Eigenvalues and Eigenvectors (review):

Definition 1. An **eigenvector** of an $n \times n$ matrix $\mathbf{A}$ is a nonzero vector $\mathbf{v}$ for some scalar λ. A scalar λ is called an **eigenvalue** of $\mathbf{A}$ if there is a nonzero (nontrivial) solution $\mathbf{v}$ of $\mathbf{A}\mathbf{v} = \lambda\,\mathbf{v}$; such a $\mathbf{v}$ is called an *eigenvector corresponding to* λ .

Remark 1. From Definition-1, λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$ if and only if the equation

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0} \tag{1}$$

has a nontrivial solution ($\mathbf{v} \neq \mathbf{0}$). Since $\mathbf{v}$ is a nonzero vector, the matrix $(\mathbf{A} - \lambda\mathbf{I})$ must be singular *(non-invertible)*. A matrix is singular if and only if its determinant is zero, thus, the eigenvalues of $\mathbf{A}$ are the scalars (λ) for which

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}\,. \tag{2}$$

The determinant in (2) is a polynominal of degree n called *characteristic polynomial* of $\mathbf{A}$. The eigenvalues of $\mathbf{A}$ are the roots of the *characteristic equation*. Since a polynomial of degree n has exactly n roots, a matrix of order n has n eigenvalues .

What does it mean for a matrix $\mathbf{A}$ to have an eigenvalue of 0?
This happens if and only if the equation

$$\mathbf{A}\,\mathbf{v} = 0\,\mathbf{v} \tag{3}$$

has a nontrivial solution (according to the Definition 1). But (3) is equivalent to $\mathbf{A}\,\mathbf{v} = \mathbf{0}$, which has a nontrivial solution if only if $\mathbf{A}$ is not *invertible*.

Remark 2. Let the matrix $\mathbf{A}$ of order n have n distinct eigenvalues as

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \quad \text{and} \quad \mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\ ,$$

and hence $\mathbf{A}$ has a complete system of eigenpairs $(\lambda_i, \mathbf{v}_i)\,, i = 1, \ldots, n$. The the individual relations $\mathbf{A}\mathbf{v}_i = \lambda_i\,\mathbf{v}_i$ can be combined in the matrix equation

$$\mathbf{A}\,\mathbf{V} = \mathbf{V}\,\mathbf{\Lambda}\,. \tag{4}$$

Because the eigenvectors $\mathbf{v}_i$ are linearly independent, the matrix $\mathbf{V}$ is nonsingular. Hence, $\mathbf{A}$ is diagonalizable and we may write

$$\mathbf{V}^{-1}\mathbf{A}\,\mathbf{V} = \mathbf{\Lambda}\,. \tag{5}$$

Proposition 1. *Let a $n \times n$ matrix $\mathbf{A}$ be diagonalizeable, it is*

$$\begin{aligned}\mathbf{A}^k &= \left(\mathbf{V}\Lambda\mathbf{V}^{-1}\right)^k \\ &= \left(\mathbf{V}\Lambda\mathbf{V}^{-1}\right)\cdots\left(\mathbf{V}\Lambda\mathbf{V}^{-1}\right) \\ &= \mathbf{V}\Lambda^k\mathbf{V}^{-1} \\ &= \mathbf{V}\,\mathbf{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right)\,\mathbf{V}^{-1}\,.\end{aligned} \tag{6}$$

For the case of $k < 0$, (6) *is hold only if $\mathbf{A}$ is invertible,* i.e. *all $\lambda_i \neq \mathbf{0}$.*

Definition 2. A matrix $\mathbf{B}_{n\times n}$ is said to be similar to a matrix $\mathbf{A}_{n\times n}$ if there exists a nonsingular square matrix $\mathbf{T}_{n\times n}$ such that $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.
The transformation $\mathbf{A} \rightarrow \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is called a *similarity transformation* by the *similarity matrix* $\mathbf{T}$. The relation "B is similar to A" is sometimes abbreviated $\mathbf{B} \sim \mathbf{A}$.

Note 1. Defining $\mathbf{Q} = \mathbf{T}^{-1}$ similarity transformation can be equivalently rewritten as $\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{-1}$, where (now) Q is the *similarity matrix*.

Note 2. Recall the following properties of determinant of square matrices,

- $\det \mathbf{A}\, \det \mathbf{B} = \det (\mathbf{A}\,\mathbf{B})$
- $\det (\mathbf{P}^{-1}) \det (\mathbf{P}) = \det (\mathbf{P}^{-1}\mathbf{P}) = \det (\mathbf{I}) = 1\,.$