Frog Puzzle

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1 Problem Statement

Suppose there exists a sequence of n lilypads in a row. A frog is on pad 1. The frog is trying to reach the n^th lilypad. The frog is very athletic, and from pad k the frog jumps to any other pad ahead of it with probability $\frac{1}{n-k}$. Seeing each pad as a state, this is formally the transition probability

$$P(S_{t+1} = i + k | S_t = i) = \frac{1}{n-i}$$

for k $\in \{1, ..., n-i\}$
Question: How many expected jumps will the frog take to get to the end?

2 Solution

Let J_k be the expected number of jumps from pad k. We have by the formulation above,

$$J_k = 1 + \frac{1}{n-k}(\sum_{i=k+1}^{n} J_i)$$

and

$$J_{k+1} = 1 + \frac{1}{n-(k+1)}(\sum_{i=k+2}^{n} J_i)$$

Thus,

$$\frac{n-k+1}{n-k} J_{k+1} = \frac{n-k+1}{n-k} + \frac{1}{n-k}(\sum_{i=k+2}^{n} J_i)$$

.

Substituting the above expression into the first equation,

$$J_k = 1 + \frac{1}{n-k} J_{k+1} + (\frac{n-k+1}{n-k} J_{k+1} - \frac{n-k+1}{n-k})$$

which simplifies to

$$J_k = \frac{1}{n-k} + J_{k+1}$$

So, with $J_n = 0$, $J_{n-1} = 1$, $J_{n-2} = \frac{1}{2} + 1$, and so on. So we see the expected number of jumps is $J_1 = \frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{2} + \frac{1}{1} = \sum_{k=1}^{n} \frac{1}{k}$.