Compute-and-Forward:

An Explicit Link between Finite Field and Gaussian Interference Networks

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Boston University

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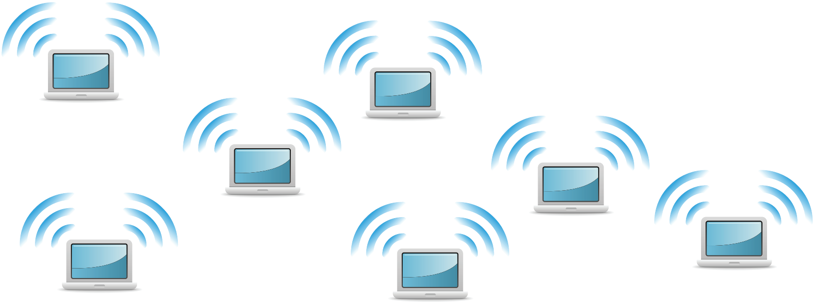
Multi-User Wireless Networks

- Must cope with interference, fading, and noise.

Multi-User Wireless Networks



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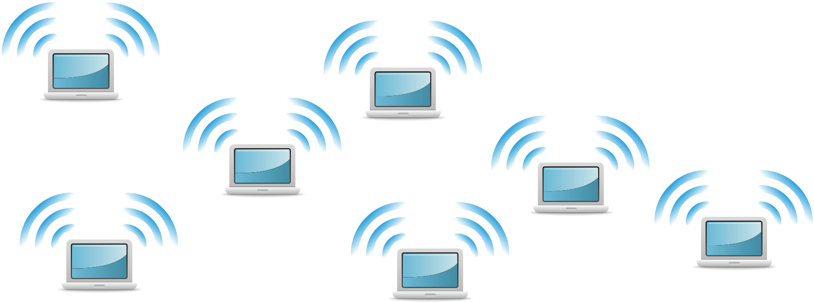

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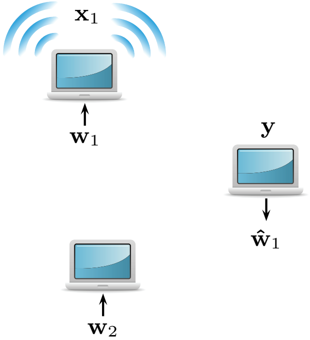
- How should we deal with interference?



Possible Coding Strategies:

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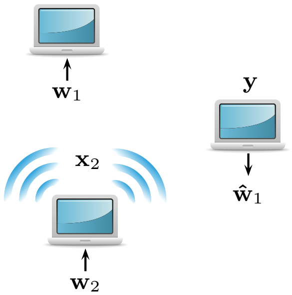

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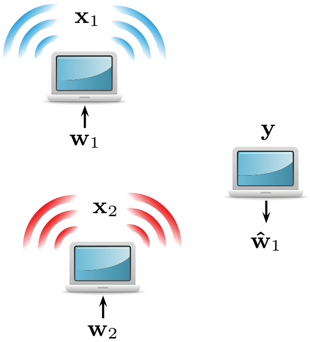

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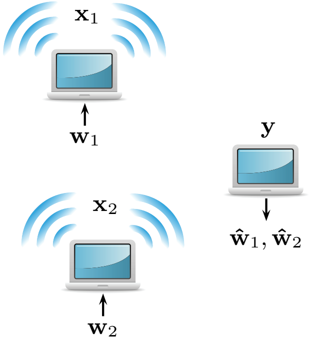

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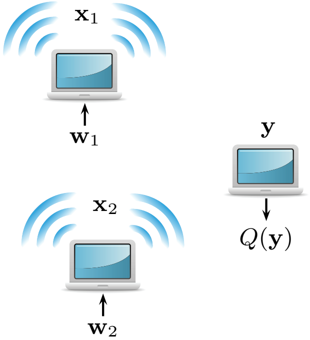

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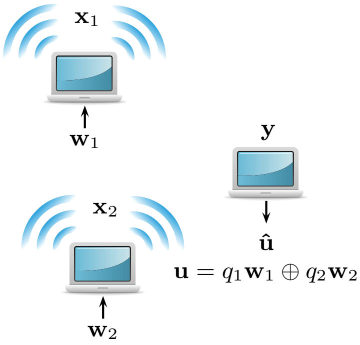

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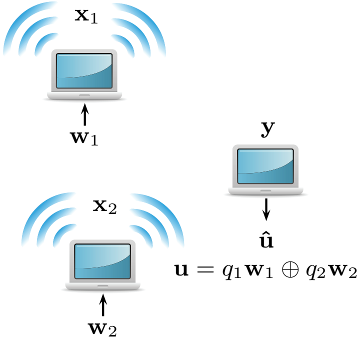

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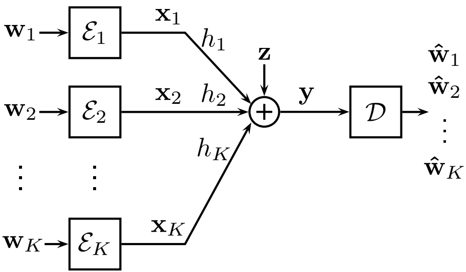

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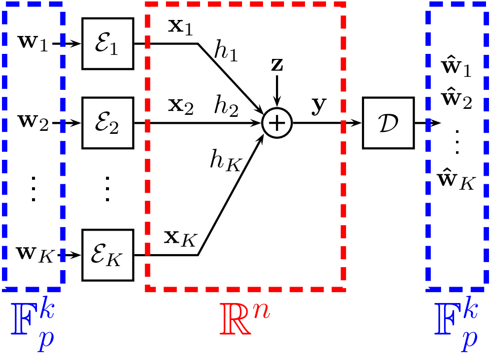

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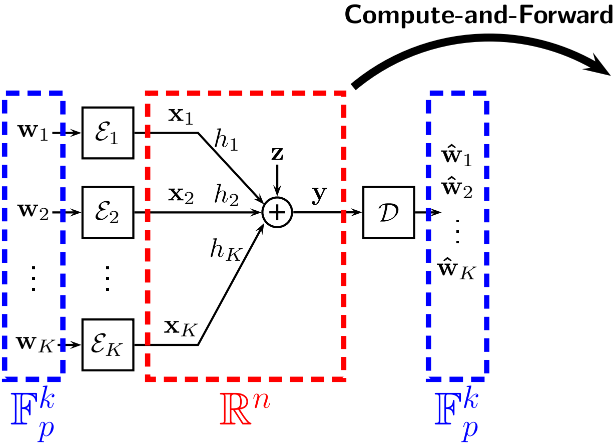

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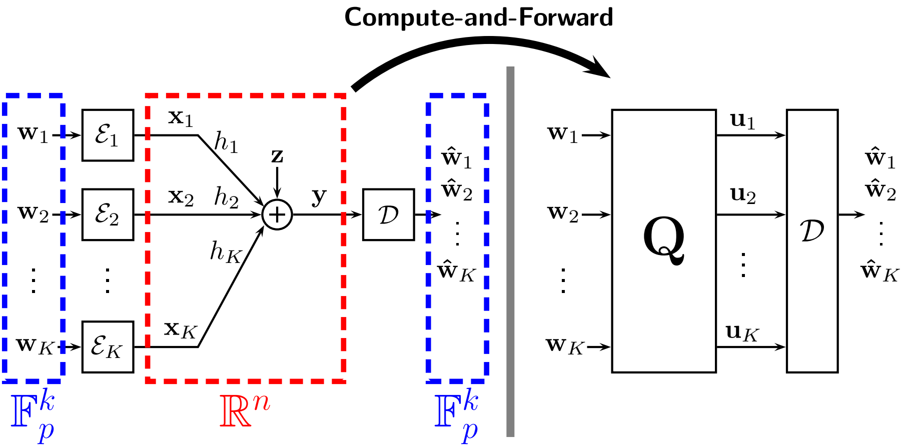

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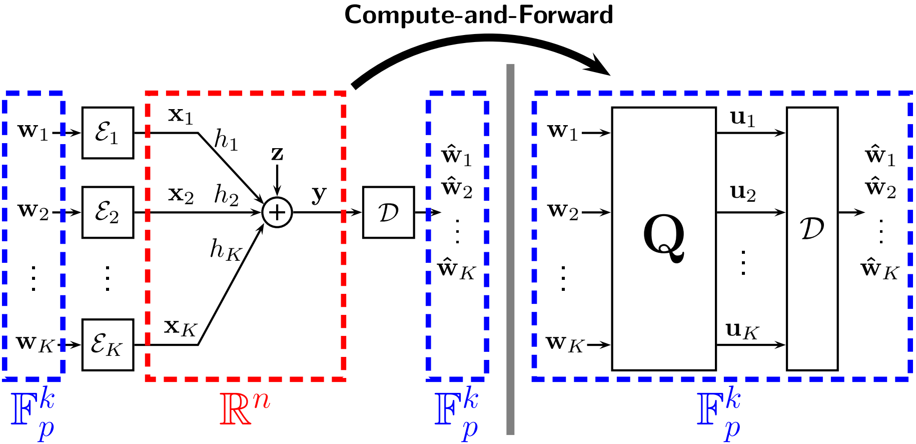

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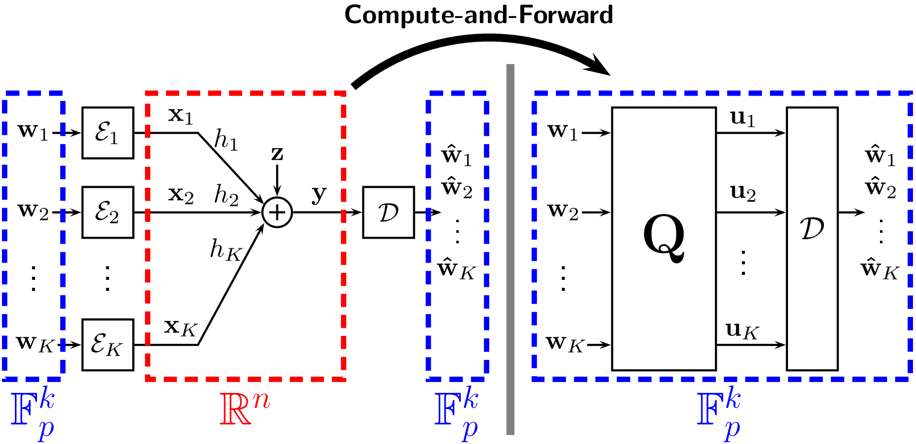

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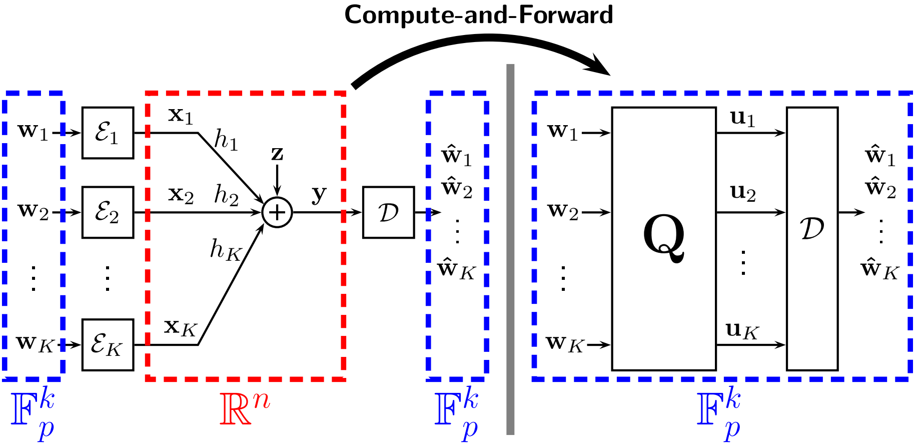

- Which linear combinations can be sent over a given channel?
- Where can this help us?

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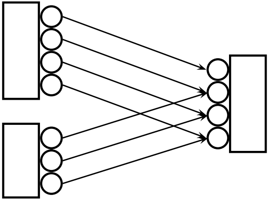

Top-Down vs. Bottom-Up

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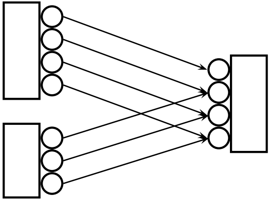

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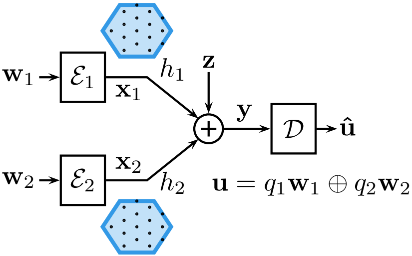

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Deterministic Model from **Avestimehr-Diggavi-Tse '11:**

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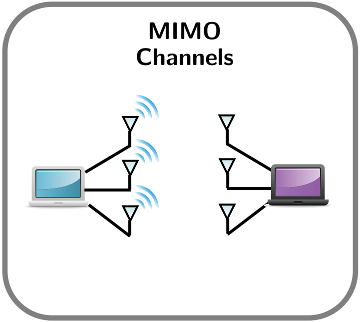

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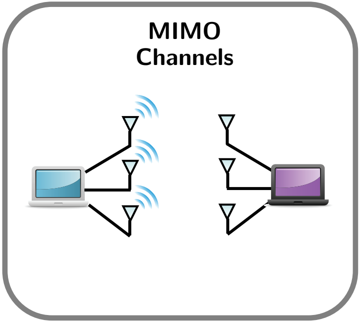

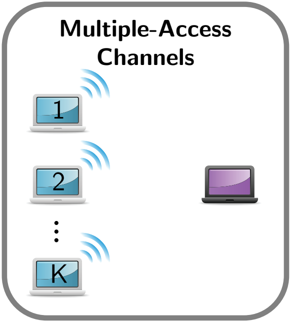

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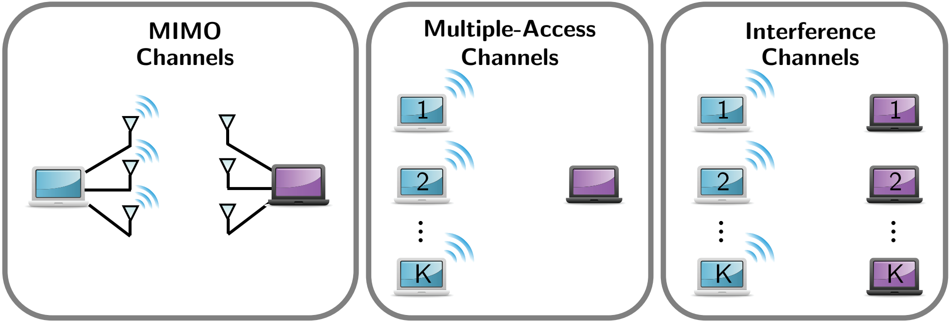
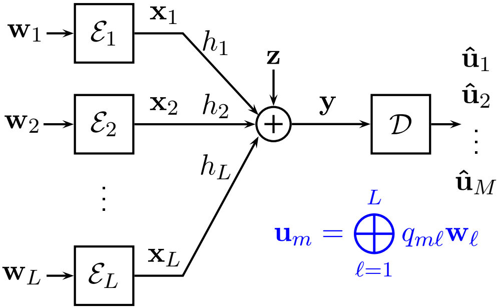


Compute-and-Forward: Problem Statement



$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m\ell} \mathbf{w}_\ell$$

- Messages are finite field vectors, $\mathbf{w}_\ell \in \mathbb{F}_p^k$.
- Real-valued inputs and outputs, $\mathbf{x}_\ell, \mathbf{y} \in \mathbb{R}^n$.
- Power constraint, $\|\mathbf{x}_\ell\|^2 \leq nP$.
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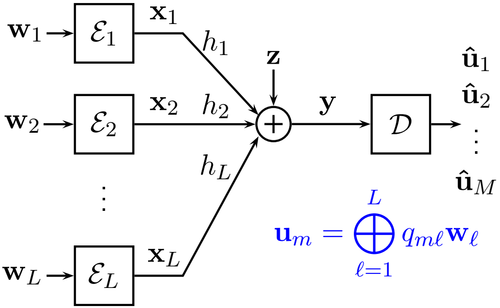

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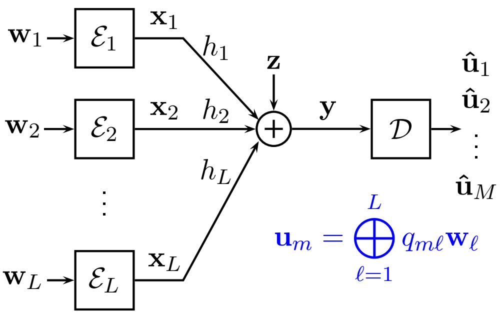

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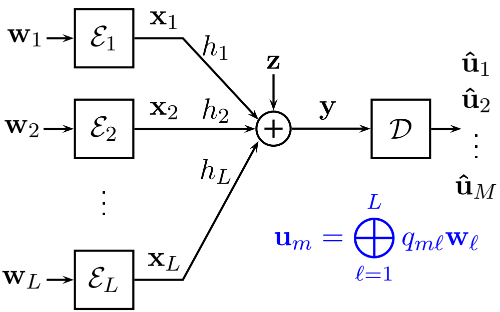

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- What rates are achievable as a function of h_ℓ and $q_{m\ell}$?

Computation Rate

- Want to characterize achievable rates as a function of h_ℓ and $q_{m\ell}$.

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- The linear combination with integer coefficient vector $\mathbf{a}_m = [a_{m1}\ a_{m2}\ \cdots\ a_{mL}]^{\mathsf{T}} \in \mathbb{Z}^L$ corresponds to

$$\mathbf{u}_m = \bigoplus_{\ell=1}^{L} q_{m\ell}\mathbf{w}_\ell \qquad \text{where } q_{m\ell} = g^{-1}\Big([a_{m\ell}] \mod p\Big)$$

  where $g(\cdot)$ is the natural mapping between $\mathbb{F}_p$ and $\mathbb{Z}$.

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where $g(\cdot)$ is the natural mapping between $\mathbb{F}_p$ and $\mathbb{Z}$.
- **Key Definition:** The computation rate region described by $R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a})$ is *achievable* if, for any $\epsilon > 0$ and n, p large enough, a receiver can decode any linear combinations with integer coefficient vectors $\mathbf{a}_1, \ldots, \mathbf{a}_M \in \mathbb{Z}^L$ for which the message rate R satisfies
$$R < \min_m R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_m)$$

Compute-and-Forward: Achievable Rates

Theorem (Nazer-Gastpar '11)

The computation rate region described by

$$R_{comp}(\mathbf{h}, \mathbf{a}) = \max_{\alpha \in \mathbb{R}} \frac{1}{2} \log^{+} \left(\frac{P}{\alpha^2 + P \|\alpha \mathbf{h} - \mathbf{a}\|^2} \right)$$

is achievable.

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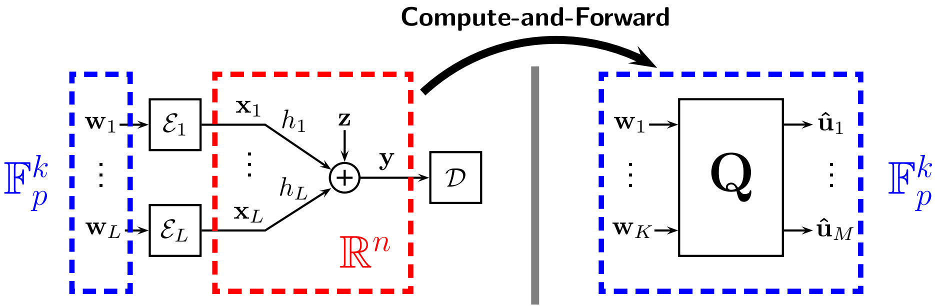

if $R < \min_m R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_m)$ for some $\mathbf{a}_m \in \mathbb{Z}^L$ satisfying $g^{-1}\big([\mathbf{a}_m] \bmod p\big) = \mathbf{q}_m$.

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Special Cases:

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Special Cases:

- Perfect Match: $R_{\text{comp}}(\mathbf{a},\mathbf{a}) = \frac{1}{2}\log^{+}\left(\frac{1}{\|\mathbf{a}\|^2} + P\right)$
- Decode a Message:

$$R_{\text{comp}}\Big(\mathbf{h}, [\underbrace{0\ \cdots\ 0}_{m-1 \text{ zeros}}\ \ 1\ 0\ \cdots\ 0]^{\mathsf{T}}\Big) = \frac{1}{2}\log\left(1 + \frac{h_m^2 P}{1 + P\sum_{\ell \neq m} h_\ell^2}\right)$$

Compute-and-Forward: Effective Noise

$$\begin{aligned}\mathbf{y} &= \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z} \\ &= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \sum_{\ell=1}^{L} (h_\ell - a_\ell)\mathbf{x}_\ell + \mathbf{z}\end{aligned}$$

Desired Codebook:

- Closed under integer linear combinations $\implies$ lattice codebook.

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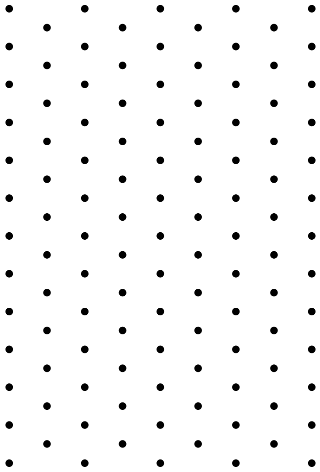
- Closed under integer linear combinations $\Longrightarrow$ lattice codebook.
- Independent effective noise $\Longrightarrow$ dithering.
- Isomorphic to $\mathbb{F}_p^k$ $\Longrightarrow$ nested lattice codebook.

Nested Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^n$.



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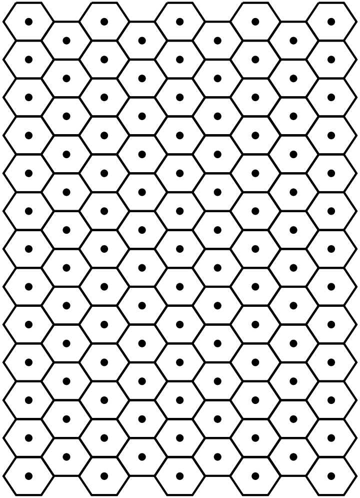

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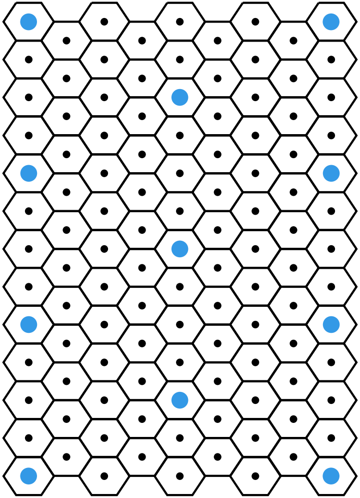

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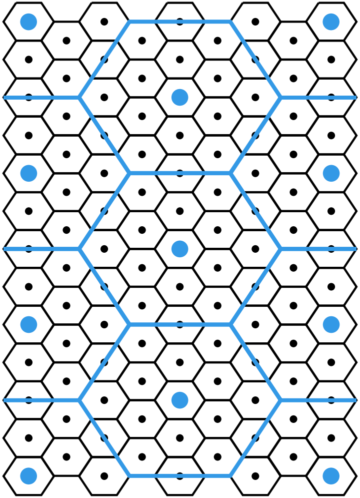

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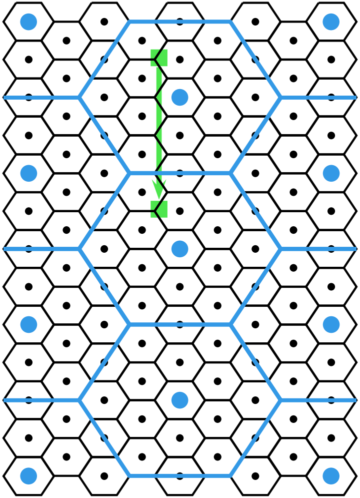
- Quantization error serves as modulo operation:

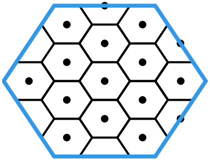
$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \ .$$



Distributive Law:

$$\big[\mathbf{x}_1 + a[\mathbf{x}_2] \bmod \Lambda\big] \bmod \Lambda = [\mathbf{x}_1 + a\mathbf{x}_2] \bmod \Lambda \quad \text{for all } a \in \mathbb{Z}.$$

- Nested Lattice Code: Formed by taking all elements of Λ_{FINE} that lie in the fundamental Voronoi region of Λ.



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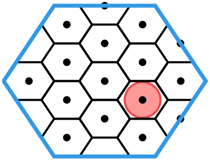

Nested Lattice Codes

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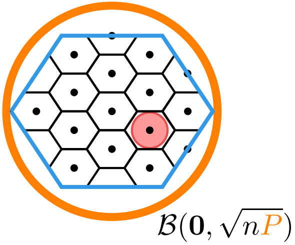

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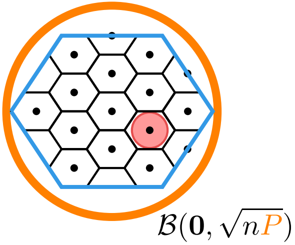
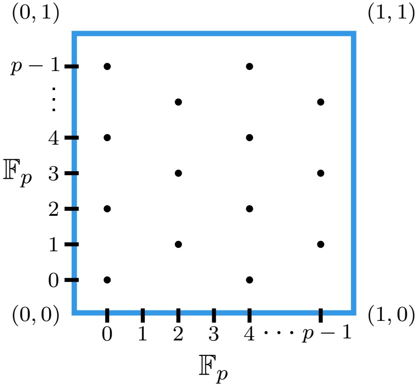
- Existence of good nested lattice codes: **Loeliger '97, Forney-Trott-Chung '00, Erez-Litsyn-Zamir '05, Ordentlich-Erez '13.**

- **Erez-Zamir '04:** Nested lattice codes can achieve the point-to-point Gaussian capacity.



Construction A: Lattice Codes from Linear Codes

- Map elements $\{0, 1, 2, \ldots, p-1\}$ to equally spaced points on $[0, 1)$.
- Choose generator matrix $\mathbf{G} \in \mathbb{F}_p^{n \times k}$ and place its codewords into the unit cube $[0, 1)^n$.



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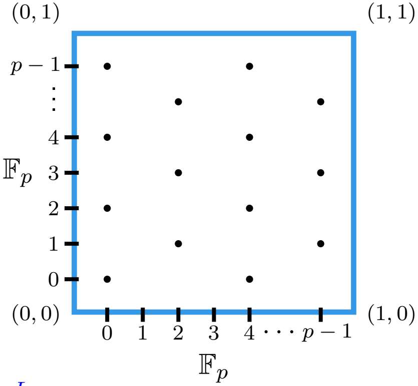
- Enables us to map between nested lattice codewords $\mathbf{t}_\ell$ and finite field messages $\mathbf{w}_\ell$:



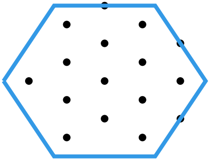
Encoding: $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$

Decoding: $\phi^{-1}\left(\left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell\right] \bmod \Lambda\right) = \bigoplus_{\ell=1}^{L} q_\ell \mathbf{w}_\ell$

where $q_\ell = g^{-1}\big([a_\ell] \bmod p\big)$.

Dithering

- The effective noise will be a mixture of the channel inputs $\mathbf{x}_\ell$ and the noise $\mathbf{z}$.
- Dithering can make the effective noise look independent from the desired lattice codeword.



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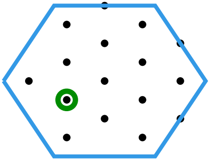

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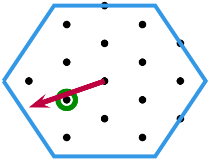

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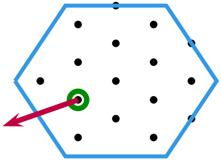
$$\mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda$$



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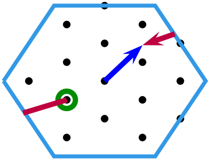

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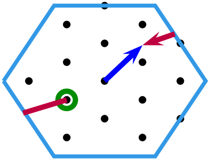

Compute-and-Forward: Encoding and Decoding

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$$
\begin{aligned}
\tilde{\mathbf{y}} &= \left[\alpha\Big(\sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}\Big) - \sum_{\ell=1}^{L} a_\ell \mathbf{d}_\ell\right] \bmod \Lambda \\
&= \left[\sum_{\ell=1}^{L} a_\ell(\mathbf{x}_\ell - \mathbf{d}_\ell) + \sum_{\ell=1}^{L} (\alpha h_\ell - a_\ell)\mathbf{x}_\ell + \alpha\mathbf{z}\right] \bmod \Lambda \\
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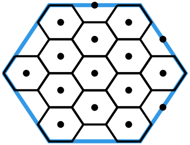
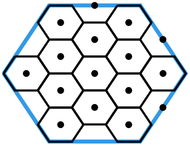
\end{aligned}$$

- Apply inverse map ϕ^{-1} to integer-linear combination to obtain linear combination $\bigoplus_\ell q_\ell \mathbf{w}_\ell$ with coefficients $q_\ell = g^{-1}\big([a_\ell] \bmod p\big)$.

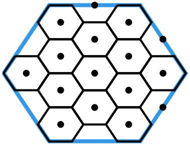
All users pick the same nested lattice code:



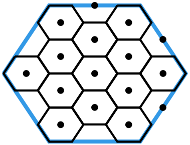


Choose message vectors over finite field $\mathbf{w}_\ell \in \mathbb{F}_p^k$:

$\mathbf{w}_1 \longrightarrow$

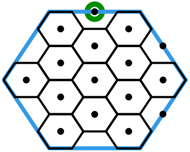


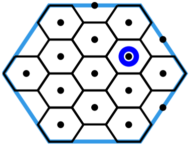
$\mathbf{w}_2 \longrightarrow$



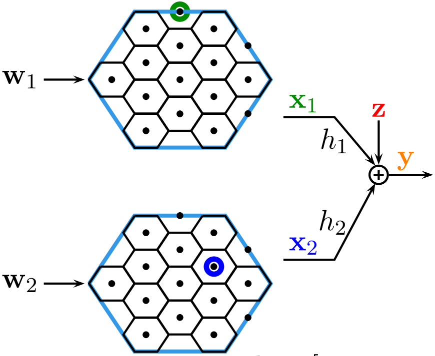
Compute-and-Forward: Illustration

Map $\mathbf{w}_\ell$ to lattice point $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$:

$\mathbf{w}_1 \longrightarrow$ 

$\mathbf{w}_2 \longrightarrow$ 

Transmit lattice points over the channel:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

$$\mathbf{a} = [\ 2 \quad 3\ ]$$

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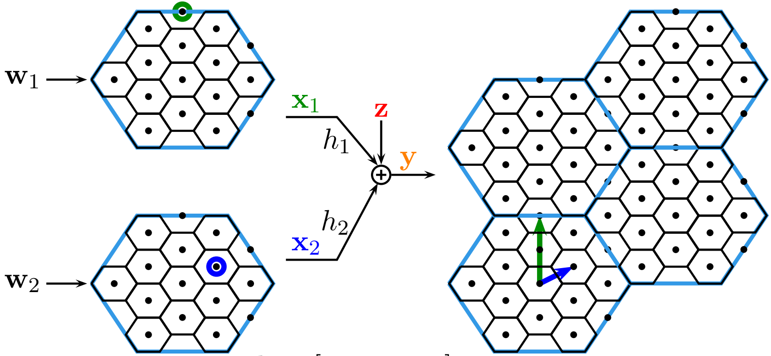

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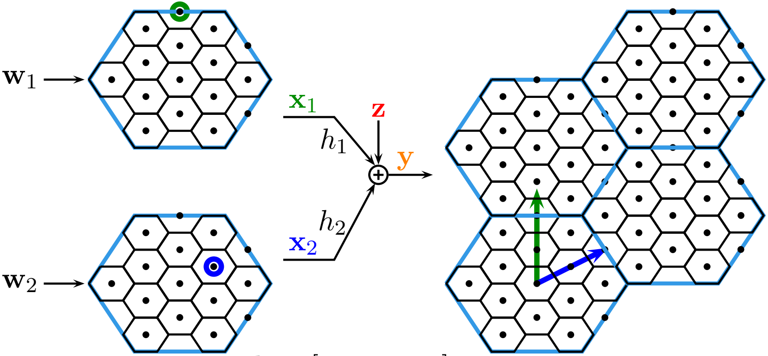
Compute-and-Forward: Illustration

Lattice codewords are scaled by channel coefficients:



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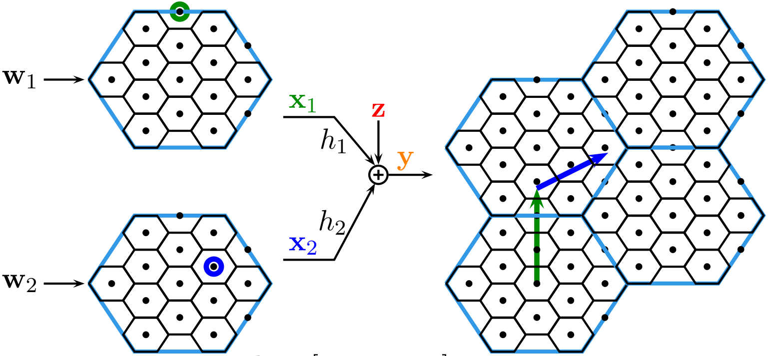
Scaled codewords added together plus noise:



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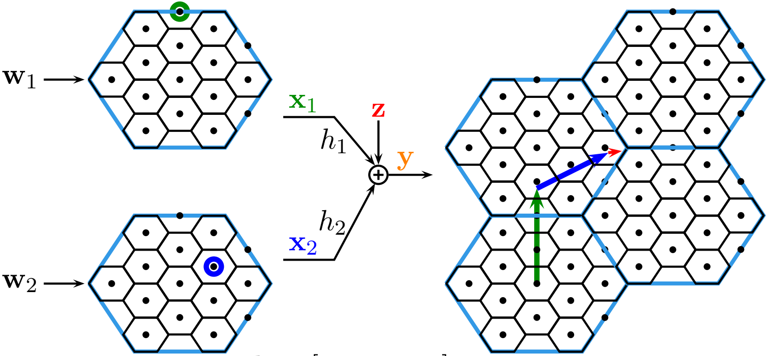

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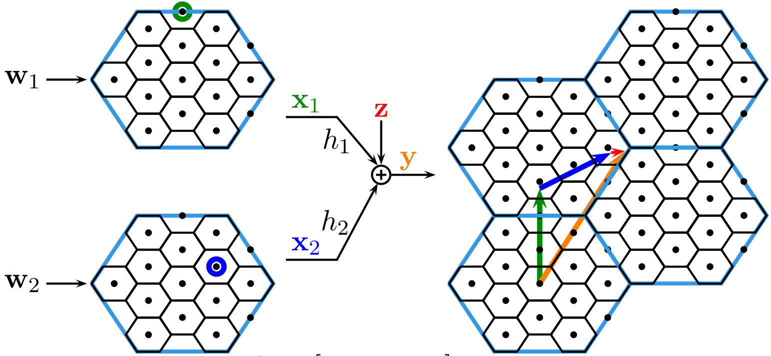
Extra noise penalty for non-integer channel coefficients:



$$\mathbf{h} = [\ 1.4 \quad 2.1\ ]$$

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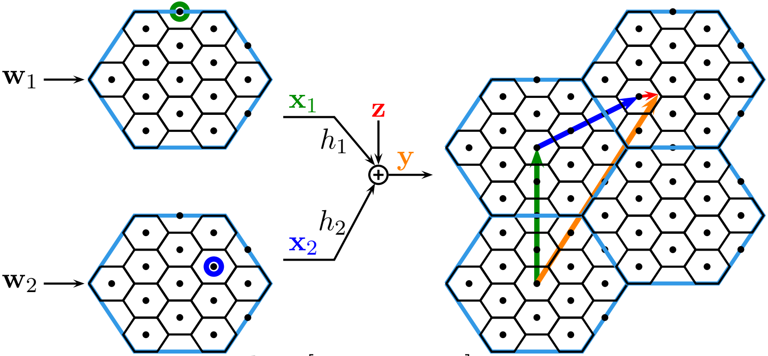
Effective noise: $1 + P\|\mathbf{h} - \mathbf{a}\|^2$

Compute-and-Forward: Illustration

Scale output by α to reduce non-integer noise penalty:



$$\alpha\mathbf{h} = [\ \alpha 1.4 \quad \alpha 2.1\ ]$$

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Effective noise: $\alpha^2 + P\|\alpha\mathbf{h} - \mathbf{a}\|^2$

Compute-and-Forward: Illustration

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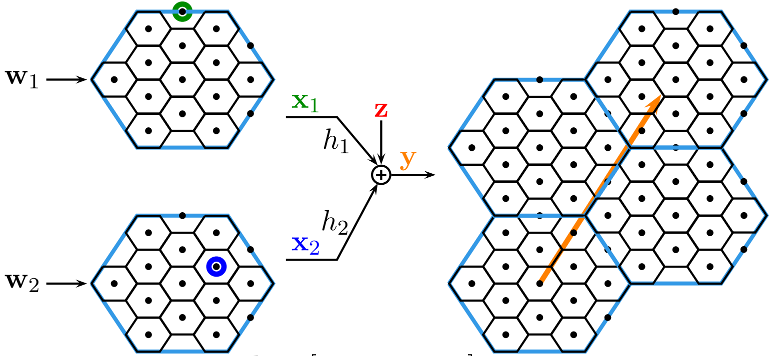

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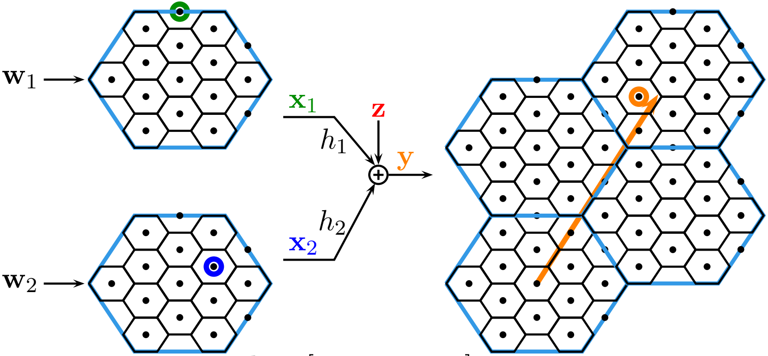
Compute-and-Forward: Illustration

Decode to the closest lattice point:



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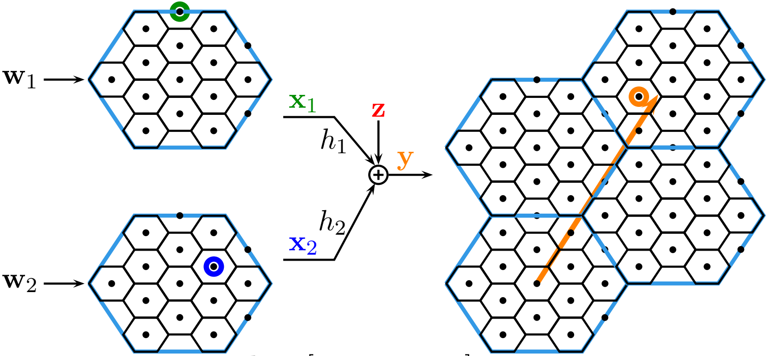
Compute-and-Forward: Illustration

Recover integer linear combination modulo Λ:



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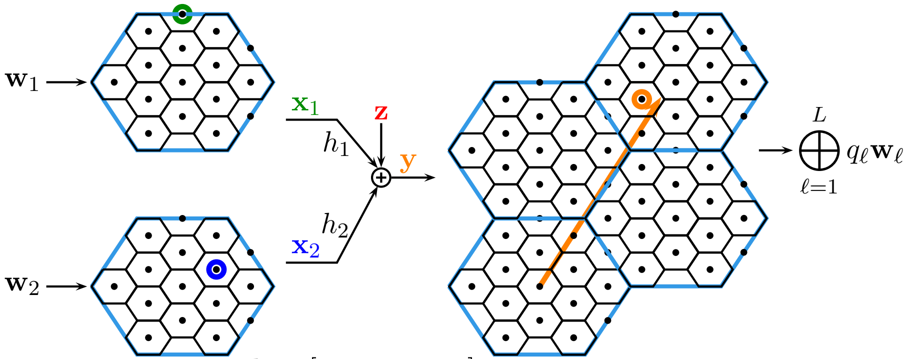
Map back to linear combination of the messages:



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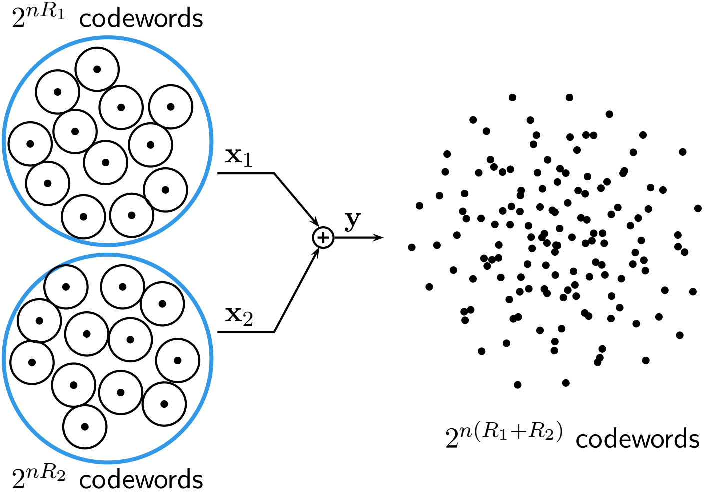

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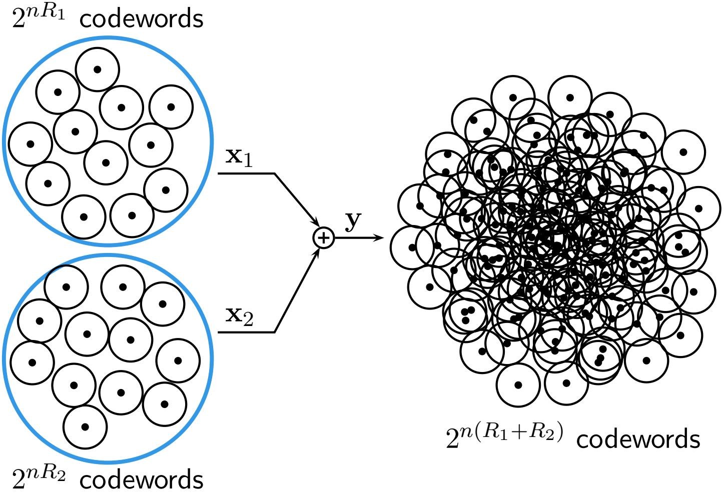
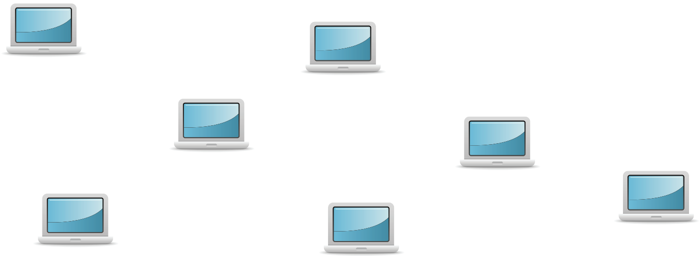


Physical-Layer Network Coding



- Usually fight interference and convert to network of bit pipes.

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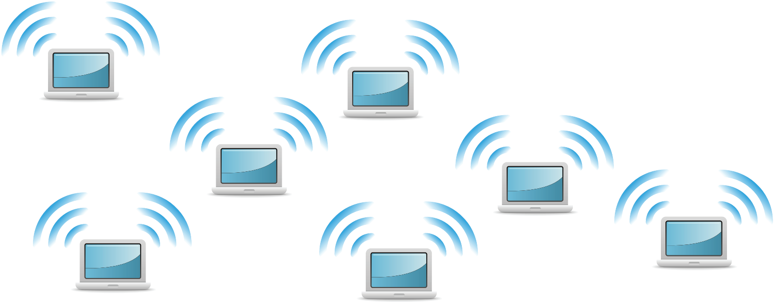

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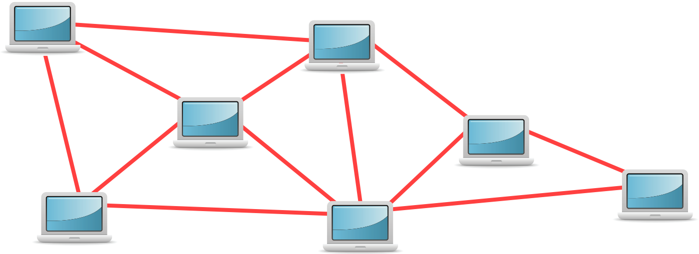

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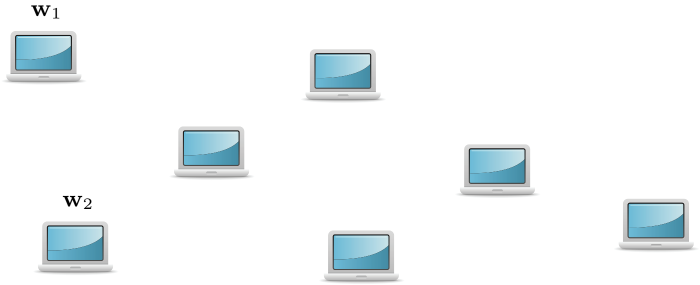

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- Physical-layer network coding: exploiting the wireless medium for network coding. Independently and concurrently proposed by **Zhang-Liew-Lam '06, Popovski-Yomo '06, Nazer-Gastpar '06.**
- Compute-and-forward is an information-theoretic approach.
- Recent surveys: **Liew-Zhang-Lu '11, Nazer-Gastpar Proc. IEEE '11.**

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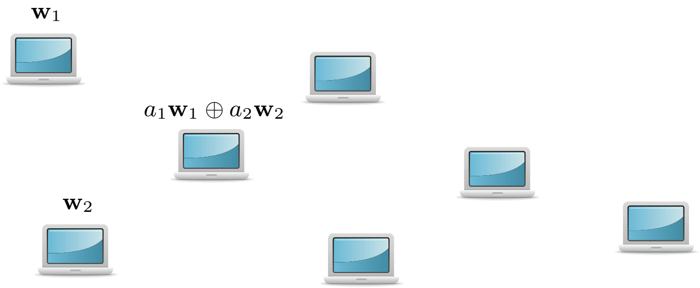

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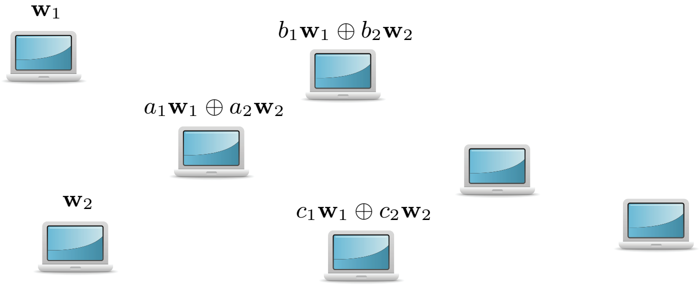

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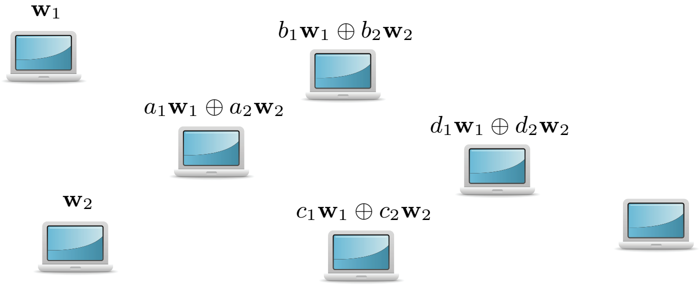

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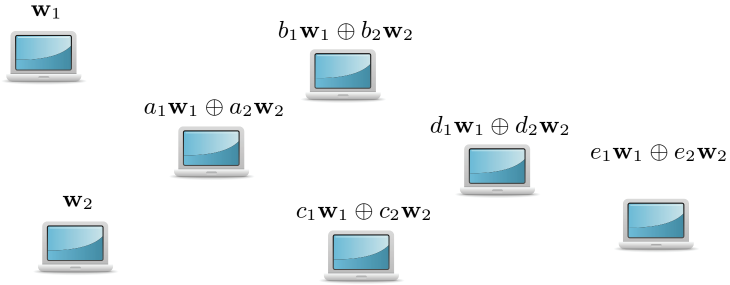

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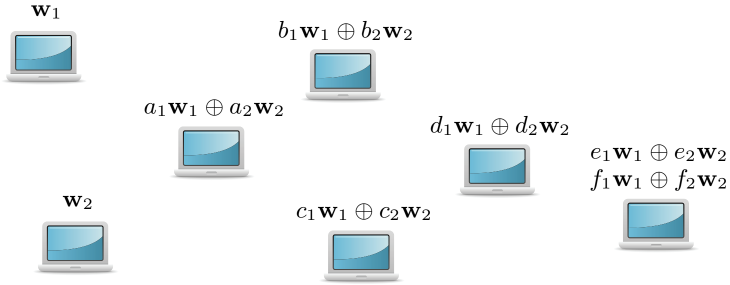

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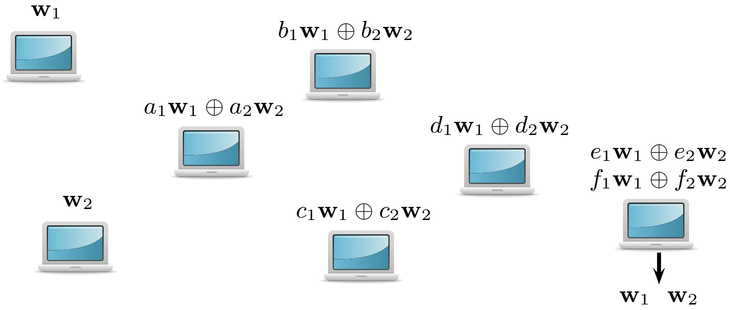

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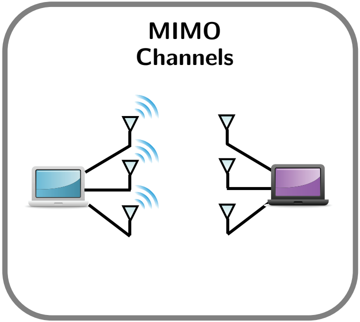
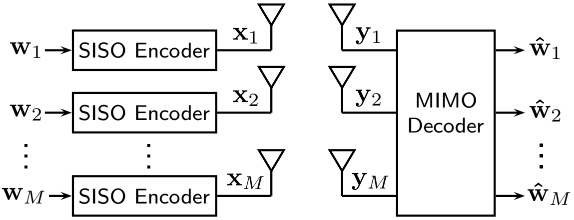


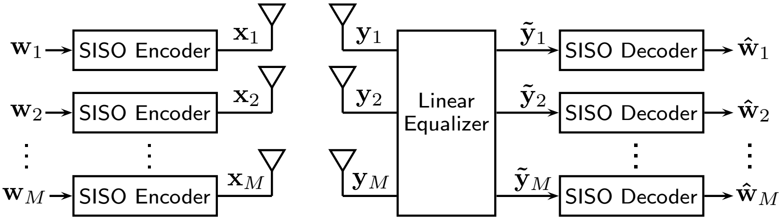
Joint work with Jiening Zhan, Uri Erez, and Michael Gastpar.

MIMO Problem Statement



- Each antenna encodes an independent data stream of rate R (e.g., V-BLAST setting, cellular uplink).
- Channel model: $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$ where $\mathbf{Z}$ is elementwise i.i.d. $\mathcal{N}(0, 1)$.
- Probability of error: $\mathbb{P}\big(\{\hat{\mathbf{w}}_1 \neq \mathbf{w}_1\} \cup \cdots \cup \{\hat{\mathbf{w}}_M \neq \mathbf{w}_M\}\big) < \epsilon$
- Joint maximum likelihood decoding is optimal but has high implementation complexity.

Zero-Forcing Linear Receivers



- Project the received signal, $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$, to eliminate interference between data streams.
- Zero-Forcing: Set $\mathbf{B} = \mathbf{H}^{-1}$ to obtain $\tilde{\mathbf{Y}} = \mathbf{X} + \mathbf{H}^{-1}\mathbf{Z}$.
- Significantly reduces complexity at the expense of performance.

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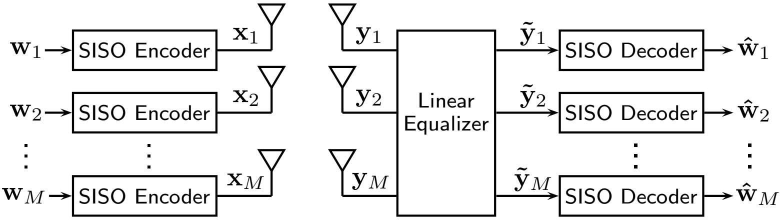

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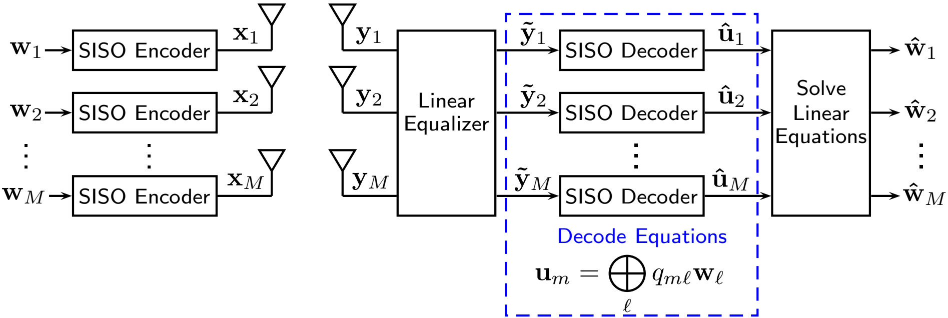

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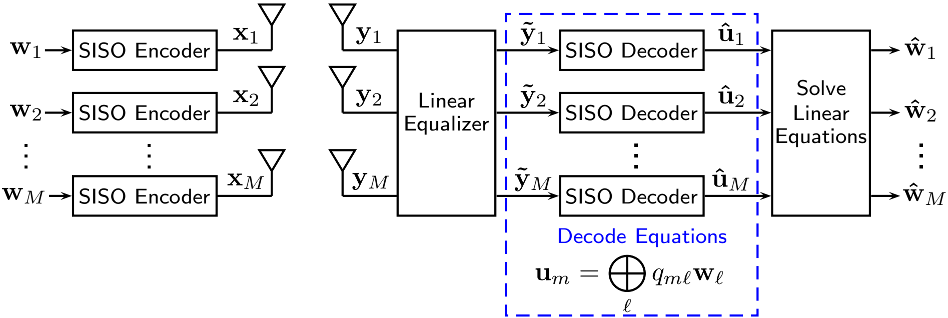

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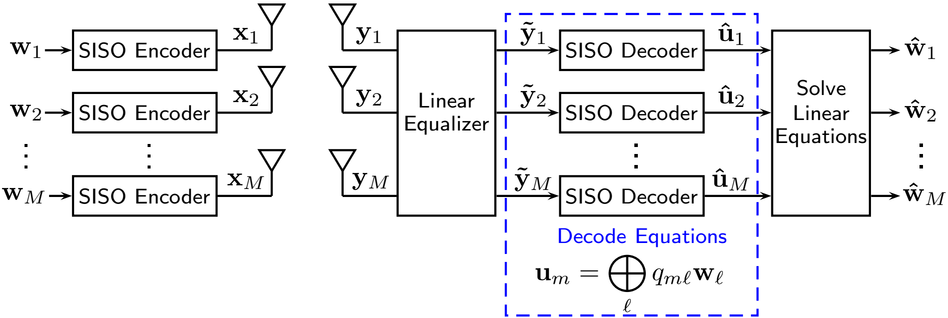

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- Receiver has an M-dimensional observation of each transmitted symbol:

$$\begin{aligned}\mathbf{Y} &= \sum_{m=1}^{M} \mathbf{h}_m \mathbf{x}_m^{\mathsf{T}} + \mathbf{Z} \\ &= \mathbf{H}\mathbf{X} + \mathbf{Z}\end{aligned}$$

MIMO Compute-and-Forward

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Theorem (Zhan-Nazer-Erez-Gastpar '12)

The computation rate region described by

$$R_{comp}(\mathbf{H}, \mathbf{a}) = \max_{\mathbf{b} \in \mathbb{R}^M} \frac{1}{2} \log^+ \left(\frac{P}{\|\mathbf{b}\|^2 + P\|\mathbf{H}^{\mathsf{T}}\mathbf{b} - \mathbf{a}\|^2} \right)$$

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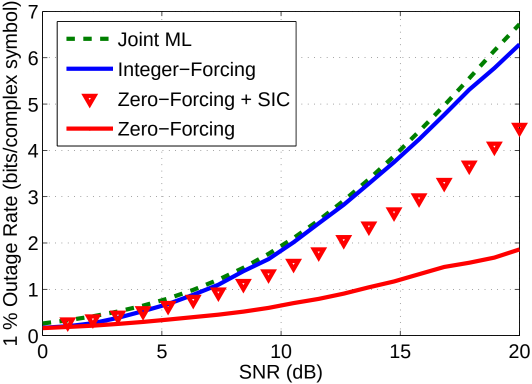
- For a fixed probability ρ, the *outage rate* is defined to be

$$R_{\mathsf{OUT}}(\rho) = \sup \left\{ \mathsf{R} : \Pr\left(R(\mathbf{H}) < \mathsf{R} \right) \leq \rho \right\}.$$



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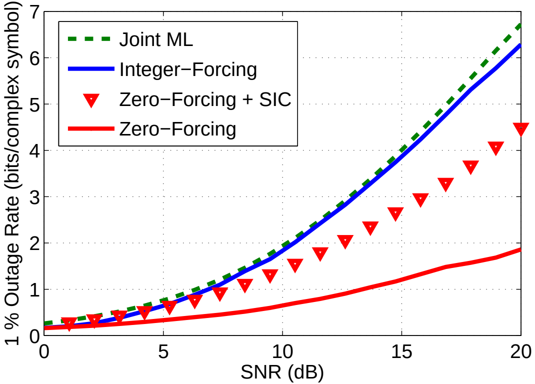
Outage Rates



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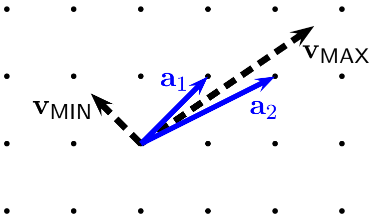
- Integer-forcing beats zero-forcing (even if it is augmented with successive interference cancellation (SIC)).

- Integer-forcing can adapt to the channel by choosing a basis (of integer vectors) close to the maximum singular vector.



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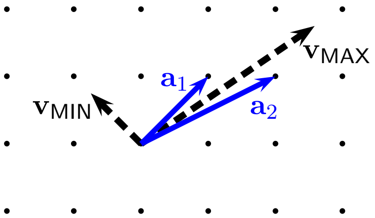

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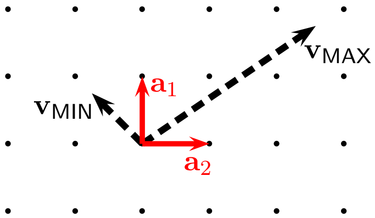

Zero-Forcing

Diversity-Multiplexing Tradeoff

- **Zheng-Tse '03:** A family of codes is said to achieve spatial multiplexing gain r and diversity gain d if the total data rate and the average probability of error satisfy

$$\lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\log \mathsf{SNR}} \geq r$$

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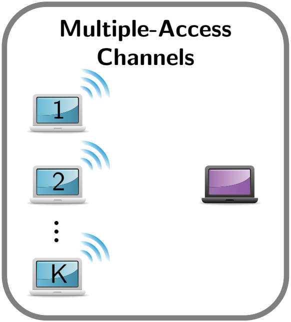
- What about space-time coding at the transmitter?

- **Ordentlich-Erez '13:** Linear dispersion codes + integer-forcing achieves the MIMO capacity universally to within a constant gap. Includes optimal DMT as a special case.

Road Map

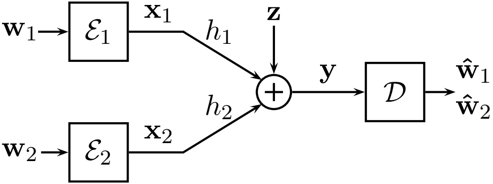
- Compute-and-Forward:
  - Achievability results for Gaussian networks.
  - Proof ideas.

- Applications to communication across single-hop Gaussian networks.



Joint work with Or Ordentlich and Uri Erez.

Gaussian Multiple-Access Channel



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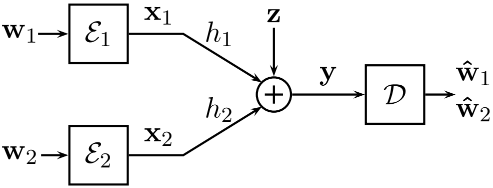

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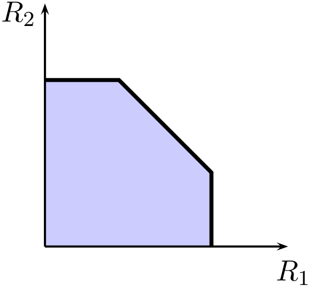

Theorem (Ahlswede '71, Liao '72, Wyner '74, Cover '75)

The capacity region is the set of all rate pairs (R_1, R_2) *satisfying:*

$$R_1 < \frac{1}{2}\log(1 + h_1^2 P) \qquad R_2 < \frac{1}{2}\log(1 + h_2^2 P)$$

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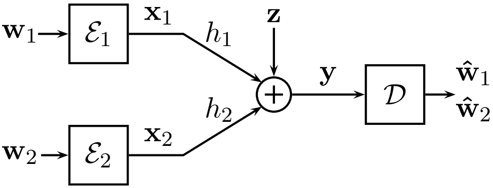

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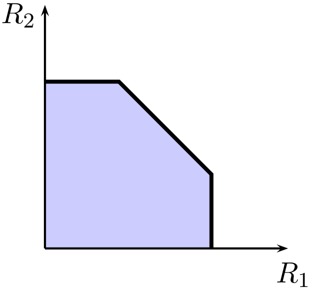

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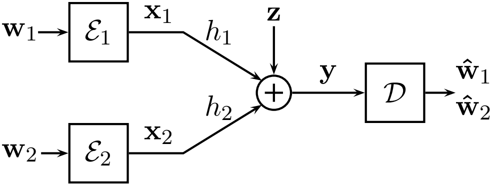
Achievable via joint decoding.

Successive Cancellation



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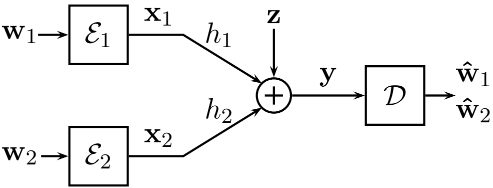

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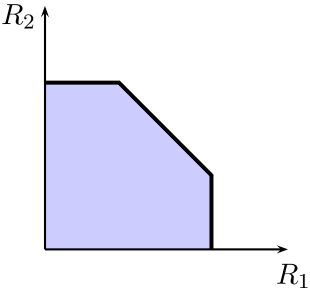

- Treat $\mathbf{x}_2$ as noise and decode $\mathbf{x}_1$, $R_1 < \dfrac{1}{2}\log\left(1 + \dfrac{h_1^2 P}{1 + h_2^2 P}\right)$.

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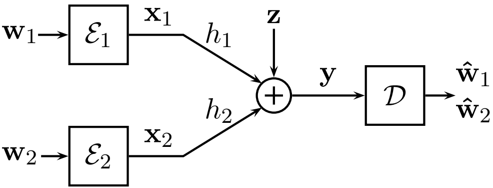

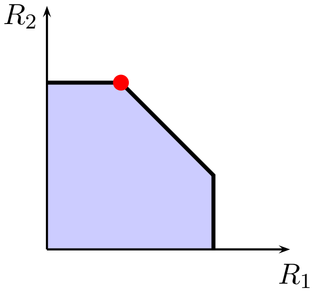

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- Treat $\mathbf{x}_2$ as noise and decode $\mathbf{x}_1$, $R_1 < \frac{1}{2} \log \left(1 + \frac{h_1^2 P}{1 + h_2^2 P}\right)$.
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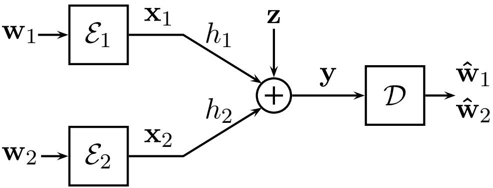
Successive Cancellation



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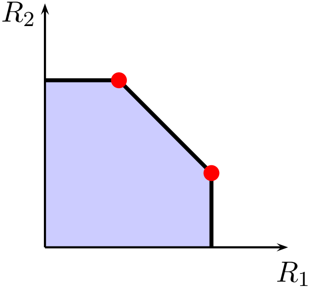

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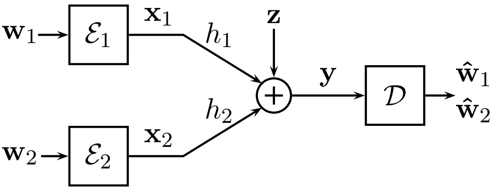

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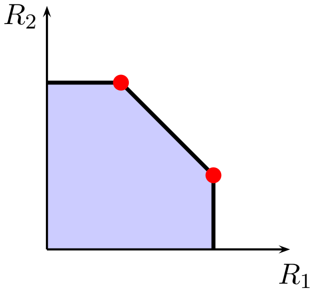

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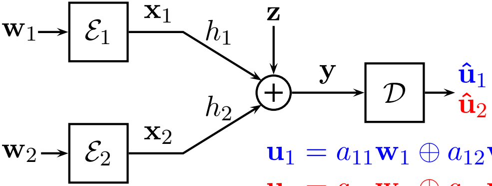
- Switch decoding order for the other corner point.
- Achieves capacity when combined with time-sharing or rate-splitting (**Rimoldi-Urbanke '96**).

Two Linear Combinations



$$\mathbf{u}_1 = a_{11}\mathbf{w}_1 \oplus a_{12}\mathbf{w}_2$$
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- Decode two linearly independent equations.

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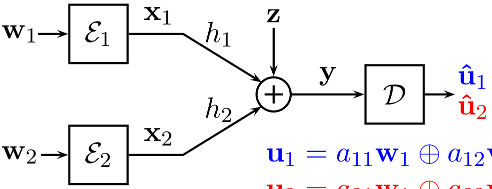

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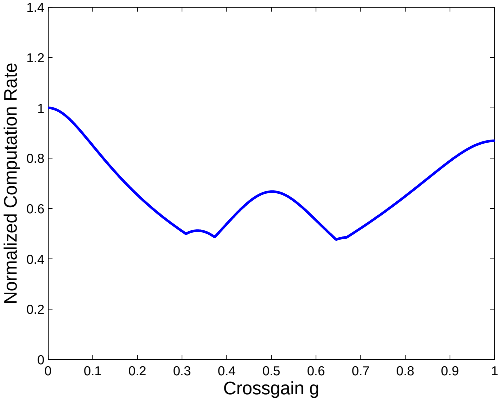

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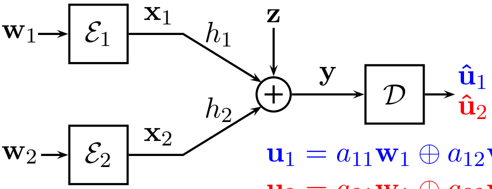

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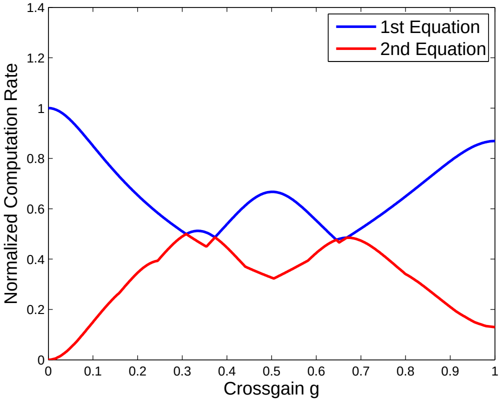

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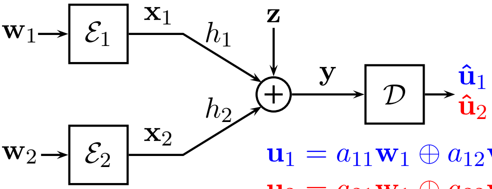

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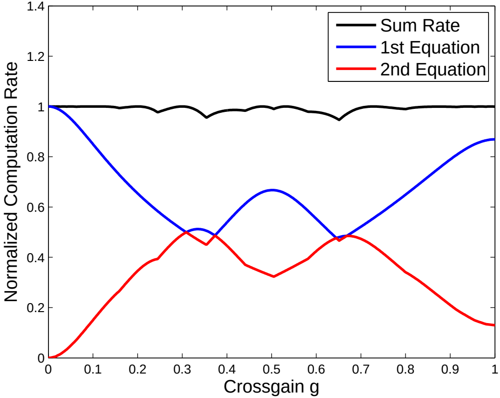

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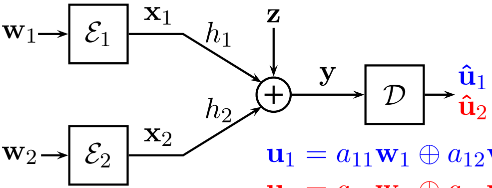

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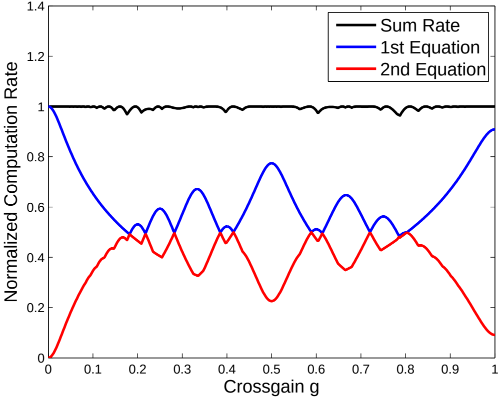

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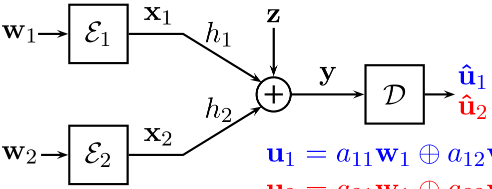

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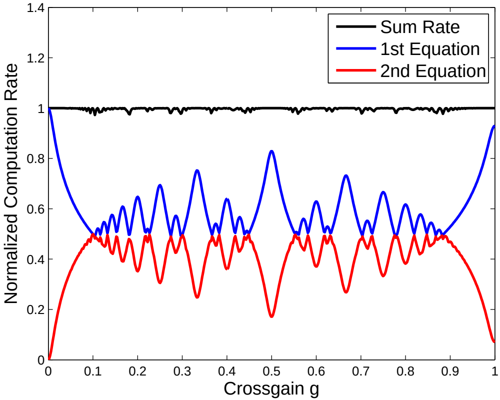


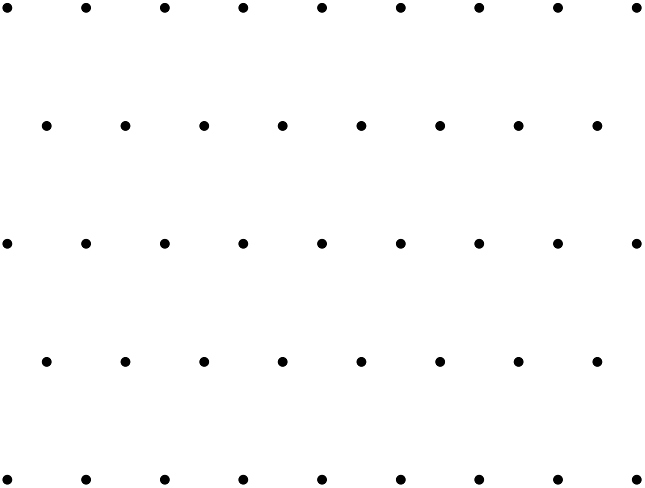
Sum of Computation Rates

- Looks as if the sum of computation rates is nearly equal to the MAC sum capacity. Why is this happening?

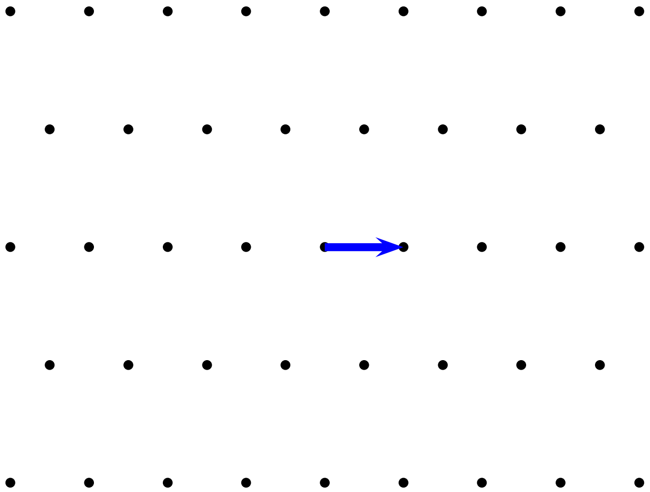
- Let $\mathbf{F} = \left(P^{-1/2}\mathbf{I} + \ \mathbf{h}\mathbf{h}^\mathsf{T}\right)^{-1/2}$. Then, each computation rate can be written as
$$R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_k) = \frac{1}{2}\log^+\left(\frac{P}{\|\mathbf{F}\ \mathbf{a}_k\|^2}\right) .$$

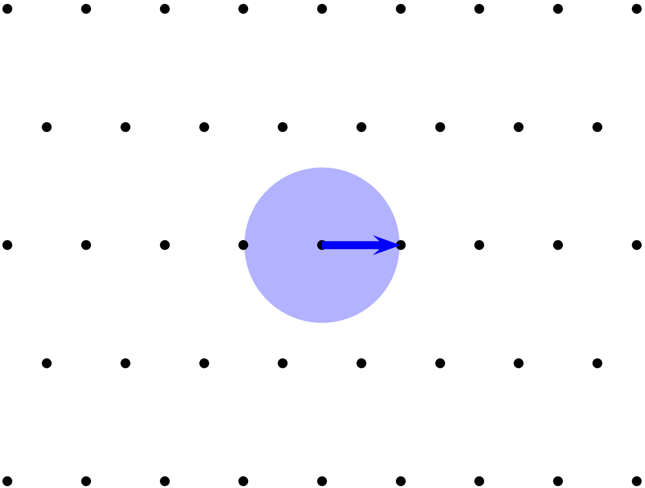
- Thus, decoding the best linear combinations is the same as finding the successive minima $\lambda_k(\mathbf{F})$ for the lattice $\Lambda(\mathbf{F}) = \mathbf{F}\mathbb{Z}^K$:

$$\lambda_k(\mathbf{F}) \triangleq \inf\left\{r : \dim\left(\operatorname{span}\left(\Lambda(\mathbf{F})\ \cap\ \mathcal{B}(\mathbf{0}, r)\right)\right) \geq k\right\} .$$



Successive Minima





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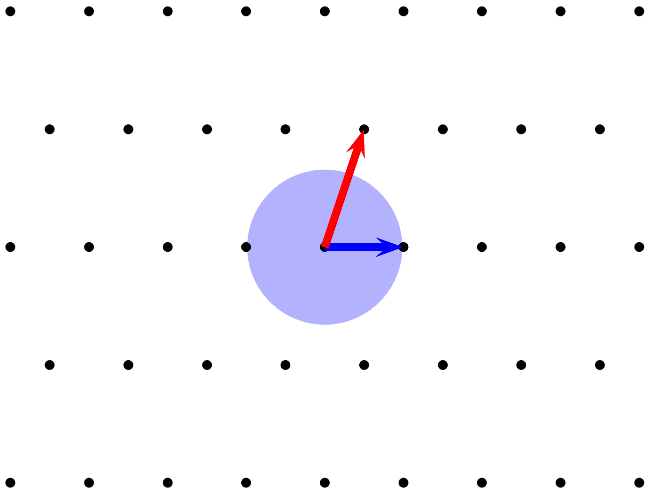

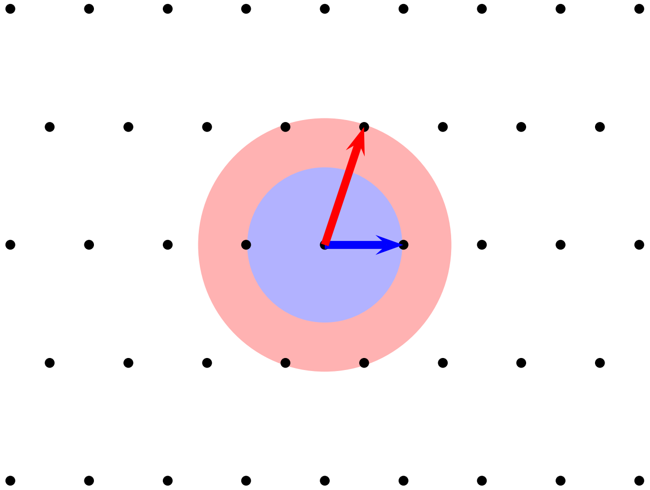

Minkowski's Theorem on Successive Minima

Theorem (Minkowski)

Let $\Lambda(\mathbf{F})$ be a lattice spanned by a full-rank $K \times K$ matrix $\mathbf{F}$. Its successive minima $\lambda_k(\mathbf{F})$ satisfy

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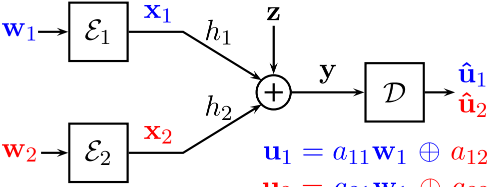
Theorem (Ordentlich-Erez-Nazer '12)

The sum of the K best linearly independent computation rates satisfies

$$\sum_{k=1}^{K} R_{comp}(\mathbf{h}, \mathbf{a}_k) \geq \frac{1}{2} \log(1 + \|\mathbf{h}\|^2 \mathsf{SNR}) - \frac{K}{2} \log K \ .$$

Operational Interpretation: Multiple-Access



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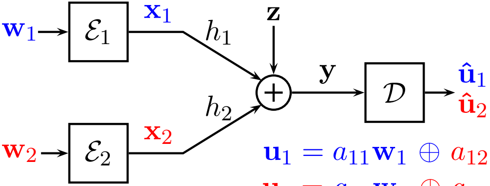

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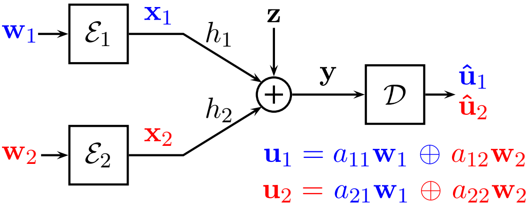

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- Decoding the second equation runs into an issue:

$$\max(R_1, R_2) > R_{\text{comp}}(\mathbf{h}, \mathbf{a}_2) \ .$$

(Algebraic) Successive Cancellation

- After decoding the first equation, the receiver knows

$$\mathbf{v}_1 = \left[a_{11}\mathbf{t}_1 + a_{12}\mathbf{t}_2\right] \mod \Lambda \ .$$

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- Using $\mathbf{v}_1$ we can cancel out $\mathbf{t}_1$ from $\tilde{\mathbf{y}}_2$ without changing the effective noise.

$$\begin{aligned} \tilde{\mathbf{y}}_2^{\mathsf{SI}} &= \Big[\mathbf{s}_2 - b_1\mathbf{v}_1\Big] \bmod \Lambda \\ &= \Big[\big(a_{22} - b_1 a_{12}\big)\mathbf{t}_2 + \mathbf{z}_{\mathsf{effec}}(\mathbf{h}, \mathbf{a}_2)\Big] \bmod \Lambda \ . \end{aligned}$$

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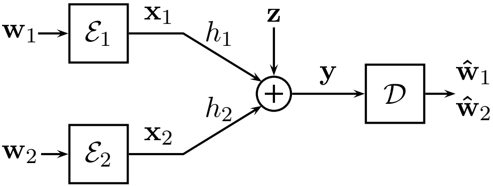
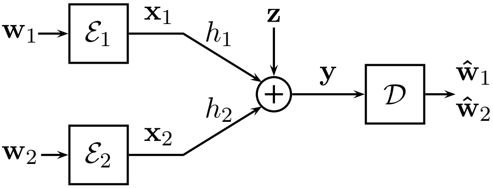
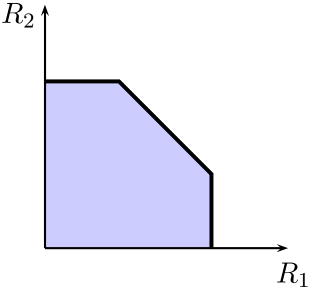
- Now, the receiver can decode since $R_2 < R_{\mathsf{comp}}(\mathbf{h}, \mathbf{a}_2)$.

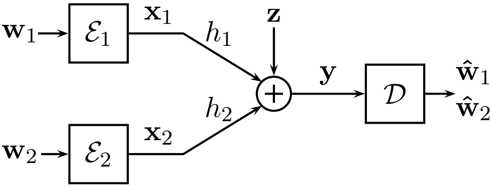
Multiple-Access via Computation

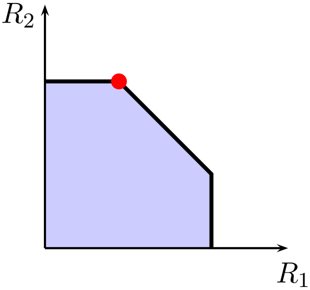






- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.





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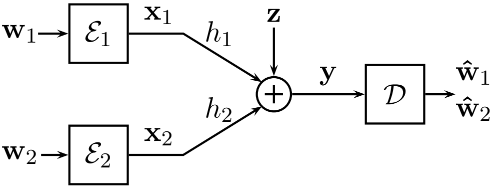

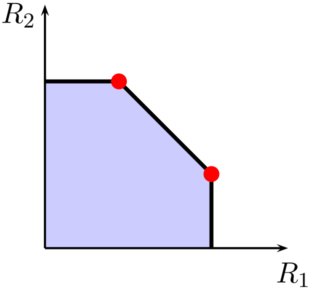

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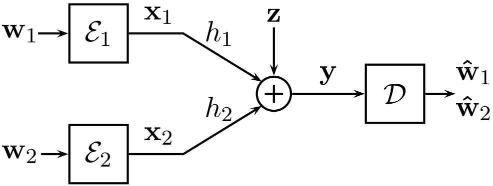
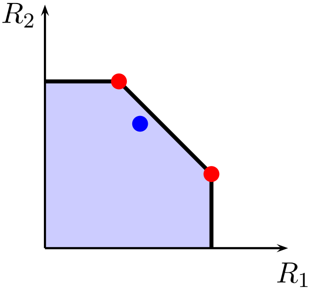
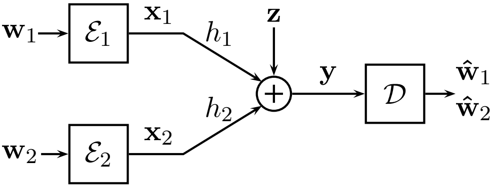
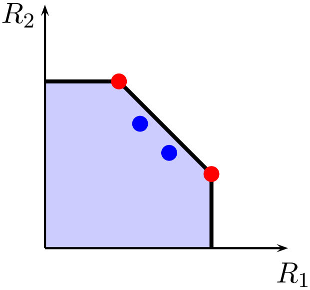




- Successive cancellation (without time-sharing or rate-splitting) achieves corner points.
- Compute-and-forward achieves another set of points near the sum rate boundary. Often closer to the symmetric capacity.





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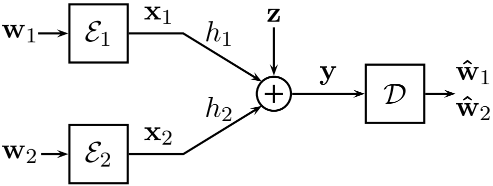

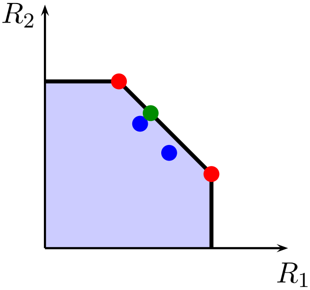

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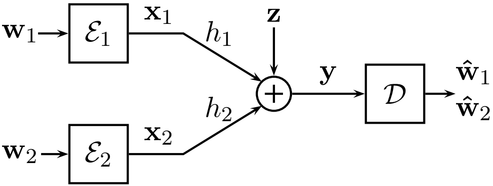

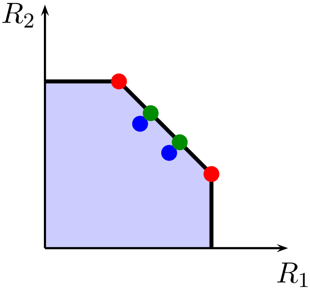

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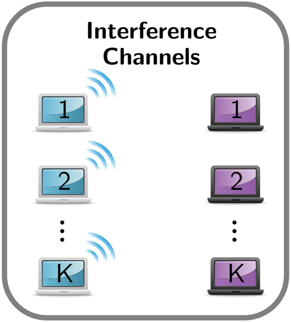
Road Map

- Decoding linear combinations.
  - Achievability results for Gaussian networks.

- Applications to communication across single-hop Gaussian networks.



Joint work with:

Symmetric case: Or Ordentlich and Uri Erez.

Stream-by-stream case: Vasilis Ntranos, Viveck Cadambe, and Giuseppe Caire.

Interference-Free Capacity

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Time Division

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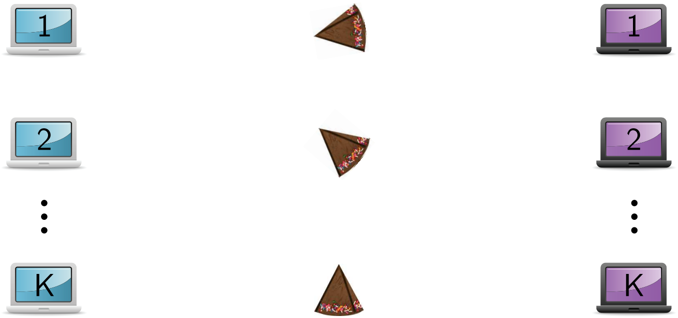

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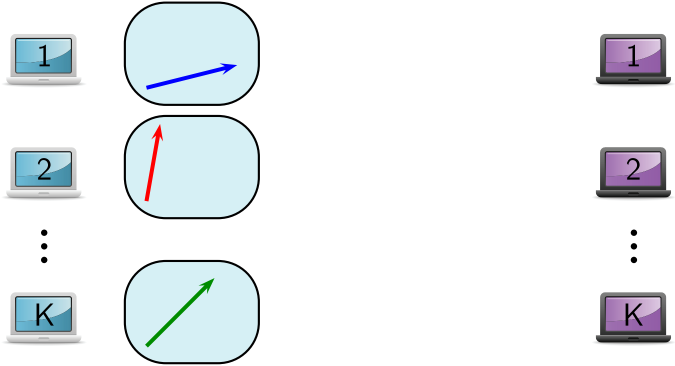

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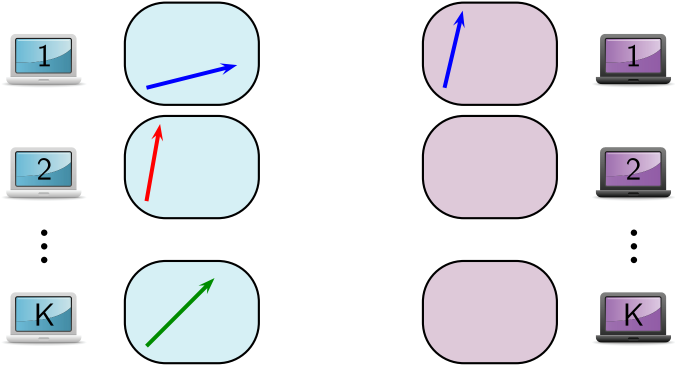

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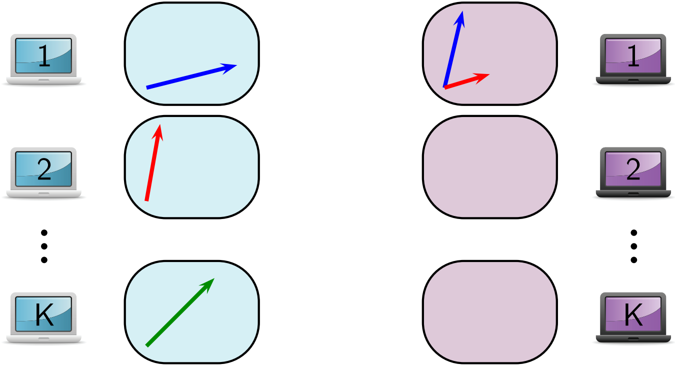

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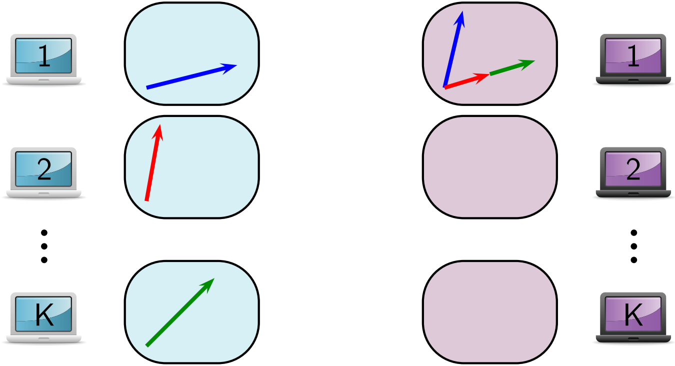

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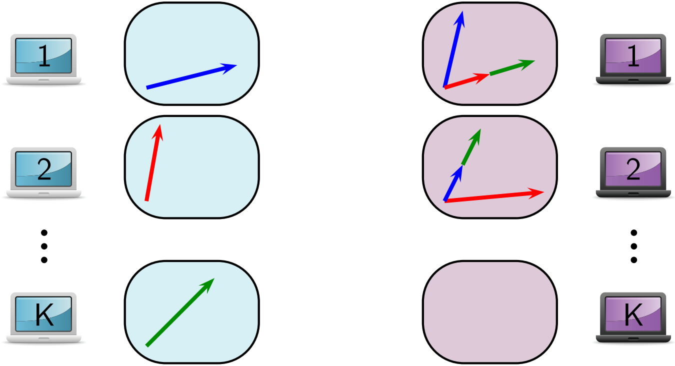

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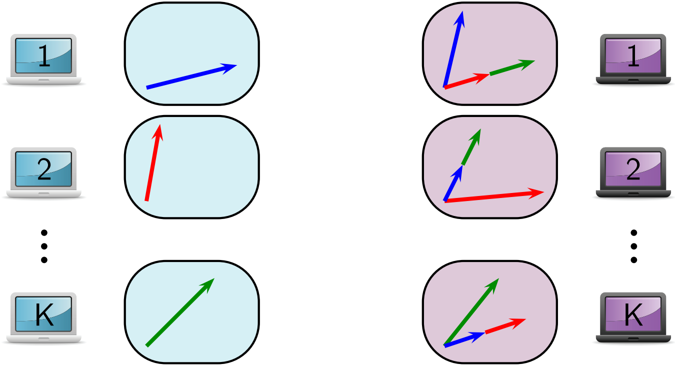

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- Very high SNR:
  - **Motahari, Gharan, Maddah-Ali, Khandani '09:** Real alignment. Achieves $\frac{K}{2}$ DoF over one channel realization using roughly 2^{K^2} codeword layers. Signal scale alignment.

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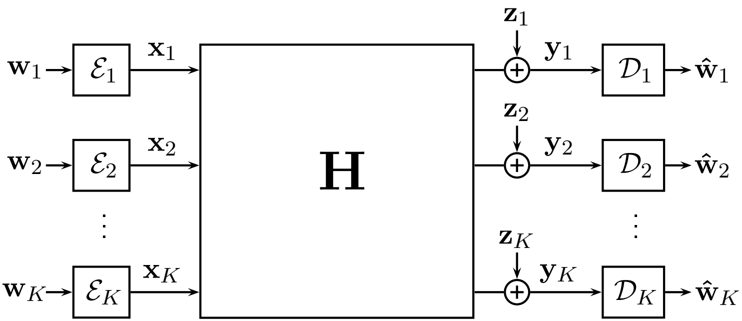
- Yields an achievable rate region for any scenario which employs "stream-by-stream" alignment.
- First, we need to develop a finer understanding of signal scale alignment which so far is only well-understood in the high SNR regime.

Symmetric K-User Gaussian Interference Channel



- Signal space alignment (e.g., beamforming) is infeasible.
- Signal scale alignment attains $K/2$ degrees-of-freedom for almost all channel gains, **Motahari et al. '09, Wu-Shamai-Verdu '11.**
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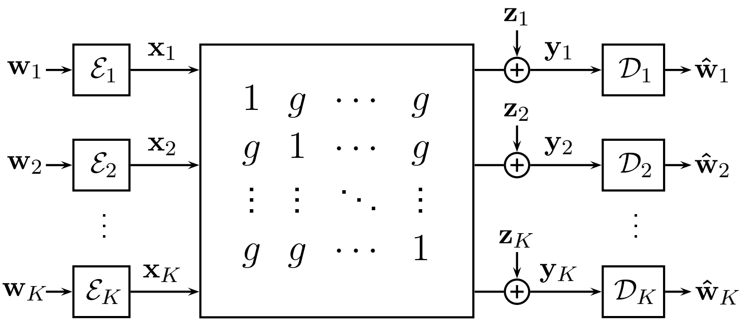

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- We will approximate the sum capacity for the symmetric case.

Effective Multiple-Access Channel

- Lattice codes can enable signal scale alignment.
- Each receiver sees an effective two-user multiple-access channel,

$$\mathbf{y}_k = \mathbf{x}_k + g \sum_{\ell \neq k} \mathbf{x}_\ell + \mathbf{z}_k \ .$$

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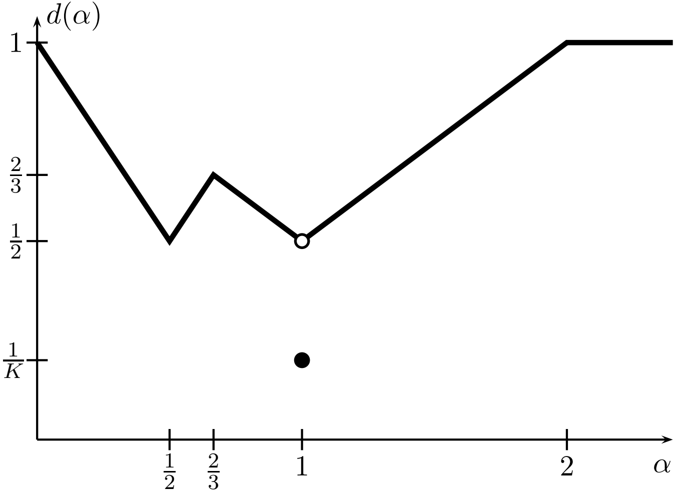
- Only optimal when the interference is very strong, **Sridharan et al. '08**.
- Unfortunately, direct analysis of joint decoding is hindered by dependencies between codeword pairs. Existing work only applies at very high SNR, **Ordentlich-Erez '13.**



- Capacity understood in the high SNR regime. **Jafar-Vishwanath '10.**

$$\alpha = \frac{\log g^2 \mathsf{SNR}}{\log \mathsf{SNR}} \qquad\qquad d(\alpha) = \lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\frac{1}{2}\log \mathsf{SNR}}$$

Alignment via Two Equations

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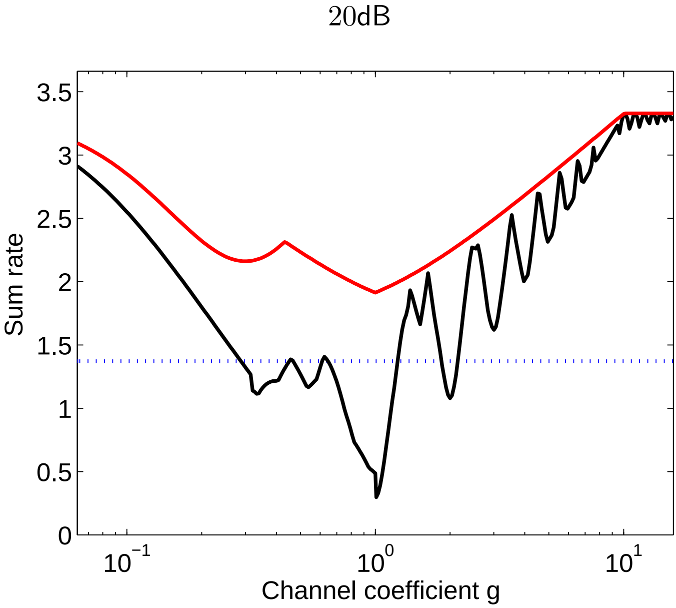
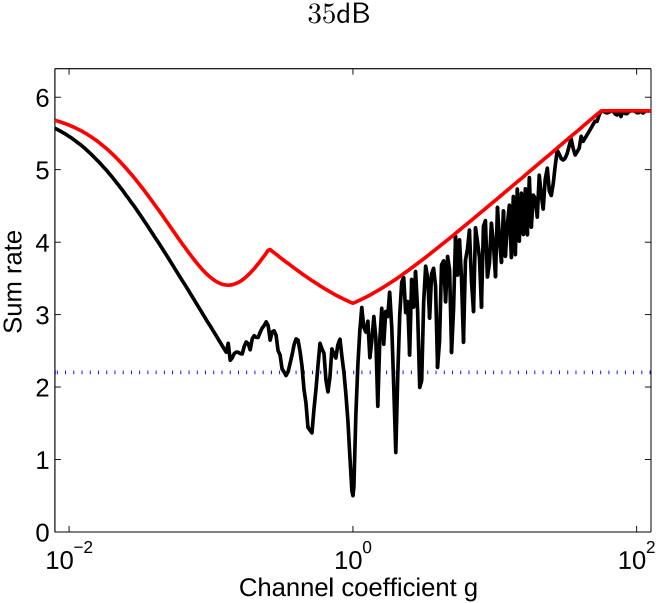
- **Ordentlich-Erez-Nazer '12:** Decode two linear combinations:

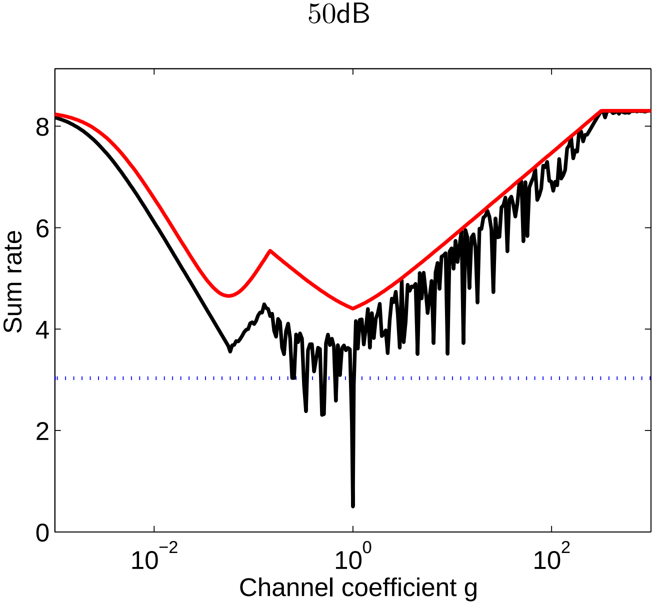
$$a_1 \mathbf{x}_k + a_2 \sum_{\ell \neq k} \mathbf{x}_\ell \qquad\qquad b_1 \mathbf{x}_k + b_2 \sum_{\ell \neq k} \mathbf{x}_\ell \ .$$

- Using the fact that the sum of the computation rates is nearly equal to the multiple-access sum capacity, we can approximate the sum capacity of the symmetric K-user Gaussian interference channel in all regimes.

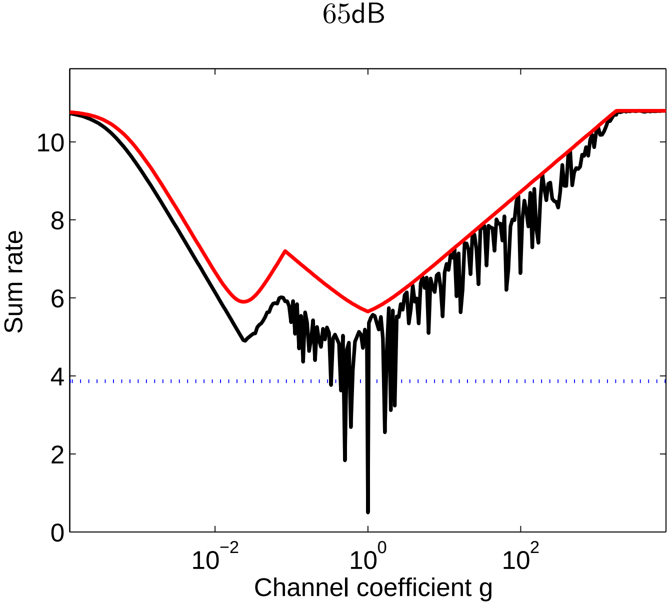
Symmetric K-User Gaussian Interference Channel




35dB
Sum rate
Channel coefficient g
0
1
2
3
4
5
6
10^{-2}
10^{0}
10^{2}



50dB
Sum rate
0
2
4
6
8
10^{-2}
10^{0}
10^{2}
Channel coefficient g



65dB
Sum rate
Channel coefficient g
0
2
4
6
8
10
10^{-2}
10^{0}
10^{2}

Beyond Symmetric Channels

- Would like to combine signal scale alignment (e.g., lattice codes) with signal space alignment (e.g., beamforming vectors).
- **Ntranos-Cadambe-Nazer-Caire '13:** New framework, integer-forcing interference alignment, that can be applied to any scenario with "stream-by-stream" alignment.

Problem Setting:

- Multiple data streams (i.e., codewords) $\mathbf{s}^{[\ell]} \in \mathbb{C}^T$, each assigned to its own beamforming vector $\mathbf{v}^{[\ell]} \in \mathbb{C}^M$.

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$$\mathbf{Y} = \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{D}}^{[\ell]} \mathbf{v}_{\mathrm{D}}^{[\ell]} \big(\mathbf{s}_{\mathrm{D}}^{[\ell]}\big)^{\mathsf{T}} + \sum_{j=1}^{J} \sum_{\ell=1}^{L} \mathbf{H}_{\mathrm{I}}^{[j,\ell]} \mathbf{v}_{\mathrm{I}}^{[j,\ell]} \big(\mathbf{s}_{\mathrm{I}}^{[j,\ell]}\big)^{\mathsf{T}} + \mathbf{Z} \ .$$

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- Different powers and rates allowed across data streams.

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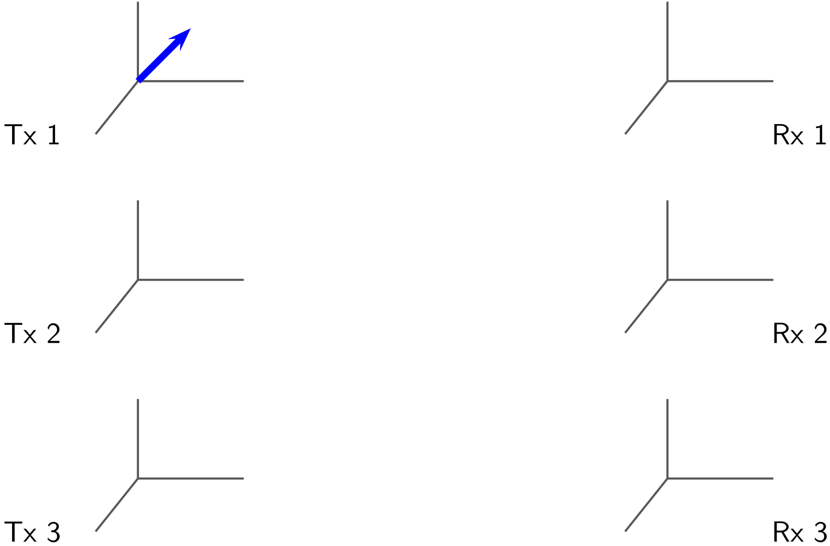

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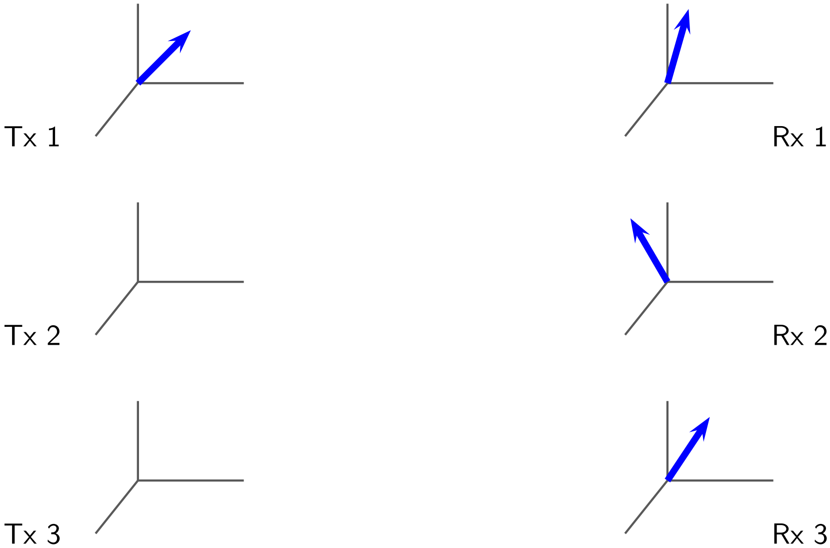

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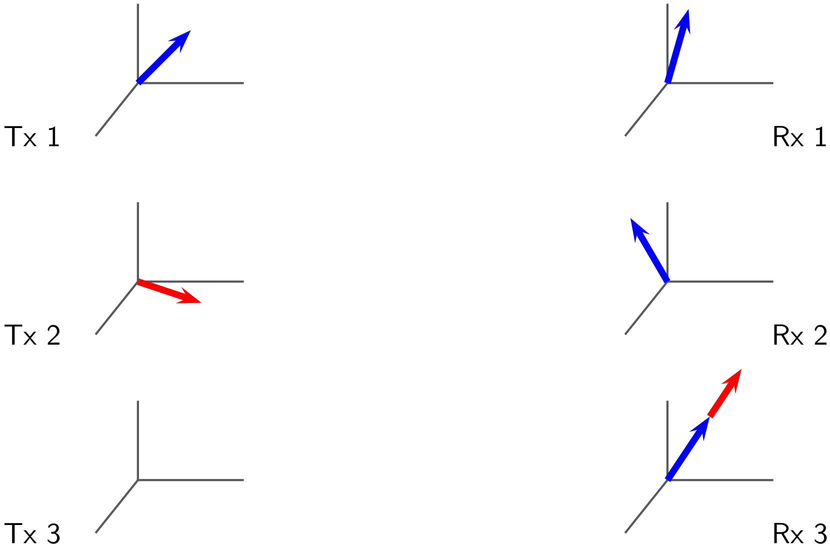

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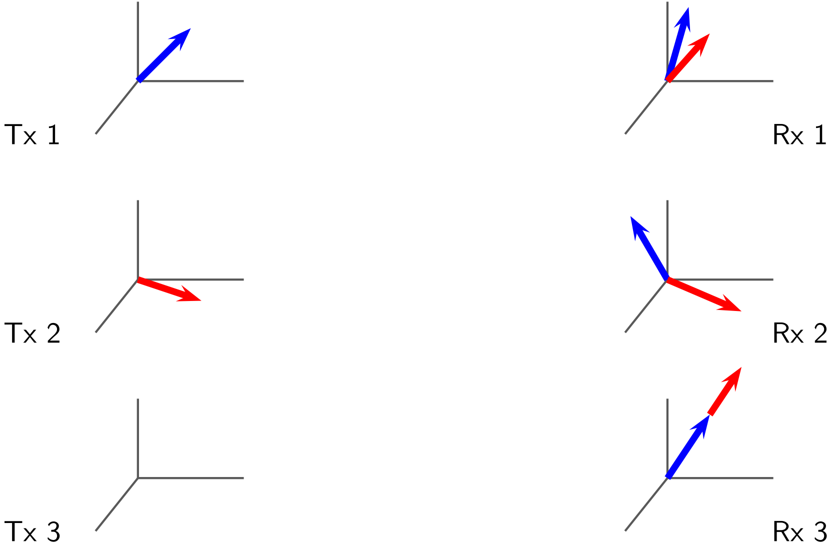

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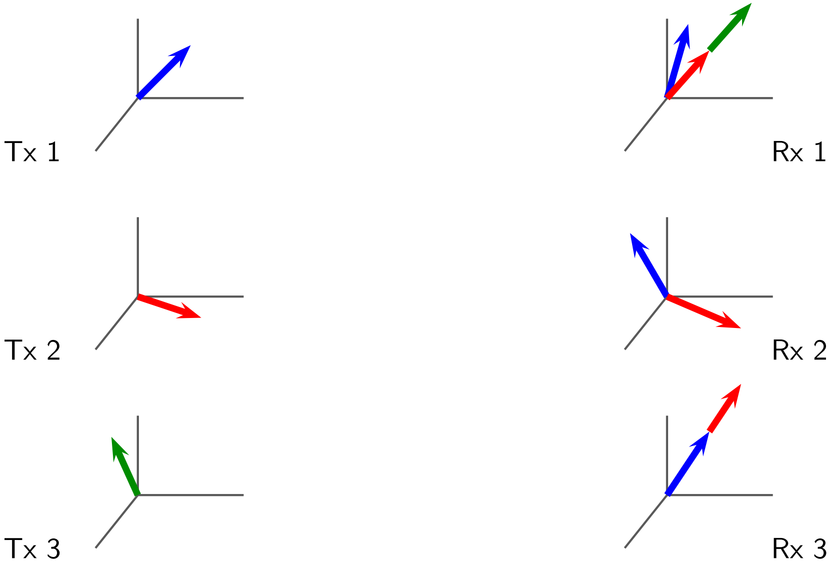

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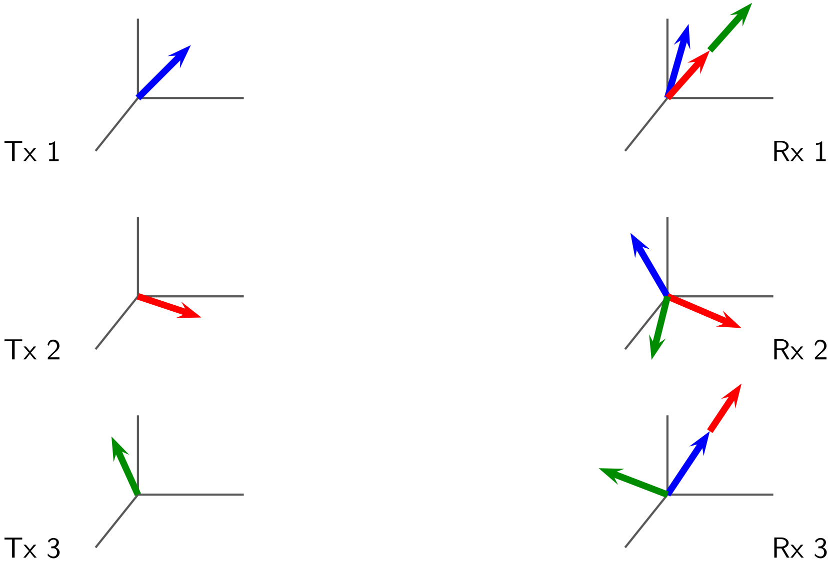

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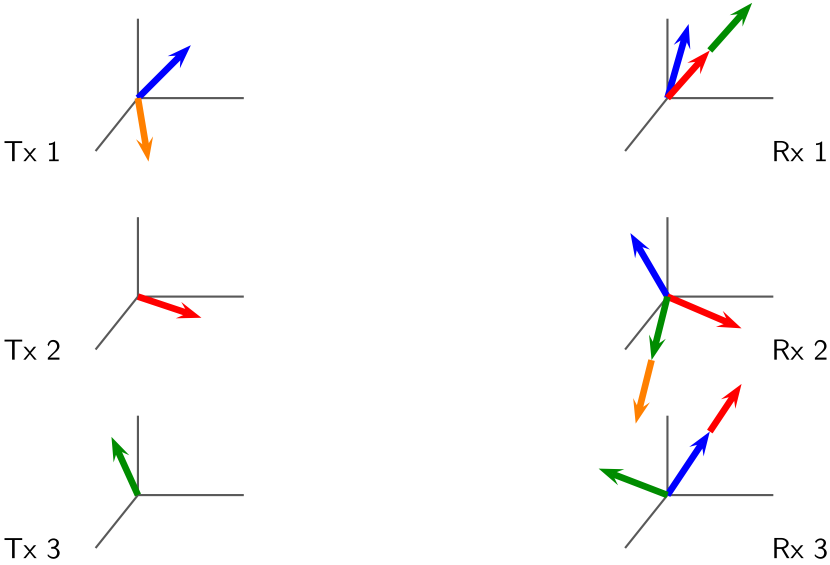

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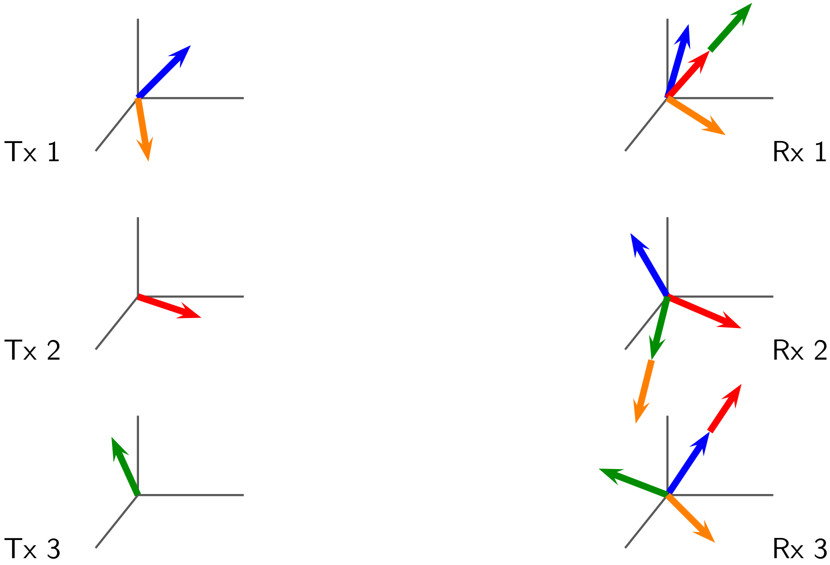

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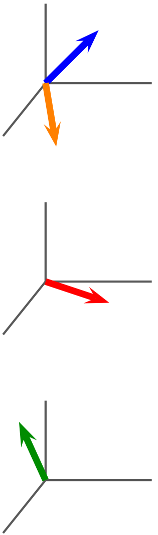


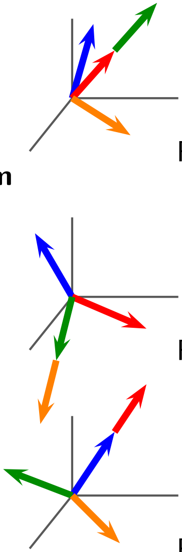
Tx 1

Tx 2

Tx 3

Total Degrees of Freedom

$$\text{DoF} = \frac{4 \text{ vectors}}{3 \text{ channel uses}}$$
$$= \frac{4}{3}$$



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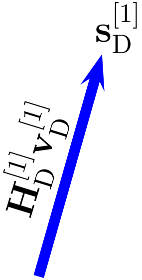
Received Signal



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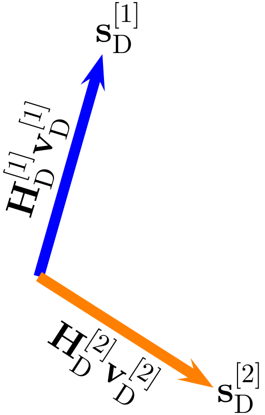

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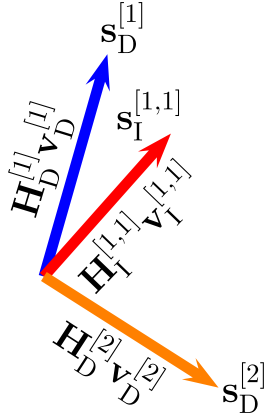

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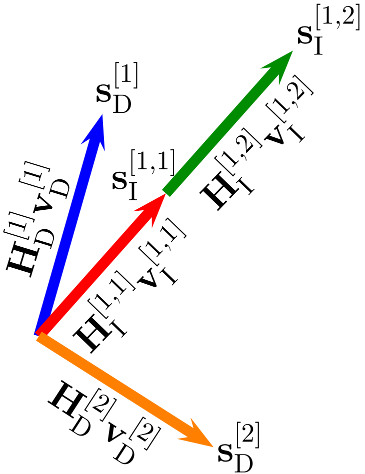

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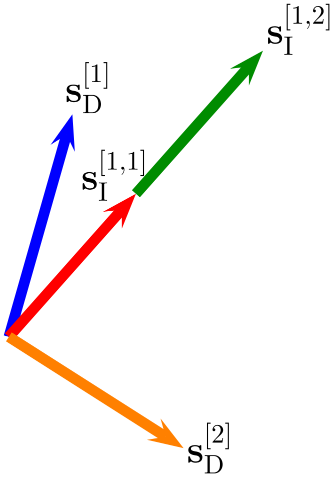

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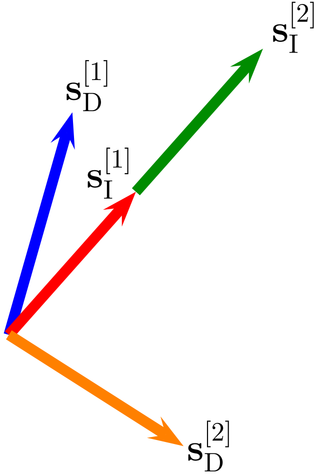
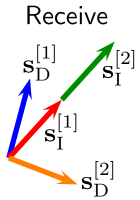


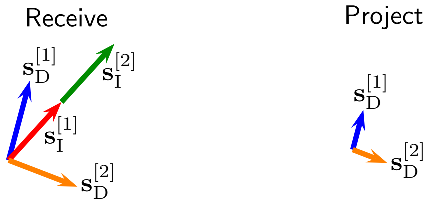


How should each receiver decoder its desired data streams?

Zero-Forcing Interference Alignment:

- Generate the data streams using i.i.d. random coding.

Zero-Forcing Decoding



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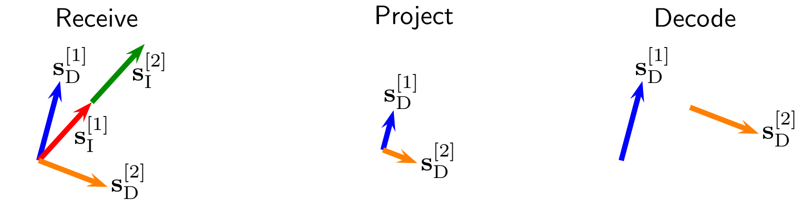

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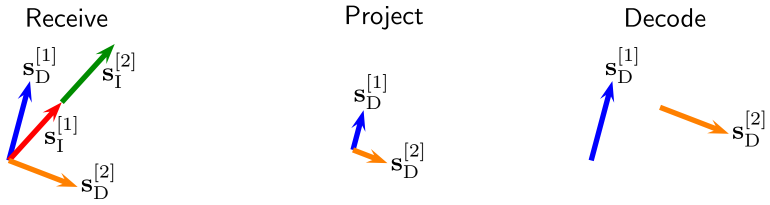

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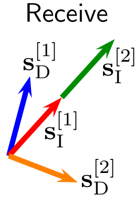
- Suffices from a degrees-of-freedom perspective.

Joint Decoding (with i.i.d Random Codes)



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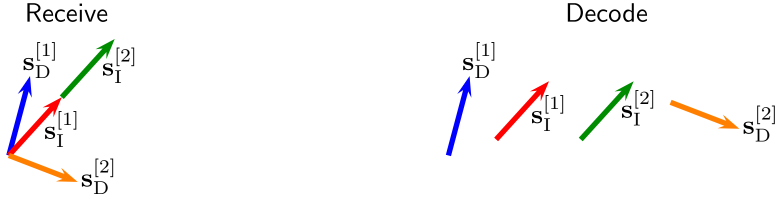

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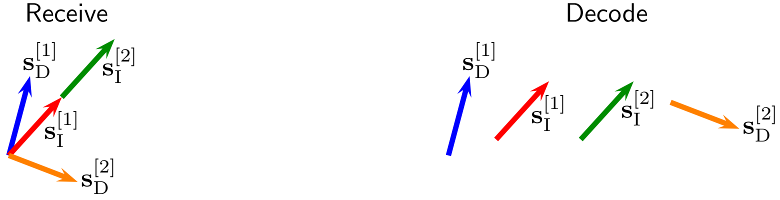

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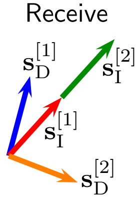
- Analyzing lattice-coded data streams is beyond the reach of current techniques owing to dependencies.



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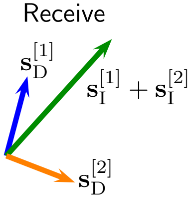

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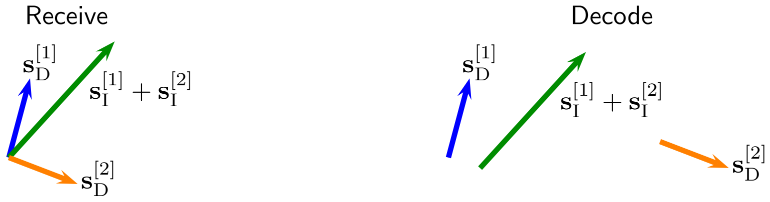

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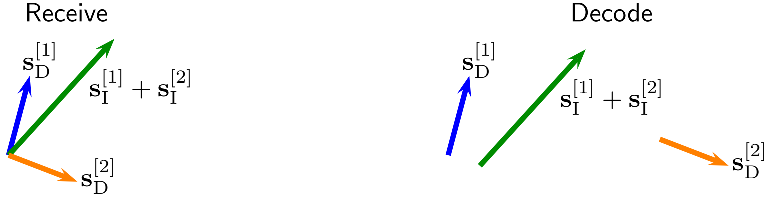
- Data streams are encoded using nested lattice codes according to some power allocation. This induces signal scale alignment.
- Receiver decodes linear combinations and solves for its desired data streams.



How should each receiver decoder its desired data streams?

Integer-Forcing Interference Alignment:

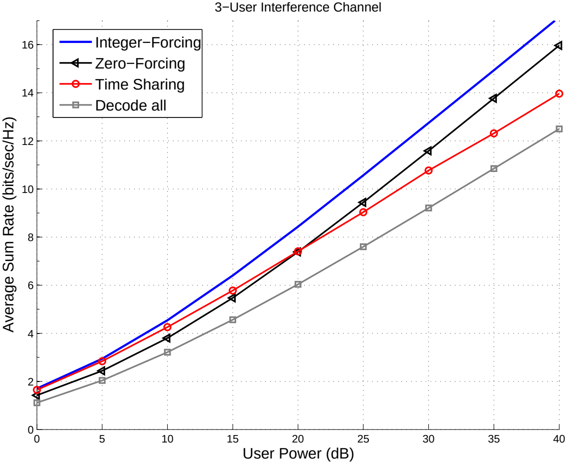
- Beamforming directions chosen to induce signal space alignment.
- Data streams are encoded using nested lattice codes according to some power allocation. This induces signal scale alignment.
- Receiver decodes linear combinations and solves for its desired data streams.
- Requires extension of compute-and-forward to asymmetric powers.

Performance Comparison

- 3-user Gaussian interference channel.
- Can code over 3 independent fading realizations from an i.i.d. Rayleigh distribution.

Strategies:

- CJ '08 Beamforming + Zero-Forcing Decoding.
- CJ '08 Beamforming + Integer-Forcing Decoding.



Recent coding perspectives on compute-and-forward:

- **Feng-Silva-Kschischang '13:** General algebraic framework in terms of lattice partitions and R-modules.
- **Hern-Narayanan '11:** Multilevel binary codes.
- **Ordentlich-Erez '10, Yang et al. '12:** Binary convolutional codes.
- **Hong and Caire '11, Ordentlich et al. '11:** Binary and p-ary LDPC codes.
- **Feng-Silva-Kschischang '11, Belfiore-Ling '12:** Code design criteria.

Many other scenarios where lattice codes can help:

- Two-Way Relaying: **Wilson-Narayanan-Pfister-Sprintson '10**, **Nam-Chung-Lee '10**
- Distributed Source Coding: **Krithivasan-Pradhan '08,'09**, **Wagner '11**, **Tse-Maddah-Ali '10**
- Decentralized Processing: **Sanderovich-Peleg-Shamai '11**, **Nazer-Sanderovich-Gastpar-Shamai '09**, **Hong-Caire '12**
- Distributed Dirty-Paper Coding: **Philosof-Zamir '09**, **Philosof-Zamir-Erez-Khisti '11**, **Wang '12**
- Joint Source-Channel Coding: **Kochman-Zamir '09**, **Nazer-Gastpar '07, '08**, **Soundararajan-Viswanath '09**
- Physical-Layer Secrecy: **He-Yener '09**, **Agrawal-Vishwanath '09**
- MIMO Broadcast/Downlink: **Hong-Caire '12**

Concluding Remarks

- Even if you only want to recover messages, it can help to first decode linear combinations.
- Compute-and-forward creates a direct link between Gaussian interference networks and finite field ones.
- Enables more efficient encoding/decoding for networks where the capacity is already known.
- Yields new achievable rates for interference channels.
- Broader story: Algebraic Structure in Network Information Theory. ISIT '11 Tutorial. Survey on physical-layer network coding in Proceedings of the IEEE, March 2011.