

Scale-free networks beyond power-law degree distribution

Xiangyi Meng^{a,b}, Bin Zhou^{c,*}

^a Network Science Institute and Department of Physics, Northeastern University, Boston, MA 02115, USA

^b Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA

^c School of Management Science and Engineering, Nanjing University of Information Science and Technology, Nanjing, 210044, China

ARTICLE INFO

Dataset link: <https://athinagroup.eng.uci.edu, https://vlado.fmf.uni-lj.si/pub/networks/data/>

Keywords:

Complex networks

Scale-free

Power-law

Degree distribution

Degree-degree distance distribution

ABSTRACT

Complex networks across various fields are often considered to be scale free—a statistical property usually solely characterized by a power-law distribution of the nodes' degree k . However, this characterization is incomplete. In real-world networks, the distribution of the degree-degree distance η , a simple link-based metric of network connectivity similar to k , appears to exhibit a stronger power-law distribution than k . While offering an alternative characterization of scale-freeness, the discovery of η raises a fundamental question: do the power laws of k and η represent the same scale-freeness? To address this question, here we investigate the exact asymptotic relationship between the distributions of k and η , proving that every network with a power-law distribution of k also has a power-law distribution of η , but *not* vice versa. This prompts us to introduce two network models as counterexamples that have a power-law distribution of η but not k , constructed using the preferential attachment and fitness mechanisms, respectively. Both models show promising accuracy by fitting only one model parameter each when modeling real-world networks. Our findings suggest that η is a more suitable indicator of scale-freeness and can provide a deeper understanding of the universality and underlying mechanisms of scale-free networks.

1. Introduction

A variety of real-world complex systems, from virtual social platforms to physical neuronal connectomes, can be modeled as complex networks [1]. Although the precise definition may vary, a complex network is typically a graph or network (as physicists call it) with the nontrivial complexity that is not present in simple, regular graphs like lattices but is often present in real systems, described by some general, appealing laws [2]. One of such laws is the *scale-freeness*, denoting the self-similarity of a complex system across various scales [3]. When the term “scale-free” was first adopted to networks [4], it was used to denote the unique property of a power-law probability distribution of some variable k , $f(k) \propto k^{-\alpha}$, such that $f(k)$ is invariant (free) under the continuous scale transformation $k \rightarrow k + \varepsilon k$. This is reminiscent of the idea of renormalization group developed in the statistical field theory [5,6]. In contrast to statistical field theory, however, where a “scale” explicitly refers to system size, the “scale” k in networks refers to connectivity, usually specified as the degree of each node, i.e., the number of links connected to the node. A network with more high-degree nodes has higher connectivity. A power-law degree distribution (DD) allows us to make “scale-free” statements on the relative abundance of high-degree nodes, such as “nodes with triple the normal connectivity are half as common as nodes with normal connectivity”.

The actual value of “normal connectivity” is irrelevant—whether it is a hundred or a million degrees [7].

The concept of scale-free networks has since seen significant theoretical advancements in the last two decades [8–15]. Due to their simplicity and general applicability, scale-free networks have been adopted as a testing ground for theories such as percolation [16] or dynamical system [17] on networks. An example is percolation on networks that have a power-law DD, which predicts that when the power-law exponent α is less than 3, a cluster spanning the whole network will almost always exist, regardless of the fraction of nodes that break down. Applying the result to the Internet (for which $\alpha \approx 2.5$ [4]), one can estimate that over 99% of the nodes must be destroyed before the spanning cluster collapses [16], demonstrating the extreme robustness of the Internet under random attacks.

When applying the theoretical results to the empirical realm, however, there is a caveat: an ideal, continuous power-law DD would not be normalizable in the full domain $k \in [0, \infty)$. Thus, the DD cannot be truly power law but requires a small- k “ultraviolet” (UV) cutoff, in terms of either a k_{\min} (the minimum degree each node can have) or other nontrivial corrections near a finite positive k . The DD only regains its scale invariance asymptotically in the “infrared” (IR) limit as $k \rightarrow$

* Corresponding author.

E-mail address: binzhou@mail.ustc.edu.cn (B. Zhou).

∞ . Therefore, a finite-size network can only be approximately scale-free [18]. This raises concerns about whether real-world networks, which are finite by nature, can still be considered as “scale-free” networks.

Real-world networks from diverse fields, such as biology [19,20], social science [21,22], and information technology [23,24], have been commonly regarded as scale-free, mostly based on the observation of their power-law DD. This supports the claim that “scale-free networks are universal [4]”. Yet, this claim remains a topic of debate in recent literature [25–28]. The controversy is rooted in the challenge of accurately determining the range of k values to be used for fitting a power law, which requires prior knowledge of the small- k UV cutoff. Fitting a range of k values that is either too large or too small leads to substantial inaccuracies [29]. One approach to simplify the analysis is to statistically test the full domain of k , ignoring possible UV cutoff. However, this oversimplified approach resulted in that more than two thirds of real-world networks do not have a statistically significant power-law DD [30].

To circumvent the challenge of determining the fitting range, an alternative solution was proposed [30]. Instead of the degree k , a link-based metric, the *degree-degree distance* [30], was adopted. The degree-degree distance η is defined for every link $i \leftrightarrow j$ that connects two nodes i and j , given by

$$\eta = \exp \left| \ln k_i - \ln k_j \right|, \quad (1)$$

which is determined solely by the network topology. Unlike k , which measures the degree of connectivity, η measures the imbalance of connectivity. Increased presence of high- η links (i.e., high-degree nodes adjacent to low-degree nodes) corresponds to more imbalanced connectivity in the network, which can lead to higher heterogeneity or lower flow capacity. The characteristics of η have since been linked to broader topics [31–34], such as network assortativity [31] and closeness [34]. In addition, many node-based metrics that were originally defined through k , such as centrality and clustering coefficient, can be redefined or generalized using η , offering new possibilities for designing network metrics.

When fitting to power laws, η is found to be less susceptible to the UV cutoff. It turned out that around 60% of real-world networks have a statistically significant power-law degree-degree distance distribution (DDDD) when fitting the full range of η [30]. The higher statistical significance hints that many real-world networks can, indeed, be considered as scale free, when characterized by the power law of DDDD instead of DD. This finding raises a fundamental question: *do the power laws of DD and DDDD represent the same scale-freeness?* Or, more generally speaking, is there only one scale to determine whether a network is scale-free or not? If the answer is affirmative, then the power law of DDDD, $g(\eta) \sim \eta^{-\beta}$, would just act as a “proxy” for the power law of DD, $f(k) \sim k^{-\alpha}$, both representing the same underlying scale-free property of complex networks.

Here, however, we show that the power law of DDDD is more than a “proxy”. Our main result [Eq. (2)] is that the set of networks with an asymptotic power-law DD is a proper subset of those with an asymptotic power-law DDDD. When both DD and DDDD are power laws, we find an exact relationship between their power-law exponents, $\beta = \alpha - 1$; meanwhile, we also find that there are networks whose DD is not power law, but DDDD is. Our result indicates that the current understanding of scale-free networks, which is based solely on the power law of DD, is incomplete.

Recently, there have been an increasing number of scholars [26, 30,35] advocating for the attribution of the scale-free property to a network mechanism, rather than a specific metric. Two extensively discussed scale-free mechanisms in the literature [36] are the preferential attachment [4] and the fitness [37] mechanisms. Interestingly, we will show that by employing either of these mechanisms, network models that do not have a power-law DD but have a power-law DDDD can be built. The resulted counterexamples—representing scale-free networks that go beyond having a power-law DD—show promising accuracy when modeling real-world networks.

2. Results

Our main result can be written as

$$\mathcal{D}^2|_{\text{power-law}} \subsetneq \mathcal{D}^4|_{\text{power-law}}, \quad (2)$$

where $\mathcal{D}^2|_{\text{power-law}}$ denotes the set of all network models that have a power-law DD in the asymptotic limit $k \rightarrow \infty$, and $\mathcal{D}^4|_{\text{power-law}}$ for power-law DDDD in the asymptotic limit $\eta \rightarrow \infty$. We derive Eq. (2) in two steps, proving its (i) inclusion and (ii) strict inequality as follows:

(i) $\mathcal{D}^2|_{\text{power-law}} \subseteq \mathcal{D}^4|_{\text{power-law}}$, i.e., *every network with a power-law DD also has a power-law DDDD*. To prove this, we start by showing that given any power-law DD $f(k) \sim k^{-\alpha}$, the DDDD of the network can always be represented by two identical copies of the DD using the copula theory [38]. Let $\mathcal{P}(\{k_i, k_j\} | i \leftrightarrow j)$ be the conditional joint probability of sequentially selecting two nodes i and j that have degrees $k_i, k_j \in [k_{\min}, \infty)$, respectively, conditioned on i and j being connected. This implies that $\mathcal{P}(\{k_i, k_j\} | i \leftrightarrow j)$ can be understood as half the probability of selecting a link (from all links) that connects two nodes of degrees k_i and k_j (half because of the symmetry between k_i and k_j). Its marginal distribution on k_i can be obtained by integrating over k_j . This marginal distribution can be understood as the probability of selecting a link that is connected to a node of degree k_i . As a result, the marginal distribution is proportional to the DD, $f(k_i)$, times k_i , since a node with degree k_i has a k_i times greater chance of being selected. When $f(k_i) \propto k_i^{-\alpha}$, one has $\int dk_j \mathcal{P}(\{k_i, k_j\} | i \leftrightarrow j) \propto k_i f(k_i) \sim k_i^{-\alpha+1}$. Thus, by Sklar’s theorem [39], one can reversely write

$$\mathcal{P}(\{k_i, k_j\} | i \leftrightarrow j) \sim k_i^{-\alpha+1} k_j^{-\alpha+1} c\left(\frac{k_i^{-\alpha+2}}{k_{\min}^{-\alpha+2}}, \frac{k_j^{-\alpha+2}}{k_{\min}^{-\alpha+2}}\right), \quad (3)$$

in terms of the marginal distributions $k_i^{-\alpha+1}$ and $k_j^{-\alpha+1}$ multiplied by a copula density $c(x, y) \geq 0$ defined on $(x, y) \in [0, 1]^2$. Following Ref. [30] (also see SI), the DDDD is given by

$$g(\eta) = \int_{k_{\min}}^{\infty} 2\mathcal{P}(\{k_i, \eta k_i\} | i \leftrightarrow j) k_i dk_i. \quad (4)$$

Inserting Eq. (3) into Eq. (4) yields the DDDD,

$$g(\eta) \sim \int_{k_{\min}}^{\infty} k_i^{-2\alpha+2} \eta^{-\alpha+1} \left(c\left(\frac{k_i^{-\alpha+2}}{k_{\min}^{-\alpha+2}}, 0\right) + O(\eta^{-\alpha+2}) \right) k_i dk_i$$

for large η , which always contains a positive power-law leading term $\sim \eta^{-\alpha+1} = \eta^{-\beta}$ when $\alpha > 2$, provided that the marginal copula density $c(x, 0) > 0$ on $x \in (0, 1)$, satisfied as long as the copula density is smooth. We thus confirm the relationship $\beta = \alpha - 1$ as empirically observed [30].

(ii) $\mathcal{D}^2|_{\text{power-law}} \neq \mathcal{D}^4|_{\text{power-law}}$, i.e., *there are networks that do not have a power-law DD but exhibit a power-law DDDD*. To prove this, it suffices to provide a counterexample with a power-law DDDD but not a power-law DD. Here, we present two network models that serve as counterexamples. These models provide theoretical insights through two very distinct mechanisms (preferential attachment vs. fitness), both having practical applications for modeling real-world networks.

2.1. Preferential attachment of internal links

Our first model is a “no-growth” preferential attachment model, where the number of nodes N is fixed, and only internal links (connections between existing nodes) are added to the network during its evolution. At every time step t , the model randomly selects two nodes i and j , creating a link between them. The probability of selecting these two nodes is $\propto (k_i + b)(k_j + b)$, where b is a small constant. The network, initially empty, acquires T links after T time steps. Although we allow multilinks so that T can technically be larger than $N(N-1)/2$, our focus is on the limiting regime $T \sim N^{1+\delta}$ where δ is a small nonnegative number. In this regime, we assume that multilinks become negligible as $N \rightarrow \infty$. Similar to the Barabási-Albert (BA) model [4], the no-growth model exhibits the “rich-gets-richer” social phenomenon

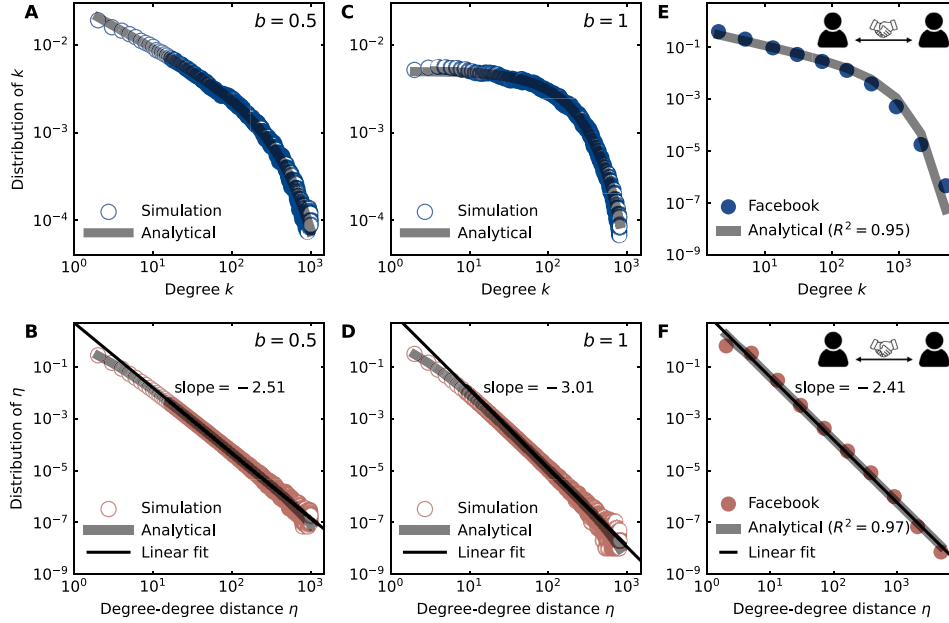


Fig. 1. Distributions of (a,c) k and (b,d) η for the “no-growth” model, a network model of a fixed number of nodes, between which each internal link is added at each time step following the preferential attachment rule. The model simulation results (circle) are given by the average of 10^2 runs on $N = 10^4$ nodes and $T = 10^6$ links. (e,f) Distributions of k and η of a subnetwork of Facebook, consisting of $N \sim 10^6$ nodes (users) and $T \sim 10^8$ links (user followerships). Both k and η are fitted by the no-growth model with $b = 0.41$ under the same N and T , revealing that the Facebook network exhibits a power-law distribution of η rather than k .

due to the linear dependence of the preferential attachment probability on the node degree. However, unlike the BA model, it does not capture the “first-mover advantage” phenomenon that is typically observed in marketing and business [40].

To solve the model, we denote its DD by $f(T, k)$, the probability that a node has degree k at time step T . The Markovian rate equation for $f(T, k)$ in the continuum limit can be written as (see SI)

$$T' \frac{\partial f}{\partial T'} = -\frac{\partial}{\partial k'} (k' f), \quad (5)$$

where $T' = T + bN/2$ and $k' = k + b$, both converging to T and k as $b \rightarrow 0$. When b is small, we find that the solution of Eq. (5) approximately follows a Gamma distribution for $k \in [0, \infty)$,

$$f(k; A, B(T)) = e^{-B(T)k} k^{A-1} B(T)^A / \Gamma(A). \quad (6)$$

The parameters $A = b$ and $B(T) = bN/(2T)$ are fixed by matching the boundary condition $f(T, 0)$ at $k = 0$ (see SI). When b is small, both the numerical solution of the rate equation [Eq. (5)] and the approximate analytical solution [Eq. (6)] accurately reproduce the model simulation result [Fig. 1(a,c)].

The DD of the no-growth model is far from being a power law: when $b \geq 1$, Eq. (6) exhibits no heavy tail but an exponential upper tail [Fig. 1(c)]; when $b < 1$, Eq. (6) might be interpreted as an almost flat power law plus a finite-size exponential cutoff [Fig. 1(a)], but a power-law exponent $1 - b$ that is less than one is often considered nonphysical [41]. Our result differs from the previous finding that preferential attachment of internal links would yield a power-law DD with an exponent of $2 + b/2$ [42]. This is because the previous study took a different limit under the assumption of stationary growth, which is not applicable to the no-growth model.

Note that Eq. (5) admits dilation symmetries [43] under two scale transformations $T' \rightarrow T' + \epsilon T'$ and $k' \rightarrow k' + \epsilon k'$, respectively. This suggests that the rate equation [Eq. (5)] is intrinsically scale-free. The observation that $f(T, k)$ is not power law is thus a result of spontaneous symmetry breaking: when the initial condition is set to a normalized Dirac function, $f(0, k) = \delta(k)$, it breaks the dilation symmetry in the solution of Eq. (5).

To derive the DDDD, recall that the probability of selecting a link that is connected to a node of degree k is proportional to $k f(k; A, B(T))$. When the DD follows a Gamma distribution [Eq. (6)], this probability is simplified to $f(k; A + 1, B(T))$. Similar to Eq. (3), we can write

$$\mathcal{P}(\{k_i, k_j\} | i \leftrightarrow j) = f(k_i; A + 1, B(T)) f(k_j; A + 1, B(T))$$

as the product of two marginal distributions, where now the copula density term disappears as k_i and k_j are independent from each other. The DDDD at time step T , denoted by $g(T, \eta)$, is thus given by [Eq. (4)]

$$g(T, \eta) = \int_1^\infty 2f(\eta k; A + 1, B(T)) f(k; A + 1, B(T)) k dk \\ = 2B(T)^{2A+2} \eta^A E_{-2A-1}((\eta + 1) B(T)) / \Gamma(A + 1)^2, \quad (7)$$

where $E_n(x)$ is the exponential integral function. This is not yet a power law. When the network is dense, however, expanding Eq. (7) around $T/N \rightarrow \infty$ yields

$$g(T, \eta) \simeq \frac{2\eta^b (\eta + 1)^{-2(b+1)} \Gamma(2(b+1))}{\Gamma(b+1)^2} + O(T^{-2-2b}) \\ \sim \eta^{-2-b} \quad \text{for large } T/N, \text{ large } \eta.$$

In contrast to DD, the DDDD of the no-growth model exhibits a strong power law (when b is small). The analytical solution [Eq. (7)] matches the model simulation result [Fig. 1(b,d)], with a power-law exponent of $2 + b$ as expected.

We tested the empirical validity of the no-growth model by applying it to the social media network, Facebook [44]. We find that both its DD and DDDD are accurately reproduced by the no-growth model under the same N and T [Fig. 1(e,f)]. Remarkably, the accuracy of the model is maintained over the full range of k and η by fitting only one parameter, b . Our result suggests that the no-growth model better captures the underlying mechanism of the Facebook network than the BA model—indeed, while preferential attachment is evident (as users with higher degrees are more attractive to others [45]), as a mature social platform, Facebook is unlikely to experience further growth in its number of users. Therefore, the no-growth model, which does not rely on network growth, is a better fit for the Facebook network.

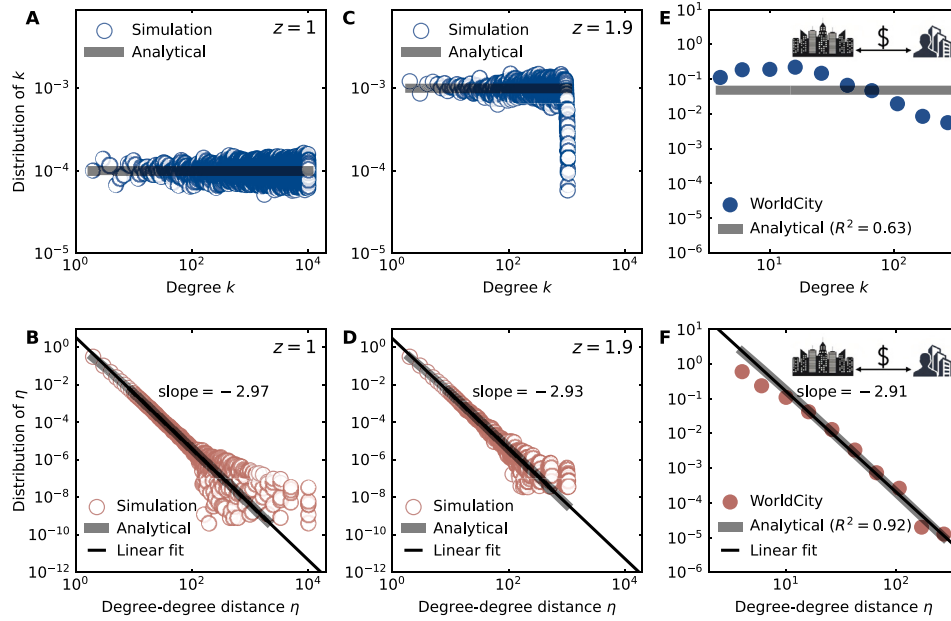


Fig. 2. Distributions of (a,c) k and (b,d) η for the “hard-threshold” model, a network model that assigns a uniformly distributed fitness $\omega \in [0, \omega_{\max} = 1]$ to each node, where two nodes are linked if and only if their fitness sum is larger than a threshold z . The model simulation results (circle) are given by the average of 10^3 runs on $N = 10^4$ nodes. (e,f) Distributions of k and η of the WorldCity network, an urban service network consisting of $N = 415$ nodes (global cities and service firms) and $T = 7518$ links (demand-supply relations between cities and firms). Both k and η are fitted by the hard-threshold model (solid lines) with $z = 1.95\omega_{\max}$ under the same N and T , revealing that the WorldCity network exhibits a power-law distribution of η rather than k .

Note that both the DD and DDDD of the no-growth model depend on the ratio N/T , not on N or T separately. This self-similarity allows us to estimate the properties of the entire network (e.g., Facebook) by analyzing a relatively small subnetwork, as long as the ratio N/T remains the same. The proportionality between N and T is reminiscent of the self-similarity of the BA model, where N/T also remains the same during the growth of the network. Yet, while the BA model only remains in the sparse regime ($N \sim T$), the no-growth model also extends to the dense regime ($N \ll T$). Indeed, the denser the network is, the closer N/T is to zero, and hence the DDDD of the no-growth model is closer to a power law.

Another connection between the no-growth model and the BA model is that when taking $b \rightarrow 0$ in the no-growth model, we find a DDDD of $\sim \eta^{-2}$, which is the same as the DDDD of the BA model (see SI). It seems that both models share the same scale-free property, characterized by the same power-law exponent of DDDD, which is solely attributed to the preferential attachment mechanism. This is different from the power-law DD in the BA model, which is attributed to both preferential attachment and growth [40]. Indeed, a BA model without growth (“model B” [40]) cannot have a power-law DD.

2.2. Fitness with threshold

Our second model is defined by assigning a random fitness ω that follows a fitness distribution $\rho(\omega)$ to each node of the network [46]. For every two nodes i and j , let a link be drawn with probability $\sigma(\omega_i, \omega_j)$ that depends only on the fitness of the nodes, not their degrees [37]. By choosing $\sigma(\omega_i, \omega_j) = \theta(\omega_i + \omega_j - z)$, where $\theta(x)$ is the Heaviside step function, a “hard threshold” of fitness is introduced, such that a link will be deterministically drawn if and only if the sum of the fitnesses of the two nodes is larger than a threshold z . As a result, the network topology is only determined by quenched disorder from the randomness of fitness, not by disorder of network evolution [37]. This hard thresholding rule simplifies success or failure as a binary event and emphasizes the collaborative nature of success, which depends solely on the combined capabilities of both parties involved.

The hard-threshold model was first solved for $\rho(\omega) = e^{-\omega}$ (denoting an exponential fitness distribution), for which the DD is found to be $f(k) \sim k^{-2}$, with a power-law exponent of two [47]. However, the DD is not power law in general for arbitrary $\rho(\omega)$. Here, we consider a uniform distribution $\rho(\omega)$ for $\omega \in [0, \omega_{\max}]$ and assume that $\omega_{\max} \leq z \leq 2\omega_{\max}$. The conditional probability that a node has degree k given fitness ω_i is given by

$$\mathcal{P}(k|\omega_i) \simeq \binom{N}{k} p(\omega_i)^k (1 - p(\omega_i))^{N-k}, \quad (8)$$

where $p(\omega_i) = \max\{1 - (z - \omega_i) / \omega_{\max}, 0\}$ is the probability that another selected node j has a fitness ω_j larger than $z - \omega_i$ so that nodes i and j are linked. Approximating Eq. (8) by a Gaussian distribution and taking the large N limit, we find that the DD is given by (see SI)

$$f(k) = \int_0^{\omega_{\max}} \mathcal{P}(k|\omega_i) \rho(\omega_i) d\omega_i \simeq \frac{N^{-1}}{2 - z/\omega_{\max}} + O(N^{-2}), \quad (9)$$

which is flat and independent of k [Fig. 2(a,c)].

To derive the DDDD, let $\mathcal{P}(\{\omega_i, \omega_j\} | i \leftrightarrow j)$ be the conditional joint probability of sequentially selecting two nodes i and j of fitnesses ω_i and ω_j , conditioned on i and j being connected. Using Bayes’ rule, we have $\mathcal{P}(\{\omega_i, \omega_j\} | i \leftrightarrow j) = \mathcal{P}(i \leftrightarrow j | \{\omega_i, \omega_j\}) \cdot \mathcal{P}(\{\omega_i, \omega_j\}) / \mathcal{P}(i \leftrightarrow j) \simeq \theta(\omega_i + \omega_j - z) \cdot \omega_{\max}^{-2} N^2 T^{-1} / 2$. Thus, by Eq. (4) we have (see SI)

$$\begin{aligned} g(\eta) &= \frac{\omega_{\max}^{-2} N^2}{2T} \int_1^\infty 2kd k \int_{z-\omega_{\max}}^{\omega_{\max}} d\omega_i \int_{z-\omega_i}^{\omega_{\max}} d\omega_j \mathcal{P}(\eta k | \omega_i) \mathcal{P}(k | \omega_j) \\ &\simeq \frac{N^2 (2 - z/\omega_{\max})^2 (2\eta + 1)}{2T \eta^2 (\eta + 1)^2} + O(N^1), \end{aligned} \quad (10)$$

which yields

$$g(\eta) \sim \eta^{-3}, \quad \text{for large } \eta$$

a strong power law that is independent of z [Fig. 2(b,d)].

Applying the hard-threshold model to real-world networks, we find that the model is in good agreement with the WorldCity network [48], a network of global cities and service firms, which also exhibits an almost

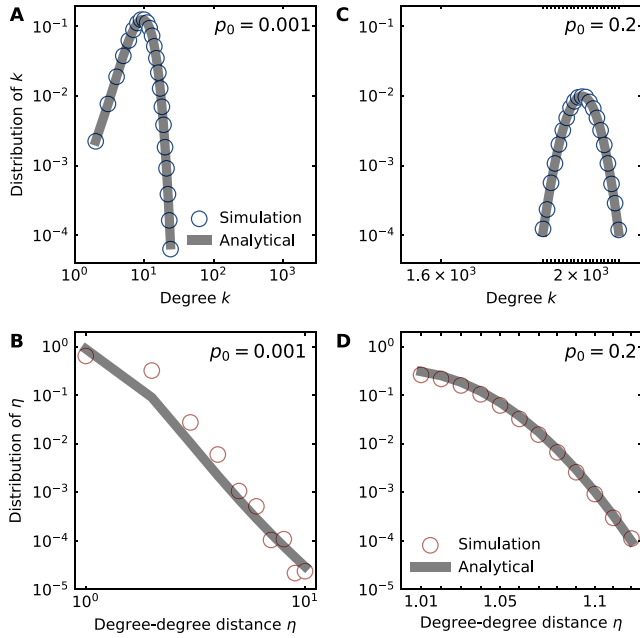


Fig. 3. Distributions of (a,c) k and (b,d) η for the Erdős-Rényi model, given the linking probability p_0 fixed. The model simulation results (circle) are given by the average of 10^2 runs on $N = 10^4$ nodes.

flat DD and a power-law DDDD of an exponent equal to 3 [Fig. 2(e,f)]. Again, the agreement is maintained over the full range of k and η by fitting only one parameter, z/ω_{\max} . This indicates that the hard-threshold model captures how business relationships are formed globally. Our result highlights the *threshold effect* in international trade [49], that a business relationship is established only when the potential combined benefits of the two parties exceed a particular threshold. Furthermore, the use of the fitness model reflects the decision-making process of global business, which is arguably more based on intrinsic values (fitness) rather than random evolution (e.g., preferential attachment).

2.3. The Erdős-Rényi model revisited

For both the no-growth and the hard-threshold models, we have shown that the DDDD is power law even if the DD is not. Now we give a case where neither DD nor DDDD is power law. Considering the Erdős-Rényi (ER) model [50], we know that its DD $f(k) \simeq \binom{N}{k} p_0^k (1-p_0)^{N-k}$ is not a power law but approximately a Gaussian distribution when taking the large N limit while fixing p_0 —the probability to link two random nodes. We find that the DDDD of the ER model (see SI),

$$g(\eta) \simeq \left[\frac{2Np_0}{\pi(1-p_0)} \right]^{1/2} \frac{\eta(\eta+1)^3}{(\eta^2+1)^{7/2}} e^{-\frac{Np_0(\eta-1)^2}{2(1-p_0)(\eta^2+1)}}, \quad (11)$$

is not power law when fixing p_0 . This is since $\eta \leq N$ by definition [Eq. (1)], which sets up an upper bound on η . As a result, the exponential term in Eq. (11) can never vanish for large η (Fig. 3). Note that the ER model can also be derived from both of our models, either by taking $b \rightarrow \infty$ and letting $T = N^2 p_0 / 2$ in the no-growth model, or by completely softening the threshold so that $\sigma(\omega_i, \omega_j) \rightarrow p_0$ becomes independent of ω_i and ω_j in the hard-threshold model. This suggests that the DDDD is power law only in specific regimes, depending on the range of the parameters and/or the order of taking their separate limits. Hence, there could be multiple regimes, characterized by different power laws or no power law of η at all. This leads to an open question to identify the *crossover* [51,52] of power laws between these regimes.

3. Discussion and conclusions

From a data science perspective, notions on heavy-tail distributions seem to be more relevant than power-law distributions for pure quantitative purposes such as statistical analysis and data processing [53]. Knowing the heaviness of distribution tails can certainly help estimating the frequency of extreme events, beyond which fitting a power law may not provide much additional interpretation. Yet, there is an equally important question to *how*—and that is *why*. Understanding the underlying mechanisms that drive network generation is just as important as knowing how to analyze the data. Simply labeling a distribution as heavy-tailed does not reveal these mechanisms, and this is where the scale-free property of networks comes in. We show that the DDDD is a valuable tool for uncovering such mechanisms. Indeed, without considering the power-law DDDD, we would not have been able to infer the scale-free mechanisms of a variety of empirical complex systems, including the Facebook and the WorldCity examples.

We suggest that similar to DD, the power-law exponent of the DDDD could also be incorporated into network theories, such as percolation or dynamical systems on networks, particularly when the DD of the network is not power law. This could expand the applicability of scale-free theories to a broader scope of networks that were previously not considered scale-free.

Furthermore, Eq. (2) establishes a strictly inclusive order between the power laws of DD and DDDD. This raises the question of whether a “hierarchy” of power laws can be established for other known or unknown network metrics, providing new insights into the origins of scale-freeness.

In conclusion, we focus on the relationship between the degree distribution (DD) and the degree-degree distance distribution (DDDD) of network models, demonstrating that power-law DDDD is more universal than power-law DD. This prompts us to consider the DDDD as a more suitable indicator of scale-free networks and reach a deeper understanding of their universality and underlying mechanisms.

CRedit authorship contribution statement

Xiangyi Meng: Conceived the research, Conducted the analysis and wrote the manuscript. **Bin Zhou:** Conceived the research, Conducted the analysis and wrote the manuscript.

Declaration of competing interest

The authors declare no competing interests.

Data availability

The Facebook and WorldCity data are publicly available at <https://athinagroup.eng.uci.edu> and <http://vlado.fmf.uni-lj.si/pub/networks/data/>.

Acknowledgments

This work started from a discussion with H. E. Stanley. We thank H. E. Stanley, P. Holme, B. Chen, and J. Ma for fruitful discussion. X.M. was supported by the NetSeed: Seedling Research Award of Northeastern University. B.Z. was supported by the Startup Foundation for Introducing Talent of NUIST and the Qinglan Project of Jiangsu Universities, and was supported by National Natural Science Foundation of China with Grant Number 72074101.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.chaos.2023.114173>.

References

- [1] Newman MEJ. Networks: An introduction. 1st ed.. New York, USA: Oxford University Press; 2010.
- [2] Albert R, Barabási A-L. Statistical mechanics of complex networks. Rev Modern Phys 2002;74(1):47–97.
- [3] Christensen K, Moloney NR. Complexity and criticality. Imperial college press advanced physics texts, 1st ed.. vol. 1, London, UK: Imperial College Press; 2005.
- [4] Barabási A-L, Albert R. Emergence of scaling in random networks. Science 1999;286(5439):509–12.
- [5] Itzykson C, Drouffe J-M. Statistical field theory: Volume 1, from Brownian motion to renormalization and lattice gauge theory. 1st ed.. New York, USA: Cambridge University Press; 1989.
- [6] Itzykson C, Drouffe J-M. Statistical field theory: Volume 2, strong coupling, Monte Carlo methods, conformal field theory and random systems. 1st ed.. New York, USA: Cambridge University Press; 1989.
- [7] Tanaka R. Scale-rich metabolic networks. Phys Rev Lett 2005;94(16):168101.
- [8] Adamic LA, Huberman BA. Power-law distribution of the world wide web. Science 2000;287(5461):2115.
- [9] Krapivsky PL, Rodgers GJ, Redner S. Degree distributions of growing networks. Phys Rev Lett 2001;86(23):5401–4.
- [10] Song C, Havlin S, Makse HA. Self-similarity of complex networks. Nature 2005;433(7024):392–5.
- [11] Evans TS, Saramäki J. Scale-free networks from self-organization. Phys Rev E 2005;72(2):026138.
- [12] Dorogovtsev SN, Goltsev AV, Mendes JFF. Critical phenomena in complex networks. Rev Modern Phys 2008;80(4):1275–335.
- [13] Szell M, Lambiotte R, Thurner S. Multirelational organization of large-scale social networks in an online world. Proc Natl Acad Sci 2010;107(31):13636–41.
- [14] Lambiotte R, Rosvall M, Scholtes I. From networks to optimal higher-order models of complex systems. Nat Phys 2019;15(4):313–20.
- [15] Nesti T, Sloothaak F, Zwart B. Emergence of scale-free blackout sizes in power grids. Phys Rev Lett 2020;125(5):058301.
- [16] Cohen R, Erez K, ben-Avraham D, Havlin S. Resilience of the Internet to random breakdowns. Phys Rev Lett 2000;85(21):4626–8.
- [17] Gao J, Barzel B, Barabási A-L. Universal resilience patterns in complex networks. Nature 2016;530(7590):307–12.
- [18] Serafino M, Cimini G, Maritan A, Rinaldo A, Suweis S, Banavar JR, Caldarelli G. True scale-free networks hidden by finite size effects. Proc Natl Acad Sci 2021;118(2):e2013825118.
- [19] Goh K-I, Oh E, Jeong H, Kahng B, Kim D. Classification of scale-free networks. Proc Natl Acad Sci 2002;99(20):12583–8.
- [20] Kovács IA, Luck K, Spirohn K, Wang Y, Pollis C, Schlabach S, Bian W, Kim D-K, Kishore N, Hao T, Calderwood MA, Vidal M, Barabási A-L. Network-based prediction of protein interactions. Nature Commun 2019;10(1):1240.
- [21] Garcia D, Mavrodiev P, Schweitzer F. Social resilience in online communities: The autopsy of friendster. In: Proceedings of the first ACM conference on online social networks. 2013, p. 39–50.
- [22] Myers SA, Sharma A, Gupta P, Lin J. Information network or social network? The structure of the Twitter follow graph. In: Proceedings of the 23rd international conference on world wide web. 2014, p. 493–8.
- [23] Goh K-I, Kahng B, Kim D. Universal behavior of load distribution in scale-free networks. Phys Rev Lett 2001;87(27):278701.
- [24] Chen Q, Chang H, Govindan R, Jamin S. The origin of power laws in Internet topologies revisited. In: Proceedings. twenty-first annual joint conference of the IEEE computer and communications societies, Vol. 2. IEEE; 2002, p. 608–17.
- [25] Broido AD, Clauset A. Scale-free networks are rare. Nature Commun 2019;10(1):1017.
- [26] Holme P. Rare and everywhere: Perspectives on scale-free networks. Nature Commun 2019;10(1):1016.
- [27] Voitalov I, van der Hoorn P, van der Hofstad R, Krioukov D. Scale-free networks well done. Phys Rev Res 2019;1(3):033034.
- [28] Artico I, Smolyarenko I, Vinciotti V, Wit E. How rare are power-law networks really? Proc R Soc Lond Ser A Math Phys Eng Sci 2020;476(2241):20190742.
- [29] Clauset A, Shalizi CR, Newman MEJ. Power-law distributions in empirical data. SIAM Rev 2009;51(4):661–703.
- [30] Zhou B, Meng X, Stanley HE. Power-law distribution of degree–degree distance: A better representation of the scale-free property of complex networks. Proc Natl Acad Sci 2020;117(26):14812–8.
- [31] Farzam A, Samal A, Jost J. Degree difference: A simple measure to characterize structural heterogeneity in complex networks. Sci Rep 2020;10(1):1–12.
- [32] Wang B, Zhu J, Wei D. The self-similarity of complex networks: From the view of degree–degree distance. Modern Phys Lett B 2021;35(18):2150331.
- [33] Maren AJ. The 2-D cluster variation method: Topography illustrations and their enthalpy parameter correlations. Entropy 2021;23(3):319.
- [34] Evans TS, Chen B. Linking the network centrality measures closeness and degree. Commun Phys 2022;5(1):1–11.
- [35] Serafino M, Cimini G, Maritan A, Rinaldo A, Suweis S, Banavar JR, Caldarelli G. True scale-free networks hidden by finite size effects. Proc Natl Acad Sci 2021;118(2):e2013825118.
- [36] Langendorf RE, Burgess MG. Empirically classifying network mechanisms. Sci Rep 2021;11(1):20501.
- [37] Caldarelli G, Capocci A, De Los Rios P, Muñoz MA. Scale-free networks from varying vertex intrinsic fitness. Phys Rev Lett 2002;89(25):258702.
- [38] Durante F, Sempì C. Copula theory: An introduction. In: Jaworski P, Durante F, Härdle WK, Rychlik T, editors. Copula theory and its applications. Lecture notes in statistics, vol. 198, Berlin, Germany: Springer; 2010, p. 3–31.
- [39] Sklar A. Fonctions de répartition à n dimensions et leurs marges. Publ Inst Stat Univ Paris 1959;8:229–31.
- [40] Barabási A-L. Network science. 1st ed.. Boston, USA: Cambridge University Press; 2016.
- [41] Cohen R, Havlin S, ben-Avraham D. Structural properties of scale free networks. In: Bornholdt S, Schuster HG, editors. Handbook of graphs and networks: From the genome to the Internet. 1st ed.. Berlin, Germany: Wiley-VCH; 2003, p. 85–110.
- [42] Ghoshal G, Chi L, Barabási A-L. Uncovering the role of elementary processes in network evolution. Sci Rep 2013;3(1):2920.
- [43] Rychkov S. EPFL lectures on conformal field theory in $D \geq 3$ dimensions. Springerbriefs in physics, 1st ed.. Geneva, Switzerland: Springer; 2017.
- [44] Gjoka M, Kuran M, Butts CT, Markopoulou A. Walking in Facebook: A case study of unbiased sampling of OSNs. In: 2010 proceedings IEEE infocom. IEEE; 2010, p. 1–9.
- [45] Kunegis J, Blattner M, Moser C. Preferential attachment in online networks: Measurement and explanations. In: Proceedings of the 5th annual ACM web science conference. WebSci '13, New York, USA: Association for Computing Machinery; 2013, p. 205–14.
- [46] Bianconi G, Barabási A-L. Competition and multiscaling in evolving networks. Europhys Lett 2001;54(4):436.
- [47] Servedio VDP, Caldarelli G, Buttà P. Vertex intrinsic fitness: How to produce arbitrary scale-free networks. Phys Rev E 2004;70(5):056126.
- [48] Taylor PJ, Derudder B. World city network: A global urban analysis. 2nd ed.. London, UK: Routledge; 2015.
- [49] Grossman GM, Helpman E, Kircher P. Matching, sorting, and the distributional effects of international trade. J Political Econ 2017;125(1):224–64.
- [50] Erdős P, Rényi A. On random graphs. Publ Math Debr 1959;6(26):290–7.
- [51] Ma J, Valdez LD, Braunstein LA. Role of bridge nodes in epidemic spreading: Different regimes and crossovers. Phys Rev E 2020;102(3):032308.
- [52] Ma J, Meng X, Braunstein LA. Peak fraction of infected in epidemic spreading for multi-community networks. J Complex Netw 2022;10(3):cnac021.
- [53] Stumpf MPH, Porter MA. Critical truths about power laws. Science 2012;335(6069):665–6.