A6Q3: Kelly Criterion

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1 Problem

Derive the kelly criterion with a bet x that returns $x(1+\alpha)$ with probability p and $x(1-\beta)$ with probability $q = 1-p$.

With wealth W_0, the wealth at the next timestep is either $W_0(1+f\alpha)$ or $W_0(1-f\beta)$. Thus the log utility is either $log(W_0(1+f\alpha))$ or $log(W_0(1-f\beta))$. The expected log utility is thus

$$E[U(W_1)] = p\log(W_0(1+f\alpha)) + q\log(W_0(1-f\beta)) \tag{1}$$

The derivative w.r.t f is

$$\frac{dE[U(W_1)]}{df} = p\frac{1}{W_0(1+f\alpha)}W_0\alpha + q\frac{1}{W_0(1-f\beta)} * -W_0\beta \tag{2}$$

which is equal to

$$\frac{dE[U(W_1)]}{df} = p\frac{\alpha}{1+f\alpha} + q\frac{-\beta}{1-f\beta} \tag{3}$$

solving for f, we set this equal to 0 and get the next expression after moving denominators up.

$$\alpha p(1-f\beta) = q\beta(1+f\alpha) \tag{4}$$

which simplifies to

$$f^* = \frac{\alpha p - q\beta}{\alpha\beta} = \frac{p}{\beta} - \frac{q}{\alpha} \tag{5}$$

The fractional bet size goes up as the the probability of a win increases relative to the loss incurred by losing the bet, and decreases when the ratio of the losing probability to the return on winning the bet increases, so this makes sense intuitively.