Homework 4

Ben Smith

February 2023

Homework #4: (Due 2/28/2023).

1. Please prove the following identity using a combinatorial argument:

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \quad \text{where} \quad r, s, n \in \mathbb{N}.$$

Claim: The above identity can be proven with a combinatorial argument.

Proof: We have n kids, r is fruit snacks, and s is granola bars. We are counting how many possibilities there are of how many kids get fruit snacks and how many get granola bars.
LHS: For each kid choose between 0 and n kids that get fruit snacks. Then choose the total number of kids minus the ones that already have fruit snacks and give them granola bars.
RHS: With all the fruit snacks and granola bars there will be enough for everyone.

2. Please prove the the equation $\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}$ in two ways: (1) using a combinatorial argument, and (2) algebraically.

Claim (1): The above equation can be proven using a combinatorial argument.

Proof (1): We are counting how many ways we can choose a team and from that team, choose a team leader.
LHS: We can first choose the i members for the team in $\binom{n}{i}$ ways. Then, we can choose the leader from the remaining n-i people in n-i ways.
RHS: We can choose the leader in n ways. For the team, we can either include or exclude each of the remaining n-1 people. Therefore, there are 2^{n-1} ways to choose the team.

Claim (2): The above equation can be proven algebraically.

Proof (2): Consider the identity: $\binom{n}{i} = \frac{n}{i}(\frac{n-1}{i-1})$

$$\begin{aligned}
\binom{n}{i} =& \frac{n!}{i!(n-i)!} \\
=& \frac{n(n-1)(n-2)\cdots(n-n+1)}{i(i-1)(i-2)\cdots(i-i+1)(n-i)(n-i-1)\cdots(n-1-(n-i)+1)} \\
=& \frac{n}{i}\left(\frac{(n-1)(n-2)\cdots(n-n+1)}{(i-1)(i-2)\cdots(i-i+1)(n-i)\cdots(n-i-(n-i)+1)}\right) \\
=& \frac{n}{i}\left(\frac{(n-1)!}{(i-1)!(n-i)!}\right)
\end{aligned}$$

Therefore, $\binom{n}{i} = \frac{n}{i}\left(\frac{(n-1)!}{(i-1)!(n-i)!}\right) = \frac{n}{i}\binom{n-1}{i-1}$

Apply that identity:

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i}i = 0^* + &\sum_{i=1}^{n} i\binom{n}{i} \\
=& \sum_{i=1}^{n} i\frac{n}{i}\binom{n-1}{i-1} \\
=& \sum_{i=1}^{n} n\binom{n-1}{i-1} \\
=& n\sum_{i=1}^{n}\binom{n-1}{i-1} \\
=& n2^{n-1}
\end{aligned}$$

*Note that i=0 is irrelevant as selecting none doesn't matter

3. Please prove that the n^{th} Fibonacci number F_n is given by the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Claim: The n^{th} Fibonacci number is given by the above formula can be proven using an inductive proof.

Proof:
Base Cases:
n=0: $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) = \frac{1}{\sqrt{5}}(1-1) = 0$
n=1: $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{1}{\sqrt{5}}(1-1) = 1$
Inductive step: Note that $F_{n+1} = F_n + F_{n-1}$ as a definition of the Fibonacci sequence.

Let $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{1}{\sqrt{5}}(\phi-(1-\phi))$ Assume true for $n = k$.

Then,

$$
\begin{aligned}
F_{k+1} =& F_k + F_{k-1} \\
=& \frac{1}{\sqrt{5}}(\phi^k - (1-\phi^k)) + \frac{1}{\sqrt{5}}(\phi^{k-1} - (1-\phi^{k-1})) \\
=& \frac{1}{\sqrt{5}}(\phi^k - (1-\phi)^k + \phi^{k-1} - (1-\phi))^{k-1} \\
=& \frac{1}{\sqrt{5}}(\phi^k + \phi^{k-1} - ((1-\phi)^k + (1-\phi)^{k-1})) \\
=& \frac{1}{\sqrt{5}}(\phi^{k-1}(\phi+1) - (1-\phi)^{k-1}((1-\phi)+1))
\end{aligned}
$$

ϕ^2 can be defined as

$$
\begin{aligned}
\phi^2 =& \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\
=& \frac{1+2\sqrt{5}+5}{4} \\
=& \frac{6+2\sqrt{5}}{4} \\
=& \frac{3+\sqrt{5}}{2} \\
=& \frac{2}{2} + \frac{1+\sqrt{5}}{2} \\
=& 1+\phi
\end{aligned}
$$

So,

$$
\begin{aligned}
=& \frac{1}{\sqrt{5}}\left(\phi^{k-1}\phi^2 - (1-\phi)^{k-1}(1-\phi)^2\right) \\
F_{k+1} =& \frac{1}{\sqrt{5}}\left(\phi^{k+1} - \left(1-\phi^{k+1}\right)\right) \\
F_{k+1} =& \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right) \\
F_{k+1} =& F_n
\end{aligned}
$$

4. Please prove the identity $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ using (a) a combinatorial argument, and (b) the principle of mathematical induction.

Claim (a): The identity can be proven using a combinatorial argument.

Proof (a): Suppose we want to climb a ladder with n rungs. At each step, we can either take one or two steps. We are counting the number of ways to climb a ladder.
LHS: F is whether our next rung will be one or two steps away.
RHS: Let i be the number of times we take two steps. Then, the number of times we take one step is $(n-2i)$

Claim (b): The identity can be proven using the principle of mathematical induction.

Proof (b):

Pascal's Rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Base Cases:

$n = 0$, then $F_{0+1} = \sum_{i=0}^{\lfloor \frac{0}{2} \rfloor} \binom{0-i}{i} = 1$

$n = 1$, then $F_{1+1} = \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} \binom{1-i}{i} = 1$

Induction Hypothesis: $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ is true for all $n \geq 1$

Inductive step:

Assume $F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ holds for $n = k$ and recall that $F_{k+1+1} = F_{k+2} = F_{k+1} + F_k$. Thus,

$$
\begin{aligned}
F_{k+2} =& F_{k+1} + F_k \\
=& \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} \\
=& \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} + \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor + 1} \binom{k-i}{i-1}
\end{aligned}
$$

Now take out the last term of the summation:

$$
\begin{aligned}
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} + \binom{k - \lfloor \frac{k-1}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor + 1 - 1} \\
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} + \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} + \binom{k - 1 - \lfloor \frac{k-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor}
\end{aligned}
$$

Note that if k is even, there are no additional terms in the sum for $\lfloor \frac{k+1}{2} \rfloor$

if k is odd, there is one additional term which ends in 0*

So we can ignore the last term because any value eliminates it or is included in the sum

$$
\begin{aligned}
&= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i} + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} \\
&= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i} + \binom{k-i}{i-1}
\end{aligned}
$$

Apply Pascal's Rule:

$$
\begin{aligned}
F_{k+2} &= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} \\
F_{n+1} &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}
\end{aligned}
$$

Therefore the given equation is true through this process of mathematical induction.

* if $k = 7$, $\binom{7 - \lfloor \frac{7+1}{2} \rfloor}{\lfloor \frac{7+1}{2} \rfloor} = \binom{3}{4} = 0$
