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How to Not Fail
Probability & Random Processes

While never going to class

Computer Engineering 2020 $\boxed{\mathcal{BM}}$

Contents

1 Descriptive Statistics

1.1 Statistics

Statistics are the summarization of a set of data that has been collected, which demonstrates random variation. *Extracting meaning from data.*

1.2 Inferential Statistics

Making inferences abotu a situation based on data, such as forecasting. Descriptive statistics can be the basis for inferences.

1.3 Representative Values

1. Mean
2. Median
3. Mode
4. Range - *[Min, Max]*
5. Variance - *Average of deviation squared from the mean*
6. Standard Deviation - *Measure of average absolute deviation*
7. Skewness - *Measure of the shape of the distribution funciton*
8. Quantiles - *Generalizatino of the median to percentiles*

1.4 Observational vs. Experimental Data

Experimental involves manipulation objects to determine cause and effect in data. Observational refers to naturally occurring events.

2 Basic Probability

2.1 Probability Calculus

Probability events have a total probability between zero and one.

$$Pr[AnEvent] = 1 \tag{1}$$

1: An event which is sure to happen

The definition of probability for how often an event is observed can be related to the number of repetiions of the experiment.

$$Pr[Heads] = \frac{\text{number } k \text{ of Heads in } N \text{ coin tosses}}{\text{coin tosses}} \tag{2}$$

2: Counting the probability of heads in a set of coin tosses

The larger the number of repetiions, the higher accuracy with which we can predict the likelihood of an event happening.

2.2 Probability Model

2.2.1 Events

Events are elements in the set of possible outcomes in an experiment.

2.2.2 Sample Space

The set of all possible outcomes for an experiment.

$$S = \{1, 2, 3, 4, 5, 6\} \tag{3}$$

3: The sample space for a dice roll

$$A_1 = \{1\}, A_2 = \{1, 2, 5\} \tag{4}$$

4: Subsets containing events in the sample space

The complement of a subset A^C is the subset of all other events in the sample space which are not contained in A.

2.3 Event Algebra

2.3.1 Or

The combination of two or more sets.

For $A = \{1, 2, 5\}$ and $B = \{3, 4\}$, $A_1 + A_2 = \{1, 2, 3, 4, 5\}$

2.3.2 And

The set of events which occur in two or more sets.

For $A = \{1, 2, 3, 5\}$ and $B = \{3, 4\}$, $A_1 A_2 = \{3\}$

Axioms

1. Mutual exclusion: $AA^c = 0$
2. Inclusion: $AS = A$
3. Double complement: $(A^C)^C = A$
4. Commutation: $A_1 + A_2 = A_2 + A_1$
5. Associativity: $A_1 + (A_2 + A_3) = (A_1 + A_2) + A_3$
6. Distributivity: $A_1(A_2 + A_3) = A_1 A_2 + A_1 A_3$
7. DeMorgans Law: $(A_1 A_2)^c = (A_1)^C + (A_2)^C$

2.4 Probability of Events

Axioms

1. For any event A: $Pr[A] \geq 0$
2. $Pr[S] = 1$
3. If A and B are **Mutually Exclusive** then $Pr[A + B] = Pr[A] + Pr[B]$

Mutual Exclusivity refers to the fact that A and B will never occur simultaneously, ie $AB = 0$.

4. Axiom 3 can be extended: $Pr[A_1 + A_2 + ...] = Pr[A_1] + Pr[A_2] + ...$

2.4.1 Non-Mutually Exclusive

In cases where A and B can occur in the same set, Axiom 3 will not apply. This is due to the fact that in overlapping events, the same area of probability will be counted twice, though it has no statistical importance.

$$Pr[A_1 + A_2] = Pr[A_1] + Pr[A_2] - Pr[A_1 A_2] \tag{5}$$

5: Non mutually exclusive OR

2.4.2 Complement of an event

From expanding on these axioms, it can be seen that the complement of an event has a probability related to subtraction of itself from the sample space probability. If the chance of the event happening is known, the chance of an event not happening is found by subtracting this from absolute certainty.

$$Pr[A^c] = 1 - Pr[A] \tag{6}$$

6: The complement of a set versus the whole

2.4.3 Statistical Independence

Two events A and B are said to be statiscally independent if

$$Pr[AB] = Pr[A]Pr[B] \tag{7}$$

7: Statistical independence

This refers to the fact that statistical data will happen independent of the preceding events. If you flip a coin, the probability of the next coin will be the same. If you take items from a bin, the probability of the next item being picked will go up, and is therefore dependent.

2.5 Repeated Independent Trials

From the rule of statistical independence, we can process repeated trials:

Coin Flips

$Pr[H] = p$

$Pr[T] = Pr[H^c] = 1 - p = q$

$Pr[HHH] = ppp = p^3$

$Pr[HTH] = pqp = p^2q$

Knowing that the probability of either event is 0.5, we can take the list of possible outcomes and calculate the probability.

$Pr[\textit{At least two heads}] = Pr[HHH] + Pr[HHT] + Pr[HTH] + Pr[THH]$

$= ppp + ppq + pqp + qpp$

$= (0.5)^3 + (0.5)^2 * (0.5) + (0.5)^2 * (0.5) + (0.5) * (0.5)^2$

$= 0.5$

$Pr[\textit{No heads}] = Pr[TTT] = q^3 = 0.125]$

2.5.1 Sampling With Replacement

In this case, we consider an event where an event occurring does not subtract from a finite amount of events, ie a coin flip. In the case of heads or tails, there aren't one less heads or tails. So it is as if we replace the event in our sample space.

2.5.2 Sampling Without Replacement

Finite amounts of events that can be subtracted from the whole. If this event happens, it won't happen again, as if we have taken our card from a deck of cards and not placed it back in the deck.

2.5.3 Order of Outcomes When Sampling Without Replacement

Cases when its important what order events occur in. Did we draw the Ace of Spades within the first 3 draws? The number of each event is not important.

K-Tuples

When a trial is repeated k times, we form a sample space of outcomes made up of k number of events.

2.5.4 The Rule of Product

How many possibilities are there for the formation of k-tuples, if there are N_i choices for the ith element?

$$N_1 * N_2 * ...N_k \tag{8}$$

8: The Rule of Product

By this rule, the number of possibilities when rolling a dice, then flipping a coin, will be $6 * 2$. In the case where same number of outcomes are possible with each experiment, the number of k-tuples is N^k, or 2^k in the case of repeated coin tosses.

2.5.5 Permutations or Unordered Outcomes

We no longer care in what order outcomes occur, we are only concerned with the number of outcomes of a certain sort across all trials.

This involves the number of ways we can choose k objects in N choices.

Permutations Without Replacement

- Experiments with two or more possible outcomes
- These trials can be repeated independently for N times
- For each kth trial the outcome from the previous is removed

- Probabilities change for each consecutive trial

The resulting set is ordered, but as mentioned before, we only care about the number of possible permutations from these elements.

The number of possible sets is $N!$ or $N * (N-1) * (N-2) * ...2 * 1$

Example - The 13 cards of a suit in a deck of cards can be laid out in $13!$ or 6227020800 different ordered sequences.

If you want just k draws from N possible ways to draw the object, the number of sets will instead be from N to $(N-k+1)$

Consider this problem - Lisa has 13 different ornaments and wants to put 4 ornaments on her mantle. In how many ways is this possible?

Using the product rule, Lisa has 13 choices for which ornament to put in the first position, 12 for the second position, 11 for the third position, and 10 for the fourth position. So the total number of choices she has is $13*12*11*10$. Using the factorial notation, the total number of choices is $\frac{13!}{9!}$.

From this example, we can see that if we have N objects and want to arrange k of them in a row, there are $\frac{N!}{(N-k)!}$ ways to do this.

The notation for permutations is P_3^{13}

2.5.6 Combinations or Non-Unique Outcomes

A *combination* is a way of choosing elements from a set in which order does not matter.

Consider the following example: Lisa has 13 different ornaments and she wants to give 3 ornaments to her mom as a birthday gift (the order of the gifts does not matter). How many ways can she do this?

We can think of Lisa giving her mom a first ornament, a second ornament, a third ornament, etc. This can be done in P_3^{13} ways. However, Lisa's mom is receiving all three ornaments at once, so the order Lisa decides on the ornaments does not matter. There are $3!$ reorderings of the chosen ornaments, implying the total number of ways for Lisa to give her mom an unordered set of 5 ornaments is $\frac{13!}{10!3!}$.

$$\frac{N!}{N!(N!-k!)} \tag{9}$$

9: Rule of Combinations or Unordered Permutations

The notation for combinations is C_k^N **=** $\binom{N}{k}$

The number of ways to choose k objects *in any order* from a set of N objects.

2.6 Conditional Probability

A **conditional probability** is a probability that a certain event will occur given some knowledge about the outcome or some other event.

$P[A|B]$ is a conditional probability, it is read as "Probability of A given B".

$$Pr[A|B] = \frac{Pr[AB]}{Pr[B]} \tag{10}$$

10: Rule of Conditional Probability

A simple example - A fair 12-sided die is rolled. What is the probability that the roll is a 3 given that the roll is odd?

This is $Pr[3|Odd]$ or $\frac{Pr[3]Pr[Odd]}{Pr[Odd]}$

Because B has already happened, the intersection of B and A can have the B probability removed, because it is statistically redundant.

$$Pr[A|B] = \frac{Pr[AB]}{Pr[B]} = \frac{Pr[B]Pr[A]}{Pr[A]} = Pr[B] \tag{11}$$

11: Conditional Probability if statiscally independent

2.6.1 Bayes Theorem

When attempting to compute the conditional probability of two events, when only one event is known, the Bayes Theorem allows for a workaround.

Consider H to be Hypothesis, and E to be Evidence

$$Pr[H|E] = \frac{Pr[E|H]Pr[H]}{Pr[E]} \tag{12}$$

12: Bayes Theorem

We can expand the equation in the numerator to demonstrate fully:

$Pr[E|H]Pr[H] = \frac{Pr[EH]}{Pr[H]} * Pr[H] = Pr[EH] = Pr[HE]$

Therefore, $Pr[H|E]$ or $\frac{Pr[HE]}{Pr[E]}$ can be found from $Pr[E|H]$ and vice versa.

2.6.2 Total Probability

If A_1, A_2, and A_3 form a partition of the sample space, for each A_i

$$Pr[A_i|B] = \frac{Pr[B|A_i]Pr[A_i]}{Pr[B]}, i = 1, 2, 3 \tag{13}$$

13: Total Probability

Knowing this, $Pr[B]$ can be found from $Pr[B|A_1] + Pr[B|A_2] + Pr[B|A_3]$