

Classical Mechanics & Special Relativity for Starters

Robert F. Mudde, Bernd Rieger, C. Freek. J. Pols

Preface

A book on mechanics at AP of TUD

Contents

| | |
|---|----------|
| 1. Introduction | 1 |
| 1.1 About this book | 1 |
| 1.1.1 Special features | 1 |
| 1.1.2 Feedback | 2 |
| 1.2 Authors | 2 |
| 1.2.1 About the authors | 2 |
| 1.3 Open Educational Resource | 2 |
| 1.3.1 Software and license | 3 |
| 1.3.2 Images, videos, apps, intermezzos | 3 |
| 1.3.3 How to cite this book | 3 |
| 2. Mechanics | 5 |
| 3. Mechanics | 7 |
| 3.1 The language of Physics | 7 |
| 3.1.1 Representations | 7 |
| 3.1.1.1 Verbal descriptions | 7 |
| 3.1.1.2 Visual representations | 8 |
| 3.1.1.3 Equations | 9 |
| 3.1.2 How to solve a physics problem? | 9 |
| 3.1.2.1 Example | 10 |
| 3.1.3 Calculus | 13 |
| 3.1.3.1 Differentiation | 13 |
| 3.1.4 Definition of velocity, acceleration and momentum | 14 |
| 3.1.4.1 Vectors - more dimensional case | 15 |
| 3.1.5 Numerical computation and simulation | 19 |
| 3.1.6 Examples, exercises and solutions | 22 |
| 3.1.6.1 Exercises | 22 |
| 3.1.6.2 Solutions | 25 |
| 3.2 Newton's Laws | 32 |
| 3.2.1 Newton's Three Laws | 32 |
| 3.2.2 Conservation of Momentum | 34 |
| 3.2.3 Newton's laws applied | 36 |
| 3.2.3.1 Force addition, subtraction and decomposition | 36 |
| 3.2.3.2 Acceleration due to gravity | 38 |
| 3.2.3.3 Frictional forces | 41 |
| 3.2.3.4 Momentum example | 46 |
| 3.2.4 Forces & Inertia | 46 |
| 3.2.4.1 Inertia | 49 |
| 3.2.4.2 Eötvös experiment on mass | 52 |
| 3.2.5 Examples, exercises and solutions | 55 |
| 3.2.5.1 Exercises set 1 | 55 |
| 3.2.5.2 Answers set 1 | 57 |
| 3.2.5.3 Exercises set 2 | 64 |
| 3.3 Work & Energy | 71 |
| 3.3.1 Work | 71 |
| 3.3.2 Kinetic Energy | 71 |
| 3.3.3 Worked Examples | 72 |

| | | |
|-------------------|--|-----------|
| 3.3.4 | Gravitational potential energy | 75 |
| 3.3.5 | Conservative force | 76 |
| 3.3.5.1 | Stokes' Theorem | 77 |
| 3.3.5.2 | Conservative force and $\vec{\nabla} \times \vec{F}$ | 81 |
| 3.3.6 | Potential Energy | 81 |
| 3.3.6.1 | Energy versus Newton's Second Law | 82 |
| 3.3.7 | Stable and Unstable Equilibrium | 83 |
| 3.3.7.1 | Taylor Series Expansion of the Potential | 84 |
| 3.3.8 | Examples, exercises and solutions | 86 |
| 3.3.8.1 | Exercise set 1 | 86 |
| 3.3.8.2 | Answers set 1 | 96 |
| 3.3.8.3 | Exercise set 2 | 96 |
| 3.3.8.4 | Answers set 2 | 96 |
| References | | 97 |

1. Introduction

1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and presumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version. Note that we made good use of other open educational resources, several exercises stem from such resources.

1.1.1 Special features

In this book you will find some 'special' features. Some of these are emphasized by their own style:

Exercise 1:

Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3 

Intermezzo: Intermezzos

Intermezzos contain background information on the topic, of the people working on the concepts.

Experiment: Experiments

We include some basic experiments that can be done at home.

Example: Examples

We provide various examples showcasing, e.g., calculations.

Python

We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to ‘enable compute’. Try it by pushing the ON-button and subsequently the play button and see the output in de code-cell below.

```
print("The square root of 2 is: ", 2 ** 0.5)
```

New concepts, such as *Free body diagram*, are introduced with a hoover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

The book can be read online. There is, however, the opportunity to download the materials and play with the code offline as well. We advise to use Jupyter Lab in combination with MyST.

1.1.2 Feedback

This is the first version of this book. Although many have work on it and several iterations have been made, there might still be issues. Do you see a mistake, do you have suggestions for exercises, are parts missing or abundant. Tell us! You can use the Feedback button at the top right button. You will need a (free) GitHub account to report an issue!

1.2 Authors

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Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files. Also thanks to Vebe Helmes, Alexander Lopes-Cardozo, Sep Schouwenaar and Winston de Greef for their comments and suggestions.

1.2.1 About the authors

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1.3 Open Educational Resource

This book is licensed under a Creative Commons Attribution 4.0 International License unless stated otherwise. It is part of the collection of Interactive Open Textbooks of TU Delft Open.

This website is a Jupyter Book. Source files are available for download using the button on the top right.

1.3.1 Software and license

This website is a Jupyter Book. Markdown source files are available for download using the button on the top right, licensed under CC-BY-NC (unless stated otherwise). All python codes / apps are freely reusable, adaptable and redistributable (CC0).

1.3.2 Images, videos, apps, intermezzos

The cover image is inspired by the work of 3blue1brown developer Grant Sanderson.

All vector images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistribution, adhere to the CC-BY licences.

We embedded several clips from 3blue1brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

The Intermezzos, which elaborate on the lives of various scientists and the efforts behind key physical discoveries, are composed by drawing from a range of different sources. Rather than directly reproducing any one account, these stories have been reworked into a narrative that fits the context and audience of this book.

1.3.3 How to cite this book

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*, CC BY-NC

2. Mechanics

3. Mechanics

3.1 The language of Physics

Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, both colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

3.1.1 Representations

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.



Figure 3.1: A base jumper is used as context to get familiar with representation, picture from <https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg>

3.1.1.1 Verbal descriptions

Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

Example: Base jumper: Verbal description

Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

3.1.1.2 Visual representations

Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more concrete.

- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.
- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.

Example: Base jumper: Free body diagram

The situation is sketched in Figure 2 using a Free body diagram. Note that all details of the jumper are ignored in the sketch.

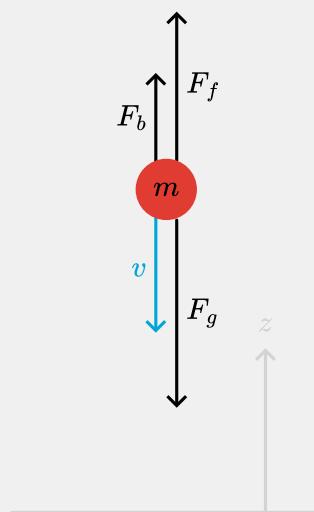


Figure 3.2: Force acting on the jumper.

- m = mass of jumper (in kg);
- v = velocity of jumper (in m/s);
- F_g = gravitational force (in N);
- F_f = drag force by the air (in N);

- F_b = buoyancy (in N): like in water also in air there is an upward force, equal to the weight of the displaced air.

3.1.1.3 Equations

Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

Example: Base jumper: equations

The forces acting on the jumper are already shown in Figure 2. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \quad (3.1)$$

We can specify each of the force:

$$\begin{aligned} F_g &= -mg = -\rho_p V_p g \\ F_f &= \frac{1}{2} \rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \quad (3.2)$$

with g the acceleration of gravity, ρ_p the density of the jumper ($\approx 10^3 \text{ kg/m}^3$), V_p the volume of the jumper, ρ_{air} the density of air ($\approx 1.2 \text{ kg/m}^3$), C_D the so-called drag coefficient, A the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering which representations suits best for the given situation. We will practice these when solving problems.

Danger

Note that in the example above we neglected directions. In our equation we should have been using vector notation, which we will cover in one of the next sections in this chapter.

3.1.2 How to solve a physics problem?

One of the most common mistakes made by ‘novices’ when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you apply this way of working automatically. It is called IDEA, an acronym that stands for:

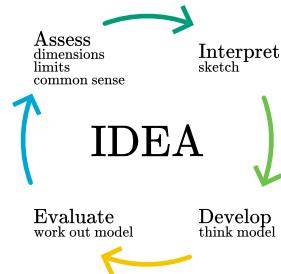


Figure 3.3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. Actually *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?).

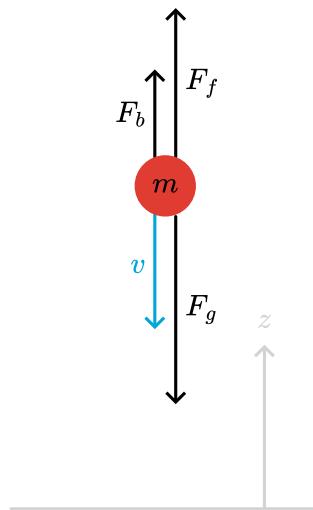
We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spent on a problem.

3.1.2.1 Example

Interpret

Three forces act on the jumper, shown in the figure below. Finding the terminal velocity implies that all forces are balanced ($\sum F = 0$).



The buoyancy force is much smaller than the force of gravity (about 0.1%) and we neglect it.

Develop

We know all forces: gravitational force equals the drag force

$$F_g = F_f$$

$$mg = \frac{1}{2} \rho_{air} C_D A v^2 \quad (3.3)$$

Evaluate

Assume a mass of 75 kg, an acceleration due to gravity of 9.81 m/s^2 , and air density of 1.2 kg/m^3 , a drag coefficient of 1, a frontal surface area of 0.7 m^2 .

$$mg = \frac{1}{2} \rho_{air} C_D A v^2 \quad (3.4)$$

Rewriting:

$$v = \sqrt{\frac{2mg}{\rho_{air} C_D A}}$$

$$v = \sqrt{\frac{2 \cdot 75 \text{ (kg)} \cdot 9.81 \text{ (m/s}^2\text{)}}{1.2 \text{ (kg/m}^3\text{)} \cdot 1 \cdot 0.7 \text{ (m}^2\text{)}}}$$

$$v = 40 \text{ m/s} \quad (3.5)$$

Assess

We may know that raindrops typically reach a terminal velocity of less than 10 m/s. A terminal velocity of 40 m/s seems therefore plausible.

Note that we didn't solve the problem entirely! We only calculated the terminal velocity, where the question was how long it would roughly take to reach such a velocity.

Good Practice

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or multiple steps in one step. If you make a mistake, it will be very hard to trace it back.

Next: check constantly the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π . The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

Powers of ten

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms (10^{-10} m) to the distance between stars

(10^{16} m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milli seconds, GB for gigabyte that is 1 billion bytes. Here is a list of prefixes.

| Prefix | Symbol | Math | Prefix | Symbol | Math |
|--------|--------|------------|--------|--------|-----------|
| Yocto | y | 10^{-24} | Base | • | 10^0 |
| Zepto | z | 10^{-21} | Deca | da | 10^1 |
| Atto | a | 10^{-18} | Hecto | h | 10^2 |
| Femto | f | 10^{-15} | Kilo | k | 10^3 |
| Pico | p | 10^{-12} | Mega | M | 10^6 |
| Nano | n | 10^{-9} | Giga | G | 10^9 |
| Micro | μ | 10^{-6} | Tera | T | 10^{12} |
| Milli | m | 10^{-3} | Peta | P | 10^{15} |
| Centi | c | 10^{-2} | Exa | E | 10^{18} |
| Deci | d | 10^{-1} | Zetta | Z | 10^{21} |
| Base | • | 10^0 | Yotta | Y | 10^{24} |

On quantities and units

Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter m? Quantities are expressed in italics (*m*) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

- Meter (m) – length
- Kilogram (kg) – mass
- Second (s) – time
- Ampere (A) – electric current
- Kelvin (K) – temperature
- Mole (mol) – amount of substance
- Candela (cd) – luminous intensity

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned}
 W &= F \cdot s = ma \cdot s = m \frac{\Delta v}{\Delta t} \cdot s \\
 &= [N] \cdot [m] = [\text{kg}] \cdot [\text{m}/\text{s}^2] \cdot [\text{m}] = [\text{kg}] \cdot \left[\frac{\text{m}/\text{s}}{\text{s}} \right] \cdot [\text{m}] = \left[\frac{\text{kg}\text{m}^2}{\text{s}^2} \right]
 \end{aligned} \tag{3.6}$$

Tip

For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

3.1.3 Calculus

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features, that help us understanding the problem and its solution much deeper.

Given the example of the base jumper, we would like to know how the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m \frac{dv}{dt} \quad (3.7)$$

$$m \frac{dv}{dt} = mg - \frac{1}{2} \rho_{air} C_D A v^2 \quad (3.8)$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself. Before we even try to solve this problem ($v(t) = \dots$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

3.1.3.1 Differentiation

Many physical phenomena are described by differential equations. That may be because a system evolves in time or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2nd law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time; velocity in itself is the derivative of position with respect to time. Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad (3.9)$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \quad (3.10)$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \quad (3.11)$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \quad (3.12)$$

It is just a matter of notation.

3.1.4 Definition of velocity, acceleration and momentum

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that x . In general, x will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time t . How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in the time interval Δt . The (momentary) velocity is defined as:

Velocity

$$\text{definition velocity : } v \equiv \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \frac{dx}{dt} \quad (3.13)$$

Similarly, we define the acceleration as the change of the velocity over a time interval Δt : $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

Acceleration

$$\text{definition acceleration : } a \equiv \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{(t + \Delta t) - t} = \frac{dv}{dt} \quad (3.14)$$

Consequently,

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2} \quad (3.15)$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

Momentum

$$\text{definition momentum : } p \equiv mv = m \frac{dx}{dt} \quad (3.16)$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.



Figure 3.5: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

3.1.4.1 Vectors - more dimensional case

Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

Tip

See the linear algebra book on vectors.

Position vector

We write the position **vector** of the particle as \vec{r} . This vector is a ‘thing’, it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

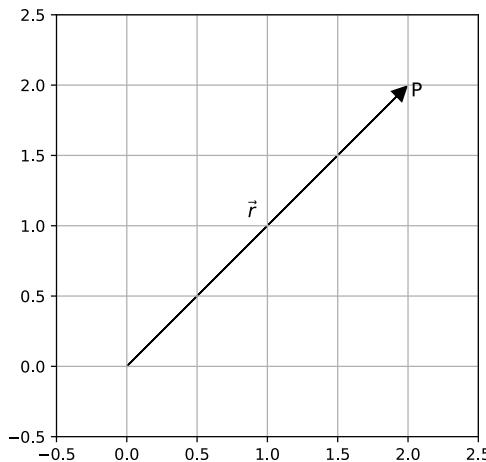


Figure 3.6: Some physical quantities (velocity, force etc) can be represented as a vector.
They have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent \vec{r} as a column.

$$\vec{r} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (3.17)$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} \hat{x}, \hat{i}, \vec{e}_x \\ \hat{y}, \hat{j}, \vec{e}_y \\ \hat{z}, \hat{k}, \vec{e}_z \end{aligned} \quad (3.18)$$

With this notation, we can write the position vector \vec{r} as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \quad (3.19)$$

Note that these representations are completely equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: \hat{x} and \hat{y} are perpendicular **vectors**.

Other textbooks

Note that other textbooks may use bold symbols to represent vectors:

$$\vec{F} = m\vec{a} \quad (3.20)$$

is the same as

$$\mathbf{F} = m\mathbf{a} \quad (3.21)$$

```
interactive(children=(FloatSlider(value=2.0, description='a_x', max=4.0,
min=-2.0, step=0.5), FloatSlider(valu...
<function __main__.plot_vector_standaard_form(a_x=2.0, a_y=2.0, b_x=4.0,
b_y=1.0)>
```

Differentiating a vector

We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position.

As an example, let's take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

Velocity vector

$$\text{definition velocity : } \vec{v} \equiv \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \quad (3.22)$$

What does it mean? Differentiating is looking at the change of your ‘function’ when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (3.23)$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to ‘the next point’ $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction.

Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\hat{x} + dy\hat{y} + dz\hat{z} \\ &= (x + dx)\hat{x} + (y + dy)\hat{y} + (z + dz)\hat{z} \end{aligned} \quad (3.24)$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \quad (3.25)$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} &\equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \quad (3.26)$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \quad (3.27)$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

Momentum Vector

$$\text{definition momentum : } \vec{p} \equiv m\vec{v} = m \frac{d\vec{r}}{dt} \quad (3.28)$$

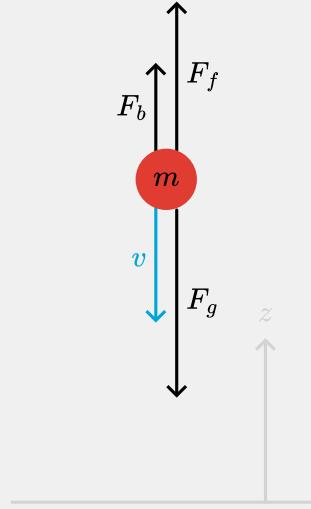
We can use the same reasoning and notation for acceleration:

Acceleration Vector

$$\text{definition acceleration : } \vec{a} \equiv \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (3.29)$$

Example: The base jumper

Given the above explanation, we can now reconsider our description of the base jumper.



We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F}_g = -mg\hat{z} \quad (3.30)$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F}_b = \rho_{air}Vg\hat{z} \quad (3.31)$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2 . To solve this, we can write

$$\vec{F}_f = -\frac{1}{2}\rho_{air}C_D A |v| \vec{v} \quad (3.32)$$

Note that \hat{z} is missing in (32) on purpose. That would be a simplification that is valid in the given situation, but not in general.

3.1.5 Numerical computation and simulation

In simple cases we can come to an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}} t\right) \quad (3.33)$$

with

$$k = \frac{1}{2} \rho_{air} C_D A \quad (3.34)$$

In this case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. (Figure 8) presents a script for the software Coach, which you might have encountered in secondary school.

| | | |
|----------------------------------|-----------------|--------|
| 'Stop condition is set | t1 := 0 | 's |
| 'Computations are based on Euler | Δt1 := 0.01 | 's |
| x := x + flow_1*Δt1 | x := 0 | 'm |
| v := v + flow_2*Δt1 | v := 0 | 'm/s |
| t1 := t1 + Δt1 | m := 75 | 'kg |
| flow_1 := v | g := 9.81 | 'm/s^2 |
| Fz := m*g | d := 2.5 | 'm |
| Fw := 6*d*d*v*v | flow_1 := v | 'm/s |
| f := Fz - Fw | Fz := m*g | 'N |
| a := f/m | Fw := 6*d*d*v*v | 'N |
| flow_2 := a | f := Fz - Fw | 'N |
| | a := f/m | 'm/s^2 |
| | flow_2 := a | 'm/s^2 |

Figure 3.8: An example of a numerical simulation made in Coach. At the left the iterative calculation process, at the right the initial conditions.

Example: The base jumper

Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. Newton's second law then becomes:

$$m\vec{a} = -m\vec{g} - k | v | \vec{v} \quad (3.35)$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \quad (3.36)$$

with $\vec{F}_{net} = -m\vec{g} - k | v | \vec{v}$

On a computer, we can not literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t + \Delta t) = \vec{v}(t) + \frac{\vec{F}}{m} \Delta t \quad (3.37)$$

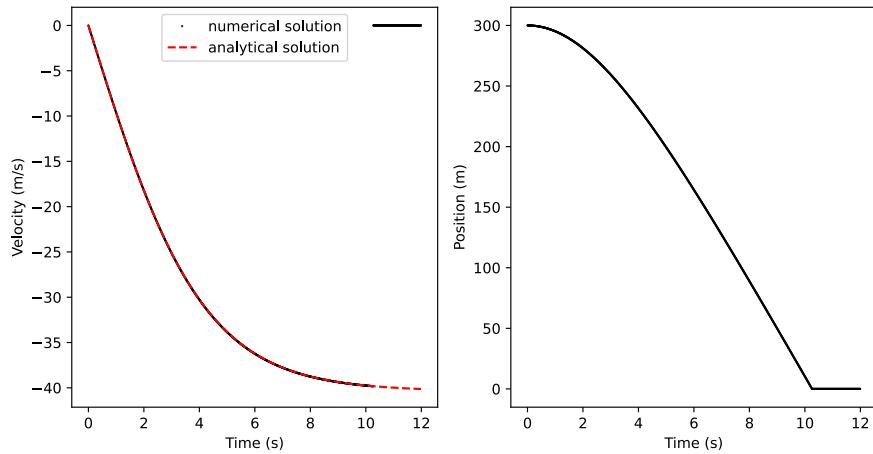
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, \dots$. We will call these values v_i . Note that the force can be calculated at time t_i once we have v_i .

$$\begin{aligned} F_i &= mg - k | v_i | v_i \\ v_{i+1} &= v_i + \frac{F_i}{m} \Delta t \end{aligned} \quad (3.38)$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i \Delta t \quad (3.39)$$

With the above rules, we can write an iterative code:



Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $| v | v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

Exercise 3.10: Base jumper with initial velocity 

Change the code so that the base jumper starts with an initial velocity along the z-direction.

Is the acceleration in the z-direction with and without initial velocity the same?
Elaborate.

To come back to our initial problem:

It roughly takes 10 s to get close to terminal velocity (note that without friction the velocity would be 98 m/s). The building is not high enough to reach this velocity (safely).

Exercise 3.11: Unit analysis

Given the formula $F = kv^2$. Derive the unit of k , expressed only in SI-units .

Exercise 3.12: Units based on physical constants¹

In physics, we assume that quantities like the speed of light (c) and Newton's gravitational constant (G) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant (\hbar , h with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

1. Combine c , G and \hbar into a quantity that has the dimensions of length.
2. Calculate the numerical value of this length in SI units (this is known as the Planck length).
3. Similarly, combine c , G and \hbar into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

3.1.6 Examples, exercises and solutions

3.1.6.1 Exercises

Your code

Your code

¹Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 3.13: Reynolds numbers²

Physicists often use *dimensionless quantities* to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the *Reynolds number* in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$Re = \frac{\text{inertialforces}}{\text{dragforces}} = \frac{\rho v L}{\mu} \quad (3.40)$$

where ρ is the density of the fluid (either a liquid or a gas), v the speed of the object, L its size, and μ the viscosity of the fluid. Typical values of the viscosity are $1.0 \text{ mPa} \cdot \text{s}$ for water, $50 \text{ mPa} \cdot \text{s}$ for ketchup, and $1.8 \cdot 10^{-5} \text{ Pa} \cdot \text{s}$ for air.

1. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
2. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
3. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
4. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the diameter D of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) Q , and the kinematic viscosity $\nu \equiv \eta/\rho$.
5. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow in a cylindrical pipe is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a 1.0 cm diameter in the case that the flow is just laminar.

Exercise 3.14: Powers of ten

Calculate:

1. $10^{-4} \cdot 10^{-8} =$
2. $\frac{10^6}{10^{-19} \cdot 10^4} =$
3. $10^{12} \cdot 10^{-15} =$

²Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 3.15: Moving a box

A box is on a frictionless incline of 10° . It is pushed upward with a force F_i for $\Delta t = 0.5$ s. It is then moving upward (inertia) but slows down due to gravity.

Below is a part of the python code. However, some essential elements of the code are missing indicated by (...).

1. Include the correct code and run it.
2. Explain the two graphs, highlighting all essential features of the graph by relating these to the given problem.
3. At what time is the acceleration 0? At what time is the box back at its origin?

The above context is not very realistic as friction is neglected. We, however, can include friction easily as it is given by $\vec{F}_w = \mu \vec{F}_N$, with $\mu = 0.05$. Note that the direction of friction changes when the direction of the velocity changes!

4. Extend the code so that friction is included.

Exercise 3.16: Basejumper with parachute

Our base jumper has yet not a soft landing. Luckily she has a working parachute. The parachute opens in 3.8 s reaching a total frontal area of 42.6 m^2 . We can model the drag force using $\vec{F}_{drag} = k |v| \vec{v}$ with $k = 0.37$.

Write the code that simulates this jump of the base jumper with deploying the parachute. Show the (F_{drag}, t) -diagram and the (v, t) -diagram. What is the minimal height at which the parachute should be deployed?

Exercise 3.17: Circular motion

Remember from secondary school circular motion, where the required force is given by $F = \frac{mv^2}{r}$. The corresponding vector form is: $\vec{F} = -\frac{mv^2}{r}\hat{r}$, or equivalent: $\vec{F} = -\frac{mv^2}{r^2}\vec{r}$. Now let's simulate that motion.

Assume:

- $m = 1 \text{ kg}$
- $\vec{r}_0 = (3, 0) \text{ m}$
- $\vec{v}(0) = (0, 7) \text{ m/s}$

Write the code. You know the output already (a circle with radius of 3)!

Solution 3.18: Solution to Exercise 1

$$\begin{aligned}
 F &= kv^2 \\
 &= [.] \left[\frac{\text{m}^2}{\text{s}^2} \right] \Rightarrow [.] = \left[\frac{\text{kg}}{\text{m}} \right]
 \end{aligned} \tag{3.41}$$

3.1.6.2 Solutions

Solution 3.19: Solution to Exercise 2

The physical constants c , G and \hbar have the following numerical values and SI-units:

$$\begin{aligned} c &= 2.99792458 \cdot 10^8 \text{ m/s} \\ G &= 6.674 \cdot 10^{-11} \text{ m}^3 / (\text{kg} \cdot \text{s}^2) \\ \hbar &= 1.054 \cdot 10^{-34} \text{ kgm}^2/\text{s} \end{aligned} \quad (3.42)$$

Note: the value of c is precise, i.e. by definition given this value. The second is defined via the frequency of radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

If we want to combine these three units into a length scale, \mathcal{L} , we try the following:

$$[\mathcal{L}] = [c]^A [G]^B [\hbar]^C \quad (3.43)$$

What we mean here, is that the units of the quantities (denoted by [.]) left and right should be the same. Thus, we get:

$$m^1 = \left(\frac{m}{s}\right)^A \left(\frac{m^3}{\text{kg} \cdot \text{s}^2}\right)^B \left(\frac{\text{kg} \cdot \text{m}^2}{\text{s}}\right)^C \quad (3.44)$$

We try to find A, B, C such that the above equation is valid. We can write this equation as:

$$m^1 = m^{A+3B+2C} \cdot \text{kg}^{-B+C} \cdot \text{s}^{-A-2B-C} \quad (3.45)$$

If we split this into requirements for m, kg, s we get:

$$\begin{aligned} m : 1 &= A + 3B + 2C \\ \text{kg} : 0 &= C - B \\ \text{s} : 0 &= -A - 2B - C \end{aligned} \quad (3.46)$$

From the second equation we get $B = C$. Substitute this into the first and third and we find:

$$\begin{aligned} m : 1 &= A + 5B \\ \text{s} : 0 &= -A - 3B \end{aligned} \quad (3.47)$$

Add these two equations: $1 = 2B \rightarrow B = \frac{1}{2}$ and thus $C = \frac{1}{2}$ and $A = -\frac{3}{2}$.

So if we plug these values into our starting equation we see:

$$\mathcal{L} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35} \text{ m} \quad (3.48)$$

We can repeat this for energy, \mathcal{E} :

$$[\mathcal{E}] = [c]^\alpha [G]^\beta [\hbar]^\gamma \quad (3.49)$$

Note: the unit of energy, [J] needs to be written in terms of the basic units: $[J] = \text{kg} \cdot \text{m}^2/\text{s}^2$.

The outcome is: $\alpha = \frac{5}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$ and thus our energy is:

$$\mathcal{E} = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \cdot 10^9 \text{ J} \quad (3.50)$$

Solution 3.20: Solution to Exercise 3

1. The size of a duck is on the order of 30 cm. It flies at a speed of about 70 km/h, that is 20 m/s. Thus we compute for the Reynolds number of a flying duck:

$$Re \equiv \frac{\rho v L}{\mu} = 4.0 \cdot 10^5 \quad (3.51)$$

Clearly, the inertial force is dominant.

What about a swimming duck? Now the velocity is much smaller: $v \approx 1 \text{ m/s} = 3.6 \text{ km/h}$. The viscosity of water is $\mu_w = 1.0 \text{ mPa} \cdot \text{s}$ and the water density is $1.0 \cdot 10^3 \text{ kg/m}^3$. We, again, calculate the Reynolds number:

$$Re_w \equiv \frac{\rho v L}{\mu} = 3.0 \cdot 10^5 \quad (3.52)$$

Hence, also in this case inertial forces are dominant. This perhaps comes as a surprise, after all the velocity is much smaller and the viscosity much larger. However, the water density is also much larger!

2. For a swimming bacterium the numbers change. The size is now about $1 \mu\text{m}$ and the velocity $60 \mu\text{m/s}$ (numbers taken from internet). That gives:

$$Re_b \equiv \frac{\rho v L}{\mu} = 6.0 \cdot 10^{-5} \quad (3.53)$$

and we see that here viscous forces are dominating.

3. For an oil tanker the Reynolds number is easily on the order of 10^8 . Obviously, viscous forces don't do much. An oil tanker that wants to slow down can not do so by just stopping the motors and let the drag force decelerate them: the Reynolds number shows that the viscous drag is negligible compared to the inertial forces. Thus, the tanker has to use its engines to slow down. Again the inertia of the system is so large, that it will take a long time to slow down. And a long time, means a long trajectory.
4. For the flow of water through a (circular) pipe the Reynolds number uses as length scale the pipe diameter. We can relate the velocity of the water in the pipe to the total volume that is flowing per second through a cross section of the pipe:

$$Q = \frac{\pi}{4} D^2 v \rightarrow v = \frac{4Q}{\pi D^2} \quad (3.54)$$

Thus we can also write Re as:

$$Re \equiv \frac{\rho v D}{\mu} = \frac{4Q}{\pi \frac{\mu u}{\rho} D^2} = \frac{4Q}{\pi \nu D^2} \quad (3.55)$$

1. If $Re = 2300$ for the pipe flow, we have:

$$Re = \frac{vD}{\nu} = 2300 \rightarrow v = \frac{2300\nu}{D} \quad (3.56)$$

with $\nu = 1.0 \cdot 10^{-6} \text{ m}^2/\text{s}$ and $D = 1.0 \cdot 10^{-2} \text{ m}$ we find: $v = 0.23 \text{ m/s}$ and $Q = 1.8 \cdot 10^{-5} \text{ m}^3/\text{s} = 0.018 \text{ liter/s}$.

Solution 3.21: Solution to Exercise 4

1. $= 10^{-12}$

2. $= 10^{21}$

3. $= 10^{-3}$

Solution 3.22: Solution to Exercise 5

```

# Moving a box

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

part_4 = 1 # Turn to 0 for first part

## Constants
m = 2 #kg
F = 30 #N
g = 9.81 #m/s^2
theta = np.deg2rad(10) #degrees

mu = 0.02
F_N = m*g*np.cos(theta) #N

## Time step
dt = 0.01 #s
t = np.arange(0, 10, dt) #s
t_F_stop = 0.5

## Initial conditions
x = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    if t[i] < t_F_stop:
        a = F/m - g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    else:
        a = -g*np.sin(theta) - F_N*mu*np.where(v[i] != 0, np.sign(v[i]),
0)*part_4
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i]*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')
axs[0].plot(t, v, 'k.', markersize=1)

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
axs[1].plot(t, x, 'k.', markersize=1)

plt.show()

```

Solution 3.23: Solution to Exercise 6

```
# Simulation of a base jumper

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

## Constants
A = 0.7 #m^2
m = 75 #kg
k = 0.37 #kg/m
g = 9.81 #m/s^2

## Time step
dt = 0.01 #s
t = np.arange(0, 12, dt) #s

## Initial conditions
z = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s
z[0] = 300 #m

## Deploy parachute
A_max = 42.6 #m^2
t_deploy_start = 2 #s
dt_deploy = 3.8 #s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    F = - m*g - k*A*abs(v[i])*v[i] #N
    v[i+1] = v[i] + F/m*dt #m/s
    z[i+1] = z[i] + v[i]*dt #m
    # Check if the jumper is on the ground
    if z[i+1] < 0:
        break
    # Deploy parachute
    if t[i] > t_deploy_start and t[i] < t_deploy_start + dt_deploy:
        A += (A_max - A)/dt_deploy*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')

axs[0].plot(t, v, 'k.', markersize=1, label='numerical solution')
axs[0].vlines(t_deploy_start, v[t==t_deploy_start], 0, color='gray',
linestyle='--', label='parachute deploy')

axs[0].legend()

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')

axs[1].plot(t, z, 'k.', markersize=1)
axs[1].vlines(t_deploy_start, 150, 300, color='gray', linestyle='--',
label='parachute deploy')

plt.show()
```

Solution 3.24: Solution to Exercise 7

```

import numpy as np
import matplotlib.pyplot as plt

F = 49/3
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7

for i in range(0, len(t)-1):
    ax = -F*(x1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    ay = -F*(y1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt

plt.figure(figsize=(4,4))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()

```

3.2 Newton's Laws

Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton. These laws provide a systematic framework for understanding how and why objects move, and form the backbone of classical mechanics. Using these three laws we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

3.2.1 Newton's Three Laws

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it:
N1 is included in N2, after all:

if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = \text{constant} \rightarrow \vec{v} = \text{constant}$, provided m is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \quad (3.57)$$

For particles of constant mass, the two are equivalent:

if $m = \text{constant}$, then

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \quad (3.58)$$

Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law, that we will derive in a minute. Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \quad (3.59)$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

Note

Momentum is in Dutch **impuls**; the English **impulse** is in Dutch **stoot**.

In Newton's Laws, velocity, acceleration and momentum are key quantities. We repeat here their formal definition.

Definition

Exercise 3.25:

Consider a point particle of mass m , moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the positive x-direction. The force lasts for a small time interval Δt .

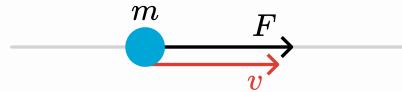
What is the velocity of the particle for $t > \Delta t$?

$$\begin{aligned} \text{velocity : } \vec{v} &\equiv \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \\ \text{acceleration : } \vec{a} &\equiv \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \\ \text{momentum : } \vec{p} &\equiv m\vec{v} = m\frac{d\vec{r}}{dt} \end{aligned} \quad (3.60)$$

Solution 3.26: Solution to Exercise 1

Interpret

First we make a sketch.



This is obviously a 1-dimensional problem. So, we can leave out the vector character of e.g. the force.

Develop

We will use $dp = Fdt$:

$$dp = Fdt \Rightarrow \Delta p = \int_0^{\Delta t} Fdt = F\Delta t \rightarrow \quad (3.61)$$

$$p(\Delta t) = p(0) + F\Delta t = mv_0 + F\Delta t \rightarrow \quad (3.62)$$

$$v(\Delta t) = v_0 + \frac{F}{m}\Delta t \quad (3.63)$$

Note that this example could also be solved by N2 in the form of $F = ma$. It is merely a matter of taste.

Exercise 3.28: A pushing contest



Exercise 3.30: Newton's third law

The base jumper from chapter 1 just jumped from the tall building. According to Newton's third law there are two coupled forces. Which are these, and what is the consequence of these two forces?

Solution 3.31: Solution to Exercise 3

The gravitational force acts from the earth on the jumper. Newton's law states that the jumper thus acts a gravitational force on the earth. Hence, the earth accelerates towards the jumper!

Although this sounds silly, when comparing this idea to the sun and the planets, we must draw the conclusion that the sun is actually wobbling as it is pulled towards the various planets! See also this animated explanation

3.2.2 Conservation of Momentum

From Newton's 2nd and 3rd law we can easily derive the law of conservation of momentum.

Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\begin{aligned}\frac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12}\end{aligned}\tag{3.64}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12}\tag{3.65}$$

And, thus by adding the two momentum equations we get:

$$\begin{aligned}\frac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} = -\vec{F}_{21}\end{aligned}\Rightarrow\tag{3.66}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0\tag{3.67}$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = \text{const i. e. does not depend on time}\tag{3.68}$$

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects can not change**. No matter what the interaction is. It is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with \vec{F}_{ij} the force from particle i felt by particle j):

$$\begin{aligned}
 \frac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\
 \frac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32} \\
 \frac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32}
 \end{aligned} \tag{3.69}$$

Sum these three equations:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \rightarrow \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = 0 \tag{3.70}$$

$\Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \text{const. i. e. does not depend on time}$

For a system of N particles, extension is straight forward.

Intermezzo: Isaac Newton

At the end of the year of Galilei's death, Isaac Newton was born in Woolsthorpe-by-Colsterworth in England. He is regarded as the founder of classical mechanics and with that he can be seen as the father of physics.

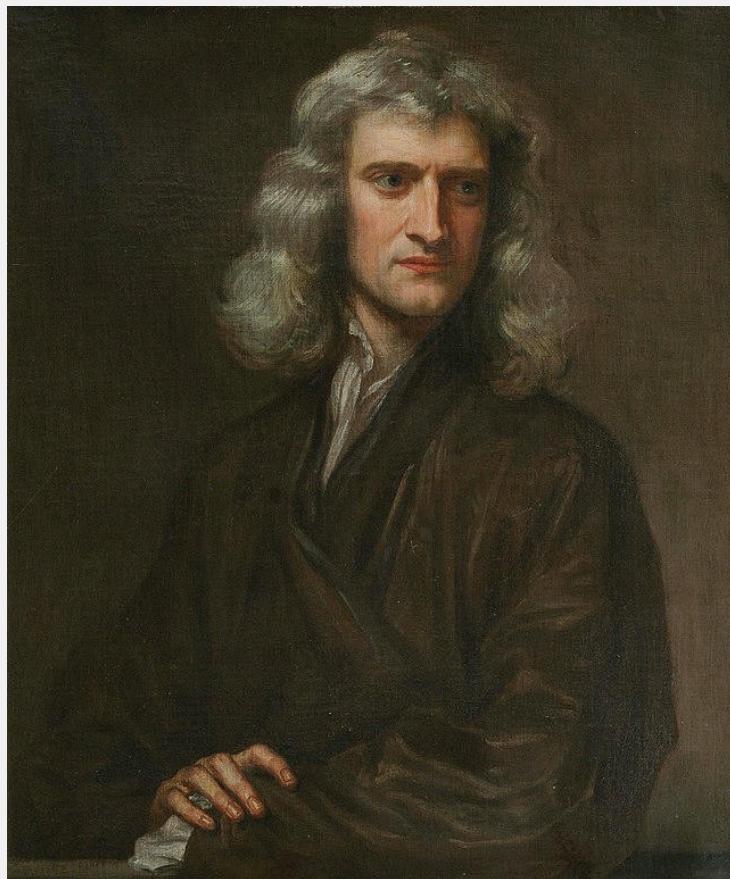


Figure 3.32: Isaac Newton (1642-1727). From Wikimedia Commons, public domain.

In 1661, he started studying at Trinity College, Cambridge. In 1665, the university temporarily closed due to an outbreak of the plague. Newton returned to his home and started working on some of his breakthroughs in calculus, optics and gravitation. Newton's list of discoveries is unsurpassed. He invented calculus (at about the same time and independent of Leibniz). Newton is known for 'the binomium of Newton', the cooling law of Newton. He proposed that light is made of particles. And he

formulated his law of gravity. Finally, he postulated his three laws that started classical mechanics and worked on several ideas towards energy and work. Much of our concepts in physics are based on the early ideas and their subsequent development in classical mechanics. The laws and rules apply to virtually all daily life physical phenomena and only do they require adaptation when we go to very small scale or extreme velocities and cosmology. In what follows, we will follow his footsteps, but in a modern way that we owe to many physicist and mathematicians that over the years shaped the theory of classical mechanics in a much more comprehensive form. We do not only stand on shoulders of giants, we stand on a platform carried by many.

Interesting to know is that his mentioning of *standing on shoulders* can be interpreted as a sneer towards Robert Hooke (1635-1703), with he was in a fight with over several things. Hooke was a rather short man... See, e.g., Gribbin (2019).

Important

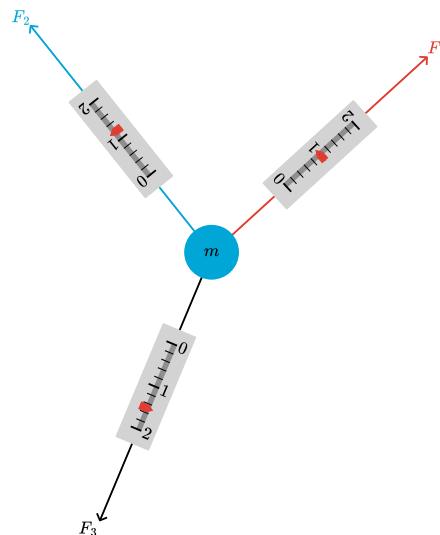
In Newtonian mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace t by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

3.2.3 Newton's laws applied

3.2.3.1 Force addition, subtraction and decomposition

Newton's laws describe how forces affect motion, and applying them often requires combining multiple forces acting on an object, see Figure 4. This is done through vector addition, subtraction, and decomposition—allowing us to find the net force and analyze its components in different directions, see this chapter in the book on linear algebra for a full elaboration on vector addition and subtraction.



Exercise 3.34: Forces acting on a particle in 3D

Three forces act on a particle with mass m :

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \text{ and } \vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} \quad (3.71)$$

Determine the acceleration of this particle.

Solution 3.35: Solution to Exercise 4

$$\begin{aligned} \vec{F}_{net} &= \sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1-1 \\ 0+1+-0.5 \\ -4+3+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} \end{aligned} \quad (3.72)$$

Hence, the net force acting on the particle is $\sqrt{1^2 + .5^2} = 1.1\text{N}$ and the particle will accelerate in the direction $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$, in essence just like in the previous example. The magnitude of the acceleration is $a = F/m$ and can only be calculated when the mass of the particle is specified.

Figure 3.33: Three forces acting on a particle. In which direction will it accelerate?

Example: Three forces acting on a particle

Consider three forces acting on a particle:

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{F}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \text{ and } \vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \\ 1 \end{pmatrix}$$

What is the net force acting on the particle and in which direction will the particle accelerate?

Example: Incline

The box in Figure 5 is at rest. Calculate the frictional force acting on the box.

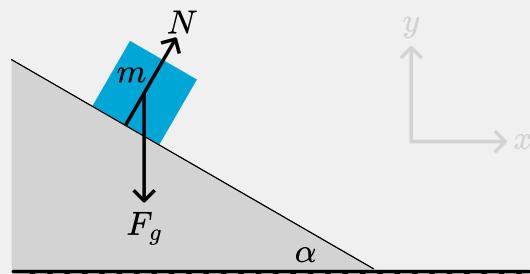


Figure 3.36: A box is at rest on an incline.

Develop

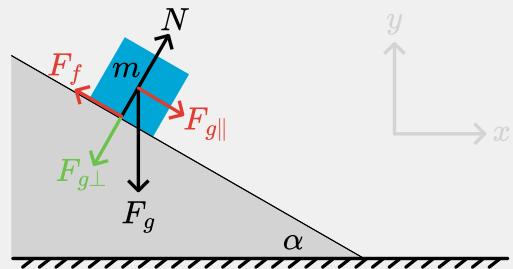
As the box is not moving (i.e. it has a constant velocity) the sum of forces on the box must be equal to zero. In the sketch we see two forces that clearly do not add up to zero. A third force is needed.

Evaluate

If we assume that only friction as a third force is present, we require:

$$\sum_i \vec{F}_i = 0 \Rightarrow \vec{F}_g + \vec{F}_N + \vec{F}_f = 0 \Rightarrow \vec{F}_f = -\vec{F}_g - \vec{F}_N \quad (3.73)$$

We can progress further by assuming that the friction force acts parallel to the slope. With this assumption, we can decompose gravity in its components perpendicular to the slope and parallel to the slope.



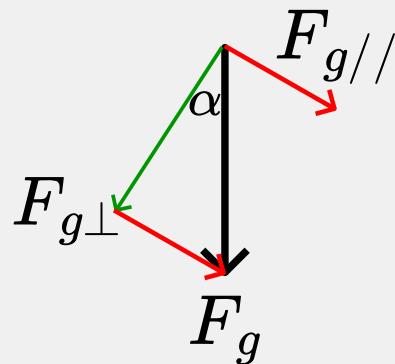
$$\vec{F}_g = \vec{F}_{g//} + \vec{F}_{g\perp} \quad (3.74)$$

The normal force exactly balances the perpendicular component: that is what a normal force does. Friction balances the parallel component of gravity:

$$\vec{F}_f + \vec{F}_{g//} = 0 \rightarrow \vec{F}_f = -\vec{F}_{g//} \quad (3.75)$$

and its magnitude is thus $F_f = F_g \sin \alpha$

Reminder



Remember from secondary school how to break down a force vector into components.

3.2.3.2 Acceleration due to gravity

In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one,

Exercise 3.39: Tossing a stone in the air 🍎

At a height of 1.5m a stone is tossed in the air with a velocity of 10m/s.

1. Calculate the maximum height that it reaches.
2. Calculate the time it takes to reach this point.
3. Calculate with which velocity it hits the ground.

constant force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

$$a = \frac{F}{m} = \text{const.} \quad (3.76)$$

Hence, for the velocity:

$$v(t) = v_0 + \int_{t_0}^{t_e} a dt = a(t_e - t_0) + v_0 \quad (3.77)$$

assuming $t_0 = 0$ and $t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t) dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \quad (3.78)$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \quad (3.79)$$

Example: 2D-motion

We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

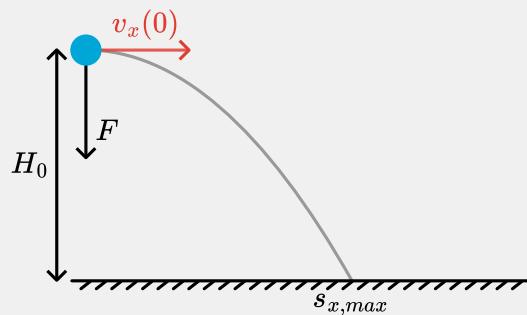


Figure 3.42: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

In the situation given in Figure 9 the object is thrown with a horizontal velocity of v_{x0} . As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_{x0}t \quad (3.80)$$

Solution 3.40: Solution to Exercise 5

Interpret

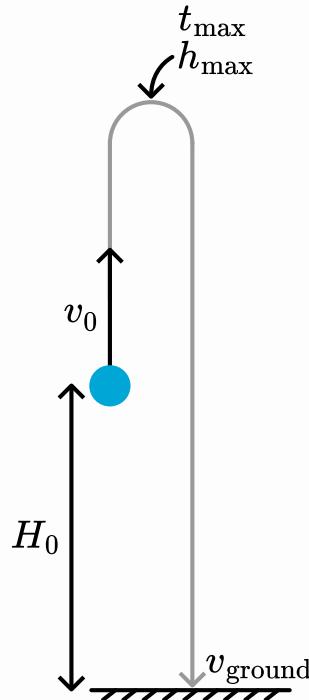


Figure 3.41: A free body diagram of the situation with all relevant quantities.

Only gravity acts on the stone (in the downward direction). We will call the position of the stone at time t : $s(t)$

Initial conditions: $t = 0 \rightarrow s(0) = s_0 = 1.5 \text{ m}$ and $\dot{s} = v = v_0 = 10 \text{ m/s}$

Develop

1. $s(t) = \frac{1}{2}at^2 + v_0t + s_0$ Highest point reached when $\dot{s} = 0$
2. $\Delta t = \frac{\Delta v}{a}$
3. $s(t) = \frac{1}{2}at^2 + v_0t + s_0$. We are interested in the stone hitting the ground.
Thus, solve for $s(t) = 0$ to find at what time this happens.

Evaluate

$$1. \dot{s} = at + v_0 = -gt + v_0 = 0 \Rightarrow t = 1.02 \text{ s}$$

$$s(1.02) = -\frac{1}{2} * 9.81 * 1.02^2 + 10 * 1.02 + 1.5 = 6.6 \text{ m}$$

1. See above.

$$2. s(t) = \frac{1}{2}at^2 + v_0t + s_0 = s_e$$

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(\frac{1}{2}a(s_0 - s_e))}}{2\frac{1}{2}a} = \frac{-10 \pm \sqrt{10^2 - 4(\frac{1}{2}(-9.81)(1.5))}}{-9.81} = 2.18 \text{ s}$$

$$v(2.18) = \dot{s}(2.18) = v_0 + at = 10 - 9.81 * 2.18 = -11.3 \text{ m/s}$$

Note that $t = -0.14 \text{ s}$ is another solution, but not physically realistic.

Assess

The times we calculated are in the right order: First stone is tossed (at $t_0 = 0$), reaches the highest point (at $t_1 = 1.02 \text{ s}$) and then hits the ground (at $t_2 = 2.18 \text{ s}$). This is consistent with the conservation of energy which will be covered in one of the next chapters.

Furthermore, the velocity upon impact with the earth is negative as it should: the stone is falling downward. Its magnitude is on the order of the initial upward velocity, which makes sense. Finally, note that the initial velocity was positive.

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (26). Taking the y -direction upward, a starting height $y(0) = H_0$ and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \quad (3.81)$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x \sqrt{\frac{2H_0}{g}} \quad (3.82)$$

This motion is visualized in Figure 10. The trajectory is shown with s_x on the horizontal axis and s_y on the vertical axis. At regular time intervals Δt , velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, v_x and v_y , vary independently throughout the trajectory. Moreover, $\vec{v}(t)$ is the tangent of $s(t)$.

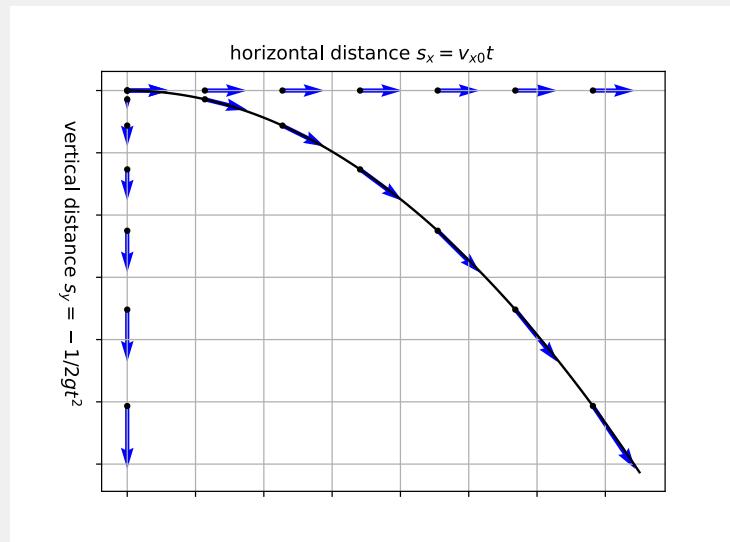


Figure 3.43: The parabolic motion is visualized with blue velocity vectors v , v_x and v_y shown at various points along the trajectory.

Danger

Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0 t + 1/2 a t^2$

3.2.3.3 Frictional forces

There are two main types of frictional force:

Exercise 3.44: Horizontal throw

Derive the above expression (29) yourselves.

Exercise 3.45: Projectile motion 

Watch the recording below. What happens with the horizontal distance traveled per time unit? And with the vertical distance traveled?

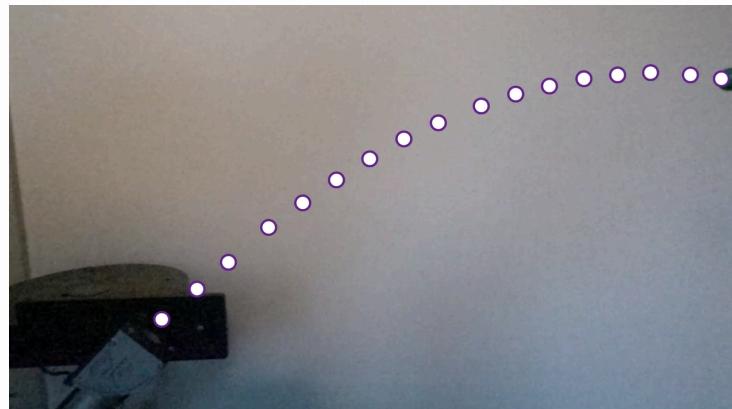


Figure 3.46: A parabolic motion visualized, with the position stored per time unit :alt: A short video of a small ball being shot upward at an angle. For each frame, its position is marked by a dot. The dots make up a parabola.

Assume the object with mass m_1 is shot from the ground with a velocity of v_0 at an angle of θ . Derive where the object hits the ground in terms of m_1 , v_0 and θ .

How does the distance traveled change when the mass of the object is doubled $m_2 = 2m_1$?

- **Static friction** prevents an object from starting to move. It adjusts in magnitude up to a maximum value, depending on how much force is trying to move the object. This maximum is given by

$$F_{static,max} = \mu_s N \quad (3.98)$$

where μ_s is the coefficient of static friction and N is the normal force. If the applied force exceeds this maximum, the object begins to slide.

- **Kinetic (dynamic) friction** opposes motion once the object is sliding. Its magnitude is generally constant and given by

$$F_{kinetic} = \mu_k N \quad (3.99)$$

where μ_k is the coefficient of kinetic friction. This force does not depend on the velocity of the object, only on the normal force and surface characteristics.

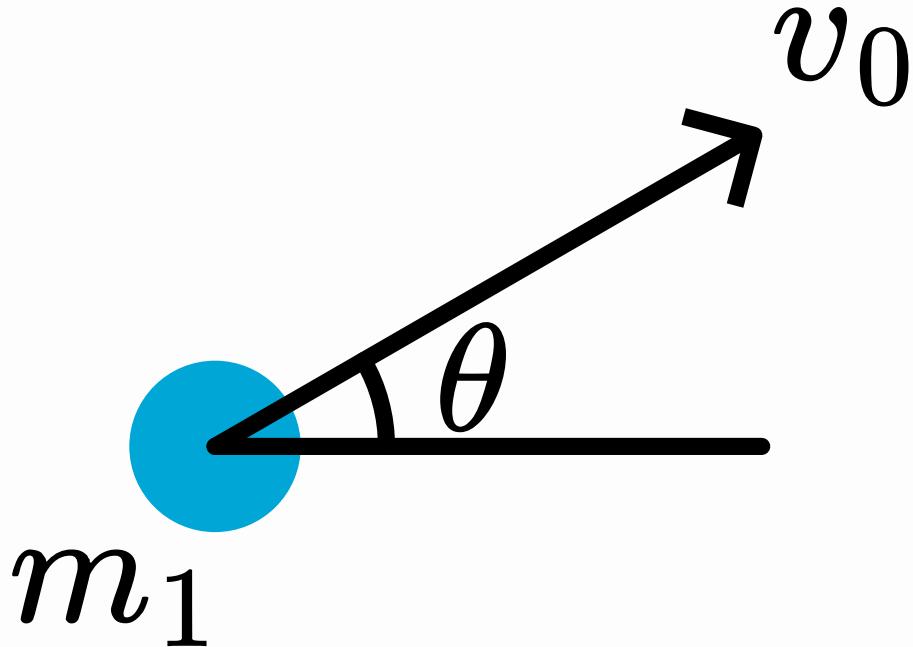
Friction always acts opposite to the direction of intended or actual motion and is essential in both preventing and controlling movement.

| Material Pair | Static Friction (μ_s) | Kinetic Friction (μ_k) |
|------------------------|-----------------------------|------------------------------|
| Rubber on dry concrete | 1.0 | 0.8 |
| Steel on steel (dry) | 0.74 | 0.57 |
| Wood on wood (dry) | 0.5 | 0.3 |
| Aluminum on steel | 0.61 | 0.47 |
| Ice on ice | 0.1 | 0.03 |

Solution 3.47: Solution to Exercise 7

The horizontal traveled distance is the same per time unit. For the vertical traveled distance it decreases until $v_y = 0$ and then increases.

Interpret



Develop

The basic formulas are:

$$s_x(t) = v_x t \quad (3.83)$$

and

$$s_y(t) = v_y t - 1/2 g t^2 \quad (3.84)$$

Evaluate

The horizontal traveled distance is given by:

$$s_x(t) = v_x t = v_0 \cos(\theta) t \quad (3.85)$$

The time the object stays in the air is

$$s_y(t) = v_y t - 1/2 g t^2 = 0 \Rightarrow t = 0 \& t = \frac{2v_y}{g} = \frac{2v_0 \sin(\theta)}{g} \quad (3.86)$$

Hence, the maximum distance traveled is:

$$s_x(t) = v_x t = v_0 \cos(\theta) \frac{2v_0 \sin(\theta)}{g} = \frac{2v_0^2 \sin(\theta) \cos(\theta)}{g} \quad (3.87)$$

Note that the distance traveled is independent of the mass!

| | | |
|----------------|------|-----|
| Glass on glass | 0.94 | 0.4 |
|----------------|------|-----|

Exercise 3.49: Constant acceleration due to gravity

We assumed a constant acceleration due to gravity. However, the gravitational force is given by $F = -G \frac{mM}{r^2}$.

Calculate at what height above the earth the acceleration due to gravity has ‘significantly’ changed from 9.81m/s^2 , say to 9.80m/s^2 .

Solution 3.50: Solution to Exercise 8

The acceleration of gravity is found by setting the gravitation force equal to $-mg$:

$$-G \frac{mM}{r^2} = -mg(r) \Rightarrow g(r) = G \frac{M}{r^2} \quad (3.88)$$

with M the mass of the earth.

At the surface of the earth, $r = R_e$ we have for the value of $g_e = 9.81 \text{ m/s}^2$. We look for the height above the earth surface where g has dropped to 9.80 m/s^2 . If we call this height H , we write for the distance to the center of the earth $r = R_e + H$.

Thus, we look for $\frac{g(r)}{g_e} = \frac{9.80(\text{m/s}^2)}{9.81(\text{m/s}^2)} = 0.999$:

$$\frac{g(r)}{g_e} = \frac{GM/r^2}{GM/R_e^2} \rightarrow \frac{R_e^2}{r^2} = \frac{R_e^2}{(R_e + H)^2} = \frac{9.80}{9.81} = 0.999 \quad (3.89)$$

If we solve H from this equation we find: $H = 3.25 \text{ km}$ (we used $R_e = 6378 \text{ km}$).

Note

We could have also looked at the ratios (between g and r), and found that $R_2 = \sqrt{.999} \cdot 6378 = 6374.8 \text{ km}$. Hence, $H = 3.2 \text{ km}$.

If we would have said: ‘significant change’ in means $g \rightarrow 9.81 \rightarrow 9.71 \text{ m/s}^2$, we would have found $H = 32.8 \text{ km}$.

| | | |
|-----------------|------|------|
| Copper on steel | 0.53 | 0.36 |
|-----------------|------|------|

Exercise 3.51: A rocket in space

A rocket moves freely horizontal through space. At position $x = 2$ it turns on its propulsion. At position $x = 4$ it turns off its propulsion. The force due to this propulsion is directed perpendicular to the x-direction.

Provide a sketch of its movement highlighting all important parts.

Exercise 3.52: Particle movement

Consider a particle which will travel a distance x . Find two different mathematical expressions for a force acting on the particle in such a way that the particle will travel the same distance in the same time for each $F(t)$ compared to a particle which travels at constant speed. Assume no initial velocity for the two particles.

Solution 3.53: Solution to Exercise 10

Uniform motion ($F = ma = 0 \rightarrow s = v_0 t$).

Constant acceleration $a = \text{const} \rightarrow s = 1/2at^2$, with $a = \frac{2v_0^2}{s}$.

Consider the third being a harmonic oscillating force field: $F(t) = A \sin(2\pi ft)$ Then the equation of motion becomes:

$$a = F/m = \frac{A}{m} \sin(2\pi ft) \quad (3.90)$$

$$v = \int a dt = \frac{A}{m2\pi f} \cos(2\pi ft) + C_0 \quad (3.91)$$

Assuming $v(0) = 0 \rightarrow C_0 = -\frac{A}{m2\pi f}$

And,

$$x = \int v dt = \frac{A}{m(2\pi f)^2} \sin(2\pi ft) + C_0 t + C_1 \quad (3.92)$$

Assuming $x(0) = 0 \rightarrow C_1 = 0$

Hence:

$$x = \frac{A}{m(2\pi f)^2} \sin(2\pi ft) - \frac{A}{m2\pi f} t \quad (3.93)$$

Now, finding traveling the same distance in the same time AND the harmonic oscillation is complete (hence, $f = \frac{1}{t_e}$):

$$v_0 t_e = \frac{At_e^2}{m(2\pi)^2} \sin(2\pi) - \frac{At_e}{m2\pi} t_e \quad (3.94)$$

$$v_0 t_e = -\frac{At_e^2}{m2\pi} \quad (3.95)$$

$$v_0 = -\frac{At_e}{m2\pi} \quad (3.96)$$

$$\frac{m}{A} = -\frac{t_e}{v_0 \pi} \quad (3.97)$$

| | | |
|------------------------|------|------|
| Teflon on Teflon | 0.04 | 0.04 |
| Rubber on wet concrete | 0.6 | 0.5 |
| Leather on wood | 0.56 | 0.4 |

Values are approximate and can vary depending on surface conditions.

Note

Not always are the friction coefficients constants. They may, for instance, depend on the relative velocity between the two materials.

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, step=...
```

Exercise 3.54: Block on an incline

A block with mass m is put on an inclined plane of which we can change the inclination angle θ .

1. Determine the angle at which it starts to slide in terms of mass m , inclination angle θ , acceleration due to gravity g and coefficient of static friction μ_s .
2. Once it starts to slide, it will accelerate. Determine its acceleration in terms of mass m , inclination angle θ , acceleration due to gravity g and coefficient of kinetic friction μ_f .

Solution 3.55: Solution to Exercise 11

1. There are two forces acting on m parallel to the inclined plane: friction and gravity's component parallel to the slope. These two determine the motion along the slope: if we tilt the plane the component of gravity parallel to the slope gets bigger. The particle will start moving when we pass: $F_{g_x} = F_s \rightarrow mg \sin(\theta) = mg\mu_s \cos(\theta) \Rightarrow \theta_{max} = \tan^{-1}(\mu_s)$
2. Once the particle is sliding downward, gravity and the kinetic friction determine how fast:

$$F_{net} = F_{g_x} - F_f \rightarrow ma = mg \sin(\theta) - mg\mu_k \cos(\theta) \Rightarrow \quad (3.100)$$

and

$$a = g(\sin(\theta) - \mu_k \cos(\theta)) \quad (3.101)$$

```
<function __main__.update(theta, mu)>
```

3.2.3.4 Momentum example

The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action = - reaction.

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

3.2.4 Forces & Inertia

Newton's laws introduce the concept of force. Forces have distinct features:

Exercise 3.56:

A point particle (mass m) is dropped from rest at a height h above the ground. Only gravity acts on the particle with a constant acceleration g ($= 9.813 \text{ m/s}^2$).

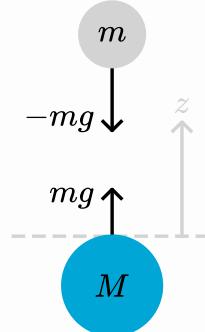
- Find the momentum when the particle hits the ground.
- What would be the earth's velocity upon impact?

Solution 3.57: Solution to Exercise 12

Let's do this one together. We follow the standard approach of IDEA: Interpret (and make your sketch!), develop (think 'model'), evaluate (solve your model) and assess (does it make any sense?).

Interpret

First a sketch: draw what is needed, no more, no less.



Develop

Actually this is half of the work, as when deciding what is needed we need to think what the problem really is. Above, is a sketch that could work. Both the object m and the earth (mass M) are drawn schematically. On each of them acts a force, where we know that on m standard gravity works. As a consequence of N3, a force equal in strength but opposite in direction acts on M .

Why do we draw forces? Well, the question mentions 'momentum the particle hits the ground'. Momentum and forces are coupled via N2.

We have drawn a z -coordinate: might be handy to remind us that this looks like a 1D problem (remember: momentum and force are both vectors).

As a first step, we ignore the motion of the earth. Argument? The magnitude of the ratio of the acceleration of earth over object is given by:

$$\frac{a_e}{a_o} = \frac{|F_{o \rightarrow e}| / m_e}{|F_{e \rightarrow o}| / m_o} = \frac{m_o}{m_e} \quad (3.102)$$

here for the second equality we used N3.

For all practical purposes, the mass of the object is many orders of magnitude smaller than that of the earth. Hence, we can conclude that the acceleration of the earth is many orders of magnitude less than that of the object. The latter is of the order of g , gravity's acceleration constant at the earth. Thus, the acceleration of the earth is next to zero and we can safely assume our lab system, that is connected to the earth, can be treated as an inertial system with, for us, zero velocity.

Evaluate

The remainder is straightforward. Now we have an object, that moves under a

We found that the particle changed its momentum from $p_i = 0$ to $p_f = -mv$. The earth compensates for this, ~~we keep momentum conserved. That gave that earth got a tiny, tiny upwards velocity. We could estimate the displacement of the earth. Suppose the particle has mass $m=1\text{kg}$ and is dropped from a height $H=100\text{m}$. Then we get for the velocity of the mass upon impact: $v = -44.3\text{m/s}$ and a falling~~ (3.103)

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:
 - ▶ they change the velocity, i.e. they accelerate the object

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \quad (3.106)$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \quad (3.107)$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Connecting physics and calculus

Let's look at a particle of mass m , that has initially (say at $t = 0$) a velocity v_0 . For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m \frac{dv}{dt} = F \Rightarrow m \frac{dv}{dt} + bv = 0 \quad (3.108)$$

Clearly, we have to solve a differential equation which states that if you take the derivative of v you should get something like $-v$ back. From calculus we know, that exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with A and μ to be determined constants.

We substitute our trial v :

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \quad (3.109)$$

This should hold for all t . Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \quad (3.110)$$

We see, that also our unknown constant A drops out. And, thus, we find

$$\mu = \frac{b}{m} \quad (3.111)$$

Next we need to find A : for that we need an initial condition, which we have: at $t = 0$ is $v = v_0$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} \text{ and } v(0) = v_0 \quad (3.112)$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \quad (3.113)$$

From the solution for v , we easily find the position of m as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b} e^{-\frac{b}{m}t} \right) + B \quad (3.114)$$

We find B with the initial condition and get as final solution:

$$x(t) = \frac{mv_0}{b} \left(1 - e^{-\frac{b}{m}t} \right) \quad (3.115)$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

3.2.4.1 Inertia

Inertia is denoted by the letter m for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like m_i , with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass m_1 and m_2 that are separated by a distance r_{12} attract each other via the so-called gravitational force (\hat{r}_{12} is a unit vector along the line connecting m_1 and m_2):

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12} \quad (3.116)$$

Here, we should have used a different symbol, rather than m . Something like m_g , as it is by no means obvious that the two 'masses' m_i and m_g refer to the same property. If you find that confusing, think about inertia and electric forces. Two particles with each an electric charge, q_1 and q_2 , respectively exert a force on each other known as the Coulomb force:

$$\vec{F}_{C,12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} \quad (3.117)$$

We denote the property associated with electric forces by q and call it charge. We have no problem writing

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \vec{F}_C &= \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r} \end{aligned} \quad (3.118)$$

We do not confuse q by m or vice versa. They are really different quantities: q tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on m . If m is big, the particle will only get a small acceleration; the strength of the force does not depend on m at all. So far, so good. But what about m_g ? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, m_i ?

$$\vec{F} = m_i \vec{a}$$

$$\vec{F}_g = -G \frac{m_g M_g}{r^2} \hat{r} \quad (3.119)$$

As far as we can tell (via experiments) m_i and m_g are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

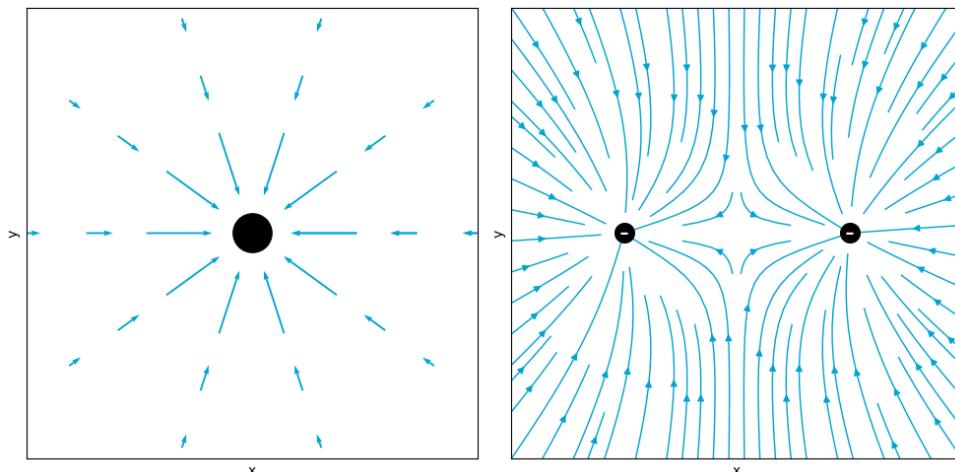
Force field

We have seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance — they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, the graph below at the left shows a gravitational field, described by $\vec{F}_g = G \frac{mM}{r^2} \hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.

The figure on the right shows the force field that a positively charged particle would feel due to the presence of 2 negatively charged particles (both of the same charge). Clearly this is a much more complicated force field.



Measuring mass or force

So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

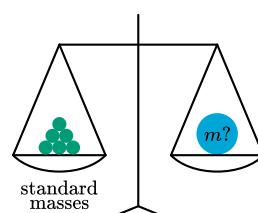


Figure 3.59: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a ‘yes’.

As on second thought: the balance compares the pull of gravity. Hence, it ‘measures’ gravitational mass, rather than inertia. Luckily, Newton’s laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\begin{cases} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{cases} \quad (3.120)$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \left| \frac{a_2}{a_1} \right| \quad (3.121)$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

Note

We will not use this method to measure mass. We came to the conclusion that we can’t find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop ‘force measuring equipment’ that we can calibrate using the method discussed above.

Intermezzo: kilogram, unit of mass

In 1795 it was decided that 1 gram is the mass of 1 cm³ of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the *kilogramme des Archives* was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4°C (when water has its maximum density).



Figure 3.60: The International Prototype of the Kilogram, whose mass was defined to be one kilogram from 1889 to 2019. Picture by BIPM, CC BY-SA 3.0 igo, <https://commons.wikimedia.org/w/index.php?curid=117707466>

In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly $h = 6.62607015 \cdot 10^{-34} \text{ kgm}^2\text{s}^{-1}$, the kilogram is now defined via h , the meter and second.

3.2.4.2 Eötvös experiment on mass

The question whether inertial mass and gravitational mass are the same has put experimentalists to work. It is by no means an easy question. Gravity is a very weak force. Moreover, determining that two properties are identical via an experiment is virtually impossible due to experimental uncertainty. Experimentalist can only tell the outcome is 'identical' within a margin. Newton already tried to establish experimentally that the two forms of mass are the same. However, in his days the inaccuracy of experiments was rather large. Dutch scientist Simon Stevin concluded in 1585 that the difference must be less than 5%. He used his famous 'drop masses from the church' experiments for this (they were primarily done to show that every mass falls with the same acceleration).

A couple of years later, Galilei used both fall experiments and pendula to improve this to: less than 2%. In 1686, Newton using pendula managed to bring it down to less than 1% .

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848-1918). We will here briefly introduce the experiment. For a full analysis, we need knowledge about angular momentum and centrifugal forces that we do not deal with in this book.

The experiment

The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to the gravitational mass) and some inertial force (sensitive to the inertial mass) are present. Obviously, gravitational forces between two objects out of our daily life are extremely small. These will be very difficult to detect and thus introduce a large error if the experiment relies on measuring them. Eötvös came up with a different idea. He connected two different objects with different masses, m_1 and m_2 , via a (almost) massless rod. Then, he attached a thin wire to the rod and let it hang down.

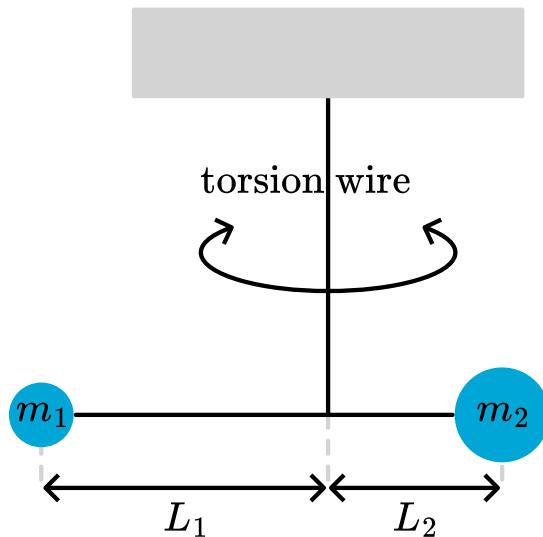


Figure 3.61: Torsion balance used by Eötvös.

This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to m_g , but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

Note

The question is not if m_i/m_g is different from 1. If that was the case but the ratio would always be the same, then we would just rescale m_g , that is redefine the value of the gravitational const G to make m_g equal to m_i . No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if m_i and m_g are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

Tip

Want to know more about this experiment? Watch this videoclip.

3.2.5 Examples, exercises and solutions

Here are some examples and exercises that deal with forces. Make sure you practice IDEA.

3.2.5.1 Exercises set 1

Exercise 1: Force on a particle

Consider a point particle of mass m , moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the negative x-direction. The force lasts for a small time interval Δt .

What is the strength of the force, if it brings the particle exactly to a zero-velocity?
Start by making a drawing.

Exercise 2: Shooting a ball

A ball is shot from a 10m high hill with a velocity of 10m/s under an angle of 30° , see Figure 1.

1. How long is the ball in the air?
2. How far does it travel in the horizontal direction?
3. With what velocity does the ball hit the ground?

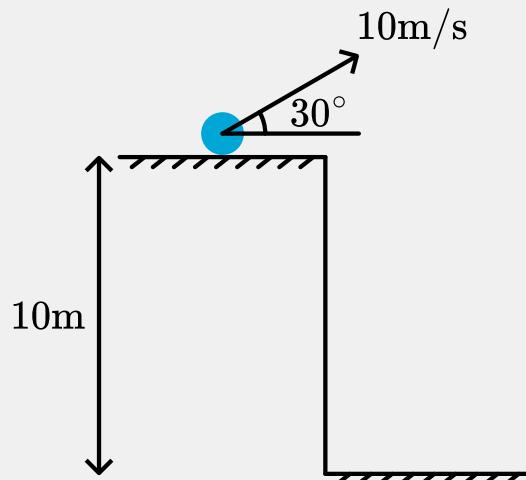


Figure 3.62: A ball on a hill launched under an angle.

Exercise 3: Constant force on a particle

A particle of mass m moves along the x -axis. At time $t = 0$ it is at the origin with velocity v_0 . For $t > 0$, a constant force acts on the particle. This is a 1-dimensional problem.

- Derive the acceleration of the particle as a function of time.
- Derive the velocity of the particle as a function of time.
- Derive the position of the particle as a function of time.

Exercise 4: Time dependent force on a particle

A particle of mass m moves along the x -axis. At time $t = 0$ it is at the origin with velocity v_0 . For $t > 0$ the particle is subject to a force $F_0 \sin(2\pi f_0 t)$. This is a 1-dimensional problem.

- Calculate the acceleration of the particle as a function of time.
- Calculate the velocity of the particle as a function of time.
- Calculate the position of the particle as a function of time.

Exercise 5: Particle trajectory

A particle follows a straight path with a constant velocity. At $t = 0$ the particle is at point A with coordinate $(0, y_A)$, while at $t = t_1$ it is at B with coordinate $(x_B, 0)$. The coordinates are given in a Cartesian system. The problem is 2-dimensional.

1. Make a sketch.
2. Find the position of the particle at arbitrary time $0 < t < t_1$.
3. Derive the velocity of the particle from position as function of time.

Represent vectors in a Cartesian coordinate system using the unit vectors \hat{i} and \hat{j} .

Exercise 6: Different coordinate systems

In Classical Mechanics we often use a coordinate system to describe motion of object. In this exercise, you will look at two Cartesian coordinate systems. System S has coordinates (x, y) and corresponding unit vectors \hat{x} and \hat{y} .

The second system, S', uses (x', y') and corresponding unit vectors. The x' -axis makes an angle of 30° with the x -axis (measured counter-clockwise).

1. Make a sketch.
2. Determine the relations between \hat{x}' and \hat{x}, \hat{y} as well as between \hat{y}' and \hat{x}, \hat{y}

An object has, according to S, a velocity of $\vec{v} = 3\hat{x} + 5\hat{y}$.

1. Determine the velocity according to S'.

Exercise 7: Rotating unit vectors

According to your observations, a particle is located at position $(1,0)$ (you use a Cartesian coordinate system). The particle has no velocity and no forces are acting on it.

Another observer, S', uses a Cartesian coordinate system described by (x', y') . You notice that her unit vectors rotate at a constant speed compared to your unit vectors:

$$\hat{x}' = \cos(2\pi ft)\hat{x} + \sin(2\pi ft)\hat{y} \quad (3.122)$$

$$\hat{y}' = -\sin(2\pi ft)\hat{x} + \cos(2\pi ft)\hat{y} \quad (3.123)$$

1. Find the position of the particle according to the other observer, S'.
2. Calculate the velocity of the particle according to S'.

Exercise 8: Moving over a frictionless table

A particle of mass m moves at a constant velocity v_0 over a frictionless table. The direction it is moving in, is at 45° with the positive x -axis. At some point in time, the

particle experiences a force $\vec{F} = -b\vec{v}$ with $b > 0$.

We call this time $t = 0$ and take the position of the particle at that time as our origin.

1. Make a sketch.
2. Determine whether this problem needs to be analyzed as a 1D or a 2D problem.
3. Set up N2 in the form $m \frac{d\vec{v}}{dt} = \vec{F}$
4. Solve N2 and find the velocity of the particle as a function of time.
5. What happens to the particle for large t ?

Exercise 9: Parabolic trajectory with maximum area³

A ball is thrown at speed v from zero height on level ground. We want to find the angle θ at which it should be thrown so that the area under the trajectory is maximized.

1. Sketch of the trajectory of the ball.
2. Use dimensional analysis to relate the area to the initial speed v and the gravitational acceleration g .
3. Write down the x and y coordinates of the ball as a function of time.
4. Find the total time the ball is in the air.
5. The area under the trajectory is given by $A = \int y dx$. Make a variable transformation to express this integral as an integration over time.
6. Evaluate the integral. Your answer should be a function of the initial speed v and angle θ .
7. From your answer at (6), find the angle that maximizes the area, and the value of that maximum area.

Exercise 10: Two attracting particles⁴

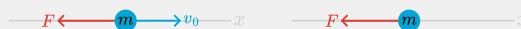
Two particles on a line are mutually attracted by a force $F = -ar$, where a is a constant and r the distance of separation. At time $t = 0$, particle A of mass m is located at the origin, and particle B of mass $m/4$ is located at $r = 5.0$ cm. Both particles have zero velocity at $t = 0$. If the particles are at rest at $t = 0$, at what value of r do they collide?

3.2.5.2 Answers set 1

Solution to Exercise 1: Force on a particle

$$\begin{aligned} t &= 0 \\ v &= v_0 \end{aligned}$$

$$\begin{aligned} t &= \Delta t \\ v &= 0 \end{aligned}$$



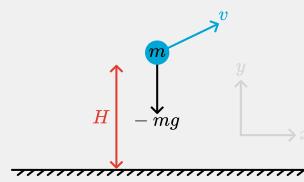
$$\vec{F} = -\frac{mv_0}{\Delta t} \hat{x}$$

Solution to Exercise 2: Shooting a ball

Interpret

³Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

⁴Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)



Develop

We know $v_y = v \sin(\theta)$ and $v_x = v \cos(\theta)$.

The motion of the ball can be split in two components: horizontal, i.e. x-direction, and vertical, that is y-direction.

In the vertical direction gravity acts: $F_y = -mg$. Thus the equation of motion in the y-direction is: $ma_y = F_y = -mg$. The vertical position can thus be expressed as $s_y(t) = s_{y0} + v_{y0}t - \frac{1}{2}gt^2$.

In the horizontal direction no force is active, thus: $ma_x = 0 \rightarrow s_x(t) = s_{x0} + v_{x0}t$

The magnitude of the velocity of the ball hitting the ground can be expressed in terms of v_x and v_y as $v_e = \sqrt{v_x^2 + v_y^2}$

Evaluate

We have as initial velocity: $v_{y0} = v \sin(\theta) = 10 * \sin(30) = 5 \text{ m/s}$
 $v_{x0} = v \cos(\theta) = 10 * \cos(30) = 5\sqrt{3} \text{ m/s}$

Solving $s_y(t) = s_{y0} + v_{y0}t - \frac{1}{2}gt^2$ for $s_y = 0$ with $s_{y0} = H$ gives for the time the ball is in the air:

$$t_{air} = \frac{v_{y0}}{g} + \sqrt{\frac{v_{y0}^2}{g^2} + \frac{2H}{g}} = 2.77 \text{ s} \quad (3.124)$$

Next, we realize that $v_x = \text{const} = v_{x0}$ as there is no force acting in the x-direction. Thus the horizontal distance traveled is

$$\Delta x = v_{x0} t_{air} = 24.0 \text{ m}$$

For the velocity when hitting the ground is (that is, its magnitude), we need both the x and y-component:

$$v_x = v_{x0} = 8.66 \text{ m/s}$$

$$v_y = v_{y0} - gt \rightarrow v_y(t_{air}) = \sqrt{v_{y0}^2 + 2gH} = 14.9 \text{ m/s} \quad (3.125)$$

$$v_{ground} = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{x0}^2 + v_y^2} = 17.2 \text{ m/s} \quad (3.126)$$

Assess

The velocity upon impact is larger than the initial velocity. This makes sense. The ball first travels upwards, then downwards and will pass $s_y = H$ again on the

downward motion. Then it will further accelerate to the ground and thus have a larger y-component of the velocity than at the start.

Solution to Exercise 3: Constant force on a particle 🌶

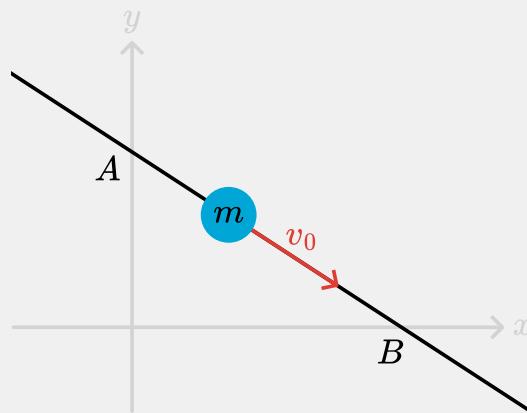
1. $a = \frac{F}{m}$ is constant
2. $v(t) = v_0 + at$
3. $x(t) = v_0 t + \frac{1}{2}at^2$

Solution to Exercise 4: Time dependent force on a particle 🌶

1. $a = \frac{F}{m} = \frac{F_0}{m} \sin(2\pi f_0 t)$ is **not** constant
2. $v(t) = v_0 + \frac{F_0}{2\pi f_0 m} (1 - \cos(2\pi f_0 t))$
3. $x(t) = v_0 t + \frac{F_0}{2\pi f_0 m} t - \frac{F_0}{4\pi^2 f_0^2 m} \sin 2\pi f_0 t$

Solution to Exercise 5: Particle trajectory 🌶

1.



2. Particle moves at constant velocity, thus path is a straight line:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t = x_0 \hat{i} + y_0 \hat{j} + v_{0x} t \hat{i} + v_{0y} t \hat{j} \quad (3.127)$$

At $t = 0$: $\vec{r}(0) = 0\hat{i} + y_A \hat{j} \rightarrow \vec{r}(0) = \vec{r}_0 = 0\hat{i} + y_A \hat{j} \rightarrow x_0 = 0$ and $y_0 = y_A$

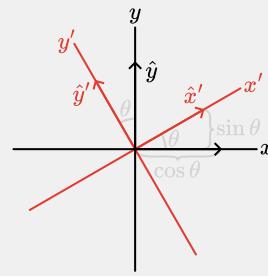
At $t = t_1$:

$$\begin{aligned} \vec{r}(t_1) &= x_B \hat{i} + 0(\hat{j}) \rightarrow \\ \vec{r}(t_1) &= \vec{r}_0 + \vec{v}_0 t_1 \\ &= (0 + v_{0x} t_1) \hat{i} + (y_A + v_{0y} t_1) \hat{j} \rightarrow \\ v_{0x} &= \frac{x_B}{t_1} \text{ and } v_{0y} = -\frac{y_A}{t_1} \end{aligned} \quad (3.128)$$

3. Thus, we find $\vec{v} = \frac{x_B}{t_1} \hat{i} - \frac{y_A}{t_1} \hat{j}$

Solution to Exercise 6: Different coordinate systems 🌶 🌶

1.



2.

$$\begin{aligned}\hat{x}' &= \cos \theta \hat{x} + \sin \theta \hat{y} = \frac{1}{2} \sqrt{3} \hat{x} + \frac{1}{2} \hat{y} \\ \hat{y}' &= -\sin \theta \hat{x} + \cos \theta \hat{y} = -\frac{1}{2} \hat{x} + \frac{1}{2} \sqrt{3} \hat{y}\end{aligned}\quad (3.129)$$

2. Invert:

$$\begin{aligned}\hat{x} &= \cos \theta \hat{x}' - \sin \theta \hat{y}' = \frac{1}{2} \sqrt{3} \hat{x}' - \frac{1}{2} \hat{y}' \\ \hat{y} &= \sin \theta \hat{x}' + \cos \theta \hat{y}' = \frac{1}{2} \hat{x}' + \frac{1}{2} \sqrt{3} \hat{y}'\end{aligned}\quad (3.130)$$

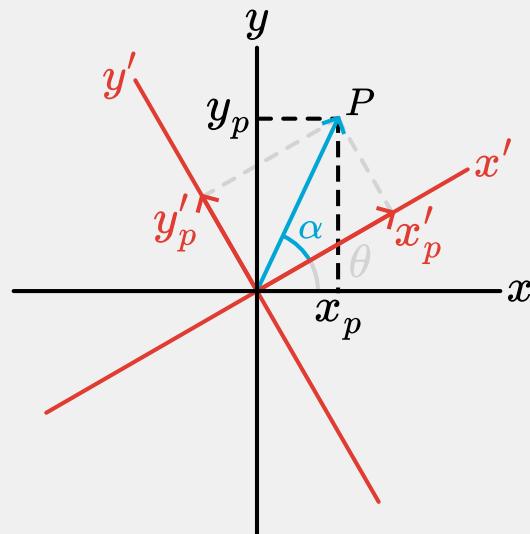
velocity:

$$\begin{aligned}\vec{v} &= v_x \hat{x} + v_y \hat{y} \\ &= v_x (\cos \theta \hat{x}' - \sin \theta \hat{y}') + v_y (\sin \theta \hat{x}' + \cos \theta \hat{y}') \\ &= (v_x \cos \theta + v_y \sin \theta) \hat{x}' + (-v_x \sin \theta + v_y \cos \theta) \hat{y}'\end{aligned}\quad (3.131)$$

from which we find

$$\vec{v} = \left(\frac{3}{2} \sqrt{3} + \frac{5}{2} \right) \hat{x}' + \left(-\frac{3}{2} + \frac{5}{2} \sqrt{3} \right) \hat{y}' \quad (3.132)$$

Solution to Exercise 7: Rotating unit vectors 🍎



$$\begin{aligned}\hat{x}' &= \cos(2\pi ft) \hat{x} + \sin(2\pi ft) \hat{y} \\ \hat{y}' &= -\sin(2\pi ft) \hat{x} + \cos(2\pi ft) \hat{y}\end{aligned}\quad (3.133)$$

The unit vectors of S' rotate with a frequency f with respect to the unit vectors of S. This means, that the coordinate system of S' rotates: the rotation angle is a function of time, i.e. $\theta(t) = 2\pi ft$

From the figure we see, that the coordinates of a point P, (x_p, y_p) according to S, are related to those used by S', (x'_p, y'_p) via:

$$\begin{aligned} x_p &= OP \cos(\alpha + \theta) = OP(\cos \alpha \cos \theta - \sin \alpha \sin \theta) = x'_p \cos \theta - y'_p \sin \theta \\ y_p &= OP \sin(\alpha + \theta) = OP(\cos \alpha \sin \theta + \sin \alpha \cos \theta) = x'_p \sin \theta + y'_p \cos \theta \end{aligned} \quad (3.134)$$

or written as the coordinate transformation:

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta \\ y &= x' \sin \theta + y' \cos \theta \end{aligned} \quad (3.135)$$

with its inverse

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta \end{aligned} \quad (3.136)$$

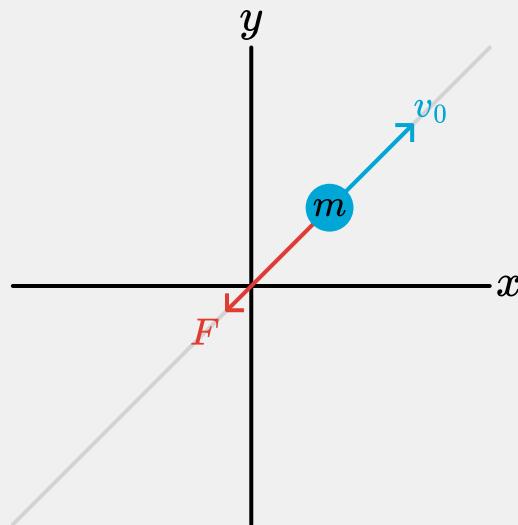
Note that in this case $\theta = 2\pi ft$, that is: it is a function of t .

- a) From the above relation we find that the point $(1,0)$ in S will be denoted by S' as $(x'(t), y'(t)) = (\cos(2\pi ft), -\sin(2\pi ft))$
- b) the velocity of the point $(1,0)$ in S is according to S of course zero: $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$ S' will say:

$$\begin{aligned} x'(t) &= \cos(2\pi ft) \rightarrow \frac{dx'}{dt} = -2\pi f \sin(2\pi ft) \\ y'(t) &= -\sin(2\pi ft) \rightarrow \frac{dy'}{dt} = 2\pi f \cos(2\pi ft) \end{aligned} \quad (3.137)$$

Solution to Exercise 8: Moving over a frictionless table

1.



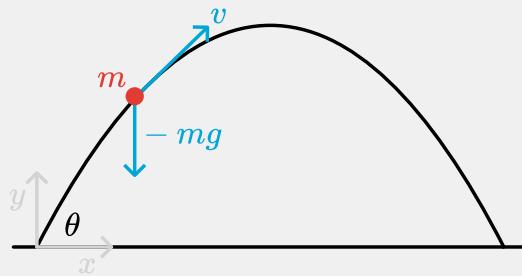
2. Since \vec{v}_0 and \vec{F} are parallel, the particle will not deviate from the line $x=y$. Hence, we are dealing with a 1-dimensional problem. The original coordinate

system, (x, y) , is not wrong: it is just not handy as it makes the problem look like 2D. Thus, we change our coordinate system, such that the new x -axis coincides with the original $x=y$ line.

3. N2: $m \frac{dv}{dt} = -bv$ with initial conditions: $t = 0 \rightarrow x = 0$ and $t = 0 \rightarrow v = v_0$
4. $\frac{dv}{dt} - \frac{b}{m}v = 0 \rightarrow v = Ae^{-\frac{b}{m}t}$ initial condition: $t = 0 \rightarrow v = v_0 \Rightarrow A = v_0$ Thus: $v(t) = v_0 e^{-\frac{b}{m}t}$
5. for $t \rightarrow \infty : v \rightarrow 0$. The particle comes to rest and then, obviously, the friction force is zero.

Solution to Exercise 9: Parabolic trajectory with maximum area

1.



2. We expect that the area, A , under the trajectory of the ball is a function of v , g , and θ . In a dimensional analysis we write this as ‘product of powers’:

$$A = v^a \cdot g^b \cdot \theta^c \quad (3.138)$$

and we make this expression dimensional correct. (Note: we don’t mean that the final outcome of a full analysis is a product of powers, it can be any function but the units should be related in the right way and that is what this ‘trick’ with powers ensures.)

The area has units m^2 , velocity m/s , g m/s^2 and θ is dimensionless (radians don’t count as a dimension or unit). Thus:

$$\begin{aligned} m : 2 &= a + b \\ s : 0 &= -a - 2b \end{aligned} \quad (3.139)$$

This yields: $a = 4$, $b = -2$. Thus on dimensional grounds we may expect: $A \sim \frac{v^4}{g^2}$.

3. In the x -direction: no forces, hence $m \frac{v_x}{dt} = 0 \rightarrow x(t) = v \cos \theta t$

In the y -direction: $m \frac{dv_y}{dt} = -mg \rightarrow y(t) = v \sin \theta t - \frac{1}{2}gt^2$. Where we have used the initial conditions: $x(0) = 0$, $y(0) = 0$, $v_x(0) = v \cos \theta$, $v_y(0) = v \sin \theta$

4. Total time in the air: $v_y(t^*) = 0 \rightarrow t^* = \frac{2v}{g} \sin \theta$

5+6. Evaluate the area under the trajectory:

$$\begin{aligned}
 A &= \int_0^{x_{max}} y dx \\
 &= \int_0^{t^*} \left(v \sin \theta t - \frac{1}{2} g t^2 \right) v \cos \theta dt \\
 &= v^2 \sin \theta \cos \theta \frac{1}{2} (t^*)^2 - \frac{1}{3} g v \cos \theta (t^*)^3 \\
 &= \frac{2}{3} \frac{v^4}{g^2} \cos \theta \sin^3 \theta
 \end{aligned} \tag{3.140}$$

7. We maximize the function $f(\theta) = \cos \theta \sin^3 \theta$:

$$\frac{df}{d\theta} = \sin^2 \theta (-\sin^2 \theta + 3 \cos^2 \theta) \tag{3.141}$$

$$\frac{df}{d\theta} = 0 \rightarrow \sin \theta = 0 \text{ or } \sin^2 \theta = 3 \cos^2 \theta \tag{3.142}$$

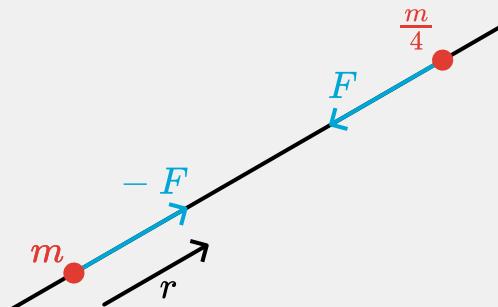
The first solution give a minimum for the area ($A = 0$). So we need the second solution:

$$\frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta = 3 \rightarrow \tan \theta = \sqrt{3} \rightarrow \theta = \frac{\pi}{3} \tag{3.143}$$

Solution to Exercise 10: Two attracting particles

Interpret

We start with a sketch.



This is a 1-dimensional problem. We will use r as the coordinate. Moreover, it is a problem involving two particles, that both can move. This makes it more difficult than 1-dimensional cases with only one particle.

Develop

We have to set up two equations of motion, one for particle 1 with mass m and position r_1 and one for particle 2 with mass $m/4$ and position r_2 . When doing so, we should realize that the mutual force obeys Newton's third law: $F_{12} = -F_{21}$

$$\begin{aligned} m \frac{dv_1}{dt} &= a(r_2 - r_1) \\ \frac{m}{4} \frac{dv_2}{dt} &= -a(r_2 - r_1) \end{aligned} \quad (3.144)$$

We see that the two equations are coupled: we can't solve one without information from the other.

Evaluate

So, how do we proceed? First, let's think about the question. We are not asked to solve the equation of motion and find the trajectory. What we need to find is the position of the collision.

From the two equation of motion we can find important information about the velocities of both particles. Just add to two equations:

$$m \frac{dv_1}{dt} + \frac{m}{4} \frac{dv_2}{dt} = 0 \rightarrow \frac{dv_1}{dt} = -\frac{1}{4} \frac{dv_2}{dt} \quad (3.145)$$

Since both particles start rest, we find from the last equation: $v_1 = -\frac{1}{4}v_2$ at any time. Thus particle 2 will travel 4 times a distance than particle 1 in the same time interval. Consequently: if particle 1 has moved 1cm, particle 2 has moved 4cm. Thus the particles (originally separated by 5cm) will collide at $r = 1\text{cm}$.

Assess

It makes sense that the heavy particle has traveled less than the light one: they both feel at any moment the same force (apart from a sign). The light particle will accelerate faster than the heavy one. Moreover, they should collide somewhere on the line element originally separating them as they are attracted to each other.

We found both these elements in our solution.

3.2.5.3 Exercises set 2

```
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, min=0...
<function __main__.update(theta, F_girl)>
interactive(children=(FloatSlider(value=0.7853981633974483,
description='theta', max=1.5707963267948966, step=...
<function __main__.update(theta, mu)>
interactive(children=(IntSlider(value=1, description='force_num', max=3,
min=1), Output(), _dom_classes='wid...
<function __main__.update(force_num)>
interactive(children=(FloatSlider(value=9.81, description='g (m/s²)', max=15.0, min=1.5), IntSlider(value=1, d...
<function __main__.run_animation(g=9.81, M=1)>
```

Exercise 3.71: Who is strongest? 🍕

Who is strongest? Two strong boys try to keep a rope straight by each pulling hard at one end. A not so strong third person is pulling in the middle of the rope, but at an angle of 90° to the rope. The two strong boys have the task to keep the deviation of the rope to a small value, set by you.

How does the force and the angle depends on the force exerted by the girl?



Figure 3.72: Picture taken from Show the Physics

Exercise 3.73: Dropping a stone from a church tower 🍕

You drop a stone from a height of 50m the tower of the church. Calculate the velocity of the stone when it hits the ground (ignore friction). In the video you will see on the left a quick and dirty solution, NOT using IDEA. The right hand side uses IDEA and Newton's 2nd law.

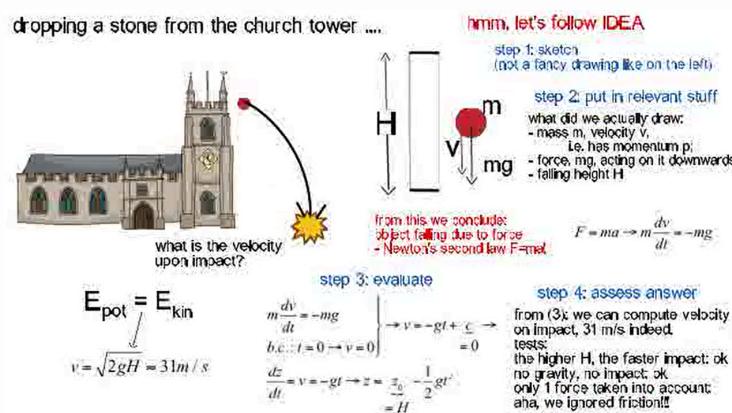


Figure 3.74: The worked out exercise

Exercise 3.75: Sliding down a slope

Two point particles slide down a slope: one feels friction the other doesn't. Can you analyse the situation and understand the graphs?

Exercise 3.76:

Below are three forces and their resultant (v, t)- and (s, t)-diagrams. What kind of forces are acting?

Exercise 3.77:

A mass $m = 1\text{kg}$ (the red one in the drawing) is attached to a massless string. The string can move freely over a massless pulley. At the other end of the string a variable mass M (the grey one) is hanging. At $t = 0$ mass m is released, while the string is stretched to its full length.

The graph on the right side of the screen shows the velocity of m as a function of time.

- ‘Play’ with the acceleration and mass M , predict every time first what will happen to the motion.
- Describe the motion of m and M .
- Write down Newton’s equation of motion for m and for M .

Exercise 3.78:

A point particle (mass m) is from position $z = 0$ shot with a velocity v_0 straight upwards into the air. On this particle only gravity acts, i.e. friction with the air can be ignored. The acceleration of gravity, g , may be taken as a constant.

The following questions should be answered.

- What is the maximum height that the particle reaches?
- How long does it take to reach that highest point?

Solve this exercise using IDEA.

- Sketch the situation and draw the relevant quantities.
- Reason that this exercise can be solved using $\vec{F} = m\vec{a}$ (or $d\vec{p}/dt = \vec{F}$).
- Formulate the equation of motion (N2) for m .
- Classify what kind of mathematical equation this is and provide initial or boundary conditions that are needed to solve the equation.
- Solve the equation of motion and answer the two questions.
- Check your math and the result for dimensional correctness. Inspect the limit: $F_{zw} \rightarrow 0$.

Exercise 3.79: Acceleration of Gravity

- Find an object that you can safely drop from some height.
- Drop the object from any (or several heights) and measure using a stop watch or your mobile the time from dropping to hitting the ground.
- Measure the dropping height.

Find from these data the value of gravity's acceleration constant.

Don't forget to first make an analysis of this experiment in terms of a physical model and make clear what your assumptions are.

Tip

Think about the effect of air resistance: is dropping from a small, a medium or a high height best? Any arguments?

Exercise 3.80: Use numerical analysis to assess influence of air friction

If you want to learn also how to use numerical methods ...

Try using an air drag force: $F_{drag} = -A_{\perp} C_D \frac{1}{2} \rho_{air} v^2$. With A_{\perp} the cross-sectional area of your object perpendicular to the velocity vector and $C_D \approx 1$ the drag coefficient (in real life it is actually a function of the velocity). ρ_{air} is the density of air which is about 1.2 kg/m^3 .

Write a computer program (e.g. in python) that calculates the motion of your object. See Solution with Python how you could do that.

Exercise 3.81: Forces on your bike 🚴



Figure 3.82: Riding a bicycle. Adapted from InjuryMap, from Wikimedia Commons, licensed under CC BY-SA 4.0.

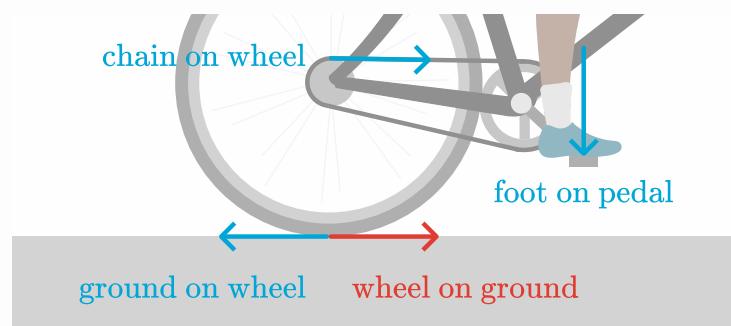
On a bicycle you will have to apply a force to the pedals to move forward, right? What force actually moves you forward, where is it located and who/what is providing that force?

- Make sketch and draw the relevant force. Give the force that actually propels you a different color.
- Think for a minute about the nature of this force: are you surprised?

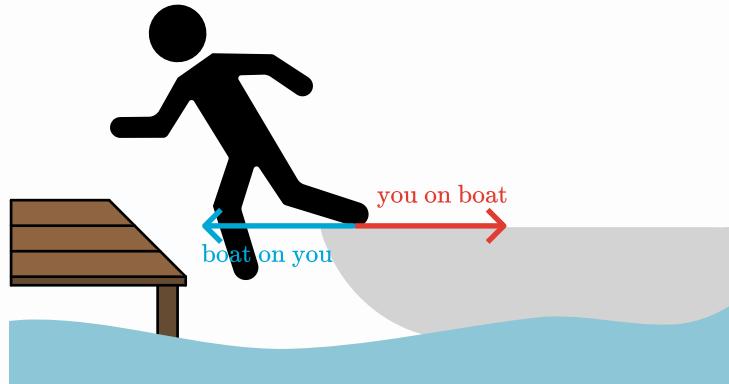
N.B. Consider while thinking about this problem: what would happen if you were biking on an extremely slippery floor?

Solution 3.83: Solution to Exercise 9

When you push with your foot on the pedal, that force is transferred to the chain of your bike. That chain exerts a force on the gear of your bike's rear wheel, trying to get it to rotate. Your wheel touches the ground and, because of the force on the gear, the wheel exerts a force in the ground, trying to push the ground backwards. Due to action--reaction, the ground exerts a forward force on your wheel. So actually, biking means "making the ground push you forward"!



Exercise 3.85: Getting off the boat 🚤



You are stepping from a boat onto the shore. Use Newton's laws to describe why you will end up in the water.

N.B. A calculation is not required, but focus on the physics and describe in words why you didn't make it to the jetty.

Solution 3.87: Solution to Exercise 10

When you try to step on the jetty, a force needs to be exerted on you, otherwise you can't move forward. The way you achieve that: you push with your back foot on the boat. And as a result of Newton 3, the boat will push back, but the force from the boat on you is forward directed. That is exactly what you need!

However, while you push, the boat will move backwards due to the force you exert on it. Consequently, your point of contact with the boat shifts away from the jetty. Either you let the boat go and no force from the boat is acting on you. Now gravity will do its work and if your forward velocity is not sufficient, you will not reach the jetty. Or your foot will try to follow the boat and that requires a force to the wrong direction acting on you.

Pushing harder seems an option: your forward velocity might increase more. However, the boat will also be pushed harder and moves quicker away from you. Consequently, the time interval of contact with the boat decreases. Thus, with Newton 2: $\Delta p = F \cdot t$ your increase in velocity due to the larger force might be compensated by a smaller duration that the force can do so. And you may still end up in the water.

Exercise 3.88: Newton's Laws 🍎

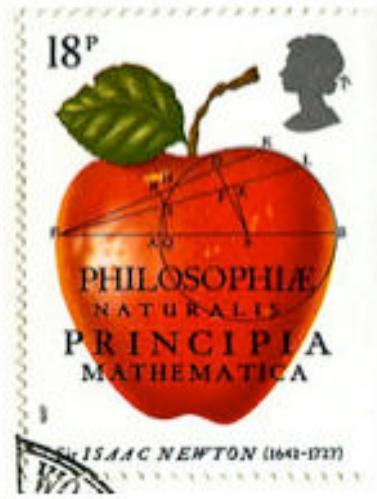


Figure 3.89: Stamp designs © Royal Mail Group Ltd^[^1].

Close this book (or don't peak at it ;-)) and write down Newton's laws. Explain in words the meaning of each of the laws. Try to come up with several, different ways of describing what is in these equations.

3.3 Work & Energy

3.3.1 Work

Work and energy are two important concepts. Work is the transfer of energy that occurs when a force is applied to an object and causes displacement in the direction of that force, calculated as ‘force times path’. However, we need a formal definition:

if a point particle moves from \vec{r} to $\vec{r} + d\vec{r}$ and during this interval a force \vec{F} acts on the particle, then this force has performed an amount of work equal to:

$$dW = \vec{F} \cdot d\vec{r} \quad (3.146)$$

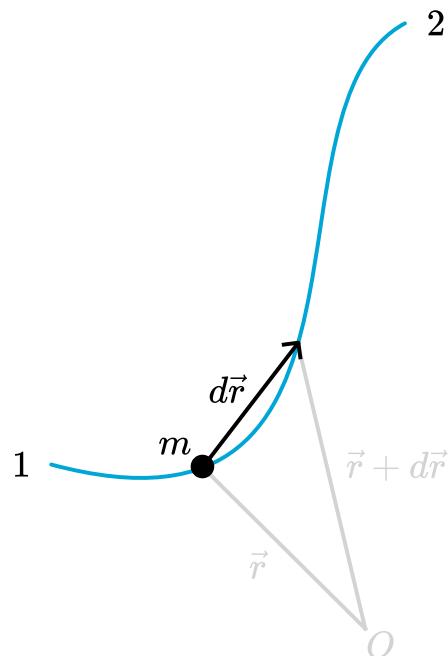


Figure 3.90: Path of a particle.

Note that this is an *inner product* between two vectors, resulting in a *scalar*. In other words, work is a number, not a vector. It has no direction. That is one of the advantages over force.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \quad (3.147)$$

Work done on m by F during motion from 1 to 2 over a prescribed trajectory:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \quad (3.148)$$

Keep in mind: in general the work depends on the starting point 1, the end point 2 and on the trajectory. Different trajectories from 1 to 2 may lead to different amounts of work.

Tip

See also the chapter in the linear algebra book on the inner product

3.3.2 Kinetic Energy

Kinetic energy is defined and derived using the definition of work and Newton's 2nd Law.

The following holds: if work is done on a particle, then its kinetic energy must change. And vice versa: if the kinetic energy of an object changes, then work must have been done on that particle. The following derivation shows this.

$$\begin{aligned}
 W_{12} &= \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_1^2 \vec{F} \cdot \vec{v} dt \\
 &= \int_1^2 m \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_1^2 \vec{v} \cdot d\vec{v} = m \left[\frac{1}{2} \vec{v}^2 \right]_1^2 \\
 &= \frac{1}{2} m \vec{v}_2^2 - \frac{1}{2} m \vec{v}_1^2
 \end{aligned} \tag{3.149}$$

It is from the above that we indicate $\frac{1}{2} m \vec{v}^2$ as kinetic energy. It is important to realize that the concept of kinetic energy does not bring anything that is not contained in N2 to the table. But it does give a new perspective: kinetic energy can only be gained or lost if a force performs work on the particle. And vice versa: if a force performs work on a particle, the particle will change its kinetic energy.

Obviously, if more than one force acts, the net work done on the particle determines the change in kinetic energy. It is perfectly possible that force 1 adds an amount W to the particle, whereas at the same time force 2 will take out an amount $-W$. This is the case for a particle that moves under the influence of two forces that cancel each other: $\vec{F}_1 = -\vec{F}_2$. From Newton 2, we immediately infer that if the two forces cancel each other, then the particle will move with a constant velocity. Hence, its kinetic energy stays constant. This is completely in line with the fact that in this case the net work done on the particle is zero:

$$W_1 + W_2 = \int_1^2 \vec{F}_1 \cdot d\vec{r} + \int_1^2 \vec{F}_2 \cdot d\vec{r} = \int_1^2 \vec{F}_1 \cdot d\vec{r} - \int_1^2 \vec{F}_1 \cdot d\vec{r} = 0 \tag{3.150}$$

3.3.3 Worked Examples

Reminder of path/line integral from Analysis

As long as the path can be split along coordinate axis the separation above is a good recipe. If that is not the case, then we need to turn back to the way how things have been introduced in the Analysis class. We need to make a 1D parameterization of the path.

Line integral of a vector valued function $\vec{F}(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ over a curve $\text{cal } C$ is given as

$$\int_{\text{cal } C} \vec{F}(x, y, z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(\tau)) \cdot \frac{d\vec{r}(\tau)}{d\tau} d\tau \tag{3.156}$$

We integrate in the definition of the work from point 1 to 2 over an implicitly given path. To compute this actually, you need to parameterize the path by $\vec{r}(\tau) = (x(\tau), y(\tau), z(\tau))$. The integration variable τ tells you where you are on the path, $\tau \in$

Exercise 3.91: Carrying a weight

You carry a heavy backpack $m = 20\text{kg}$ from Delft to Rotterdam (20km). What is the work that you have done against the gravitational force?

Solution 3.92: Solution to Exercise 1

The answer is, of course, zero! That is because the path (from Delft to Rotterdam) is perpendicular to the gravitational force. Therefore the inner product $\vec{F}_g \cdot d\vec{r} = 0$ over the whole way. Let us look at it more formally, this will help us when things get more complicated later.

The force is $\vec{F}_g(x, y, z) = (0, 0, -mg) = -mg\hat{z}$ and we choose our coordinate system such that the path be along the x -axis, the y -coordinate is zero and we the backpack is at height $z = 1$ m.

$$W_g = \int_{Delft}^{Rott} F_x dx + F_y dy + F_z dz = \int F_x dx \mid_{y=0, z=1} = \int 0 dx = 0 \quad (3.151)$$

So gravity has not performed work on your backpack. Similarly, you have exercised a force \vec{F}_N on the backpack. As the backpack doesn't change its vertical coordinate, we know $\vec{F}_N + \vec{F}_g = 0$. And immediately, we see:

$$W_N = \int_{Delft}^{Rott} F_{Nx} dx + F_{Ny} dy + F_{Nz} dz = \int F_x dx \mid_{y=0, z=1} = \int 0 dx = 0 \quad (3.152)$$

You didn't perform any work either. This may feel strange or even wrong. After all, you will probably be pretty tired after the walk. However, that is due to the internal working of our muscles and body. In order to sustain the force \vec{F}_N humans do use energy: work is done in their muscles. But from a physics point of view: no work is done on the backpack.

$[a, b] \in \mathbb{R}$. The derivative of \vec{r} with respect to τ gives the tangent vector to the curve, the "speed" of walking along the curve. In the analysis class you used $\vec{v}(\tau) \equiv \frac{d\vec{r}(\tau)}{d\tau}$ for the speed. The value of the line integral is independent of the chosen parameterization. However, it changes sign when reversing the integration boundaries.

Example 4.3

Exercise 3.93: Compressing a spring⁵

You're compressing an uncompressed spring with spring constant k over a distance x . How much work do you need to do?

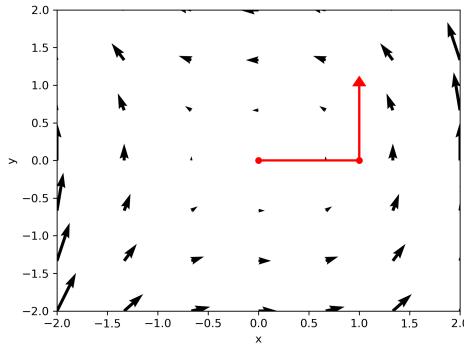
Solution 3.94: Solution to Exercise 2

$$W = \int_{x_1}^{x_2} F dx = \int_0^x kx dx = \frac{1}{2} kx^2 \quad (3.153)$$

⁵Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 3.95: Work in a force field

Now we consider a force field $\vec{F}(x, y) = (-y, x^2) = -y\hat{x} + x^2\hat{y}$. We compute the work done over a path from the origin $(0, 0)$ to $(1, 0)$ and then to $(1, 1)$ first along the x -axis and then parallel to the y -axis.



Now we integrate from $(0, 0) \rightarrow (1, 1)$ but along the diagonal. A parameterization of this path is $\vec{r}(\tau) = (0, 0) + (1, 1)\tau = (\tau, \tau)$, $\tau \in [0, 1]$. The derivative is $\frac{d\vec{r}(\tau)}{d\tau} = (1, 1)$. Therefore we can write the work of $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ along the diagonal as

$$\begin{aligned} \int_0^1 \vec{F}(\tau, \tau) \cdot (1, 1) d\tau &= \int_0^1 (-\tau, \tau^2) \cdot (1, 1) d\tau = \\ &\quad \int_0^1 (-\tau + \tau^2) d\tau = -\frac{1}{6} \end{aligned} \tag{3.157}$$

Solution 3.97: Solution to Exercise 3

We can split up the integral in these two parts as the direction in both parts is constant, therefore the inner product can be separated out.

$$\begin{aligned} W &= \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} \\ &= \int_{(0,0)}^{(1,0)} F_x dx \Big|_{y=0} + \int_{(1,0)}^{(1,1)} F_y dy \Big|_{x=1} \\ &= \int_{(x=0)}^{(x=1)} -y dx \Big|_{y=0} + \int_{(y=0)}^{(y=1)} x^2 dy \Big|_{x=1} \\ &= -yx \Big|_{x=0}^{x=1} \Big|_{y=0} + x^2 y \Big|_{y=0}^{y=1} \Big|_{x=1} = 1 \end{aligned} \tag{3.154}$$

Try to integrate the force field yourself along a different path $(0, 0) \rightarrow (0, 1) \rightarrow (1, 1)$ to the same end point.

$$\begin{aligned} W &= \int_{y=0}^{y=1} F_y dy \Big|_{x=0} + \int_{x=0}^{x=1} F_x dx \Big|_{y=1} \\ &= \int_{y=0}^{y=1} x^2 dy \Big|_{x=0} + \int_{x=0}^{x=1} -y dx \Big|_{y=1} \\ &= -1 + 0 = -1 \end{aligned} \tag{3.155}$$

The work done is not the same over this path. This is already obvious from the graph showing the path and the force field: the second path clearly moves against the force, where the first is moving with direction of the force.

Integration of the same force $\vec{F}(x, y) = -y\hat{x} + x^2\hat{y}$ from $(0, 0) \rightarrow (1, 1)$ but along a normal parabola. A parameterization of the path is $\vec{r}(\tau) = (0, 0) + (\tau, \tau^2)$, $\tau \in [0, 1]$ and the derivative is $\frac{d\vec{r}}{d\tau} = (1, 2\tau)$. The work then is

$$\begin{aligned} \int_0^1 \vec{F}(\tau, \tau^2) \cdot (1, 2\tau) d\tau &= \\ \int_0^1 (-\tau^2, \tau^2) \cdot (1, 2\tau) d\tau &= \\ \int_0^1 (-\tau^2 + 2\tau^3) d\tau &= \frac{1}{6} \end{aligned} \tag{3.158}$$

3.3.4 Gravitational potential energy

Let's consider an object close to the surface of any planet, where the acceleration due to gravity can be described by $F_g = -mg$. Raising the object to a height H requires us to do work: we will have to apply a force $F = +mg$ to the object to lift it to position H . Thus, with two forces acting - each doing work on the object we get:

$$\begin{aligned} W_g &= \int_0^H F_g dx = \int_0^H -mg dx = -mgH \\ W_+ &= \int_0^H -F_g dx = \int_0^H mg dx = mgH \end{aligned} \tag{3.159}$$

The net effect is of course $W_{net} = 0$ as the object started without kinetic energy and ends without kinetic energy, thus we knew in advance $0 = \Delta E_{kin} = W_g + W_+$

We can also take a slightly different view on this. Suppose we only concentrate on the work done by gravity: $W_g = -mgH$. Note that there is a minus sign, the gravitational force works in the opposite direction of the movement of the object. As energy is a conservative quantity, someone or something has supplied the object with some 'gained' energy. We call this potential energy, more particular in this case gravitational potential energy.

Why is it called 'potential'? When the object is released from that height H , this gravitational potential energy is converted to kinetic energy. The gravitational force does work on the object:

$$W = \int_H^0 F dx = \int_H^0 mg dx = mgH = \Delta E_{kin} \tag{3.160}$$

From this, it follows that the object will reach a velocity of $v = \sqrt{2gH}$. This is an example of a situation where an object loses potential energy and gains kinetic energy.

```
interactive(children=(FloatSlider(value=9.81, description='g (m/s²)', max=15.0, min=1.5), IntSlider(value=1, d...
```

Exercise 3.98: Potential & kinetic energy

Proof that the velocity of an object released from a height H will reach the velocity $v = \sqrt{2gH}$.

Exercise 3.99:

A point particle of mass $m = 1\text{kg}$ is at $t = 0$ at position $x = 0$. It has initial velocity v_0 . From $t = 0$ to $t_{stop} = 2\text{s}$ it is under the influence of a constant force F . This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Exercise 3.100:

Use the Python app below, and answer the following questions:

- does the acceleration double when the mass of the falling box doubles?
- the position time diagram is made using kinematics, how would the code look like when based on energy conservation?
- how would you include friction in the code?

```
<function __main__.run_animation(g=9.81, M=1)>
```

3.3.5 Conservative force

As we saw, work done on m by F during motion from 1 to 2 over a prescribed trajectory, is defined as:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \quad (3.161)$$

In general, the amount of work depends on the path followed. That is, the work done when going from \vec{r}_1 to \vec{r}_2 over the red path in the figure below, will be different when going from \vec{r}_1 to \vec{r}_2 over the blue path. Work depends on the specific trajectory followed.

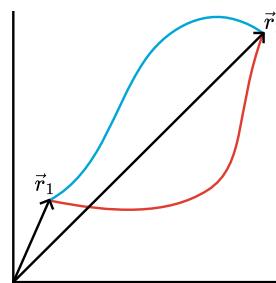


Figure 3.102: Two different paths.

Exercise 3.101:

Look at the following roller coaster app.

Change the various graph settings (what is on the x/y axis). Change the starting position of the ball, and try to change the path.

Can you make sense of the motion and the graphs?

However, there is a certain class of forces for which the path does not matter, only the start and end point do. These forces are called conservative forces. As a consequence, the work done by a conservative force over a closed path, i.e. start and end are the same, is always zero. No matter which closed path is taken.

$$\text{conservative force} \Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0 \text{ for ALL closed paths} \quad (3.162)$$

3.3.5.1 Stokes' Theorem

It was George Stokes who proved an important theorem, that we will use to turn the concept of conservative forces into a new and important concept.



Figure 3.103: Sir George Stokes (1819-1903). From Wikimedia Commons, public domain.

His theorem reads as:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\sigma \quad (3.163)$$

In words: the integral of the force over a closed path equals the surface integral of the curl of that force. The surface being ‘cut out’ by the close path. The term $\vec{\nabla} \times \vec{F}$ is called the curl of F , which is a vector. The meaning of the curl and some words on the theorem are given below.

Intermezzo: intuitive proof of Stokes' Theorem

Consider a closed curve Γ in the xy -plane. We would like to calculate the work done when going around this curve. In other words: what is $\oint \vec{F} \cdot d\vec{r}$ if we move along this curve?

We can visualize what we need to do: we cut the curve in small part; compute $\vec{F} \cdot d\vec{r}$ for each part (i.e. the red, green, blue, etc. in Figure 5 and sum these to get the total

along the curve. If we make the parts infinitesimally small, we go from a (Riemann) sum to an integral.

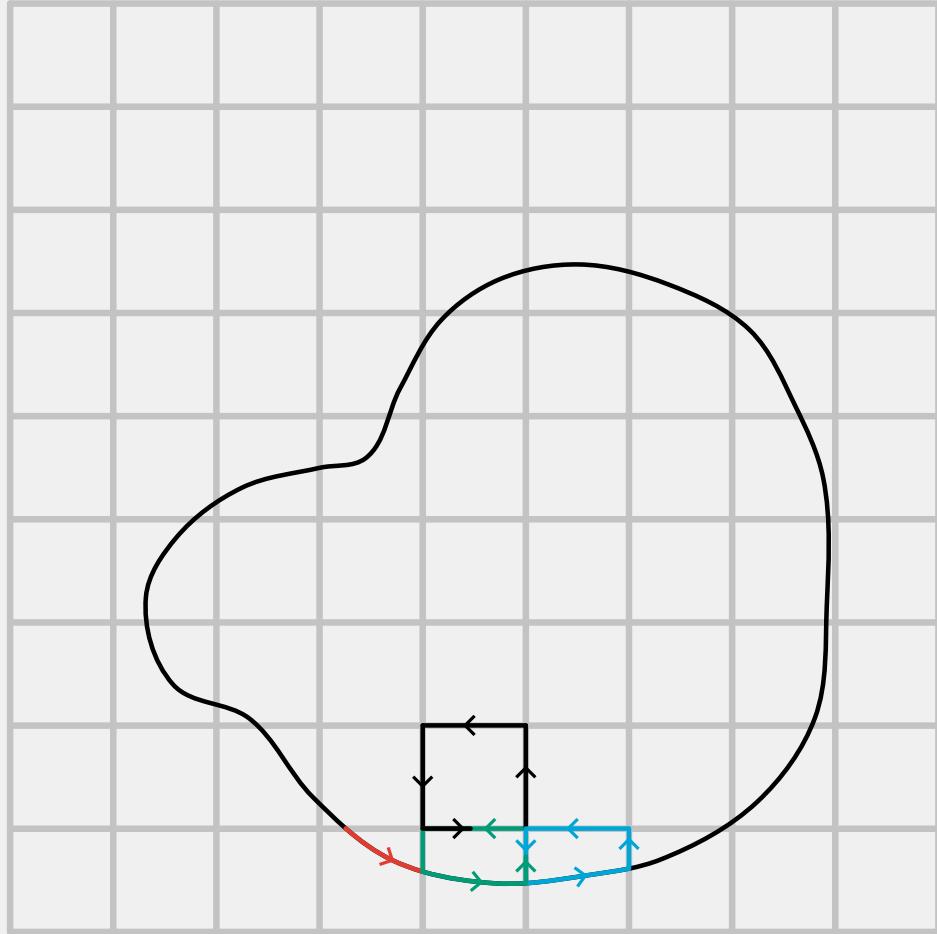


Figure 3.104: Closed path on a grid.

However, we are going to compute much more: take a look at Figure 5. We have put a grid in the xy -plane over a closed curve Γ . Hence, the interior of our curve is full of squares. We are not only computing the parts along the curve, but also along the sides of all curves. This will sound like way too much work, but we will see that it actually is a very good idea.

See Figure 5: we calculate $\oint \vec{F} \cdot d\vec{r}$ counter clockwise for the green square. Then we have at least the green part of our $\oint \vec{F} \cdot d\vec{r}$ done in the right direction. Hence, we compute $\int \vec{F} \cdot d\vec{r}$ along the right side of the green square. We do that from bottom to top as we go counter clockwise along the green square. Let's call that $\int_g \vec{F} \cdot d\vec{r}$.

Then we move to the blue square and repeat in counter clockwise direction our calculation. But this means that we compute along the left side of blue the square from top to bottom. We will call this $\int_b \vec{F} \cdot d\vec{r}$.

Next, we will add all contributions. Thus we get $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r}$. But these two cancel each other as they are exactly the same but done in opposite directions. Thus if we use that $\int_1^2 f dx = - \int_2^1 f dx$ for any integration, it becomes obvious that $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r} = 0$.

Note that this will happen for all side of the squares that are in the interior of our curve. Thus, the integral over all squares is exactly the integral along the curve Γ .

It seems, we do a lot of work for nothing. But there is another way of looking at the path-integrals along the squares. If we make the square small enough, the calculation along one square can be approximated:

$$\begin{aligned} \oint_{\text{square}} \vec{F} \cdot d\vec{r} &\approx F_x(x, y)dx + F_y(x + dx, y)dy - F_x(x, y + dy)dx - F_y(x, y)dy \\ &\approx \frac{F_x(x, y) - F_x(x, y + dy)}{dy} dxdy + \frac{F_y(x + dx, y) - F_y(x, y)}{dx} dxdy \\ &\approx \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \end{aligned} \quad (3.164)$$

The results get more accurate the smaller we make the square.

If we now sum up all squares and make these squares infinitesimally small, the sum becomes an integral, but now an integral over the surface enclosed by the curve:

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \quad (3.165)$$

The right hand side of the above equation is a surface integral of the ‘vector’ $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. Obviously, we did not provide a rigorous proof, but only an intuitive one. For a mathematical proof, see your calculus classes.

Moreover, we only worked in the xy -plane. If we would extend our reasoning to a closed curve in 3 dimensions, we would get Stokes theorem, which reads as:

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \quad (3.166)$$

Here, $d\vec{\sigma}$ is a small element out of the surface. Note that it is a vector: it has size and directions. The latter is perpendicular to the surface element itself. Furthermore, we have the vector $\vec{\nabla} \times \vec{F}$. This is the cross-product of the nabla-operator and our vector field \vec{F} . The nabla operator is a strange kind of vector. Its components are: partial differentiation. In a Cartesian coordinate system this can be written as:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \quad (3.167)$$

or if you prefer a column notation:

$$\vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad (3.168)$$

The curl of F can be found from e.g.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \quad (3.169)$$

Note of warning: do be careful with the nabla-operator. It is not a standard vector. For instance, ordinary vectors have the property $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. This does not hold for the nabla-operator.

Second note of warning: the representation of the nabla-operator does change quite a bit when using other coordinate systems like cylindrical or spherical. For instance, in cylindrical coordinates it is **not** equal to $\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}$. This can be easily seen as both r, z have units length, i.e. meters, but ϕ has no units.

Example: Work done in a vectorfield

Suppose we need to calculate the integral of the vectorfield $\vec{F}(x, y) = y\hat{x} - x\hat{y}$ over the closed curve formed by a square from $(0, 0)$ to $(1, 0)$, $(1, 1)$, $(0, 1)$ and back to $(0, 0)$.

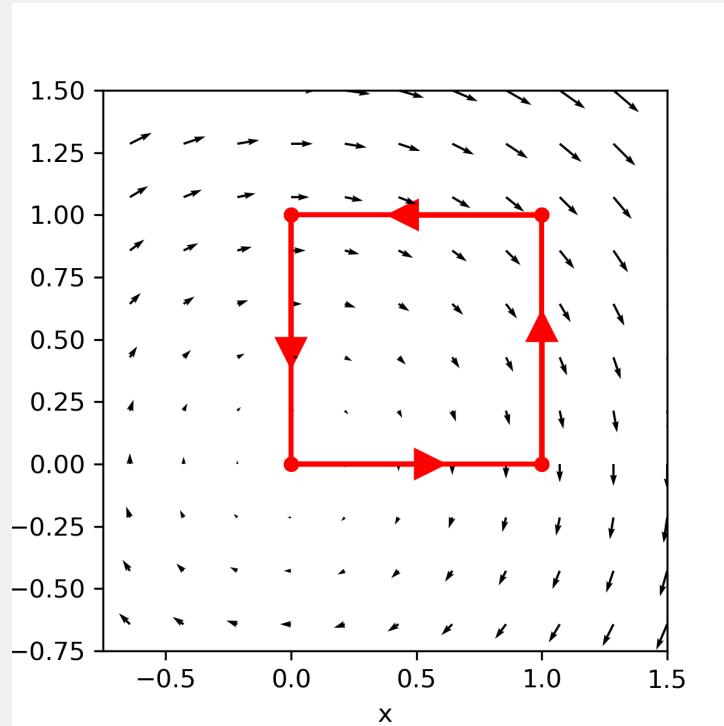


Figure 3.105: Integrating along the unit square.

We go counter clockwise.

$$\begin{aligned}
 \oint \vec{F} \cdot d\vec{r} &= \int_{x=0}^1 F_x(x, y=0) dx + \int_{y=0}^1 F_y(x=1, y) dy + \\
 &\quad + \int_{x=1}^0 F_x(x, y=1) dx + \int_{y=1}^0 F_y(x=0, y) dy \\
 &= \int_0^{10} dx + \int_0^1 -1 dy + \int_1^{01} dx + \int_1^0 -0 dx \\
 &= 0 - [y]_0^1 + [x]_1^0 - 0 \\
 &= -2
 \end{aligned} \tag{3.170}$$

Now we try this using Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \quad (3.171)$$

We first calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y} \right) \hat{z} = -2\hat{z} \quad (3.172)$$

Thus, in this example $\vec{\nabla} \times \vec{F}$ has only a z -component.

An elementary surface element of the square is: $d\vec{\sigma} = dx dy \hat{z}$. This also has only a z -component. Note that it points in the positive z -direction. This is a consequence of the counter clockwise direction that we use to go along the square.

According to Stokes Theorem, we this find:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_{x=0}^1 \int_{y=0}^1 (-2) dx dy = -2 \quad (3.173)$$

Indeed, we find the same outcome.

3.3.5.2 Conservative force and $\vec{\nabla} \times \vec{F}$

For a conservative force the integral over the closed path is zero for any closed path. Consequently, $\vec{\nabla} \times \vec{F} = 0$ everywhere. How do we know this? Suppose $\vec{\nabla} \times \vec{F} \neq 0$ at some point in space. Then, since we deal with continuous differentiable vector fields, in the close vicinity of this point, it must also be non-zero. Without loss of generality, we can assume that in that region $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$. Next, we draw a closed curve around this point, in this region. We now calculate the $\oint \vec{F} \cdot d\vec{r}$ along this curve. That is, we invoke Stokes Theorem. But we know that $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$ on the surface formed by the closed curve. Consequently, the outcome of the surface integral is non-zero. But that is a contradiction as we started with a conservative force and thus the integral should have been zero.

The only way out, is that $\vec{\nabla} \times \vec{F} = 0$ everywhere.

Thus we have:

$$\text{conservative force} \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere} \quad (3.174)$$

3.3.6 Potential Energy

This function V is called the potential energy or the potential for short and has a direct connection to the work. A direct consequence of the above is:

if $\vec{\nabla} \times \vec{F} = 0$ everywhere, a function $V(\vec{r})$ exists such that $\vec{F} = -\vec{\nabla}V$

$$\text{conservative force} \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere}$$

⇓

$$\vec{F} = -\vec{\nabla}V \Leftrightarrow V(\vec{r}) = - \int_{ref}^{\vec{r}} \vec{F} \cdot d\vec{r} \quad (3.175)$$

where in the last integral, the lower limit is taken from some, self picked, reference point. The upper limit is the position \vec{r} .

Next to its direct connection to work, the potential is also connected to kinetic energy.

$$E_{kin,2} - E_{kin,1} = W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1) \quad (3.176)$$

or rewritten:

$$E_{kin,1} + V(\vec{r}_1) = E_{kin,2} + V(\vec{r}_2) \quad (3.177)$$

In words: for a conservative force, the sum of kinetic and potential energy stays constant.

3.3.6.1 Energy versus Newton's Second Law

We, starting from Newton's Laws, arrived at an energy formulation for physical problems.

Question: can we also go back? That is: suppose we would start with formulating the energy rule for a physical problem, can we then back out the equation of motion?

Answer: yes, we can!

It goes as follows. Take a system that can be completely described by its kinetic plus potential energy. Then: take the time-derivative and simplify, we will do it for a 1-dimensional case first.

$$\begin{aligned} \frac{1}{2}mv^2 + V(x) &= E_0 \Rightarrow \\ \frac{d}{dt} \left[\frac{1}{2}mv^2 + V(x) \right] &= \frac{dE_0}{dt} = 0 \Rightarrow \\ mv\dot{v} + \underbrace{\frac{dV}{dx} \frac{dx}{dt}}_{=v} &= 0 \Rightarrow \\ v \left(m\dot{v} + \frac{dV}{dx} \right) &= 0 \end{aligned} \quad (3.178)$$

The last equation must hold for all times and all circumstances. Thus, the term in brackets must be zero.

$$m\dot{v} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} = -\frac{dV}{dx} = F \quad (3.179)$$

And we have recovered Newton's second law.

In 3 dimensions it is the same procedure. What is a bit more complicated, is using the chain rule. In the above 1-d case we used $\frac{dV}{dt} = \frac{dV(x(t))}{dt} = \frac{dV}{dx} \frac{dx(t)}{dt}$. In 3-d this becomes:

$$\frac{dV}{dt} = \frac{dV(\vec{r}(t))}{dt} = \frac{dV}{d\vec{r}} \cdot \frac{d\vec{r}(t)}{dt} = \vec{\nabla}V \cdot \vec{v} \quad (3.180)$$

Thus, if we repeat the derivation, we find:

$$\begin{aligned} \frac{1}{2}mv^2 + V(\vec{r}) &= E_0 \Rightarrow \\ \frac{d}{dt} \left[\frac{1}{2}mv^2 + V(\vec{r}) \right] &= 0 \Rightarrow \\ m\vec{v} \cdot \dot{\vec{v}} + \vec{\nabla}V \cdot \vec{v} &= 0 \Rightarrow \\ v \left(m\vec{a} + \vec{\nabla}V \right) &= 0 \Rightarrow \\ m\vec{a} &= -\vec{\nabla}V = \vec{F} \end{aligned} \quad (3.181)$$

And we have recovered the 3-dimensional form of Newton's second Law. This is a great result. It allows us to pick what we like: formulate a problem in terms of forces and momentum, i.e. Newton's second law, or reason from energy considerations. It doesn't matter: they are equivalent. It is a matter of taste, a matter of what do you see first, understand best, find easiest to start with. Up to you!

3.3.7 Stable and Unstable Equilibrium

A particle (or system) is in equilibrium when the sum of forces acting on it is zero. Then, it will keep the same velocity, and we can easily find an inertial system in which the particle is at rest, at an equilibrium position.

The equilibrium position (or more general: state) can also be found directly from the potential energy.

Potential energy and (conservative) forces are coupled via:

$$\vec{F} = -\vec{\nabla}V \quad (3.182)$$

The equilibrium positions ($\sum_i \vec{F}_i = 0$) can be found by finding the extremes of the potential energy:

$$\text{equilibrium position} \Leftrightarrow \vec{\nabla}V = 0 \quad (3.183)$$

Once we find the equilibrium points, we can also quickly address their nature: is it a stable or unstable solution? That follows directly from inspecting the characteristics of the potential energy around the equilibrium points.

For a stable equilibrium, we require that a small push or a slight displacement will result in a force pushing back such that the equilibrium position is restored (apart from the inertia of the object that might cause an overshoot or oscillation).

However, an unstable equilibrium is one for which the slightest push or displacement will result in motion away from the equilibrium position.

The second derivative of the potential can be investigated to find the type of extremum. For 1D functions that is easy, for scalar valued functions of more variables that is a bit more complicated. Here we only look at the 1D case $V(x) : \mathbb{R} \rightarrow \mathbb{R}$

$$\text{equilibrium} : \vec{\nabla}V = 0 \quad (3.184)$$

Luckily, the definition of potential energy is such that these rules are easy to visualize in 1D and to remember, see Figure 7

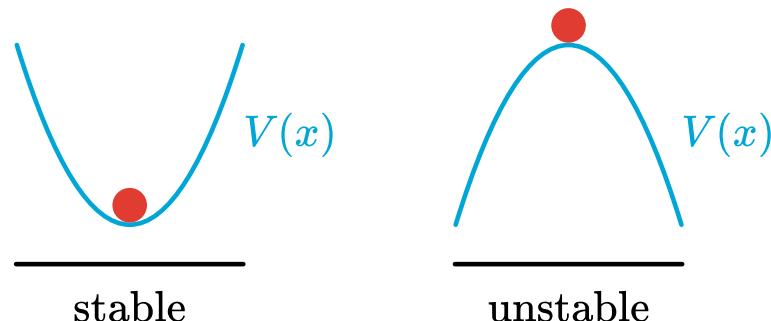


Figure 3.106: Stable and unstable position of a particle in a potential.

A valley is stable; a hill top is unstable.

NB: Now the choice of the minus sign in the definition of the potential is clear. Otherwise a hill would be stable, but that does not feel natural at all.

It is also easy to visualize what will happen if we distort that particle from the equilibrium state:

- The valley, i.e., the stable system, will make the particle move back to the lowest point. Due to inertia, it will not stop but will continue to move. As the lowest position is one of zero force, the particle will ‘climb’ toward the other end of the valley and start an oscillatory motion.
- The top, i.e., the unstable point, will make the particle move away from the stable point. The force acting on the particle is now pushing it outwards, ‘down the slope of the hill’.

3.3.7.1 Taylor Series Expansion of the Potential

The Taylor expansion or Taylor series is a mathematical approximation of a function in the vicinity of a specific point. It uses an infinite series of polynomial terms with coefficients given by value of the derivative of the function at that specific point: the more terms you use, the better the approximation. If you use all terms, then it is exact.

Mathematically, it reads for a 1D scalar function $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 \quad (3.185)$$

For our purpose here, it suffices to stop after the second derivative term:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \quad (3.186)$$

A way of understanding why the Taylor series actually works is the following.

Imagine you have to explain to someone how a function looks around some point x_0 , but you are not allowed to draw it. One way of passing on information about $f(x)$ is to start by giving the value of $f(x)$ at the point x_0 :

$$f(x) \approx f(x_0) \quad (3.187)$$

Next, you give how the tangent at x_0 is: you pass on the first derivative at x_0 . The other person can now see a bit better how the function changes when moving away from x_0 :

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (3.188)$$

Then, you tell that the function is not a straight line but curved, and you give the second derivative. So now the other one can see how it deviates from a straight line:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \quad (3.189)$$

Note that the prefactor is placed back. But the function is not necessarily a parabola; it will start deviating more and more as we move away from x_0 . Hence we need to correct that by invoking the third derivative that tells us how fast this deviation is. And this process can continue on and on.

Important to note: if we stay close enough to x_0 the terms with the lowest order terms will always prevail as higher powers of $(x - x_0)$ tend to zero faster than a lower powers (for instance: $0.5^4 \ll 0.5^2$).

This 3Blue1Brown clip explains the 1D Taylor series nicely.

Figure 3.107: A 3blue1brown clip on Taylor series.

For scalar valued functions as our potentials $V(\vec{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ the extension of the Taylor series is not too difficult. If we expand the function around a point

$$\begin{aligned} V(\vec{r}) &\approx V(\vec{r}_0) + \vec{\nabla}V(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) \\ &+ \frac{1}{2}(\vec{r} - \vec{r}_0) \cdot (\partial^2 V)(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) + \mathcal{O}(r^3) \end{aligned} \tag{3.190}$$

The second derivative of the potential indicated by $\partial^2 V$ is the Hessian matrix. Right now, this all sound a bit hocus pocus. But don't worry: you won't need it right away in its full glory. In the rest of your physics and math classes, this will all come back and start to make sense.

Conceptually the extrema of the function are again the hills and valleys. The classification of the extrema has next to hills and valleys also saddle points etc. In this course we will not bother about these more dimensional cases, but only stick to simple ones.

Exercise 3.108: Gravity, a conservative force? 🌶

Is gravity $\vec{F}_g = m\vec{g}$ a conservative force? If yes, what is the corresponding potential energy?

To find the answer:

- Show $\vec{\nabla} \times m\vec{g} = 0$
- Find a V that satisfies $-m\vec{g} = -\vec{\nabla}V$

Exercise 3.109: 🌶

A point particle of mass $m = 1\text{kg}$ is at $t = 0$ at position $x = 0$. It has initial velocity v_0 . From $t=0$ to $t_{stop} = 2\text{s}$ it is under the influence of a constant force F . This is a 1D problem.

The top graph shows the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

3.3.8 Examples, exercises and solutions

3.3.8.1 Exercise set 1

```
interactive(children=(FloatSlider(value=5.0, description='v₀ (m/s)', max=10.0, step=0.5), FloatSlider(value=5...<function __main__.run_animation(v₀, F, m)>
```

Exercise 3.110: 🌶

A simple model for the frictional force experienced by a body sliding over a horizontal, smooth surface is $F_f = -\mu F_g$ with F_g the gravitational force on the object. The friction force is opposite the direction of motion of the object.

- Show that this frictional force is not conservative (and, consequently, a potential energy associated does not exist!).

Tip

Think of two different trajectories to go from point 1 to point 2 and show that the amount of work along these trajectories is not the same.

Or: find a closed loop for which the work done by the frictional force is non-zero.

Exercise 3.111:

A force is given by: $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$

- Show that this force is conservative.
- Find the corresponding potential energy.

A second force is given by: $\vec{F} = y\hat{x} + x\hat{y} + z\hat{z}$

- Show that this force is also conservative.
- Find the corresponding potential energy.

Exercise 3.112:

Another force is given by: $\vec{F} = y\hat{x} - x\hat{y}$

- Show that this force is not conservative.
- Compute the work done when moving an object over the unit circle in the xy-plane in an anti-clockwise direction. (Hint: use Stokes theorem.)
- Discuss the meaning of your answer: is it positive or negative? And what does that mean in terms of physics?

Exercise 3.113:

Given a potential energy $E_{pot} = xy$.

- Find the corresponding force (field).
- Make a plot of \vec{F} as a function of (x,y,z).
- Describe the force and comment on what the potential itself already reveals about the force.

Exercise 3.114:

Given a force field $\vec{F} = -xy\hat{x} + xy\hat{y}$. A particle moves from $(x, y) = (0, 0)$ over the x-axis to $(x, y) = (1, 0)$ and then parallel to the y-axis to $(x, y) = (1, 1)$. In a second motion, the same particle goes from $(x, y) = (0, 0)$ over the y-axis to $(x, y) = (0, 1)$ and then parallel to the x-axis to end also in $(x, y) = (1, 1)$.

- Show that not necessarily the work done over the two paths is equal.
- Compute the amount of work done over each of the paths.

Exercise 3.115: 

A particle of mass m is initially at position $x = 0$. It has zero velocity. On the particle a force is acting. The force can be described by $F = F_0 \sin \frac{x}{L}$ with F_0 and L positive constants.

1. Show that this force is conservative and find the corresponding potential. Take as reference point for the potential energy $x = \frac{\pi}{2}L$.
2. The particle gets a tiny push, such that it starts moving in the positive x-direction. Its initial velocity is so small that, for all practical calculations, it can be set to zero. What will happen to the particle after the push?
3. Find the maximum velocity that the particle can get. At which location(s) will this take place?

Note: this is a 1-dimensional problem.

Solution 3.116: Solution to Exercise 1

a. Show $\vec{\nabla} \times m\vec{g} = 0$

$\vec{\nabla} \times m\vec{g} = 0$? How to compute it? For **Cartesian** coordinates there is an easy to remember rule:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \quad (3.191)$$

If we chose our coordinates such that $\vec{g} = -g\hat{z}$ we get:

$$\vec{\nabla} \times \vec{F}_g = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = 0 \quad (3.192)$$

Thus \vec{F}_g is conservative.

b. Find a V that satisfies $-m\vec{g} = -\vec{\nabla}V$

Does $-m\vec{g} = -\vec{\nabla}V$ have a solution for V ? Let's try, using the same coordinates as above.

$$\begin{aligned} -\vec{\nabla}V &= -m\vec{g} \Rightarrow \\ \frac{\partial V}{\partial x} &= 0 \rightarrow V(x, y, z) = f(y, z) \\ \frac{\partial V}{\partial y} &= 0 \rightarrow V(x, y, z) = g(x, z) \\ \frac{\partial V}{\partial z} &= mg \rightarrow V(x, y, z) = mgz + h(x, y) \end{aligned} \quad (3.193)$$

f,g,h are unknown functions. But all we need to do, is find one V that satisfies $-m\vec{g} = -\vec{\nabla}V$.

So, if we take $V(x, y, z) = mgz$ we have shown, that gravity in this form is conservative and that we can take $V(x, y, z) = mgz$ for its corresponding potential energy.

By the way: from the first part ($\text{curl } \vec{F} = 0$), we know that the force is conservative and we know that we could try to find V from

$$\begin{aligned} V(x, y, z) &= - \int_{ref} m\vec{g} \cdot d\vec{r} = \int_{ref} mg\hat{z} \cdot d\vec{r} \\ &= \int_{ref} mgdz = mgz + const \end{aligned} \quad (3.194)$$

Solution 3.117: Solution to Exercise 3

Click for the solution Friction Not Conservative.

Solution 3.118: Solution to Exercise 4

Click for the solution Conservative Force.

Solution 3.119: Solution to Exercise 5

Click for the solution Non-Conservative Force.

Solution 3.120: Solution to Exercise 6

Click for the solution Potential energy & Force.

Solution 3.121: Solution to Exercise 7

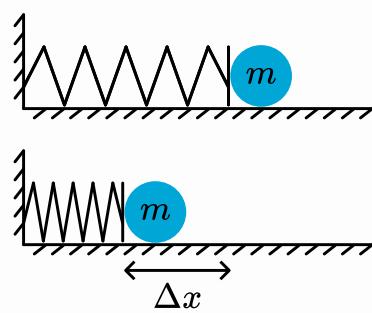
Click for the solution Force Field.

Solution 3.122: Solution to Exercise 8

Click for the solution Sinusoidal Force Field.

Exercise 3.123: Shooting a ball using a spring

A ball with mass m is horizontally pressed against a spring with spring constant k , compressing the spring by Δx .



1. Express the velocity of the ball when released.
2. Why is in real life the actual velocity of the ball less (friction is not the answer)?
3. Why is the velocity of the ball less when shot vertically?

Exercise 3.125: Firing a cannon ball⁶

1. Show that, if you ignore drag, a projectile fired at an initial velocity v_0 and angle θ has a range R given by

$$R = \frac{v_0^2 \sin 2\theta}{g} \quad (3.195)$$

1. A target is situated 1.5km away from a cannon across a flat field. Will the target be hit if the firing angle is 42° and the cannonball is fired at an initial velocity of 121 m/s? (Cannonballs, as you know, do not bounce).
2. To increase the cannon's range, you put it on a tower of height h_0 . Find the maximum range in this case, as a function of the firing angle and velocity, assuming the land around is still flat.

Exercise 3.126: Pushing a box uphill⁷

You push a box of mass m up a slope with angle θ and kinetic friction coefficient μ . Find the minimum initial speed v you must give the box so that it reaches a height h . Use energy and work to find the answer.

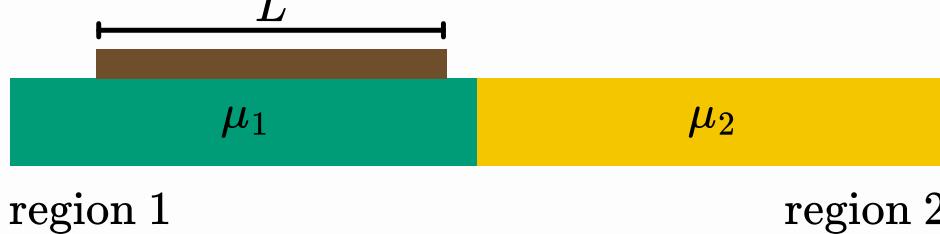
⁶Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

⁷Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 3.127: Work done dragging a board⁸

A uniform board of length L and mass M lies near a boundary that separates two regions. In region 1, the coefficient of kinetic friction between the board and the surface is μ_1 , and in region 2, the coefficient is μ_2 . Our objective is to find the net work W done by friction in pulling the board directly from region 1 to region 2, under the assumption that the board moves at constant velocity.

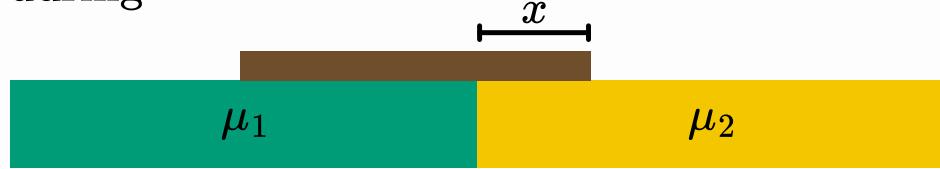
before



region 1

region 2

during



after



1. Suppose that at some point during the process, the right edge of the board is a distance x from the boundary, as shown. When the board is at this position, what is the magnitude of the force of friction acting on the board, assuming that it's moving to the right? Express your answer in terms of all relevant variables (L, M, g, x, μ_1 , and μ_2).
2. As we have seen, when the force is not constant, you can determine the work by integrating the force over the displacement, $W = \int F(x)dx$. Integrate your answer from (1) to get the net work you need to do to pull the board from region 1 to region 2.

⁸Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 3.129: 

A point particle (mass m) drops from a height H downwards. It starts with zero initial velocity. The only force acting is gravity (take gravity's acceleration as a constant).

- Set up the equation of motion (i.e. N2) for m .
- Calculate the velocity upon impact with the ground.
- Calculate the kinetic energy of m when it hits the ground.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.

Solve this exercise using IDEA.

 Exercise 3.130: 

A hockey puck ($m = 160$ gram) is hit and slides over the ice-floor. It starts at an initial velocity of 10m/s . The hockey puck experiences a frictional force from the ice that can be modeled as $-\mu F_g$ with F_g the gravitational force on the puck. The friction force eventually stops the motion of the puck. Then the friction is zero (otherwise it would accelerate the puck from rest to some velocity :smiley:). Constant $\mu = 0.01$.

- Set up the equation of motion (i.e. N2) for m .
- Solve the equation of motion and find the trajectory of the puck.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.
- Solve this exercise using IDEA.

 Exercise 3.131: 

An electron (mass m , charge $-e$) is in a static electric field. The electric field is of the form $\vec{E} = E_0 \sin(2\pi \frac{x}{L}) \hat{x}$. As a consequence, the electron experiences a force $\vec{F} = -e\vec{E}$. Due to this force, the electron moves from position $x = \frac{1}{4}L$ to $x = 0$.

- Calculate the amount of work done by the electric field.
- Assuming that the electron was initially at rest, what is the velocity at $x = 0$?

Exercise 3.132: 

A force $F = F_0 e^{-t/\tau}$ is acting on a particle of mass m . The particle is initially at position $x = 0$. It is, starting at $t = 0$, moving at a constant velocity $v_0 > 0$ to $x = L$, ($L > 0$).

- Calculate the amount of work done by F .
- The amount of work done is equal to the change in kinetic energy. However, the particle doesn't change its kinetic energy. Why is this general rule not violated in this case?

 Exercise 3.133: Work by a linear force 

A point particle of mass $m = 2\text{kg}$ is at $t = 0$ at position $x = 0$. It has initial velocity v_0 . From $t = 0$ to $t_{stop} = 4\text{s}$ it is under the influence of a force $F(x)$ that linearly increases with the position: $F(x) = kx$ with $k > 0$. This is a 1D problem.

The top graph show the position of the particle. The bottom graph shows the Work done on the particle by the force and the kinetic energy of the particle.

Analyse this situation and calculate the work done by the force at any time. Is the work done in this case always sufficient to account for the change in kinetic energy? What does it mean if the work is positive or negative?

Are the red (W) line and the green (E_{kin}) parallel? What does that mean?

Solution 3.134: Solution to Exercise 9

- $W = \Delta E_{kin} = \int_0^x F dx = \int_0^x kx dx = 1/2kx^2 = 1/2mv^2 \Rightarrow v = \sqrt{\frac{kx^2}{m}}$
- The spring has mass as well.
- The gravitational does work as well ($W = F_g dx < 0$)

Solution 3.135: Solution to Exercise 13

• 2.5 Testje (1/1)

$$F = ma \rightarrow m \frac{dv}{dt} = -mg \quad \left| \begin{array}{l} \text{e.g. of motion} \\ \text{v} = -gt \end{array} \right.$$

$$\frac{dv}{dt} = -g \Rightarrow v(t) = -gt + C_1 \quad \left| \begin{array}{l} t=0 \rightarrow v=0 \rightarrow C_1=0 \\ v = -gt \end{array} \right.$$

$$\frac{dt}{dt} = v = -gt \Rightarrow t(t) = -\frac{1}{g}t^2 + C_2 \quad \left| \begin{array}{l} t=H \rightarrow C_2=H \\ t=0 \rightarrow t=H \quad \therefore C_2=H \end{array} \right. \quad t = H - \frac{1}{g}t^2$$

ground: $t=0 \rightarrow 0 = H - \frac{1}{g}t^2 \Rightarrow t^* = \sqrt{\frac{2H}{g}}$

$$\Rightarrow v(t^*) = -gt^* = -\sqrt{2gH}$$

$$E_{kin} = \frac{1}{2}mv^2 = mgH \Rightarrow \Delta E_{kin,12} = mg(H-0)$$

$$W_{12} = \int_{-H}^{0} -mg \, dt = -mg \cdot \frac{1}{g}t^2 \Big|_{-H}^0 = mgH \quad \text{and } W_{12} = \Delta E_{kin,12}$$

Solution 3.137: Solution to Exercise 15

Work done by electric field when the electron moves from $x = \frac{1}{4}L$ to $x = 0$:

$$W = \int_{\frac{1}{4}L}^0 \vec{F} \cdot d\vec{s} = -eE_0 \int_{\frac{1}{4}L}^0 \sin\left(2\pi \frac{x}{L}\right) dx = -eE_0 \frac{L}{2\pi} \left[-\cos\left(2\pi \frac{x}{L}\right) \right]_{\frac{1}{4}L}^0 = \frac{1}{2\pi} eE_0 L \quad (3.196)$$

Work done is gain in kinetic energy: $\Delta E_{kin} = W$. Assuming the only work done is by the electric field and using initial velocity is zero: $v_i = 0$:

$$\frac{1}{2}mv^2 = \frac{1}{2\pi} eE_0 L \Rightarrow v = \sqrt{\frac{eE_0 L}{\pi m}} \quad (3.197)$$

Note that indeed the work done is positive, as it should in this case since the electron starts with zero velocity.

Solution 3.138: Solution to Exercise 16

$$W = \int_0^L \vec{F} \cdot d\vec{s} = \int_0^L F_0 e^{-\frac{t}{\tau}} dx \quad (3.198)$$

Particle velocity is $v_0 = \text{const}$. Thus, trajectory $x(t) = v_0 t$ since at $t = 0 \rightarrow x = 0$
 Consequently: $x = L \rightarrow t = \frac{L}{v_0}$

Thus, we can write for the amount of work done:

$$\begin{aligned} W &= \int_0^{\frac{L}{v_0}} F_0 e^{-\frac{t}{\tau}} \cdot v_0 dt = \\ F_0 v_0 \left[-\tau e^{-\frac{t}{\tau}} \right]_0^{L/v_0} &= F_0 v_0 \tau \left(1 - e^{-\frac{L}{v_0 \tau}} \right) \end{aligned} \quad (3.199)$$

We note: $W > 0$ and naively, we could expect that the kinetic energy of the particle would have increased. But that isn't the case: it started with $E_{kin} = \frac{1}{2}mv_0^2$ and it kept this along the entire path as it is given that the particle is traveling with a constant velocity.

From this last statement, we immediately learn, that there must be a second force acting on the particle. This force is exactly equal and opposite to F at all times! Otherwise, the particle would accelerate and change its velocity. Consequently, this second force also perform work on m , the amount is exactly $-W$ and thus the total work done on the particle is zero which reflects that the particle does not change its kinetic energy.

3.3.8.2 Answers set 1

3.3.8.3 Exercise set 2

3.3.8.4 Answers set 2

References

- Gribbin, J. (2019). *The scientists: A history of science told through the lives of its greatest inventors*. Random House.
- Idema, T. (2023). *Introduction to particle and continuum mechanics*. TU Delft OPEN Publishing. <https://doi.org/10.59490/tb.81>