1 Time domain signals

The difference between outputs should be measured at the same n time instances and for the same input signal $u(t)$. Let $x(t)$ and $y(t)$ be the two time signals sampled at the same N discrete time instances in the interval T, $t_i \in \mathbf{T}$, given as two (real-valued) time series, i.e.,

$$\mathbf{x}(\cdot) = \{x(t_n) \in \mathbb{R} \mid \forall \ 1 \leq n \leq N\}, \text{ and} \tag{1}$$

$$\mathbf{y}(\cdot) = \{y(t_n) \in \mathbb{R} \mid \forall \ 1 \leq n \leq N\} \tag{2}$$

1.1 Absolute Error

Absolute Error (AE) is defined as the absolute-vale of the instantaneous difference between two time signals measured at the same time instances (and logically for the same excitation).

$$\zeta(t_n) = |x(t_n) - y(t_n)|, \qquad 1 \leq n \leq N \tag{3}$$

The absolute-error for two signals is a time series as

$$\zeta(\cdot) = \left\{e(t_n) \in \mathbb{R}^+ \mid \forall \ 1 \leq n \leq N\right\} \tag{4}$$

Absolute Error may be seen as a more natural way of quantifying the instantaneous error.

Implementation considerations:

If $x(t)$ and $y(t)$ are not given at the same time points in the same time span, i.e.,

$$\mathbf{x}(.) = \{x(t_{n_i}) \mid \forall \ 1 \leq n_i \leq N_i, \ \ t_{n_i} \in \mathbf{T}_i\} \tag{5}$$

$$\mathbf{y}(.) = \left\{y(t_{n_j}) \mid \forall \ 1 \leq n_j \leq N_j, \ \ t_{n_j} \in \mathbf{T}_j\right\} \tag{6}$$

Given that measurement of the signals are not available at the same time points ($t_{n_i} \neq t_{n_j}, \ \text{ for } \ n_i = n_j$), number of time points are different ($N_i \neq N_j$), and (/or) the evaluations are not in the same time span ($\mathbf{T}_i \neq \mathbf{T}_j$). The absolute error can be evaluated through the following steps

1. Take the overlapping part of the time spans, $\mathbf{T} = \mathbf{T}_i \bigcap \mathbf{T}_j$
2. Select the signal with the larger number of samples in $\mathbf{T}$.
   Let it be $\mathbf{x}(.)$ having N samples at $t_n \in \mathbf{T} \; \forall \; \leq n \leq N$
3. Interpolate the other signal($y(t)$) at $t_n \in \mathbf{T} \; \forall \; \leq n \leq N$ to obtain $\mathbf{y}(.)$ with entries coincide with the ones in $\mathbf{x}(.)$
4. Use (4) to evaluate the absolute error.

It is to be noted that

- The evaluated error is computed only in the overlaping time span as $T \subsetneqq \ \mathbf{T}_i, \mathbf{T}_j$.
- Approximating signal(s) using interpolation will lead to an estimation of absolute errors
- On the other hand the erroneous results from extrapolation can provide a misleading measure for accuracy.

Table 1 presents a summary of the commonly used measures to quantify the errors.

Table 1: Measuring reduction accuracy in time domain

Error	**Definition**
Absolute error:	$\boldsymbol{\zeta}(t_n) = \lvert x(t_n) - y(t_n) \rvert, \qquad \boldsymbol{\zeta}(\cdot) = \{e(t_n) \vert\ \forall t_n \in \mathbf{T}\}$
Average (mean) absolute error:	$\dfrac{\lVert \boldsymbol{\zeta}(\cdot) \rVert_1}{N} = \dfrac{1}{N} \sum_{n=1}^{N} \boldsymbol{\zeta}(t_n)$, where $\lVert \cdot \rVert_1$ denotes $\mathcal{L}_1$ norm
Relative error:	$\dfrac{\boldsymbol{\zeta}(\cdot)}{\lvert x(t_n) \rvert}$
mean squared error:	$\dfrac{\lVert \boldsymbol{\zeta} \rVert_e^2}{n}$, where $\lVert \cdot \rVert_e$ is Euclidean norm
normalized mean squared error:	$\dfrac{\lVert \boldsymbol{\zeta} \rVert_e^2}{\mathrm{var}(y)}$, where "**var**" denotes the *variance** of data set
root mean squared error:	$\dfrac{\lVert \boldsymbol{\zeta} \rVert_e}{\sqrt{n}}$
normalized root mean squared error:	$\dfrac{\lVert \boldsymbol{\zeta} \rVert_e}{\sqrt{\mathrm{var}(y)}}$
mean absolute error:	$\dfrac{\lVert \boldsymbol{\zeta} \rVert_1}{n}$
mean absolute relative error:	$\dfrac{\left\lVert \dfrac{\boldsymbol{\zeta}}{y} \right\rVert_1}{n}$

*- For a data set $y = \{y_i\}$ including N data points, variance is computed as $\mathrm{var}(y) \triangleq \dfrac{1}{N} \sum_{i=1}^{N} (y_i - \overline{y})^2$, where $\overline{y}$ is the data mean $\overline{y} \triangleq \dfrac{1}{N} \sum_{i=1}^{N} y_i$.

2 Frequency domain signals

The difference between outputs should be measured at the same n time instances and for the same input signal $u(t)$. ONLY for the case of linear systems such deviation can also be judged comparing the frequency response of the original system and the one from the reduced transfer function at the n frequency points throughout the frequency spectrum of interest. The results for "single-input and single-output" (SISO) systems will be a vector and for "multi-input and multi-output" (MIMO) cases it is a matrix containing the instantaneous errors at different "ports".

The error space (the space, where error resides) is considered as metric space (definition **??**) endowed with different norms that can be properly used to characterize the error (Sec. **??**). Table 1 presents a summary of the commonly used measures to quantify the error in the context of (linear / nonlinear) MOR.

Notes: Matlab-like pseudo definition

```
%Absolute Error
e= abs(abs(Y1)-abs(Y2));

%Relative Error
nzidx = find(abs(Y1)>1e-10);
re = e(nzidx)./abs(Y1(nzidx));

%Root mean squarred (RMS) error
RMSe = sqrt(mean(e.^2));  %Root Mean Square Error


% Relative root mean squarred (RMS) error
%or: RMSe12 = norm(e)/sqrt(numel(e));
RMSre = sqrt(mean(re.^2));
```