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How to Not Fail
Control Systems

While never going to class

Computer Engineering 2020 $\boxed{\mathcal{BM}}$

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1 Modeling in the Frequency Domain

2 Modeling in the Time Domain

3 Time Response

4 Reduction of Multiple Systems

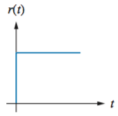
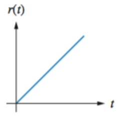
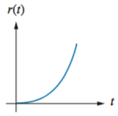
5 Stability

5.1 Routh-Hurwitz Criteria

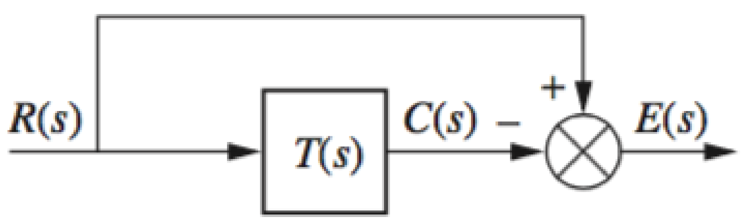
5.2 Routh-Hurwitz Special Cases

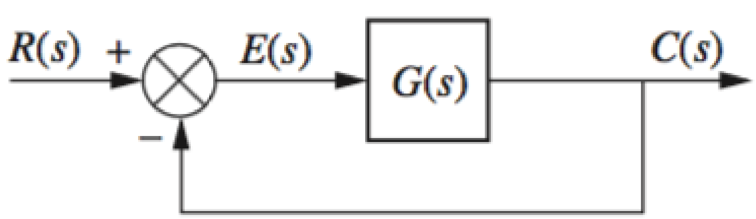
6 Steady State Errors

Steady State Error is defined as the difference between the input and output as t $\rightarrow \infty$. When testing for factors such as constant position, constant velocity and constant acceleration, inputs such as unit steps $u(t)$, ramps $r(t)$ and parabolas are used. This discussion is limited to stable systems.

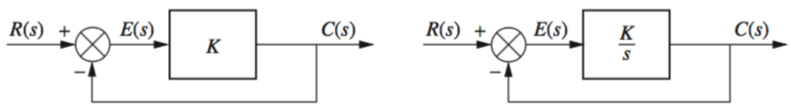
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Most steady state errors $E(s)$ arise from the input and/or the configuration of the system, as seen in the diagrams below for general closed loop and unity feedback systems.





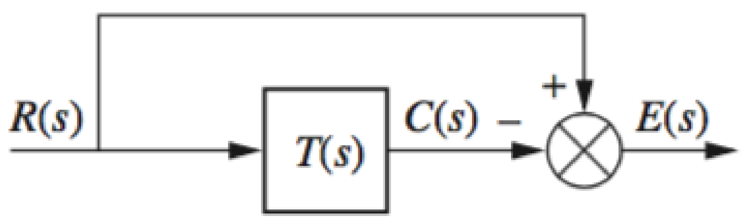
In the first case $E(s) = R(s) - C(s)$ is the error. If the input $R(s)$ is a step input, then $C(s)$ should $= R(s)$ and $E(s) = 0$. However, if gain K is introduced, $C(s) = KR(s)$ and $E(s)$ must be finite and non-zero.



From these systems we see that $C(s) = KE(s)$, or $E(s) = \frac{1}{K}C(s)$

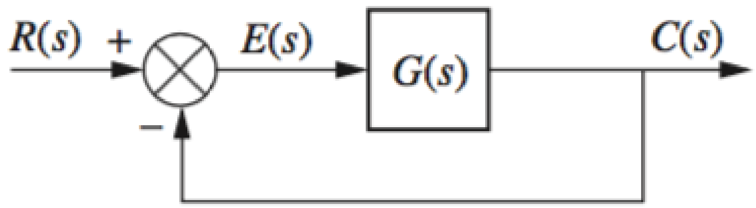
6.1 Steady State Error for Unity Feedback Systems

Steady-state error can be calculated from transfer function $T(s)$ or the open loop transfer function $G(s)$. Once $E(s)$ is found, the steady state error can be found using the *Final Value Theorem*, which states that the value at infinity is equal to the Laplace as $s \to 0$.



$$E(s) = R(s) - C(s)$$
$$C(s) = R(s)T(s)$$
$$E(s) = R(s)[1 - T(s)]$$
$$e(\infty) = \text{lim}_{s\to\infty} e(t) = \text{lim}_{s\to 0} sE(s)$$
$$e(\infty) = \text{lim}_{s\to 0} sR(s)[1 - T(s)]$$

6.1.1 Steady State Error in Terms of G(s)



$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s)$$
$$E(s) = \frac{R(s)}{1+G(s)}$$
$$e(\infty) = \lim_{s\to\infty} \frac{sR(s)}{1+G(s)}$$

6.2 Static Error Constants and System Type

The steady-state error for unit step inputs is

$e(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)}$

The steady-state error for ramp inputs of unit velocity is

$e(\infty) = \frac{1}{\lim_{s\to 0} sG(s)}$

The steady-state error for parabolic inputs of unit acceleration is

$e(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)}$

The terms in the denominator are known as k_p, k_v, k_a, the **static error constants**, representing position, velocity and acceleration, respectively.

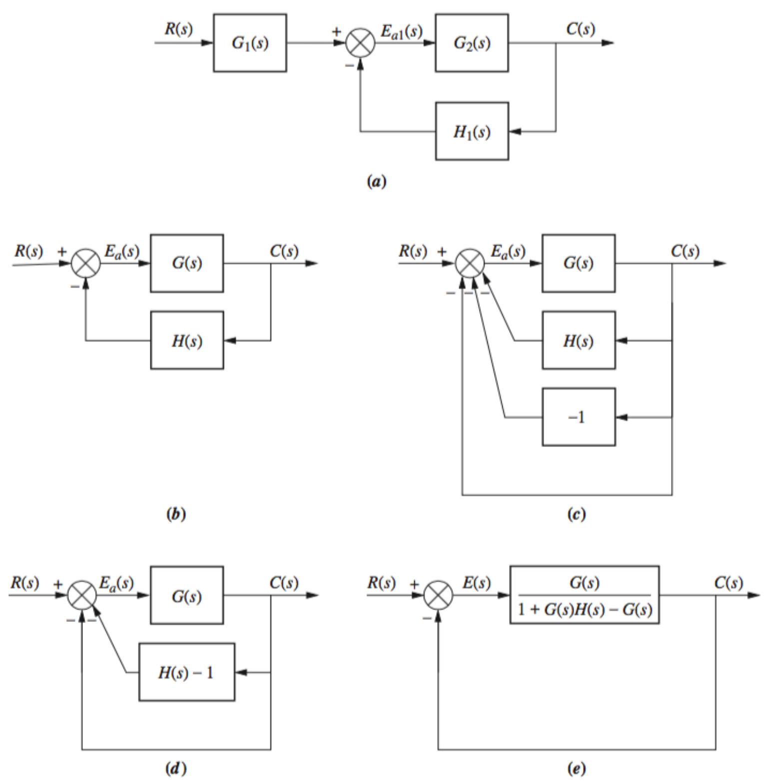
Systems can also be defined by **system type**. This defines the number of pure integrations in the forward path, assuming a unity feedback system. Increasing the system type decreases the steady-state error as long as the system is stable.

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

6.3 Steady-State Error Specifications

The steady-state error is inversely proportional to the static error constant - the larger the constant, the smaller the steady-state erorr. Increasing gain increases the static error constant, thus, increasing the gain decreases the steady-state error if the system is stable.

6.4 Forming Equivalent Unity from Non-Unity Systems



7 Root Locus Techniques

The following sections apply to **Negative Feedback Closed Loop** systems.

7.1 Sketching the Root Locus

7.1.1 Number of Branches

The number of branches in a root locus equal the number of poles.

7.1.2 Symmetry

The root locus is symmetrical about the real axis.

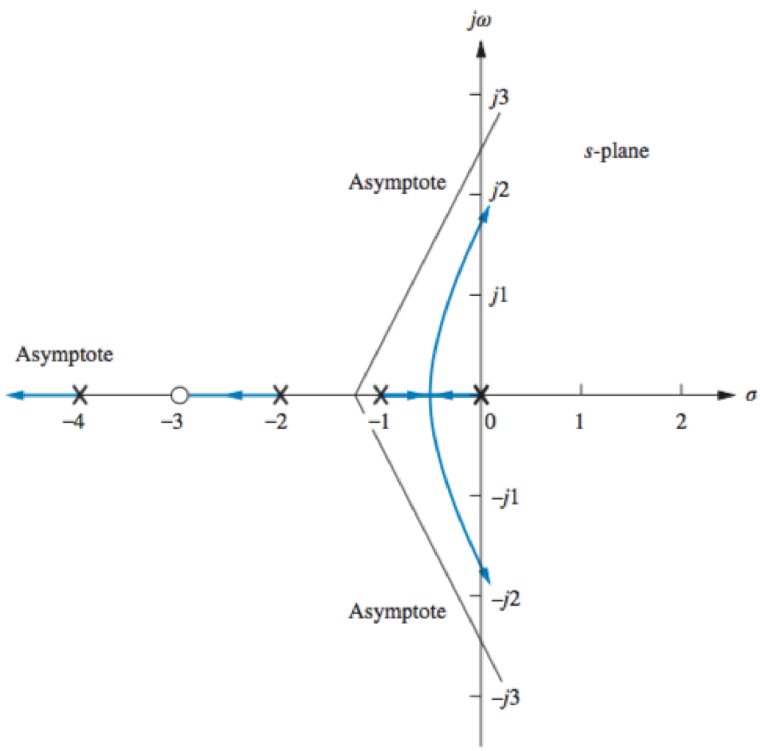
7.1.3 Real Axis Segments

On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

7.1.4 Starting and Ending Points

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

7.1.5 Behavior at Infinity



7.2 Refining the Sketch

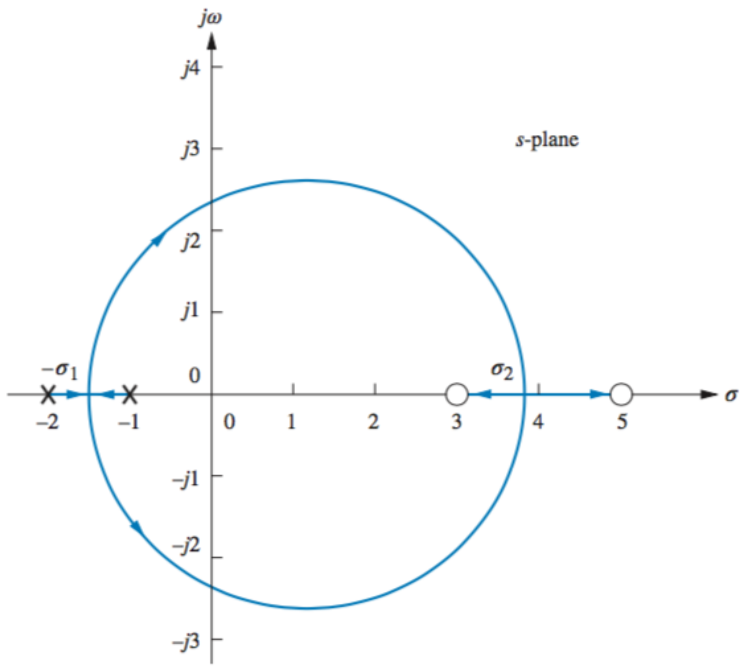
7.2.1 Breakaway/Break-in Point

At the breakaway or break-in point, the branches of the root locus form an angle of $180/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis. These are the points when gain is at its minimum and maximum, respectively.

For all points on the root locus,

$K = -\frac{1}{G(s)H(s)}, \frac{1}{G(s)H(s)} = -1$ and by differential calculus,

$K = -\frac{1}{G(\sigma)H(\sigma)}$ where breakpoints occur - setting σ to 0 will produce K.



Or, conversely,

$\sum \frac{1}{\sigma+z_i} = \sum \frac{1}{\sigma+p_i}$

Where z and p are the zero and pole values. By equating the two sides and simplifying to a single equation, factoring can produce σ.

7.2.2 $j\omega$-Axis Crossings

The crossing of the $j\omega$ axis defines when the system becomes unstable. The crossing of the ω axis deines the frequency of oscillation, while the gain at the $j\omega$ axis yields the maximum positive gain for system stability.

The $j\omega$ axis crossings can be found using the Routh-Hurwitz criterion. Forcing a row of zeros yields the gain, then going back a row and solving for the roots yields the frequency at the imaginary axis crossing.

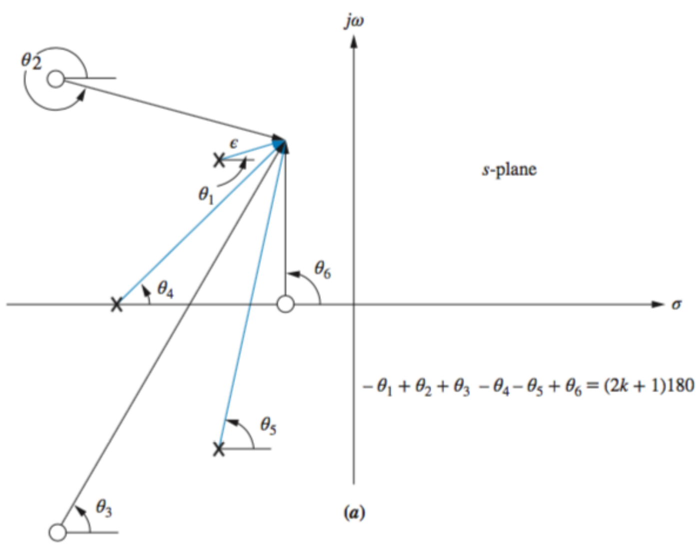
7.2.3 Angles of Departure and Arrival

If we assume a point on the root locus ϵ close to a complex **pole**, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the **pole** that is ϵ close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the **pole** that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the **pole** that is ϵ close. We can solve for this unknown angle, which is also the angle of departure from this complex **pole**.

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180^\circ$$

or

$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)180^\circ$$

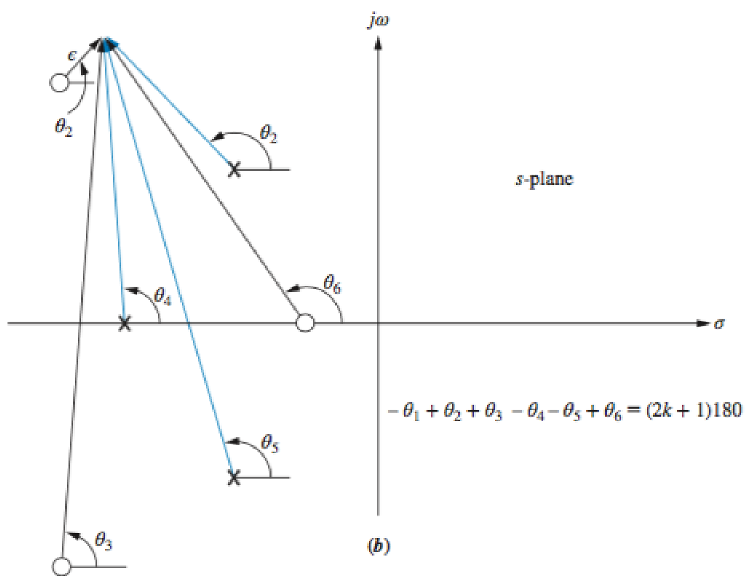


If we assume a point on the root locus ϵ close to a complex **zero**, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the **zero** that is ϵ close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the **zero** that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the **zero** that is ϵ close. We can solve for this unknown angle, which is also the angle of arrival to this complex **zero**.

$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180°$

or

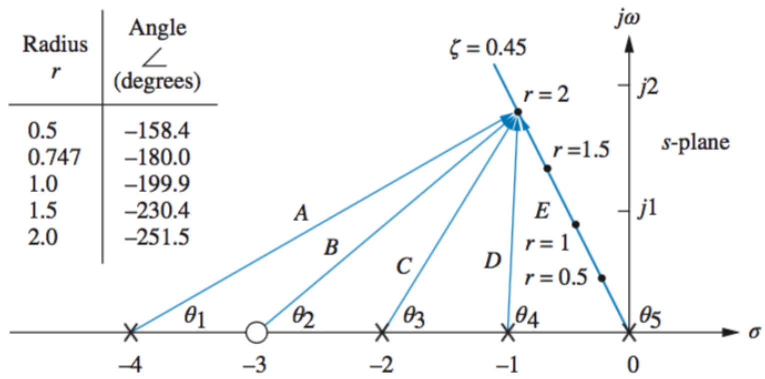
$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 - (2k+1)180°$



7.3 Plotting and Calibrating the Root Locus

When locating points on the root locus and finding their specified gain, for example as it crosses the radial line representing 20% overshoot, such as the below graph where $\zeta = 0.45$

Evaluating the graph at points along the line, and summing the angles from poles and zeros, it can be determined if a point is on the root locus if the angles are a multiple of 180°.

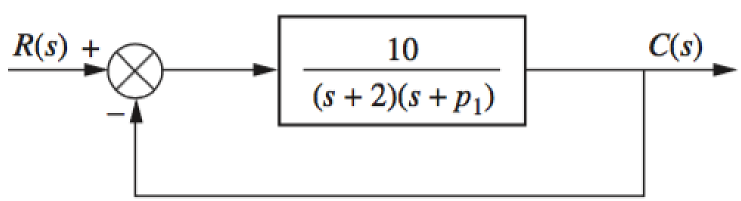


Radius r	Angle $\angle$ (degrees)
0.5	–158.4
0.747	–180.0
1.0	–199.9
1.5	–230.4
2.0	–251.5

7.4 Generalized Root Locus

If finding the root locus of a system concerning a single parameter instead of gain K, an equivalent system can be used where the denominator is represented as $1 + p_1 G(s)H(s)$. From the below system,

$$T(s) = \frac{KG(s)H(s)}{1+KG(s)H(s)} = \frac{10}{s^2+(p_1+2)s+2p_1+10} = \frac{10}{s^2+2s+10+p_1(s+2)}$$



7.5 Positive Feedback Systems

$KG(s)H(s) = 1 = 1\angle k360°$ $\qquad k = 0, 1, 2, 3...$

1. **Number of Branches**
   No change
2. **Symmetry**
   No change

3. **Real Axis Segments**
   On the real axis, the root locus for positive-feedback systems exists to the left of an **even** number of real-axis, finite open-loop poles and/or finite open-loop zeros.

4. **Starting and Ending Points**
   The root locus for positive-feedback systems begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

5. **Behavior at Infinity**
   The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes for positive-feedback systems are given by the real-axis intercept, σ_a, and angle, θ_a, as follows:

$$\sigma_a = \frac{\sum FinitePoles - \sum FiniteZeros}{\#FinitePoles - \#FiniteZeros}$$
$$\theta_a = \frac{k2\pi}{\#FinitePoles - \#FiniteZeros}$$

8 Design via Root Locus

9 Frequency Response Techniques

10 Design via Frequency Response