Classical Definition

Historically, the data for a *manifold* was specified as a collection of *coordinate domains* related by changes of coordinates. The manifold itself could be obtained by gluing the domains in accordance with the transition functions, provided the changes of coordinates were free of inconsistencies.

In this formulation, a $\mathcal{C}^k$ manifold is specified by two types of information. The first item of information is a collection of open sets

$$V_\alpha \subset \mathbb{R}^n, \quad \alpha \in \mathcal{A},$$

indexed by some set $\mathcal{A}$. The second item is a collection of transition functions, that is to say $\mathcal{C}^k$ *diffeomorphisms*

$$\sigma_{\alpha\beta} : V_{\alpha\beta} \to \mathbb{R}^n, \quad V_{\alpha\beta} \subset V_\alpha, \text{ open}, \quad \alpha, \beta \in \mathcal{A},$$

obeying certain consistency and topological conditions.

We call a pair

$$(\alpha, x), \quad \alpha \in \mathcal{A},\ x \in V_\alpha$$

the coordinates of a point relative to chart α, and define the manifold M to be the set of equivalence classes of such pairs modulo the relation

$$(\alpha, x) \simeq (\beta, \sigma_{\alpha\beta}(x)).$$

To ensure that the above is an equivalence relation we impose the following hypotheses.

- For $\alpha \in \mathcal{A}$, the transition function $\sigma_{\alpha\alpha}$ is the identity on V_α.
- For $\alpha, \beta \in \mathcal{A}$ the transition functions $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ are inverses.
- For $\alpha, \beta, \gamma \in \mathcal{A}$ we have for a suitably restricted domain

$$\sigma_{\beta\gamma} \circ \sigma_{\alpha\beta} = \sigma_{\alpha\gamma}$$

We topologize M with the least coarse topology that will make the mappings from each V_α to M continuous. Finally, we demand that the resulting topological space be paracompact and Hausdorff.

Notes

To understand the role played by the notion of a differential manifold, one has to go back to classical differential geometry, which dealt with geometric objects such as curves and surface only in reference to some ambient geometric setting typically a 2-dimensional plane or 3-dimensional space. Roughly speaking, the concept of a manifold was created in order to treat the intrinsic geometry of such an object, independent of any embedding. The motivation for a theory of intrinsic geometry can be seen in results such as Gauss's famous Theorema Egregium, that showed that a certain geometric property of a surface, namely the scalar curvature, was fully determined by intrinsic metric properties of the surface, and was independent of any particular embedding. Riemann [1] took this idea further in his habilitation lecture by describing intrinsic metric geometry of n-dimensional space without recourse to an ambient Euclidean setting. The modern notion of manifold, as a general setting for geometry involving differential properties evolved early in the twentieth century from works of mathematicians such as Hermann Weyl [3], who introduced the ideas of an atlas and transition functions, and Elie Cartan, who investigation global properties and geometric structures on differential manifolds. The modern definition of a manifold was introduced by Hassler Whitney [4] (For more foundational information, follow this link to some old notes by Matthew Frank).

Note: The above text is (mainly) from [5]

References

[1] Riemann, B., "Uber die Hypothesen welche der Geometrie zu Grunde liegen (On the hypotheses that lie at the foundations of geometry)" in M. Spivak, *A comprehensive introduction to differential geometry*, vol. II.

[2] Spivak, M., *A comprehensive introduction to differential geometry*, vols I & II.

[3] Weyl, H., *The concept of a Riemann surface*, 1913

[4] Whitney, H., *Differentiable Manifolds*, Annals of Mathematics, 1936.

[5] Robert Milson, matte, *"notes on the classical definition of a manifold" (version 8)*, PlanetMath. Freely available at `http://planetmath.org/?op=getobj;from=objects;id=5691`