)}(s)|_{s=0} = F(0)\mathbf{A}^{-1}F(0) = \mathbf{A}^{-1}$

iii . $F^{(2)}(s) = F^{(1)}(s)\mathbf{A}^{-1}F(s) + F(s)\mathbf{A}^{-1}F^{(1)}(s) \longrightarrow F^{(2)}(s)|_{s=0} = F^{(1)}(0)\mathbf{A}^{-1}F(0) + F(0)\mathbf{A}^{-1}F^{(1)}(0) = 2\mathbf{A}^{-2}$

iv . $F^{(3)}(s) = F^{(2)}(s)\mathbf{A}^{-1}F(s) + F^{(1)}(s)\mathbf{A}^{-1}F^{(1)}(s) + F^{(1)}(s)\mathbf{A}^{-1}F^{(1)}(s) + F(s)\mathbf{A}^{-1}F^{(2)}(s) \longrightarrow F^{(3)}(s)|_{s=0} = 2\mathbf{A}^{-3} + \mathbf{A}^{-3} + \mathbf{A}^{-3} + 2\mathbf{A}^{-3} = 6\mathbf{A}^{-3}$

v . and so on so force!

Plunging all above evaluated derivatives in the general form of the Taylor expansion for (1-24) results in (1-25) below.

$$F(s) = \left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1} \approx \mathbf{I} + \mathbf{A}^{-1}s + \mathbf{A}^{-2}s^2 + \mathbf{A}^{-3}s^3 + \dots . \tag{1-25}$$

$\square$