MATH 3310 – Midterm Experience

Spring 2023

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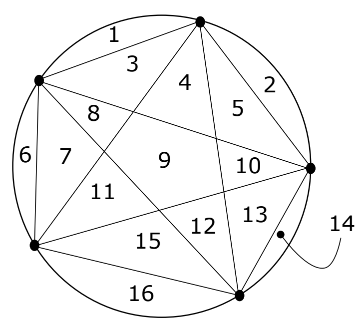
Constraints: You may discuss the problems below with the following set of people:

$$P = \{\text{Anyone currently enrolled in MATH 3310, Sam Powell, Erin Pitts, Xavier Parent, David Brown}\}.$$

The same rules as in the recitations apply: no one from P should give or confirm any solution to any problem on this Experience. Unless otherwise specified in the prompt with something like "no outside resources, please", electronic or literary resources may be used (not plagiarized), but must be cited somehow. The solutions to this Experience should be created by groups of at most four and are to be submitted March 17, 2023, by 11:59pm. One submission per group, with all group members clearly indicated.

✠ Sub-Experience One: *Pizza Cutting with Lasers*

No outside resources, please Given the difficulty of organizing and sequentially lining up cuts to get the maximum number of pieces, you invent a laser cutting system that allows you to make all cuts at once. You place n of your devices on the crust with spacing that will maximize the number of pieces made and, once activated (ideally remotely because they are dangerous), *laser sabers*® shoot between every pair of devices simultaneously cutting the pizza into $R(n)$ pieces Below is an illustration representing 5 of your devices cutting the pizza into 16 pieces.

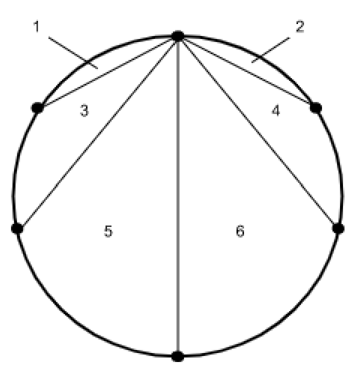


Please determine a formula for $R(n)$.

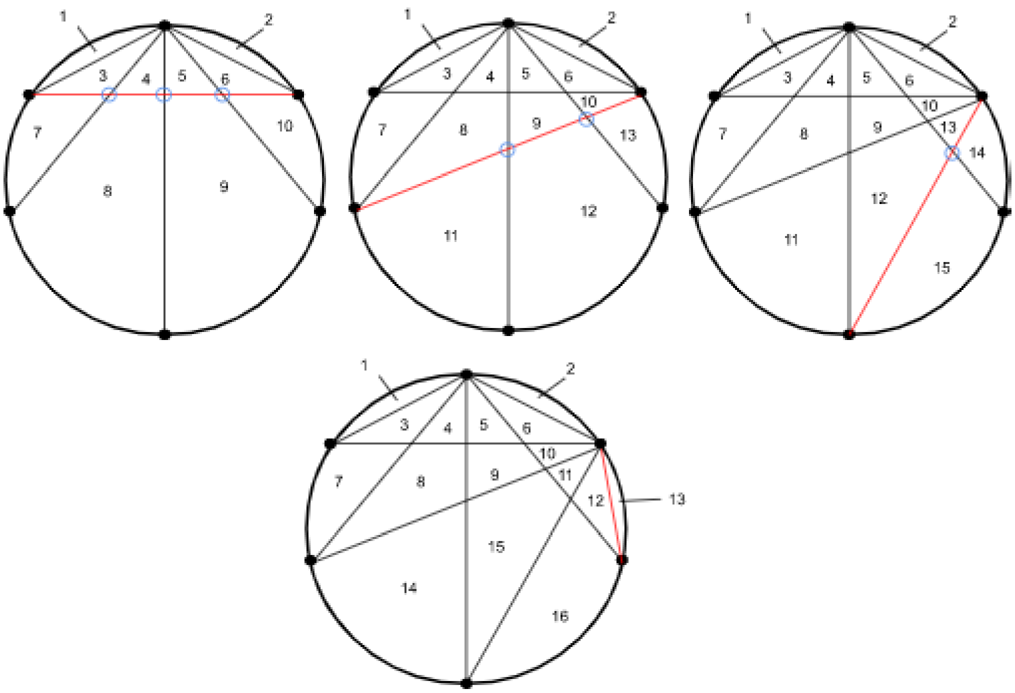
Claim: R(n) can be defined as:

$$n + \sum_{i=2}^{n-1}\left[\sum_{j=1}^{n-i} j(i-1) - (i-2)\right]$$

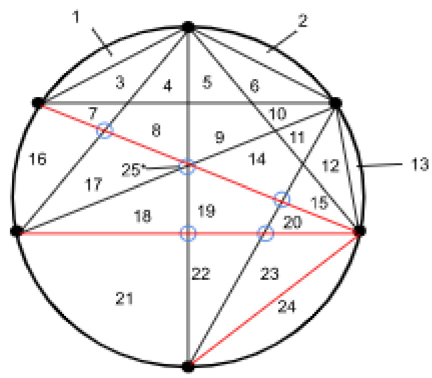
Proof: Take for example a pizza with n=6 slicers. By connecting only one point to the others you get n pieces.



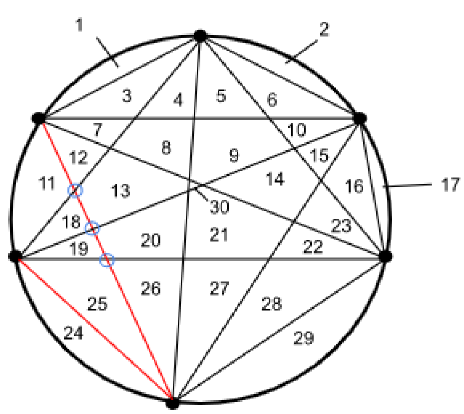
Moving on to the next point, you create 10 more spaces. Note that this is #intersections + 1. Also note that the intersections decrement (3,2,1,0). 3 intersections adds 4, 2 intersections adds 3, one intersection adds 2, and zero intersections adds 1 space.



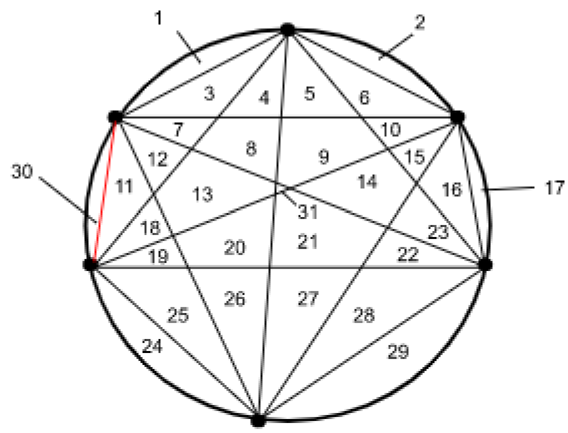
The next point creates 9 more spaces, still intersections + 1



The next point creates 4 more spaces intersections + 1



The final point only adds one space.



Assuming there are no three lines that intersect at exactly the same point see the *, this pattern follows:

$$n + (1 + 2 + 3 + 4 \ldots + n - 2) + (1 + 3 + 5 + 7 \ldots + n - 1) + (1 + 4 + 7 + 10 \ldots + n - 1).$$

Note that the $n - 2$ in the first sequence is due to the first image having two lines that can't be crossed.

To visualize:

$$\begin{aligned}
&6 + (4 + 3 + 2 + 1) + (5 + 3 + 1) + (4 + 1) + 1 = 31 \\
&6 + \sum_{j=1}^{6-2} j + \sum_{j=1}^{6-3} j(2) - 1 + \sum_{j=1}^{6-4} j(3) - 2 + \sum_{j=1}^{6-5} j(4) - 3 \\
&n + \sum_{j=1}^{n-2} j + \sum_{j=1}^{n-3} j(2) - 1 + \sum_{j=1}^{n-4} j(3) - 2 + \cdots + \sum_{j=1}^{n-(n-1)} j(n-1) - (n-2) \\
&n + \sum_{i=2}^{n-1} \left[\sum_{j=1}^{n-i} j(i-1) - (i-2) \right]
\end{aligned}$$

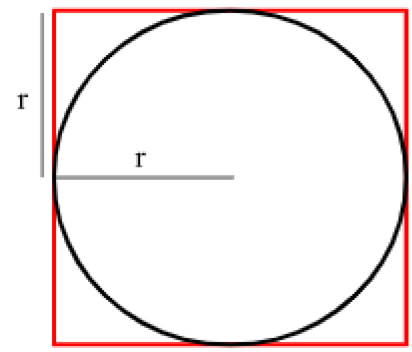
✠ Sub-Experience Two: *Strange Walks*

After the pizza, you decide to take a walk.
Number of shortest paths using the *city block metric*. Starting at Center and Main, you decide to walk to X North and Y East. The distance in the "*city block metric*" is measured in blocks and so the distance from X_1 North and Y_1 East to X_2 North and Y_2 East is $|X_2 - X_1| + |Y_2 - Y_1|$.

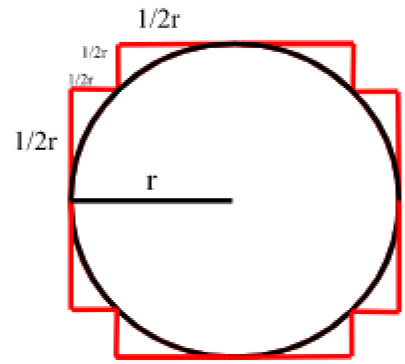
SE 2.1. For yuks, determine π in the city block metric. Remember that π is defined to be the ratio of the circumference to the diameter of a circle. Also remember that a circle is the set of points equidistant from a given point (that distance is typically called the *radius* of the circle and the point is called its *center*), but distance in this prompt is measured differently than is typically done.

Claim: π in the city block metric is 4.

Proof: π in the city block metric is defined as the ratio of the circumference to the diameter of a circle in this metric ($\pi = \frac{C}{d}$). With the city block metric, a 'circle' is really going to be a square:



Even a more resolute circle using the city block metric would look like:



Any number of turns made to reproduce the curve of the circle would result in a 90° curve to have a length of 2r. Therefore, the circumference of a circle with radius r in the city block metric is given by 8r, and the diameter is given by 2r. Thus, the ratio of the circumference to the diameter is:

$$\begin{aligned}\pi =& \text{circumference/diameter}\\ =& 8r/2r\\ \pi =& 4\end{aligned}$$

SE 2.2. Count the number of shortest paths from Center and Main to X North and Y East. A more erudite way to pose this problem is as follows: count paths from $(0, 0)$ to (X, Y) using steps of the form $R : (x, y) \mapsto (x + 1, y)$ or $U : (x, y) \mapsto (x, y + 1)$. So paths trace rectilinear "curves" in the first quadrant of the Cartesian coordinate system that visit only integer- valued coordinates.

Claim: The number of shortest paths from Center and Main to X North and Y East is $\binom{X+Y}{X}$

Proof: To count the number of shortest paths from (0,0) to (X,Y) using only steps R (moving one block right) or U (moving one block up), The total distance to (X,Y) will be X+Y steps. We need to count how many ways the R and U steps can be ordered such that the number of steps is X+Y. This relationship can be described as:

$$\binom{X+Y}{X}$$
$$\text{OR}$$
$$\binom{X+Y}{Y}$$

Therefore, the number of shortest paths from (0,0) to any (X,Y) using only steps R and U is defined as either of those binomials. It can be either because combinatorially, choosing the order of X steps in X+Y steps is the same as choosing the order of Y steps that are left over.

SE 2.3. Walk in a strange way. Start at Center and Main, thinking of it as the origin $(0, 0)$ and take n steps, each of type R, L, or U, with R never followed by L and vice-versa. The steps are defined via $R : (x, y) \mapsto (x + 1, y)$, $L : (x, y) \mapsto (x - 1, y)$, and $U : (x, y + 1)$. Determine the number of different paths with n steps.

Claim: The number of different paths with n steps can be defined as$F(n) = 2F(n - 1) + F(n - 2)$

Proof: Simply by drawing out the number of paths on a grid, we find:

n	0	1	2	3	4	5	6	$n?$
$\#ofpaths$	1	3	7	17	41	99	239	...

Another way to visualize this:

$$\begin{aligned}
&0 &&= 1\\
&3 &&= 3\\
&3+4 &&= 7\\
&3+4+10 &&= 17\\
&3+4+10+24 &&= 41\\
&3+4+10+24+58 &&= 99\\
&3+4+10+24+58+140 &&= 239
\end{aligned}$$

We can also see that the next number in the pattern appears to be $2 \cdot F(n-2)+F(n-1)$ where $F(0) = 1$ and $F(1) = 3$. For example:

$$\begin{aligned}
&0 &&= 1\\
&3 &&= 3\\
&3+(3+1) &&= 7\\
&3+4+(7+3) &&= 17\\
&3+4+10+(17+7) &&= 41\\
&3+4+10+24+(41+17) &&= 99\\
&3+4+10+24+58+(99+41) &&= 239
\end{aligned}$$

For any n, F(n) can be defined by that equation, assuming you have F(n-1) and F(n-2)

✠ Sub-Experience Three: The Master Table of Finite Functions.

Below is the table of distribution functions $f : A \to B$, where A is a finite set with n elements and B is a finite set with x elements. The goal: determine a formula for the number of functions with properties described by the table for each of the twelve possibilities. So, for example, the entry in cell 5 of the table is to be the number of injective functions that map a set of n indistinguishable objects into a set of x distinguishable objects.

A	B	unrestricted	injective	onto
distinguishable	distinguishable	1.	2.	3.
indistinguishable	distinguishable	4.	5.	6.
distinguishable	indistinguishable	7.	8.	9.
indistinguishable	indistinguishable	10.	11.	12.

Please determine, with an argument for each, a formula for each entry in the table.

Claim: The following table shows an accurate argument for each cell.

Proof:

A	B	unrestricted	injective	onto
distinguishable	distinguishable	x^n	$x^{\underline{n}}$	$x^n - \sum_{i=1}^{x} [-1^{i-1} \cdot \binom{x}{i} \cdot (x-i)^n]$
indistinguishable	distinguishable	$\left(\binom{x}{n}\right)$	$\binom{x}{n}$	$\left(\binom{x}{n-x}\right)$
distinguishable	indistinguishable	$\sum_{i=1}^{x} \left\{ {n \atop i} \right\}$	$\binom{x-n}{x-n}$	$\left\{ {n \atop x} \right\}$
indistinguishable	indistinguishable	$\sum_{i=1}^{x} p_i n$	$\binom{x-n}{x-n}$	$p_x n$

1. We have n distinguishable balls and we need to find how many ways we can fill x distinguishable boxes putting any amount of balls in each box. Every ball has x possible boxes to be put in. This can be shown algebraically as follows: $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ or more succinctly represented by x^n.

2. We have n distinguishable balls and we need to find how many ways we can fill x distinguishable boxes putting only one ball in each box. The first ball denoted by b_1 has x possible boxes to be placed, the subsequent balls b_k have $x-k$ possible boxes to be placed. There are n balls so the final ball would have $x-n$ boxes to choose from. Algebraically shown as follows: $x \cdot x - 1 \cdot ... \cdot x - n$ or more succinctly represented by x^n.

3. We'll start with all the possible combinations if there were no restrictions, x^n. Since this includes possibilities where a box is empty, we have to take out all the possibilities where at least one box is empty. We start with one box being empty, which we will choose with $\binom{x}{1}$. We then find all the possible ways to get a combination with only x-1 boxes and take those out. We'll continue to do this until we have no more boxes to assume are empty. However, since we'll have repeated combinations that we are taking out, we have to add those back in, thus the -1^{i-1}. We'll end up with all the combinations total minus all the combinations that include at least one empty box.

4. We have n number of indistinguishable balls and we find how many ways we can fill x labeled boxes with the balls. Any box can have between 0 and n balls.

5. We have x number of distinguishable boxes and n number of balls. Each box can only have 1 ball in it, so we choose a number of boxes equal to the number of balls we have, choosing n boxes.

6. Since each box must have at least one ball and the balls are indistinguishable, we start by placing one ball in each box. With the remaining balls, we find how many ways we can fill x boxes with the balls, and each of the boxes can have between 0 and n of those remaining balls since each box already has one.

7. Since the boxes are indistinguishable, we'll split the balls into groups. We'll start with one group and all the rest of the groups will be 0. Then, we'll have 2 groups and the rest will be 0. Then 3, then 4, etc. until we have n groups and any remaining boxes will have 0. After that, if x is larger than n

8. Since each box can only have one ball and we don't distinguish between different boxes, there is only one way to place the balls in the boxes and that is to have n number of boxes each with one ball. However, if there are more balls than boxes, it is impossible and there are 0 ways, therefore we have x-n.

9. Each box must have at least one ball but can have as many balls as needed. However, since the specific bx doesnt matter, we can just sort the balls into a number of groups equal to that of the boxes.

10. Starting with just 1 box being filled, we'll find how many ways to split n balls into that noe box (1). Then, we'll do the same with 2 boxes, then 3, up until we find it for x boxes. If we end up with more boxes than balls, the partition will equal 0 for those parts of the sum as there are no ways to split the balls into more boxes than we have.

11. Since each box can only have one ball and we don't distinguish between different boxes, there is only one way to place the balls in the boxes and that is to have n number of boxes each with one ball. However, if there are more balls than boxes, it is impossible, therefore we have x-n.

12. We need to split n into x groups that are all non zero

✠ **Sub-Experience Four:** ***Game Time!***

SE 4.1: Modified Towers of Hanoi. The *Towers of Hanoi* are mythical diamond needles, three of them, with 64 gold discs impaled on one of the needles. God told the monks of Brahma to transfer all the discs to another needle with the constraints that no larger disc ever be placed atop a smaller disc, and only one disc at a time is to be moved. Once the monks finish moving the discs the World will end. Evidently the monks have not yet finished. Label the needles of the Towers of Hanoi L, M, and R, for the left, middle, and right, respectively. Consider the original Towers of Hanoi game, but with the additional constraint that a disk can only be moved to an adjacent peg; that is, a disk can only be moved to M from L or R, and can only be moved to L or R from M. Assume all discs are initially on L.

Determine the minimum number of moves (where a move *is defined to be the transfer of one disc from one needle to another) required to transfer* n *discs from* L *to* R.

How Long? Assume the monks of Brahma were given this game with 64 discs, and can move a disc at a rate of one per second. How long in centuries will it take for the monks to complete their task?

Claim: Minimum number of moves is $3n - 1$ where n is the number of disks on the first tower When there is 64 disks it will take 3433683820292512484657849089280 moves to complete the game. If the monks can move the disks at a rate of one second per tower then it would take $1.0881 * 10^{21}$ centuries to complete the game.

Proof: The number of moves given size n can be represented by the recursive statement R(n) = 3R(n-1) + 2. To understand this recursive statement, any given game can be explained combinatorially as follows: To win the game the top n-1 discs need to be moved from L to the R needle which would take R(n-1) moves. Then the largest remaining disc on L would be moved to M consuming 1 move. So now we are at R(n-1) + 1 moves. The n-1 discs on R now need to be transferred back to L consuming another R(n-1) moves. We are now at 2R(n-1) + 1 moves. The largest disk on M is to be moved to R adding 1 move. We are now at 2R(n-1) + 2 moves. Finally, we can move the stack of n-1 discs from L to R consuming another R(n-1) moves. The final moves required is 3R(n-1) + 2.

We want to convert the recursive statement 3R(n-1) + 2 to be an explicit statement. Because we are multiplying the previous iteration by 3 this will be a geometric equation. The formula for a geometric sequence is $(1-r^n)/(1-r)$ where n is the first term (number of moves when n = 1 being in our case 2) and r is the common ratio (in our case it is 3).

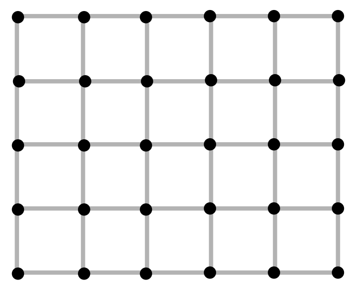
$$\begin{aligned}
2(1-3^n)/(1-3) \\
2(1-3^n)/-2 \\
-(1-3^n) \\
3^{n-1}
\end{aligned}$$

Therefore, Minimum number of moves is 3^{n-1} where n is the number of disks on the first tower.

SE 4.2: A Two-Player Game with no Dire Consequences.

The Game

Two players, A and B, alternately select an *edge* (line segment connecting two dots) on the grid graph shown below and color it red. The loser of the game is the player who is forced to select an edge that creates a red C_4 — a red cycle on 4 vertices.



The Fun

Confirm or deny, with proof, whether player A (the first player) can always win if she employs a particular strategy for each move.

Claim: Player A can always win if starting in the center line then reflecting what player B does across an axis

Proof: By selecting the center edge first, Player A has consumed the only edge that does not have a corresponding reflection across both axes. Then after Player B moves, Player A can reflect that move across both axes. By doing so, it protects Player A from ever closing a box because if Player A has to close a box, it would mean that Player B has already closed a box on the reflection of the axes thus Player B loses the game.

✠ **Sub-Experience Five – A Matter of Life and Death**

No outside resources You are among a group of $n-1$ of your friends (pretend you have $n-1$ friends even if you don't, where n is possibly very large) and are captured by a horde of theater students who will force you and your $n-1$ friends to enact the play *Cats* with them over and over and over and ... . Your friends choose death over this fate and decide to form a circle and have every other person commit suicide (someone has a pistol and n bullets, which is quite reasonable to assume here at USU) until only one person survives, who will supposedly kill themselves. You want no part of this suicide madness since, you figure, the theater people will tire eventually and you can make your escape by taking Jennyanydots hostage with the pistol and remaining bullet and escape (or something like that).

Number you and your friends 1 to n and assume you all form a vicious circle to facilitate the suicide.

Question: *In which position (call it $S(n)$) must you be in order to survive?*

Examples: $S(4) = 1$, $S(7) = 7$, and $S(24) = 17$, as the reader should verify.

Claim: the Equation $S(n) = 2(n - 2^{\lfloor \log_2 n \rfloor}) + 1$ will give the position of the person that will survive where n is the number of people in a circle.

Proof: To prove this statement I will proceed by way of induction. Assume that $n = k$.

Base Case:

When k=1,

$$\begin{aligned} S(k) = S(1) &= 2(1 - 2^{\lfloor \log_2 1 \rfloor}) + 1 \\ &= 2(1 - 2^0) + 1 \\ &= 1 \end{aligned}$$

from here we have two other cases. When n is even we will use 2k to show that, and when n is odd we will use 2k+1.

Lemma: $\log_2 2 = 1$

Even Case:

$$\begin{aligned} S(2k) =& 2S(k) - 1 \\ =& 2(2(k - 2^{\lfloor \log_2 (k) \rfloor}) + 1) - 1 \\ =& 2(2k - 2^1 * 2^{\lfloor \log_2 (k) \rfloor} + 1) - 1 \end{aligned}$$

Now we will use the lemma and the rule of exponent multiplication to exchange the 2^1 into the other 2

$$\begin{aligned} &= 2(2k - 2^{\lfloor \log_2 (k) \rfloor + 1} + 1) - 1 \\ &= 2(2k - 2^{\lfloor \log_2 (2k) \rfloor} + 1) - 1 \\ &= 4k - 4^{\lfloor \log_2 (2k) \rfloor} + 2 - 1 \\ &= 4k - 4^{\lfloor \log_2 (2k) \rfloor} + 1 \end{aligned}$$

Finally we get:

$$S(2k) = 2(2k^{\lfloor \log_2 (2k) \rfloor}) + 1$$

From here we can see that the even equation is proved using induction because we successfully turned the equation into a form equivalent to S(2k)

Odds Case:

$$\begin{aligned} S(2k+1) &= 2S(k) + 1 \\ &= 2(2(k - 2^{\lfloor \log_2 (k) \rfloor}) + 1) + 1 \\ &= 4k(-4(2^{\lfloor \log_2 (k) \rfloor})) + 2 + 1 \\ &= 4k - 2^2 * 2^{\lfloor \log_2 (k) \rfloor} + 2 + 1 \end{aligned}$$

Here we distribute one of the powers in 2^2 to the other power in the $2^{\log}$ creating the following function:

$$= 4k - 2^1 * 2^{\lfloor \log_2 (k) \rfloor + 1} + 2 + 1$$

Now using the rules of floor discussed in class for homework 4 we know we can move the +1 into the floor statement:

$$= 4k - 2^1 * 2^{\lfloor \log_2 (k) + 1 \rfloor} + 2 + 1$$

For the next step we need to use the lemma shown above.
using the lemma discussed we turn the +1 into a multiple of 2 for the log function getting:

$$= 4k - 2^1 * 2^{\lfloor \log_2 (2k) \rfloor} + 2 + 1$$

We can add a +1 inside the log of the equation $2^{\lfloor \log_2 (2k) \rfloor}$ because it will never change the outcome of the equation. (see the following sub-proof for reasoning)

Sub-proof:
Using the rules of logarithms combined with how floor works we can prove that adding a plus 1 to the log equation

will never change the outcome.
To start, we add the +1 into the log and get the following:

$$2^{\lfloor \log_2 (2k+1) \rfloor}$$

Using the rule of logarithmic addition this function can be written as:

$$2^{\lfloor \log_2 (2k) + \log_2 (1) \rfloor}$$

If we calculate $\log_2(1)$ we get 0. This then proves that adding a +1 into the original log equation will not change the outcome.

Now we have:

$$\begin{aligned}
&= 4k - 2^1 * 2^{\lfloor \log_2 (2k+1) \rfloor} + 2 + 1 \\
&= 4k - 4^{\lfloor \log_2 (2k+1) \rfloor} + 2 + 1 \\
&= 4k + 2 - 4^{\lfloor \log_2 (2k+1) \rfloor} + 1 \\
S(2k+1) &= 2((2k+1) - 2^{\lfloor \log_2 (2k+1) \rfloor}) + 1
\end{aligned}$$

From here we can see that the odd equation is proved using induction because we successfully turned the equation into a form equivalent to $S(2k+1)$

✠ Sub-Experience Six – Not Numbing Number Theory

Prove or disprove whether you can order the years $1985, \ldots, 1995$ so that the resulting 44-digit number is prime.

Claim: Any combination of $1985, \ldots, 1995$ is divisible by 11.

Proof: The rule for divisibility by 11 is adding the odd-numbered digits (1st, 3rd, 5th, etc) and then adding the even-numbered digits (2nd, 4th ,6th, etc) and then finding the difference between those two sums. If the difference is divisible by 11, the number itself is also divisible by 11. As the years 1985-1995 are 4-digit numbers, the 1st and 3rd digits of each year will always be odd-numbered digits in the 44-digit number and the 2nd and 4th digits will likewise always be even-numbered digits. By taking a sum of all the 1st and 3rd digits of each year, we get a sum of 105. By taking a sum of all the 2nd and 4th digits of each year, we get a sum of 149. The difference between these sums is 44, which is divisible by 11. Since this works regardless of order, we can conclude that any combination of the years 1985-1995 is divisible by 11, and therefore no combination of these years is a prime number. (Fun fact: any sequential set of 11 years between 1000 and 9999, when put together to make a 44-digit number, will never be prime regardless of order.
