Chapter 2 : PQs

2024

Question 1:

ⓐ use bisection method to find n_3 for
$f(x) = \sqrt{x} - \cos(x)$ on $[0, 1]$

$f(0) = -1$ -ve

$f(1) = 0.459$ +ve

i	a	b	c	f(c)	error
1	0	1	0.5	-0.17	0.5
2	0.5	1	0.75	0.134	0.25
3	0.5	0.75	0.625	-0.0204	0.125

$n_3 = 0.625$

ⓑ use bisection method to find solutions accurate to within 10^{-2} for $f(x) = x^3 - 7x^2 + 14x - 6 = 0$

$\downarrow$
0.01

$f(a) = -ve$ $f(b) = +ve$

ⓘ $[0, 1]$ $f(0) = -6$ $f(1) = 2$

i	a	b	c	f(c)	error
1	0	1	0.5	-0.625	0.5
2	0.5	1	0.75	0.9844	0.25
3	0.5	0.75	0.625	0.259	0.125
4	0.5	0.625	0.5625	-0.16187	0.0625
5	0.5625	0.625	0.59375	0.054	0.03125
6	0.5625	0.59375	0.578125	-0.0526	0.015625
7	0.578125	0.59375	0.5859375	0.0011	0.0078125

less than 0.01 so can stop

Root = 0.5859375

(ii) [1, 3.2] 0.01 $f(1) = 2$ $f(3.2) = -0.112$

$f(a) = +ve$ $f(b) = -ve$

i	a	b	c	f(c)	error
1	1	3.2	2.1	1.791	1.1
2	2.1	3.2	2.65	0.552125	0.55
3	2.65	3.2	2.925	0.0858	0.275
4	2.925	3.2	3.0625	-0.0544	0.1375
5	2.925	3.0625	2.99375	-0.0063	0.06875

Root = 2.99375

(iii) [3.2, 4] 0.01 $f(3.2) = -0.112$ $f(4) = 2$

$f(a) = -ve$ $f(b) = +ve$

i	a	b	c	f(c)	error
1	3.2	4	3.6	0.336	0.4
2	3.2	3.6	3.4	-0.016	0.2
3	3.4	3.6	3.5	0.125	0.1
4	3.4	3.5	3.45	0.046125	0.05
5	3.4	3.45	3.425	0.01301	0.025
6	3.4	3.425	3.4125	-0.002	0.0125

Root = 3.4125

c) use bisection method to find solutions accurate to 10^{-5} for $f(x) = x\cos(x) - 2x^2 + 3x - 1$ for

$f(0.2) = -0.284$ $\quad f(0.3) = 6.6 \times 10^{-3}$ $\quad 0.2 \le x \le 0.3$

$f(a) = -ve$ $\quad f(b) = +ve$

0.00001

i	a	b	c	f(c)	error
1	0.2	0.3	0.25	-0.1327	0.05
2	0.25	0.3	0.275	-0.062	0.025
3	0.275	0.3	0.2875	-0.0271	0.0125
4	0.2875	0.3	0.29375	-0.0101	0.00625
5	0.29375	0.3	0.296875	-0.0018	0.003125
6	0.296875	0.3	0.2984375	2.428×10^{-3}	0.0015625
7	0.296875	0.2984375	381/1280	3.375×10^{-4}	0.0007812
8	0.296875	381/1280	761/2560	-7.01×10^{-4}	0.0003906
9	761/2560	381/1280	1523/5120	-1.856×10^{-4}	0.0001953
10	1523/5120	381/1280	3047/10240	7.596×10^{-5}	0.00009765
11	1523/5120	3047/10240	6093/20480	-5.48×10^{-5}	0.00004883
12	6093/20480	3047/10240	0.2975341797	1.057×10^{-5}	0.000024414
13	6093/20480	0.2975341797	0.2975219727	-2.213×10^{-5}	0.00001221
14	0.2975219727	0.2975341797	0.2975280762	-5.779×10^{-6}	0.0000061

Root = 0.2975280762

Question 2:

ⓐ Approximate the zeros of the given functions. use Newton's Method and continue until 2 successive approximations differ by less than 0.001

ⓘ $f(x) = x^3 + 4$, $[-2, 1]$ $f'(x) = 3x^2$

$f'(-2) = 3(-2)^2 = 12$ $f'(1) = 3(1)^2 = 3$

Taking $x_n = -2$ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

i	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	error
1	-2	-4	12	-5/3	1/3
2	-5/3	-17/27	25/3	-1.591	0.077
3	-1.591	-0.0272	7.59	-1.587	0.004
4	-1.587	3.03×10^{-3}	7.555	-1.587	0

Root = -1.587

Taking $x_n = 1$

i	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	error
1	1	5	3	-2/3	5/3
2	-2/3	3.704	4/3	-3.445	2.778
3	-3.445	-36.89	35.6	-2.409	1.036
4	-2.409	-9.98	17.41	-1.836	0.573
5	-1.836	-2.189	10.11	-1.619	0.217
6	-1.619	-0.243	7.863	-1.588	0.031
7	-1.588	-4.53×10^{-3}	7.565	-1.587	0.001
8	-1.587	3.03×10^{-3}	7.555	-1.587	0

ROOT = -1.587

(ii) $f(x) = x^3 + x - 1$ $[-1, 1]$ $f'(x) = 3x^2 + 1$

$f'(-1) = 4$ $f'(1) = 4$ 0.001

Taking $x_n = 1$ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

i	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	error
1	1	1	4	3/4	0.25
2	3/4	0.17188	2.6875	0.686	0.064
3	0.686	0.00883	2.411	0.6823	0.0037
4	0.6823	-0.0000666	2.3966	0.6823	0

Root = 0.6823

Taking $x_n = 2.5$

(iii) $f(x) = 1 - x + \sin(x)$ $[-2, 2.5]$ $f'(x) = -1 + \cos(x)$

$f'(-2) = -1 + \cos(-2) = -1.416$ $f'(2.5) = -1.8011$

i	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	error
1	2.5	-0.90153	-1.8011	1.9995	0.5005
2	1.9995	-0.08999	-1.4157	1.9359	0.0636
3	1.9359	-0.001813	-1.357	1.9345	0.0014
4	1.9345	0.0000857	-1.3557	1.9346	0.0001
5	1.9346	-0.00004987	-1.3558	1.9346	0

Root = 1.9346

(b) Apply newton's method to approximate the intersection point between both graphs. Continue until 2 successive approximations differ by less than 0.001

$f(x) = 2x+1$ $g(x) = \sqrt{x+4}$ $f(x) = g(x)$

$2x+1 = \sqrt{x+4} \implies h(x) = 2x+1 - \sqrt{x+4}$

$h'(x) = 2 - \dfrac{1}{2\sqrt{x+4}}$

Take $x_n = 0.5$ → chosen from the graph, any point that is close to the intersection point. Also, because $f'(0.5) \neq$ zero.

$f'(0.5) = 2 - \dfrac{1}{2\sqrt{0.5+4}}$

$f'(0.5) = 1.7643$

i	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	error
1	0.5	-0.12132	1.7643	0.5687	0.0687
2	0.5687	-5.1754×10^{-5}	1.7661	0.56873	0.00003
3	0.56873	1.2285×10^{-6}	1.7661	0.56873	0

Intersection point = 0.56873

(21)

(c) Apply Newton's method using the given initial guess and explain why it fails $y = 2x^3 - 6x^2 + 6x - 1$

$x_1 = 1$

$y' = 6x^2 - 12x + 6$ at $x = 1$

$y'(1) = 6(1)^2 - 12(1) + 6 = 0$

The method fails bc $y'(1) = 0$, so x_{n+1} is undefined at the initial guess $x = 1$

Question 3:

$|x_{n+1} - x_n|$

(a) use secant method to find n_3 for $f(x) = x^2 - 6$ with $x_0 = 3$ and $x_1 = 2$

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$$

i	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	error
1	3	3	2	-2	2.4	0.4
2	2	-2	2.4	-0.24	2.45455	0.05455
3	2.4	-0.24	2.45455	0.02482	2.449	0.05

$n_3 = 2.449$

(b) use Secant method to find solutions accurate to within $10^{-4} \Rightarrow 0.0001$

(i) $x^3 - 2x^2 - 5 = 0$ $[1, 4]$

i	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	error
1	1	-6	4	27	1.545455	2.454545
2	4	27	1.545455	-6.0856	1.9969	0.451445
3	1.545455	-6.0856	1.9969	-5.0124	4.1184	2.1215
4	1.9969	-5.0124	4.1184	30.931	2.2919	1.8265
5	4.1184	30.931	2.2919	-10.002	2.738	0.4461
6	2.2919	-10.002	2.738	0.5325	2.7155	0.0225
7	2.738	0.5325	2.7155	0.2761	2.6913	0.0242
8	2.7155	0.2761	2.6913	7.152×10^{-3}	2.69065	0.00065
9	2.6913	7.152×10^{-3}	2.69065	2.796×10^{-5}	2.69064	0.00001

Root = 2.69064

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$$

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(ii) $x^3 + 3x^2 - 1 = 0 \quad [-3, -2]$

i	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	error
1	-3	-1	-2	3	-2.75	0.75
2	-2	3	-2.75	0.890625	-3.06667	0.31667
3	-2.75	0.890625	-3.06667	-1.6269	-2.862	0.20467
4	-3.06667	-1.6269	-2.862	0.1304	-2.8772	0.0152
5	-2.862	0.1304	-2.8772	0.0166	-2.8794	0.00222
6	-2.8772	0.0166	-2.8794	-1.211×10^{-4}	-2.8794	0

Root = -2.8794

(iii) $x - \cos(x) = 0 \quad [0, \pi/2]$

i	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	error
1	0	-1	$\pi/2$	$\pi/2$	0.611	-0.9597
2	$\pi/2$	$\pi/2$	0.611	-0.2081	0.72327	0.11227
3	0.611	-0.2081	0.72327	-0.026376	0.73956	0.01629
4	0.72327	-0.026376	0.73956	0.0008	0.73908	0.002652
5	0.73956	0.0008	0.73908	-8.591×10^{-6}	0.73908	0

Root = 0.73908

Question 4:

$$V = L\left[0.5\pi r^2 - r^2 \arcsin\left(\frac{h}{r}\right) - h(r^2 - h^2)^{1/2}\right]$$

~~sy~~ suppose $L = 10$ ft, $r = 1$ ft, $V = 12.4$ ft^3. find the depth of the water to within 0.01 ft

looking at the provided figures, h is the distance from water level to the top and d is the depth of the water is = 1 (radius) − h. so we need to approximate h.

using Bisection method.

$$12.4 = 10\left[0.5\pi(1^2) - 1^2\left(\arcsin\left(\frac{h}{1}\right) - h(1^2 - h^2)^{1/2}\right]\right.$$

$$f(h) = 10\left[0.5\pi - \arcsin(h) - h(1^2 - h^2)^{1/2}\right] - 12.4 = 0$$

using $[0, 1]$ as the interval for Bisection method

$$f(0) = 10\left[0.5\pi - \arcsin(0) - 0(1^2 - 0^2)^{1/2}\right] - 12.4 = 3.308$$

$$f(1) = 10\left[0.5\pi - \arcsin(1) - 1(1 - 1)^{1/2}\right] - 12.4 = -12.4$$

$$\left.\begin{array}{l} f(a) \longrightarrow +ve \\ f(b) \longrightarrow -ve \end{array}\right\}$$

Since one is positive and the other is negative we can use $[0, 1]$ as the interval ~~&~~ to approximate h using Bisection method

$$f(h) = 10\left[0.5\pi - \arcsin(h) - h(1^2 - h^2)^{1/2}\right] - 12.4$$

$[0, 1] \longrightarrow$ interval $\quad f(a) \to +ve$

$f(b) \to -ve$

$c = \dfrac{a+b}{2}$ $\qquad$ error $= |c - \text{previous } c|$

i	a	b	c	f(c)	error
1	0	1	0.5	-6.258	0.5
2	0	0.5	0.25	-1.6395	0.25
3	0	0.25	0.125	0.8145	0.125
4	0.125	0.25	0.1875	-0.4199	0.0625
5	0.125	0.1875	0.15625	0.1957	0.03125
6	0.15625	0.1875	0.171875	-0.1125	0.015625
7	0.15625	0.171875	0.1640825	0.0415	0.007812

$$0.007812 < \underbrace{0.01}_{\text{tolerance}}$$

so $h = 0.1640625$

recall $d = 1 - h$

$d = 1 - 0.1640625$

$d = 0.8359 \simeq 0.836$ within 0.01

Question 5: $f(x) = x^3 - 4x^2 + 3$ $[1,3]$

$f(1) = 1^3 - 4(1)^2 + 3$ $f(3) = 3^3 - 4(3)^2 + 3$

$f(1) = 0$ $f(3) = -6$

Thus the root within the interval $[1,3]$ is $x = 1$

Question 6:

design requires a water flow rate of 500 L/min

flow rate $(Q) = \frac{A}{\sqrt{R(F)}}$ $A \rightarrow$ constant $= 10^4$

$R(F) \longrightarrow$ resitance function

$R(F) = 2f^2 - 20f + 100$

$f \longrightarrow$ friction factor

(ALWAYS POSITIVE)

find f such that the flow rate is exactly 500 L/min using bisection method

$$Q = \frac{10^4}{\sqrt{2f^2 - 20f + 100}} \qquad Q = 500$$

$$500 = \frac{10^4}{\sqrt{2f^2 - 20f + 100}}$$

$\longrightarrow$

$$f(f) = 500 - \frac{10^4}{\sqrt{100+2f^2-20f}}$$

f ↓ friction factor

using [18,19] as root interval and 0.001 as the tolerance

$$f(18) = 500 - \frac{10^4}{\sqrt{2(18)^2-20(18)+100}} = -7.673$$

$$f(19) = 500 - \frac{10^4}{\sqrt{2(19)^2-20(19)+100}} = 24.35$$

$f(a) \to -ve \qquad f(b) \to +ve$

i	a	b	c	f(c)	error
1	18	19	18.5	8.823	0.5
2	18	18.5	18.25	0.702	0.25
3	18	18.25	18.125	-3.453	0.125
4	18.125	18.25	18.1875	-1.3678	0.0625
5	18.1875	18.25	18.21875	-0.3311	0.03125
6	18.21875	18.25	18.234375	0.1857	0.015625
7	18.21875	18.234375	18.2265625	-0.0725	0.0078125
8	18.2265625	18.234375	18.23046875	0.0566	0.003906
9	18.2265625	18.23046875	18.22851563	-7.968×10^{-3}	0.0019531
10	18.22851563	18.23046875	18.22949219	0.0243	0.0009766 ↑could stop here
~~11~~	~~18.22851563~~	~~18.230468-~~			
11	18.22851563	18.22949219	18.22900391	8.18×10^{-3}	0.0004883
12	18.22851563	18.22900391	18.22875977	1.063×10^{-4} <0.001	0.0002442 <0.001

friction factor that satisfies the flow rate condion is $\boxed{f = 18.22875977}$