chapter 8

1. 求下列序列的z变换，并标明收敛域

(1) $(\frac{1}{2})^{n-2}u(-n+3)$

解: $\sum_{n=-\infty}^{\infty}(\frac{1}{2})^{n-2}u(3-n)z^{-n}$

$= \sum_{n=-\infty}^{3}(\frac{1}{2})^{n-2}z^{-n}$

$= \sum_{n=-3}^{\infty}2^{n+2}z^{n}$

$= 4\sum_{n=-3}^{\infty}(2z)^{n}$

$= \frac{4(2z)^{-3}}{1-2z}$

$= \frac{1}{2(1-2z)z^3}$

ROC: $|2z|<1 \Rightarrow |z|<\frac{1}{2}$

(2) $n^2 a^{n-1} u(n)$

解 $\mathcal{Z}[a^{n-1}u(n)] = \dfrac{z}{a(z-a)}$, $|z|>|a|$

$$\Rightarrow \mathcal{Z}[n^2 a^{n-1} u(n)] = -z\frac{d}{dz}\cdot\left\{-z\frac{d}{dz}\left[\frac{z}{a(z-a)}\right]\right\}$$

$$= -z\frac{d}{dz}\,\frac{z}{(z-a)^2}$$

$$= -z\,\frac{(z-a)^2-2z(z-a)}{(z-a)^4}$$

$$= \frac{z(z^2-a^2)}{(z-a)^4}$$

$$= \frac{z(z+a)}{(z-a)^3}\ ,\quad |z|>|a|$$

(3) $\cos\frac{\pi n}{4}\cdot u(n)$

解: $\cos\frac{\pi n}{4} = \frac{1}{2}\left(e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}\right)$

$\mathcal{Z}[e^{j\frac{\pi n}{4}}u(n)] = \dfrac{z}{z-e^{j\frac{\pi}{4}}}$, $|z|>1$

$$\Rightarrow \mathcal{Z}[\cos\tfrac{\pi n}{4}\cdot u(n)]$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{j\frac{\pi}{4}}} + \frac{z}{z-e^{-j\frac{\pi}{4}}}\right)$$

$$= \frac{1}{2}\cdot\frac{z[2z-(e^{j\frac{\pi}{4}}+e^{-j\frac{\pi}{4}})]}{z^2-(e^{j\frac{\pi}{4}}+e^{-j\frac{\pi}{4}})z+1}$$

$$= \frac{z(z-\cos\frac{\pi}{4})}{z^2-2z\cos\frac{\pi}{4}+1}$$

$$= \frac{z(z-\frac{\sqrt{2}}{2})}{z^2-\sqrt{2}z+1}$$

ROC: $|z|>1$

2、求下列函数的z逆变换

(1) $X(z)=\frac{z}{z^2+1}$, $|z|>1$

解

$$X(z)=\frac{1}{2j}\left(\frac{z}{z-j}-\frac{z}{z+j}\right)$$

$$\Rightarrow x(n)=\frac{1}{2j}\left[(j)^n-(-j)^n\right]u(n)$$

(2) $X(z)=\frac{z^3+2z^2+1}{z^3-1.5z^2+0.5z}$, $|z|>1$

解

$$\frac{X(z)}{z}=\frac{1}{z}\left[1+\frac{3.5z^2-0.5z+1}{(z-1)(z-\frac{1}{2})z}\right]$$

$$=\frac{1}{z}\left(1+\frac{2}{z}-\frac{\frac{13}{2}}{z-\frac{1}{2}}+\frac{8}{z-1}\right)$$

$$=\frac{1}{z}+\frac{2}{z^2}-\frac{13/2}{z(z-\frac{1}{2})}+\frac{8}{z(z-1)}$$

$$=\frac{1}{z}+\frac{2}{z^2}-13\left(\frac{1}{z-\frac{1}{2}}-\frac{1}{z}\right)+8\left(\frac{1}{z-1}-\frac{1}{z}\right)$$

$$=\frac{6}{z}+\frac{2}{z^2}-\frac{13}{z-\frac{1}{2}}+\frac{8}{z-1}$$

$$X(z)=6+\frac{2}{z}-13\frac{z}{z-\frac{1}{2}}+8\frac{z}{z-1}, \quad |z|>1$$

$$\Rightarrow x(n)=6\delta(n)+2\delta(n-1)+\left[8-13\left(\frac{1}{2}\right)^n\right]u(n)$$

(3) $X(z)=\dfrac{z^{-1}-0.5z^{-2}}{1+0.75z^{-1}+0.125z^{-2}}, |z|>0.5$

解:

$$X(z)=\frac{z-\frac{1}{2}}{z^2+0.75z+0.125}=\frac{z-\frac{1}{2}}{(z+\frac{1}{2})(z+\frac{1}{4})}$$

$$\frac{X(z)}{z}=\frac{z-\frac{1}{2}}{z(z+\frac{1}{2})(z+\frac{1}{4})}=\frac{12}{z+\frac{1}{4}}-\frac{8}{z+\frac{1}{2}}-\frac{4}{z}$$

$$X(z)=12\cdot\frac{z}{z+\frac{1}{4}}-8\cdot\frac{z}{z+\frac{1}{2}}-4, \quad |z|>\frac{1}{2}$$

$$\Rightarrow x(n)=\left[12\cdot(-\tfrac{1}{4})^n-8\cdot(-\tfrac{1}{2})^n\right]u(n)-4\delta(n)$$

3. 已知因果序列 $x(n)$ 的 z 变换 $X(z)$, $X(z)=\dfrac{z^{N+1}}{(z-1)(z-0.5)^N}$, 求序列的初值和终值

解: ① $x(0)=\lim\limits_{z\to\infty}X(z)=\lim\limits_{z\to\infty}\dfrac{z^{N+1}}{(z-1)(z-0.5)^N}=1$

② $X(z)$ 的 N 阶极点 $z=0.5$ 位于单位圆内, 一阶极点 $z=1$ 位于单位圆上

$\therefore$ $x(n)$ 有终值

$$\lim_{n\to\infty}x(n)=\lim_{z\to1}\frac{(z-1)z^{N+1}}{(z-1)(z-0.5)^N}=\lim_{z\to1}\frac{z^{N+1}}{(z-0.5)^N}$$

$$=2^N$$

4. 求卷积 $y(n)=3^n u(n-3) * 2^n u(n+1)$

解：记 $x_1(n)=3^n u(n-3)$，$x_2(n)=2^n u(n+1)$

$$X_1(z)=3^3\cdot z^{-3}\cdot\frac{z}{z-3}=\frac{27}{(z-3)z^2}, \quad |z|>3$$

$$X_2(z)=\frac{1}{2}\cdot z\cdot\frac{z}{z-2}=\frac{z^2}{2(z-2)}, \quad |z|>2$$

$$\Rightarrow Y(z)=X_1(z)\cdot X_2(z)=\frac{27}{(z-3)z^2}\cdot\frac{z^2}{2(z-2)}=\frac{27}{2(z-2)(z-3)}, \quad |z|>3$$

$$\frac{Y(z)}{z}=\frac{27}{2}\ \frac{1}{z(z-2)(z-3)}=\frac{9}{4}\left(\frac{1}{z}-\frac{3}{z-2}+\frac{2}{z-3}\right)$$

$$Y(z)=\frac{9}{4}\left(1-3\cdot\frac{z}{z-2}+2\cdot\frac{z}{z-3}\right), \quad |z|>3$$

$$\Rightarrow y(n)=\frac{9}{4}\left[\delta(n)+(2\cdot 3^n-3\cdot 2^n)u(n)\right]$$

5. 利用单边z变换求解差分方程

$y(n)-3y(n-1)+2y(n-2)=u(n-1)-2u(n-2)$，$y(0)=y(1)=1$，

并试求出零输入响应和零状态响应

解：记 $\mathcal{Z}[y(n)]=Y(z)$

$$Y(z)-3\left[z^{-1}Y(z)+y(-1)\right]+2\left[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right]$$

$$=\frac{z}{z-1}(z^{-1}-2z^{-2})=\frac{z^{-1}(1-2z^{-1})}{1-z^{-1}}$$

$$\begin{cases} y(1)-3y(0)+2y(-1)=u(0)-2u(-1)=1 \Rightarrow y(-1)=\frac{3}{2} \\ y(0)-3y(-1)+2y(-2)=u(-1)-2u(-2)=0 \Rightarrow y(-2)=\frac{7}{4} \end{cases}$$

$$\Rightarrow (1-3z^{-1}+2z^{-2})Y(z)=(1-3z^{-1})+\frac{z^{-1}(1-2z^{-1})}{1-z^{-1}}$$

$$Y(z) = \frac{1-3z^{-1}}{(1-z^{-1})(1-2z^{-1})} + \frac{z^{-1}}{(1-z^{-1})^2}$$

$$= \left(\frac{2}{1-z^{-1}} - \frac{1}{1-2z^{-1}}\right) + \frac{z^{-1}}{(1-z^{-1})^2}$$

$$= \left(\frac{2z}{z-1} - \frac{z}{z-2}\right) + \frac{z}{(z-1)^2}$$

$$Y_{zi}(z) = \frac{2z}{z-1} - \frac{z}{z-2}$$

$$\Rightarrow y_{zi}(n) = (2-2^n)u(n)$$

$$Y_{zs}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow y_{zs}(n) = nu(n)$$

$$\Rightarrow y(n) = y_{zi}(n) + y_{zs}(n)$$

$$= (2-2^n+n)u(n)$$

6. 已知某离散系统的单位样值响应为 $h(n)=[2(-2)^n-(-1)^n]\cdot u(n)$,

(1) 求系统函数和差分方程

解(1) $h(n)=[2(-2)^n-(-1)^n]u(n)$

$$H(z)=\frac{2z}{z+2}-\frac{z}{z+1},\quad |z|>2$$

$$=\frac{z^2}{(z+1)(z+2)},\quad |z|>2$$

又 $H(z)=\dfrac{Y(z)}{X(z)}$

$$\Rightarrow (z^2+3z+2)Y(z)=z^2X(z)$$

$$(1+3z^{-1}+2z^{-2})Y(z)=X(z)$$

$$\Rightarrow y(n)+3y(n-1)+2y(n-2)=x(n)$$

(2) 当激励为 $3u(n)$ 时，求系统的零状态响应

解(2) $X(z)=\mathcal{Z}[3u(n)]=\dfrac{3z}{z-1},\quad |z|>1$

$$Y(z)=H(z)\cdot X(z)=\frac{z^2}{z^2+3z+2}\cdot\frac{3z}{z-1}=\frac{3z^3}{(z+1)(z+2)(z-1)}$$

$$\frac{Y(z)}{z}=\frac{3z^2}{(z+1)(z+2)(z-1)}=\frac{1}{2}\cdot\frac{1}{z-1}+4\cdot\frac{1}{z+2}-\frac{3}{2}\cdot\frac{1}{z+1}$$

$$Y(z)=\frac{1}{2}\cdot\frac{z}{z-1}+4\frac{z}{z+2}-\frac{3}{2}\cdot\frac{z}{z+1}$$

$$y_{zs}(n)=\left[\frac{1}{2}+4\cdot(-2)^n-\frac{3}{2}(-1)^n\right]u(n)$$

$$=\left[\frac{1}{2}+(-2)^{n+2}-\frac{3}{2}(-1)^n\right]u(n)$$

7、一个LTI因果离散系统具有非零初始状态，当输入$x_1(n)=\delta(n)$时，系统的全响应为$y_1(n)=2(0.25)^n u(n)$，在相同初始条件下，输入$x_2(n)=(0.5)^n u(n)$时，全响应为$y_2(n)=[(0.25)^n+0.5^n]u(n)$，求系统函数

解：

$$\delta(n) \longrightarrow \boxed{系统} \longrightarrow 2\left(\tfrac{1}{4}\right)^n u(n)$$

$$\left(\tfrac{1}{2}\right)^n u(n) \longrightarrow \boxed{系统} \longrightarrow \left[\left(\tfrac{1}{4}\right)^n+\left(\tfrac{1}{2}\right)^n\right] u(n)$$

$$\begin{cases} y_1(n) = h(n) + y_{zi}(n) = 2\left(\tfrac{1}{4}\right)^n u(n) \\ y_2(n) = x_2(n)*h(n) + y_{zi}(n) = \left[\left(\tfrac{1}{4}\right)^n+\left(\tfrac{1}{2}\right)^n\right] u(n) \end{cases}$$

$$\Rightarrow \begin{cases} H(z)+Y_{zi}(z) = \dfrac{2z}{z-\frac{1}{4}} & ① \\ \dfrac{z}{z-\frac{1}{2}}H(z)+Y_{zi}(z) = \dfrac{z}{z-\frac{1}{4}}+\dfrac{z}{z-\frac{1}{2}} & ② \end{cases}$$

$$②-①: \quad \frac{\frac{1}{2}}{z-\frac{1}{2}}H(z) = -\frac{z}{z-\frac{1}{4}}+\frac{z}{z-\frac{1}{2}}$$

$$H(z) = -\frac{2z(z-\frac{1}{2})}{z-\frac{1}{4}}+2z$$

$$= 2z\left(1-\frac{z-\frac{1}{2}}{z-\frac{1}{4}}\right)$$

$$= 2z\cdot\frac{\frac{1}{4}}{z-\frac{1}{4}}$$

$$= \frac{2z}{4z-1}$$