Bounded-Real Lemma for s-parameters

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1 Mathematical Preliminaries

This section presents the mathematical backgrounds and theories required for studying the Bounded-Real Lemma (BRL) in scattering-parameters of linear systems.
Let,

- \- *spec* $(\mathbf{A})$ be spectrum of $\mathbf{A}$, the set of eigenvalues of the matrix $\{\lambda_i\}$,
- \- *mspec* $(\cdot)$ be read as $\{\lambda_i\}$, even if it is multiple,
- \- *tr* $(\cdot)$ denote the trace, i.e., the sum of the diagonal elements of its argument,
- \- For $\mathbf{A} \in \mathbb{F}^{n\times n}$, $\mathbf{A}^*$ denotes the conjugate (Hermitian) transpose of $\mathbf{A} = [a_{ij}]$ as $a^*_{ij} = (a_{ji})^*$, where $\mathbf{A}^* = \left[a^*_{ij}\right]$.

Definition 1.1 *For Square matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$ define the following types:*

(i) $\mathbf{A}$ *is* ***Hermitian*** *if* $\mathbf{A} = \mathbf{A}^*$.

(ii) $\mathbf{A}$ *is* ***positive-semidefinite*** $(\mathbf{A} \geq 0)$ *if* $\mathbf{A}$ *is Hermitian and* $\mathbf{x}^*\mathbf{A}\mathbf{x} \geq 0$ *for all* $\mathbf{x} \in \mathbb{F}^n$.

[1], Definition 3.1.1, pp. 81-82

Fact 1.2 *Consider $\mathbf{A} \in \mathbb{F}^{n\times n}$ (e.g. a scattering parameter matrix of a n-port linear system), $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ are Hermitian.*

Proof:

$$(\mathbf{A}^*\mathbf{A})^* = \mathbf{A}^*(\mathbf{A}^*)^* = \mathbf{A}^*\mathbf{A}$$

For $\mathbf{A}\mathbf{A}^*$ matrix it is proved similarly. ∎

Fact 1.3 *Consider* $\mathbf{A} \in \mathbb{F}^{n\times n}$, *matrix* $\mathbf{A}^*\mathbf{A}$ *is positive-definite.*

Proof: let $\mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{F}^n$, then

$$\mathbf{x}^*\left(\mathbf{A}^*\mathbf{A}\right)\mathbf{x} = \left(\mathbf{x}^*\mathbf{A}^*\right)\left(\mathbf{A}\mathbf{x}\right) = \left(\mathbf{A}\mathbf{x}\right)^*\left(\mathbf{A}\mathbf{x}\right)$$

It is noted that $(\mathbf{A}\mathbf{x})$ is a column vector with complex entries ($\in \mathbb{F}^n$). Therefore, we have

$$\left(\mathbf{A}\mathbf{x}\right)^*\left(\mathbf{A}\mathbf{x}\right) =$$

$$\begin{bmatrix} A^*_{x_1} & \cdots & A^*_{x_n} \end{bmatrix}\begin{bmatrix} a_{x_1} \\ \vdots \\ a_{x_n} \end{bmatrix} = \left|a_{x_1}\right|^2 + \cdots + \left|a_{x_n}\right|^2 \geq 0\,. \quad \blacksquare$$

Proposition 1.4 *Let* $\mathbf{A} \in \mathbb{F}^{n\times n}$ *and* $\alpha \in \mathbb{F}$, *then, the following statements hold:*

(i) $mspec\left(\alpha\mathbf{A}\right) = \alpha mspec\left(\mathbf{A}\right)$.

(ii) $mspec\left(\beta I_n + \alpha\mathbf{A}\right) = \beta + \alpha mspec\left(\mathbf{A}\right)$.

(iii) if $\mathbf{A}$ *is Hermitian,* $spec\left(\mathbf{A}\right) \subset \mathbb{R}$.

[1], Proposition 4.4.4, pp. 131

In a general form, let $\mathbf{F} \in \mathbb{F}^{n\times m}$, Noting (1.3) and (1.4), it is concluded

Fact 1.5 *Matrices* $\mathbf{A}^*\mathbf{A} \in \mathbb{F}^{m\times m}$ *and* $\mathbf{A}\mathbf{A}^* \in \mathbb{F}^{n\times n}$ *have* ***positive-real*** *eigenvalues.*

$$spec\left(\mathbf{A}^*\mathbf{A}\right) \subset \mathbb{R}^+$$

Definition 1.6 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times m)}$. *Then, the* ***singular values*** *of* $\mathbf{A}$ *are the* $min\{m, n\}$ *nonnegative number* $\sigma_1\left(A\right), \cdots, \sigma_{min\{m,n\}}\left(A\right)$, *where, for all* $i = 1, \cdots, min\{m, n\}$,

$$\sigma_i(\mathbf{A}) \triangleq \left[\lambda\left(\mathbf{A}\mathbf{A}^*\right)\right]^{1/2} = \left[\lambda\left(\mathbf{A}^*\mathbf{A}\right)\right]^{1/2}.$$

Then,

$$\sigma_1\left(\mathbf{A}\right) \geq \cdots \geq \sigma_{min\{n,m\}}\left(\mathbf{A}\right) \geq 0\,.$$

[1], Definition 5.6.1, pp. 181-182

Fact 1.7 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times m)}$*, and let* $r = rank\mathbf{A}$*. Then, for all* $i = 1, \cdots, r$*,*

$$\sigma_i(\mathbf{A}^*\mathbf{A}) = \sigma_i(\mathbf{A}\mathbf{A}^*) = \sigma_i^2\left(\mathbf{A}\right).$$

In particular,

$$\sigma_1\left(\mathbf{A}^*\mathbf{A}\right) \;=\; \sigma_{max}^2\left(\mathbf{A}\right).$$

[1], Fact 5.10.18, pp. 198

Fact 1.8 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times n)}$*,*

$$\|\mathbf{A}\|_F^2 \;=\; tr\left(\mathbf{A}\mathbf{A}^*\right) \;=\; tr\left(\mathbf{A}^*\mathbf{A}\right) \;=\; \sum_{i,j} |a_{ij}|^2$$

Proof: e.g.:

$$\mathbf{A}^*\mathbf{A} = \begin{bmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} |a_{11}^2| + |a_{21}^2| & X \\ X & |a_{12}^2| + |a_{22}^2| \end{bmatrix}$$
$$tr(\mathbf{A}^*\mathbf{A}) = \left|a_{11}^2\right| + \left|a_{21}^2\right| + \left|a_{12}^2\right| + \left|a_{22}^2\right| = \|\mathbf{A}\|_F^2\,. \quad \blacksquare$$

Fact 1.9 *For the norm of the matrix* $\mathbf{A}$ *we have*

(i) $\|A\|_2 = \sqrt{\max(eig(A^*A))}$.

(ii) $\|A\|_\infty = \max_i \sum_j |A_{ij}|$.

[2], ch. 10.3

Fact 1.10 *For the (ii) in fact. 1.9 and def. 1.6, it is norm of the matrix* $\mathbf{A}$ *we have*

$$\|A\|_2 = \sigma_{max}(A). \tag{1-1}$$

2 Bounded Real Lemma (BRL)

As a (passivity) checking criterion for bounded-real-ness of s-parameter matrix $\mathbf{Q} \in \mathbb{F}^{n \times n}$, which is the scattering-parameters for a n-port linear system, we have:

$$I_n - \mathbf{Q}^*\mathbf{Q} \geq 0\,. \tag{2-2}$$

From the mathematical elaborations in previous chapter, it is concluded that, eq. (2-2) requires

$$mspec\,(I - \mathbf{Q}^*\mathbf{Q}) = 1 - mspec\,(\mathbf{Q}^*\mathbf{Q}) \geq 0\,, \tag{2-3}$$

which is equivalently shortened as

$$mspec\,(\mathbf{Q}^*\mathbf{Q}) \leq 1\,. \tag{2-4}$$

The above notation in (2-4) shows that all eigenvalues of Hermitian matrix $\mathbf{Q}^*\mathbf{Q}$ requires to be bounded to one, while they are known from previous section as real and positive values. This is logically means that

$$\lambda_{max}\,(\mathbf{Q}^*\mathbf{Q}) \leq 1\,. \tag{2-5}$$

According to the Definition. 1.6 we have $[\lambda\,(\mathbf{A}^*\mathbf{A})] = \sigma_i^2(\mathbf{A})$, by subsituting which in (2-4) it is

$$\sigma^2_{max}\,(\mathbf{Q}) \leq 1\,, \tag{2-6}$$

or equivalently

$$\sigma_{max}\,(\mathbf{Q}) \leq 1\,. \tag{2-7}$$

Fact 2.1 *Considering fact 1.10 and (2-7) the following inequalities are equivalent.*

(i) $\sigma_{max}\,(\mathbf{Q}) \leq 1$.

(ii) $\|\mathbf{Q}\|_2 \leq 1$.

References

[1] D. S. Bernstein, *Matrix Mathematics: Theorey, Facts, and Formulas with Application to Linear System Theory,* NJ:Priceton University Press, 2005.

[2] K. B. Petersen and M. S. Pedersen, *The Matrix Cookbook*, available:[http://matrixcookbook.com], Nov. 14, 2008.