EECS 545 HW3. Po-Jen Lai.

1. $l(w) = \sum_{i=1}^{n} -y_i \log h(x_i) - (1-y_i)\log(1-h(x_i))$

$= \sum_{i=1}^{n} -y_i w^T x_i + \log(e^{w^T x_i} + 1)$

(a) $\frac{\partial l(w)}{\partial w_k} = \sum_{i=1} -X_k^{(i)}\left(Y_i^{(i)} - \frac{1}{1+e^{-w^T x^{(i)}}}\right) \Rightarrow \nabla l(w) = \begin{bmatrix} \frac{\partial l(w)}{\partial w_1} \\ \vdots \\ \frac{\partial l(w)}{\partial w_m} \end{bmatrix}$ #

(b) $\frac{\partial^2 l(w)}{\partial w_k^2} = \sum_{i=1}^{n} (X_k^{(i)})^2 \cdot \frac{e^{-w^T x^{(i)}}}{(1+e^{-w^T x^{(i)}})^2}$, $\frac{\partial^2 l(w)}{\partial w_k w_j} = \sum_{i=1}^{n} X_k^{(i)} \cdot X_j^{(i)} \frac{e^{-w^T x^{(i)}}}{(1+e^{-w^T x^{(i)}})^2}$

$\Rightarrow H = \begin{bmatrix} \frac{\partial^2 l(w)}{\partial w_1^2} & \cdots & \frac{\partial^2 l(w)}{\partial w_1 \partial w_m} \\ \vdots & & \vdots \\ \frac{\partial^2 l(w)}{\partial w_m \partial w_1} & & \frac{\partial^2 l(w)}{\partial w_m^2} \end{bmatrix} = c \cdot U \cdot U^T$ where $c \in \mathbb{R}, c \geq 0, U \in \mathbb{R}^m$. #

$\Rightarrow$ for all $v \in \mathbb{R}^m$, $v^T H v = v^T (cUU^T) v = c\|v^T U\|^2 \Rightarrow H$ is PSD. #

2.

(a) $f(x; \alpha, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma^2}\right)$

$l(\alpha, \sigma) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \alpha)^2$

$\frac{\partial l}{\partial \alpha} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \alpha) = 0 \Rightarrow \hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} X_i$ #.

$\frac{\partial l}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \alpha)^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_j - \hat{\alpha})^2$ #.

(b) $l(\mu, \Sigma) = \frac{-nd}{2} \cdot \ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^T \Sigma^{-1}(x_i - \mu)$.

$\frac{\partial l(\mu, \Sigma)}{\partial \mu} = \sum_{i=1}^{n} \Sigma^{-1}(x_i - \mu) \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$. #

3.

(a) $H(X) + H(Y|X) = -\int p(X,Y)\ln p(X)\,dX\,dY - \int p(X,Y)\ln p(Y|X)\,dX\,dY$.

$= -\int p(X,Y) \cdot \ln p(X,Y)\,dX\,dY = H(Y) + H(X|Y)$

$\Rightarrow H(X) - H(X|Y) = H(Y) - H(Y|X)$.

(b) $H(X|Y) = H(f(Y)|Y) = -\int p(f(Y), Y)\ln p(f(Y)|Y)\,dX\,dY$.

$\begin{cases} P(f(Y)=k \mid Y=y) = 1, & k = f(y) \\ P(f(Y)=k \mid Y=y) = 0, & k \neq f(y) \end{cases} \Rightarrow H(X|Y) = 0$. same as $H(Y|X)$

$\Rightarrow I(X,Y) = H(X) = H(Y)$ #