One-sided DFTs and DFT Matrices for Real Sequences

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Why?

- In audio, we often use real sequences.
- The two-sided DFT $X[k]$ of a length-N real sequence $x[n]$ is conjugate symmetric:

$$X[m] = X^*[N - m] \tag{1}$$

- Hence, it contains redundant information.
- Instead, **we can consider using one-sided DFTs**.

Conjugate Symmetry? Uneven Sequences I

- What does it mean for something to be conjugate symmetric?

$$X[m] = X^*[N-m]$$

- Even example:

$$\mathrm{FFT}\left(\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\right) = \begin{bmatrix}15\\-2.5+3.44j\\-2.5+0.81j\\-2.5-0.81j\\-2.5-3.44j\end{bmatrix} \tag{2}$$

Conjugate Symmetry? Uneven Sequences II

- The corresponding real FFT is:

$$\mathrm{RFFT}\left(\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\right) = \begin{bmatrix}15\\-2.5+3.44j\\-2.5+0.81j\end{bmatrix} \tag{3}$$

Conjugate Symmetry? Even Sequences I

- What does it mean for something to be conjugate symmetric?

$$X[m] = X^*[N-m]$$

- Even example:

$$\mathrm{FFT}\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) = \begin{bmatrix}21\\-3+5.2j\\-3+1.73j\\-3\\-3-1.73j\\-3-5.2j\end{bmatrix} \tag{4}$$

Conjugate Symmetry? Even Sequences II

- The corresponding real FFT is:

$$\mathrm{RFFT}\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) = \begin{bmatrix}\color{red}{21}\\ \color{blue}{-3+5.2j}\\ \color{blue}{-3+1.73j}\\ \color{green}{-3}\end{bmatrix} \tag{5}$$

DFT matrices I

- In our algorithms, we often use DFT matrices.
  Let $\omega = \exp(-2\pi j/N)$

$$\mathbf{W}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} \end{bmatrix} \tag{6}$$

- As we saw, only first 4 rows contain information; hence:

$$\mathbf{V}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \end{bmatrix} \tag{7}$$

DFT matrices II

- As you can see, the real DFT matrix has a lower dimension:

$$\mathbf{V}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \end{bmatrix} \tag{8}$$

- $(N \times N)$ to $(N/2 + 1 \times N)$, yielding sped-up algorithms.

IDFT matrices I

- Can we do the same for IDFT matrices?

$$\mathbf{W}_6^{-1} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \omega^{-8} & \omega^{-10} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \omega^{-12} & \omega^{-15} \\ \omega^0 & \omega^{-4} & \omega^{-8} & \omega^{-12} & \omega^{-16} & \omega^{-20} \\ \omega^0 & \omega^{-5} & \omega^{-10} & \omega^{-15} & \omega^{-20} & \omega^{-25} \end{bmatrix} \tag{9}$$

- Turns out, there's a problem. . .

IDFT matrices II

- Consider the one-sided DFT of our previous example:

$$\mathrm{RFFT}\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) = \begin{bmatrix}21\\-3+5.2j\\-3+1.73j\\-3\end{bmatrix} \tag{10}$$

- We are looking for a matrix that transforms the ones-sided DFT back to the time-domain sequence.

IDFT matrices III

- **Problem:** We need the conjugate of the blue terms to reconstruct the time domain sequence.
- **Claim:** There is no *general* way to represent taking the conjugate with a matrix.
  - ... Feel free to disagree ;)

My Solution I

- I usually work with **optimization problems**. In general, these solvers don't work with imaginary numbers
- Given some complex matrix $\mathbf{A} \in \mathbb{C}^{N \times M}$, I perform the following transformation to make the matrix real:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathrm{Re}\left\{\mathbf{A}\right\} & -\mathrm{Im}\left\{\mathbf{A}\right\} \\ \mathrm{Im}\left\{\mathbf{A}\right\} & \mathrm{Re}\left\{\mathbf{A}\right\} \end{bmatrix} \tag{11}$$

- Similarly some complex vector $\mathbf{x} \in \mathbb{C}^{K}$:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathrm{Re}\left\{\mathbf{x}\right\} \\ \mathrm{Im}\left\{\mathbf{x}\right\} \end{bmatrix} \tag{12}$$

My Solution II

- The real DFT matrix $\mathbf{V}_6$ introduced before is always applied to a real vector. Hence it can be expressed even more compactly:

$$\tilde{\mathbf{V}}_6 = \begin{bmatrix} \mathrm{Re}\left\{\mathbf{V}_6\right\} \\ \mathrm{Im}\left\{\mathbf{V}_6\right\} \end{bmatrix} \tag{13}$$

- What about the IDFT matrix, though?

My Solution III

- Consider the following multiplication between a special matrix and a vector in the new representation:

$$\mathbf{x} = \mathbf{S}\tilde{\mathbf{x}}$$

$$\begin{bmatrix} \mathrm{Re}\{x_0\} + j\mathrm{Im}\{x_0\} \\ \mathrm{Re}\{x_1\} + j\mathrm{Im}\{x_1\} \\ \mathrm{Re}\{x_2\} + j\mathrm{Im}\{x_2\} \\ \mathrm{Re}\{x_3\} + j\mathrm{Im}\{x_3\} \\ \mathrm{Re}\{x_2\} - j\mathrm{Im}\{x_2\} \\ \mathrm{Re}\{x_1\} - j\mathrm{Im}\{x_1\} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & j & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & j \\ 0 & 0 & 1 & 0 & 0 & 0 & -j & 0 \\ 0 & 1 & 0 & 0 & 0 & -j & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{x_0\} \\ \mathrm{Re}\{x_1\} \\ \mathrm{Re}\{x_2\} \\ \mathrm{Re}\{x_3\} \\ \mathrm{Im}\{x_0\} \\ \mathrm{Im}\{x_1\} \\ \mathrm{Im}\{x_2\} \\ \mathrm{Im}\{x_3\} \end{bmatrix}$$

- Hence, we can take $\tilde{\mathbf{V}}_6^{-1} = \mathbf{W}_6^{-1}\mathbf{S}$

My Solution IV

- So, what would $\tilde{\mathbf{V}}_6^{-1} = \mathbf{W}_6^{-1}\mathbf{S}$ look like?

$$\begin{bmatrix} \omega^0 & 2\omega^0 & 2\omega^0 & \omega^0 & j\omega^0 & 0 & 0 & j\omega^0 \\ \omega^0 & \omega^{-1}+\omega^{-5} & \omega^{-2}+\omega^{-4} & \omega^{-3} & j\omega^0 & j\omega^{-1}-j\omega^{-5} & j\omega^{-2}-j\omega^{-4} & j\omega^{-3} \\ \omega^0 & \omega^{-2}+\omega^{-10} & \omega^{-4}+\omega^{-8} & \omega^{-6} & j\omega^0 & j\omega^{-2}-j\omega^{-10} & j\omega^{-4}-j\omega^{-8} & j\omega^{-6} \\ \omega^0 & \omega^{-3}+\omega^{-15} & \omega^{-6}+\omega^{-12} & \omega^{-9} & j\omega^0 & j\omega^{-3}-j\omega^{-15} & j\omega^{-6}-j\omega^{-12} & j\omega^{-9} \\ \omega^0 & \omega^{-4}+\omega^{-20} & \omega^{-8}+\omega^{-16} & \omega^{-12} & j\omega^0 & j\omega^{-4}-j\omega^{-20} & j\omega^{-8}-j\omega^{-16} & j\omega^{-12} \\ \omega^0 & \omega^{-5}+\omega^{-25} & \omega^{-10}+\omega^{-20} & \omega^{-15} & j\omega^0 & j\omega^{-5}-j\omega^{-25} & j\omega^{-10}-j\omega^{-20} & j\omega^{-15} \end{bmatrix}$$

- Which is actually real if we compute the exponentials and use the fact that $\mathrm{Im}\{x_0\} = \mathrm{Im}\{x_3\} = 0$:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -\sqrt{2} & -\sqrt{2} & 0 \\ 1 & -1 & -1 & -1 & 0 & -\sqrt{2} & \sqrt{2} & 0 \\ 1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 & -1 & 0 & \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$