Romberg Integration with Richardson Extrapolation

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this document was designed to be printed in color

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1 Introduction

In this document, numerical integration is a routine to determine the area under the curve of a definite integral. Typically, the function, in this case f(x), is too complex to determine a closed form solution. The only alternative is to solve it numerically. A typical function and definite integral would be:

$$\int_a^b f(x)\,dx \tag{1}$$

We wish to find the area under the curve of f(x) between the limits of a and b.

An example of integrals which can only be solved numerically are:

$$\left.\begin{aligned} J_s{}^* &= \int_0^\infty \frac{u^s e^{-2u}}{\sinh(2u)+2u}\,du \qquad s = 1 \text{ to } 20 \\ I_s &= \int_0^\infty \frac{u^s}{sinh(2u)+2u}\,du \qquad s = 1 \text{ to } 20 \end{aligned}\right\} \tag{2}$$

It was while trying to integrate these forty integrals, more than 20 years ago, that I found Simplson's rule was worthless and that prompted me to learn how Romberg integration worked. Romberg integration processed these easily and I became a fan.

We wish to show Romberg Integration†, with Richardson Extrapolation, is a robust, stable, efficient, consistent, and is a relatively simple routine for performing numerical integration.

*this document was typeset in LaTeX; romberg_final_tex/romberg.tex

*On the Stresses in the Neighbourhood of a Circular Hole in a Strip under Tension, R.C.J. Howland, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character Vol. 229 (1930), pp. 49-86. Also, please note, in the paper these had to be integrated without a computer.

†Applied Numerical Methods, Carnahan, Luther and Wilkes, page 90; this book is a great source for all things concerning numerical integration

The definite integral we will use as an example

Calculate the natural log[‡] of 10:

$$log_e(10.) = \int_{a=1}^{b=10} \frac{1}{x}\, dx = \ 2.302585092994045 \tag{3}$$

This equation was picked because $\frac{1}{x}$ is simple and everyone is familiar with the natural log.

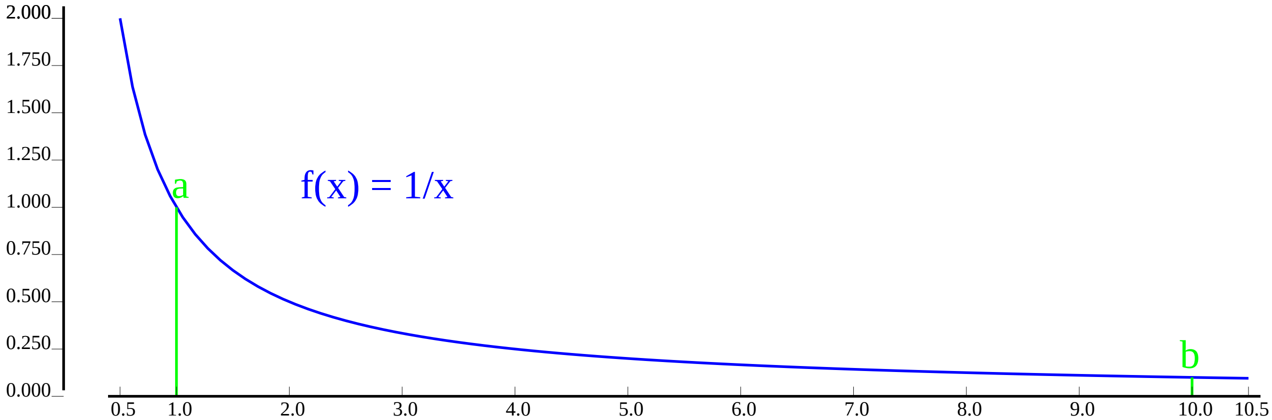

Figure 1: Natural log: f(x) = 1/x a = 1 b = 10 h=b-a

It is vital that the function to be integrated is plotted. The above cheesy plot is from a Fortran program I wrote to create an Encapsulated Postscript File (.eps) file which shows the function.

2 Romberg Integration: The Parts

Note: all the results are stored in a two-dimensional T[§]

We are going to present Romberg integration in three Parts:

1. Part 1: Trapezoidal rule with repeated interval halving - these values will go into column 1 of T During the derivation, the reader will see that everything calculated in the previous iteration, is used in the next iteration - this is incredibly efficient.

2. Part 2: Richardson extrapolation to enhance convergence - these values will go into columns 2 through ... of T

3. Part 3: How to determine convergence with a convergence tolerance and what is the solution. The solution will also show to how many digits the solution is accurate to.

As you go through the development, it becomes clear that it is straight forward to learn.

3 Something for the Future

As you go through this development, you will begin to wonder what happens if you set the convergence tolerance too tight. You can have a good solution, and not know it. You will have to reduce the convergence tolerance and run it again. This is not efficient. Remember, all of the values are in the T matrix.

In my 128-bit version, I added a feature where if the run does not converge, the convergence toleranace is lowered by 10. and the T matrix is processed again to see if there is a solution - (Repeat as necessary). As you become familiar with this algorithm, I am sure you will add the same feature.

[‡]All calculations in this document will be using 8 byte (64 bit), double precision arithmetic. This means all results will be accurate to about 16 digits. Also, all software in this document is written in FORTRAN -see Appendix III

[§]A brief description of the T matrix. The Fortran is: T(0:50,50) - this means there needs to be a 0 row.

4 Part 1: Trapezoidal rule with repeated interval halving - these values go into column 1 of T

This piece is really magical - watch the way the algebra comes together to make a simple algorithm, and also how the value we calculate in each interval is used in the next interval.

4.1 Preamble

Create the matrix T(0:50,50)
and define

$$\left.\begin{aligned} a &= 1 \\ b &= 10 \\ h &\equiv b - a = 10 - 1 = 9 \\ tol &= 1. \times 10^{-15} \qquad We\, will\, use\, this\, for\, convergence \\ max\_iter &= 26 \qquad Max\, number\, of\, iterations \end{aligned}\right\} \tag{4}$$

4.2 [$n = 0$; row 0 of T] : number of intervals = $2^n = 1$: [j = 1 (column 1 of T)]

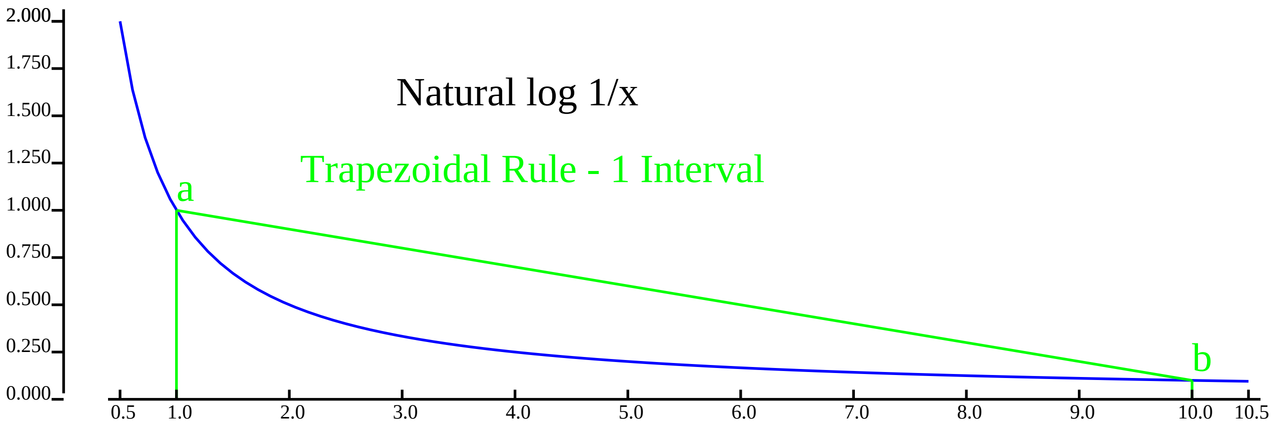

Figure 2: Natural log: f(x) = 1/x a = 1 b = 10 One Interval

$$T(n = 0, j = 1) = h \left[\frac{f(a) + f(b)}{2} \right] \tag{5}$$

4.3 [$n=1$; row 1 of T] : number of intervals $= 2^n = 2$: [j = 1 (column 1 of T)]

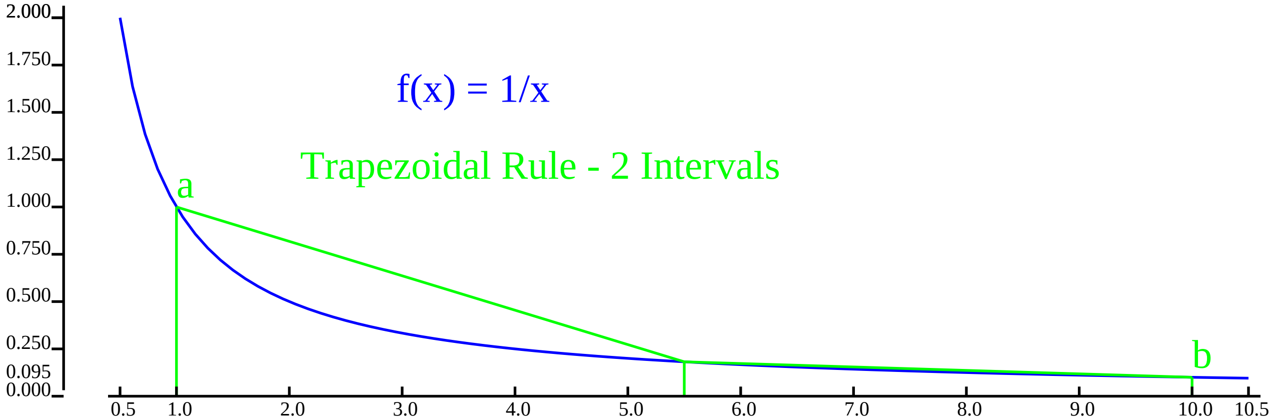

Figure 3: Natural log: f(x) = 1/x a = 1 b = 10 Two Intervals

$$
\begin{aligned}
T(n=1, j=1)^{\P} &= \frac{h}{2}\left[\frac{f(a)+f(a+\frac{h}{2})}{2} + \frac{f(a+\frac{h}{2})+f(b)}{2}\right] \qquad (6) \\
&= \frac{h}{2}\left[\frac{f(a)+f(b)}{2} + \frac{\cancel{2}}{\cancel{2}} f\left(a+\frac{h}{2}\right)\right] \\
&= \frac{1}{2}\left[\underbrace{h\frac{f(a)+f(b)}{2}}_{T(0,1)} + hf\left(a+\frac{h}{2}\right)\right]
\end{aligned}
$$

¶I did equation 6, 7, 8 etc. in Feb of 2008. There was a really bad snowstorm, it was blowing and cold so I worked on this. If you find a mistake, please contact me and I will give you credit.

4.4 [$n = 2$; row 2 of T] : number of intervals $= 2^n = 4$: [j = 1 (column 1 of T)]

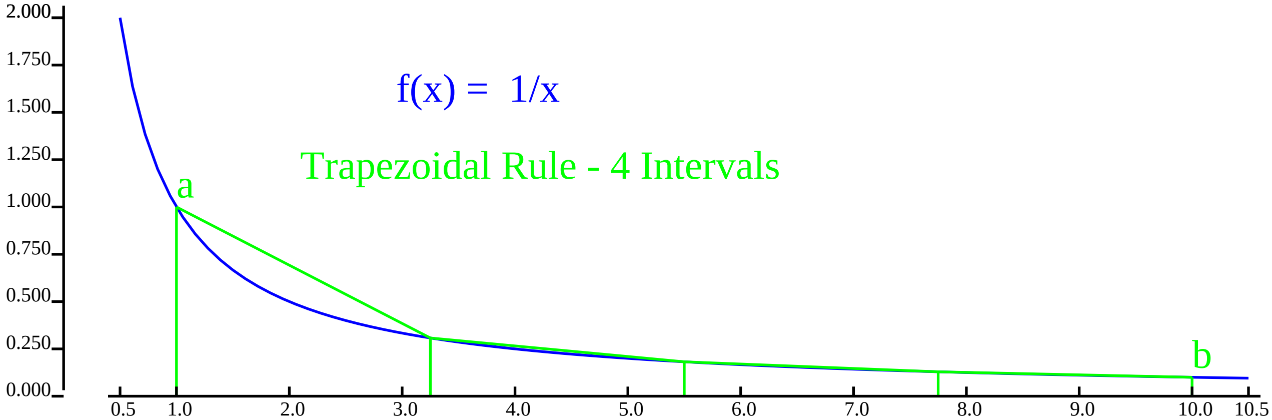

Figure 4: Natural log: f(x) = 1/x a = 1 b = 10 Four Intervals

$$
\begin{aligned}
T(n=2, j=1) \quad &= \quad \frac{h}{4}\left[\frac{f(a)+f(a+\frac{1}{4}h)}{2} + \frac{f(a+\frac{1}{4}h)+f(a+\frac{2}{4}h)}{2}\right. \\
&\quad + \left.\frac{f(a+\frac{2}{4}h)+f(a+\frac{3}{4}h)}{2} + \frac{f(a+\frac{3}{4}h)+f(b)}{2}\right] \\
&= \quad \frac{h}{4}\left[\frac{f(a)+f(b)}{2} + \frac{2}{2}f(a+\frac{1}{4}h) + \frac{2}{2}f(a+\frac{2}{4}h) + \frac{2}{2}f(a+\frac{3}{4}h)\right] \\
&= \quad \frac{h}{4}\left[\frac{f(a)+f(b)}{2} + f(a+\frac{1}{2}h) + \sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)\right] \\
&= \quad \frac{1}{2}\left[\underbrace{\frac{1}{2}\left[h\frac{f(a)+f(b)}{2} + hf\left(a+\frac{1}{2}h\right)\right]}_{T(1,1)} + \frac{h}{2}\sum_{i=1,\Delta i=2}^{3} f\left(a+\frac{i}{4}h\right)\right]
\end{aligned} \tag{7}
$$

4.5 [$n = 3$; row 3 of T] : number of intervals $= 2^n = 8$: [j = 1 (column 1 of T)]

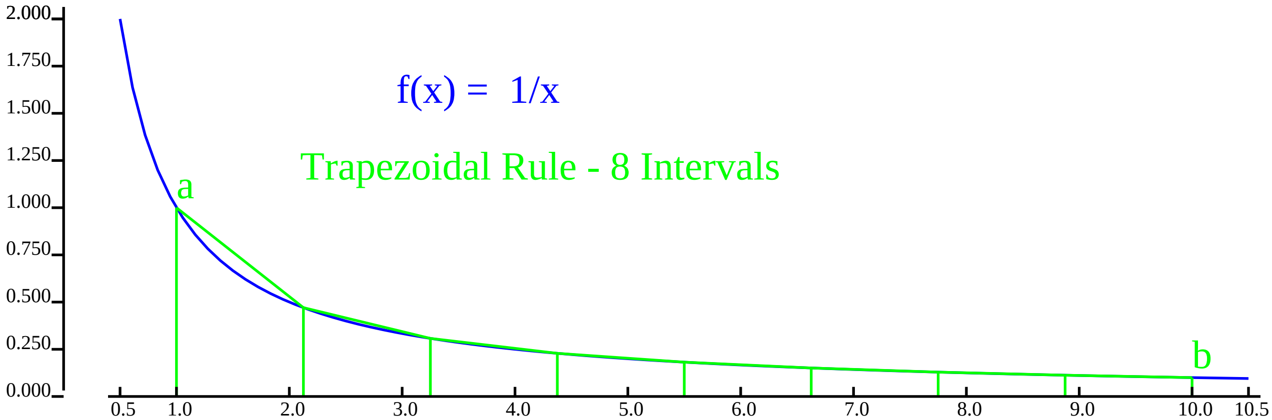

Figure 5: Natural log: f(x) = 1/x a = 1 b = 10; Eight Intervals

$$
\begin{aligned}
T(n,1) \quad &= \frac{h}{8}\left[\frac{f(a)+f(a+\frac{1}{8}h)}{2}+\frac{f(a+\frac{1}{8}h)+f(a+\frac{2}{8}h)}{2}\right. \\
&+ \frac{f(a+\frac{2}{8}h)+f(a+\frac{3}{8}h)}{2}+\frac{f(a+\frac{3}{8}h)+f(a+\frac{4}{8}h)}{2} \\
&+ \frac{f(a+\frac{4}{8}h)+f(a+\frac{5}{8}h)}{2}+\frac{f(a+\frac{5}{8}h)+f(a+\frac{6}{8}h)}{2} \\
&+ \left.\frac{f(a+\frac{6}{8}h)+f(a+\frac{7}{8}h)}{2}+\frac{f(a+\frac{7}{8}h)+f(b)}{2}\right] \\
&= \frac{h}{8}\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{8}h)+f(a+\frac{2}{8}h)\right. \\
&+ f(a+\frac{3}{8}h)+f(a+\frac{4}{8}h)+f(a+\frac{5}{8}h)+f(a+\frac{6}{8}h)+f(a+\frac{7}{8}h)\,\Big] \\
&= \frac{h}{8}\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{2}h)+f(a+\frac{1}{4}h)+f(a+\frac{3}{4}h)\right. \\
&+ \left.f(a+\frac{1}{8}h)+f(a+\frac{3}{8}h)+f(a+\frac{5}{8}h)+f(a+\frac{7}{8}h)\right] \\
&= \frac{h}{8}\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{2}h)+\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)\right. \\
&+ \left.f(a+\frac{1}{8}h)+f(a+\frac{3}{8}h)+f(a+\frac{5}{8}h)+f(a+\frac{7}{8}h)\right] \\
&= \frac{h}{8}\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{2}h)+\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)+\sum_{i=1,\Delta i=2}^{7} f(a+\frac{i}{8}h)\right] \\
&= \frac{1}{8}\left[h\frac{f(a)+f(b)}{2}+hf(a+\frac{1}{2}h)+h\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)+h\sum_{i=1,\Delta i=2}^{7} f(a+\frac{i}{8}h)\right] \\
&= \frac{1}{4}\left[\frac{1}{2}\left[h\frac{f(a)+f(b)}{2}+hf(a+\frac{1}{2}h)\right]+\frac{1}{2}h\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)\frac{1}{2}h\sum_{i=1\Delta i=2}^{7} f(a+\frac{i}{8}h)\right] \\
&= \frac{1}{2}\left[\underbrace{\frac{1}{2}\left[\frac{1}{2}\left[h\frac{f(a)+f(b)}{2}+hf\left(a+\frac{h}{2}\right)\right]+\frac{h}{2}\sum_{i=1,\Delta i=2}^{3} f\left(a+\frac{i\,h}{4}\right)\right]}_{T(2,1)}+\frac{h}{4}\sum_{i=1,\Delta i=2}^{7} f\left(a+\frac{i\,h}{8}\right)\right]
\end{aligned}
\tag{8}
$$

4.6 [$n = 4$; row 4 of T] : number of intervals $= 2^n = 16$: [j = 1 (column 1 of T)]

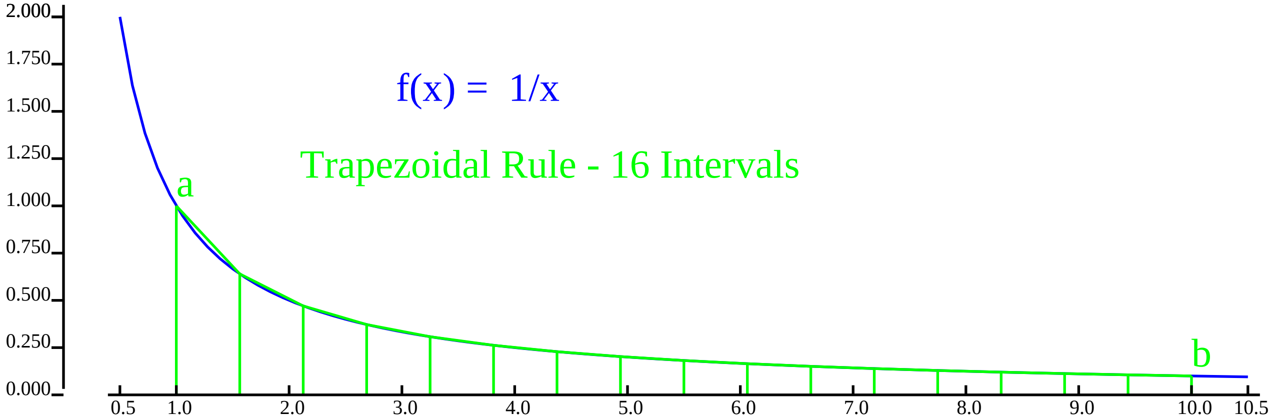

Figure 6: Natural log: f(x) = 1/x a = 1 b = 10; Sixteen Intervals

$$
\begin{aligned}
T(n=4,1) \quad &= \frac{h}{16}\left[\frac{f(a)+f(a+\frac{1}{16}h)}{2}+\frac{f(a+\frac{1}{16}h)+f(a+\frac{2}{16}h)}{2}+\frac{f(a+\frac{2}{16}h)+f(a+\frac{3}{16}h)}{2}+\frac{f(a+\frac{3}{16}h)+f(a+\frac{4}{16}h)}{2}\right.\\
&+ \frac{f(a+\frac{4}{16}h)+f(a+\frac{5}{16}h)}{2}+\frac{f(a+\frac{5}{16}h)+f(a+\frac{6}{16}h)}{2}+\frac{f(a+\frac{6}{16}h)+f(a+\frac{7}{16}h)}{2}+\frac{f(a+\frac{7}{16}h)+f(a+\frac{8}{16}h)}{2}\\
&+ \frac{f(a+\frac{8}{16}h)+f(a+\frac{9}{16}h)}{2}+\frac{f(a+\frac{9}{16}h)+f(a+\frac{10}{16}h)}{2}+\frac{f(a+\frac{10}{16}h)+f(a+\frac{11}{16}h)}{2}+\frac{f(a+\frac{11}{16}h)+f(a+\frac{12}{16}h)}{2}\\
&+ \left.\frac{f(a+\frac{12}{16}h)+f(a+\frac{13}{16}h)}{2}+\frac{f(a+\frac{13}{16}h)+f(a+\frac{14}{16}h)}{2}+\frac{f(a+\frac{14}{16}h)+f(a+\frac{15}{16}h)}{2}+\frac{f(a+\frac{15}{16}h)+f(b)}{2}\right]\\
&= \frac{h}{16}\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{2}h)+\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)\sum_{i=1,\Delta i=2}^{7} f(a+\frac{i}{8}h)+\sum_{i=1,\Delta i=2}^{15} f(a+\frac{i}{16}h)\right]\\
&= \frac{1}{16}\left[h\left[\frac{f(a)+f(b)}{2}+f(a+\frac{1}{2}h)\right]+h\sum_{i=1,\Delta i=2}^{3} f(a+\frac{i}{4}h)+h\sum_{i=1,\Delta i=2}^{7} f(a+\frac{i}{8}h)+h\sum_{i=1,\Delta i=2}^{15} f(a+\frac{i}{16}h)\right]\\
&= \frac{1}{2}\left[\underbrace{\frac{1}{2}\left[\frac{1}{2}\left[\frac{1}{2}\left[h\frac{f(a)+f(b)}{2}+hf\left(a+\frac{1}{2}h\right)\right]+\frac{1}{2}h\sum_{i=1,\Delta i=2}^{3} f\left(a+\frac{i}{4}h\right)\right]+\frac{1}{4}h\sum_{i=1,\Delta i=2}^{7} f\left(a+\frac{i}{8}h\right)\right]}_{T(3,1)}+\frac{1}{8}h\sum_{i=1,\Delta i=2}^{15} f\left(a+\frac{i}{16}h\right)\right]
\end{aligned}
\tag{9}
$$

4.7 The general recursion for Romberg integration: Trapezoidal Rule with Interval Halving

First:

$$n = 0 \qquad j = 1 \tag{10}$$

$$T(n=0, j=1) = h\left[\frac{f(a)+f(b)}{2}\right] \tag{11}$$

Begin iteration: $n = 1$ to a max iteration (max_iter) using:

$$T(n, j=1) = \frac{1}{2}\left[T(n-1,1) + \frac{h}{2^{n-1}} \sum_{i=1,\Delta i=2}^{2^n-1} f\left(a + \frac{i}{2^n}h\right)\right] \tag{12}$$

$$log_e(10) = \underline{2.302585092994045}^{\|}$$

Romberg Integration: Part 1 $log_e(10) = \underline{2.302585092994045}$

Trapezoidal Rule with Interval Halving

n	number of intervals: 2^n	T(n,j=1)
0	1	4.950000000000000
1	2	3.293181818181818
2	4	$\underline{2}.629221182043763$
3	8	$\underline{2.3}97737097005620$
4	16	$\underline{2.3}27952104982484$
5	32	$\underline{2.30}9060655357341$
6	64	$\underline{2.30}4213334235463$
7	128	$\underline{2.302}992757242847$
8	256	$\underline{2.302}687047130696$
9	512	$\underline{2.302}610583913093$
10	1024	$\underline{2.3025}91465872944$
11	2048	$\underline{2.30258}6686223092$
12	4096	$\underline{2.302585}491301889$
13	8192	$\underline{2.302585}192571041$
14	16384	$\underline{2.302585}117888296$
15	32768	$\underline{2.3025850}99217609$
16	65536	$\underline{2.30258509}4549926$
17	131072	$\underline{2.30258509}3383006$
18	262144	$\underline{2.30258509}3091277$
19	524288	$\underline{2.30258509}3018368$
20	1048576	$\underline{2.30258509}3000163$
21	2097152	$\underline{2.30258509299}5553$
22	4194304	$\underline{2.302585092994}309$
23	8388608	$\underline{2.302585092994}068$
24	16777216	$\underline{2.30258509299}3993$
25	33554432	$\underline{2.30258509299}3818$
26	67108864	$\underline{2.30258509299}3699$

This shows the inability of the trapezoidal rule, even with interval halving, to converge to the correct results. Even after 67,108,864 intervals, the solution is only good to 14 digits and takes 23 iterations to get it.. Also, after 8,388,608 intervals it appears that round-off error has crept into the solution, and is dominating the solution.

At this point, it is time to employ Richardson Extrapolation.

‖ The underlined numbers show to how many digits the number is accurate

5 Part 2: Richardson Extrapolation - these values go into columns 2 through ... of T and Part 3: convergence check

The following shows the general case of Richardson Extrapolation, which applies to all columns $j > 1$ of T.

$$T_{n,j} = \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1} - 1} \qquad j > 1 \tag{13}$$

6 Romberg Integration - Go through the steps to see how it works

6.1 Preamble: n = 0; j = 1

Process equation (4) on page 3

$$\text{using equation (11) on page 8:} \qquad T(0,1) = 4.950000000000000 \tag{14}$$

Romberg Integration

Trapezoidal Rule with Interval Halving		
n (row) of T	number of intervals: 2^n	T(n,j=1)
0	1	4.950000000000000

6.2 Begin iterations – ITERATION 1: n = 1

$$\text{using equation (12) on page 8:} \qquad T(1,1) = 3.293181818181818 \tag{15}$$

Richardson Extrapolation routine for n=0, j = 2:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1} - 1} \qquad j > 1 \\ T_{0,2} &= \frac{4\,T_{1,1} - T_{0,1}}{3} \\ &= \frac{4 \times 3.293181818181818 - 4.950000000000000}{3} \\ &= 2.740909090909091 \end{aligned} \tag{16}$$

Romberg Integration: Part 1

Trapezoidal Rule with Interval Halving + Richardson Extrap			
n	number of intervals: 2^n	T(n,j=1)	T(n,j=2)
0	1	4.950000000000000	2.740909090909091
1	2	3.293181818181818	

convergence check: check two values in column 1

$$\left.\begin{array}{ll} if \qquad T(1,1) - T(0,1) \leq tol^{**} & \text{then convergence: T(1,1) is the solution} \\ 3.293181818181818 - 4.950000000000000 = 1.6568\ldots >> tol & \end{array}\right\} \tag{17}$$

**tol was defined in equation (4) on page 3; also we need the absolute value for convergence

6.3 ITERATION 2: n = 2

$$\text{using equation (12) on page 8:} \qquad T(n=2,1) = 2.629221182043763 \tag{18}$$

Richardson Extrapolation routine for n=1, j = 2:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1} - 1} \qquad j > 1 \\ T_{1,2} &= \frac{4\,T_{2,1} - T_{1,1}}{3} \\ &= \frac{4 \times 2.629221182043763 - 3.293181818181818}{3} \\ &= 2.407900969997745 \end{aligned} \tag{19}$$

Richardson Extrapolation routine for n=0, j = 3:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1} - 1} \qquad j > 1 \\ T_{0,3} &= \frac{16 \times T_{1,2} - T_{0,2}}{15} \\ &= \frac{16 \times 2.407900969997745 - 2.740909090909091}{15} \\ &= 2.385700428603655^{\dagger\dagger} \end{aligned} \tag{20}$$

Romberg Integration: Part 2

	Richardson Extrapolation: Column 3			
n	number of intervals: 2^n	T(n,j=1)	T(n,j=2)	T(n,j=3)
0	1	4.950000000000000	2.740909090909091	2.385700428603655
1	2	3.293181818181818	2.407900969997745	
2	4	2.629221182043763		

convergence check: check two values in a column for columns 1 and 2

$$\left.\begin{array}{l} \text{column 1: if} \quad T(2,1) - T(1,1) \leq tol \quad \text{then convergence T(2,1) is the solution} \\ 2.629221182043763 - 3.293181818181818 = .66396\ldots >> tol \end{array}\right\} \tag{21}$$

$$\left.\begin{array}{l} \text{column 2: if} \quad T(1,2) - T(0,2) \leq tol \quad \text{then convergence T(1,2) is the solution} \\ 2.407900969997745 - 2.740909090909091 = .33300\ldots >> tol \end{array}\right\} \tag{22}$$

†† This is value you get by using the 5 Point Newton-Cotes Integration Rule, with n=0 (one interval) (See Appendix II). This value has to go in the n=0, one interval row.

6.4 ITERATION 3: n = 3

$$\text{using equation (12) on page 8:} \qquad T(n=3,1) = 2.397737097005620 \tag{23}$$

Richardson Extrapolation routine for n=2, j = 2:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1}-1} \qquad j>1 \\ T_{2,2} &= \frac{4\,T_{3,1} - T_{2,1}}{3} \\ &= \frac{4 \times 2.397737097005620 - 2.629221182043763}{3} \\ &= 2.320575735326239 \end{aligned} \tag{24}$$

Richardson Extrapolation routine for n=1, j = 3:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1}-1} \qquad j>1 \\ T_{1,3} &= \frac{16 \times T_{2,2} - T_{1,2}}{15} \\ &= \frac{16 \times 2.320575735326239 - 2.407900969997745}{15} \\ &= 2.314754053014805 \end{aligned} \tag{25}$$

Richardson Extrapolation routine for n=0, j = 4:

$$\begin{aligned} T_{n,j} &= \frac{4^{j-1} \times T_{n+1,j-1} - T_{n,j-1}}{4^{j-1}-1} \qquad j>1 \\ T_{0,4} &= \frac{64 \times T_{1,3} - T_{0,3}}{15} \\ &= \frac{64 \times 2.314754053014805 - 2.385700428603655}{63} \\ &= 2.313627920068950 \end{aligned} \tag{26}$$

Romberg Integration: Part 2

Richardson Extrapolation:

n	number of intervals: 2^n	T(n,j=1)	T(n,j=2)	T(n,j=3)	(T(n,j=4))
0	1	4.950000000000000	2.740909090909091	2.385700428603655	2.313627920068950
1	2	3.293181818181818	2.407900969997745	2.314754053014805	
2	4	2.629221182043763	2.320575735326239		
3	8	2.397737097005620			

convergence check: check last two values in a column for columns 1 through 3

$$\left.\begin{array}{l} \text{column 1: if} \quad T(3,1) - T(2,1) \le tol \quad \text{then convergence: T(3,1) is the solution} \\ 2.397737097005620 - 2.629221182043763 = .23149\ldots >> tol \end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l} \text{column 2: if} \quad T(2,2) - T(1,2) \le tol \quad \text{then convergence: T(2,2) is the solution} \\ 2.320575735326239 - 2.407900969997745 = .08732\ldots >> tol \end{array}\right\} \tag{28}$$

$$\left.\begin{array}{l} \text{column 3: if} \quad T(1,3) - T(0,3) \le tol \quad c\text{then convergence: T(1,3) is the solution} \\ 2.314754053014805 - 2.385700428603655 = .07094\ldots >> tol \end{array}\right\} \tag{29}$$

In the following table, column 2 is filled out.

Romberg Integration: Part 2

Richardson Extrapolation: Column 2

		Trapezoidal	Simpsonś
n	number of intervals: 2^n	T(n,j=1)	T(n,j=2)
0	1	4.950000000000000	2.740909090909091
1	2	3.293181818181818	2.407900969997745
2	4	2.629221182043763	2.320575735326239
3	8	2.397737097005620	2.304690440974772
4	16	2.327952104982484	2.302763505482294
5	32	2.309060655357341	2.302597560528170
6	64	2.304213334235463	2.302585898245308
7	128	2.302992757242847	2.302585143759979
8	256	2.302687047130696	2.302585096173893
9	512	2.302610583913093	2.302585093192894
10	1024	2.302591465872944	2.302585093006475
11	2048	2.302586686223092	2.302585092994821
12	4096	2.302585491301889	2.302585092994092
13	8192	2.302585192571041	2.302585092994048
14	16384	2.302585117888296	2.302585092994047
15	32768	2.302585099217609	2.302585092994032
16	65536	2.302585094549926	2.302585092994033
17	131072	2.302585093383006	2.302585092994033
18	262144	2.302585093091277	2.302585092994065
19	524288	2.302585093018368	2.302585092994095
20	1048576	2.302585093000163	2.302585092994017
21	2097152	2.302585092995553	2.302585092993894
22	4194304	2.302585092994309	2.302585092993988
23	8388608	2.302585092994068	2.302585092993967
24	16777216	2.302585092993993	2.302585092993759
25	33554432	2.302585092993818	2.302585092993659
26	67108864	2.302585092993699	

Column 2 (Simpson's Rule), converges to 15 digits at n = 12 (4096 intervals). Column 1 converges to 14 digits and takes n=23 iterations (8388608 intervals) to achieve it. Richardson's Extrapolation takes a trivial number of calculations to determine each value in column 2. When you see Table I in Appendix I, column 2 is composite Simpson's Rule.

In the following table, column 3 is filled out.

$$log_e(10) = \underline{2.302585092994045} \tag{30}$$

Romberg Integration: Part 2

Richardson Extrapolation: Column 3

		Trapezoidal	Simpsons	5 pt NewtonCotes
n	number of intervals: 2^n	T(n,j=1)	T(n,j=2)	T(n,j=3)
0	1	4.950000000000000	2.740909090909091	2.385700428603655
1	2	3.293181818181818	2.407900969997745	2.314754053014805
2	4	2.629221182043763	2.320575735326239	2.303631421351341
3	8	2.397737097005620	2.304690440974772	2.302635043116128
4	16	2.327952104982484	2.302763505482294	2.302586497531228
5	32	2.309060655357341	2.302597560528170	2.302585120759784
6	64	2.304213334235463	2.302585898245308	2.302585093460957
7	128	2.302992757242847	2.302585143759979	2.302585093001487
8	256	2.302687047130696	2.302585096173893	2.302585092994161
9	512	2.302610583913093	2.302585093192894	2.302585092994046
10	1024	2.302591465872944	2.302585093006475	2.302585092994044
11	2048	2.302586686223092	2.302585092994821	2.302585092994043
12	4096	2.302585491301889	2.302585092994092	2.302585092994045
13	8192	2.302585192571041	2.302585092994048	2.302585092994047
14	16384	2.302585117888296	2.302585092994047	2.302585092994031
15	32768	2.302585099217609	2.302585092994032	2.302585092994033
16	65536	2.302585094549926	2.302585092994033	2.302585092994033
17	131072	2.302585093383006	2.302585092994033	2.302585092994067
18	262144	2.302585093091277	2.302585092994065	2.302585092994097
19	524288	2.302585093018368	2.302585092994095	2.302585092994012
20	1048576	2.302585093000163	2.302585092994017	2.302585092993886
21	2097152	2.302585092995553	2.302585092993894	2.302585092993994
22	4194304	2.302585092994309	2.302585092993988	2.302585092993966
23	8388608	2.302585092994068	2.302585092993967	2.302585092993746
24	16777216	2.302585092993993	2.302585092993759	2.302585092993652
25	33554432	2.302585092993818	2.302585092993659	
26	67108864	2.302585092993699		

The third column (n=12), is correct to 16 digits. However, we do not yet have a solution.

6.5 Column Four (j = 4) of the T matrix

In the following table, column 4 is filled out.

$$T_{n,4} = \frac{64 \times T_{n+1,3} - T_{n,3}}{63} \tag{31}$$

$$log_e(10) = \underline{2.302585092994045}$$

Romberg Integration: Part 2

Richardson Extrapolation: Column 4

n	No. of intervals: 2^n	T(n,j=1)	T(n,j=2)	T(n,j=3)	T(n,j=4)
0	1	4.950000000000000	2.740909090909091	2.385700428603655	2.313627920068950
1	2	3.293181818181818	2.407900969997745	2.314754053014805	2.303454871642397
2	4	2.629221182043763	2.320575735326239	2.303631421351341	2.302619227588585
3	8	2.397737097005620	2.304690440974772	2.302635043116128	2.302585726966388
4	16	2.327952104982484	2.302763505482294	2.302586497531228	2.302585098906269
5	32	2.309060655357341	2.302597560528170	2.302585120759784	2.302585093027643
6	64	2.304213334235463	2.302585898245308	2.302585093460957	2.302585092994194
7	128	2.302992757242847	2.302585143759979	2.302585093001487	2.302585092994045
8	256	2.302687047130696	2.302585096173893	2.302585092994161	2.302585092994045
9	512	2.302610583913093	2.302585093192894	2.302585092994046	2.302585092994044
10	1024	2.302591465872944	2.302585093006475	2.302585092994044	2.302585092994043
11	2048	2.302586686223092	2.302585092994821	2.302585092994043	2.302585092994045
12	4096	2.302585491301889	2.302585092994092	2.302585092994045	2.302585092994047
13	8192	2.302585192571041	2.302585092994048	2.302585092994047	2.302585092994031
14	16384	2.302585117888296	2.302585092994047	2.302585092994031	2.302585092994033
15	32768	2.302585099217609	2.302585092994032	2.302585092994033	2.302585092994033
16	65536	2.302585094549926	2.302585092994033	2.302585092994033	2.302585092994067
17	131072	2.302585093383006	2.302585092994033	2.302585092994067	2.302585092994098
18	262144	2.302585093091277	2.302585092994065	2.302585092994097	2.302585092994010
19	524288	2.302585093018368	2.302585092994095	2.302585092994012	2.302585092993884
20	1048576	2.302585093000163	2.302585092994017	2.302585092993886	2.302585092993996
21	2097152	2.302585092995553	2.302585092993894	2.302585092993994	2.302585092993966
22	4194304	2.302585092994309	2.302585092993988	2.302585092993966	2.302585092993742
23	8388608	2.302585092994068	2.302585092993967	2.302585092993746	2.302585092993651
24	16777216	2.302585092993993	2.302585092993759	2.302585092993652	
25	33554432	2.302585092993818	2.302585092993659		
26	67108864	2.302585092993699			

Note how rows n=8 and n=7 values of column 4 have converged. This shows an advantage in that we can see to how many digits the solution has converged to.

```
                T( 7, 4) =    2.302585092994045
                T( 8, 4) =    2.302585092994045
     T( 8, 4) - T( 7, 4) =    0.000000000000000  accurate to 16 digits
                              1 234567890123456
           solution      =    2.302585092994045 =   T( 8, 4)
```

```
This shows us how accurate our solution is.
```

$$log_e(10) = \underline{2.302585092994045}^{\ddagger\ddagger}$$

[‡‡]Now we are only going to underline numbers that are equal to this exact solution

Romberg Integration: Part 2

Richardson Extrapolation: Columns 1-7 of the T matrix

n	T(n,j=1)	T(n,j=2)	T(n,j=3)	T(n,j=4)	T(n,j=5)	T(n,j=6)	T(n,j=7)
0	4.950000000000000	2.740909090909091	2.385700428603655	2.313627920068950	2.303414977334842	2.302615169490732	2.302585558689705
1	3.293181818181818	2.407900969997745	2.314754053014805	2.303454871642397	2.302615950553080	2.302585565918905	2.302585095840596
2	2.629221182043763	2.320575735326239	2.303631421351341	2.302619227588585	2.302585595591399	2.302585095955362	2.302585093000507
3	2.397737097005620	2.304690440974772	2.302635043116128	2.302585726966388	2.302585096443288	2.302585093001228	2.302585092994051
4	2.327952104982484	2.302763505482294	2.302586497531228	2.302585098906269	2.302585093004589	2.302585092994053	2.302585092994045
5	2.309060655357341	2.302597560528170	2.302585120759784	2.302585093027643	2.302585092994063	2.302585092994045	2.302585092994045
6	2.304213334235463	2.302585898245308	2.302585093460957	2.302585092994194	2.302585092994045	2.302585092994045	2.302585092994044
7	2.302992757242847	2.302585143759979	2.302585093001487	2.302585092994045	2.302585092994045	2.302585092994044	2.302585092994043
8	2.302687047130696	2.302585096173893	2.302585092994161	2.302585092994045	2.302585092994044	2.302585092994043	2.302585092994045
9	2.302610583913093	2.302585093192894	2.302585092994046	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047
10	2.302591465872944	2.302585093006475	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031
11	2.302586686223092	2.302585092994821	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033
12	2.302585491301889	2.302585092994092	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033
13	2.302585192571041	2.302585092994048	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067
14	2.302585117888296	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098
15	2.302585099217609	2.302585092994032	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010
16	2.302585094549926	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884
17	2.302585093383006	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884	2.302585092993996
18	2.302585093091277	2.302585092994065	2.302585092994097	2.302585092994010	2.302585092993884	2.302585092993996	2.302585092993965
19	2.302585093018368	2.302585092994095	2.302585092994012	2.302585092993884	2.302585092993996	2.302585092993965	2.302585092993741
20	2.302585093000163	2.302585092994017	2.302585092993886	2.302585092993996	2.302585092993965	2.302585092993741	2.302585092993650
21	2.302585092995553	2.302585092993894	2.302585092993994	2.302585092993966	2.302585092993741	2.302585092993650	
22	2.302585092994309	2.302585092993988	2.302585092993966	2.302585092993742	2.302585092993651		
23	2.302585092994068	2.302585092993967	2.302585092993746	2.302585092993651			
24	2.302585092993993	2.302585092993759	2.302585092993652				
25	2.302585092993818	2.302585092993659					
26	2.302585092993699						

At this point, the assiduous reader can see how convergence presents itself.

Romberg Integration: Part 2

Richardson Extrapolation: Columns 8-14 of the T matrix

n	T(n,j=8)	T(n,j=9)	T(n,j=10)	T(n,j=11)	T(n,j=12)	T(n,j=13)	T(n,j=14)
0	2.302585095812345	2.302585093000290	2.302585092994050	2.302585092994045	2.302585092994045	2.302585092994044	2.302585092994043
1	2.302585093000333	2.302585092994050	2.302585092994045	2.302585092994045	2.302585092994044	2.302585092994043	2.302585092994045
2	2.302585092994050	2.302585092994045	2.302585092994045	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047
3	2.302585092994045	2.302585092994045	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031
4	2.302585092994045	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033
5	2.302585092994044	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033
6	2.302585092994043	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067
7	2.302585092994045	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098
8	2.302585092994047	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010
9	2.302585092994031	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884
10	2.302585092994033	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884	2.302585092993996
11	2.302585092994033	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884	2.302585092993996	2.302585092993965
12	2.302585092994067	2.302585092994098	2.302585092994010	2.302585092993884	2.302585092993996	2.302585092993965	2.302585092993741
13	2.302585092994098	2.302585092994010	2.302585092993884	2.302585092993996	2.302585092993965	2.302585092993741	2.302585092993650
14	2.302585092994010	2.302585092993884	2.302585092993996	2.302585092993965	2.302585092993741	2.302585092993650	
15	2.302585092993884	2.302585092993996	2.302585092993965	2.302585092993741	2.302585092993650		
16	2.302585092993996	2.302585092993965	2.302585092993741	2.302585092993650			
17	2.302585092993965	2.302585092993741	2.302585092993650				
18	2.302585092993741	2.302585092993650					
19	2.302585092993650						
20							
21							
22							
23							
24							
25							
26							

7 Appendix I - Simpson's Rule

$$\text{Simpson's Rule: 1 interval } = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{32}$$

$$\text{Simpson's Rule: } N = 2^n \text{ intervals } = \frac{b-a}{6N}\left[f(a) + f(b) + 2\sum_{i=1}^{N-1} f\left(a + (\frac{b-a}{N}i\right) + 4\sum_{i=1}^{2N-1} f\left(a + \frac{b-a}{2N}i\right)\right] \tag{33}$$

Simpson's Rule

This is Column 2 of the T Matrix

n	number of intervals: $N = 2^n$	Simpson's Rule
0	1	2.740909090909091
1	2	2.407900969997745
2	4	2.320575735326238
3	8	2.304690440974772
4	16	2.302763505482294
5	32	2.302597560528170

This table is to show that Column 2 of the T matrix is actually Simpson's rule.

Romberg Integration: Column 2 is Simpson's Rule

		columns of T:	
n	number of intervals	Simpson's Rule calculated using equation 32	Column 2 from Table Above
0	1	2.74090909090909	2.74090909090909
1	2	2.40790096999774	2.40790096999774
2	4	2.32057573532623	2.32057573532623
3	8	2.30469044097477	2.30469044097477
4	16	2.30276350548229	2.30276350548229
5	32	2.30259756052817	2.30259756052817
6	64	2.30258589824530	2.30258589824530
7	128	2.30258514375997	2.30258514375997
8	256	2.30258509617389	2.30258509617389
9	512	2.30258509319289	2.30258509319289
10	1024	2.30258509300647	2.30258509300647
11	2048	2.30258509299482	2.30258509299482
12	4096	2.30258509299409	2.30258509299409
13	8192	2.30258509299404	2.30258509299404
14	16384	2.30258509299404	2.30258509299404
15	32768	2.30258509299403	2.30258509299403
16	65536	2.30258509299404	2.30258509299403
17	131072	2.30258509299403	2.30258509299403
18	262144	2.30258509299406	2.30258509299406
19	524288	2.30258509299407	2.30258509299409
20	1048576	2.30258509299399	2.30258509299401
21	2097152	2.30258509299390	2.30258509299389
22	4194304	2.30258509299411	2.30258509299396
23	8388608	2.30258509299405	2.30258509299375
24	16777216	2.30258509299369	2.30258509299365

8 Appendix II - 5 Point Newton-Cotes Integration

5 Point Newton-Cotes		
This is Column 3 of the T Matrix		
n	number of intervals: 2^n	5 Point Newton-Cotes
0	1	2.385700428603654
1	2	2.314754053014804
2	4	2.303631421351341
3	8	2.302635043116128
4	16	2.302586497531228
5	32	2.302585120759784
6	64	2.302585093460957
7	128	2.302585093001488
8	256	2.302585092994163
9	512	2.302585092994048
10	1024	2.302585092994046
11	2048	2.302585092994044
12	4096	2.302585092994046
13	8192	2.302585092994047
14	16384	2.302585092994053
15	32768	2.302585092994049
16	65536	2.302585092994035
17	131072	2.302585092994063
18	262144	2.302585092994052
19	524288	2.302585092994035
20	1048576	2.302585092994101
21	2097152	2.302585092994100
22	4194304	2.302585092994233
23	8388608	2.302585092994006
24	16777216	2.302585092994135

9 Appendix III - Computer Languages and Arithmetic

Computer Languages, such as Fortran, Perl, Python, C etc. are, tools. You have to pick the tool that works best for you. Just because your neighbor uses a 20 ounce hammer, doesn't mean a 20 ounce hammer is for you. The best choice for a tool, is the one that works best for you.

Think of it, when Michaelangelo was sculpting the David, he used the hammer(s) and chisel(s) that worked best for him. It is what you do with the tools, not which tools you use.

I use Fortran because it is simple, consistent, and I can do whatever I want to do with it. Other people will use Perl, or Python, for the same reasons.

If someone tells you that you are using the wrong language, tell them to pound sand.

I used 64 bit Arithmetic in this document for obvious reasons. If you are going to do serious numerical integration and use the results in other, perhaps, production codes, you should consider using 128-bit Arithmetic.