1 Network Scaling in Time Domain

In many practical applications, the units of ohm and hertz are too small whereas Farads and Henrys are too large. Typical designs generally include inductors measured in mH or μH, capacitors in μF to pF, resistors in $k\Omega$ and currents in mA to μA. In frequency domain, the typical circuit analyses are performed in a typical frequency range of few Hz to tens of GHz or more. In time domain, the range of analyses has typically dropped to sub-$nsec$ range. These spread of values is inconvenient and leads to over- and under-flows in computer applications [1].

For practical calculations, as well as for theoretical considerations, it is convenient to scale networks e.g. such that:

In FD:
1. The components with frequency-dependent admittances (/ impedances) are scaled such that some frequency of interest is reduced to 1 rad/sec (frequency scaling).
2. In Model order reduction frequency scaling is managed to make DC moment (m_0) equal to the first moment (m_1) by selecting $K_f = \frac{m0}{m1}$ to obtain scaled frequency as $f_{scaled} = K_f \times f$.

In TD:
1. One (any) resistor is scaled to the value of one ohm (impedance scaling), this directly leads to scaling the current.
2. The components such as capacitors and inductors are scaled such that some time of interest is reduced to 1 sec (time scaling).

Generally, in any attempt for scaling (either in frequency or time domain) the dimension of the currents and voltages should properly decided to be uniformly consistent. It is such that, both Kirchhoff's Voltage and current Law hold. This requires properly scaling admittance and impedance parameters in both parallel and serial connections, respectively.

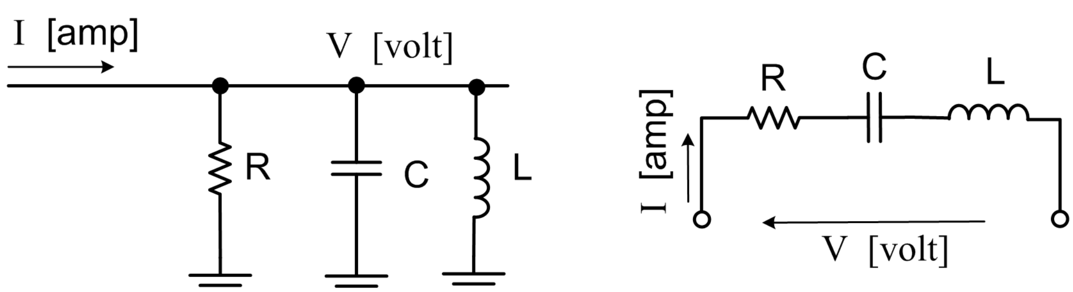

Figure 1: Parallel and Serial connections in Networks

2 Scaling Scheme for Time Domain

1) Resistor:

The original element V-I equation (before scaling) is

$$V_R(t)^{[V]} = R \times I_R(t)^{[A]} \tag{2.1}$$

$$V^{[V]} = \frac{R}{K_i} \times \left(K_i \, I^{[A]}\right) \ , \tag{2.2}$$

where k_i is scaling both current and resistor.

$$I_{scaled} \triangleq K_i \, I^{\,[A]} . \tag{2.3}$$

When we get I_{scaled} from simulation to convert it to $I^{\,[A]}$ one can consider that,

$$I\,[A] = I_{scaled} \left[\frac{A}{K_i}\right] . \tag{2.4}$$

Eq. (2.4) means as soon as we obtain I_{scaled} we can consider the actual answer but in the scale of $\left[\frac{A}{K_i}\right]$.

$$V^{\,[V]} = \underbrace{\frac{R}{K_i}}_{R_{scaled}} \times I_{scaled} \tag{2.5}$$

In (2.5), we have:

$$\boxed{\begin{array}{l} K_R \triangleq \dfrac{1}{K_i}, \quad R_{scaled} \triangleq K_R \, R, \quad I_{scaled} = K_i \, I_R , \\ \qquad \text{Then, to obtain } I^{\,[A]} \text{ from } I_{scaled}, \text{(scaling-back) it is } I\,[A] = I_{scaled} \left[\dfrac{A}{K_i}\right] \quad (2.6) \end{array}}$$

Example: Given $R = 1k\Omega$ and $V_R = 10V$. let $K_i = 1000$ we have:
Before scaling:

$$10V = 1k\Omega \; I_R \rightarrow I_R = 10m[A] . \tag{2.7}$$

By scaling:

$$V^{\,[V]} = R_{scaled} \times I_{scaled} \rightarrow 10V = \frac{1k\Omega}{1000} \times I_{scaled} \rightarrow I_{scaled} = 10 ,$$
$$\xrightarrow{\text{To scale bk}:} I_R\,[A] = \frac{I_{scaled}}{K_i} = 10\,[mA] . \tag{2.8}$$

2) Capacitor:

The original element V-I equation (before scaling) is

$$I_C(t)^{\,[A]} = C \, \frac{dV_C(t)}{dt} \left[\frac{V}{sec}\right] \tag{2.9}$$

Considering the fact that the current flowing in the network has been already scaled with K_i, from (2.6) we will have $I = \frac{I_{scaled}}{K_i}$ and

$$\frac{I_{scaled}}{K_i} = C \, \frac{dV}{dt} \left[\frac{V}{sec}\right] . \tag{2.10}$$

Next, scaling time by K_t, it is

$$\frac{I_{scaled}}{K_i} = (K_t \times C) \, \frac{dV}{d\,(K_t \times t)} \left[\frac{V}{sec}\right] , \tag{2.11}$$

where K_t is scaling both time and capacitor.

$$I_{scaled} \;=\; \underbrace{(K_i \times K_t)}_{\triangleq K_C}\, C\, \frac{dV}{d\,\underbrace{(K_t \times t)}_{\triangleq t_{scaled}}} \;\left[\frac{V}{sec}\right]\,. \tag{2.12}$$

Eq (2.12) can be equivalently written as:

$$I_{scaled} \;=\; C_{scaled}\, \frac{dV}{dt_{scaled}}\,. \tag{2.13}$$

In (2.13), we have:

$$\boxed{K_C \;\triangleq\; K_i\, K_t\,, \quad C_{sacled} \;\triangleq\; K_C\, C\,, \quad t_{scaled} \;\triangleq\; K_t\; t\,, \quad \text{having,}\;\; I_{scaled} \;=\; K_i\, I_C\,. \quad (2.14)}$$

Example: Given $C = 1pF$ and $V_C \;=\; 10\,\sin(t)\;V$. let $K_i \;=\; 1000$ and $K_t = 1e9$ we have:
Before scaling:

$$I_C\;[A] \;=\; C\,\frac{d}{dt}(10\sin(t)) \;\rightarrow\; I_C \;= 1e-12 \times 10\cos(t)\,. \tag{2.15}$$

By scaling:

$$\begin{aligned} &K_C \;=\; K_i\,K_t \;=\; 1000 \times 1e9 \;=\; 1e12\,, \qquad C_{scaled} \;=\; K_C\,C \;=\; 1\,,\\ &I_{scaled} \;=\; C_{scaled}\,\frac{d}{dt_{scaled}}\,(10\sin(t_{scaled}))\;\;, \rightarrow\;\; I_{scaled} \;=\; 1 \times 10\cos(t_{scaled})\\ &\xrightarrow{\text{To scale bk}}\;\; I\;[A] \;=\; \frac{I_{scaled}}{K_i} \;=\; 10\cos(t_{scaled})\;[mA]\,. \end{aligned} \tag{2.16}$$

3) Inductor:

The original element V-I equation (before scaling) is

$$V_L(t)^{\;[V]} \;=\; L\,\frac{dI_L(t)}{dt}\;\left[\frac{A}{sec}\right] \tag{2.17}$$

Considering the fact that the current flowing in the network has been already scaled with K_i, from (2.6) we will have $I \;=\; \frac{I_{scaled}}{K_i}$ and

$$V \;=\; L\,\frac{d}{dt}\left(\frac{I_{scaled}}{K_i}\right)\left[\frac{A}{sec}\right]\,. \tag{2.18}$$

Next, scaling time by K_t, it is

$$V \;=\; (K_t \times L)\;\frac{d}{d\,(K_t \times t)}\left(\frac{I_{scaled}}{K_i}\right)\;\left[\frac{A}{sec}\right]\,, \tag{2.19}$$

where K_t is scaling both time and inductor.

$$V \;=\; \underbrace{\left(\frac{K_t}{K_i}\right)}_{\triangleq K_L}\,L\,\frac{dI_{scaled}}{d\,\underbrace{(K_t \times t)}_{\triangleq t_{scaled}}}\;\left[\frac{V}{sec}\right]\,. \tag{2.20}$$

Eq (2.20) can be equivalently written as:

$$V \;=\; L_{scaled}\,\frac{dI_{scaled}}{dt_{scaled}}\,. \tag{2.21}$$

In (2.21), we have:

$$\boxed{K_L \;\triangleq\; \frac{K_t}{K_i}\,, \quad L_{sacled} \triangleq K_L\,L\,, \quad t_{scaled} \triangleq K_t\,t\,, \quad \text{having,}\;\; I_{scaled} \;=\; K_i\,I_L\,. \qquad (2.22)}$$

Example: Given $L = 1\mu H$ and $V_L \;=\; 10\sin(t)\;V$. let $K_i \;=\; 1000$ and $K_t = 1e9$ we have:
<u>Before scaling:</u>

$$V_L\;[V] \;=\; L\,\frac{dI_L}{dt}\,\left[\frac{A}{t}\right] \;\to\; 10\sin(t) \;= 1e-6\frac{dI_L}{dt} \;\to\; I_L = -1e7\,cos(t)\;. \tag{2.23}$$

<u>By scaling:</u>

$$\begin{aligned} K_L \;=\; \frac{K_t}{K_i} \;=\; \frac{1e9}{1000} \;=\; 1e6\,, \qquad L_{scaled} \;=\; K_L\,L \;=\; 1\,, & \\ V \;=\; L_{scaled}\,\frac{dI_{scaled}}{dt_{scaled}}\,,\to\; 10\sin(t_{scaled}) \;=\; 1\times\frac{I_{scaled}}{t_{scaled}}\,,\to & \\ I_{scaled} \;=\; -10\,\cos(t_{scaled}) \xrightarrow{\text{To scale bk:}} I\;[A] \;=\; \frac{I_{scaled}}{K_i} \;=\; -10\cos(t_{scaled})\;[mA]\,. & \end{aligned} \tag{2.24}$$

Summary:

After deciding scaling factors for current K_i and time K_t, *i.e.*

$$K_R \;\triangleq\; \frac{1}{K_i}\,, \qquad K_C \;\triangleq\; K_i\,K_t\,, \qquad K_L \;\triangleq\; \frac{K_t}{K_i}\,,$$

$$R_{scaled} \;\triangleq\; K_R\,R\,, \qquad C_{scaled} \;\triangleq\; K_C\,C\,, \qquad L_{scaled} \;\triangleq\; K_L\,L\,,$$

$$t_{scaled} \;\triangleq\; K_t\,t\,, \qquad I_{scaled} \;=\; K_i\,I\,, \qquad V_{scaled} \;=\; V\,,$$

$$\text{having}\;\; I_{scaled}\;\; \text{to obtain:}\; I\;[A] \;=\; \frac{I_{scaled}}{K_i} \;=\; I_{scaled}\;\left[\frac{A}{K_i}\right]$$

References

[1] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design (Van Nostrand Reinhold Electrical/Computer Science and Engine)*. Springer, 1983.