RESOURCE LETTER

Behzad Nouri (PhD student)
Electronics Dep., Carleton University

1 Time-Series Analysis, Phase-Space Reconstruction and Embedology

---

- **1936**

Hassler Whitney, In [1] Whitney a MIT professor showed that a generic smooth map $\mathbf{F}$ from a d-dimensional smooth compact manifold $\mathbf{M}$ to $\mathbf{R}^{2d+1}$ is actually a diffeomorphisim on $\mathbf{M}$. That is, $\mathbf{M}$ and $\mathbf{F}(\mathbf{M})$ are diffeomorphic [2].

- **1981**

Floris Takens, Takens dealt with a restricted class of maps called *delay-coordinate* maps. A delay-coordinate map is constructed from a time series of a single observed quantity from an experiment. Because of this, a typical delay-coordinate map is much more likely to be accessible to an experimentalist than a typical map.
Takens in [3] showed that if the dynamical system and the observed quantity are generic, then the delay-coordinate map from a d-dimensional smooth compact manifold $\mathbf{M}$ to $\mathbf{R}^{2d+1}$ is a diffeomorphism on $\mathbf{M}$ [2] (CF. Casdagli[1990]).

Ricardo Mañé, The results achieved in the theory of chaotic systems point out very relevant elements which can be extracted from the measurement of time series of one variable of the non-linear dynamic system. One of these results is given by the Takens-Mañé theorem [3, 4] about the sufficient dimension of an Euclidean space to secure a fair representation of the true strange attractor of the underlying system [5].

This fact is known as *Takens-Mañé Theorem.*

- **1991**

Tim Sauer Embedology [2].

- **1992**

Martin Casdagli The motivation for using a constructed-space to non-linear systems identification based on the observed data goes under the rubric of Non-Linear Time Series Analysis (NLTSA) [6]. The key idea is to embed the measured or simulated stimulus and response variables in a higher dimensional space built not only from the measured data but also transforms of the measured data, e.g., their time derivatives. The suggestion to use this approach for describing input-output systems is due to Casdagli [7]. In particular, deterministic prediction is possible from an embedded model that will mimic the dynamics of the actual system. Due to a theorem of Takens even when it is extended to the driven case, these embedded models can be faithful to the dynamics of the original system. [8] Casdagli[1992] [7] suggested an extension of the Takens' embedding theorem [Takens, 1981; Dauer et al. 1991] to the input-output case. His argument was heuristic but appears to be backed up by successive work which aims at rigor [stark et al., 1996]. As in Takens' theorem, the condition $m > 2d$ and $l > 2d$ is sufficient but may not be necessary [9].

Norman F. Hunter, Jr. In [10] also outlines some elaboration on the application of nonlinear time series models to driven systems.

- **1997**

J. Stark Initially, Takens theorem was limited to systems governed by autonomous systems of equations. Fortunately, it has recently been extended to driven systems by J. Stark [11,12] and later to stochastically forced systems [13].
These works have been originally presented in 1997. However the proceeding was published in 1999 and they may dated so.

2 Minimum Embedding Dimension

- **1992**

M. Kennel (FNN) The method of false neighbors [14]. It was developed based on the fact that choosing too low an embedding dimension results in points that are far apart in the original phase space being moved closer together in the reconstruction space. Certainly this method is a good approach.

- **1994**

Edward Ott, et al. There have been many discussions on how to determine the optimal embedding dimension from a scalar time series based on Takens' theorem [3] or its extensions in Tim Sause's work [2]. Ott et-al. in [15] provides a survey of the methods for deciding the minimum embedding dimension.

References

[1] H. Whitney, "Differentiable manifolds," *Annals of Mathematics (Ann. Math.)*, vol. 37, pp. 579–616, 1936, jSTOR 1968482. [Online]. Available: http://dx.doi.org/10.2307/196848

[2] T. Sauer, J. A. Yorke, and M. Casdagli, "Embedology," *Journal of Statistical Physics*, vol. 65, pp. 579–616, 1991, 10.1007/BF01053745. [Online]. Available: http://dx.doi.org/10.1007/BF01053745

[3] F. Takens, "Detecting strange attractors in turbulence," in *Dynamical Systems and Turbulence, Warwick 1980*, ser. Lecture Notes in Mathematics, D. Rand and L.-S. Young, Eds. Springer Berlin / Heidelberg, 1981, vol. 898, pp. 366–381, 10.1007/BFb0091924. [Online]. Available: http://dx.doi.org/10.1007/BFb0091924

[4] R. Mañé, "On the dimension of the compact invariant sets of certain non-linear maps," in *Dynamical Systems and Turbulence, Warwick 1980*, ser. Lecture Notes in Mathematics, D. Rand and L.-S. Young, Eds. Springer Berlin / Heidelberg, 1981, vol. 898, pp. 230–242, 10.1007/BFb0091916. [Online]. Available: http://dx.doi.org/10.1007/BFb0091916

[5] F. Masulli, L. Studer, R. Parenti, and A. Ricerche, "Neural modeling of non-linear processes: The relevance of the takens-mañé theorem," *International Journal on Chaos Theory and Applications*, vol. 4, no. 2-3, pp. 59–74, 1999.

[6] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis.* Cambridge, Massachusetts: Cambridge Univ. Press, 1997.

[7] M. Casdagli, "A dynamical systems approach to modeling input-output systems," in *Proceedings of the Workshop on Nonlinear Modeling and Forecasting (SANTA FE, New Mexico, 1990)*, M. Casdagli and S. Eubank, Eds., 1992.

[8] J. Wood and D. Root, "The behavioral modeling of microwave/rf ics using non-linear time series analysis," in *Microwave Symposium Digest, 2003 IEEE MTT-S International*, vol. 2, 2003, pp. 791 – 794 vol.2.

[9] L. Cao, A. Mees, K. Judd, and G. Froyland, "Determining the minimum embedding dimensions of input- output time series data," *International Journal of Bifurcation and Chaos (IJBC)*, vol. 8, no. 7, pp. 1491–1504, 1998.

[10] N. J. Hunter, "Application of nonlinear time-series models to driven systems," in *Proceedings of the Workshop on Nonlinear Modeling and Forecasting*, M. Casdagli and S. Eubank, Eds., vol. XII. SANTA FE, New Mexico: Addison-Wesley Publication Company, September 1990, pp. 467–491.

[11] J. Stark, D. S. Broomhead, M. E. Davies, and J. Huke, "Takens embedding theorems for forced and stochastic systems," *Nonlinear Anal.*, vol. 30, pp. 5303–5314, December 1997. [Online]. Available: http://portal.acm.org/citation.cfm?id=283577.283733

[12] J. Stark, "Delay embeddings for forced systems. i. deterministic forcing," *Journal of Nonlinear Science*, vol. 9, pp. 255–332, 1999, 10.1007/s003329900072. [Online]. Available: http://dx.doi.org/10.1007/s003329900072

[13] J. Stark, D. Broomhead, M. Davies, and J. Huke, "Delay embeddings for forced systems. ii. stochastic forcing," *Journal of Nonlinear Science*, vol. 13, pp. 519–577, 2003, 10.1007/s00332-003-0534-4. [Online]. Available: http://dx.doi.org/10.1007/s00332-003-0534-4

[14] M. B. Kennel, R. Brown, and H. D. I. Abarbanel, "Determining embedding dimension for phase-space reconstruction using a geometrical construction," *Phys. Rev. A*, vol. 45, no. 6, pp. 3403–3411, Mar 1992.

[15] E. Ott, T. Sauer, and Yorke, *Coping with Choas.* New York: Wiley, 1994.