



湖南大學
HUNAN UNIVERSITY

分治

—— 湖南大学信息科学与工程学院 ——



目录

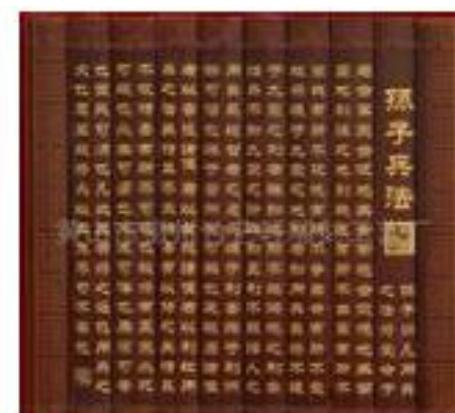
- 第一节 分治法的设计思想
- 第二节 最大子段和
- 第三节 求解递归式
- 第四节 组合问题中的分治法
- 第五节 几何问题中的分治法



分治法的设计思想

凡治众如治寡， 分数是也；
斗众如斗寡， 形名是也。

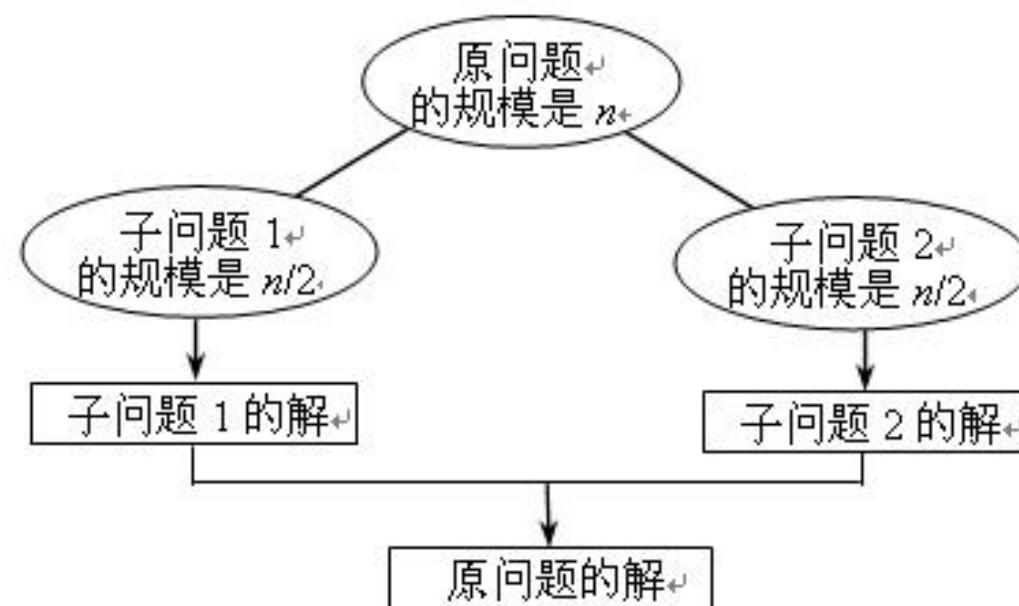
Divide and conquer





分治法的设计思想

将一个难以直接解决的大问题，分割成一些规模较小的相同问题，以便各个击破，分而治之。





分治法的设计思想

分治法的求解过程是由以下三个阶段组成：

(1) Divide : 划分

(2) Conquer : 求解子问题

(3) Combine : 合并



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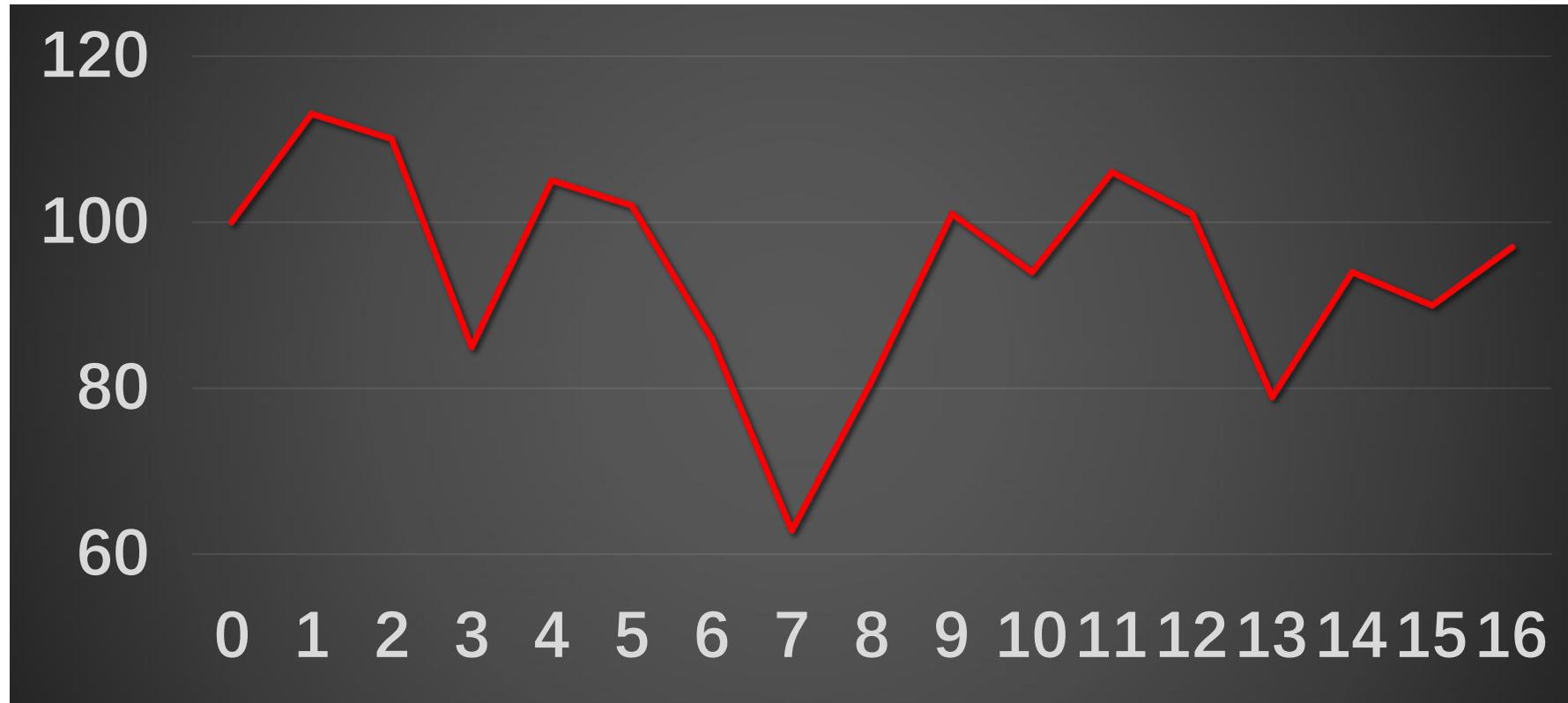
组合问题中的分治法-最大子段和问题

如何购买股
票？





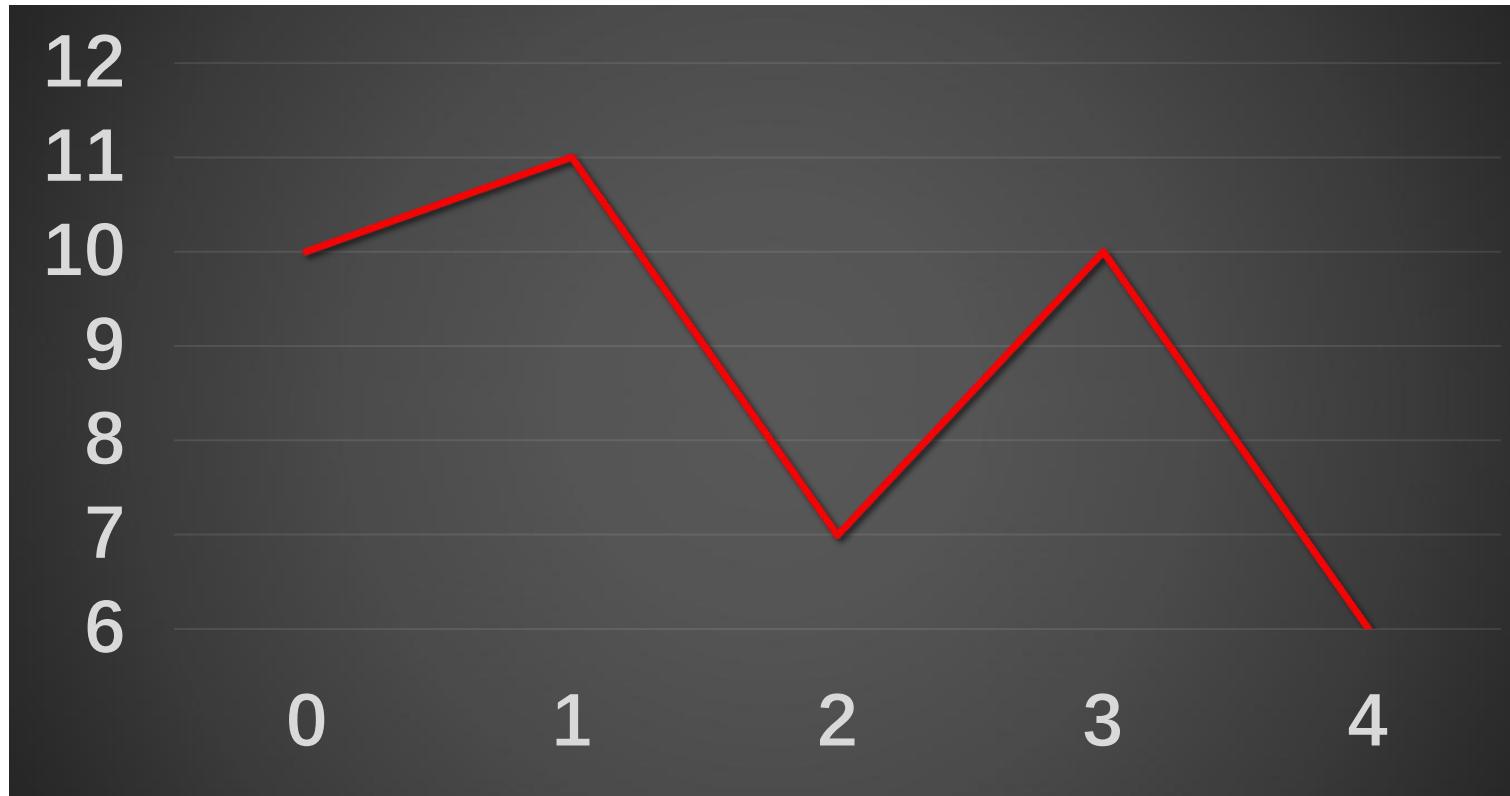
组合问题中的分治法-最大子段和问题



| 天 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----|-----|-----|-----|-----|-----|-----|-----|-----|----|-----|----|-----|-----|-----|----|----|----|
| 价格 | 100 | 113 | 110 | 85 | 105 | 102 | 86 | 63 | 81 | 101 | 94 | 106 | 101 | 79 | 94 | 90 | 97 |
| 变化 | | 13 | -3 | -25 | 20 | -3 | -16 | -23 | 18 | 20 | -7 | 12 | -5 | -22 | 15 | -4 | 7 |



组合问题中的分治法-最大子段和问题

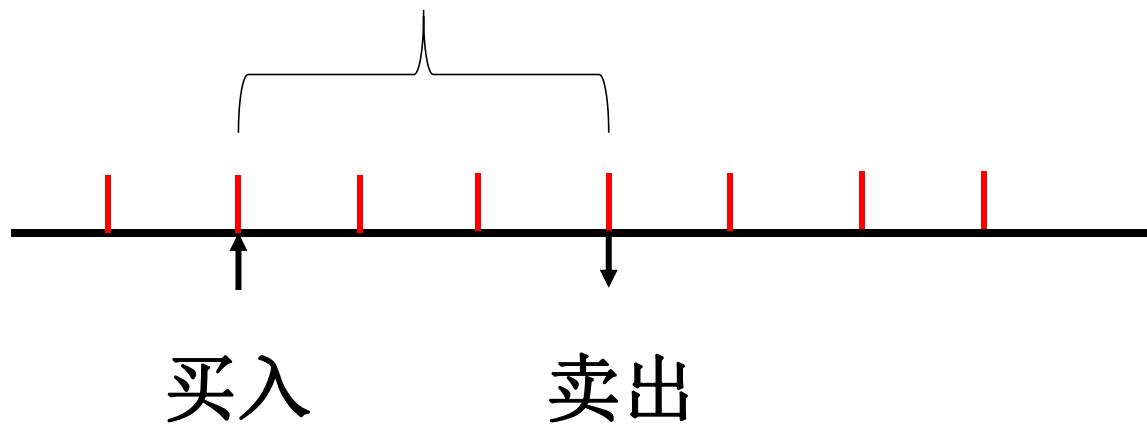


| 天 | 0 | 1 | 2 | 3 | 4 |
|------|----|----|---|----|---|
| 价格变化 | 10 | 11 | 7 | 10 | 6 |
| | 1 | -4 | 3 | -4 | |

组合问题中的分治法-最大子段和问题



最简单方法：暴力法解决



代码及时间复杂度分析



组合问题中的分治法-最大子段和问题

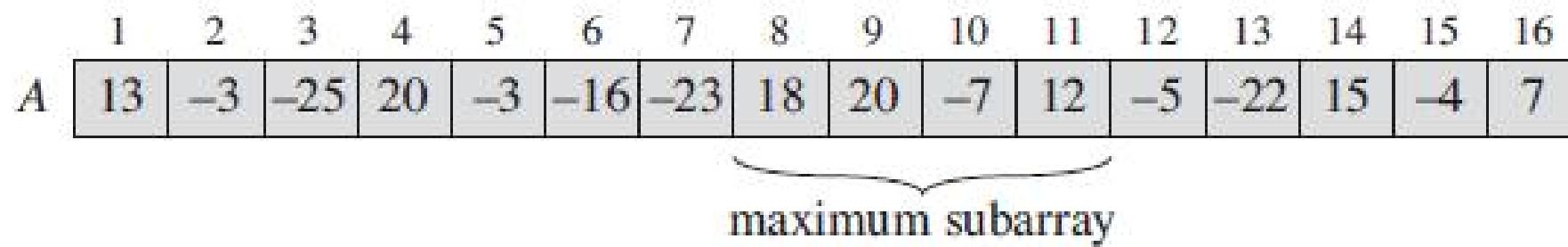
问题变换(最大字段和问题或最大子数组问题)

给定由n个整数（可能有负整数）组成的序列(a_1, a_2, \dots, a_n)，最大子段和问题要求该序列形如 $\sum_{k=i}^j a_k$ 的最大值 ($1 \leq i \leq j \leq n$)，当序列中所有整数均为负整数时，其最大子段和为0。例如，序列(-20, 11, -4, 13, -5, -2)的最大子段和为 $\sum_{k=2}^4 a_k = 20$ 。

| 天 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----|-----|-----|-----|-----|-----|-----|-----|-----|----|-----|----|-----|-----|-----|----|----|----|
| 价格 | 100 | 113 | 110 | 85 | 105 | 102 | 86 | 63 | 81 | 101 | 94 | 106 | 101 | 79 | 94 | 90 | 97 |
| 变化 | | 13 | -3 | -25 | 20 | -3 | -16 | -23 | 18 | 20 | -7 | 12 | -5 | -22 | 15 | -4 | 7 |



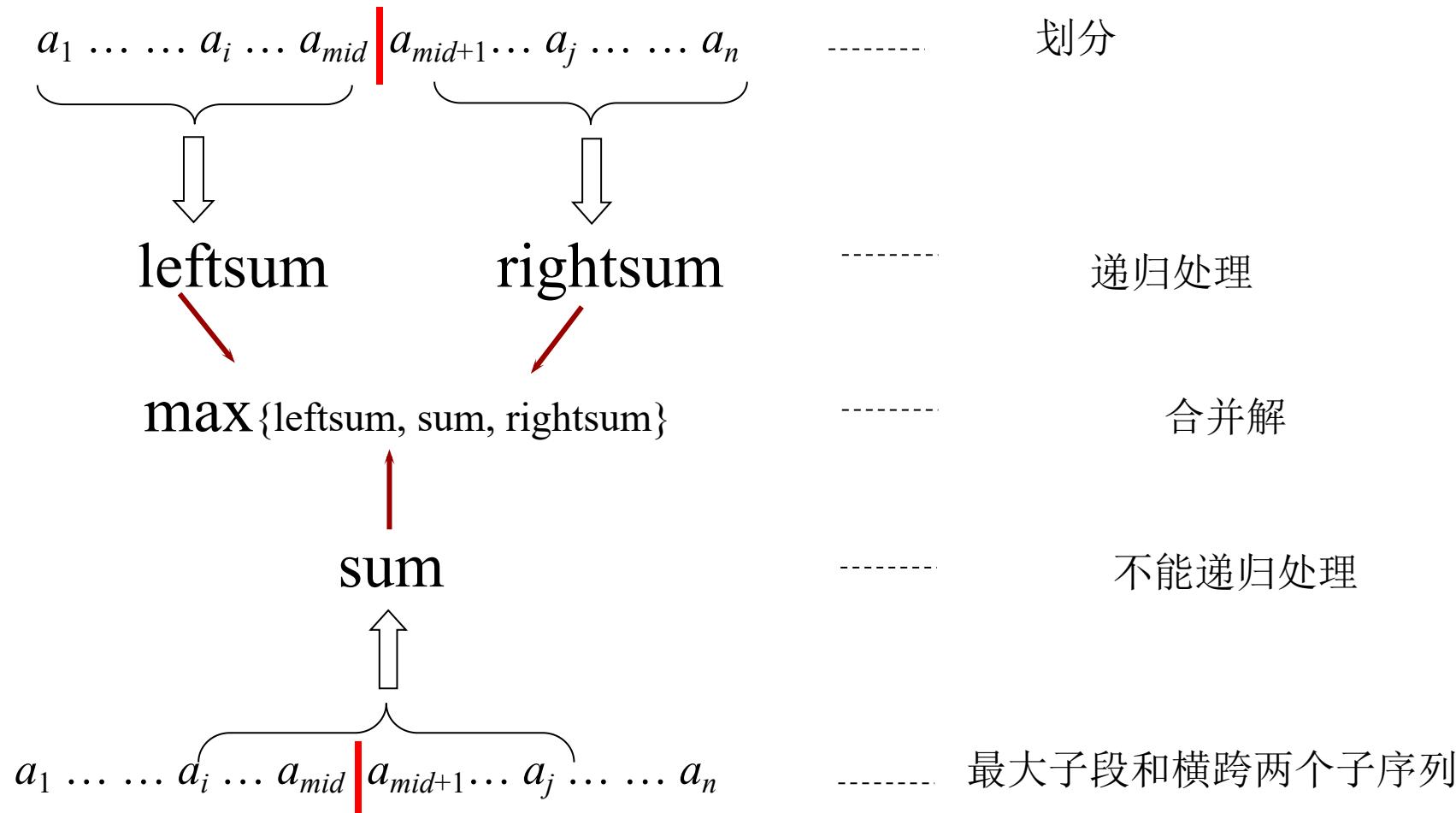
组合问题中的分治法-最大子段和问题





组合问题中的分治法-最大子段和问题

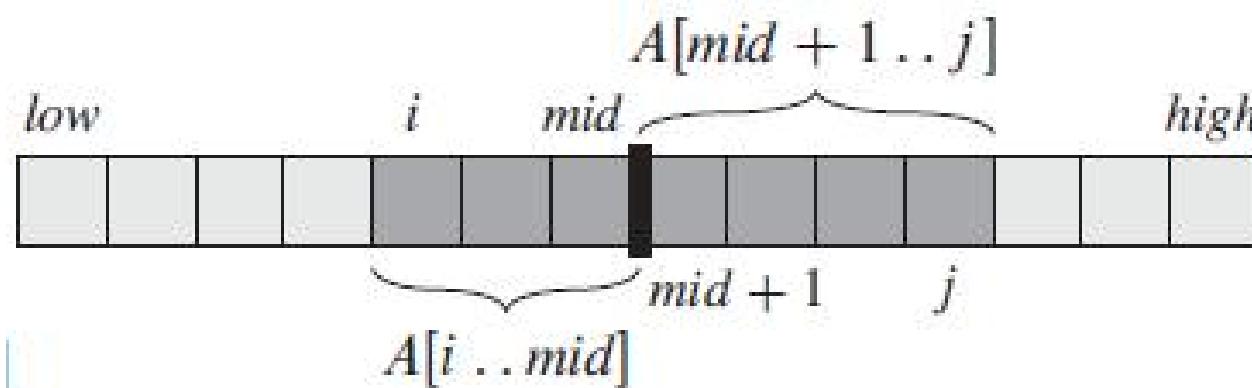
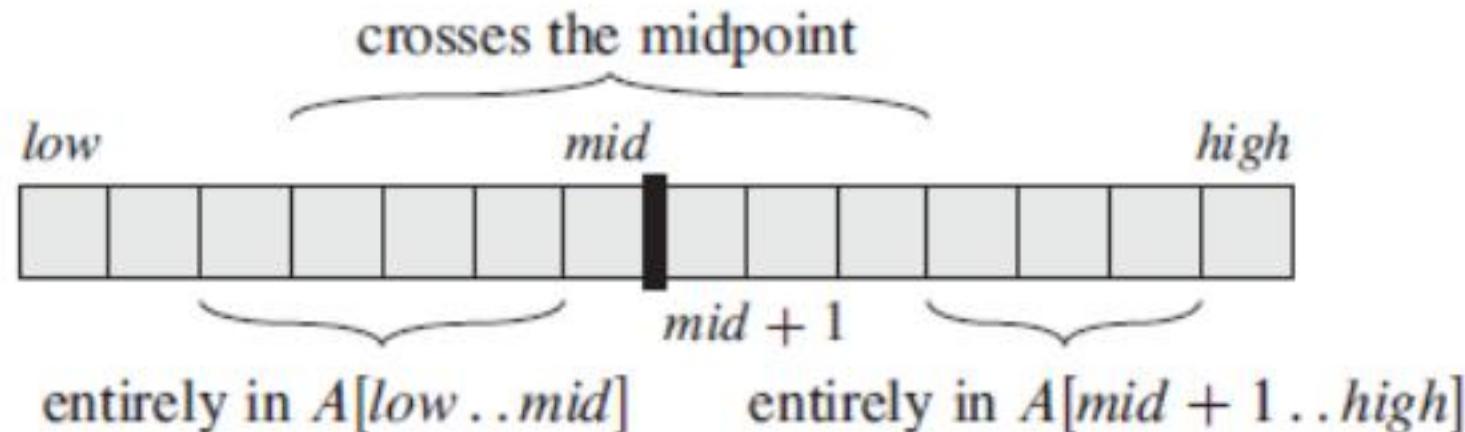
使用分治策略的求解方法





组合问题中的分治法-最大子段和问题

使用分治策略的求解方法





组合问题中的分治法-最大子段和问题

伪代码

FIND-MAX-CROSSING-SUBARRAY($A, low, mid, high$)

```
1  left-sum = -∞
2  sum = 0
3  for i = mid downto low
4      sum = sum + A[i]
5      if sum > left-sum
6          left-sum = sum
7          max-left = i
8  right-sum = -∞
9  sum = 0
10 for j = mid + 1 to high
11     sum = sum + A[j]
12     if sum > right-sum
13         right-sum = sum
14         max-right = j
15 return (max-left, max-right, left-sum + right-sum)
```



组合问题中的分治法-最大子段和问题

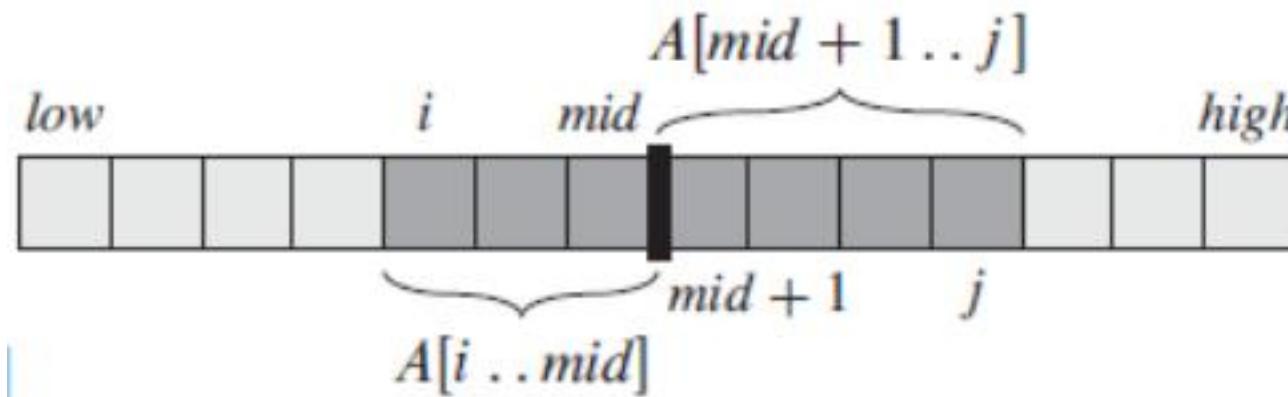
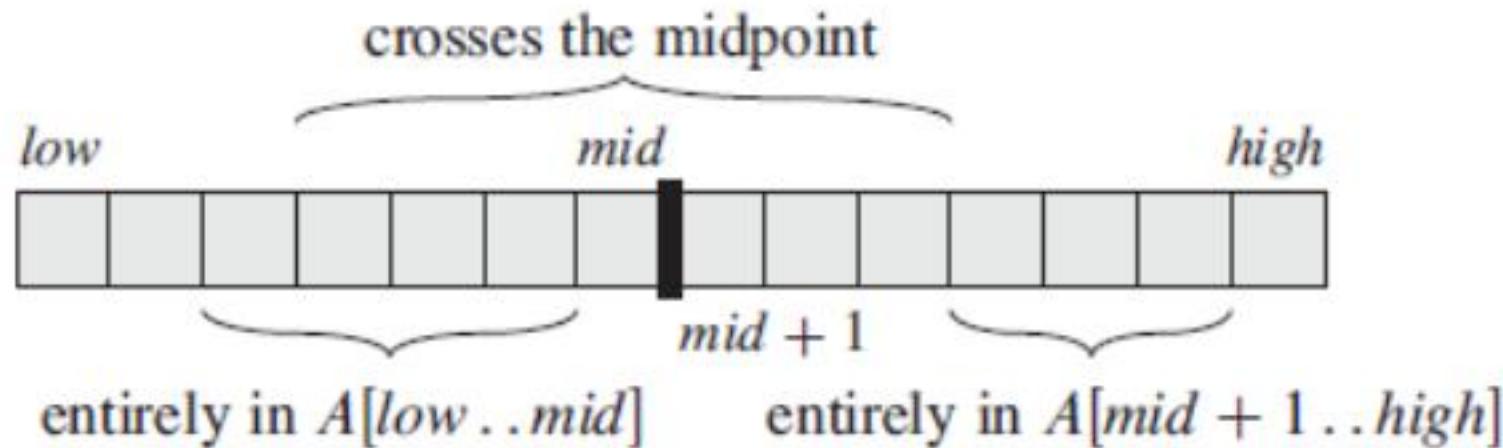
伪代码

```
FIND-MAXIMUM-SUBARRAY( $A, low, high$ )
1  if  $high == low$ 
2    return ( $low, high, A[low]$ )           // base case: only one element
3  else  $mid = \lfloor (low + high)/2 \rfloor$ 
4    ( $left-low, left-high, left-sum$ ) =
      FIND-MAXIMUM-SUBARRAY( $A, low, mid$ )
5    ( $right-low, right-high, right-sum$ ) =
      FIND-MAXIMUM-SUBARRAY( $A, mid + 1, high$ )
6    ( $cross-low, cross-high, cross-sum$ ) =
      FIND-MAX-CROSSING-SUBARRAY( $A, low, mid, high$ )
7    if  $left-sum \geq right-sum$  and  $left-sum \geq cross-sum$ 
8      return ( $left-low, left-high, left-sum$ )
9    elseif  $right-sum \geq left-sum$  and  $right-sum \geq cross-sum$ 
10   return ( $right-low, right-high, right-sum$ )
11   else return ( $cross-low, cross-high, cross-sum$ )
```



组合问题中的分治法-最大子段和问题

分治法时间复杂度分析





最大子段和

时间复杂度: $T(n)=2T(n/2)+\Theta(n)=\Theta(n\log n)$



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求解递归式

三种求解递归式的方法：

- 1、代入法
- 2、递归树法
- 3、主方法



代入法

求解: $T(n)=4T(n/2)+n$

We first guess $T(n)=O(n^3)$

(Proof by induction) Assume $T(k)\leq ck^3$, for $k < n$

Then $T(n)=4T(n/2)+n\leq 4c(n/2)^3+n=cn^3/2+n=cn^3-(cn^3/2-n)$

$cn^3-(cn^3/2-n)\leq cn^3$, if $(cn^3/2-n)\geq 0$, e.g. $c > 1$ and $n \geq 1$



代入法

求解: $T(n)=4T(n/2)+n$

Then, we prove $T(n)=O(n^2)$.

(Proof by induction) Assume $T(k)\leq ck^2$, for $k < n$

Then $T(n)\leq 4c(n/2)^2+n=cn^2+n$



代入法

求解: $T(n)=4T(n/2)+n$

Assume $T(k) \leq c_1 k^2 - c_2 k$, for $k < n$

$$T(n) \leq 4(c_1(n/2)^2 - c_2n/2) + n \leq c_1n^2 - 2c_2n + n = c_1n^2 - c_2n - (c_2 - 1)n$$

If $c_2 \geq 1$, $(c_2 - 1)n \geq 0$. Then $T(n) \leq c_1n^2 - c_2n$



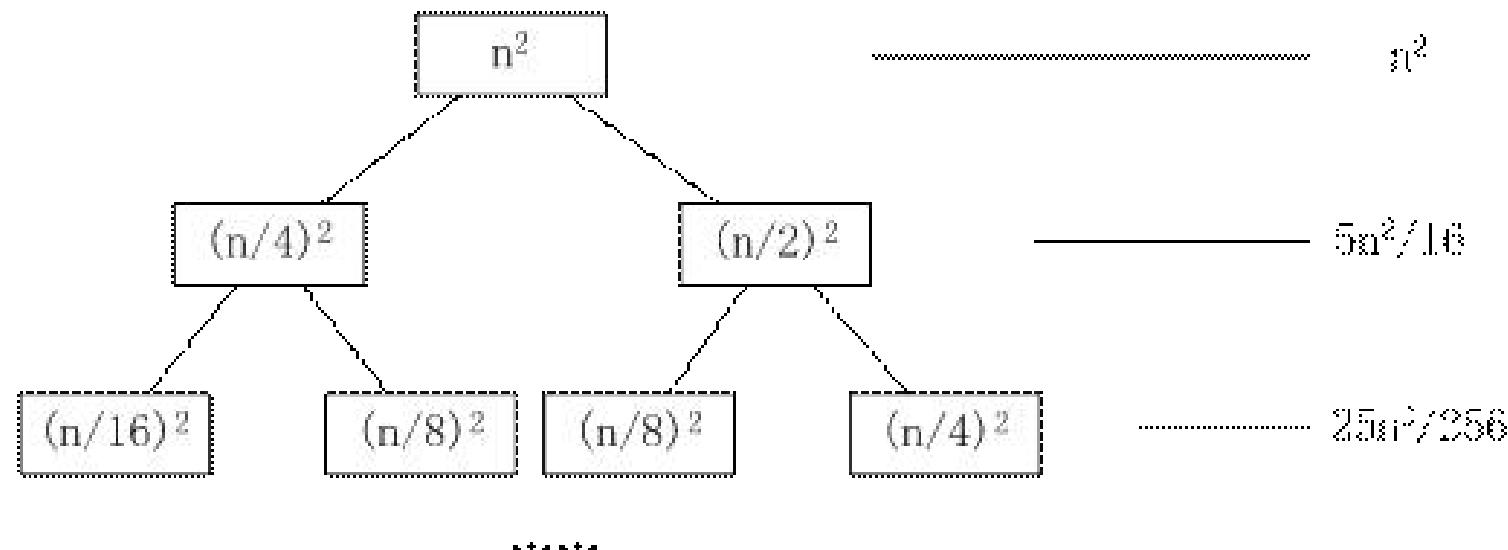
求解递归式

$$T(n) = T(n/4) + T(n/2) + n^2$$



递归树法

- Example: $T(n)=T(n/4)+ T(n/2)+n^2$



$$T(n) \leq n^2(1 + 5/16 + (5/16)^2 + \dots + (5/16)^k + \dots) < 2n^2 = O(n^2)$$



求解递归式-递归树法

- (1) 归并排序关键代码分析
- (2) 递归调用过程分析（图例）
- (3) 时间复杂度分析？



求解递归式-主方法

主方法适用于下面的递归形式

$$T(n) = a T(n/b) + f(n) ,$$

$a \geq 1, b > 1, f$ 为正的渐进函数



求解递归式-主方法

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$. ■



求解递归式-主方法

1. 若对某个常数 $\varepsilon > 0$ 有 $f(n) = O(n^{\log_b^{a-\varepsilon}})$ ， 则 $T(n) = \Theta(n^{\log_b^a})$ 。
2. 若 $f(n) = \Theta(n^{\log_b^a})$ ， 则 $T(n) = \Theta(n^{\log_b^a} \lg n)$ 。
3. 若对某个常数 $\varepsilon > 0$ ， 有 $f(n) = \Omega(n^{\log_b^{a+\varepsilon}})$ ， 且对某个常数 $c < 1$ 和足够大的 n 有 $a f(n/b) \leq c f(n)$ ， 则 $T(n) = \Theta(f(n))$ 。



求解递归式-主方法

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

$$T(n) = T\left(\frac{2n}{3}\right) + 1$$

$$T(n) = 3T\left(\frac{n}{4}\right) + n \log n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n \log n$$

求解递归式-主定理



自学，算法导论：P55–60



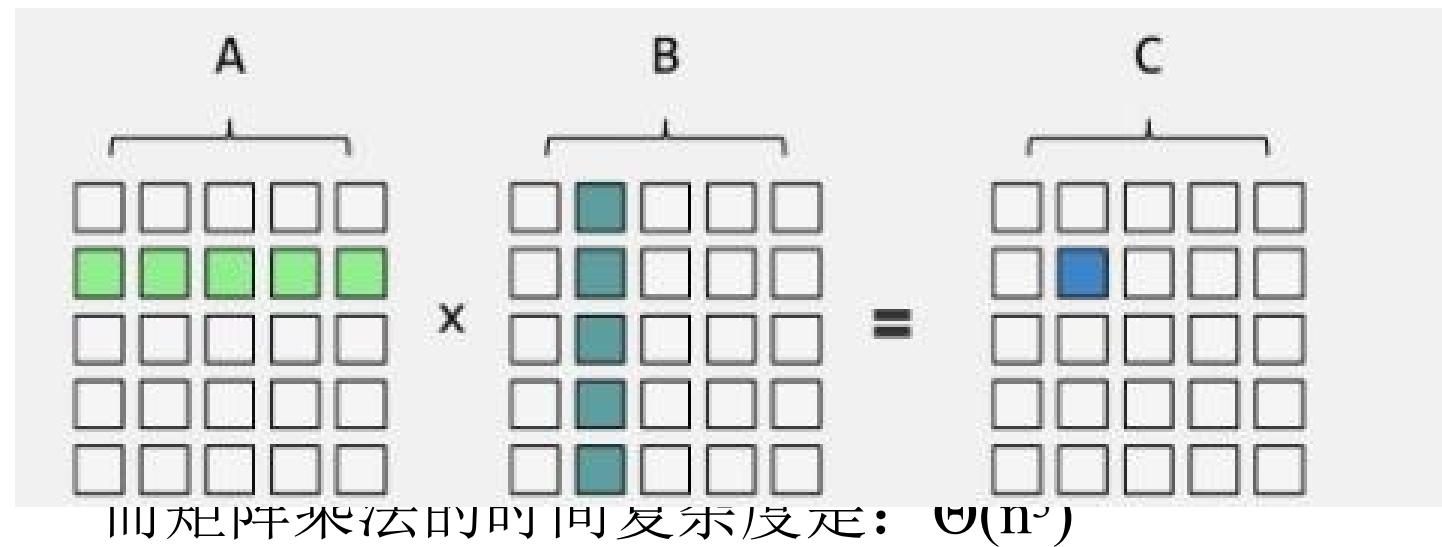
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组合问题中的分治法-矩阵乘法

$$S(i, j) = A(i, j) + B(i, j) \quad 1 \leq i, j \leq n$$



求每个元素 $C(i,j)$ 都需要 n 次乘法和 $n-1$ 次加法运算，且 C 共有 n^2 个元素。



组合问题中的分治法-矩阵乘法

用分治法解决矩阵的乘法问题，先假定 n 是2的 k 次幂，即 $n=2^k$ 。

1) 第一步：递归维度半分。

如果 $n=2$ ，则递归结束。

如果 $n>2$ ，将A 和B 两个 n 维方阵各分解为4 份，每份 $n/2$ 维。当子矩阵的阶还是大于2时，继续将子矩阵分块，直到子矩阵的阶降为2。

第一步将会产出很多个2阶的方阵！

$$C_{11} = A_{11} * B_{11} + A_{12} * B_{21}$$

下面用一个动画来展示分矩阵的过程

$$B_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B_{12} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

A矩阵 B矩阵

| | | | | | |
|----------|----------|----------|----------|----------|----------|
| a11 | a12 | | | | |
| a13 | a14 | A_{11} | A_{12} | | |
| | | | | | |
| A_{21} | A_{22} | | | | |
| | | | | | |
| | | | | B_{11} | B_{12} |
| b13 | b14 | | | | |
| | | | | | |
| | | | | B_{21} | B_{22} |
| | | | | | |



组合问题中的分治法-矩阵乘法

2) 第二步：计算两个2阶方阵的乘积之后再合并。

假如A，B两个矩阵是由第一步得出的维数为2的方阵，则：

$$A * B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21} \quad C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} \quad C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

通过上面4条式可以将 $A * B$ 的值存到C矩阵中。



组合问题中的分治法-矩阵乘法

分治法求矩阵乘法的效率分析

如果用 $T(n)$ 记两个 n 阶矩阵相乘所用的时间，则有如下递归关系式：

$$T(n) = \begin{cases} b & n \leq 2 \\ 8T(n/2) + dn^2 & n > 2 \end{cases}$$

$$T(n)=\Theta(n^3)$$



组合问题中的分治法-矩阵乘法

Strassen矩阵乘法

Strassen提出了一种新的算法来计算2个2阶方阵的乘积。他的算法只用了7次乘法运算，但增加了加、减法的运算次数。这7次乘法是：

$$M_1 = A_{11}(B_{12} - B_{22})$$

$$M_2 = (A_{11} + A_{12})B_{22}$$

$$M_3 = (A_{21} + A_{22})B_{11}$$

$$M_4 = A_{22}(B_{21} - B_{11})$$

$$M_5 = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$M_6 = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$M_7 = (A_{11} - A_{21})(B_{11} + B_{12})$$



组合问题中的分治法-矩阵乘法

Strassen矩阵乘法

做了7次乘法后，再做若干次加、减法：

$$C_{11} = M_5 + M_4 - M_2 + M_6$$

$$C_{12} = M_1 + M_2$$

$$C_{21} = M_3 + M_4$$

$$C_{22} = M_5 + M_1 - M_3 - M_7$$



组合问题中的分治法-矩阵乘法

Strassen算法效率分析

Strassen矩阵乘积分治算法中，用了7次对于n/2阶矩阵乘积的递归调用和18次n/2阶矩阵的加减运算。由此可知，该算法的所需的计算时间T(n)满足如下的递归方程：

$$T(n) = \begin{cases} b & n \leq 2 \\ 7T(n/2) + an^2 & n > 2 \end{cases}$$

$$T(n)=O(n^{\log_2 7})$$



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几何问题中的分治法-最近对问题

设 $p_1=(x_1,y_1), p_2=(x_2,y_2), \dots, p_n=(x_1,y_1)$, 是平面上 n 个点构成的集合 S , 最近对问题就是找出集合 S 中距离最近的点对。



你想到些什么？



几何问题中的分治法-最近对问题

最近对问题的分治策略是：

(1)划分：将集合S分成两个子集 S_1 和 S_2 ，设集合S的最近点对是 p_i 和 p_j
($1 \leq i, j \leq n$)，则会出现以下三种情况：

① p_i 和 p_j 在 S_1

② p_i 和 p_j 在 S_2

③ p_i 和 p_j 分别在 S_1 和 S_2

几何问题中的分治法-最近对问题



- (2)求解子问题：对于划分阶段的情况①和②可递归求解，如果最近点对分别在集合 S_1 和 S_2 中，问题就比较复杂了。
- (3)合并：比较在划分阶段三种情况下最近点对，取三者之中较小者为原问题的解。

几何问题中的分治法-最近对问题

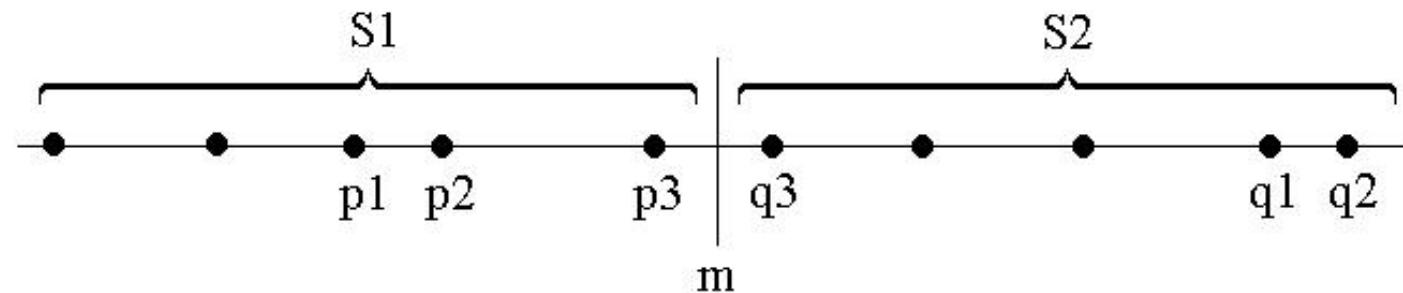


- ◆ 一维空间
- ◆ 二维空间（平面）

几何问题中的分治法-最近对问题



一维空间

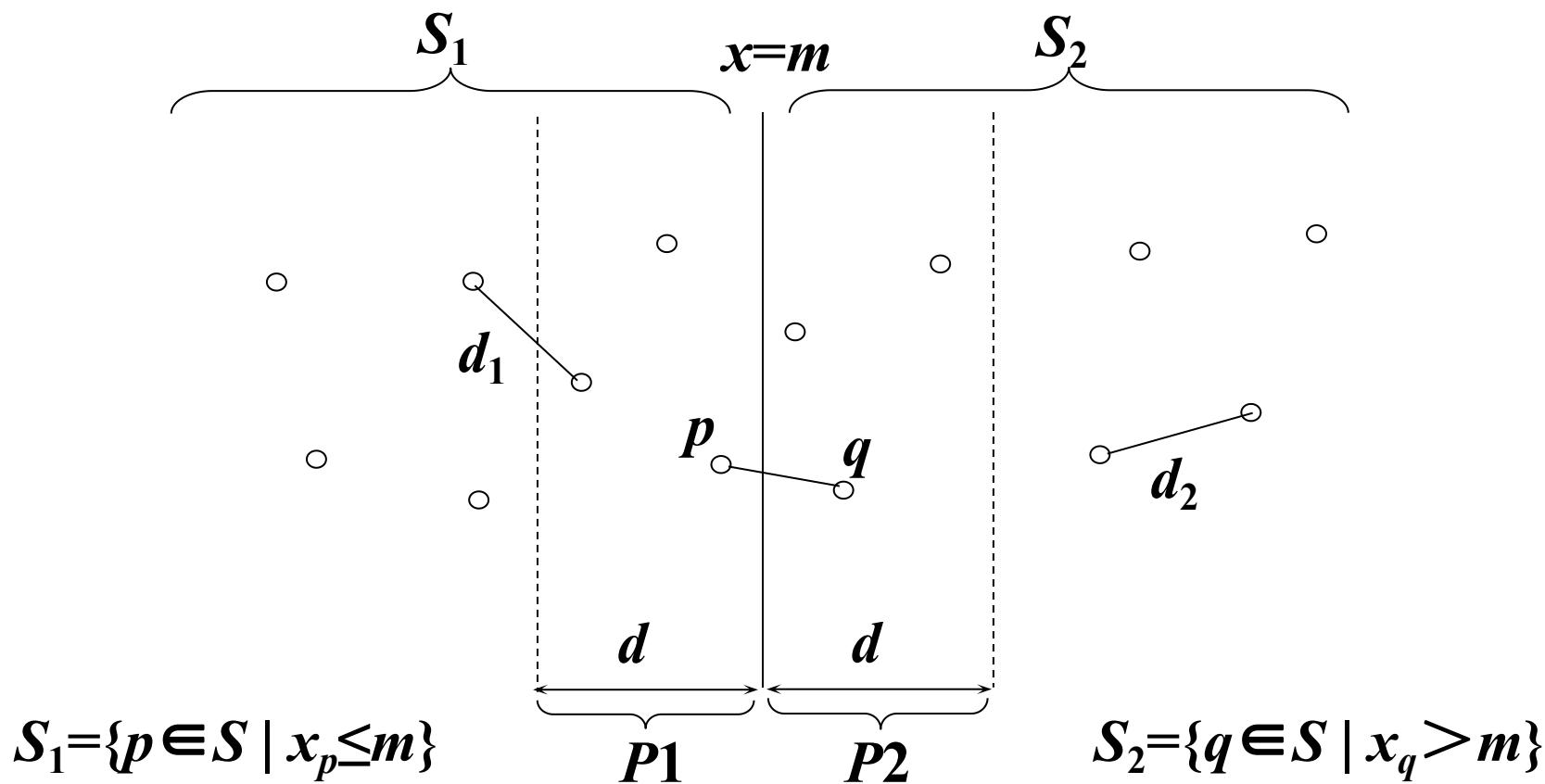


$$T(n)=O(n \log n)$$

几何问题中的分治法-最近点对问题



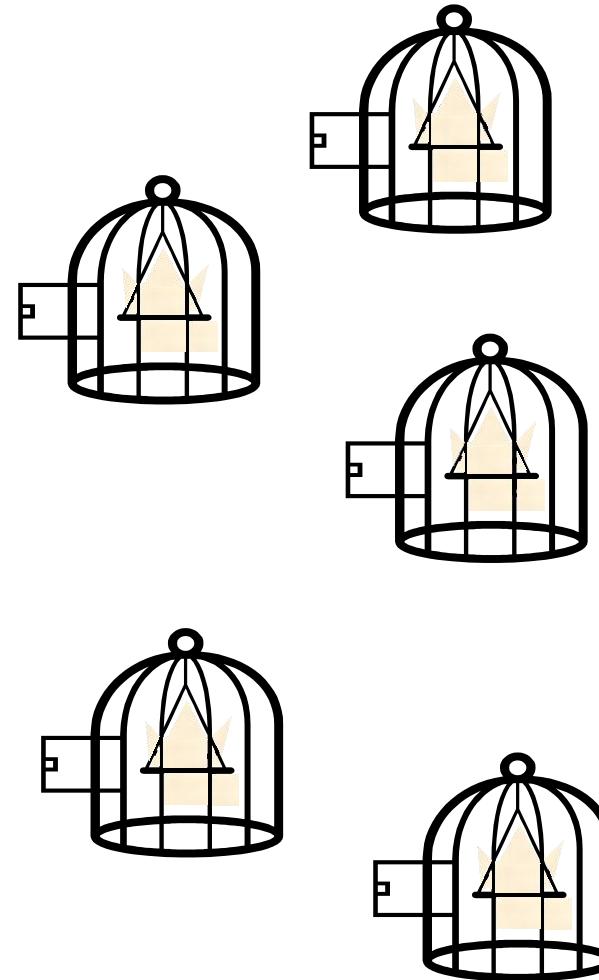
二维空间



几何问题中的分治法-最近点对问题

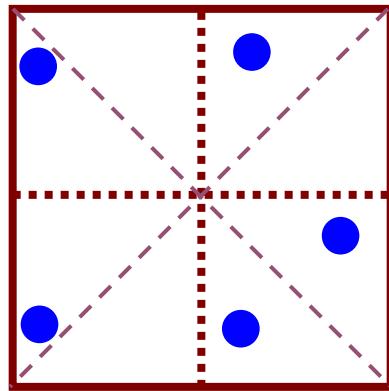


鸽笼原理





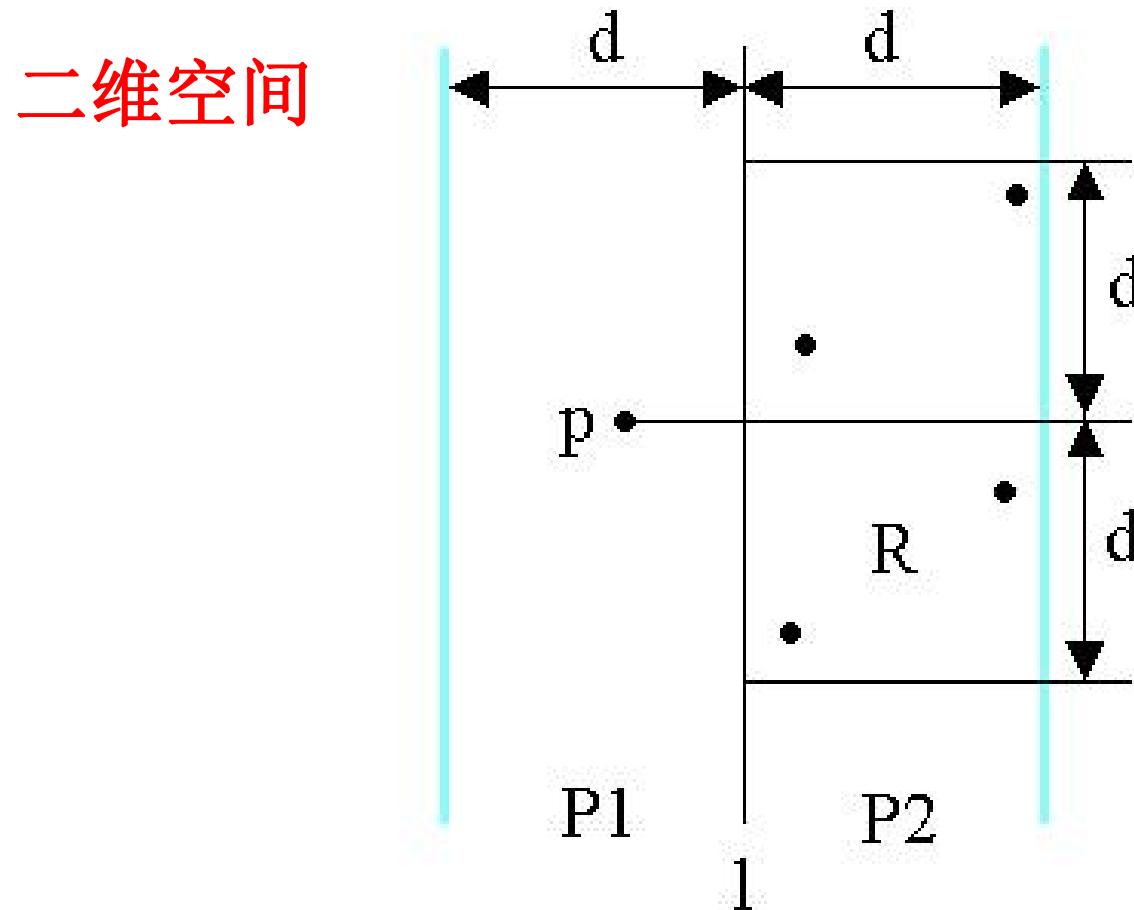
几何问题中的分治法-最近点对问题



证明：把5个顶点放到边长为2的正方形中，至少存在两个顶点，它们之间的距离小于或等于 $\sqrt{2}$ 。



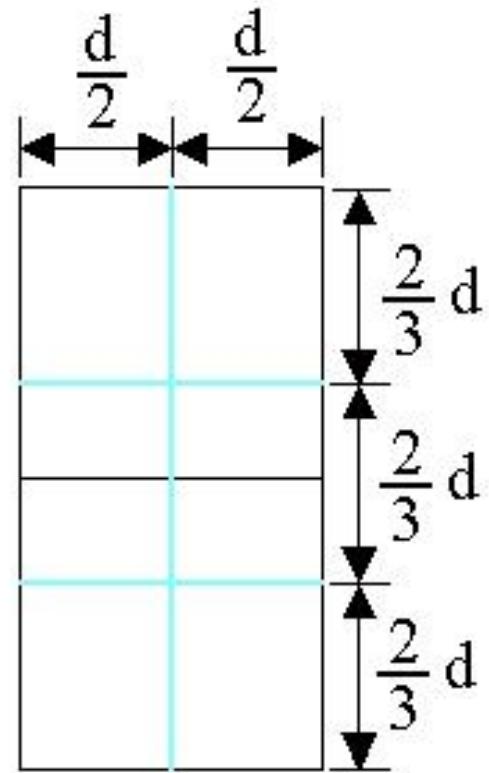
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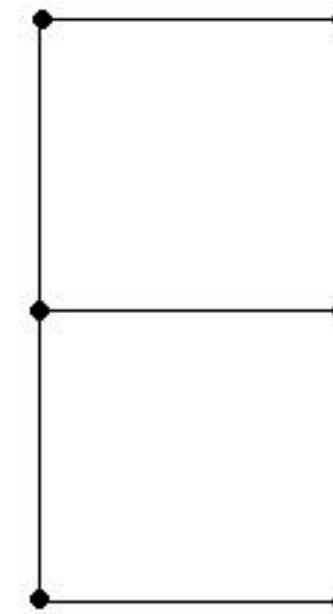
几何问题中的分治法-最近点对问题



二维空间



(a)



(b)



几何问题中的分治法-最近点对问题

二维空间（伪代码）

输入：按x坐标排列的n($n \geq 2$)个点的集合 $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$

输出：最近点的距离

1. 如果 $n==2$, 则返回 (x_1, y_1) 和 (x_2, y_2) 之间的距离, 算法结束;
2. 如果 $n==3$, 则返回 (x_1, y_1) 、 (x_2, y_2) 和 (x_3, y_3) 之间的最小距离, 算法结束; (此步必要在于, 若 $n==3$ 划分之后必然有一半为 $n==1$, 导致无法正确执行递归)
3. 划分: $m=S$ 中各点x坐标的中位数;
4. $d_1 =$ 计算 $\{(x_1, y_1), \dots, (x_m, y_m)\}$ 的最近对距离;
5. $d_2 =$ 计算 $\{(x_m, y_m), \dots, (x_n, y_n)\}$ 的最近对距离;
6. $d = \min(d_1, d_2);$
7. 依次考察集合 S 中的点 $p(x, y)$, 如果 $(x \leq x_m \text{ 并且 } x \geq x_m - d)$, 则将点 p 放入集合 P_1 中;
如果 $(x > x_m \text{ 并且 } x \leq x_m + d)$, 则将点 p 放入集合 P_2 中;
8. 将集合 P_1 和 P_2 按y坐标升序排列;
9. 对集合 P_1 和 P_2 中的每个点 $p(x, y)$, 在y坐标区间 $[y, y+d]$ 内最对取出6个候选点, 计算与点 p 的最近距离 d_3 ;
10. 返回 $\min\{d, d_3\};$

几何问题中的分治法-最近点对问题



(1) 代码分析

(2) 算法时间复杂度分析

$$T(n)=O(n \log n)$$



Finding Peak...

IN SPACE

- Problem: Find a local minimum or maximum in a terrain by sampling





1D Peak Finding

- Given an array $A[0..n - 1]$:



- $A[i]$ is a **peak** if it is not smaller than its neighbor(s):

$$A[i - 1] \leq A[i] \geq A[i + 1]$$

where we imagine

$$A[-1] = A[n] = -\infty$$

- Goal: Find *any* peak



“Brute Force” Algorithm

- Test all elements for peakyness

```
for i in range(n):  
    if A[i - 1] ≤ A[i] ≥ A[i + 1]:  
        return i
```

$\{O(1)\}^n \} O(n)$

| | | | | | | | |
|------|---|---|---|---|---|---|---|
| $A:$ | 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |



Algorithm 1½

- $\max(A)$
 - Global maximum is a local maximum

```
m = 0
for i in range(1, n):
    if A[i] > A[m]:
        m = i
return m
```

$\left. \begin{matrix} \Theta(1) \\ \Theta(n) \end{matrix} \right\} \Theta(n)$

| | | | | | | | |
|------|---|---|---|---|---|---|---|
| $A:$ | 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |



Clever Idea

- Find a subarray containing at least one elements larger than the boundary
 - Find the largest element of the first, last and central elements
 - If it's a peak, return it; otherwise, find the subarray containing its larger neighbor



- Recurse in the subarray



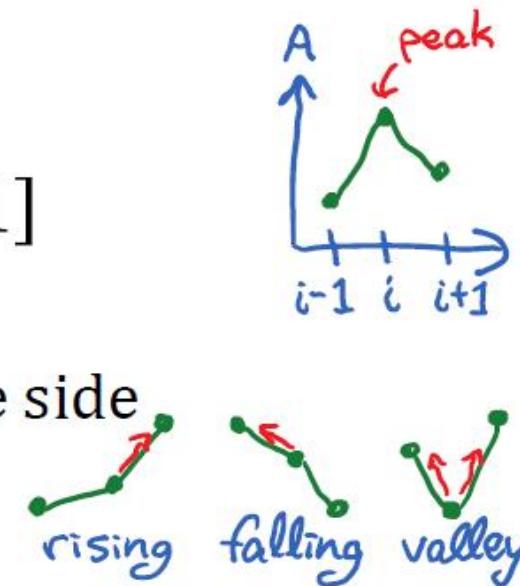
1D Peak Finding Analysis

- Divide problem into 1 problem of size $\sim -$
- Divide cost: $O(1)$
- Combine cost: $O(1)$
- Recurrence:

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

Cleverer Idea

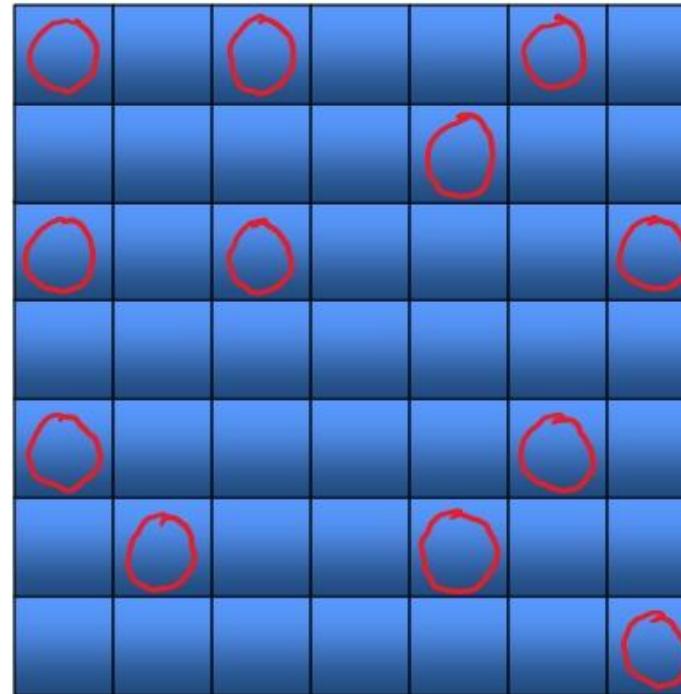
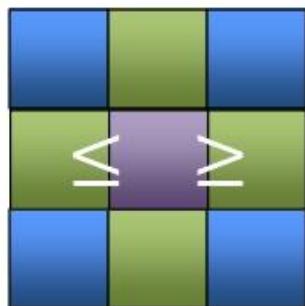
- Look at any element $A[i]$ and its neighbors $A[i - 1]$ & $A[i + 1]$
 - If peak: return i
 - Otherwise: locally rising on some side
 - Must be a peak in that direction
 - So can throw away rest of array, leaving $A[:i]$ or $A[i + 1:]$





2D Peak Finding

- Given $n \times n$ matrix of numbers
- Want an entry not smaller than its (up to) 4 neighbors:





Peak Finding: 2D

- Consider a 2D array $A[1\dots n, 1\dots m]$:

| | | |
|----|----|---|
| 10 | 8 | 5 |
| 3 | 2 | 1 |
| 7 | 13 | 4 |
| 6 | 8 | 3 |

- An element $A[i]$ is a *2D peak* if it is not smaller than its (at most 4) neighbors.
- Problem: find any 2D peak.

Algorithm I: use the 1D algorithm

- Algorithm I:
 - For each column j , find its *global* maximum $B[j]$
 - Apply 1D peak finder to find a peak (say $B[j]$) of $B[1\dots m]$
- Running time ?
....is $\Theta(n \cdot m)$
- Correctness:
 - $B[j]$ not smaller than $B[j-1], B[j+1]$
 - For any k , $B[k]$ not smaller than any element from the k -th column of A
 - Therefore, $B[j]$ not smaller than any element from the columns $j-1, j$ and $j+1$ of A
 - But this includes all neighbors of $B[j]$ in A , so $B[j]$ is a peak in A

| | | |
|----|---|---|
| 12 | 8 | 5 |
| 11 | 3 | 6 |
| 10 | 9 | 2 |
| 8 | 4 | 1 |
| 12 | 9 | 6 |

Algorithm I': use the 1D algorithm

- Observation: 1D peak finder uses only $O(\log m)$ entries of B
- We can modify Algorithm I so that it only computes $B[j]$ when *needed* !
- Total time ?
 - ...only $O(n \log m)$!
 - Need $O(\log m)$ entries $B[j]$
 - Each computed in $O(n)$ time

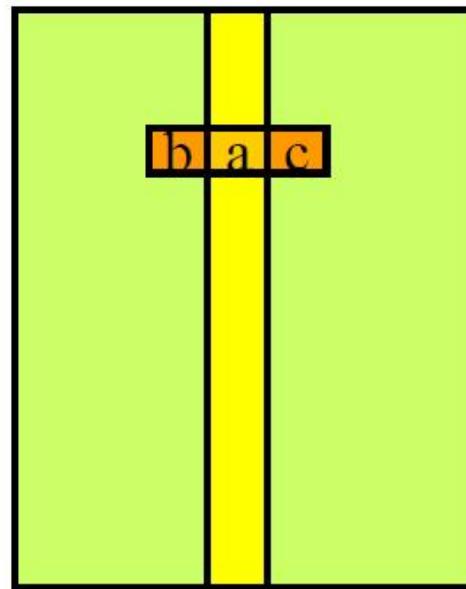
| | | |
|----|---|---|
| 12 | 8 | 5 |
| 11 | 3 | 6 |
| 10 | 9 | 2 |
| 8 | 4 | 1 |

| | | |
|----|---|---|
| 12 | 9 | 6 |
|----|---|---|



Algorithm II

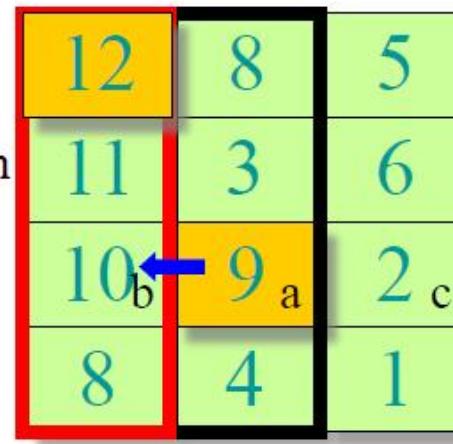
- Pick middle column ($j=m/2$)
- Find *global* maximum $a=A[i,m/2]$ in that column
(and quit if $m=1$)
- Compare a to $b=A[i,m/2-1]$ and $c=A[i,m/2+1]$
- If $b>a$
then recurse on left columns
- Else, if $c>a$
then recurse on right columns
- Else a is a 2D peak!





Algorithm II: Example

- Pick middle column ($j=m/2$)
- Find *global* maximum $a=A[i,m/2]$ in that column
(and quit if $m=1$)
- Compare a to $b=A[i,m/2-1]$ and $c=A[i,m/2+1]$
- If $b>a$
then recurse on left columns
- Else, if $c>a$
then recurse on right columns
- Else a is a 2D peak!





Algorithm II: Correctness

- Claim: If $b > a$, then there is a peak among the left columns
- Proof (by contradiction):
 - Assume no peak on the left
 - Then b must have a neighbor b_1 with higher value
 - And b_1 must have a neighbor b_2 with higher value
 - ...
 - We have to stay on the left side – why?
 - (because we cannot enter the middle column)
 - But at some point, we would run out the elements of the left columns
 - Hence, we have to find a peak at some point

| | | |
|----|---|---|
| 12 | 8 | 5 |
| 11 | 3 | 6 |
| 10 | 9 | 2 |
| 8 | 4 | 1 |
| | | |

Algorithm II: Complexity



- We have

$$T(n,m) = T(n,m/2) + \Theta(n)$$

Recursion

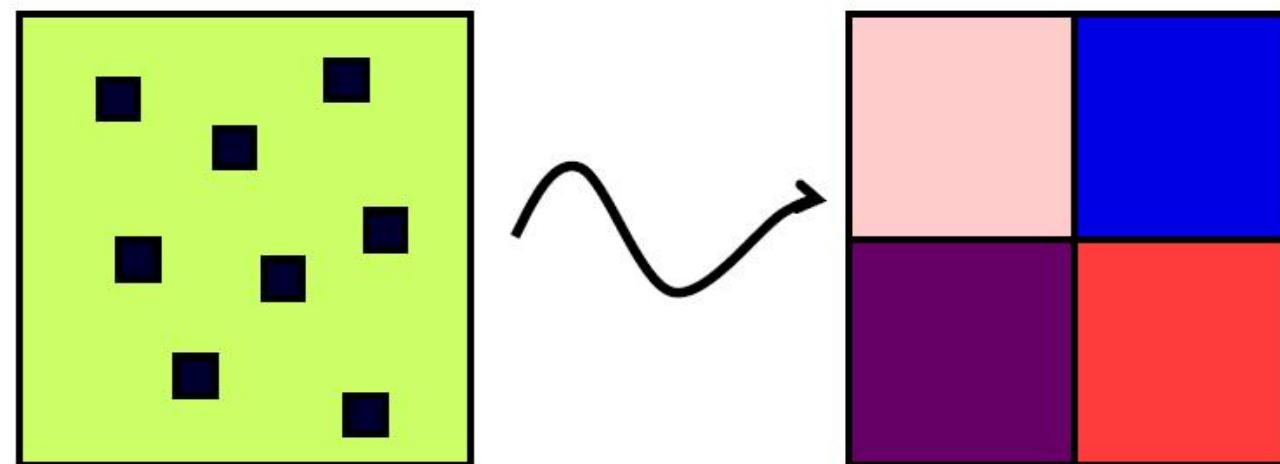
Scanning middle column

- Hence:
 - $T(n,n) = \underbrace{\Theta(n) + \Theta(n) + \dots + \Theta(n)}_{\log_2 m} = \Theta(n \log m)$



Faster than $O(n \log n)$?

- Idea:
Reading only $O(n + m)$ elements, reduce an array of $n \times m$ candidates to an array of $n/2 \times m/2$ candidates
- Pictorially:



read only $O(n + m)$ elements



Faster than $O(n \log n)$?

- Hypothetical algorithm has recursion:

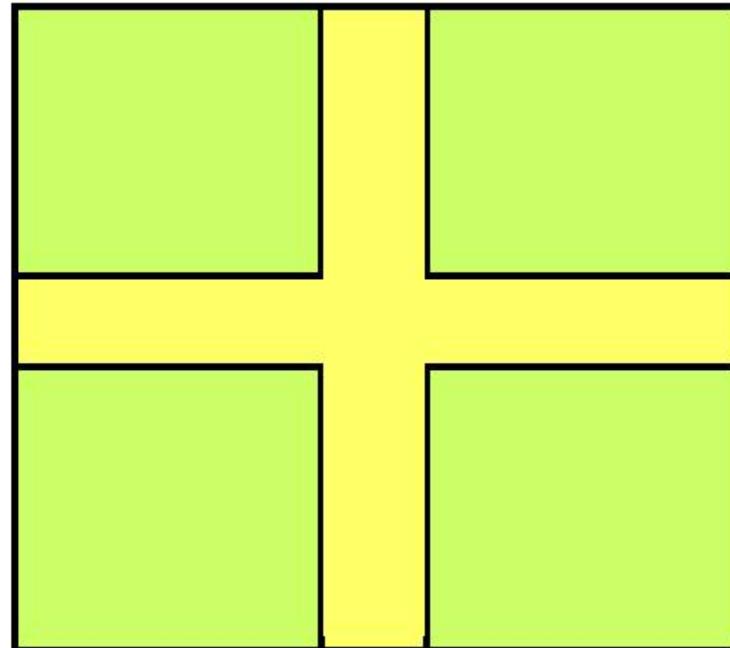
$$T(n, m) = T\left(\frac{n}{2}, \frac{m}{2}\right) + \Theta(n + m)$$

- Hence:
$$\begin{aligned} T(n, m) &= \Theta(n + m) + \Theta\left(\frac{n + m}{2}\right) \\ &\quad + \Theta\left(\frac{n + m}{4}\right) \\ &\quad + \dots + \Theta(1) \\ &= \Theta(n + m) ! \end{aligned}$$

Towards a linear-time algorithm



What elements are useful to check?

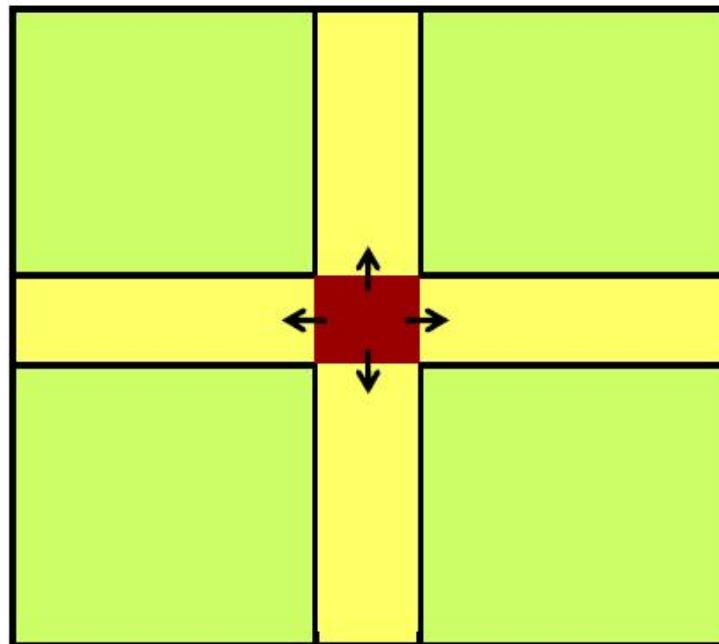


- suppose we find global
max on the cross



Towards a linear-time algorithm

What elements are useful to check?



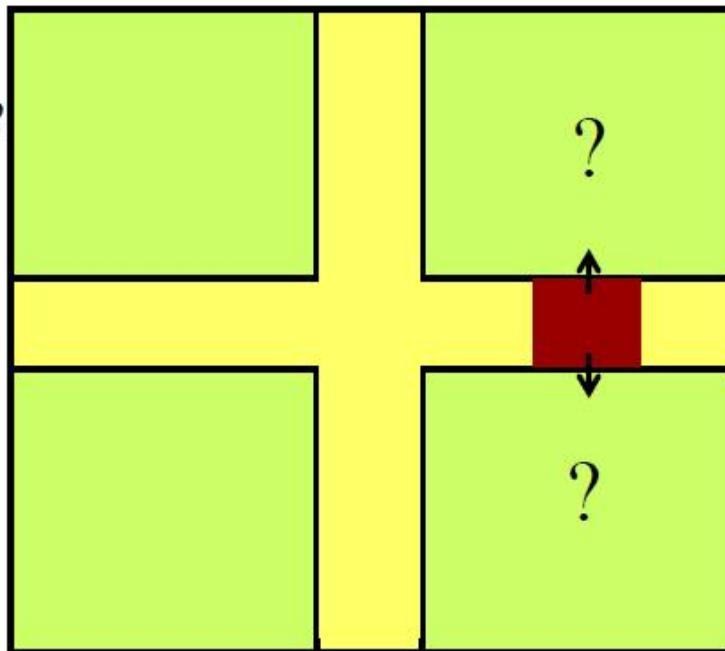
- suppose we find global max on the cross
- if middle element done!

Towards a linear-time algorithm



What elements are useful to check?

*proof of claim 2
and fix to this
algorithm
provided in
recitation*



- find global max on the cross
- if middle element done!
- o.w. two candidate sub-squares
- determine which one to pick by looking at its neighbors not on the cross (as in Algorithm II)

Claim: The sub-square chosen by the above procedure (if any), always contains a peak of the large square.

BUT: Claim 2: Not every peak of the chosen sub-square is necessarily a peak of the large square. Hence, it is hard to recurse...



Divide & Conquer #0

- Looking at center element doesn't split the problem into pieces...

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 4 | 5 | 6 | 4 | 9 | 8 |
| 3 | 2 | 3 | 1 | 4 | 0 | 3 |
| 9 | 3 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |



Divide & Conquer #1/2

- Consider max element in each column
- 1D algorithm would solve max array in $O(\lg n)$ time
- But $\Theta(n)$ time to compute max array

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |





Divide & Conquer #1

- Look at center column
- Find global max within
- If peak: return it
- Else:
 - Larger left/right neighbor
 - Larger max in that column
 - Recurse in left/right half
- Base case: 1 column
 - Return global max within

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 3 | 5 | 9 | 8 |
|---|---|---|---|---|---|---|



Analysis #1

- $O(n)$ time to find max in column
 - $O(\lg n)$ iterations (like binary search)
 - $O(n \lg n)$ time total
-
- Can we do better?

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

Divide & Conquer #2

- Look at boundary, center row, and center column (**window**)
 - Find global max within
 - If it's a peak: return it
 - Else:
 - Find larger neighbor
 - Can't be in window
 - Recurse in quadrant, including green bounda



Correctness

- Lemma: If you enter a quadrant, it contains a peak of the overall array [climb up]
- Invariant: Maximum element of window never decreases as we descend in recursion
- Theorem: Peak in visited quadrant is also peak in overall array

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 3 | 5 | 2 | 4 | 9 | 8 | 0 |
| 0 | 7 | 2 | 5 | 1 | 4 | 0 | 3 | 0 |
| 0 | 9 | 8 | 9 | 3 | 2 | 4 | 8 | 0 |
| 0 | 7 | 6 | 3 | 1 | 3 | 2 | 3 | 0 |
| 0 | 9 | 0 | 6 | 0 | 4 | 6 | 4 | 0 |
| 0 | 8 | 9 | 8 | 0 | 5 | 3 | 0 | 0 |
| 0 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

→ proofs in recitation

Analysis #2



- Reduce $n \times n$ matrix to $\sim \frac{n}{2} \times \frac{n}{2}$ submatrix in $O(n)$ time (|window|)

$$T(n) = T\left(\frac{n}{2}\right) + c n$$

$$T(n) = T\left(\frac{n}{4}\right) + c \frac{n}{2} + c n$$

$$T(n) = T\left(\frac{n}{8}\right) + c \frac{n}{4} + c \frac{n}{2} + c n$$

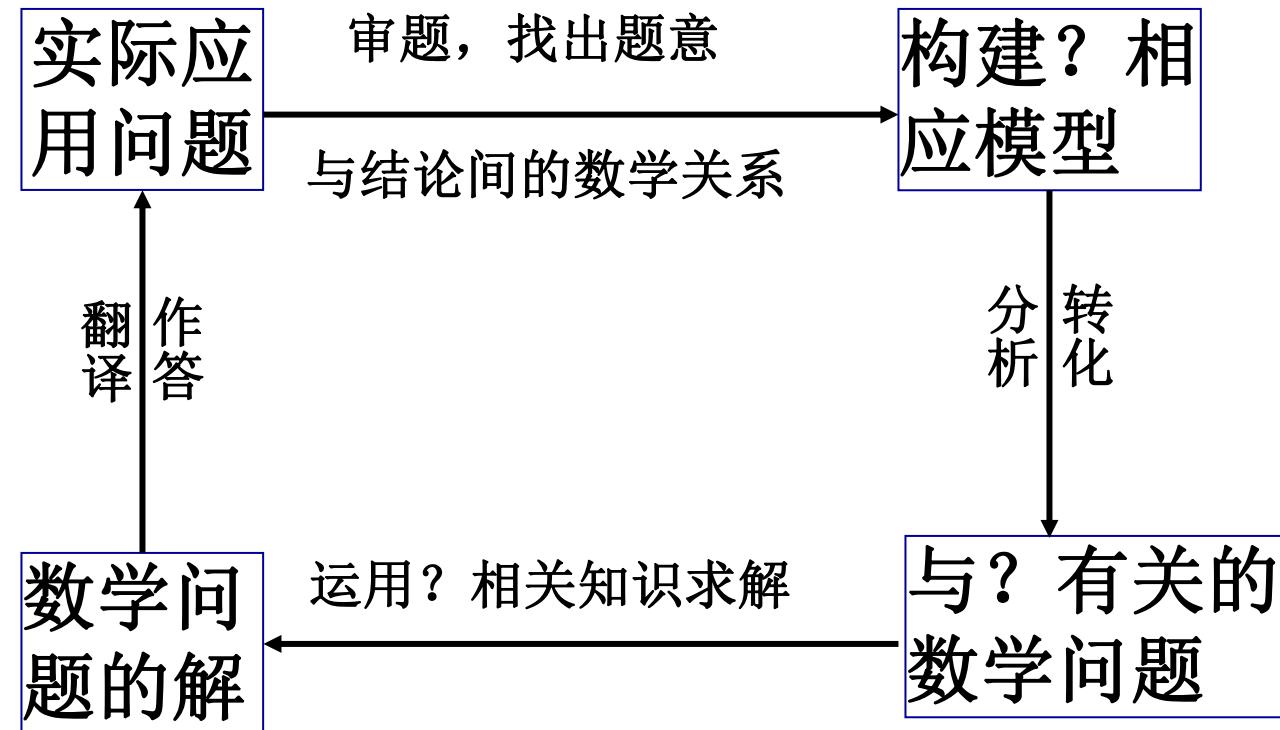
$$T(n) = T(1) + c \left(1 + 2 + 4 + \dots + \frac{n}{4} + \frac{n}{2} + n\right)$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 3 | 5 | 2 | 4 | 9 | 8 | 0 | 0 |
| 0 | 7 | 2 | 5 | 1 | 4 | 0 | 3 | 0 | 0 |
| 0 | 9 | 8 | 9 | 3 | 2 | 4 | 8 | 0 | 0 |
| 0 | 7 | 6 | 3 | 1 | 3 | 2 | 3 | 0 | 0 |
| 0 | 9 | 0 | 6 | 0 | 4 | 6 | 4 | 0 | 0 |
| 0 | 8 | 9 | 8 | 0 | 5 | 3 | 0 | 0 | 0 |
| 0 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$\Theta(n)$



第四章：小结





湖南大學
HUNAN UNIVERSITY

感谢观赏

—— 实事求是 敢为人先 ——