

One-sided DFTs and DFT Matrices for Real Sequences

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Why?

- ▶ In audio, we often use real sequences.
- ▶ The two-sided DFT $X[k]$ of a length- N real sequence $x[n]$ is conjugate symmetric:

$$X[m] = X^*[N - m] \quad (1)$$

- ▶ Hence, it contains redundant information.
- ▶ Instead, **we can consider using one-sided DFTs.**

Conjugate Symmetry? Uneven Sequences I

- ▶ What does it mean for something to be conjugate symmetric?

$$X[m] = X^*[N - m]$$

- ▶ Even example:

$$\text{FFT} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 15 \\ -2.5 + 3.44j \\ -2.5 + 0.81j \\ -2.5 - 0.81j \\ -2.5 - 3.44j \end{bmatrix} \quad (2)$$

Conjugate Symmetry? Uneven Sequences II

- The corresponding real FFT is:

$$\text{RFFT} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 15 \\ -2.5 + 3.44j \\ -2.5 + 0.81j \end{bmatrix} \quad (3)$$

Conjugate Symmetry? Even Sequences I

- ▶ What does it mean for something to be conjugate symmetric?

$$X[m] = X^*[N - m]$$

- ▶ Even example:

$$\text{FFT} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 21 \\ -3 + 5.2j \\ -3 + 1.73j \\ -3 \\ -3 - 1.73j \\ -3 - 5.2j \end{bmatrix} \quad (4)$$

Conjugate Symmetry? Even Sequences II

- The corresponding real FFT is:

$$\text{RFFT} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 21 \\ -3 + 5.2j \\ -3 + 1.73j \\ -3 \end{bmatrix} \quad (5)$$

DFT matrices I

- In our algorithms, we often use DFT matrices.

Let $\omega = \exp(-2\pi j/N)$

$$\mathbf{W}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} \end{bmatrix} \quad (6)$$

- As we saw, only first 4 rows contain information; hence:

$$\mathbf{V}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \end{bmatrix} \quad (7)$$

DFT matrices II

- ▶ As you can see, the real DFT matrix has a lower dimension:

$$\mathbf{V}_6 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \end{bmatrix} \quad (8)$$

- ▶ $(N \times N)$ to $(N/2 + 1 \times N)$, yielding sped-up algorithms.

IDFT matrices I

- Can we do the same for IDFT matrices?

$$\mathbf{W}_6^{-1} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \omega^{-8} & \omega^{-10} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \omega^{-12} & \omega^{-15} \\ \omega^0 & \omega^{-4} & \omega^{-8} & \omega^{-12} & \omega^{-16} & \omega^{-20} \\ \omega^0 & \omega^{-5} & \omega^{-10} & \omega^{-15} & \omega^{-20} & \omega^{-25} \end{bmatrix} \quad (9)$$

- Turns out, there's a problem...

IDFT matrices II

- Consider the one-sided DFT of our previous example:

$$\text{RFFT} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 21 \\ -3 + 5.2j \\ -3 + 1.73j \\ -3 \end{bmatrix} \quad (10)$$

- We are looking for a matrix that transforms the one-sided DFT back to the time-domain sequence.

IDFT matrices III

- ▶ **Problem:** We need the conjugate of the blue terms to reconstruct the time domain sequence.
- ▶ **Claim:** There is no *general* way to represent taking the conjugate with a matrix.
 - ▶ ... Feel free to disagree ;)

My Solution I

- ▶ I usually work with **optimization problems**. In general, these solvers don't work with imaginary numbers
- ▶ Given some complex matrix $\mathbf{A} \in \mathbb{C}^{N \times M}$, I perform the following transformation to make the matrix real:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \text{Re}\{\mathbf{A}\} & -\text{Im}\{\mathbf{A}\} \\ \text{Im}\{\mathbf{A}\} & \text{Re}\{\mathbf{A}\} \end{bmatrix} \quad (11)$$

- ▶ Similarly some complex vector $\mathbf{x} \in \mathbb{C}^K$:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \text{Re}\{\mathbf{x}\} \\ \text{Im}\{\mathbf{x}\} \end{bmatrix} \quad (12)$$

My Solution II

- ▶ The real DFT matrix \mathbf{V}_6 introduced before is always applied to a real vector. Hence it can be expressed even more compactly:

$$\tilde{\mathbf{V}}_6 = \begin{bmatrix} \text{Re} \{ \mathbf{V}_6 \} \\ \text{Im} \{ \mathbf{V}_6 \} \end{bmatrix} \quad (13)$$

- ▶ What about the IDFT matrix, though?

My Solution III

- Consider the following multiplication between a special matrix and a vector in the new representation:

$$\mathbf{x} = \mathbf{S}\tilde{\mathbf{x}}$$

$$\begin{bmatrix} \text{Re}\{x_0\} + j\text{Im}\{x_0\} \\ \text{Re}\{x_1\} + j\text{Im}\{x_1\} \\ \text{Re}\{x_2\} + j\text{Im}\{x_2\} \\ \text{Re}\{x_3\} + j\text{Im}\{x_3\} \\ \text{Re}\{x_2\} - j\text{Im}\{x_2\} \\ \text{Re}\{x_1\} - j\text{Im}\{x_1\} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & j & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & j \\ 0 & 0 & 1 & 0 & 0 & 0 & -j & 0 \\ 0 & 1 & 0 & 0 & 0 & -j & 0 & 0 \end{bmatrix} \begin{bmatrix} \text{Re}\{x_0\} \\ \text{Re}\{x_1\} \\ \text{Re}\{x_2\} \\ \text{Re}\{x_3\} \\ \text{Im}\{x_0\} \\ \text{Im}\{x_1\} \\ \text{Im}\{x_2\} \\ \text{Im}\{x_3\} \end{bmatrix}$$

- Hence, we can take $\tilde{\mathbf{V}}_6^{-1} = \mathbf{W}_6^{-1}\mathbf{S}$

My Solution IV

- So, what would $\tilde{\mathbf{V}}_6^{-1} = \mathbf{W}_6^{-1} \mathbf{S}$ look like?

$$\begin{bmatrix} \omega^0 & 2\omega^0 & 2\omega^0 & \omega^0 & j\omega^0 & 0 & 0 & j\omega^0 \\ \omega^0 & \omega^{-1} + \omega^{-5} & \omega^{-2} + \omega^{-4} & \omega^{-3} & j\omega^0 & j\omega^{-1} - j\omega^{-5} & j\omega^{-2} - j\omega^{-4} & j\omega^{-3} \\ \omega^0 & \omega^{-2} + \omega^{-10} & \omega^{-4} + \omega^{-8} & \omega^{-6} & j\omega^0 & j\omega^{-2} - j\omega^{-10} & j\omega^{-4} - j\omega^{-8} & j\omega^{-6} \\ \omega^0 & \omega^{-3} + \omega^{-15} & \omega^{-6} + \omega^{-12} & \omega^{-9} & j\omega^0 & j\omega^{-3} - j\omega^{-15} & j\omega^{-6} - j\omega^{-12} & j\omega^{-9} \\ \omega^0 & \omega^{-4} + \omega^{-20} & \omega^{-8} + \omega^{-16} & \omega^{-12} & j\omega^0 & j\omega^{-4} - j\omega^{-20} & j\omega^{-8} - j\omega^{-16} & j\omega^{-12} \\ \omega^0 & \omega^{-5} + \omega^{-25} & \omega^{-10} + \omega^{-20} & \omega^{-15} & j\omega^0 & j\omega^{-5} - j\omega^{-25} & j\omega^{-10} - j\omega^{-20} & j\omega^{-15} \end{bmatrix}$$

- Which is actually real if we compute the exponentials and use the fact that $\text{Im}\{x_0\} = \text{Im}\{x_3\} = 0$:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -\sqrt{2} & -\sqrt{2} & 0 \\ 1 & -1 & -1 & -1 & 0 & -\sqrt{2} & \sqrt{2} & 0 \\ 1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 & -1 & 0 & \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$