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Classical Mechanics & Special Relativity for Starters

Robert F. Mudde, **Bernd Rieger**, and **C. Freek. J. Pols**

Abstract

some abstract

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Contents

1. Introduction

1.a. About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and pressumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version. Note that we made good use of other open educational resources, several exercises stem from such resources.

1.a.i. *Special features:*

In this book you will find some 'special' features. Some of these are emphasized by their own style:

> **Exercise 1:** 🌶️
>
> Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.
>
> To indicate the level of difficulty, each exercise is marked with 1, 2, or 3 🌶️

> **intermezzo: Intermezzos**

> Intermezzos contain background information on the topic, of the people working on the concepts.

> **Experiment: Experiments**
>
> We include some basic experiments that can be done at home.

> **Example: Examples**
>
> We provide various examples showcasing, e.g., calculations.

> **Python**
>
> We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'. Try it by pushing the ON-button and subsequently the play button and see the output in de code-cell below.

```
print("The square root of 2 is: ", 2 ** 0.5)
```

New concepts, such as *Free body diagram*, are introduced with a hoover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

The book can be read online. There is, however, the opportunity to download the materials and play with the code offline as well. We advise to use Jupyter Lab in combination with MyST.

1.a.ii. *Feedback:*

This is the first version of this book. Although many have work on it and several iterations have been made, there might still be issues. Do you see a mistake, do you have suggestions for exercises, are parts missing or abundant. Tell us! You can use the `Feedback` button at the top right button. You will need a (free) GitHub account to report an issue!

1.b. Authors

- Prof.Dr. Robert F. Mudde, Department of Chemical Engineering, Delft University of Technology
- Prof.Dr. Bernd Rieger, Department of Imaging Physics, Delft University of Technology
- Dr. Freek Pols, Science & Engineering Education, Delft University of Technology

Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files. Also thanks to Vebe Helmes, Alexander Lopes-Cardozo, Sep Schouwenaar and Winston de Greef for their comments and suggestions.

1.b.i. *About the authors:*

Robert Mudde is Distinguished Professor of Science Education at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

Bernd Rieger is Antoni van Leeuwenhoek Professor in the Department of Imaging Physics at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

Freek Pols is an assistant professor in the Science & Engineering Education group at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

1.c. *Open Educational Resource*

This website is a Jupyter Book. Source files are available for download using the button on the top right.

1.c.i. *Software and license:*

This website is a Jupyter Book. Markdown source files are available for download using the button on the top right, licensed under CC-BY-NC (unless stated otherwise). All python codes / apps are freely reusable, adaptable and redistributable (CC0).

1.c.ii. *Images, videos, apps, intermezzos:*

The cover image is inspired by the work of 3blue1brown developer Grant Sanderson.

All vector images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistribution, adhere to the CC-BY licences.

We embedded several clips from 3blue1brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

The Intermezzos, which elaborate on the lives of various scientists and the efforts behind key physical discoveries, are composed by drawing from a range of different sources. Rather than directly reproducing any one account, these stories have been reworked into a narrative that fits the context and audience of this book.

1.c.iii. *How to cite this book:*

2. Mechanics

3. Mechanics

3.a. The language of Physics

Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, both colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

3.a.i. *Representations:*

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.

Figure 1: A base jumper is used as context to get familiair with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

Verbal descriptions:

Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

Example: Base jumper: Verbal description

Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

Visual representations:

Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more concrete.

- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.
- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.

Example: Base jumper: Free body diagram

The situation is sketched in Figure 2 using a Free body diagram. Note that all details of the jumper are ignored in the sketch.

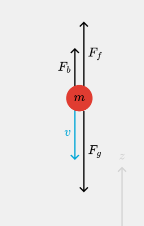


Figure 2: Force acting on the jumper.

- m = mass of jumper (in kg);

- v = velocity of jumper (in m/s);
- F_g = gravitational force (in N);
- F_f = drag force by the air (in N);
- F_b = buoyancy (in N): like in water also in air there is an upward force, equal to the weight of the displaced air.

Equations:

Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

Example: Base jumper: equations

The forces acting on the jumper are already shown in Figure 2. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \tag{1}$$

We can specify each of the force:

$$\begin{aligned} F_g &= -mg = -\rho_p V_p g \\ F_f &= \frac{1}{2}\rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \tag{2}$$

with g the acceleration of gravity, ρ_p the density of the jumper ($\approx 10^3 \mathrm{kg/m^3}$), V_p the volume of the jumper, ρ_{air} the density of air ($\approx 1.2 \mathrm{kg/m^3}$), C_D the so-called drag coefficient, A the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering which representations suits best for the given situation. We will practice these when solving problems.

Danger

Note that in the example above we neglected directions. In our equation we should have been using vector notation, which we will cover in one of the next sections in this chapter.

3.a.ii. *How to solve a physics problem?:*

One of the most common mistakes made by ‘novices’ when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you apply this way of working automatically. It is called IDEA, an acronym that stands for:

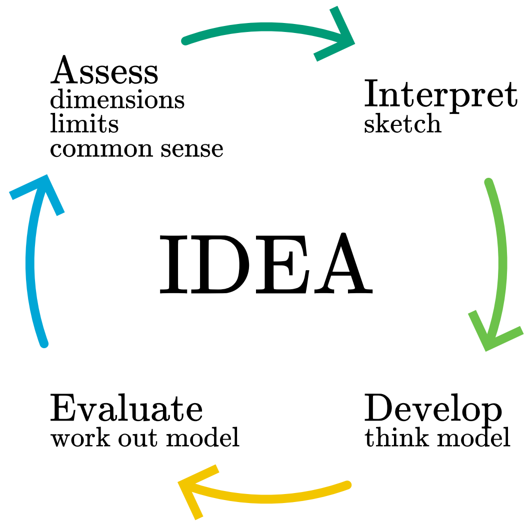


Figure 3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. Actually *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?).

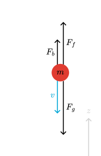
We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

Example:

Interpret

Three forces act on the jumper, shown in the figure below. Finding the terminal velocity implies that all forces are balanced ($\sum F = 0$).



The buoyancy force is much smaller than the force of gravity (about 0.1%) and we neglect it.

Develop

We know all forces: gravitational force equals the drag force

$$\begin{aligned} F_g &= F_f \\ mg &= \frac{1}{2}\rho_{air} C_D A v^2 \end{aligned} \tag{3}$$

Evaluate

Assume a mass of 75 kg, an acceleration due to gravity of 9.81 $\mathrm{m/s^2}$, and air density of 1.2 $\mathrm{kg/m^3}$, a drag coefficient of 1, a frontal surface area of 0.7 $\mathrm{m^2}$.

$$mg = \frac{1}{2}\rho_{air} C_D A v^2 \tag{4}$$

Rewriting:

$$\begin{aligned} v &= \sqrt{\frac{2mg}{\rho_{air} C_D A}} \\ v &= \sqrt{\frac{2 \cdot 75\ (\mathrm{kg}) \cdot 9.81\ (\mathrm{m/s^2})}{1.2\ (\mathrm{kg/m^3}) \cdot 1 \cdot 0.7\ (\mathrm{m^2})}} \\ v &= 40\ \mathrm{m/s} \end{aligned} \tag{5}$$

Assess

We may know that raindrops typically reach a terminal velocity of less than 10 m/s. A terminal velocity of 40 m/s seems therefore plausible.

Note that we didn't solve the problem entirely! We only calculated the terminal velocity, where the question was how long it would roughly take to reach such a velocity.

Good Practice

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or multiple steps in one step. If you make a mistake, it will be very hard to trace it back.

Next: check constantly the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

Powers of ten

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms (10^{-10} m) to the distance between stars (10^{16} m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milli seconds, GB for gigabyte that is 1 billion bytes. Here is a list of prefixes.

Prefix	Symbol	Math	Prefix	Symbol	Math
Yocto	y	10^{-24}	Base	•	10^{0}
Zepto	z	10^{-21}	Deca	da	10^{1}
Atto	a	10^{-18}	Hecto	h	10^{2}
Femto	f	10^{-15}	Kilo	k	10^{3}
Pico	p	10^{-12}	Mega	M	10^{6}
Nano	n	10^{-9}	Giga	G	10^{9}
Micro	µ	10^{-6}	Tera	T	10^{12}
Milli	m	10^{-3}	Peta	P	10^{15}
Centi	c	10^{-2}	Exa	E	10^{18}
Deci	d	10^{-1}	Zetta	Z	10^{21}
Base	•	10^{0}	Yotta	Y	10^{24}

On quantities and units

Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter m? Quantities are expressed in italics (m) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

- Meter (m) – length
- Kilogram (kg) – mass
- Second (s) – time

- Ampere (A) – electric current
- Kelvin (K) – temperature
- Mole (mol) – amount of substance
- Candela (cd) – luminous intensity

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned} W &= F \cdot s = ma \cdot s = m\frac{\Delta v}{\Delta t} \cdot s \\ &= [\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [\mathrm{m/s^2}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m/s}]}{[\mathrm{s}]} \cdot [\mathrm{m}] = \left[\frac{\mathrm{kgm^2}}{\mathrm{s^2}}\right] \end{aligned} \tag{6}$$

Tip

For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

3.a.iii. *Calculus:*

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features, that help us understanding the problem and its solution much deeper.

Given the example of the base jumper, we would like to know how the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{7}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air} C_D A v^2 \tag{8}$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself. Before we even try to solve this problem ($v(t) = \ldots$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

Differentiation:

Many physical phenomena are described by differential equations. That may be because a system evolves in time or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2^{nd} law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time; velocity in itself is the derivative of position with respect to time. Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{9}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \tag{10}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{11}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{12}$$

It is just a matter of notation.

3.a.iv. *Definition of velocity, acceleration and momentum:*

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that x. In general, x will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time t. How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in the time interval Δt. The (momentary) velocity is defined as:

Velocity

$$\text{definition velocity} : v \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = \frac{dx}{dt} \tag{13}$$

Similarly, we define the acceleration as the change of the velocity over a time interval Δt: $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

Acceleration

$$\text{definition acceleration} : a \equiv \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{(t+\Delta t) - t} = \frac{dv}{dt} \tag{14}$$

Consequently,

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{15}$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

Momentum

$$\text{definition momentum} : p \equiv mv = m\frac{dx}{dt} \tag{16}$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.

Figure 5: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

Vectors - more dimensional case:

Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

> **Tip**
>
> See the linear algebra book on vectors.

Position vector

We write the position **vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

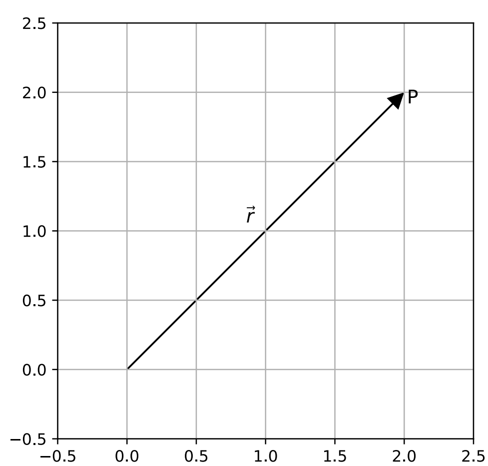


Figure 6: Some physical quantities (velocity, force etc) can be represented as a vector. The have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{17}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \hat{\imath},\ \vec{e}_x \\ &\hat{y},\ \hat{\jmath},\ \vec{e}_y \\ &\hat{z},\ \hat{k},\ \vec{e}_z \end{aligned} \tag{18}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{19}$$

Note that these representations are completely equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

Other textbooks

Note that other textbooks may use bold symbols to represent vectors:

$$\vec{F} = m\vec{a} \tag{20}$$

is the same as

$$\boldsymbol{F} = m\boldsymbol{a} \tag{21}$$

```
interactive(children=(FloatSlider(value=2.0, description='a x', max=4.0,
min=-2.0, step=0.5), FloatSlider(valu…
```

```
<function __main__.plot_vector_standaard_form(a_x=2.0, a_y=2.0, b_x=4.0,
b_y=1.0)>
```

Differentiating a vector

We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position.
As an example, let's take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

Velocity vector

$$\text{definition velocity} : \vec{v} \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{22}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{23}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction.
Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\widehat{dx} + dy\hat{y} + dz\hat{z} \\ &= (x + dx)\hat{x} + (y + dy)\hat{y} + (z + dz)\hat{z} \end{aligned} \tag{24}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{25}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{26}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{27}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

Momentum Vector

$$\text{definition momentum} : \vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \tag{28}$$

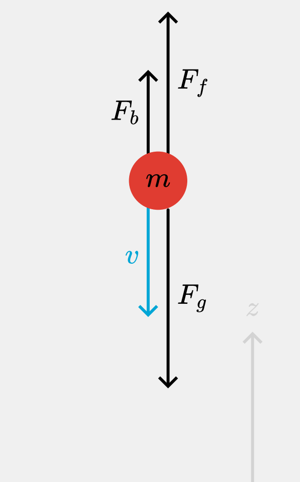
We can use the same reasoning and notation for acceleration:

Acceleration Vector

$$\text{definition acceleration} : \vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{29}$$

Example: The base jumper

Given the above explanation, we can now reconsider our description of the base jumper.



We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F_g} = -mg\hat{z} \tag{30}$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F_b} = \rho_{air} V g \hat{z} \tag{31}$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2. To solve this, we can write

$$\vec{F_f} = -\frac{1}{2}\rho_{air} C_D A \mid v \mid \vec{v} \tag{32}$$

Note that $\hat{z}$ is missing in (32) on purpose. That would be a simplification that is valid in the given situation, but not in general.

3.a.v. *Numerical computation and simulation:*

In simple cases we can come to an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right) \tag{33}$$

with

$$k = \frac{1}{2}\rho_{air} C_D A \tag{34}$$

In this case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. (Figure 8) presents a script for the software Coach, which you might have encountered in secondary school.

```
'Stop condition is set
'Computations are based on Euler

x := x + flow_1*Δt1
v := v + flow_2*Δt1

t1 := t1 + Δt1

flow_1 := v
Fz := m*g
Fw := 6*d*d*v*v
f := Fz - Fw
a := f/m
flow_2 := a
```

```
t1 := 0             's
Δt1 := 0.01         's

x := 0              'm
v := 0              'm/s
m := 75             'kg
g := 9.81           'm/s^2
d := 2.5            'm

flow_1 := v         'm/s
Fz := m*g           'N
Fw := 6*d*d*v*v     'N
f := Fz - Fw        'N
a := f/m            'm/s^2
flow_2 := a         'm/s^2
```

Figure 8: An example of a numerical simulation made in Coach. At the left the iterative calculation process, at the right the initial conditions.

Example: The base jumper

Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. Newton's second law then becomes:

$$m\vec{a} = -m\vec{g} - k \mid v \mid \vec{v} \tag{35}$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{36}$$

with $\vec{F}_{net} = -m\vec{g} - k \mid v \mid \vec{v}$

On a computer, we can not literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{37}$$

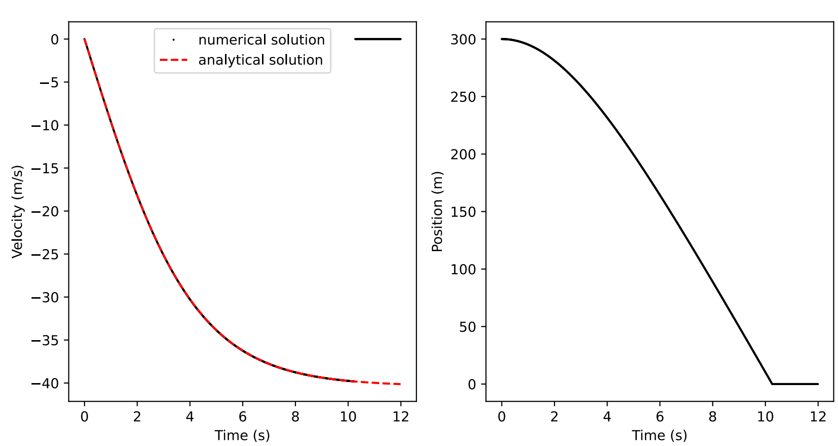
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots$. We will call these values v_i. Note that the force can be calculated at time t_i once we have v_i.

$$\begin{aligned} F_i &= mg - k \mid v_i \mid v_i \\ v_{i+1} &= v_i + \frac{F_i}{m}\Delta t \end{aligned} \tag{38}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i \Delta t \tag{39}$$

With the above rules, we can write an iterative code:



Exercise 1: Base jumper with initial velocity 🌶️

Change the code so that the base jumper starts with an initial velocity along the z-direction.

Is the acceleration in the z-direction with and without initial velocity the same? Elaborate.

> Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $\mid v \mid v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.
>
> To come back to our initial problem:
> It roughly takes 10 s to get close to terminal velocity (note that without friction the velocity would be 98 m/s). The building is not high enough to reach this velocity (safely).

Exercise 2: Unit analysis 🌶️

Given the formula $F = kv^2$. Derive the unit of k, expressed only in SI-units .

Exercise 3: Units based on physical constants[1] 🌶️ 🌶️

In physics, we assume that quantities like the speed of light (c) and Newton's gravitational constant (G) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant ($\hbar$, h an with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

1. Combine c, G and $\hbar$ into a quantity that has the dimensions of length.
2. Calculate the numerical value of this length in SI units (this is known as the Planck length).
3. Similarly, combine c, G and $\hbar$ into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

3.a.vi. *Examples, exercises and solutions:*

Exercises:

```
### Your code
```

```
### Your code
```

Solutions:

[1]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 4: Reynolds numbers[2] 🌶️ 🌶️

Physicists often use *dimensionless quantities* to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the *Reynolds number* in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$\mathrm{Re} = \frac{\mathrm{inertial forces}}{\mathrm{drag forces}} = \frac{\rho v L}{\mu} \tag{40}$$

where ρ is the density of the fluid (either a liquid or a gas), v the speed of the object, L its size, and μ the viscosity of the fluid. Typical values of the viscosity are $1.0\ \mathrm{mPa} \cdot \mathrm{s}$ for water, $50\ \mathrm{mPa} \cdot \mathrm{s}$ for ketchup, and $1.8 \cdot 10^{-5}\ \mathrm{Pa} \cdot \mathrm{s}$ for air.

1. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
2. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
3. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
4. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the diameter D of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) Q, and the kinematic viscosity $\nu \equiv \eta/\rho$.
5. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow in a cylindrical pipe is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a 1.0 cm diameter in the case that the flow is just laminar.

[2] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 5: Powers of ten 🌶️

Calculate:

1. $10^{-4} \cdot 10^{-8} =$
2. $\frac{10^6}{10^{-19} \cdot 10^4} =$
3. $10^{12} \cdot 10^{-15} =$

Exercise 6: Moving a box 🌶️

A box is on a frictionless incline of 10°. It is pushed upward with a force F_i for $\Delta t = 0.5$ s. It is then moving upward (inertia) but slows down due to gravity.

Below is a part of the python code. However, some essential elements of the code are missing indicated by **(..)**.

1. Include the correct code and run it.
2. Explain the two graphs, highlighting all essential features of the graph by relating these to the given problem.
3. At what time is the acceleration 0? At what time is the box back at its origin?

The above context is not very realistic as friction is neglected. We, however, can include friction easily as it is given by $\overrightarrow{F_w} = \mu \overrightarrow{F_N}$, with $\mu = 0.05$. Note that the direction of friction changes when the direction of the velocity changes!

4. Extend the code so that friction is included.

Exercise 7: Basejumper with parachute 🌶️ 🌶️

Our base jumper has yet not a soft landing. Luckily she has a working parachute. The parachute opens in 3.8 s reaching a total frontal area of 42.6 m^2. We can model the drag force using $\vec{F}_{drag} = k \mid v \mid \vec{v}$ with $k = 0.37$.

Write the code that simulates this jump of the base jumper with deploying the parachute. Show the (F_{drag}, t)-diagram and the (v, t)-diagram. What is the minimal height at which the parachute should be deployed?

Exercise 8: Circular motion 🌶️ 🌶️

Remember from secondary school circular motion, where the required force is given by $\vec{F} = \frac{m\vec{v}^2}{r}$. Now let's simulate that motion.

Assume:

- $m = 1 \text{ kg}$
- $\vec{r}_0 = (3, 0) \text{ m}$
- $\vec{v}(0) = (0, 7) \text{ m/s}$

Write the code. You know the output already (a circle with radius of 3)!

Solution 1: Solution to Exercise 1

$$\begin{aligned} F &= kv^2 \\ &= [.]\left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right] \Rightarrow [.] = \left[\frac{\mathrm{kg}}{\mathrm{m}}\right] \end{aligned} \tag{41}$$

Solution 2: Solution to Exercise 2

The physical constants c, G and $\hbar$ have the following numerical values and SI-units:

$$\begin{aligned} c &= 2.99792458 \cdot 10^8 \text{ m/s} \\ G &= 6.674 \cdot 10^-11 \text{ m}^3/(\text{kg} \cdot \text{s}^2) \\ \hbar &= 1.054 \cdot 10^{-34} \text{ kgm}^2/\text{s} \end{aligned} \tag{42}$$

Note: the value of c is precise, i.e. by definition given this value. The second is defined via the frequency of radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

If we want to combine these three units into a length scale, $\mathcal{L}$, we try the following:

$$[\mathcal{L}] = [c]^A [G]^B [\quad \hbar]^C \tag{43}$$

What we mean here, is that the units of the quantities (denoted by [.]) left and right should be the same. Thus, we get:

$$m^1 = \left(\frac{m}{s}\right)^A \left(\frac{m^3}{kgs^2}\right)^B \left(\frac{kgm^2}{s}\right)^C \tag{44}$$

We try to find A, B, C such that the above equation is valid. We can write this equation as:

$$m^1 = m^{A+3B+2C} \cdot kg^{-B+C} \cdot s^{-A-2B-C} \tag{45}$$

If we split this into requirements for m, kg, s we get:

$$\begin{aligned} m &: 1 = A + 3B + 2C \\ kg &: 0 = C - B \\ s &: 0 = -A - 2B - C \end{aligned} \tag{46}$$

From the second equation we get $B = C$. Substitute this into the first and third and we find:

$$\begin{aligned} m &: 1 = A + 5B \\ s &: 0 = -A - 3B \end{aligned} \tag{47}$$

Add these two equations: $1 = 2B \rightarrow B = \frac{1}{2}$ and thus $C = \frac{1}{2}$ and $A = -\frac{3}{2}$.

So if we plug these values into our starting equation we see:

$$\mathcal{L} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35} \text{ m} \tag{48}$$

We can repeat this for energy, $\mathcal{E}$:

$$[\mathcal{E}] = [c]^\alpha [G]^\beta [\quad \hbar]^\gamma \tag{49}$$

Note: the unit of energy, [J] needs to be written in terms of the basic units: $[J] = \text{kgm}^2/\text{s}^2$.

The outcome is: $\alpha = \frac{5}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$ and thus our energy is:

$$\mathcal{E} = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \cdot 10^9 \text{ J} \tag{50}$$

Solution 3: Solution to Exercise 3

1. The size of a duck is on the order of 30 cm. It flies at a speed of about 70 km/h, that is 20 m/s. Thus we compute for the Reynolds number of a flying duck:

$$Re \equiv \frac{\rho v L}{\mu} = 4.0 \cdot 10^5 \tag{51}$$

Clearly, the inertial force is dominant.

What about a swimming duck? Now the velocity is much smaller: $v \approx 1\ \mathrm{m/s} = 3.6\ \mathrm{km/h}$. The viscosity of water is $\mu_w = 1.0\ \mathrm{mPa \cdot s}$ and the water density is $1.0 \cdot 10^3\ \mathrm{kg/m^3}$. We, again, calculate the Reynolds number:

$$Re_w \equiv \frac{\rho v L}{\mu} = 3.0 \cdot 10^5 \tag{52}$$

Hence, also in this case inertial forces are dominant. This perhaps comes as a surprise, after all the velocity is much smaller and the viscosity much larger. However, the water density is also much larger!

2. For a swimming bacterium the numbers change. The size is now about 1μm and the velocity 60μ m/s (numbers taken from internet). That gives:

$$Re_b \equiv \frac{\rho v L}{\mu} = 6.0 \cdot 10^{-5} \tag{53}$$

and we see that here viscous forces are dominating.

3. For an oil tanker the Reynolds number is easily on the order of 10^8. Obviously, viscous forces don't do much. An oil tanker that wants to slow down can not do so by just stopping the motors and let the drag force decelerate them: the Reynolds number shows that the viscous drag is negligible compared to the inertial forces. Thus, the tanker has to use ots engines to slow down. Again the inertia of the system is so large, that it will take a long time to slow down. And a long time, means a long trajectory.
4. For the flow of water through a (circular) pipe the Reynolds number uses as length scale the pipe diameter. We can relate the velocity of the water in the pipe tot the total volume that is flowing per second through a cress section of the pipe:

$$Q = \frac{\pi}{4} D^2 v \to v = \frac{4Q}{\pi D^2} \tag{54}$$

Thus we can also write Re as:

$$Re \equiv \frac{\rho v D}{\mu} = \frac{4Q}{\pi \frac{mu}{\rho} D^2} = \frac{4Q}{\pi \nu D^2} \tag{55}$$

1. If $Re = 2300$ for the pipe flow, we have:

$$Re = \frac{vD}{\nu} = 2300 \to v = \frac{2300\nu}{D} \tag{56}$$

with $\nu = 1.0 \cdot 10^{-6}\ \mathrm{m^2/s}$ and $D = 1.0 \cdot 10^{-2}$ m we find: $v = 0.23$ m/s and $Q = 1.8 \cdot 10^{-5}\ \mathrm{m^3/s} = 0.018$ liter/s.

Solution 4: Solution to Exercise 4

1. $= 10^{-12}$
2. $= 10^{21}$
3. $= 10^{-3}$

Solution 5: Solution to Exercise 5

```
# Moving a box

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt


part_4 = 1 # Turn to 0 for first part

## Constants
m = 2 #kg
F = 30 #N
g = 9.81 #m/s^2
theta = np.deg2rad(10) #degrees

mu = 0.02
F_N = m*g*np.cos(theta) #N

## Time step
dt = 0.01 #s
t = np.arange(0, 10, dt) #s
t_F_stop = 0.5

## Initial conditions
x = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s


## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    if t[i] < t_F_stop:
        a = F/m - g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    else:
        a = -g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i]*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')
axs[0].plot(t, v, 'k.', markersize=1)


axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
axs[1].plot(t, x, 'k.', markersize=1)
```



```
plt.show()
```

Solution 6: Solution to Exercise 6

```
# Simulation of a base jumper

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

## Constants
A = 0.7 #m^2
m = 75 #kg
k = 0.37 #kg/m
g = 9.81 #m/s^2

## Time step
dt = 0.01 #s
t = np.arange(0, 12, dt) #s

## Initial conditions
z = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s
z[0] = 300 #m

## Deploy parachute
A_max = 42.6 #m^2
t_deploy_start = 2 #s
dt_deploy = 3.8 #s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    F = - m*g - k*A*abs(v[i])*v[i]  #N
    v[i+1] = v[i] + F/m*dt #m/s
    z[i+1] = z[i] + v[i]*dt #m
    # Check if the jumper is on the ground
    if z[i+1] < 0:
        break
    # Deploy parachute
    if t[i] > t_deploy_start and t[i] < t_deploy_start + dt_deploy:
        A += (A_max - A)/dt_deploy*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')

axs[0].plot(t, v, 'k.', markersize=1, label='numerical solution')
axs[0].vlines(t_deploy_start, v[t==t_deploy_start],0, color='gray',
linestyle='--', label='parachute deploy')

axs[0].legend()

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')

axs[1].plot(t, z, 'k.', markersize=1)
axs[1].vlines(t_deploy_start, 150,300, color='gray', linestyle='--',
label='parachute deploy')
```

Solution 7: Solution to Exercise 7

```
import numpy as np
import matplotlib.pyplot as plt

F = 49/3
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7


for i in range(0, len(t)-1):
    ax = -F*(x1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    ay = -F*(y1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt


plt.figure(figsize=(4,4))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()
```

3.b. *Newton's Laws*

Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton. These laws provide a systematic framework for understanding how and why objects move, and form the backbone of classical mechanics. Using these three laws we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

3.b.i. *Newton's Three Laws:*

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it: N1 is included in N2, after all:
if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \to \vec{p} = \text{constant} \to \vec{v} = \text{constant}$, provided m is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \tag{57}$$

For particles of constant mass, the two are equivalent:
if $m = \text{constant}$, then

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{58}$$

Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law, that we will derive in a minute. Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \tag{59}$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

> **Note**
>
> **Momentum** is in Dutch **impuls**; the English **impulse** is in Dutch **stoot**.

Exercise 9: 🌶️

Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the positive x-direction. The force lasts for a small time interval Δt.

What is the velocity of the particle for $t > \Delta t$?

In Newton's Laws, velocity, acceleration and momentum are key quantities. We repeat here their formal definition.

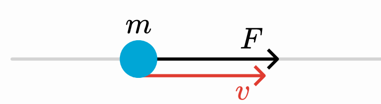
Definition

$$\begin{aligned} \text{velocity} : \vec{v} &\equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \\ \text{acceleration} : \vec{a} &\equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \\ \text{momentum} : \vec{p} &\equiv m\vec{v} = m\frac{d\vec{r}}{dt} \end{aligned} \tag{60}$$

Solution 8: Solution to Exercise 1

Interpret

First we make a sketch.



This is obviously a 1-dimensional problem. So, we can leave out the vector character of e.g. the force.

Develop

We will use $dp = Fdt$:

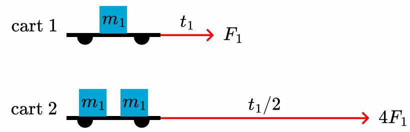
$$dp = Fdt \Rightarrow \Delta p = \int_0^{\Delta t} Fdt = F\Delta t \rightarrow \tag{61}$$

$$p(\Delta t) = p(0) + F\Delta t = mv_0 + F\Delta t \rightarrow \tag{62}$$

$$v(\Delta t) = v_0 + \frac{F}{m}\Delta t \tag{63}$$

Note that this example could also be solved by N2 in the form of $F = ma$. It is merely a matter of taste.

Exercise 10: A pushing contest 🌶️



Exercise 11: Newton's third law 🌶️

The base jumper from chapter 1 just jumped from the tall building. According to Newton's third law there are two coupled forces. Which are these, and what is the consequence of these two forces?

Solution 9: Solution to Exercise 3

The gravitational force acts from the earth on the jumper. Newton's law states that the jumper thus acts a gravitational force on the earth. Hence, the earth accelerates towards the jumper!

Although this sounds silly, when comparing this idea to the sun and the planets, we must draw the conclusion that the sun is actually wobbling as it is pulled towards the various planets! See also this animated explanation

3.b.ii. *Conservation of Momentum:*

From Newton's 2$^{\text{nd}}$ and 3$^{\text{rd}}$ law we can easily derive the law of conservation of momentum.
Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\begin{gathered} \frac{d\vec{p}_1}{dt} = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} = \vec{F}_{12} \end{gathered} \tag{64}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{65}$$

And, thus by adding the two momentum equations we get:

$$\left.\begin{array}{ll} \frac{d\vec{p}_1}{dt} & = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} & = \vec{F}_{12} = -\vec{F}_{21} \end{array}\right\} \Rightarrow \tag{66}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{67}$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = const \text{ i. e. does} not \text{ depend on time} \tag{68}$$

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects can not change.** No matter what the interaction is. It is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with $\vec{F}_{ij}$ the force from particle i felt by particle j):

$$\left.\begin{aligned} \frac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32} \\ \frac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32} \end{aligned}\right\} \tag{69}$$

Sum these three equations:

$$\begin{gathered} \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = 0 \\ \Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \text{const. i. e. does} not \text{ depend on time} \end{gathered} \tag{70}$$

For a system of N particles, extension is straight forward.

intermezzo: Intermezzo: Isaac Newton

At the end of the year of Galilei's death, Isaac Newton was born in Woolsthorpe-by-Colsterworth in England. He is regarded as the founder of classical mechanics and with that he can be seen as the father of physics.

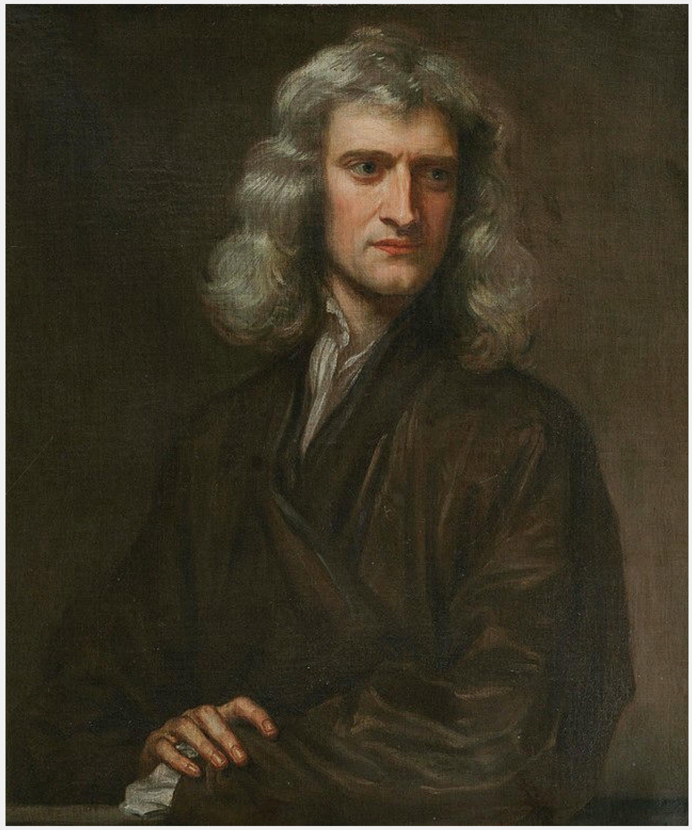


Figure 12: Isaac Newton (1642-1727). From Wikimedia Commons, public domain.

In 1661, he started studying at Trinity College, Cambridge. In 1665, the university temporarily closed due to an outbreak of the plague. Newton returned to his home and started working on some of his breakthroughs in calculus, optics and gravitation. Newton's list of discoveries is unsurpassed. He invented calculus (at about the same time and independent of Leibniz). Newton is known for 'the binomium of Newton', the the cooling law of Newton. He proposed that light is made of particles. And he formulated his law of gravity. Finally, he postulated his three laws that started classical mechanics and worked on several ideas towards energy and work. Much of our concepts in physics are based on the early ideas and their subsequent development in classical mechanics. The laws and rules apply to virtually all daily life physical phenomena and only do they require adaptation when we go to very small scale or extreme velocities and cosmology. In what follows, we will follow his footsteps, but in a modern way that we owe to many physicist and mathematicians that over the years shaped the theory of classical mechanics in a much more comprehensive form. We do not only stand on shoulders of giants, we stand on a platform carried by many.

Interesting to know is that his mentioning of *standing on shoulders* can be interpreted as a sneer towards Robert Hooke (1635-1703), with he was in a fight with over several things. Hooke was a rather short man... See, e.g., Gribbin (2019).

Important

In Newtonian mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace t by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

3.b.iii. *Newton's laws applied:*

Force addition, subtraction and decomposition:

Newton's laws describe how forces affect motion, and applying them often requires combining multiple forces acting on an object, see Figure 4. This is done through vector addition, subtraction, and decomposition—allowing us to find the net force and analyze its components in different directions, see this chapter in the book on linear algebra for a full elaboration on vector addition and subtraction.

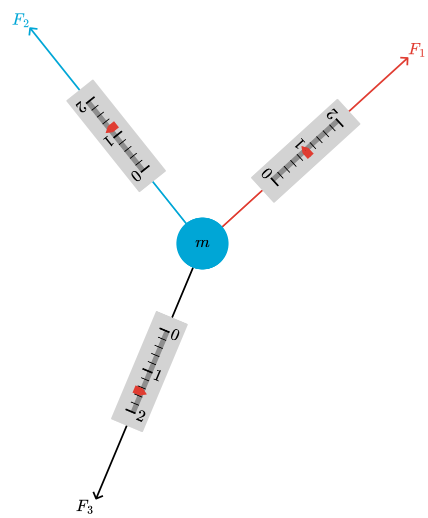


Figure 13: Three forces acting on a particle. In which direction will it accelerate?

Example: Three forces acting on a particle

Consider three forces acting on a particle:

$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{F}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$

What is the net force acting on the particle and in which direction will the particle accelerate?

Exercise 12: Forces acting on a particle in 3D

Three forces act on a particle with mass m:

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \text{and} \vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} \tag{71}$$

Determine the acceleration of this particle.

Solution 10: Solution to Exercise 4

$$\begin{aligned} \vec{F}_{net} &= \sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1-1 \\ 0+1+-0.5 \\ -4+3+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} \end{aligned} \tag{72}$$

Hence, the net force acting on the particle is $\sqrt{1^2 + .5^2} = 1.1\text{N}$ and the particle will accelerate in the direction $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$, in essence just like in the previous example. The magnitude of the acceleration is $a = F/m$ and can only be calculated when the mass of the particle is specified.

Example: Incline

The box in Figure 5 is at rest. Calculate the frictional force acting on the box.

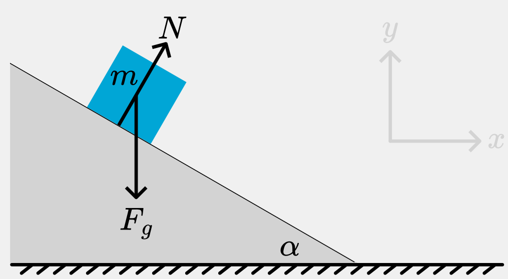


Figure 14: A box is at rest on an incline.

Develop

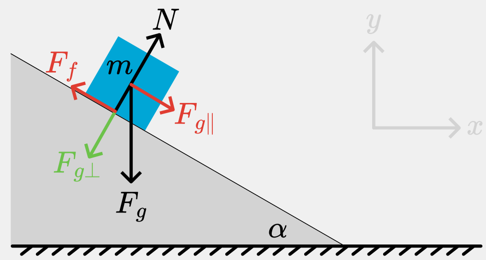
As the box is not moving (i.e. it has a constant velocity) the sum of forces on the box must be equal to zero. In the sketch we see two forces that clearly do not add up to zero. A third force is needed.

Evaluate

If we assume that only friction as a third force is present, we require:

$$\sum_i \vec{F}_i = 0 \Rightarrow \vec{F}_g + \overrightarrow{F_N} + \vec{F}_f = 0 \Rightarrow \vec{F}_f = -\vec{F}_g - \vec{F}_N \tag{73}$$

We can progress further by assuming that the friction force acts parallel to the slope. With this assumption, we can decomposed gravity in its components perpendicular to the slope and parallel to the slope.



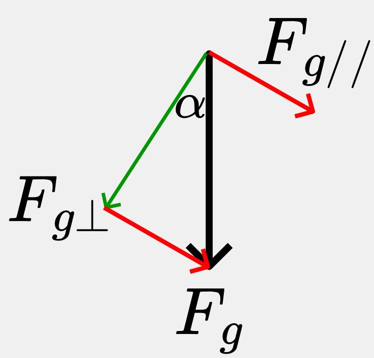
$$\vec{F}_g = \vec{F}_{g//} + \vec{F}_{g\perp} \tag{74}$$

The normal force exactly balances the perpendicular component: that is what a normal force does. Friction balances the parallel component of gravity:

$$\vec{F}_f + \vec{F}_{g//} = 0 \rightarrow \vec{F}_f = -\vec{F}_{g//} \tag{75}$$

and its magnitude is thus $F_f = F_g \sin\alpha$

Reminder



Remember from secondary school how to break down a force vector into components.

Acceleration due to gravity:

In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one, constant force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

Exercise 13: Tossing a stone in the air 🌶️

At a height of 1.5m a stone is tossed in the air with a velocity of 10m/s.

1. Calculate the maximum height that it reaches.
2. Calculate the time it takes to reach this point.
3. Calculate with which velocity it hits the ground.

$$a = \frac{F}{m} = \text{const.} \tag{76}$$

Hence, for the velocity:

$$v(t) = v_0 + \int_{t_0}^{t_e} a dt = a(t_e - t_0) + v_0 \tag{77}$$

assuming $t_0 = 0$ and$t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t) dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \tag{78}$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{79}$$

Example: 2D-motion

We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

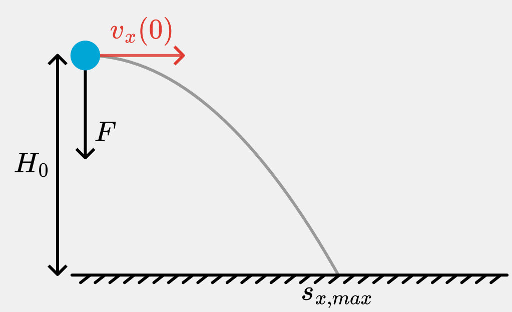


Figure 18: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

In the situation given in Figure 9 the object is thrown with a horizontal velocity of v_{x0}. As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_{x0} t \tag{80}$$

Solution 11: Solution to Exercise 5

Interpret

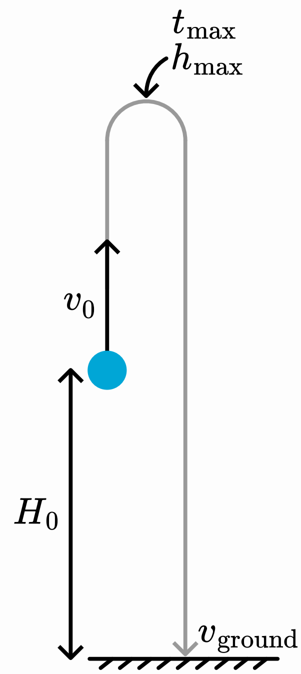


Figure 17: A free body diagram of the situation with all relevant quantities.

Only gravity acts on the stone (in the downward direction). We will call the position of the stone at time t: $s(t)$
Initial conditions: $t = 0 \rightarrow s(0) = s_0 = 1.5$ m and $\dot{s} = v = v_0 = 10$ m/s

Develop

1. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$ Highest point reached when $\dot{s} = 0$
2. $\Delta t = \frac{\Delta v}{a}$
3. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$. We are interested in the stone hitting the ground. Thus, solve for $s(t) = 0$ to find at what time this happens.

Evaluate

1. $\dot{s} = at + v_0 = -gt + v_0 = 0 \Rightarrow t = 1.02$s

$s(1.02) = -\frac{1}{2} * 9.81 * 1.02^2 + 10 * 1.02 + 1.5 = 6.6$m

1. See above.
2. $s(t) = \frac{1}{2}at^2 + v_0 t + s_0 = s_e$

$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(\frac{1}{2}a(s_0 - s_e))}}{2\frac{1}{2}a} = \frac{-10 \pm \sqrt{10^2 - 4(\frac{1}{2}(-9.81)(1.5))}}{-9.81} = 2.18$s
$v(2.18) = \dot{s}(2.18) = v_0 + at = 10 - 9.81 * 2.18 = -11.3$ m/s
Note that $t = -0.14$s is another solution, but not physically realistic.

The times we calculated are in the right order: First stone is tossed (at $t_0 = 0$); then it reaches its highest point (at $t_m = 1.02$ s). After that it falls and hits the ground at $t_e = 2.18$ s. Thus $t_0 < t_m < t_e$.

Furthermore, the velocity upon impact with the earth is negative as it should:

NOTE: Some of these solutions can be derived more easily using the concept of *conservation of energy* which will be covered in one of the next chapters.

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (26). Taking the y-direction upward, a starting height $y(0) = H_0$ and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \tag{81}$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x\sqrt{\frac{2H_0}{g}} \tag{82}$$

This motion is visualized in Figure 10. The trajectory is shown with s_x on the horizontal axis and s_y on the vertical axis. At regular time intervals Δt, velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, v_x and v_y, vary independently throughout the trajectory. Moreover, $\vec{v}(t)$ is the tangent of $s(t)$.

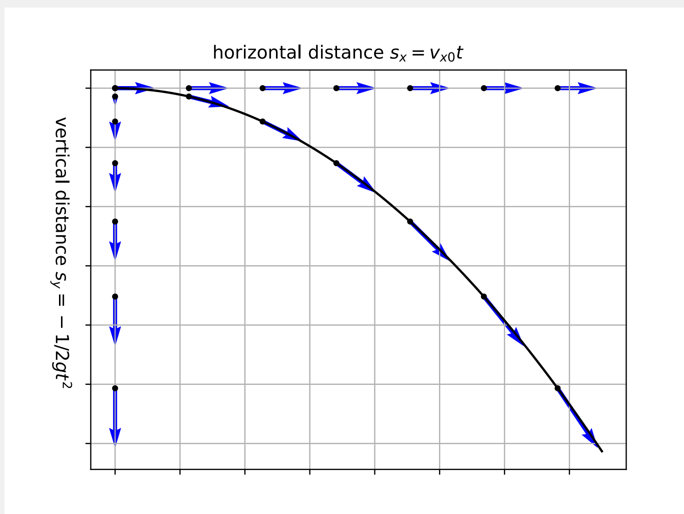


Figure 19: The parabolic motion is visualized with blue velocity vectors v, v_x andv_y shown at various points along the trajectory.

Danger

Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0t + 1/2at^2$

Frictional forces:

Exercise 14: Horizontal throw 🌶️

Derive the above expression (29) yourselves.

Exercise 15: Projectile motion 🌶️ 🌶️

Watch the recording below. What happens with the horizontal distance traveled per time unit? And with the vertical distance traveled?

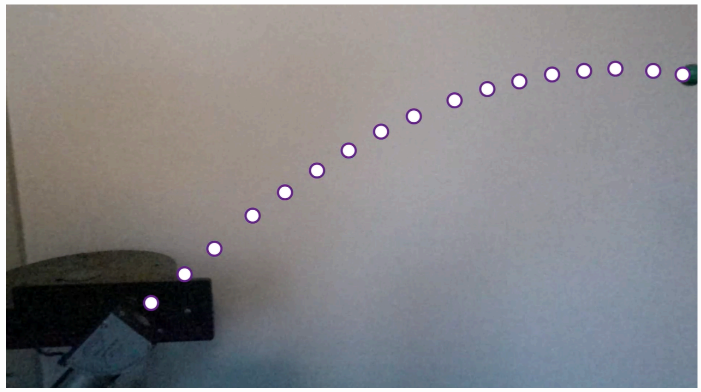


Figure 20: A parabolic motion visualized, with the position stored per time unit :alt: A short video of a small ball being shot upward at an angle. For each frame, it position is marked by a dot. The dots make up a parabola.

Assume the object with mass m_1 is shot from the ground with a velocity of v_0 at an angle of θ. Derive where the object hits the ground in terms of m_1, v_0 and θ.

How does the distance traveled changes when the mass of the object is doubled $m_2 = 2m_1$?

There are two main types of frictional force:

- **Static friction** prevents an object from starting to move. It adjusts in magnitude up to a maximum value, depending on how much force is trying to move the object. This maximum is given by

$$F_{static,max} = \mu_s N \tag{98}$$

where μ_s is the coefficient of static friction and N is the normal force. If the applied force exceeds this maximum, the object begins to slide.

- **Kinetic (dynamic) friction** opposes motion once the object is sliding. Its magnitude is generally constant and given by

$$F_{kinetic} = \mu_k N \tag{99}$$

where μ_k is the coefficient of kinetic friction. This force does not depend on the velocity of the object, only on the normal force and surface characteristics.

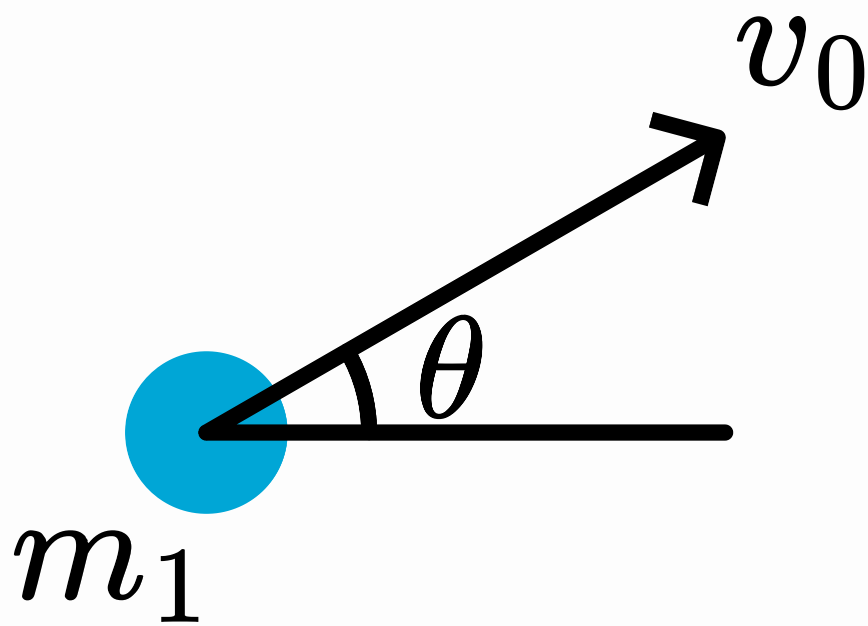
Friction always acts opposite to the direction of intended or actual motion and is essential in both preventing and controlling movement.

Material Pair	Static Friction (μ_s)	Kinetic Friction (μ_k)

Solution 12: Solution to Exercise 7

The horizontal traveled distance is the same per time unit. For the vertical traveled distance it decreases until $v_y = 0$ and then increases.

Interpret



Develop

The basic formulas are:

$$s_x(t) = v_x t \tag{83}$$

and

$$s_y(t) = v_y t - 1/2gt^2 \tag{84}$$

Evaluate

The horizontal traveled distance is given by:

$$s_x(t) = v_x t = v_0 \cos(\theta) t \tag{85}$$

The time the object stays in the air is

$$s_y(t) = v_y t - 1/2gt^2 = 0 \Rightarrow t = 0 \& t = \frac{2v_y}{g} = \frac{2v_0 \sin(\theta)}{g} \tag{86}$$

Hence, the maximum distance traveled is:

$$s_x(t) = v_x t = v_0 \cos(\theta) \frac{2v_0 \sin(\theta)}{g} = \frac{2v_o^2 \sin(\theta) \cos(\theta)}{g} \tag{87}$$

Note that the distance traveled is independent of the mass!

Exercise 16: Constant acceleration due to gravity 🌶️

We assumed a constant acceleration due to gravity. However, the gravitational force is given by $F = -G\frac{mM}{r^2}$.
Calculate at what height above the earth the acceleration due to gravity has 'significantly' changed from $9.81\mathrm{m/s^2}$, say to $9.80\mathrm{m/s^2}$.

Solution 13: Solution to Exercise 8

The acceleration of gravity is found by setting the gravitation force equal to $-mg$:

$$-G\frac{mM}{r^2} = -mg(r) \Rightarrow g(r) = G\frac{M}{r^2} \tag{88}$$

with M the mass of the earth.

At the surface of the earth, $r = R_e$ we have for the value of $g_e = 9.81\ \mathrm{m/s^2}$. We look for the height above the earth surface where g has dropped to $9.80\ \mathrm{m/s^2}$. If we call this height H, we write for the distance to the center of the earth $r = R_e + H$.

Thus, we look for $\frac{g(r)}{g_e} = \frac{9.80(\mathrm{m/s^2})}{9.81(\mathrm{m/s^2})} = 0.999$:

$$\frac{g(r)}{g_e} = \frac{GM/r^2}{GM/R_e^2} \rightarrow \frac{R_e^2}{r^2} = \frac{R_e^2}{(R_e+H)^2} = \frac{9.80}{9.81} = 0.999 \tag{89}$$

If we solve H from this equation we find: $H = 3.25$ km (we used $R_e = 6378$ km).

Note

We could have also looked at the ratios (between g and r), and found that $R_2 = \sqrt{.999} \cdot 6378 = 6374.8$ km. Hence, $H = 3.2$ km.

If we would have said: 'significant change' in means $g \rightarrow 9.81 \rightarrow 9.71\ \mathrm{m/s^2}$, we would have found $H = 32.8$ km.

Rubber on dry concrete	1.0	0.8
Steel on steel (dry)	0.74	0.57

Exercise 17: A rocket in space 🌶️

A rocket moves freely horizontal through space. At position $x = 2$ it turns on its propulsion. At position $x = 4$ it turns off its propulsion. The force due to this propulsion is directed perpendicular to the x-direction.

Provide a sketch of its movement highlighting all important parts.

Exercise 18: Particle movement 🌶️ 🌶️

Consider a particle which will travel a distance x. Find two different mathematical expressions for a force acting on the particle in such a way that the particle will travel the same distance in the same time for each $F(t)$ compared to a particle which travels at constant speed. Assume no initial velocity for the two particles.

Wood on wood (dry)	0.5	0.3

Solution 14: Solution to Exercise 10

Uniform motion ($F = ma = 0 \rightarrow s = v_0 t$).

Constant acceleration $a = const \rightarrow s = 1/2at^2$, with $a = \frac{2v_0^2}{s}$.

Consider the third being a harmonic oscillating force field: $F(t) = A\sin(2\pi f t)$ Then the equation of motion becomes:

$$a = F/m = \frac{A}{m}\sin(2\pi f t) \tag{90}$$

$$v = \int a dt = \frac{A}{m2\pi f}\cos(2\pi f t) + C_0 \tag{91}$$

Assuming $v(0) = 0 \rightarrow C_0 = -\frac{A}{m2\pi f}$

And,

$$x = \int v dt = \frac{A}{m(2\pi f)^2}\sin(2\pi f t) + C_0 t + C_1 \tag{92}$$

Assuming $x(0) = 0 \rightarrow C_1 = 0$

Hence:

$$x = \frac{A}{m(2\pi f)^2}\sin(2\pi f t) - \frac{A}{m2\pi f}t \tag{93}$$

Now, finding traveling the same distance in the same time AND the harmonic oscillation is complete (hence, $f = \frac{1}{t_e}$):

$$v_0 t_e = \frac{A t_e^2}{m(2\pi)^2}\sin(2\pi) - \frac{A t_e}{m2\pi}t_e \tag{94}$$

$$v_0 t_e = -\frac{A t_e^2}{m2\pi} \tag{95}$$

$$v_0 = -\frac{A t_e}{m2\pi} \tag{96}$$

$$\frac{m}{A} = -\frac{t_e}{v_{02}\pi} \tag{97}$$

Aluminum on steel	0.61	0.47
Ice on ice	0.1	0.03
Glass on glass	0.94	0.4
Copper on steel	0.53	0.36
Teflon on Teflon	0.04	0.04
Rubber on wet concrete	0.6	0.5
Leather on wood	0.56	0.4

Values are approximate and can vary depending on surface conditions.

Note

Not always are the friction coefficients constants. They may, for instance, depend on the relative velocity between the two materials.

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, step=…
```

```
<function __main__.update(theta, mu)>
```

Momentum example:

The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action = - reaction.

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

3.b.iv. *Forces & Inertia:*

Exercise 19: Block on an incline

A block with mass m is put on an inclined plane of which we can change the inclination angle θ.

1. Determine the angle at which it starts to slide in terms of mass m, inclination angle θ, acceleration due to gravity g and coefficient of static friction μ_s.
2. Once it starts to slide, it will accelerate. Determine its acceleration in terms of mass m, inclination angle θ, acceleration due to gravity g and coefficient of kinetic friction μ_f.

Solution 15: Solution to Exercise 11

1. There a two forces acting on m parallel to the inclined plane: friction and gravity's component parallel to the slope. These two determine the motion along the slope: if we tilt the plane the component of gravity parallel to the slope gets bigger. The particle will start moving when we pass: $F_{g_x} = F_s \rightarrow mg\sin(\theta) = mg\mu_s\cos(\theta) \Rightarrow \theta_{max} = \tan^{-1}(\mu_s)$
2. Once the particle is sliding downward, gravity and the kinetic friction determine how fast:

$$F_{net} = F_{g_x} - F_f \rightarrow ma = mg\sin(\theta) - mg\mu_k\cos(\theta) \Rightarrow \tag{100}$$

and

$$a = g(\sin(\theta) - \mu_k\cos(\theta)) \tag{101}$$

Exercise 20: 🌶️

A point particle (mass m) is dropped from rest at a height h above the ground. Only gravity acts on the particle with a constant acceleration g ($= 9.813$ m/s^2).

- Find the momentum when the particle hits the ground.
- What would be the earth' velocity upon impact?

Newton's laws introduce the concept of force. Forces have distinct features:

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:
  - they change the velocity, i.e. they accelerate the object

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \tag{106}$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \tag{107}$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Connecting physics and calculus

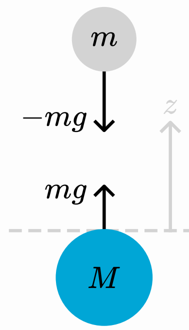
Let's look at a particle of mass m, that has initially (say at $t = 0$) a velocity v_0. For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind

Solution 16: Solution to Exercise 12

Let's do this one together. We follow the standard approach of IDEA: Interpret (and make your sketch!), develop (think 'model'), evaluate (solve your model) and assess (does it make any sense?).

Interpret

First a sketch: draw what is needed, no more, no less.



Develop

Actually this is half of the work, as when deciding what is needed we need to think what the problem really is. Above, is a sketch that could work. Both the object m and the earth (mass M) are drawn schematically. On each of them acts a force, where we know that on m standard gravity works. As a consequence of N3, a force equal in strength but opposite in direction acts on M.
Why do we draw forces? Well, the question mentions 'momentum the particle hits the ground'. Momentum and forces are coupled via N2.

We have drawn a z-coordinate: might be handy to remind us that this looks like a 1D problem (remember: momentum and force are both vectors).

As a first step, we ignore the motion of the earth. Argument? The magnitude of the ratio of the acceleration of earth over object is given by:

$$\frac{a_e}{a_o} = \frac{\mid F_{o \to e} \mid / m_e}{\mid F_{e \to o} \mid / m_o} = \frac{m_o}{m_e} \tag{102}$$

here for the second equality we used N3.

For all practical purposes, the mass of the object is many orders of magnitude smaller than that of the earth. Hence, we can conclude that the acceleration of the earth is many orders of magnitude less than that of the object. The latter is of the order of g, gravity's acceleration constant at the earth. Thus, the acceleration of the earth is next to zero and we can safely assume our lab system, that is connected to the earth, can be treated as an inertial system

Evaluate

That is rather straightforward. Now we have an object that moves under a constant force. So its velocity will increase linearly in time:

$$\frac{dp}{dt} = -mg \Rightarrow p(t) = \underbrace{mv_0}_{=0} - mgt = -mgt \tag{103}$$

We can find that the particle changed its momentum from $p_i = 0$ to $p_f = -mv$. The earth compensates for this, to keep momentum conserved. That gave that earth got a tiny, tiny upwards velocity. We could estimate the displacement of the earth. Suppose the particle has mass $m = 1$kg and is dropped from a height $H = 100$m. Then we get for the velocity of the mass upon impact: $v =$

of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m\frac{dv}{dt} = F \Rightarrow m\frac{dv}{dt} + bv = 0 \tag{108}$$

Clearly, we have to solve a differential equation which states that if you take the derivative of v you should get something like $-v$ back. From calculus we know, that exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with A and μ to be determined constants.

We substitute our trial v:

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \tag{109}$$

This should hold for all t. Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \tag{110}$$

We see, that also our unknown constant A drops out. And, thus, we find

$$\mu = \frac{b}{m} \tag{111}$$

Next we need to find A: for that we need an initial condition, which we have: at $t = 0$ is $v = v_0$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} \text{ and} v(0) = v_0 \tag{112}$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \tag{113}$$

From the solution for v, we easily find the position of m as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b}e^{-\frac{b}{m}t}\right) + B \tag{114}$$

We find B with the initial condition and get as final solution:

$$x(t) = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{115}$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

Inertia:

Inertia is denoted by the letter m for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like m_i, with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass m_1 and m_2 that are separated by a distance r_{12} attract each other via the so-called gravitational force ($\hat{r}_{12}$ is a unit vector along the line connecting m_1 and m_2):

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{116}$$

Here, we should have used a different symbol, rather than m. Something like m_g, as it is by no means obvious that the two 'masses' m_i and m_g refer to the same property. If you find that confusing, think about inertia and electric forces. Two particles with each an electric charge, q_1 and q_2, respectively exert a force on each other known as the Coulomb force:

$$\vec{F}_{C,12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \tag{117}$$

We denote the property associated with electric forces by q and call it charge. We have no problem writing

$$\begin{gathered}\vec{F} = m\vec{a} \\ \vec{F}_C = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}\end{gathered} \tag{118}$$

We do not confuse q by m or vice versa. They are really different quantities: q tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on m. If m is big, the particle will only get a small acceleration; the strength of the force does not depend on m at all. So far, so good. But what about m_g? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, m_i?

$$\begin{gathered}\vec{F} = m_i\vec{a} \\ \vec{F}_g = -G\frac{m_g M_g}{r^2}\hat{r}\end{gathered} \tag{119}$$

As far as we can tell (via experiments) m_i and m_g are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

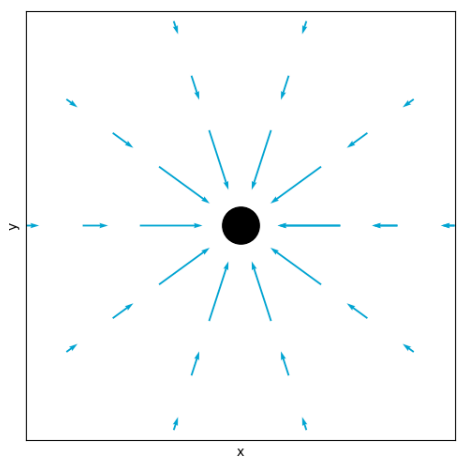
Force field

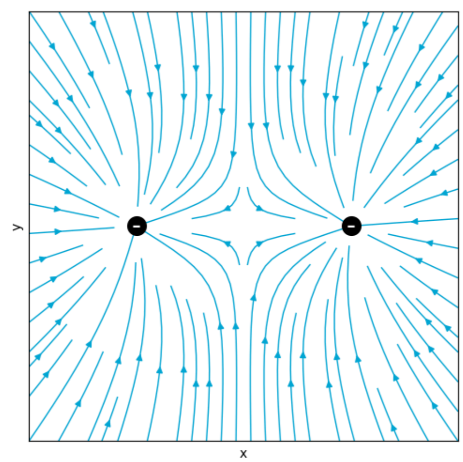
We have seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance — they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, the graph below at the left shows a gravitational field, described by $\vec{F}_g = G\frac{mM}{r^2}\hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.

The figure on the right shows the force field that a positively charged particle would feel due to the presence of 2 negatively charged particles (both of the same charge). Clearly this is a much more complicated force field.





Measuring mass or force

So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

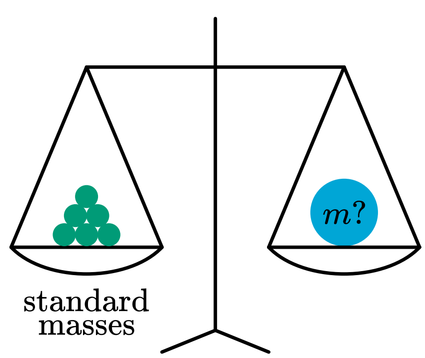


Figure 23: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a 'yes'.

As on second thought: the balance compares the pull of gravity. Hence, it 'measures' gravitational mass, rather than inertia. Luckily, Newton's laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\{\begin{matrix} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{matrix} \tag{120}$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \mid \frac{a_2}{a_1} \mid \tag{121}$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

Note

We will not use this method to measure mass. We came to the conclusion that we can't find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop 'force measuring equipment' that we can calibrate using the method discussed above.

intermezzo: Intermezzo: kilogram, unit of mass

In 1795 it was decided that 1 gram is the mass of 1 cm^3 of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the *kilogramme des Archives* was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4°C (when water has its maximum density).

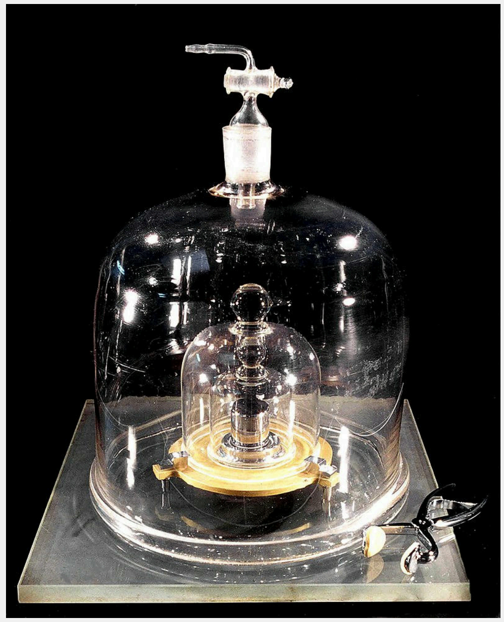


Figure 24: The International Prototype of the Kilogram, whose mass was defined to be one kilogram from 1889 to 2019. Picture by BIPM, CC BY-SA 3.0 igo, https://commons.wikimedia.org/w/index.php?curid=117707466

In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly $h = 6.62607015 \cdot 10^{-34}$ $\mathrm{kgm^2s^{-1}}$, the kilogram is now defined via h, the meter and second.

Eötvös experiment on mass:

The question whether inertial mass and gravitational mass are the same has put experimentalists to work. It is by no means an easy question. Gravity is a very weak force. Moreover, determining that two properties are identical via an experiment is virtually impossible due to experimental uncertainty. Experimentalist can only tell the outcome is 'identical' within a margin. Newton already tried to establish experimentally that the two forms of mass are the same. However, in his days the inaccuracy of experiments was rather large. Dutch scientist Simon Stevin concluded in 1585 that the difference must be less than 5%. He used his famous 'drop masses from the church' experiments for this (they were primarily done to show that every mass falls with the same acceleration).

A couple of years later, Galilei used both fall experiments and pendula to improve this to: less than 2%. In 1686, Newton using pendula managed to bring it down to less than 1‰ .

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848-1918). We will here briefly introduce the experiment. For a full analysis, we need knowledge about angular momentum and centrifugal forces that we do not deal with in this book.

The experiment
The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to the gravitational mass) and some inertial force (sensitive to the inertial mass) are present. Obviously, gravitational forces between two objects out of our daily life are extremely small. These will be very difficult to detect and thus introduce a large error if the experiment relies on measuring them. Eötvös came up with a different idea. He connected two different objects with different masses, m_1 and m_2, via a (almost) massless rod. Then, he attached a thin wire to the rod and let it hang down.

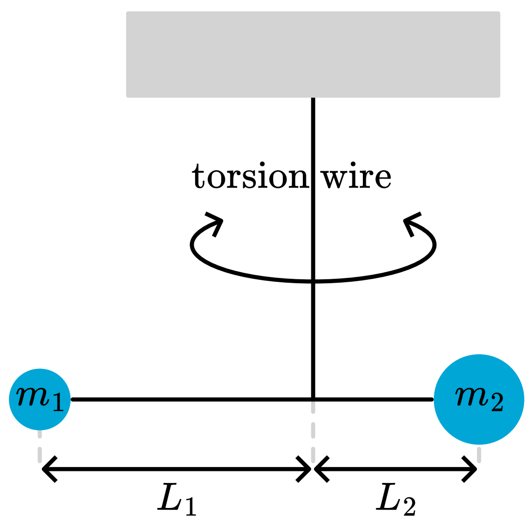


Figure 25: Torsion balance used by Eötvös.

This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to m_g, but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

Note

The question is not if m_i/m_g is different from 1. If that was the case but the ratio would always be the same, then we would just rescale m_g, that is redefine the value of the gravitational const G to make m_g equal to m_i. No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if m_i and m_g are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

Tip

Want to know more about this experiment? Watch this videoclip.

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