pstat274_hw02_aoxu

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Problem 1

E is correct, data set I and II exhibit statistically significant autocorrelations since k $\neq$ 0 and $\rho > 1$.

Problem 2

(a)

It's stationary but not invertible.

It follows MA(2) Module, so it's stationary.

From R, we could get B_1 = 1, B_2 = -3; Since B_1 = 1 lies on the unit circle, it is not invertible.

```
polyroot(c(1,-2/3,-1/3))
```

```
## [1]  1+0i -3+0i
```

(b)

It's not stationary but invertible.

It's AR(2) module, and it could converts to (1 - $\frac{2B}{3}$ - $\frac{B^2}{3}$) X_t = Z_t. It's invertible since we could write it as the form of "Z_t = ϕBX_t".

From R, we could get B_1 = 1, B_2 = -3; Since B_1 = 1 lies on the unit circle, it is not stationary.

```
polyroot(c(1,-2/3,-1/3))
```

```
## [1]  1+0i -3+0i
```

Problem 3

(a)

MA(3), θ_1 = 2, θ_2 = 0.5, θ_3 = -0.1

1. The mathematical equation for MA(3) model: X_t = Z_t + 2Z_{t-1} + 0.5Z_{t-2} - 0.1Z_{t-3}

2. By using the formula “$\rho_x(k) = \frac{\theta_k + \theta_1\theta_{k+1} + ... + \theta_{q-k}\theta_k}{1+\theta_1^2+...+\theta_q^2}$), k = 1,2,…,q, $\rho_x(k)$ = 0, k > q”, we get

$\rho(1) = \frac{\theta_1+\theta_1\theta_2+\theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \frac{2+2*0.5+0.5(-0.1)}{1+2^2+0.5^2+(-0.1)^2}$ = 0.56083650

$\rho(2) = \frac{\theta_2+\theta_1\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \frac{0.5+2(-0.1)}{1+2^2+0.5^2+(-0.1)^2}$ = 0.05703422

$\rho(3) = \frac{theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \frac{-0.1}{1+2^2+0.5^2+(-0.1)^2}$ = -0.01901141

$\rho(4)$ = 0

```
ARMAacf(ma = c(2, 0.5, -0.1), lag.max = 4, pacf = FALSE)
```

```
##          0          1          2           3          4
## 1.00000000 0.56083650 0.05703422 -0.01901141 0.00000000
```

(b)

AR(1) ϕ_1 = -0.5

1. the mathematical equation for AR(1) model: X_t = -0.5X_{t-1} + Z_t

2.

$\rho(1) = \phi_1$ = -0.5

$\rho(2) = {\phi_1}^2$ = (-0.5)^2 = 0.25

$\rho(3) = {\phi_1}^3$ = (-0.5)^3 = -0.125

$\rho(4) = {\phi_1}^4$ = (-0.5)^4 = 0.0625

```
ARMAacf(ar = -0.5, lag.max = 4, pacf = FALSE)
```

```
##      0       1      2       3      4
## 1.0000 -0.5000 0.2500 -0.1250 0.0625
```

Problem 4

Since X_t = 3 + Y + Z, Y is a mean zero random variable with variance σ_v^2, independent of the white noise Z_t, then we could get

E(X_t) = E(3 + Y + Z_t) = 3 + E(Y) + E(Z_t) = 3 + 0 + 0 = 3

Var(X_t) = Var(3 + Y + Z_t) = 0 + Var(Y) + Var(Z_t) + 2Cov(3,Y) + 2Cov(3,Z_t) + 2Cov(Y,Z_t) = $\sigma_Y^2 + \sigma_Z^2$

$\gamma(X_t, X_{t+k})$ =Cov(X_t,X_{t+k}) = E($X_t X_{t+k}$) - E(X_t)E(X_{t+k}) = E(9 + 3Y + 3Z_{t+k} + 3Y + Y^2 + YZ_{t+k} + 3Z_t + YZ_t + $Z_t Z_{t+k}$) - (3 + E(Y) + E(Z_t))(3 + E(Y) + E(Z_{t+k})) = 9 + σ_Y^2 - 9 = σ_Y^2

When k = 0, Cov(X_t,X_{t+k}) = E(X_t^2) - E(X_t)^2 = Var(X_t) = $\sigma_Y^2 + \sigma_Z^2$

$$\gamma_x(k) = \begin{cases} \sigma_Y^2 & k \neq 0 \\ \sigma_Y^2 + \sigma_Z^2 & k = 0 \end{cases}$$

Therefore, X_t is stationary since μ is constant and $\sigma_x(k)$ doesn't depend on t.

Autocovariance function: $\gamma(X_t, X_{t+k})$ = $\rho_x(k) = \begin{cases} \sigma_Y^2 + \sigma_Z^2 & k \neq 0 \\ \sigma_Y^2 & k = 0 \end{cases}$

Autocorrelation function:

$\rho_x(t, t+k) = Cov(X_t, X_{t+k}) / \sqrt{Var(X_t)Var(X_{t+k})} = \begin{cases} \sigma_Y^2/(\sigma_Y^2 + \sigma_Z^2) & k \neq 0 \\ (\sigma_Y^2 + \sigma_Z^2)/(\sigma_Y^2 + \sigma_Z^2) = 1 & k = 0 \end{cases}$

Problem 5

X_t = Z_t + 2Z_{t-1} - 8Z_{t-2}

(a)

MA(2), q = 2

(b)

The model is stationary but not invertible.

Since MA(2) is always stationary, the model in the problem is stationary.

$\theta(z)$ = 1 + θ_1z+ $\theta_2 z^2$

Since both roots are inside of the unit circle, the model is not invertible.

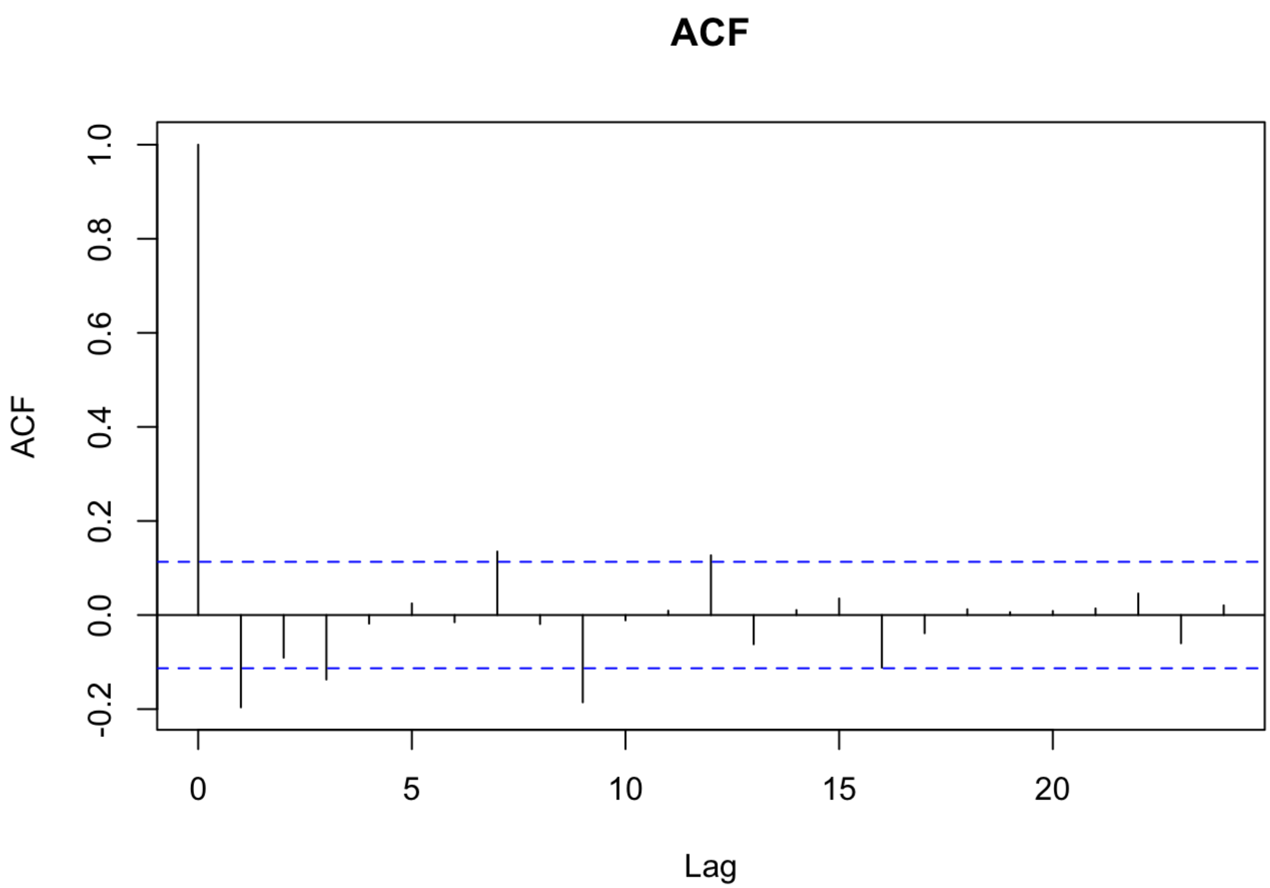
```
polyroot(c(1,2,-8))
```

```
## [1] -0.25+0i  0.50+0i
```

(c)

$\rho_x(2)$ = $\frac{\theta_2}{1+\theta_1^2+\theta_2^2}$ = $\frac{-8}{1+2^2+(-8)^2}$ = $-\frac{8}{69}$

```
xt <- arima.sim(list(ma=c(1,2,-8)),n=300)
acf(xt, main="ACF")
acf(xt, main="ACF")$acf[3]
```



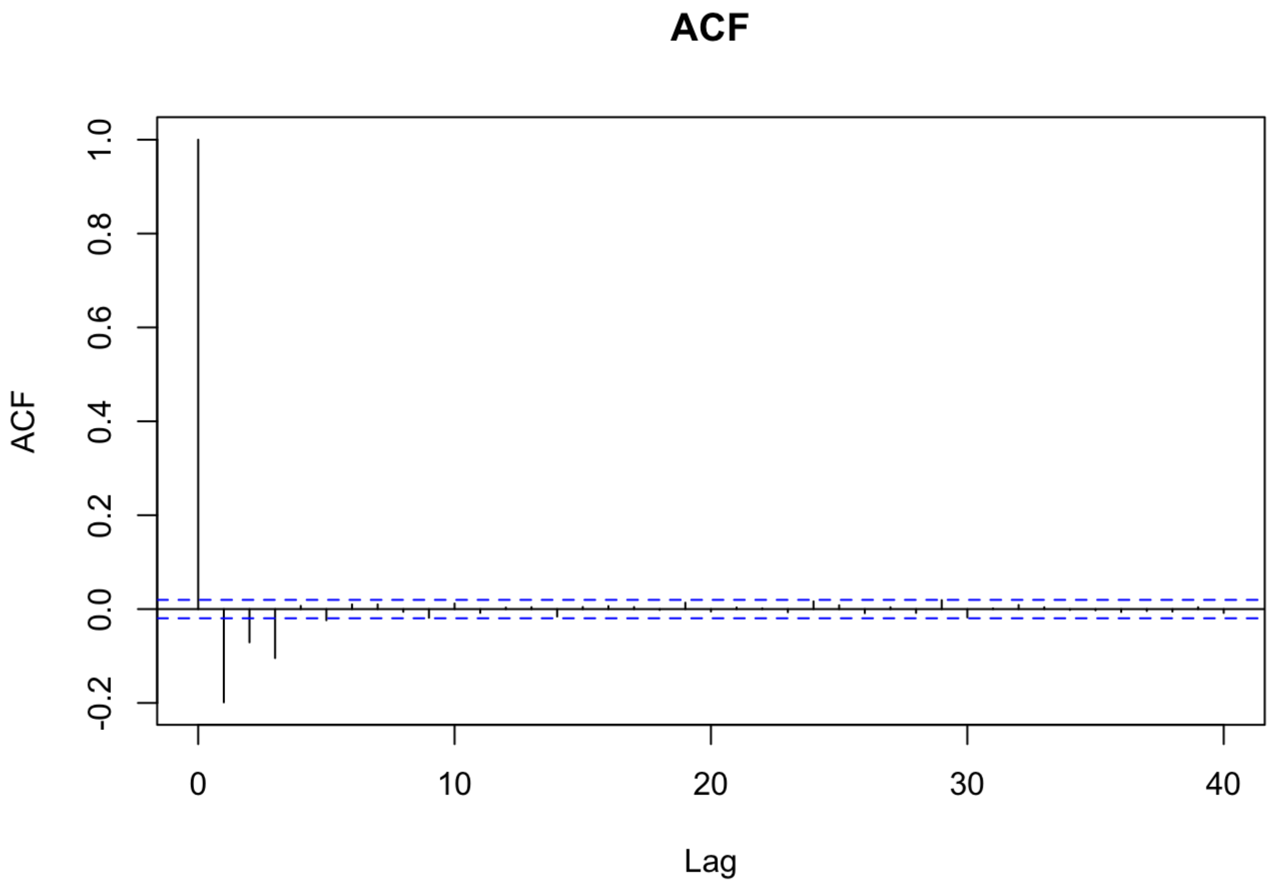

```
## [1] -0.09069785
```

```
-8/69
```

```
## [1] -0.115942
```

```
# Since -0.1424422 is nearly the same as -0.115942, then my sample estimate of ρX(2) are nearly the same as its true value found by calculations.
```

```
xt <- arima.sim(list(ma=c(1,2,-8)),n=10000)
acf(xt, main="ACF")
acf(xt, main="ACF")$acf[3]
```




```
## [1] -0.0708146
```

G1

$X_t = Z_t + \theta Z_{t-2}$

(a)

E($X_t X_{t+k}$) = E((Z_t+θ)($Z_{t+k} Z_{t-2+k}$)) = E($Z_t Z_{t+k}$) + θE($Z_{t-2} Z_{t+k-2}$) + θE($Z_{t-2} Z_{t+k}$) + θ^2 E($Z_{t-2} Z_{t+k-2}$)

$\gamma_x(t, t+k)$ = E($X_t X_{t+k}$) - E(X_t)E(X_{t+k}); When k = 0, $\gamma_x(t, t+k)$ = 1 + θ^2; When k = ± 2, $\gamma_x(t, t+k)$ = θ; When k is other numbers, $\gamma_x(t, t+k)$ = 0.

Therefore, $\rho_x(k)$ = 1 when k = 0; $\rho_x(k)$ = $\frac{\theta}{1+\theta^2}$ when k = ± 2; $\rho_x(k)$ = 0 when k is other numbers.

When θ = 0.8,

Autocovariance function: $\gamma_x(t, t+k)$ = 1.64 when k = 0; $\gamma_x(t, t+k)$ = 0,8 when k = ± 2; $\gamma_x(t, t+k)$ = 0 when k is other numbers.

Autocorrelation function: $\rho_x(k)$ = 1 when k = 0; $\rho_x(k)$ = $\frac{20}{41}$ when k = ± 2; $\rho_x(k)$ = 0 when k is other numbers.

(b)

When θ = 0.8,

Var($\frac{X_1+X_2+X_3+X_4}{4}$) = $\frac{Var(X_1+X_2+X_3+X_4)}{16}$ = $\frac{1}{16}$(($Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4)$ + 2(Cov(X_1X_2)+Cov(X_1X_3)+Cov(X_1X_4)+Cov(X_2X_3)+Cov(X_2X_4)+Cov(X_3X_4))) = $\frac{1}{16}$(4Var(X_t)+4θ) = $\frac{1}{16}$(4($Var(Z_t)$+$\theta^2 Var(Z_{t-2})$+Cov(Z_t, Z_{t-2}))+4θ) = $\frac{1}{16}$(4(1+θ^2)+4θ) = $\frac{1}{16}$(4*1.64+3.2) = 0.61

(c)

When θ = -0.8,

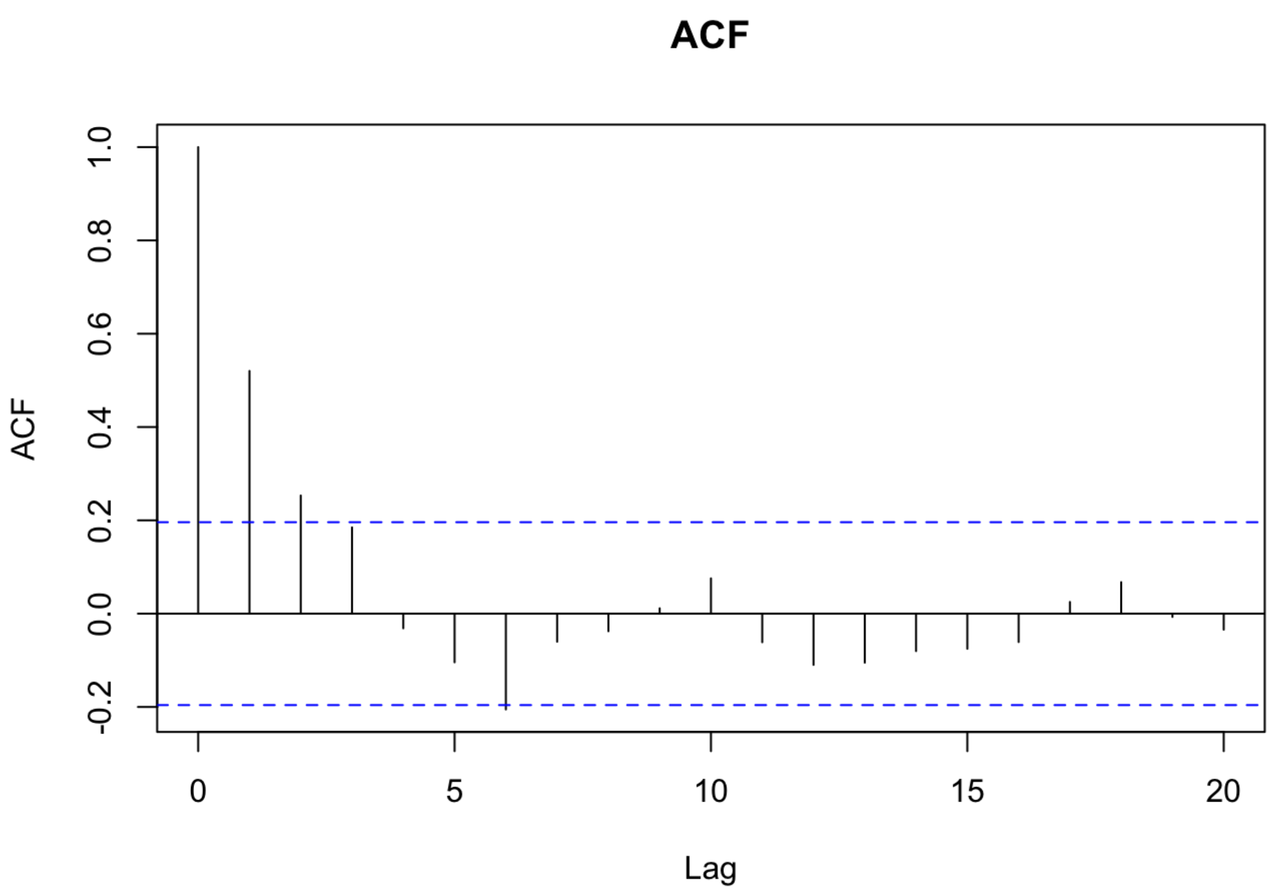
Var($\frac{X_1+X_2+X_3+X_4}{4}$) = $\frac{1}{16}$(4(1+θ^2)+4θ) = $\frac{1}{16}$(4*1.64-3.2) = 0.21

The variance in (c) is smaller than in (b) and closer to 0.

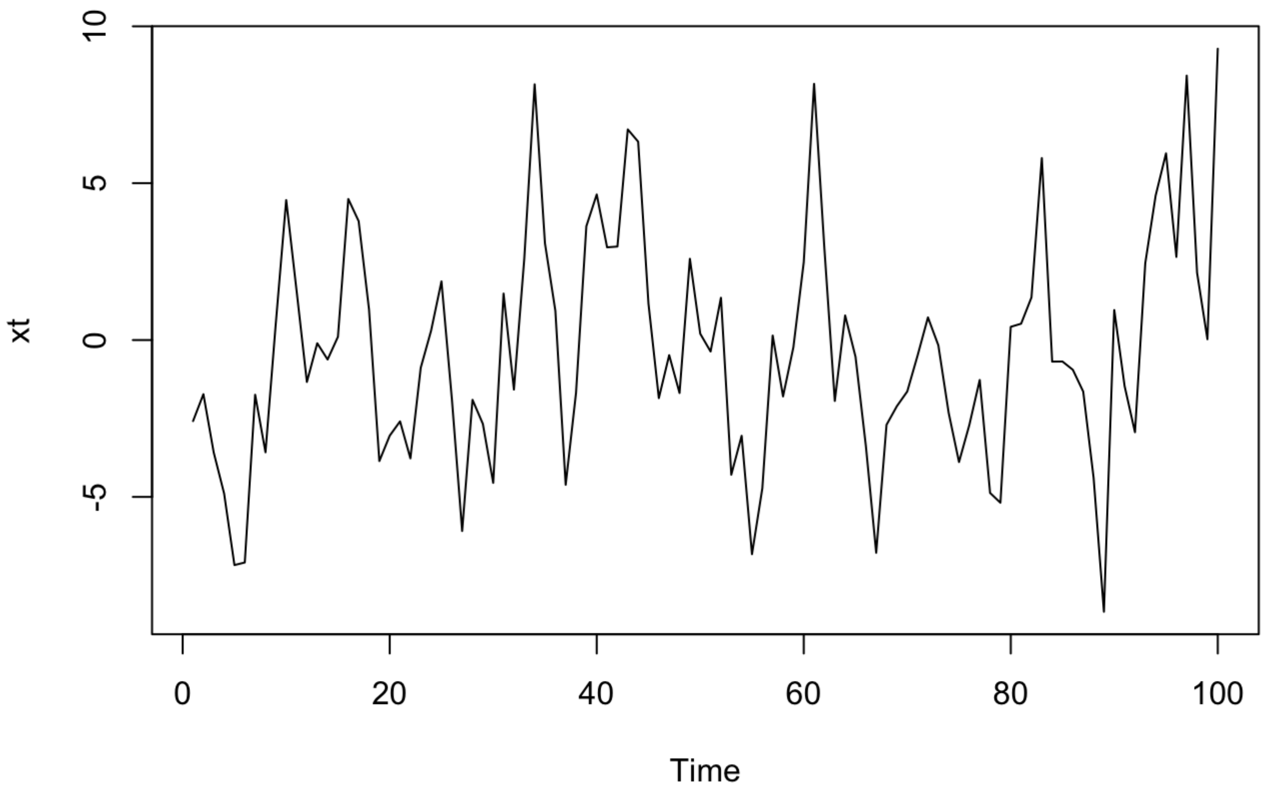
G2

Example 1: X_t = 3X_{t-1} + 3X_{t-2} + Z_t

```
xt <- arima.sim(list(ma=c(1,3,3)),n=100)
acf(xt, main="ACF")
```

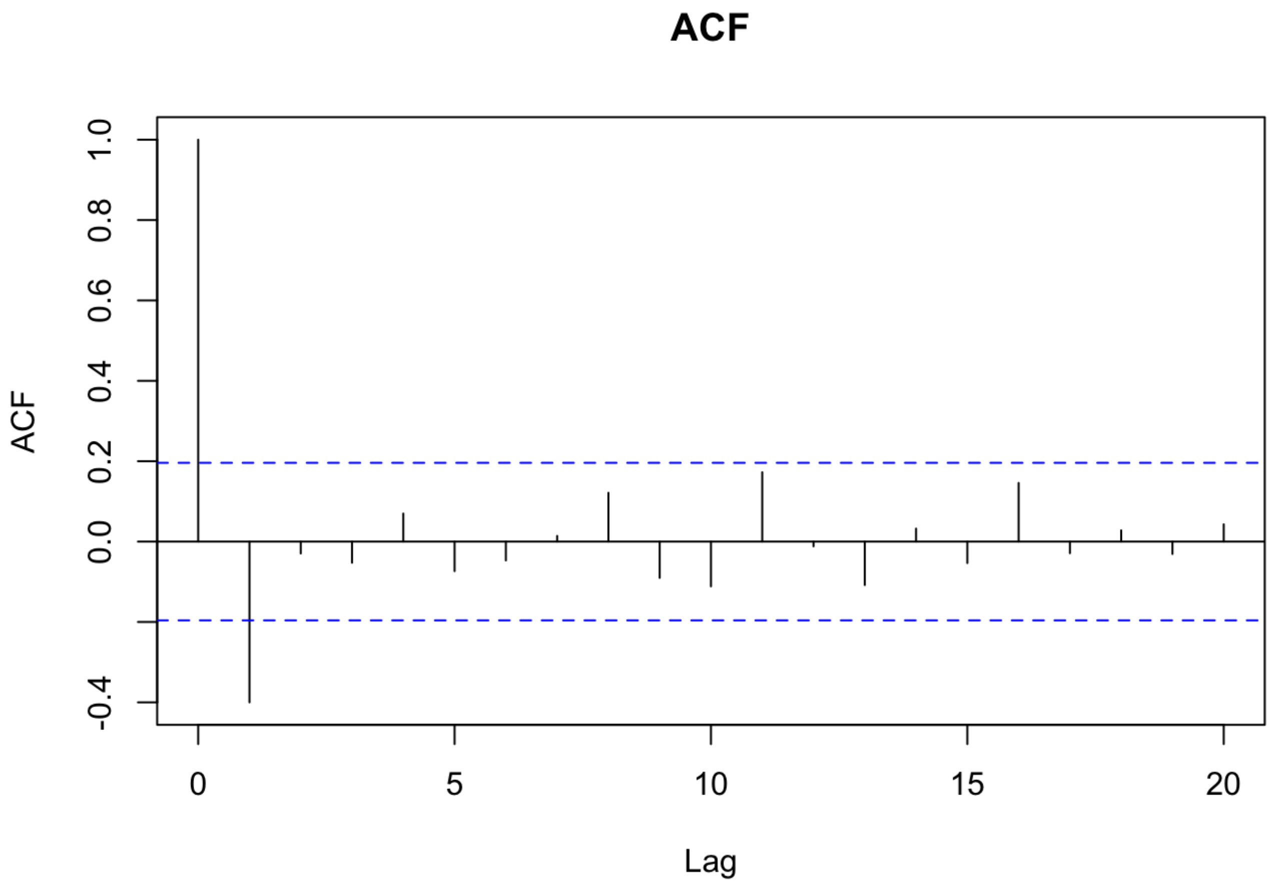



```
plot(xt)
```

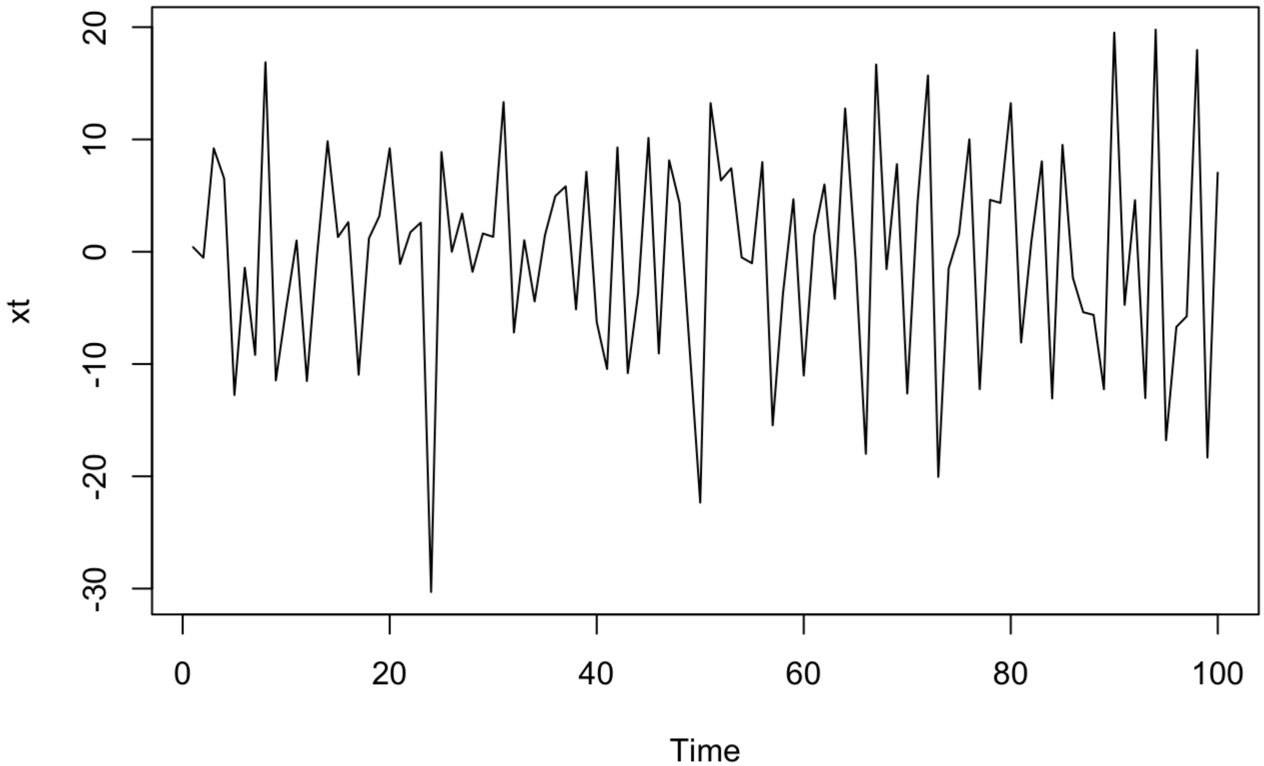


Example 2: X_t = 5X_{t-1} - 8X_{t-2} + Z_t

```
xt <- arima.sim(list(ma=c(1,5,-8)),n=100)
acf(xt, main="ACF")
```




```
plot(xt)
```



G3

$E(e^{\sum_{i=1}^{n} a_i x_i})$ = E(exp(a_1 (Z_1 +θZ_0)+a_2 (Z_2 +θZ_1)+…+a_n (Z_n +θZ_{n-1}))) = E(exp(a_1 θZ_0))E(exp(a_1 +a_2 θ)Z_1))…E(exp($a_n Z_n$)) = m(θa_1)m(a_1 +$theta a_2$)…m(a_{n-1} +θa_n)ma_n

We find that the MGF of x_1 ,x_2 ,…,x_n depends on a_1 ,a_2 …a_n and θ but not t, so X_t is strictly stationary.

E(X_t) = E(Z_t+θZ_{t-1}) = E(Z_t)+E(θZ_{t-1}) = 0 + $\theta(0)$ = 0;

Cov(X_{t+h},X_t) = Cov(Z_{t+h}+θZ_{t+h-1},Z_t+θZ_{t-1}) = Cov(Z_{t+h}, Z_t) + θCov(Z_{t+h}, Z_{t-1}) + θCov(Z_{t+h-1}, Z_t) + θ^2Cov(Z_{t+h-1}, Z_{t-1}) = E($Z_{t+h}Z_t$)+θE($Z_{t+h}Z_{t-1}$)+θE($Z_{t+h-1}Z_t$)+θ^2E(Z_{t+h-1}, Z_{t-1})

When h = 0, E(Z_t^2) = Var(Z_t) - $E(Z_t)^2$ = σ_z^2, then Cov(X_{t+h},X_t) = σ_z^2 + $\theta^2 \sigma_z^2$;

When h = ± 1, Cov(X_{t+h},X_t) = θ;

When h is other numbers, Cov(X_{t+h},X_t) = 0;

Then, we could get $\gamma_x(t+h,h)$ = Cov(X_{t+h},X_t) =

$$\begin{cases} \sigma_Z^2 + \theta^2\sigma_Z^2 & h = 0 \\ \theta & h = \pm 1 \\ 0 & othernumbers \end{cases}$$

Since μ is independent of t and $\gamma_x(t+h,h)$ is also independent of t, then X_t is weakly stationary.

Therefore, X_t is s both weakly and strictly stationary.