JOURNAL OF GEOPHYSICAL RESEARCH VOL. 68, NO. 16 AUGUST 15, 1963

A Theoretical Analysis of Groundwater Flow in Small Drainage Basins[1]

J. TÓTH

Groundwater Division, Research Council of Alberta
Edmonton, Alberta, Canada

Abstract. Theoretically, three types of flow systems may occur in a small basin: local, intermediate, and regional. The local systems are separated by subvertical boundaries, and the systems of different order are separated by subhorizontal boundaries. The higher the topographic relief, the greater is the importance of the local systems. The flow lines of large unconfined flow systems do not cross major topographic features. Stagnant bodies of groundwater occur at points where flow systems meet or branch. Recharge and discharge areas alternate; thus only part of the basin will contribute to the baseflow of its main stream. Motion of groundwater is sluggish or nil under extended flat areas, with little chance of the water being freshened. Water level fluctuations decrease with depth, and only a small percentage of the total volume of the groundwater in the basin participates in the hydrologic cycle.

Introduction. Whereas it is important to have a *general understanding* of the motion of groundwater in dealing with groundwater problems, the careless and frequent use of the expression may subvert its basic meaning. Until certain characteristics of the flow systems involved are well defined, groundwater motion in a given area cannot be conceived to be generally known. Among the numerous features, a knowledge of which is indispensable to the understanding of groundwater motion in an area, the following are thought to be the most important: the locations and extent of recharge and discharge areas, the direction and velocity of flow at any given point in the region, and the depths of penetration of the flow systems. It is easy to appreciate the value of this information if one considers only the difficulties which may arise in connection with problems such as outlining areas of potentially equal yield, tracing contaminants, estimating baseflow of rivers, and establishing groundwater budgets.

The purpose of this paper is to present a theory by means of which groundwater flow in small drainage basins can be analyzed. Some of the properties of flow derived from this analysis are obvious and may be observed in the field, but others are hidden and may not be revealed even by expensive test programs. The neglect of these latter properties could lead to entirely wrong conclusions regarding groundwater flow in small basins either in general or in any particular case. Even if the theory is not used to obtain quantitative results, the qualitative application may still contribute to the general understanding of groundwater flow in small basins.

Before starting with the development of the theory a brief account will be given of the reasons why a theoretical analysis is believed to be best suited for an initial general study of groundwater flow in a given area.

General. The methods of studying groundwater motion can be either practical or theoretical. The group of practical methods includes field investigations based on the principles of geology, geophysics, geochemistry, and hydrology, and it is thus based on observations of phenomena controlling or related to the flow of groundwater in nature. The theoretical methods, on the other hand, make use of electrical analogs, scale models, and mathematical models to investigate phenomena resulting from idealized situations. In the final analysis the conclusions drawn from the data of both groups should be considered, and they must be in agreement. Nevertheless, the writer believes the results obtained by the application of the theoretical methods to be the more useful in the initial

[1] Contribution 185 from the Research Council of Alberta, Edmonton, Alberta. Presented at the Third Canadian Hydrology Symposium, Calgary, Alberta, November 8–9, 1962.

stages of an investigation. This view is based on the presumption that an observed phenomenon is usually related to only one feature of a given flow system, whereas it might be brought about by different causes in different situations; it may be the identical particular solution of several problems.

A few examples may prove helpful in clarifying this statement. For instance, a decrease in hydraulic head with depth, commonly observed in water wells, may be produced either by head losses due to the vertical downward-component of the water motion or by the water being perched in the permeable layers of a geological formation consisting of a series of more and less pervious beds [*Meinzer*, 1923, p. 41]. Another good example is a perennial body of impounded surface water. It may owe its existence either to poor underground drainage due to geologic conditions or to continuous groundwater discharge caused by the general pattern of the flow systems. The cause of a relatively low baseflow yield for a river may be even more uncertain because it could be explained by a basin-wide low permeability, by good surface drainage, or by the stream not being the only place of groundwater discharge in the basin. It is realized, of course, that the larger the variety of independent investigations, the more precisely the characteristics of the flow can be outlined. There still may remain the uncertainty, however, of whether some of the decisive features have been overlooked and whether there may be some characteristics that cannot be measured at all. Whereas it is practically impossible to observe separately all phenomena connected with a regime of groundwater flow, a correct theory discloses every feature and draws attention to the most important properties of the flow.

It is believed that small drainage basins are the most important units in the groundwater regime. A good understanding of groundwater movement in adjacent small basins makes possible an accurate representation of the motion of groundwater within the large basin that they form. In working from larger basins to smaller basins the weight of the uncertainties increases, and a vague and possibly unreliable analysis is obtained. Apart from this, a small basin is commonly much less complicated than a large basin with respect to geology and topography; therefore, it lends itself much better to both practical and theoretical studies.

The definition of a small drainage basin as it will be understood throughout this paper is: *an area bounded by topographic highs, its lowest parts being occupied by an impounded body of surface water or by the outlet of a relatively low order stream and having similar physiographic conditions over the whole of its surface.*

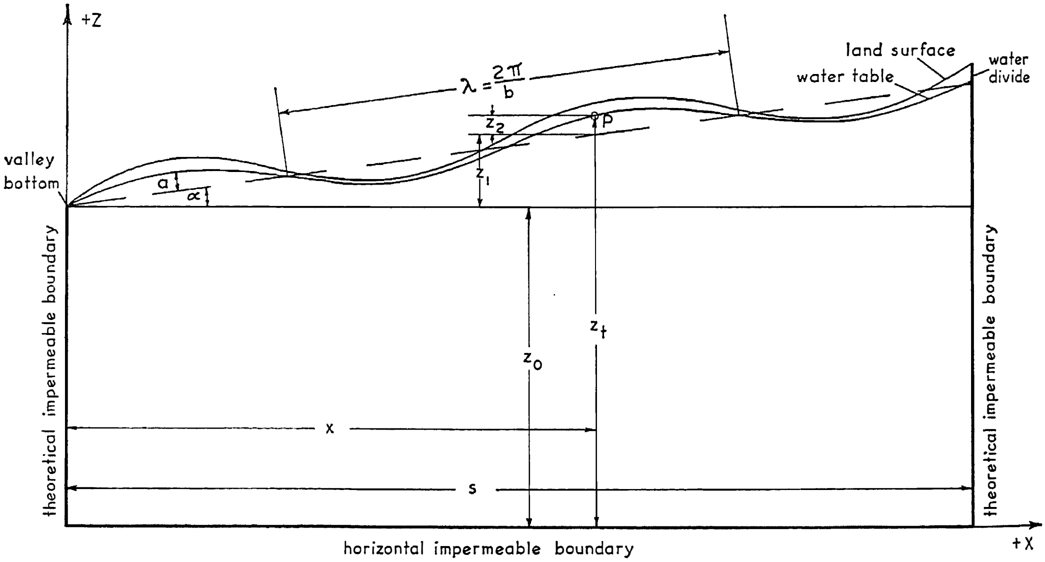

Fig. 1. Idealized cross section of a valley flank in a small drainage basin.

The upper limit of area for such basins is usually several hundred square miles.

Mathematical development. In the mathematical analysis the region of groundwater flow at one side of the valley is represented by a rectangular area (Figure 1). This area is limited by a horizontal impermeable boundary at its base, by two vertical impermeable boundaries extending downward from the stream and the water divide, and by a horizontal line at the elevation of the stream along which line the fluid-potential distribution is supposed to be the same as that for the real water table. The assumption of a horizontal impermeable boundary as the lower limit of the basin is justified because, in the interval above this in which no such boundary is known, all groundwater belongs to the flow region of the basin. If, however, a relatively impermeable boundary is present underlying the whole basin, the water systems under it will not significantly interfere with the systems within the basin. The assumption of the two vertical boundaries is, strictly speaking, correct only if the surface drainage pattern is symmetrical—that is, if the basin is bounded by two parallel and equally removed surface-water divides of equal topographic elevation. In this case the potential distribution at both sides of the stream is symmetrical and the impermeable boundaries may be drawn vertically at the stream and at the divides. It will be shown later that a small amount of asymmetry in the topography does not cause a significant deviation from the vertical of these boundaries.

The potential distribution along the theoretical surface, although identical with that of the water table, is along a horizontal surface, and this restricts the validity of the numerical results to small slopes of about 3° or less. For the topography within the basin, a sinusoidal shape has been chosen, the highs and lows of which are thought to be representative of the hills and depressions of the natural land surface.

The analysis is also based on the assumption that the geologic conditions in the basin are isotropic and homogeneous. Whether or not this assumption is justified depends on the extent to which a real case deviates from the ideal conditions.

The assumption that the problem can be treated as a two-dimensional one is supported by the recognition that in most small basins the slopes of the valley flanks greatly exceed the longitudinal slopes of the valley floors. This difference in slope causes the longitudinal component of the flow to become negligible compared with the lateral component.

The distribution of the fluid potential in a basin with boundaries as outlined above is derived from the general expression for the fluid potential [*Hubbert*, 1940, p. 802]:

$$\phi = gz + \int_{p_0}^{p} \frac{dp}{\rho} \tag{1}$$

where ϕ = fluid potential, g = acceleration due to the earth's gravity field, z = elevation above the horizontal impermeable boundary as standard datum, p_0 = pressure of the atmosphere, p = pressure in the flow region at any point, ρ = density of water.

If the water table is defined as *a specific piezometric surface in the groundwater region at which the gravity potential is a maximum and the pressure potential equals that of the atmosphere,* (1) reduces to

$$\phi_t = gz_t \tag{2}$$

for the water table, where z_t = the topographic elevation of the water table at any point in the basin. It has been observed in Alberta [*Meneley*, 1963, pp. 4–12] as well as elsewhere [*King*, 1892, pp. 15–18; *Meinzer*, 1923, p. 34; *Wisler and Brater*, 1959, Figure 85; *Wieckowska*, 1960, p. 64] that the water table is generally similar in form to the land surface. Thus z_t is found to consist of three components: z_0, z_1, and z_2 (Figure 1). z_0 is a constant, denoting the depth to the horizontal impermeable boundary from the stream bottom. $z_1 = x \tan \alpha$, where x is the horizontal distance of any point in the flow region from the valley bottom and α is the average slope of the valley flank. As long as α is small, z_2 may be approximated by

$$z_2 = a \frac{\sin (bx/\cos \alpha)}{\cos \alpha}$$

where a is the amplitude of the sine curve, $b = 2\pi/\lambda$ is the frequency, and λ is the period of the sine wave. With the three components known, the equation of the water table is obtained:

$$z_t = z_0 + x \tan \alpha + a \frac{\sin (bx/\cos \alpha)}{\cos \alpha}$$

Upon introducing the abbreviations $\tan \alpha = c'$, $a/\cos \alpha = a'$, and $b/\cos \alpha = b'$, the final form of z_t is written as

$$z_t = z_0 + c'x + a' \sin b'x \tag{3}$$

From (2) and (3) the potential at the water table is found:

$$\phi_t = g(z_0 + c'x + a' \sin b'x) \tag{4}$$

Owing to the natural equilibrium of the groundwater budget in a basin, the average level of the water table is assumed to be constant. The problem is thus a steady-state potential problem which may be solved by applying the Laplace equation:

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial_z^{\ 2} = 0$$

The four boundary conditions will be as follows:

$$\partial\phi/\partial x = 0 \quad \text{at} \quad x = 0 \text{ and } s \quad \text{for} \quad 0 \le z \le z_0 \tag{5a, 5b}$$

$$\partial\phi/\partial z = 0 \quad \text{at} \quad z = 0 \text{ for } 0 \le x \le s \tag{5c}$$

$$\phi_t = g(z_0 + c'x + a' \sin b'x) \quad \text{at} \quad z = z_0 \quad \text{for} \quad 0 \le x \le s \tag{5d}$$

where s is the horizontal distance between the valley bottom and the water divide.

The general solution of the Laplace equation can be written in the following form:

$$\phi = e^{-kz}(A \cos kx + B \sin kx) + e^{kz}(M \cos kx + N \sin kx)$$

The arbitrary constants A, B, M, and N can be found from the boundary conditions.

Upon performing the derivation we get the following final equation for the fluid potential:

$$\phi = g\left\{z_0 + \frac{c's}{2} + \frac{a'}{sb'}(1 - \cos b's) + 2\sum_{m=1}^{\infty}\left[\frac{a'b'(1 - \cos b's \cos m\pi)}{b'^2 - m^2\pi^2/s^2} + \frac{c's^2}{m^2\pi^2}(\cos m\pi - 1)\right] \cdot \frac{\cos(m\pi x/s)\cosh(m\pi z/s)}{s \cdot \cosh(m\pi z_0/s)}\right\} \tag{6}$$

Equation 6 satisfies both the boundary conditions and the Laplace equation. By means of Darcy's law, (6) can be used to obtain the specific mass discharge in the direction of r [*Hubbert,* 1940, p. 842].

$$j_r = -\rho\sigma\, \partial\phi/\partial r \tag{7a}$$

or the total flow vector:

$$\mathbf{j} = -\rho\sigma \operatorname{grad} \phi \tag{7b}$$

where $\sigma = k\rho/\eta$ (k = coefficient of permeability, η = viscosity of fluid).

Numerical computations. To analyze the effect of the geometry of the basin on the groundwater flow, (6) has been solved for various parameters. To facilitate visualization of the flow, the numerical values of the potential are expressed in 'head of water above standard datum.' The potential distributions and flow patterns for the various cases are shown in Figures 2*a* to 2*i*.

The horizontal distance between the water divide and the valley bottom is 20,000 feet in all computed cases. This distance seems to be fairly representative for the half-width of a small basin.

Three values have been assumed for the depth to the impermeable boundary at the valley bottom: 1000, 5000, and 10,000 feet. The 1000-foot case is likely to be encountered in nature, whereas a relatively homogeneous body of sediment 10,000 feet thick is a rather hypothetical case. Flow patterns have, however, been evaluated for this situation, for several reasons: first, it represents an extreme case, and therefore the general validity of the conclusions arrived at by employing (6) can be checked; second, the general features of the flow patterns are more conspicuous in the deeper boundaries than in the shallow ones; and third, the measure that is the most characteristic in determining the potential distribution is the *ratio* $n = (z_0/s)$ of the depth of the impermeable boundary to the horizontal distance between divide and stream. By employing the three values of z_0, a wide variety of potential distributions for values of n up to 0.5 can be inferred at least qualitatively.

Flow systems in small basins based on interpretation of the mathematical results. Upon inspection of Figures 2*a* to 2*i* we recognize a certain grouping of the flow lines. (In the figures the solid lines are called lines of force. Under isotropic conditions the lines of force coin-

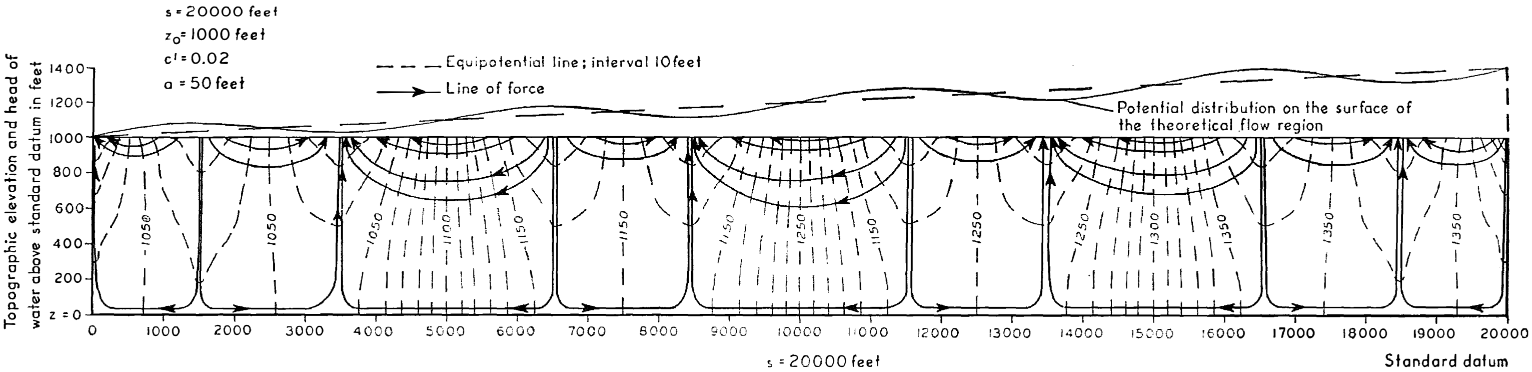

Fig. 2*a*. Potential distribution and flow pattern as obtained by equation 6.

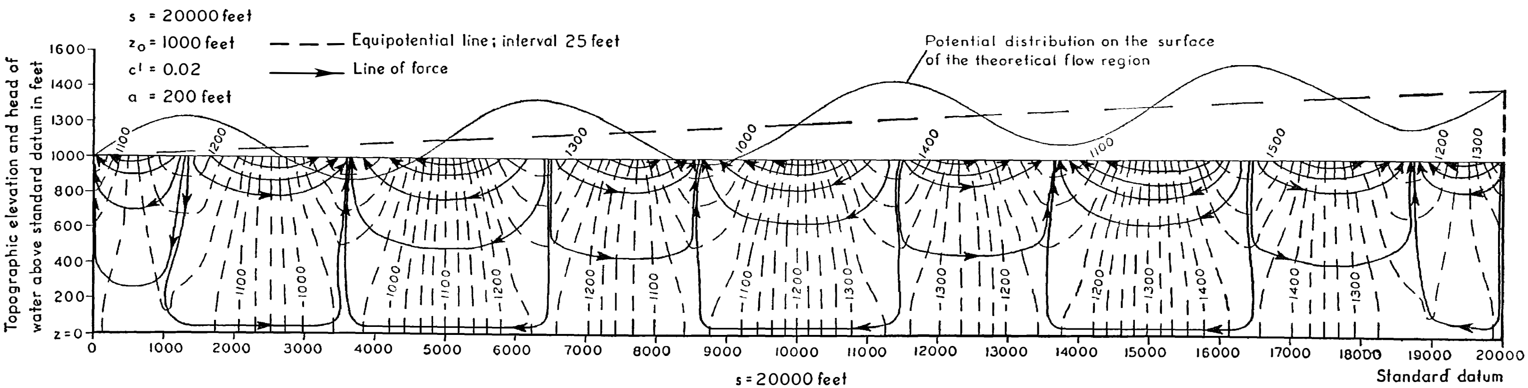

Fig. 2*b*. Potential distribution and flow pattern as obtained by equation 6.

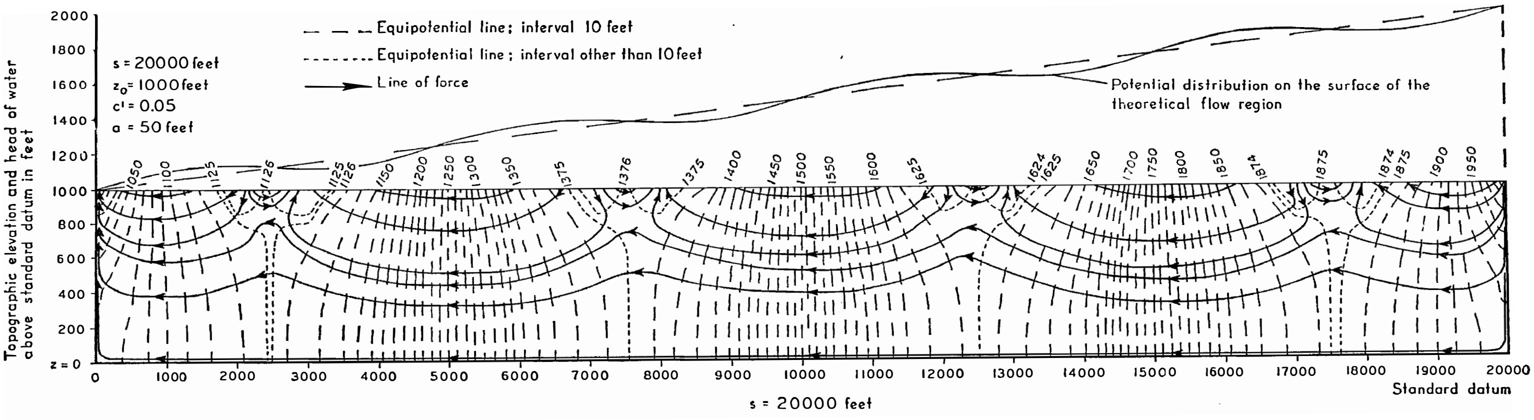

Fig. 2*c*. Potential distribution and flow pattern as obtained by equation 6.

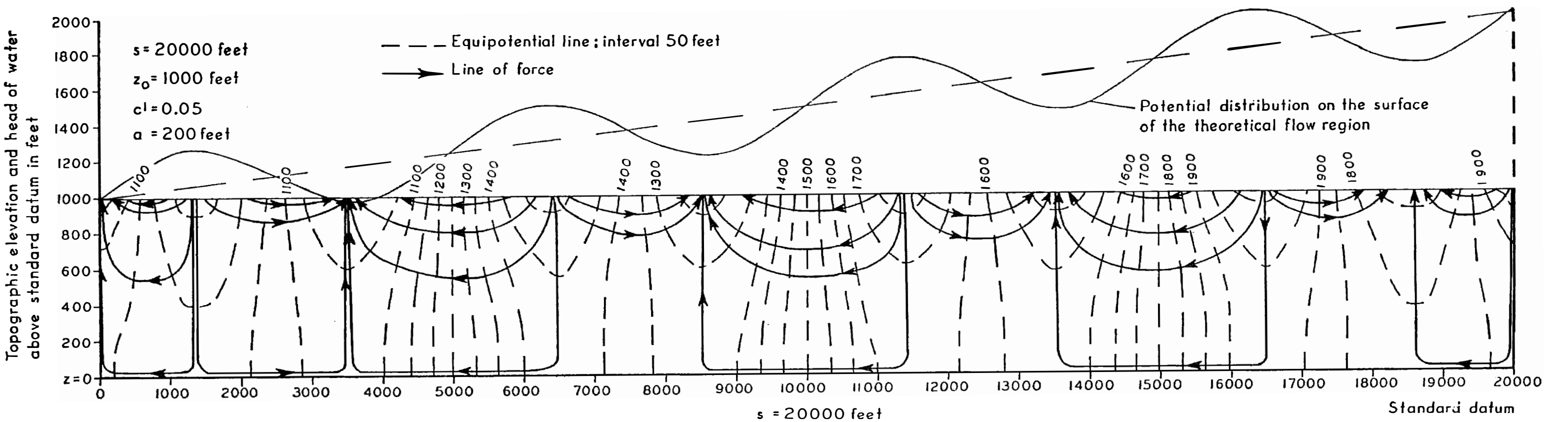

Fig. 2*d*. Potential distribution and flow pattern as obtained by equation 6.

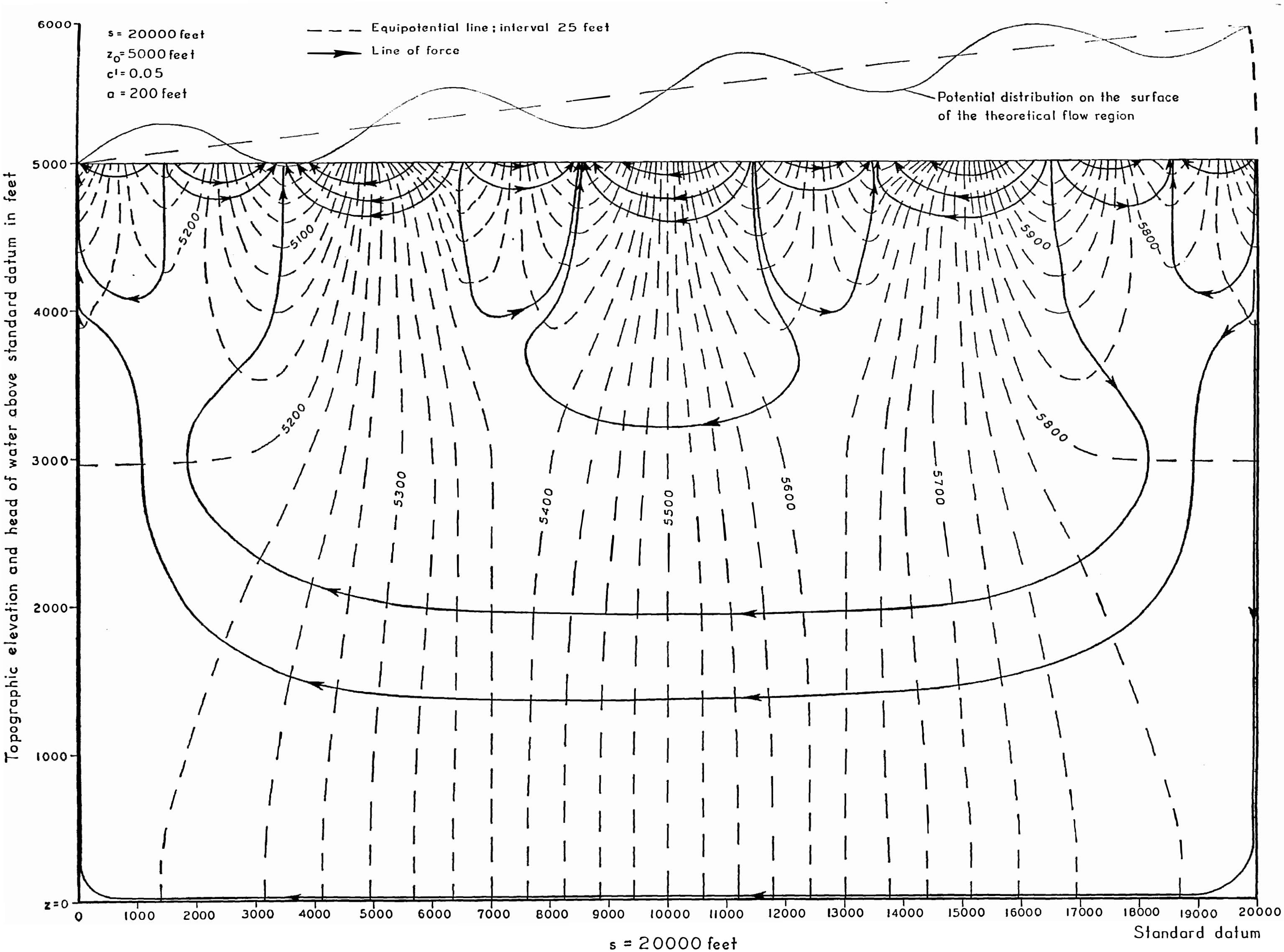

Fig. 2e. Potential distribution and flow pattern as obtained by equation 6.

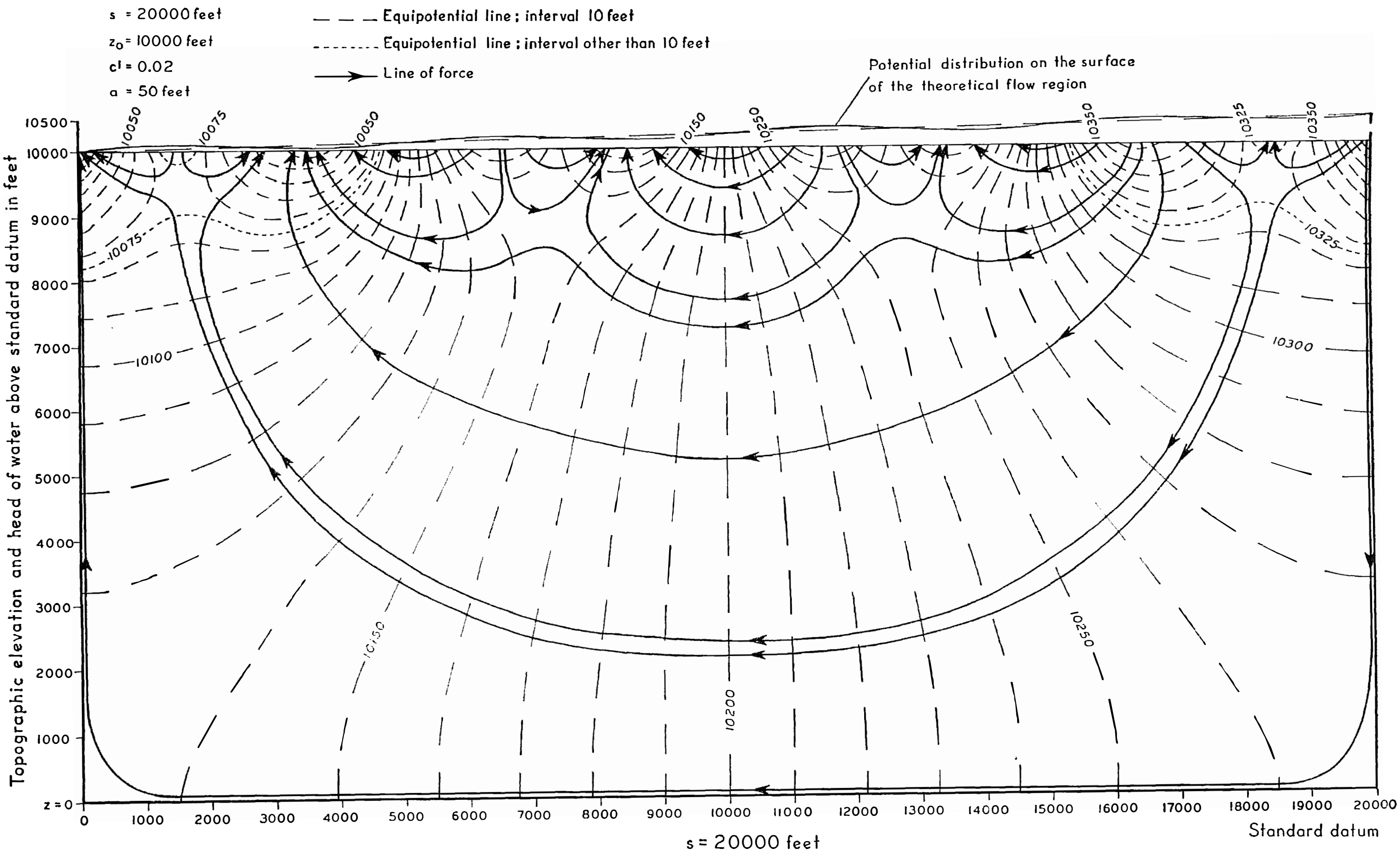

Fig. 2*f*. Potential distribution and flow pattern as obtained by equation 6.

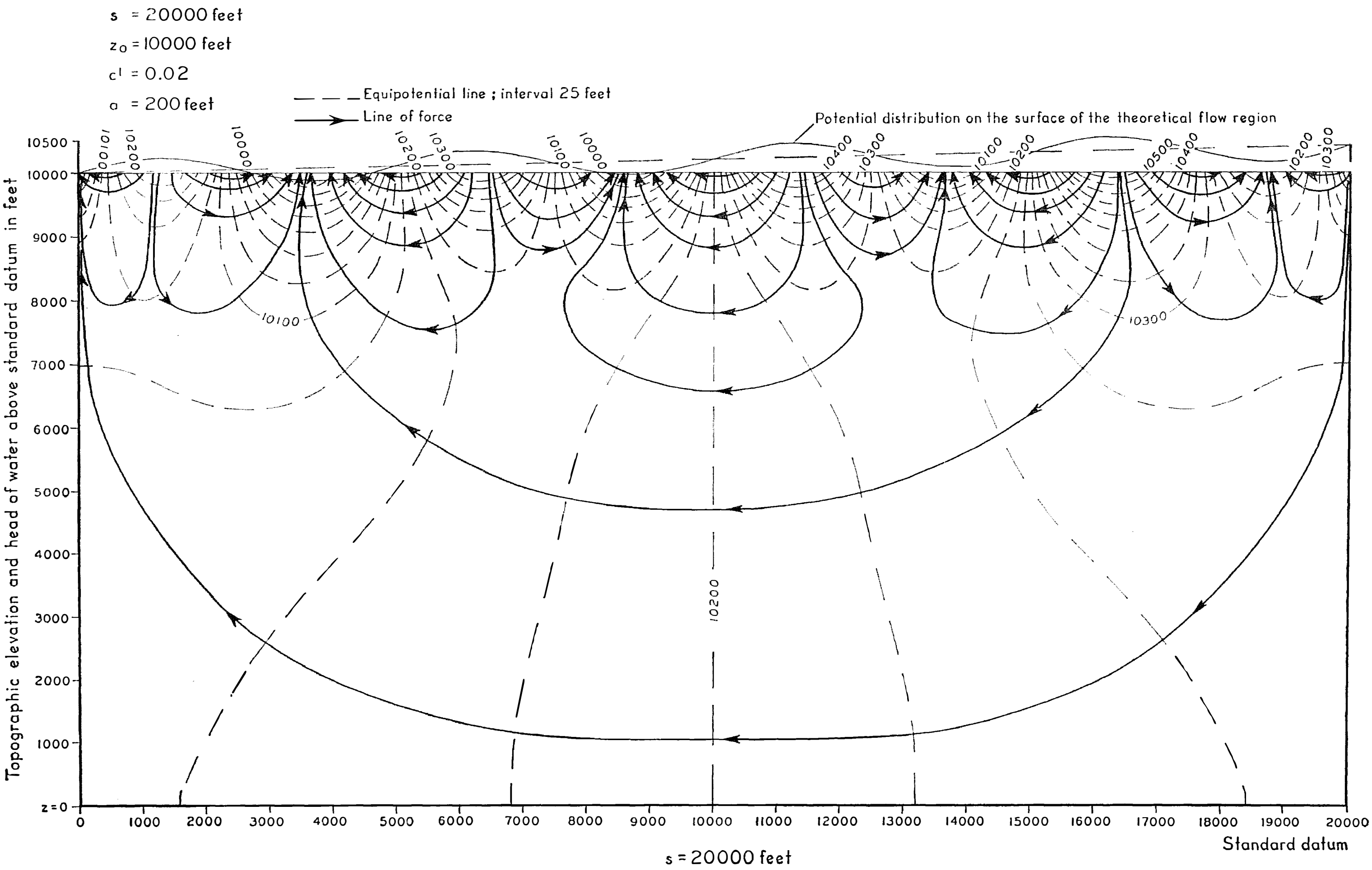

Fig. 2g. Potential distribution and flow pattern as obtained by equation 6.

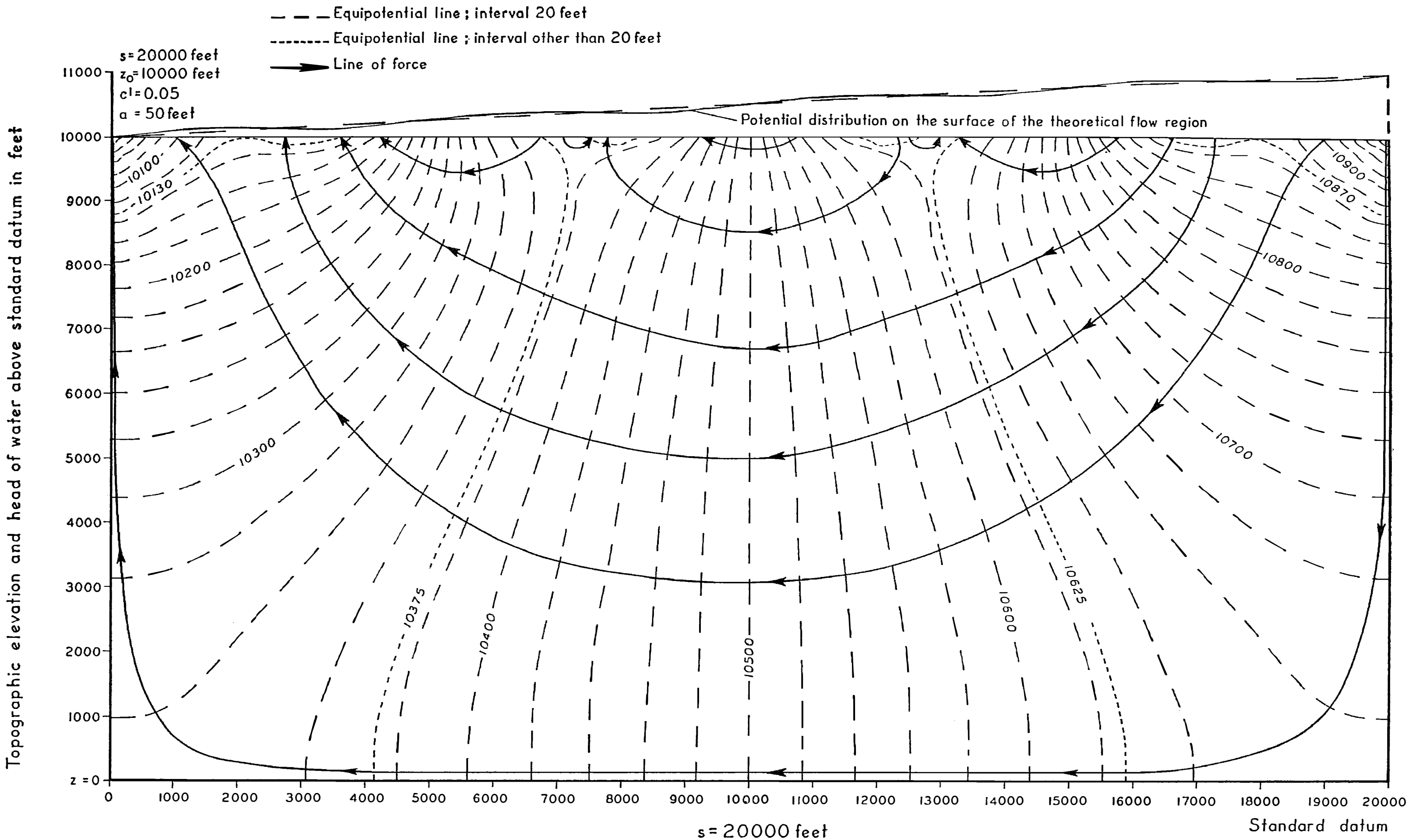

Fig. 2*h*. Potential distribution and flow pattern as obtained by equation 6.

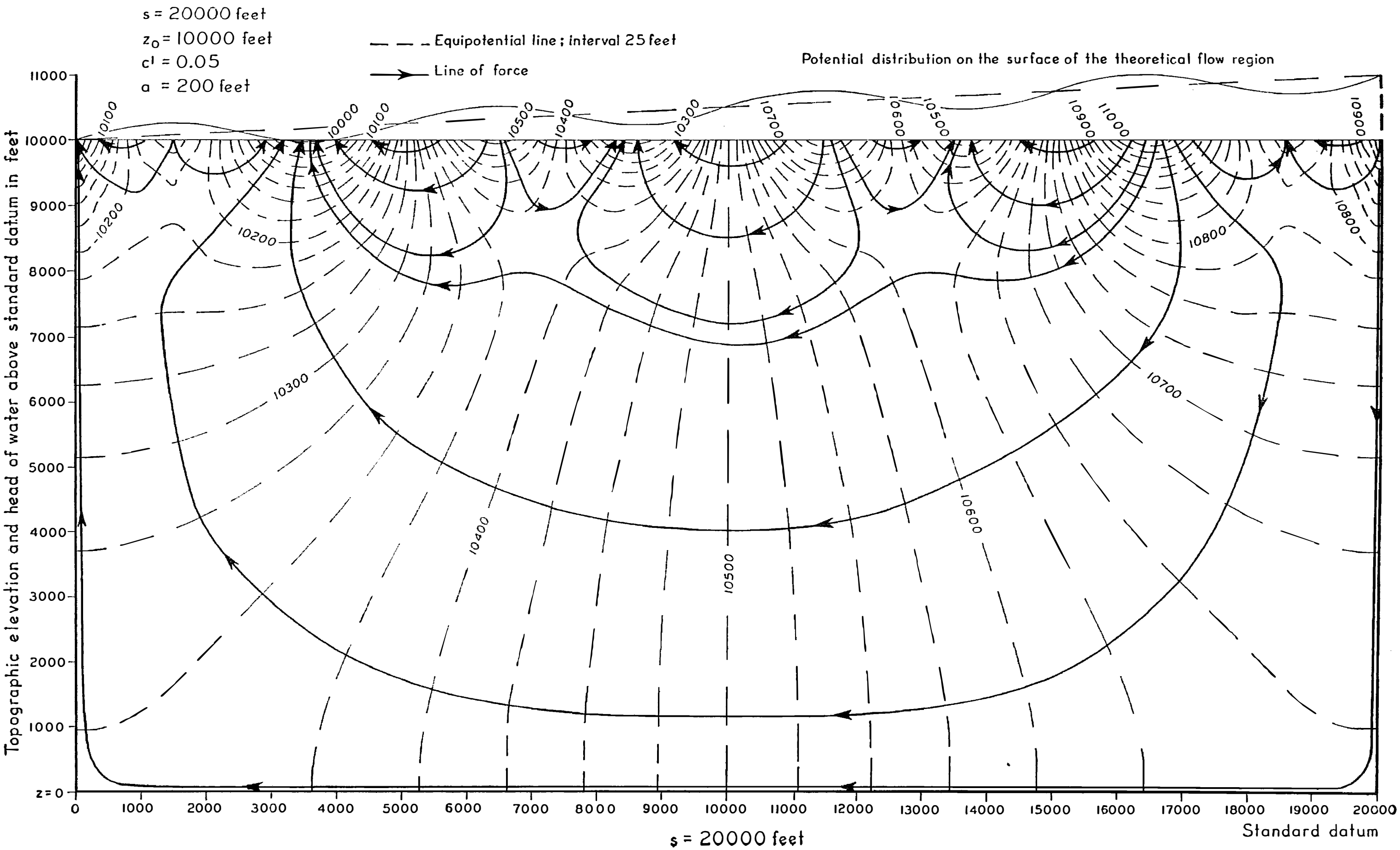

Fig. 2*i*. Potential distribution and flow pattern as obtained by equation 6.

cide with the flow lines.) Such a group of flow lines is said to form a flow system if it satisfies the following definition: *a flow system is a set of flow lines in which any two flow lines adjacent at one point of the flow region remain adjacent through the whole region; they can be intersected anywhere by an uninterrupted surface across which flow takes place in one direction only.*

Further investigation of the figures shows that three distinctly different types of flow systems can occupy a basin, namely, local, intermediate, and regional systems (Figure 3). A *local system* of groundwater flow has its recharge area at a topographic high and its discharge area at a topographic low that are located adjacent to each other. Local systems can be readily observed on each diagram of Figure 2. The major characteristic of an *intermediate system* of groundwater flow is that, although its recharge and discharge areas do not occupy the highest and lowest elevated places, respectively, in the basin, one or more topographic highs and lows may be located between them. Very-well-defined intermediate systems can be seen in Figures 2*e*, *f*, *g*, *h*, and *i*. The apparent lack of intermediate systems in those cases for which z_0 is 1000 feet does not mean that no such systems may exist in basins of relatively shallow depth. As soon as the real land surface departs from the regularity of the sine curve, the symmetrical flow pattern of Figure 2*c*, for instance, will be somewhat modified, and flow will occur between intermediate highs and lows also. A system of groundwater flow is considered to be *regional* if its recharge area occupies the water divide and its discharge area lies at the bottom of the basin. Regional systems can be observed in all the deep cases and in Figure 2*c*, where z_0 is 1000 feet.

Whereas theoretically the boundaries between different flow systems are very well defined, they do not signify an abrupt change of any of the physical properties of the flow. Relatively rapid changes of the chemical composition of the water across the boundaries, however, could be expected because of the different locations of the recharge and the different lengths of the flow paths of the different systems. In a small basin of moderate relief the amount of recharge water is directly proportional to the area of recharge. With this in mind it is obvious from Figure 2 that the greatest flow-line densities are found at shallow depths of the local systems. Except at places where local stagnant bodies of water occur, the density of the flow lines decreases rapidly with depth and with the transition from the local to the intermediate region and reaches its minimum in the regional system, provided the latter exists.

This interpretation of the theoretical results is very much in agreement with views expressed by *Norvatov and Popov* [1961, p. 21]. They recognize 'three well pronounced vertical zones of groundwater flow':

1. 'upper zone of active flow, whose geographical zonality coincides with climatic belts. The lower boundary of this zone coincides with the local base levels of rivers;
2. 'medium zone of delayed flow, subject to lesser climatic effect but also geographically zonal. The lower boundary of this zone is the base level of large rivers;
3. 'lower zone (of relatively stagnant water), geographically azonal and lying below the base level of large stream systems.'

Taking into account the extent of the recharge areas of the regional systems (which are small relative to those of the local systems), we see that flow in the regional system is influenced by climatic effects to a much lesser degree than flow in the upper zone. Climatic or geographical zonalities are, therefore, a straightforward consequence of the present theory.

In the next few paragraphs an analysis will be given of the effects of geomorphological factors on the flow of groundwater. These factors, or parameters in (6), are (*a*) the ratio n of the depth z_0 to the impermeable boundary to the half-width s of the basin (for convenience, in discussing the effect of n, only the depth to the impermeable boundary will be referred to, s being the same in all cases); (*b*) the average slope of the valley flanks; and (*c*) the local relief.

In analyzing the effect of z_0 on the flow of groundwater, a comparison of the diagrams of Figure 2 is helpful. Let those diagrams be considered for which all parameters but z_0 are equal, for instance, Figures 2*d*, *e*, and *i*. It appears that the spacing of the equipotential lines is *closer* in the shallow case than in the deeper ones. The flow lines are more arcuate as depth

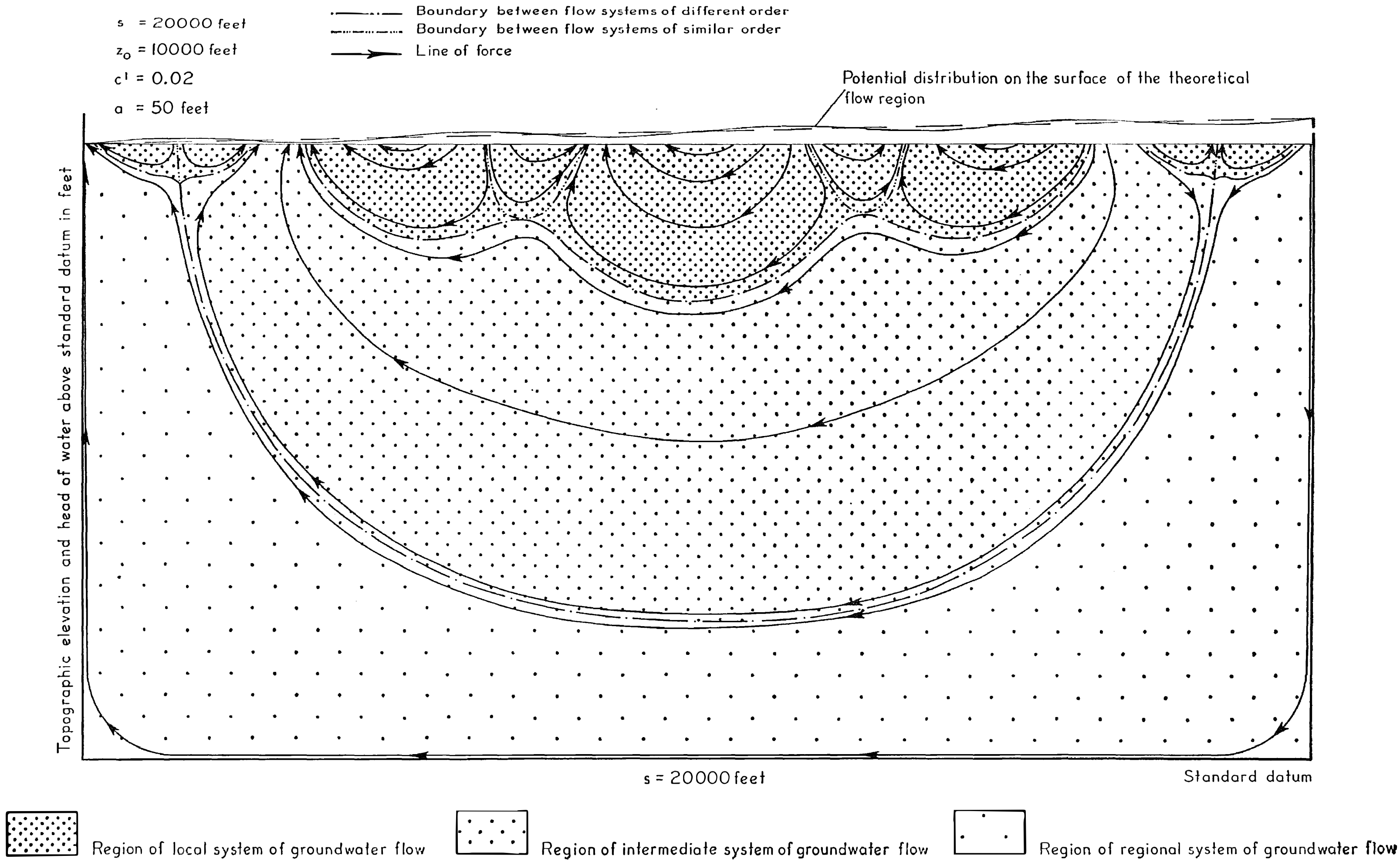

Fig. 3. Theoretical flow pattern and boundaries between different flow systems.

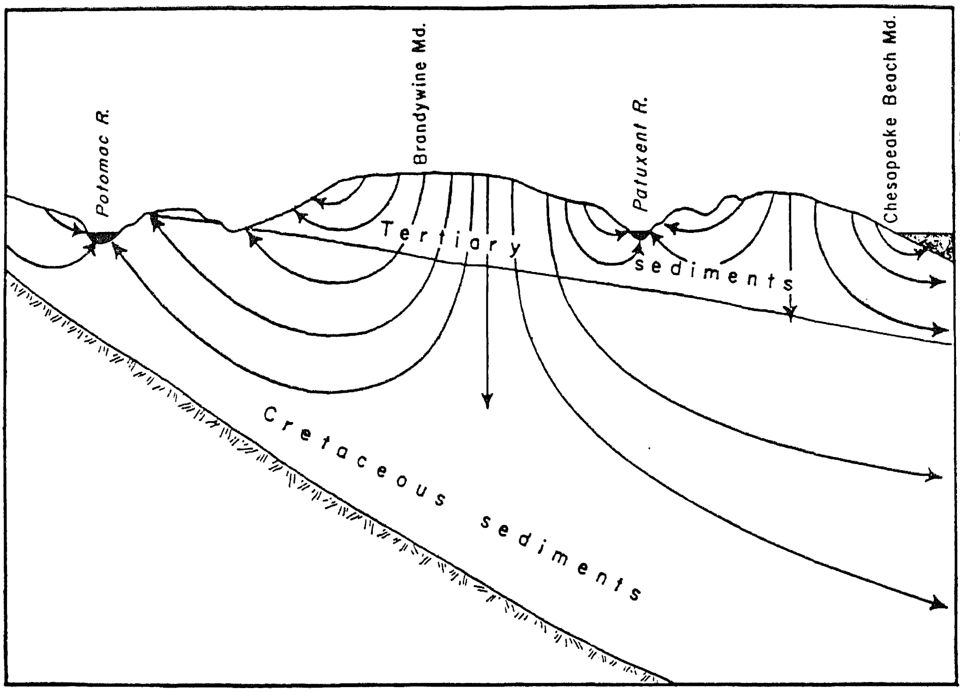

Fig. 5. Diagrammatic cross section through southern Maryland showing the lines of groundwater flow [after *Back*, 1960].

(Figure 2*c*), an area of stagnant water is formed. A high accumulation of mineral constituents is probable at these places. Below such a stagnant body of groundwater, flow occurs again and may result in a better quality of water than that from shallower depths.

7. Decreasing potential with depth in recharge areas and increasing potential in discharge areas are direct consequences of the theory and can be observed in all diagrams of Figure 2. It should be noted, however, that such a configuration of the equipotential lines is pronounced only in the immediate vicinities of the highs and lows. The midlines and their vicinities are locations of relatively straight, vertical equipotential lines.

8. From Figure 2 it can safely be stated that the major stream of the basin receives groundwater contributions only from the adjacent topographic highs and from possible regional flow. The latter is probably unimportant in most cases because of the low rate of flow. It is conceivable, then, that the methods in which baseflow data are used for computation of basin-wide characteristics (average recharge, permeability, etc.) are misleading or erroneous. Even if the two flanks of a basin are of low relief, so that there are no local systems, the bulk of the basin discharge will take place between the midline and the valley bottom and only a small portion will appear as baseflow.

9. A further consequence of the theory is that the water levels at shallow depths are the most affected by seasonal recharge and discharge. The small intake and outlet areas of the intermediate and regional zones prevent the water levels from fluctuating widely. *Plotnikov and Bogomolov* [1958, p. 90] make a distinction between two zones on the basis of fluctuation of the water levels. They call the first 'zone of oscillations of underground water levels.' According to them the volume of water that occupies the zone of oscillation undergoes seasonal variations. This volume would control groundwater discharge and therefore they call it 'control reserves of underground waters.' Their second zone includes all the water that is below the zone of oscillation, both that in the deeper, still homogeneous parts of the basin and that in artesian aquifers; these are the 'secular resources.' It seems that the 'zone of oscillation' and the portion of the 'secular resources' that

is above the first impermeable boundary coincide very well with the 'local systems' and with the 'intermediate and regional systems' of the present paper.

10. Another result of the analysis is that only a small portion of the total amount of water occupying the basin participates in the hydrologic cycle. The deeper the basin, the smaller is this portion. This is easily conceived when one considers that the greatest part of the surface of the basin is occupied by the recharge and discharge areas of local systems which are usually shallow. But even when the local systems reach the horizontal impermeable boundary (Figures 2*a*, *b*, *d*), approximately 90 per cent of the total recharge water never penetrates deeper than 250 to 300 feet. A similar view is expressed by *Ubell* [1962, p. 96] who believes that about 80 to 90 per cent of the 'static supply does not participate in the natural hydrological cycle.' His experiments, on the basis of which this conclusion was drawn, indicated that 'below a certain depth in loose sedimentary rocks . . . water does not move in the voids until their state of stress is disturbed by boring.'

Summary. It is the writer's belief that in drainage basins, down to depths at which basin-wide extended layers of contrasting low permeability are found, groundwater motion may be treated as an unconfined flow through a homogeneous medium. On the basis of this principle a mathematical model of a small drainage basin (as defined in the paper) has been constructed. Potential distributions have been computed for basins of different geometrical parameters. These computations have led to a number of conclusions regarding features of the groundwater flow.

In the most general case, groundwater flow in a basin can be thought to be apportioned among three types of flow systems, the regional, intermediate, and local systems. The three systems, being the results of combinations of three particular solutions of the Laplace equation, can be superimposed on one another. If the local variations in topography are negligible the flow consists of the combination of only two particular solutions, and no local systems occur. This case has been found in nature by *Geraghty* [1960] and has been theoretically treated in detail elsewhere [*Tóth*, 1962].

The emphasis in the present paper has been on the general situation for which local topography plays a part in controlling groundwater motion. The distribution of the flow systems will, in turn, have its effect on the chemical quality of local occurrences of groundwater. The areally unrelated origin of local systems, associated with local topographic highs and lows, may result in abrupt changes in the chemical composition of relatively shallow groundwater. Vertical changes in quality may be the result of local stagnant bodies and of the vertical arrangement of different flow systems.

It is thought that (6) may be used for obtaining quantitative information about groundwater flow in an area, the surface of which can be approximated by a harmonic function. It is hoped also that the results of the above analysis will be useful in test programs planning well fields, solving pollution and tracer problems, making baseflow studies, and setting up water budgets.

Acknowledgments. I am greatly indebted to Messrs. R. Newton and P. Redberger, of the Petroleum Division, Research Council of Alberta, who adapted the flow equation described in the paper for solution by digital computer. Only the availability of the computer solutions made possible the derivation of the many flow diagrams.

References

Back, W., Origin of hydrochemical facies of ground water in the Atlantic coastal plain, *Intern. Geol. Congr., 21st, Copenhagen, 1960, Rept. Session, Norden,* part 1, pp. 87–95, Copenhagen, 1960.

Geraghty, J. J., *Movement of Contaminants through Geologic Formations,* 90 pp., Technical Division Activities, National Water Well Association, Urbana, Ill., 1960.

Hubbert, M. K., The theory of groundwater motion, *J. Geol., 48*(8), part 1, 785–944, 1940.

King, F. H., Observations and experiments on the fluctuations in the level and rate of movement of ground-water on the Wisconsin Agricultural Experiment Station Farm, *U. S. Dept. Agr. Weather Bureau, Bull. no. 5,* 75 pp., 1892.

Meinzer, O. E., Outline of ground-water hydrology with definitions, *U. S. Geol. Surv. Water Supply Paper 494,* 71 pp., 1923.

Meneley, W. A., The occurrence and movement of groundwater in Alberta, in *Early Contributions to the Groundwater Hydrology of Alberta, Res. Council Alberta, Can., Bull. 12,* 123 pp., Edmonton, 1963.

Norvatov, A. M., and O. V. Popov, Laws of the formation of minimum stream flow, *Bull. Intern. Assoc. Sci. Hydrol.,* vol. 6, no. 1, pp. 20–27, Louvain, 1961.

Plotnikov, N. A., and G. B. Bogomolov, Classification of underground waters resources and their reflection on maps, *Intern. Assoc. Sci. Hydrol., General Assembly of Toronto,* vol. 2, no. 44, 525 pp., Gentbrugge, 1958.

Tóth, J., A theory of groundwater motion in small drainage basins in Central Alberta, Canada, *J. Geophys. Res., 67*(11), 4375–4387, 1962.

Ubell, K., A felszin alatti vízkészlet, *Hidrol. Kozl., 42*(2), 94–104, Budapest (Hungary), April 1962 (English summary).

Wieckowska, H., Zones geographiques des eaux phreatiques, *Intern. Assoc. Sci. Hydrol., General Assembly of Helsinki,* no. 52, 609 pp., Gentbrugge, 1960.

Wisler, Ch. O., and E. F. Brater, *Hydrology,* 2nd ed., 408 pp., John Wiley & Sons, New York, 1959.

(Manuscript received February 2, 1963; revised May 29, 1963.)