Classical Mechanics & Special Relativity for Starters

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Abstract

some abstract

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Contents

1. Introduction

1.a. About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and pressumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version. Note that we made good use of other open educational resources, several exercises stem from such resources.

1.a.i. *Special features:*

In this book you will find some 'special' features. Some of these are emphasized by their own style:

Exercise 1: 🌶️

Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3 🌶️

intermezzo: Intermezzos

> Intermezzos contain background information on the topic, of the people working on the concepts.

> **Experiment: Experiments**
>
> We include some basic experiments that can be done at home.

> **Example: Examples**
>
> We provide various examples showcasing, e.g., calculations.

> **Python**
>
> We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'. Try it by pushing the ON-button and subsequently the play button and see the output in de code-cell below.

```python
print("The square root of 2 is: ", 2 ** 0.5)
```

New concepts, such as *Free body diagram*, are introduced with a hoover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

The book can be read online. There is, however, the opportunity to download the materials and play with the code offline as well. We advise to use Jupyter Lab in combination with MyST.

1.a.ii. *Feedback:*

This is the first version of this book. Although many have work on it and several iterations have been made, there might still be issues. Do you see a mistake, do you have suggestions for exercises, are parts missing or abundant. Tell us! You can use the `Feedback` button at the top right button. You will need a (free) GitHub account to report an issue!

1.b. Authors

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1.b.i. *About the authors:*

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1.c. *Open Educational Resource*

This website is a Jupyter Book. Source files are available for download using the button on the top right.

1.c.i. *Software and license:*

1.c.ii. *Images, videos, apps, intermezzos:*

The cover image is inspired by the work of 3blue1brown developer Grant Sanderson.

All vector images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistribution, adhere to the CC-BY licences.

We embedded several clips from 3blue1brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

The Intermezzos, which elaborate on the lives of various scientists and the efforts behind key physical discoveries, are composed by drawing from a range of different sources. Rather than directly reproducing any one account, these stories have been reworked into a narrative that fits the context and audience of this book.

1.c.iii. *How to cite this book:*

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*, CC BY-NC

2. Mechanics

3. Mechanics

3.a. The language of Physics

Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, both colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

3.a.i. *Representations:*

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.

Figure 1: A base jumper is used as context to get familiair with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

Verbal descriptions:

Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

Example: Base jumper: Verbal description

Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

Visual representations:

Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more concrete.

- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.
- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.

Example: Base jumper: Free body diagram

The situation is sketched in Figure 2 using a Free body diagram. Note that all details of the jumper are ignored in the sketch.

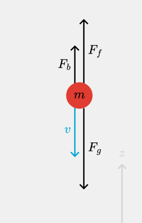


Figure 2: Force acting on the jumper.

- m = mass of jumper (in kg);

- v = velocity of jumper (in m/s);
- F_g = gravitational force (in N);
- F_f = drag force by the air (in N);
- F_b = buoyancy (in N): like in water also in air there is an upward force, equal to the weight of the displaced air.

Equations:

Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

Example: Base jumper: equations

The forces acting on the jumper are already shown in Figure 2. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \tag{1}$$

We can specify each of the force:

$$\begin{aligned} F_g &= -mg = -\rho_p V_p g \\ F_f &= \frac{1}{2}\rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \tag{2}$$

with g the acceleration of gravity, ρ_p the density of the jumper ($\approx 10^3 \mathrm{kg/m^3}$), V_p the volume of the jumper, ρ_{air} the density of air ($\approx 1.2 \mathrm{kg/m^3}$), C_D the so-called drag coefficient, A the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering which representations suits best for the given situation. We will practice these when solving problems.

Danger

Note that in the example above we neglected directions. In our equation we should have been using vector notation, which we will cover in one of the next sections in this chapter.

3.a.ii. *How to solve a physics problem?:*

One of the most common mistakes made by 'novices' when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you apply this way of working automatically. It is called IDEA, an acronym that stands for:

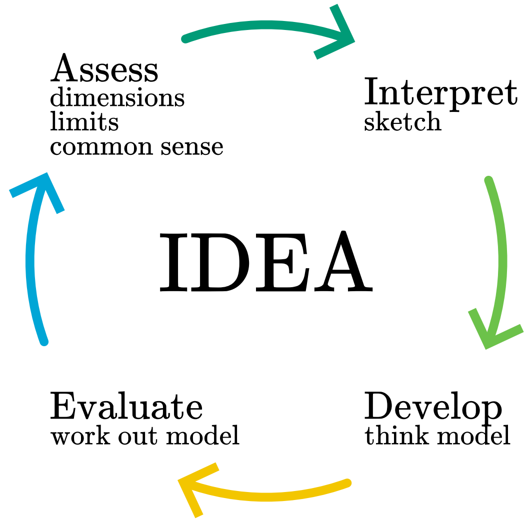


Figure 3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. Actually *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?).

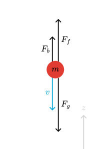
We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

Example:

Interpret

Three forces act on the jumper, shown in the figure below. Finding the terminal velocity implies that all forces are balanced ($\sum F = 0$).



The buoyancy force is much smaller than the force of gravity (about 0.1%) and we neglect it.

Develop

We know all forces: gravitational force equals the drag force

$$\begin{aligned} F_g &= F_f \\ mg &= \frac{1}{2}\rho_{air} C_D A v^2 \end{aligned} \tag{3}$$

Evaluate

Assume a mass of 75kg, an acceleration due to gravity of 9.81m/s^2, and air density of 1.2kg/m^3, a drag coefficient of 1, a frontal surface area of 0.7m^2.

$$mg = \frac{1}{2}\rho_{air} C_D A v^2 \tag{4}$$

Rewriting:

$$\begin{aligned} v &= \sqrt{\frac{2mg}{\rho_{air} C_D A}} \\ v &= \sqrt{\frac{2 \cdot 75(\text{kg}) \cdot 9.81(\text{m} \setminus/ \text{s}^2)}{1.2(\text{kg/m}^3) \cdot 1 \cdot 0.7(\text{m}^2)}} \\ v &= 40\text{m/s} \end{aligned} \tag{5}$$

Assess

We may know that raindrops typically reach a terminal velocity of less than 10m/s. A terminal velocity of 40m/s seems therefore plausible.

Note that we didn't solve the problem entirely! We only calculated the terminal velocity, where the question was how long it would roughly take to reach such a velocity.

Good Practice

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or multiple steps in one step. If you make a mistake, it will be very hard to trace it back.

Next: check constantly the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

Powers of ten

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms (10^{-10}m) to the distance between stars (10^{16}m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milli seconds, GB for gigabyte that is 1 billion bytes. Here is a list of prefixes.

Prefix	Symbol	Math	Prefix	Symbol	Math
Yocto	y	10^{-24}	Base	•	10^{0}
Zepto	z	10^{-21}	Deca	da	10^{1}
Atto	a	10^{-18}	Hecto	h	10^{2}
Femto	f	10^{-15}	Kilo	k	10^{3}
Pico	p	10^{-12}	Mega	M	10^{6}
Nano	n	10^{-9}	Giga	G	10^{9}
Micro	μ	10^{-6}	Tera	T	10^{12}
Milli	m	10^{-3}	Peta	P	10^{15}
Centi	c	10^{-2}	Exa	E	10^{18}
Deci	d	10^{-1}	Zetta	Z	10^{21}
Base	•	10^{0}	Yotta	Y	10^{24}

On quantities and units

Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter m? Quantities are expressed in italics (m) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

- Meter (m) – length
- Kilogram (kg) – mass
- Second (s) – time

- Ampere (A) – electric current
- Kelvin (K) – temperature
- Mole (mol) – amount of substance
- Candela (cd) – luminous intensity

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned} W &= F \cdot s = ma \cdot s = m\frac{\Delta v}{\Delta t} \cdot s \\ &= [\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [\mathrm{m/s^2}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m/s}]}{[\mathrm{s}]} \cdot [\mathrm{m}] = \left[\frac{\mathrm{kgm^2}}{\mathrm{s^2}}\right] \end{aligned} \tag{6}$$

Tip

For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

3.a.iii. *Calculus:*

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features, that help us understanding the problem and its solution much deeper.

Given the example of the base jumper, we would like to know how the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{7}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air} C_D A v^2 \tag{8}$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself. Before we even try to solve this problem ($v(t) = \ldots$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

Differentiation:

Many physical phenomena are described by differential equations. That may be because a system evolves in time or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2^{nd} law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time; velocity in itself is the derivative of position with respect to time. Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and} a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{9}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{or} a = \dot{v} = \ddot{x} \tag{10}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{11}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{12}$$

It is just a matter of notation.

3.a.iv. *Definition of velocity, acceleration and momentum:*

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that x. In general, x will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time t. How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in the time interval Δt. The (momentary) velocity is defined as:

Velocity

$$\text{definition velocity} : v \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = \frac{dx}{dt} \tag{13}$$

Similarly, we define the acceleration as the change of the velocity over a time interval Δt: $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

Acceleration

$$\text{definition acceleration} : a \equiv \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{(t+\Delta t) - t} = \frac{dv}{dt} \tag{14}$$

Consequently,

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{15}$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

Momentum

$$\text{definition momentum} : p \equiv mv = m\frac{dx}{dt} \tag{16}$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.

Figure 5: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

Vectors - more dimensional case:

Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

> **Tip**
>
> See the linear algebra book on vectors.

Position vector

We write the position **vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

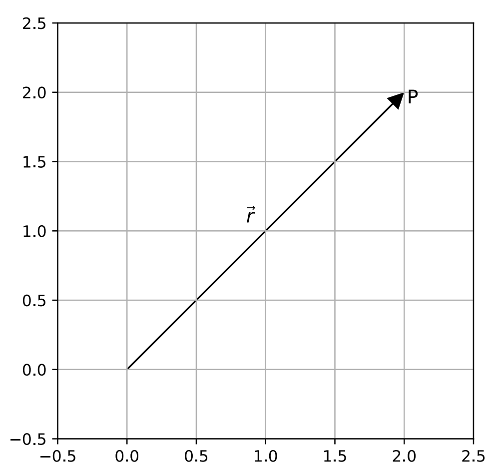


Figure 6: Some physical quantities (velocity, force etc) can be represented as a vector. The have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{17}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \hat{\imath},\ \vec{e}_x \\ &\hat{y},\ \hat{\jmath},\ \vec{e}_y \\ &\hat{z},\ \hat{k},\ \vec{e}_z \end{aligned} \tag{18}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{19}$$

Note that these representations are completely equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

Other textbooks

Note that other textbooks may use bold symbols to represent vectors:

$$\vec{F} = m\vec{a} \tag{20}$$

is the same as

$$\boldsymbol{F} = m\boldsymbol{a} \tag{21}$$

```
interactive(children=(FloatSlider(value=2.0, description='a x', max=4.0,
min=-2.0, step=0.5), FloatSlider(valu…
```

```
<function __main__.plot_vector_standaard_form(a_x=2.0, a_y=2.0, b_x=4.0,
b_y=1.0)>
```

Differentiating a vector

We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position.
As an example, let’s take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

Velocity vector

$$\text{definition velocity} : \vec{v} \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{22}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{23}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction.
Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\widehat{dx} + dy\hat{y} + dz\hat{z} \\ &= (x+dx)\hat{x} + (y+dy)\hat{y} + (z+dz)\hat{z} \end{aligned} \tag{24}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{25}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{26}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{27}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

Momentum Vector

$$\text{definition momentum} : \vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \tag{28}$$

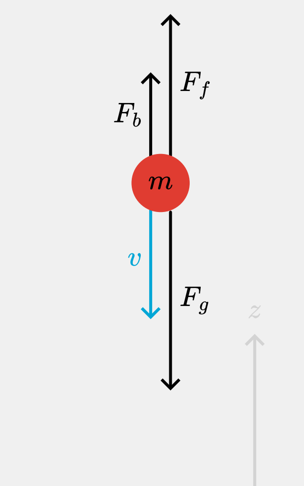
We can use the same reasoning and notation for acceleration:

Acceleration Vector

$$\text{definition acceleration} : \vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (29)$$

Example: The base jumper

Given the above explanation, we can now reconsider our description of the base jumper.



We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F_g} = -mg\hat{z} \quad (30)$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F_b} = \rho_{air} V g\hat{z} \quad (31)$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2. To solve this, we can write

$$\vec{F_f} = -\frac{1}{2}\rho_{air} C_D A \mid v \mid \vec{v} \quad (32)$$

Note that $\hat{z}$ is missing in (32) on purpose. That would be a simplification that is valid in the given situation, but not in general.

3.a.v. *Numerical computation and simulation:*

In simple cases we can come to an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right) \tag{33}$$

with

$$k = \frac{1}{2}\rho_{air} C_D A \tag{34}$$

In this case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. (Figure 8) presents a script for the software Coach, which you might have encountered in secondary school.

```
'Stop condition is set
'Computations are based on Euler

x := x + flow_1*Δt1
v := v + flow_2*Δt1

t1 := t1 + Δt1

flow_1 := v
Fz := m*g
Fw := 6*d*d*v*v
f := Fz - Fw
a := f/m
flow_2 := a
```

```
t1 := 0              's
Δt1 := 0.01          's

x := 0               'm
v := 0               'm/s
m := 75              'kg
g := 9.81            'm/s^2
d := 2.5             'm

flow_1 := v          'm/s
Fz := m*g            'N
Fw := 6*d*d*v*v      'N
f := Fz - Fw         'N
a := f/m             'm/s^2
flow_2 := a          'm/s^2
```

Figure 8: An example of a numerical simulation made in Coach. At the left the iterative calculation process, at the right the initial conditions.

Example: The base jumper

Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air} C_D A$ which eases writing. Newton's second law then becomes:

$$m\vec{a} = -m\vec{g} - k \mid v \mid \vec{v} \tag{35}$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{36}$$

with $\vec{F}_{net} = -m\vec{g} - k \mid v \mid \vec{v}$

On a computer, we can not literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t + \Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{37}$$

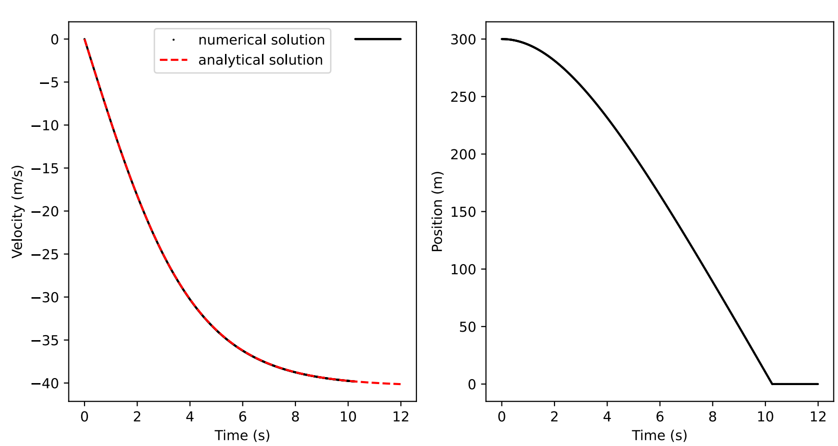
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots$. We will call these values v_i. Note that the force can be calculated at time t_i once we have v_i.

$$\begin{aligned} F_i &= mg - k \mid v_i \mid v_i \\ v_{i+1} &= v_i + \frac{F_i}{m}\Delta t \end{aligned} \tag{38}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i \Delta t \tag{39}$$

With the above rules, we can write an iterative code:



Exercise 1: Base jumper with initial velocity 🌶️

Change the code so that the base jumper starts with an initial velocity along the z-direction.

Is the acceleration in the z-direction with and without initial velocity the same? Elaborate.

Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $|\ v\ |\ v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

To come back to our initial problem:
It roughly takes 10s to get close to terminal velocity (note that without friction the velocity would be 98m/s). The building is not high enough to reach this velocity (safely).

Exercise 2: Unit analysis 🌶️

Given the formula $F = kv^2$. Derive the unit of k, expressed only in SI-units .

Exercise 3: Units based on physical constants[1] 🌶️ 🌶️

In physics, we assume that quantities like the speed of light (c) and Newton's gravitational constant (G) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant ($\hbar$, h an with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

1. Combine c, G and $\hbar$ into a quantity that has the dimensions of length.
2. Calculate the numerical value of this length in SI units (this is known as the Planck length).
3. Similarly, combine c, G and $\hbar$ into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

3.a.vi. *Examples, exercises and solutions:*

Exercises:

```
### Your code
```

```
### Your code
```

Solutions:

References

Idema, T. (2023). *Introduction to particle and continuum mechanics*. TU Delft OPEN Publishing. https://doi.org/10.59490/tb.81

[1]Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 4: Reynolds numbers[2] 🌶️ 🌶️

Physicists often use *dimensionless quantities* to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the *Reynolds number* in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$\mathrm{Re} = \frac{\mathrm{inertialforces}}{\mathrm{dragforces}} = \frac{\rho v L}{\mu} \tag{40}$$

where ρ is the density of the fluid (either a liquid or a gas), v the speed of the object, L its size, and μ the viscosity of the fluid. Typical values of the viscosity are $1.0\ \mathrm{mPa} \cdot \mathrm{s}$ for water, $50\ \mathrm{mPa} \cdot \mathrm{s}$ for ketchup, and $1.8 \cdot 10^{-5}\ \mathrm{Pa} \cdot \mathrm{s}$ for air.

1. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
2. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
3. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
4. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the diameter D of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) Q, and the kinematic viscosity $\nu \equiv \eta/\rho$.
5. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow in a cylindrical pipe is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a 1.0 cm diameter in the case that the flow is just laminar.

[2] Exercise from Idema, T. (2023). Introduction to particle and continuum mechanics. Idema (2023)

Exercise 5: Powers of ten 🌶️

Calculate:

1. $10^{-4} \cdot 10^{-8} =$
2. $\frac{10^6}{10^{-19} \cdot 10^4} =$
3. $10^{12} \cdot 10^{-15} =$

Exercise 6: Moving a box 🌶️

A box is on a frictionless incline of 10°. It is pushed upward with a force F_i for $\Delta t = 0.5$s. It is then moving upward (inertia) but slows down due to gravity.

Below is a part of the python code. However, some essential elements of the code are missing indicated by **(..)**.

1. Include the correct code and run it.
2. Explain the two graphs, highlighting all essential features of the graph by relating these to the given problem.
3. At what time is the acceleration 0? At what time is the box back at its origin?

The above context is not very realistic as friction is neglected. We, however, can include friction easily as it is given by $\overrightarrow{F_w} = \mu \overrightarrow{F_N}$, with $\mu = 0.05$. Note that the direction of friction changes when the direction of the velocity changes!

4. Extend the code so that friction is included.

Exercise 7: Basejumper with parachute 🌶️ 🌶️

Our base jumper has yet not a soft landing. Luckily she has a working parachute. The parachute opens in 3.8s reaching a total frontal area of 42.6m^2. We can model the drag force using $\vec{F}_{drag} = k \mid v \mid \vec{v}$ with $k = 0.37$.

Write the code that simulates this jump of the base jumper with deploying the parachute. Show the (F_{drag}, t)-diagram and the (v, t)-diagram. What is the minimal height at which the parachute should be deployed?

Exercise 8: Circular motion 🌶️ 🌶️

Remember from secondary school circular motion, where the required force is given by $\vec{F} = \frac{m\vec{v}^2}{r}$. Now let's simulate that motion.

Assume:

- m = 1 kg
- $\vec{r}_0$ = (3,0) m
- $\vec{v}(0)$ = (0,7) m/s

Write the code. You know the output already (a circle with radius of 3)!

Solution 1: Solution to Exercise 1

$$\begin{aligned} F &= kv^2 \\ &= [.]\left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right] \Rightarrow [.] = \left[\frac{\mathrm{kg}}{\mathrm{m}}\right] \end{aligned} \tag{41}$$

Solution 2: Solution to Exercise 2

The physical constants c, G and $\hbar$ have the following numerical values and SI-units:

$$\begin{aligned} c &= 2.99792458 \cdot 10^8 \text{ m/s} \\ G &= 6.674 \cdot 10^-11 \text{ m}^3/\text{kg} \cdot \text{s}^2 \\ \hbar &= 1.054 \cdot 10^{-34} \text{ kgm}^2/\text{s} \end{aligned} \tag{42}$$

Note: the value of c is precise, i.e. by definition given this value. The second is defined via the frequency of radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

If we want to combine these three units into a length scale, $\mathcal{L}$, we try the following:

$$[\mathcal{L}] = [c]^A[G]^B[\hbar]^C \tag{43}$$

What we mean here, is that the units of the quantities (denoted by [.]) left and right should be the same. Thus, we get:

$$m^1 = \left(\frac{m}{s}\right)^A \left(\frac{m^3}{kgs^2}\right)^B \left(\frac{kgm^2}{s}\right)^C \tag{44}$$

We try to find A, B, C such that the above equation is valid. We can write this equation as:

$$m^1 = m^{A+3B+2C} \cdot kg^{-B+C} \cdot s^{-A-2B-C} \tag{45}$$

If we split this into requirements for m, kg, s we get:

$$\begin{aligned} m &: 1 = A + 3B + 2C \\ kg &: 0 = C - B \\ s &: 0 = -A - 2B - C \end{aligned} \tag{46}$$

From the second equation we get $B = C$. Substitute this into the first and third and we find:

$$\begin{aligned} m &: 1 = A + 5B \\ s &: 0 = -A - 3B \end{aligned} \tag{47}$$

Add these two equations: $1 = 2B \to B = \frac{1}{2}$ and thus $C = \frac{1}{2}$ and $A = -\frac{3}{2}$.

So if we plug these values into our starting equation we see:

$$\mathcal{L} = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \cdot 10^{-35} \text{ m} \tag{48}$$

We can repeat this for energy, $\mathcal{E}$:

$$[\mathcal{E}] = [c]^\alpha[G]^\beta[\hbar]^\gamma \tag{49}$$

Note: the unit of energy, [J] needs to be written in terms of the basic units: $[J] = \text{kgm}^2/\text{s}^2$.

The outcome is: $\alpha = \frac{5}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$ and thus our energy is:

$$\mathcal{E} = \sqrt{\frac{\hbar c^5}{G}} = 1.96 \cdot 10^9 \text{ J} \tag{50}$$

Solution 3: Solution to Exercise 3

1. The size of a duck is on the order of 30cm. It flies at a speed of about 70km/h, that is 20m/s. Thus we compute for the Reynolds number of a flying duck:

$$Re \equiv \frac{\rho v L}{\mu} = 4.0 \cdot 10^5 \tag{51}$$

Clearly, the inertial force is dominant.

What about a swimming duck? Now the velocity is much smaller: $v \approx 1\ \mathrm{m/s} = 3.6\ \mathrm{km/h}$. The viscosity of water is $\mu_w = 1.0\ \mathrm{mPa \cdot s}$ and the water density is $1.0 \cdot 10^3\ \mathrm{kg/m^3}$. We, again, calculate the Reynolds number:

$$Re_w \equiv \frac{\rho v L}{\mu} = 3.0 \cdot 10^5 \tag{52}$$

Hence, also in this case inertial forces are dominant. This perhaps comes as a surprise, after all the velocity is much smaller and the viscosity much larger. However, the water density is also much larger!

2. For a swimming bacterium the numbers change. The size is now about 1μm and the velocity 60 μm/s (numbers taken from internet). That gives:

$$Re_b \equiv \frac{\rho v L}{\mu} = 6.0 \cdot 10^{-5} \tag{53}$$

and we see that here viscous forces are dominating.

3. For an oil tanker the Reynolds number is easily on the order of 10^8. Obviously, viscous forces don't do much. An oil tanker that wants to slow down can not do so by just stopping the motors and let the drag force decelerate them: the Reynolds number shows that the viscous drag is negligible compared to the inertial forces. Thus, the tanker has to use ots engines to slow down. Again the inertia of the system is so large, that it will take a long time to slow down. And a long time, means a long trajectory.
4. For the flow of water through a (circular) pipe the Reynolds number uses as length scale the pipe diameter. We can relate the velocity of the water in the pipe tot the total volume that is flowing per second through a cress section of the pipe:

$$Q = \frac{\pi}{4} D^2 v \to v = \frac{4Q}{\pi D^2} \tag{54}$$

Thus we can also write Re as:

$$Re \equiv \frac{\rho v D}{\mu} = \frac{4Q}{\pi \frac{mu}{\rho} D^2} = \frac{4Q}{\pi \nu D^2} \tag{55}$$

1. If $Re = 2300$ for the pipe flow, we have:

$$Re = \frac{vD}{\nu} = 2300 \to v = \frac{2300\nu}{D} \tag{56}$$

with $\nu = 1.0 \cdot 10^{-6}\ \mathrm{m^2/s}$ and $D = 1.0 \cdot 10^{-2}$ m we find: $v = 0.23\ \mathrm{m/s}$ and $Q = 1.8 \cdot 10^{-5}\ \mathrm{m^3/s} = 0.018$ liter/s.

Solution 4: Solution to Exercise 4

1. $= 10^{-12}$
2. $= 10^{21}$
3. $= 10^{-3}$

Solution 5: Solution to Exercise 5

```
# Moving a box

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt


part_4 = 1 # Turn to 0 for first part

## Constants
m = 2 #kg
F = 30 #N
g = 9.81 #m/s^2
theta = np.deg2rad(10) #degrees

mu = 0.02
F_N = m*g*np.cos(theta) #N

## Time step
dt = 0.01 #s
t = np.arange(0, 10, dt) #s
t_F_stop = 0.5

## Initial conditions
x = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s


## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    if t[i] < t_F_stop:
        a = F/m - g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    else:
        a = -g*np.sin(theta) - F_N*mu*np.where(v[i] != 0,
np.sign(v[i]), 0)*part_4
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i]*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')
axs[0].plot(t, v, 'k.', markersize=1)


axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
axs[1].plot(t, x, 'k.', markersize=1)
```



```
plt.show()
```

Solution 6: Solution to Exercise 6

```
# Simulation of a base jumper

## Importing libraries
import numpy as np
import matplotlib.pyplot as plt

## Constants
A = 0.7 #m^2
m = 75 #kg
k = 0.37 #kg/m
g = 9.81 #m/s^2

## Time step
dt = 0.01 #s
t = np.arange(0, 12, dt) #s

## Initial conditions
z = np.zeros(len(t)) #m
v = np.zeros(len(t)) #m/s
z[0] = 300 #m

## Deploy parachute
A_max = 42.6 #m^2
t_deploy_start = 2 #s
dt_deploy = 3.8 #s

## Loop to calculate position and velocity
for i in range(0, len(t)-1):
    F = - m*g - k*A*abs(v[i])*v[i]  #N
    v[i+1] = v[i] + F/m*dt #m/s
    z[i+1] = z[i] + v[i]*dt #m
    # Check if the jumper is on the ground
    if z[i+1] < 0:
        break
    # Deploy parachute
    if t[i] > t_deploy_start and t[i] < t_deploy_start + dt_deploy:
        A += (A_max - A)/dt_deploy*dt

## Plotting results
figs, axs = plt.subplots(1, 2, figsize=(10, 5))

axs[0].set_xlabel('Time (s)')
axs[0].set_ylabel('Velocity (m/s)')

axs[0].plot(t, v, 'k.', markersize=1, label='numerical solution')
axs[0].vlines(t_deploy_start, v[t==t_deploy_start],0, color='gray',
linestyle='--', label='parachute deploy')

axs[0].legend()

axs[1].set_xlabel('Time (s)')
axs[1].set_ylabel('Position (m)')
```



```
axs[1].plot(t, z, 'k.', markersize=1)
axs[1].vlines(t_deploy_start, 150,300, color='gray', linestyle='--',
label='parachute deploy')
```

Solution 7: Solution to Exercise 7

```
import numpy as np
import matplotlib.pyplot as plt

F = 49/3
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7


for i in range(0, len(t)-1):
    ax = -F*(x1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    ay = -F*(y1[i]-0)/np.sqrt(x1[i]**2 + y1[i]**2)/m1
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt


plt.figure(figsize=(4,4))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()
```