A Collection of the Preliminaries and Theorems in Linear Algebra

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Notation

0.1 Nomenclature

$\mathbb{R}$ The field of real numbers

$\mathbb{R}^N$ The set of real vectors of size N

$\mathbb{R}^{N\times N}$ The set of real matrices of size N $\times$ N

$\mathbb{C}$ The field of complex numbers, e.g.: s-plane

$\mathbb{C}^+$ The open right half plane in the complex s-plane; $\mathbb{C}^+ = \{s \in \mathbb{C} : \text{Re}(s) > 0\}$

$\mathbb{C}^-$ The open left half plane in the complex s-plane; $\mathbb{C}^- = \{s \in \mathbb{C} : \text{Re}(s) < 0\}$

$\overline{\mathbb{C}}$ The closed right half plane in the complex s-plane; $\overline{\mathbb{C}} = \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$

$\mathbb{F}$ The field of complex or real numbers

$\boldsymbol{spec}(\mathbf{A})$ spectrum of $\mathbf{A}$, the set of eigenvalues of the matrix $\{\lambda_i\}$

$\boldsymbol{mspec}(\cdot)$ is read as $\{\lambda_i\}$, even if it is multiple

$\boldsymbol{tr}(\cdot)$ the trace, i.e., the sum of the diagonal elements of its argument

$\mathbf{A}^*$ for $\mathbf{A} \in \mathbb{F}^{n\times n}$ the conjugate (Hermitian) transpose of $\mathbf{A} = [a_{ij}]$ as $a^*_{ij} = (a_{ji})^*$, where $\mathbf{A}^* = [a^*_{ij}]$

Chapter 1

Derivative of Matrices

1.1 Preliminaries

Theorem 1.1.1 *Let* $\mathbf{A} = [a_{ij}]$ *be a* **square nonsingular** *(hence, invertible) matrix of order* n*. If its inverse* $\mathbf{A}^{-1}$ *is derivable with respect to one of its parameters (e.g.* x*), it is*

$$\frac{d}{dx}\left(\mathbf{A}^{-1}\right) = -\mathbf{A}^{-1}\left(\frac{d}{dx}A\right)\mathbf{A}^{-1} \tag{1.1}$$

Proof: Consider $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$,

$$\frac{d}{dx}\left(\mathbf{A}\mathbf{A}^{-1}\right) = \left(\frac{d}{dx}\mathbf{A}\right)\mathbf{A}^{-1} + \mathbf{A}\left(\frac{d}{dx}\mathbf{A}^{-1}\right) = \mathbf{0}$$

$$\mathbf{A}\left(\frac{d}{dx}\mathbf{A}^{-1}\right) = -\left(\frac{d}{dx}\mathbf{A}\right)\mathbf{A}^{-1}$$

$$\frac{d}{dx}\left(\mathbf{A}^{-1}\right) = -\mathbf{A}^{-1}\left(\frac{d}{dx}\mathbf{A}\right)\mathbf{A}^{-1}$$

$\square$

Note: As an special case, for $\mathbf{A}$ as a scalar variable it is $\frac{d}{dx}\left(\frac{1}{\mathbf{A}}\right) = -\frac{A'}{A^2}$, which can be written in a form as $\left(\frac{1}{\mathbf{A}}\right)' = -\mathbf{A}^{-1}\mathbf{A}'\mathbf{A}^{-1}$.

Chapter 2

Taylor's Expansion for Matrix Function

2.1 Preliminaries

Remark 2.1.1 *Let $F(s) = \left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1}$ be a smooth (continuously derivable, C^∞) matrix function, its Taylor expansion in proximity of $s = 0$ is*

$$F\left(s\right) \; = \sum_{i=0}^{\infty} \mathbf{M}_i s^i \; = \sum_{i=0}^{\infty} \frac{1}{i!} F^{(i)}(s)|_{s=0} s^i \,, \tag{2.1}$$

where $\frac{1}{i!}F^{(i)}(s)|_{s=0} = \mathbf{A}^{-i}$

Proof: consider the subsequent derivatives as

i . $F(s) = \left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1} \longrightarrow F(0) = F(s)|_{s=0} = \mathbf{I},$

ii . $F^{(1)}(s) = -F(s)\frac{d}{ds}\left(\mathbf{I} - s\mathbf{A}^{-1}\right)F(s) = F(s)\mathbf{A}^{-1}F(s) \longrightarrow F^{(1)}(s)|_{s=0} = F(0)\mathbf{A}^{-1}F(0) = \mathbf{A}^{-1}$

iii . $F^{(2)}(s) = F^{(1)}(s)\mathbf{A}^{-1}F(s) + F(s)\mathbf{A}^{-1}F^{(1)}(s) \longrightarrow F^{(2)}(s)|_{s=0} = F^{(1)}(0)\mathbf{A}^{-1}F(0) + F(0)\mathbf{A}^{-1}F^{(1)}(0) = 2\mathbf{A}^{-2}$

iv . $F^{(3)}(s) = F^{(2)}(s)\mathbf{A}^{-1}F(s) + F^{(1)}(s)\mathbf{A}^{-1}F^{(1)}(s) + F^{(1)}(s)\mathbf{A}^{-1}F^{(1)}(s) + F(s)\mathbf{A}^{-1}F^{(2)}(s) \longrightarrow F^{(3)}(s)|_{s=0} = 2\mathbf{A}^{-3} + \mathbf{A}^{-3} + \mathbf{A}^{-3} + 2\mathbf{A}^{-3} = 6\mathbf{A}^{-3}$

v . and so on so force!

Plunging all above evaluated derivatives in (2.1) results in (2.2) below.

$$F(s) = \left(\mathbf{I} - s\mathbf{A}^{-1}\right)^{-1} \approx \mathbf{I} + \mathbf{A}^{-1}s + \mathbf{A}^{-2}s^2 + \mathbf{A}^{-3}s^3 + \dots . \tag{2.2}$$

$\square$

Theorem 2.1.2 *Let p be a polynomial and suppose A and B are squared matrices of the same size, then*$p(\mathbf{A}+\mathbf{B}) = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)}(\mathbf{A})\mathbf{B}^k$, *where* $n = \deg(p)$.

Proof: Since p is a polynomial, we can apply the Taylor expansion:

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)}\left(x_0\right)\left(x - x_0\right)^k \ ,$$

where $n = \deg(p)$. Now let $\mathbf{X} = \mathbf{A} + \mathbf{B}$ and $\mathbf{X}_0 = \mathbf{A}$. The Taylor expansion can be checked as follows: let $p\left(x\right) = \sum_{k=0}^{n} a_k x^k$ for coefficients a_k (note that this coefficients can be taken from the space of square matrices defined over a field). We define the formal derivative of this polynomial as $p^{(1)}\left(x\right) = \frac{dp}{dx} = \sum_{k=1}^{n} a_k k x^{k-1}$ and we define $p^{(k)} = \frac{dp^{(k-1)}}{dx}$ $\frac{1}{k!}p^{(k)}\left(x_0\right) = \sum_{i=k}^{n} a_i \frac{i!}{(i-k)!k!}\left(x_0\right)^{i-k}$.

Then $p^{(k)}\left(x\right) = \sum_{i=k}^{n} a_i \frac{i!}{(i-k)!} x^{i-k}$ and we have $\frac{1}{k!}p^{(k)}\left(x_0\right) = \sum_{i=k}^{n} a_i \frac{i!}{(i-k)!k!}\left(x_0\right)^{i-k}$. Now consider

$$\sum_{k=0}^{n} \frac{1}{k!} p^{(k)}\left(x_0\right)\left(x - x_0\right)^k = \sum_{k=0}^{n}\left(\sum_{i=k}^{n} a_i \frac{i!}{(i-k)!k!}\left(x_0\right)^{i-k}\left(x - x_0\right)^k\right)$$

$$= \sum_{i=0}^{n} a_i \left(x_0\right)^i + \sum_{i=1}^{n} a_i i \left(x_0\right)^{i-1}\left(x - x_0\right) + \dots + \sum_{i=j}^{n} a_i \frac{i!}{(i-j)!j!}\left(x_0\right)^{i-j}\left(x - x_0\right)^j + \dots + a_n \left(x - x_0\right)^n$$

$$= a_0 + a_1\left(x\right) + \dots + a_i\left(\sum_{j=0}^{i} \frac{i!}{(i-j)!j!}\left(x_0\right)^{i-j}\left(x - x_0\right)^j\right) + \dots +$$

$$a_n \left(\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} (x_0)^{n-j} (x - x_0)^j \right) = \sum_{k=0}^{n} a_k x^i = p\,(x)$$

since $\sum_{j=0}^{i} \frac{i!}{(i-j)!j!} (x_0)^{i-j} (x - x_0)^j = (x)^i$. $\square$

Note: To prove theorem 2.1.2 in above [1] has been consulted with.

Chapter 3

Eigen-Values and Diagonal Factorization

Definition 3.0.3 *For Square matrix* $\mathbf{A} \in \mathbb{F}^{n \times n}$ *define the following types:*

(i) $\mathbf{A}$ *is* ***Hermitian*** *if* $\mathbf{A} = \mathbf{A}^*$.

(ii) $\mathbf{A}$ *is* ***positive-semidefinite*** $(\mathbf{A} \geq 0)$ *if* $\mathbf{A}$ *is Hermitian and* $\mathbf{x}^*\mathbf{A}\mathbf{x} \geq 0$ *for all* $\mathbf{x} \in \mathbb{F}^n$.

[2], Definition 3.1.1, pp. 81-82

Fact 3.0.4 *Consider* $\mathbf{A} \in \mathbb{F}^{n \times n}$ *(e.g. a scattering parameter matrix of a n-port linear system),* $\mathbf{A}^*\mathbf{A}$ *and* $\mathbf{A}\mathbf{A}^*$ *are Hermitian.*

Proof:

$$\left(\mathbf{A}^*\mathbf{A}\right)^* = \mathbf{A}^* \left(\mathbf{A}^*\right)^* = \mathbf{A}^*\mathbf{A}$$

For $\mathbf{A}\mathbf{A}^*$ matrix it is proved similarly. $\square$

Fact 3.0.5 *Consider* $\mathbf{A} \in \mathbb{F}^{n \times n}$*, matrix* $\mathbf{A}^*\mathbf{A}$ *is positive-definite.*

Proof: let $\mathbf{x} = \left[x_1, \cdots, x_n\right]^T \in \mathbb{F}^n$, then

$$\mathbf{x}^* \left(\mathbf{A}^*\mathbf{A}\right) \mathbf{x} \;=\; \left(\mathbf{x}^*\mathbf{A}^*\right) \left(\mathbf{A}\mathbf{x}\right) \;=\; \left(\mathbf{A}\mathbf{x}\right)^* \left(\mathbf{A}\mathbf{x}\right)$$

It is noted that $(\mathbf{Ax})$ is a column vector with complex entries $(\in \mathbb{F}^n)$. Therefore, we have

$$(\mathbf{Ax})^* (\mathbf{Ax}) =$$

$$\begin{bmatrix} A_{x_1}^* & \cdots & A_{x_n}^* \end{bmatrix} \begin{bmatrix} a_{x_1} \\ \vdots \\ a_{x_n} \end{bmatrix} = |a_{x_1}|^2 + \cdots + |a_{x_n}|^2 \geq 0\,.$$

Proposition 3.0.6 *Let $\mathbf{A} \in \mathbb{F}^{n\times n}$ and $\alpha \in \mathbb{F}$, then, the following statements hold:*

(i) $mspec\,(\alpha\mathbf{A}) \;=\; \alpha mspec\,(\mathbf{A})$.

(ii) $mspec\,(\beta I_n + \alpha\mathbf{A}) \;=\; \beta \,+\, \alpha mspec\,(\mathbf{A})$.

(iii) *if* $\mathbf{A}$ *is Hermitian,* $spec\,(\mathbf{A}) \;\subset\; \mathbb{R}$.

[2], Proposition 4.4.4, pp. 131

In a general form, let $\mathbf{F} \in \mathbb{F}^{n\times m}$, Noting (3.0.5) and (3.0.6), it is concluded

Fact 3.0.7 *Matrices $\mathbf{A}^*\mathbf{A} \;\in\; \mathbb{F}^{m\times m}$ and $\mathbf{AA}^* \in \mathbb{F}^{n\times n}$ have* ***positive-real*** *eigenvalues.*

$$spec\,(\mathbf{A}^*\mathbf{A}) \subset \mathbb{R}^+$$

Fact 3.0.8 *Let $\mathbf{A} \;\in\; \mathbb{F}^{(n\times n)}$,*

$$\|\mathbf{A}\|_F^2 \;=\; tr\,(\mathbf{AA}^*) \;=\; tr\,(\mathbf{A}^*\mathbf{A}) \;=\; \sum_{i,j} |a_{ij}|^2$$

Proof: e.g.:

$$\mathbf{A}^*\mathbf{A} = \begin{bmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} |a_{11}^2| + |a_{21}^2| & X \\ X & |a_{12}^2| + |a_{22}^2| \end{bmatrix}$$

$$tr(\mathbf{A}^*\mathbf{A}) = \left|a_{11}^2\right| + \left|a_{21}^2\right| + \left|a_{12}^2\right| + \left|a_{22}^2\right| = \|\mathbf{A}\|_F^2\ .$$

Fact 3.0.9 *For the norm of the matrix* $\mathbf{A}$ *we have*

(i) $\|A\|_2 = \sqrt{\max(eig(A^*A))}$.

(ii) $\|A\|_\infty = \max_i \sum_j |A_{ij}|$.

[3], ch. 10.3

Fact 3.0.10 *For the (ii) in fact. 3.0.9 and def. 4.0.12, it is norm of the matrix* $\mathbf{A}$ *we have*

$$\|A\|_2 = \sigma_{max}(A). \tag{3.1}$$

Chapter 4

The Singular Value decomposition (SVD)

In this section, we briefly review the singular value decomposition method.

Definition 4.0.11 *For* $\mathbf{A} \in \mathbb{F}^{(n\times m)}$*, the SVD decomposition of A is*

$$\mathbf{A} = \mathbf{U}_{(n\times n)}\Sigma_{(n\times m)}\mathbf{V}^T_{(m\times m)}$$

or equivalently,

$$\mathbf{A} = \mathbf{U}_{(n\times n)} \begin{bmatrix} \Sigma_{(r\times r)} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \mathbf{V}^T_{(m\times m)}$$

where $r = min\{n, m\}$ *for a full (column and row) rank* $\mathbf{A}$*,* $\mathbf{U}_{(n\times n)}$ *and* $\mathbf{V}_{(\mathbf{m}\times\mathbf{m})}$ *are orthogonal matrices,* $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ *and* $\mathbf{V}^T\mathbf{V} = \mathbf{I}$*.* $\Sigma = Diag\{\sigma_1, \sigma_1, \cdots, \sigma_r\}$*,* σ_i *is called singular values and are defined in c.f. 4.0.12.*

When the matrix is not (e.g.) full column rank $r \leq m$, where $r = rank(\mathbf{A})$, a short-en (economy) form of SVD can be

$$\mathbf{A} = \begin{bmatrix} \underset{\sim}{u}_1 & \cdots & \underset{\sim}{u}_i & \cdots & \underset{\sim}{u}_r \end{bmatrix}_{(n\times r)} \Sigma_{(r\times r)} \begin{bmatrix} \underset{\sim}{v}_1 & \cdots & \underset{\sim}{v}_i & \cdots & \underset{\sim}{v}_r \end{bmatrix}^T_{(r\times m)}.$$

Definition 4.0.12 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times m)}$. *Then, the* ***singular values*** *of* $\mathbf{A}$ *are the* $min\{m,n\}$ *nonnegative number* $\sigma_1(A), \cdots, \sigma_{min\{m,n\}}(A)$, *where, for all* $i = 1, \cdots, min\{m,n\}$,

$$\sigma_i(\mathbf{A}) \triangleq [\lambda(\mathbf{A}\mathbf{A}^*)]^{1/2} = [\lambda(\mathbf{A}^*\mathbf{A})]^{1/2}.$$

Then,

$$\sigma_1(\mathbf{A}) \geq \cdots \geq \sigma_{min\{n,m\}}(\mathbf{A}) \geq 0.$$

[2], Definition 5.6.1, pp. 181-182 SVD can lead to the best approximation in terms of the 2-norm.

Fact 4.0.13 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times m)}$, *and let* $r = rank\mathbf{A}$. *Then, for all* $i = 1, \cdots, r$,

$$\sigma_i(\mathbf{A}^*\mathbf{A}) = \sigma_i(\mathbf{A}\mathbf{A}^*) = \sigma_i^2(\mathbf{A}).$$

In particular,

$$\sigma_1(\mathbf{A}^*\mathbf{A}) = \sigma_{max}^2(\mathbf{A}).$$

[2], Fact 5.10.18, pp. 198

Fact 4.0.14 *Let* $\mathbf{A} \in \mathbb{F}^{(n\times n)}$,

$$\|\mathbf{A}\|_F^2 = tr(\mathbf{A}\mathbf{A}^*) = tr(\mathbf{A}^*\mathbf{A}) = \sum_{i,j} |a_{ij}|^2$$

Fact 4.0.15 *For the (ii) in fact. 3.0.9 and def. 4.0.12, it is norm of the matrix* $\mathbf{A}$ *we have*

$$\|A\|_2 = \sigma_{max}(A). \tag{4.1}$$

Chapter 5

Bounded Real Lemma (BRL)

5.1 Preliminaries

As a (passivity) checking criterion for bounded-real-ness of s-parameter matrix $\mathbf{S} \in \mathbb{F}^{n\times n}$, which is the scattering-parameters for a n-port linear system, we have:

$$I_n - \mathbf{S}^*\mathbf{S} \geq 0\,. \tag{5.1}$$

From the mathematical elaborations in previous chapter, it is concluded that, eq. (5.1) requires

$$mspec\,(I - \mathbf{S}^*\mathbf{S}) = 1 - mspec\,(\mathbf{S}^*\mathbf{S}) \geq 0\,, \tag{5.2}$$

which is equivalently shortened as

$$mspec\,(\mathbf{S}^*\mathbf{S}) \leq 1\,. \tag{5.3}$$

The above notation in (5.3) shows that all eigenvalues of Hermitian matrix $\mathbf{S}^*\mathbf{S}$ requires to be bounded to one, while they are known from previous section as real and positive values. This is logically means that

$$\lambda_{max}\left(\mathbf{S}^{*}\mathbf{S}\right) \leq 1\,. \tag{5.4}$$

According to the Definition. 4.0.12 we have $[\lambda\left(\mathbf{A}^{*}\mathbf{A}\right)] = \sigma_i^2(\mathbf{A})$, by subsituting which in (5.3) it is

$$\sigma_{max}^{2}\left(\mathbf{S}\right) \leq 1\,, \tag{5.5}$$

or equivalently

$$\sigma_{max}\left(\mathbf{S}\right) \leq 1\,. \tag{5.6}$$

Fact 5.1.1 *Considering fact 4.0.15 and (5.6) the following inequalities are equivalent.*

(i) $\sigma_{max}\left(\mathbf{S}\right) \leq 1$.

(ii) $\left\|\mathbf{S}\right\|_2 \leq 1$.

Chapter 6

Diagonally Dominant Matrices

Definition 6.0.2 *A diagonally dominant matrix is a complex matrix* $\Phi = [\varphi_{ij}] \in \mathbb{Z}^{n\times n}$ *with the property that we have*

$$|\varphi_{ii}| \geqslant \sum_{i=1,\ i\neq j}^{n} |\varphi_{ij}| \,, \tag{6.1}$$

for all i. *When all these inequalities are strict, the matrix is called strictly diagonally dominant [4]. where* $\| \ \|$ *defines the matrix norm subordinate to the ordinary Euclidean norm for vectors.*

Lemma 6.0.3 *A symmetric diagonally dominant real matrix with nonnegative diagonal entries is positive semidefinite.*

Proof: Given any eigenvalue of matrix λ and corresponding eigenvector $\mathbf{U}$,

$$\Phi\mathbf{U} = \lambda\mathbf{U} \tag{6.2}$$

More precisely, it is

$$\begin{bmatrix} \\ \\ \varphi_{i1} & \cdots & \varphi_{ii} & \cdots & \varphi_{iK} \\ \\ \\ \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_N \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_N \end{bmatrix} \tag{6.3}$$

and

$$(\varphi_{ii} - \lambda)\, u_{\mathbf{i}} = -\sum_{i \neq j} \varphi_{ij} u_j \ . \tag{6.4}$$

From (6.4),

$$\begin{aligned} |(\varphi_{ii} - \lambda)\, u_{\mathbf{i}}| = \left| \sum_{i \neq j} \varphi_{ij} u_j \right| \\ \left| \sum_{i \neq j} \varphi_{ij} u_j \right| \leqslant \sum_{i \neq j} |\varphi_{ij} u_j| \\ \sum_{i \neq j} |\varphi_{ij} u_j| = \sum_{i \neq j} |\varphi_{ij}|\, |u_j|, \end{aligned} \tag{6.5}$$

then

$$|(\varphi_{ii} - \lambda)\, u_{\mathbf{i}}| \leqslant |u_j| \sum_{i \neq j} |\varphi_{ij}| \ . \tag{6.6}$$

let $|u_{\mathbf{i}}| > |u_j|$ for all $j \neq i$, it is

$$|u_j| \sum_{i \neq j} |\varphi_{ij}| < |u_i| \sum_{i \neq j} |\varphi_{ij}| \tag{6.7}$$

(6.6) and (6.7) lead to

$$\left|(\varphi_{ii} - \lambda)\, u_{\mathbf{i}}\right| \leqslant |u_i| \sum_{i \neq j} |\varphi_{ij}| \ . \tag{6.8}$$

It is equivalently

$$|\varphi_{ii} - \lambda|\, |u_{\mathbf{i}}| \leqslant |u_i| \sum_{i \neq j} |\varphi_{ij}| \ , \tag{6.9}$$

considering $|u_i| > 0$,

$$|\varphi_{ii} - \lambda| \leqslant \sum_{i \neq j} |\varphi_{ij}| \ . \tag{6.10}$$

Let assume there exist a negative eigenvalue $\lambda < 0$, then intuitively it is

$$|\varphi_{ii}| < |\varphi_{ii} - \lambda| \tag{6.11}$$

From (6.10) and (6.11), it is

$$|\varphi_{ii}| < \sum_{i \neq j} |\varphi_{ij}| \ . \tag{6.12}$$

Eq (6.12) is in contradiction with the presumption of diagonally dominance in (6.1), than the assumption of $\lambda < 0$ is not true and eigenvalues are confined to the non-negative values. $\square$

Lemma 6.0.4 *A strictly diagonally dominant complex matrix is nonsingular.*

Lemma 6.0.5 *The multiplicity of the eigenvalue 0 of a symmetric real matrix Φ with zero row sums and nonpositive off-diagonal entries equals the number of connected components of the graph Γ defined on the index set of the rows and columns of Φ, where two distinct indices i and j are adjacent when $\varphi_{ij} \neq 0$.*

List of References

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