4) (b) Let x_1, x_2 be global minimizer. $x' = \frac{x_1+x_2}{2}$. $\Rightarrow f(x') > f(x_1) = f(x_2)$

but $f\left(\frac{x_1+x_2}{2}\right) < \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2) \Rightarrow$ Contradiction. f has at most one global minimizer.

(c) $\nabla f(x^*) = 0$. Let $x = x^* + th$, where $t \in \mathbb{R}$, $h \in \mathbb{R}^n$

$$f(x^*+th) = f(x^*) + \langle \underbrace{\cancel{\nabla f(x^*)}}_{\to 0}, th \rangle + \frac{1}{2}\langle th, \nabla^2 f(x^*) th \rangle + o(\|th\|^2)$$

$$\Rightarrow 0 < \frac{f(x^*+th) - f(x^*)}{\|th\|^2} = \frac{(th)^T \nabla^2 f(x^*)(th)}{2\|th\|^2} + \underbrace{\cancel{\frac{o(\|th\|^2)}{\|th\|^2}}}_{\to 0}$$

$\Rightarrow 0 < (th)^T \nabla^2 f(x^*)(th)$, where $th \in \mathbb{R}^n$. $\Rightarrow t^2 h^T \cdot \nabla^2 f(x^*) \cdot h > 0$ #

(d) f is convex $\Rightarrow f(x) \geq f(y) + \langle \nabla f(y), x-y \rangle \Rightarrow \langle x-y, \nabla^2 f(y)(x-y) \rangle \geq 0$

$\Rightarrow (x-y)^T \cdot \nabla^2 f(y) \cdot (x-y) \geq 0 \Rightarrow \nabla^2 f(y)$ is PSD #

$\nabla^2 f(y)$ is PSD. $\langle x-y, \nabla^2 f(y+t(x-y))(x-y) \rangle \geq 0$.

$\Rightarrow f(x) \leq f(y) + \langle \nabla f(y), x-y \rangle \Rightarrow f$ is convex #.

(e) $f(x) = \frac{1}{2} x^T A x + b^T x + c$. $\nabla^2 f(x) = A$. # f is $\begin{cases} \text{convex} & \text{if } A \text{ is PSD} \\ \text{strictly convex} & \text{PD.} \end{cases}$ #