CS208 (Semester 1) Week 1 : Propositional Logic

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Computer & Information Sciences

Propositional Logic, Part 1

Syntax

Atomic Statements

Propositional Logic is concerned with statements that make assertions (about the world, or about some "situation"):

1. "It is raining"
2. "I am in Glasgow"
3. "Version 2.1 of *libfoo* is installed"
4. "The number in cell $(3,3)$ is 7"

usually, we abbreviate these: R, G, $\mathrm{foo}_{2.1}$, $\mathrm{C}_7^{3,3}$

These are called *atomic statements* or *atoms*.

Compound Statements

1. $R \to G$

   if it is raining, I am in Glasgow

2. $\neg R \to \neg G$

   if it is not raining, then I am not in Glasgow

3. $\neg \mathrm{foo}_{2.1} \lor \neg \mathrm{foo}_{2.0}$

   either version 2.1 or 2.0 of libfoo *is not installed*

4. $C_7^{3,3} \land C_8^{3,4}$

   cell $(3,3)$ *contains* 7*, and cell* $(3,4)$ *contains* 8

Formulas

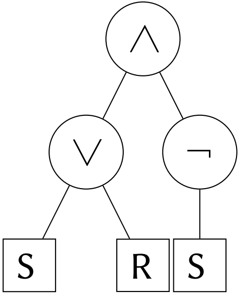
… are built from *atomic propositions* $A, B, C, \cdots$, and the *connectives* $\wedge$ (“and”), $\vee$ (“or”), $\neg$ (“not”), and $\rightarrow$ (“implies”).

As a grammar:

$$P, Q ::= A \mid P \wedge Q \mid P \vee Q \mid \neg P \mid P \rightarrow Q$$

where A stands for any atomic proposition.

Typically, formulas are written done in a “linear” notation, like in algebra. This is because it is more compact…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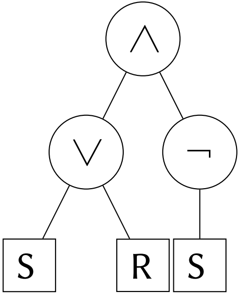
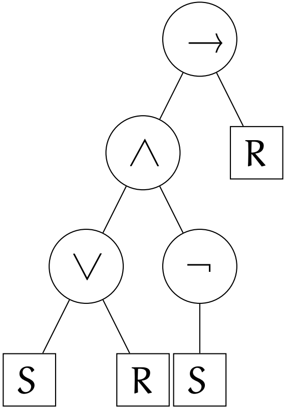


$$(S \vee R) \wedge \neg S$$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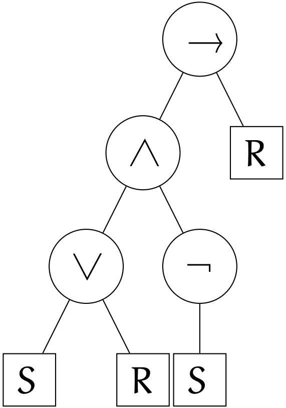

$$((S \vee R) \wedge \neg S) \to R$$

Ambiguity

For compactness, we write out formulas "linearly":

$$(S \vee R) \wedge \neg S \quad ((S \vee R) \wedge \neg S) \to R$$

However, this is ambiguous. What tree does this represent?

$$S \vee R \wedge \neg S \to R$$

we disambiguate with parentheses:

$$((S \vee R) \wedge \neg S) \to R$$

Could put parentheses around every connective, but this is messy.

Disambiguation

1. Runs of $\wedge$, $\vee$, $\rightarrow$ associate to the right:

$$P_1 \wedge P_2 \wedge P_3 \wedge P_4 \quad \text{is same as} \quad P_1 \wedge (P_2 \wedge (P_3 \wedge P_4))$$

2. For any binary connective inside another, require parentheses:

$$(P_1 \vee P_2) \wedge P_3 \quad \text{good} \qquad\qquad P_1 \vee P_2 \wedge P_3 \quad \text{bad}$$

3. For a binary connective under a $\neg$, require parentheses:

$$\neg P \wedge Q \quad \text{is not the same as} \quad \neg(P \wedge Q)$$

4. We don't put parentheses around a $\neg$:

$$\neg(P \wedge Q) \quad \text{good} \qquad\qquad (\neg(P \wedge Q)) \quad \text{bad}$$

Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

Formula	Connective	Subformulas
$A \wedge B$		
$A \wedge B \wedge C$		
$\neg(A \wedge B)$		
$A \to B \to C \to D$		
$B \to C \to D$		

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$A \wedge B$	$\wedge$	A and B
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Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

Formula	Connective	Subformulas
$A \wedge B$	$\wedge$	A and B
$A \wedge B \wedge C$	$\wedge$	A and $B \wedge C$
$\neg(A \wedge B)$		
$A \to B \to C \to D$		
$B \to C \to D$		

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$A \wedge B$	$\wedge$	A and B
$A \wedge B \wedge C$	$\wedge$	A and $B \wedge C$
$\neg(A \wedge B)$	$\neg$	$A \wedge B$
$A \to B \to C \to D$		
$B \to C \to D$		

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$A \wedge B$	$\wedge$	A and B
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$\neg(A \wedge B)$	$\neg$	$A \wedge B$
$A \to B \to C \to D$	$\to$	A and $B \to C \to D$
$B \to C \to D$		

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$A \wedge B$	$\wedge$	A and B
$A \wedge B \wedge C$	$\wedge$	A and $B \wedge C$
$\neg(A \wedge B)$	$\neg$	$A \wedge B$
$A \to B \to C \to D$	$\to$	A and $B \to C \to D$
$B \to C \to D$	$\to$	B and $C \to D$

Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

Formula	Connective	Subformulas
$(A \wedge B) \to (A \vee B)$		
$(A \wedge B) \vee (B \wedge C)$		
$A \vee B \vee C$		
$A \vee B \wedge C$		

Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

Formula	Connective	Subformulas
$(A \wedge B) \to (A \vee B)$	$\to$	$(A \wedge B)$ and $(A \vee B)$
$(A \wedge B) \vee (B \wedge C)$		
$A \vee B \vee C$		
$A \vee B \wedge C$		

Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

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$(A \wedge B) \to (A \vee B)$	$\to$	$(A \wedge B)$ and $(A \vee B)$
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$A \vee B \vee C$		
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$(A \wedge B) \to (A \vee B)$	$\to$	$(A \wedge B)$ and $(A \vee B)$
$(A \wedge B) \vee (B \wedge C)$	$\vee$	$(A \wedge B)$ and $(B \wedge C)$
$A \vee B \vee C$	$\vee$	A and $B \vee C$
$A \vee B \wedge C$		

Decomposing Formulas

Split into: *a)* toplevel connective;*b)* immediate subformulas

Formula	Connective	Subformulas
$(A \wedge B) \rightarrow (A \vee B)$	$\rightarrow$	$(A \wedge B)$ and $(A \vee B)$
$(A \wedge B) \vee (B \wedge C)$	$\vee$	$(A \wedge B)$ and $(B \wedge C)$
$A \vee B \vee C$	$\vee$	A and $B \vee C$
$A \vee B \wedge C$	?	?

Last one is ambiguous! $A \vee (B \wedge C)$ or $(A \vee B) \wedge C$?

Summary

Propositional Logic formulas comprise:

1. Atomic propositions
2. Compound formulas built from $\wedge$, $\vee$, $\rightarrow$, $\neg$

Formulas are "really" trees, but we write them linearly.

We use parentheses to disambiguate.

Propositional Logic, Part 2

Semantics

Truth Values

We define the semantics of formulas in terms of **truth values**:

- T — meaning "true", also written 1, $\top$, t, true;
- F — meaning "false", also written 0, $\bot$, f, false.

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- T — meaning "true", also written 1, $\top$, t, true;
- F — meaning "false", also written 0, $\bot$, f, false.

- Other collections of truth values are possible
  (e.g., "unknown", or values between 0 and 1)
- The truth values mean whatever we want them to mean:
  - Current or no current on a wire
  - Package is installed or not installed
  - Grid cell is filled or not

Meaning is Compositional

The Meaning of a Formula is Defined In Terms of its Parts

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To work out the meaning of $P \wedge Q$:

1. Work out the meaning of P
2. Work out the meaning of Q
3. Combine using the meaning of $\wedge$ and similar for $\rightarrow$, $\vee$, $\neg$.

Meaning is Compositional

The Meaning of a Formula is Defined In Terms of its Parts

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1. Work out the meaning of P
2. Work out the meaning of Q
3. Combine using the meaning of $\wedge$ and similar for $\rightarrow$, $\vee$, $\neg$.

This recipe leaves us to determine:

1. What is the meaning of an atom A?
2. What is the meaning of $\rightarrow$, $\wedge$, $\vee$, $\neg$?

Valuations

An assignment of truth values to atomic propositions is called a **valuation**. We use the letter v to stand for valuations.

For an atom A, we write $v(A)$ for the value assigned to A by v.

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For an atom A, we write $v(A)$ for the value assigned to A by v.

Example

$$v = \{A : \mathsf{T}, \mathrm{B} : \mathsf{F}, \mathrm{C} : \mathsf{T}\}$$

So: $v(A) = \mathsf{T}$

$v(\mathrm{B}) = \mathsf{F}$

$v(\mathrm{C}) = \mathsf{T}$

Example Valuations

1. $v = \{S : T, R : F\}$
   "It is sunny ($v(S) = T$). It is not raining ($v(R) = F$)"
2. $v = \{S : F, R : T\}$
   "It is not sunny ($v(S) = F$). It is raining ($v(R) = T$)"
3. $v = \{S : T, R : T\}$
   "It is sunny ($v(S) = T$). It is raining ($v(R) = T$)"

Example Valuations

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   "It is not sunny ($v(S) = F$). It is raining ($v(R) = T$)"
3. $v = \{S : T, R : T\}$
   "It is sunny ($v(S) = T$). It is raining ($v(R) = T$)"

Intuition: Valuations describe "states of the world"

Notes on Writing Valuations

1. Two valuations are equal if they assign the same truth values to the same atoms.

   Order of writing them down doesn't matter.

2. Each atom can only be assigned one truth value.
3. Every relevant atom must be assigned some truth value.

Semantics of the Connectives

Formula	*is true when ...*
$P \wedge Q$	both P and Q are true
$P \vee Q$	at least one of P or Q is true
$\neg P$	P isn't true
$P \to Q$	if P is true, then Q is true
	otherwise it is false.

Semantics of the Connectives I

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Semantics of the Connectives II

P	$\neg P$
F	T
T	F

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Truth Assignment

For a formula P and a valuation v, we write

$$[\![P]\!]v$$

to mean "the truth value of P at the valuation v".

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$$[\![P]\!]v$$

to mean "the truth value of P at the valuation v".

$$\begin{aligned}
[\![A]\!]v &= v(A) \\
[\![P \wedge Q]\!]v &= [\![P]\!]v \wedge [\![Q]\!]v \\
[\![P \vee Q]\!]v &= [\![P]\!]v \vee [\![Q]\!]v \\
[\![\neg P]\!]v &= \neg [\![P]\!]v \\
[\![P \to Q]\!]v &= [\![P]\!]v \to [\![Q]\!]v
\end{aligned}$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$[\![(A \vee B) \wedge \neg A]\!]v$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$\begin{aligned} & [\![(A \vee B) \wedge \neg A]\!]v \\ = \;& [\![A \vee B]\!]v \wedge [\![\neg A]\!]v \end{aligned}$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathrm{F}, B : \mathrm{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
= \;& [\![A \vee B]\!]v \wedge [\![\neg A]\!]v \\
= \;& ([\![A]\!]v \vee [\![B]\!]v) \wedge [\![\neg A]\!]v
\end{aligned}
$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
= \; & [\![A \vee B]\!]v \wedge [\![\neg A]\!]v \\
= \; & ([\![A]\!]v \vee [\![B]\!]v) \wedge [\![\neg A]\!]v \\
= \; & ([\![A]\!]v \vee [\![B]\!]v) \wedge \neg [\![A]\!]v
\end{aligned}
$$

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$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
= \;& [\![A \vee B]\!]v \wedge [\![\neg A]\!]v \\
= \;& ([\![A]\!]v \vee [\![B]\!]v) \wedge [\![\neg A]\!]v \\
= \;& ([\![A]\!]v \vee [\![B]\!]v) \wedge \neg [\![A]\!]v \\
= \;& (v(A) \vee v(B)) \wedge \neg v(A)
\end{aligned}
$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
= \;& [\![A \vee B]\!]v \wedge [\![\neg A]\!]v \\
= \;& ([\![A]\!]v \vee [\![B]\!]v) \wedge [\![\neg A]\!]v \\
= \;& ([\![A]\!]v \vee [\![B]\!]v) \wedge \neg [\![A]\!]v \\
= \;& (v(A) \vee v(B)) \wedge \neg v(A) \\
= \;& (\mathsf{F} \vee \mathsf{T}) \wedge \neg \mathsf{F}
\end{aligned}
$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
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= \;& (v(A) \vee v(B)) \wedge \neg v(A) \\
= \;& (\mathsf{F} \vee \mathsf{T}) \wedge \neg \mathsf{F} \\
= \;& \mathsf{T} \wedge \neg \mathsf{F}
\end{aligned}
$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!] v \\
= \;& [\![A \vee B]\!] v \wedge [\![\neg A]\!] v \\
= \;& ([\![A]\!] v \vee [\![B]\!] v) \wedge [\![\neg A]\!] v \\
= \;& ([\![A]\!] v \vee [\![B]\!] v) \wedge \neg [\![A]\!] v \\
= \;& (v(A) \vee v(B)) \wedge \neg v(A) \\
= \;& (\mathsf{F} \vee \mathsf{T}) \wedge \neg \mathsf{F} \\
= \;& \mathsf{T} \wedge \neg \mathsf{F} \\
= \;& \mathsf{T} \wedge \mathsf{T}
\end{aligned}
$$

Example

$(A \vee B) \wedge \neg A$ with the valuation $v = \{A : \mathsf{F}, B : \mathsf{T}\}$:

$$
\begin{aligned}
& [\![(A \vee B) \wedge \neg A]\!]v \\
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= \; & ([\![A]\!]v \vee [\![B]\!]v) \wedge \neg [\![A]\!]v \\
= \; & (v(A) \vee v(B)) \wedge \neg v(A) \\
= \; & (\mathsf{F} \vee \mathsf{T}) \wedge \neg \mathsf{F} \\
= \; & \mathsf{T} \wedge \neg \mathsf{F} \\
= \; & \mathsf{T} \wedge \mathsf{T} = \mathsf{T}
\end{aligned}
$$

Semantics of a Formula

For a formula P, its *meaning* is the collection of all valuations v that make $[\![P]\!]v = \mathsf{T}$.

For example,

$$[\![(A \vee B) \wedge \neg A]\!] = \left\{ \{A : \mathsf{F}, B : \mathsf{T}\} \right\}$$

To compute sets of valuations, we will use truth tables.

Summary

1. Semantics defines the *meaning* of formulas.
2. We use *truth values* T and F.
3. A valuation v assigns truth values to atoms.
4. We extend that assignment to whole formulas: $[\![P]\!]v$.
5. The meaning of P is the set of valuations that make it true.

Propositional Logic, Part 3

Truth Tables, Satisfiability, and Validity

Truth table for $(A \vee B) \wedge \neg A$

Name the parts: ① $= A \vee B$; ② $= \neg A$

A	B	① $A \vee B$	② $\neg A$	① $\wedge$ ② $(A \vee B) \wedge \neg A$
F	F			
F	T			
T	F			
T	T			

Truth table for $(A \vee B) \wedge \neg A$

Name the parts: ① $= A \vee B$; ② $= \neg A$

A	B	① $A \vee B$	② $\neg A$	① $\wedge$ ② $(A \vee B) \wedge \neg A$
F	F	F		
F	T			
T	F			
T	T			

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F	T	T		
T	F			
T	T			

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F	F	F		
F	T	T		
T	F	T		
T	T			

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F	F	F	T	
F	T	T		
T	F	T		
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F	T	T	T	
T	F	T		
T	T	T		

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F	F	F	T	
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Truth table for $(A \vee B) \wedge \neg A$

Name the parts: ① $= A \vee B$; ② $= \neg A$

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F	T	T	T	
T	F	T	F	
T	T	T	F	

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Name the parts: ① $= A \vee B$; ② $= \neg A$

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T	F	T	F	F
T	T	T	F	F

Truth table for $(A \vee B) \wedge \neg A$

A	B	$A \vee B$	$\neg A$	$(A \vee B) \wedge \neg A$
F	F	F	T	F
F	T	T	T	T
T	F	T	F	F
T	T	T	F	F

1. Row for every valuation
2. Intermediate columns for the subformulas
3. Final column for the whole formula

Truth table for $(A \vee B) \wedge \neg A$

A	B	$A \vee B$	$\neg A$	$(A \vee B) \wedge \neg A$
F	F	F	T	F
F	T	T	T	T
T	F	T	F	F
T	T	T	F	F

Read off the truth value assignments:

1. For $v = \{A : \mathsf{F}; B : \mathsf{F}\}$: $[\![(S \vee R) \wedge \neg S]\!]v = \mathsf{F}$.
2. For $v = \{A : \mathsf{F}; B : \mathsf{T}\}$: $[\![(S \vee R) \wedge \neg S]\!]v = \mathsf{T}$.
3. For $v = \{A : \mathsf{T}; B : \mathsf{F}\}$: $[\![(S \vee R) \wedge \neg S]\!]v = \mathsf{F}$.
4. For $v = \{A : \mathsf{T}; B : \mathsf{T}\}$: $[\![(S \vee R) \wedge \neg S]\!]v = \mathsf{F}$.

University of Strathclyde Science

Truth table for $(A \vee B) \wedge \neg A$

A	B	$A \vee B$	$\neg A$	$(A \vee B) \wedge \neg A$
F	F	F	T	F
F	T	T	T	T
T	F	T	F	F
T	T	T	F	F

The semantics of a formula can be read off from the lines of the truth table that end with T:

$$[\![(A \vee B) \wedge \neg A]\!] = \{\{A : \mathsf{F}; B : \mathsf{T}\}\}$$

Satisfiability

A formula P is **satisfiable** if there **exists at least one** valuation v such that $[\![P]\!]v = \mathsf{T}$.

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Alternatively: there is at least one row in the truth table that ends with T.

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Alternatively: there is at least one row in the truth table that ends with T.

Alternatively: the semantics of P contains at least one valuation.

Validity

A formula P is **valid** if **for all** valuations v, we have $[\![P]\!]v = T$.

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Alternatively: the semantics of P consists of all possible valuations.

A valid formula is also called a *tautology*.

Sunny and Rainy

Is the formula $(S \lor R) \land \neg S$

1. Satisfiable?

2. Valid?

Sunny and Rainy

Is the formula $(S \vee R) \wedge \neg S$

1. Satisfiable?

   Yes. $v = \{S : F, R : T\}$

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Is the formula $((S \vee R) \wedge \neg S) \rightarrow R$

1. Satisfiable?

2. Valid?

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   Yes. $v = \{S : F, R : T\}$

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Is the formula $((S \vee R) \wedge \neg S) \to R$

1. Satisfiable?

   Yes. $v = \{S : T, R : F\}$

2. Valid?

Sunny and Rainy

Is the formula $(S \lor R) \land \neg S$

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Is the formula $((S \lor R) \land \neg S) \to R$

1. Satisfiable?

   Yes. $v = \{S : T, R : F\}$

2. Valid?

   Yes. (need to check the truth table)

An observation

If a valuation v makes a formula P true, then it makes $\neg P$ false.

$$[\![P]\!]v = \mathsf{T} \qquad \Leftrightarrow \qquad [\![\neg P]\!]v = \mathsf{F}$$

Satisfiability vs Validity

A formula P is valid exactly when $\neg P$ is not satisfiable.

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Proof.

$$P \text{ valid}$$

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Proof.

$$
\begin{array}{lll}
 & P \text{ valid} & \\
\Leftrightarrow & \text{for all } v, [\![P]\!]v = T & \textit{by definition}
\end{array}
$$

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Proof.

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\Leftrightarrow & \text{for all } v, \text{not } ([\![\neg P]\!]v = \mathsf{T}) & \mathsf{T} \textit{ is not } \mathsf{F}
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A formula P is valid exactly when $\neg P$ is not satisfiable.

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\Leftrightarrow & \text{for all } v, [\![\neg P]\!]v = \mathsf{F} & \textit{by above observation} \\
\Leftrightarrow & \text{for all } v, \text{not } ([\![\neg P]\!]v = \mathsf{T}) & \mathsf{T} \textit{ is not } \mathsf{F} \\
\Leftrightarrow & \text{does not exist } v \text{ such that } [\![\neg P]\!]v = \mathsf{T} & \textit{“for all, not”} \equiv \textit{“not exists”}
\end{array}
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Satisfiability vs Validity

A formula P is valid exactly when $\neg P$ is not satisfiable.

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\Leftrightarrow & \text{does not exist } v \text{ such that } [\![\neg P]\!]v = \mathsf{T} & \textit{“for all, not”} \equiv \textit{“not exists”} \\
\Leftrightarrow & \neg P \text{ not satisfiable} & \textit{by definition}
\end{array}
$$

Satisfiability vs Validity

A formula P is valid exactly when $\neg P$ is not satisfiable.

Consequence: Counterexample finding

- If we get a valuation satisfying $\neg P$, it is a **counterexample** to the validity of P.
- If we do not find any valuation satisfying $\neg P$, then P is valid.
- So we can reduce the problem of determining validity to finding satisfying valuations.

Summary

- Truth tables enable mass production of meaning
- Satisfiability: at least one valuation makes it true.
- Validity: every valuation makes it true.
- Satisfiability and Validity related via negation.

Propositional Logic, Part 4

Entailment

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Entailment is a relation between some assumptions:

$$P_1, \ldots, P_n$$

and a conclusion:

$$Q$$

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$$P_1, \ldots, P_n$$

and a conclusion:

$$Q$$

What we want to capture is:

If we assume P_1, ..., P_n are all true, then it is safe to conclude Q.

Is it safe?

If we assume

it is sunny

then is it safe to conclude

it is sunny

Is it safe?

If we assume

it is sunny

then is it safe to conclude

it is sunny

Yes!

Is it safe?

If we assume

it is sunny

then is it safe to conclude

it is sunny

Yes! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)

Is it safe?

If we assume

it is sunny

then is it safe to conclude

it is sunny

Yes! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. ~~It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)~~

But we are assuming "it is sunny", so the second case doesn't matter.

Is it safe?

If we assume

nothing

then is it safe to conclude

it is sunny

Is it safe?

If we assume

nothing

then is it safe to conclude

it is sunny

No!

Is it safe?

If we assume

nothing

then is it safe to conclude

it is sunny

No! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)

Is it safe?

If we assume

nothing

then is it safe to conclude

it is sunny

No! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)

But we are making no assumptions, so either "world" is possible: it might not be sunny.

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

No!

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

No! There are four cases:

1. It is sunny and raining
2. It is sunny and not raining
3. It is not sunny, but is raining
4. It is not sunny and not raining

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

No! There are four cases:

1. It is sunny and raining
2. ~~It is sunny and not raining~~
3. It is not sunny, but is raining
4. ~~It is not sunny and not raining~~

Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

it is not sunny

Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

it is not sunny

Yes!

Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

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Yes! There are four cases:

1. It is sunny and raining
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Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

it is not sunny

Yes! There are four cases:

1. It is sunny and raining
2. ~~It is sunny and not raining~~
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Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

it is not sunny

Yes! There are four cases:

1. ~~It is sunny and raining~~
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3. It is not sunny, but is raining
4. ~~It is not sunny and not raining~~

Is it safe?

If we assume

nothing

then is it safe to conclude:

it is sunny or not sunny

Is it safe?

If we assume

nothing

then is it safe to conclude:

it is sunny or not sunny

Yes!. There are two cases:

1. It is sunny
2. It is not sunny

Is it safe?

If we assume

nothing

then is it safe to conclude:

it is sunny or not sunny

Yes!. There are two cases:

1. It is sunny
2. It is not sunny

In either case the conclusion is true: $A \vee B$ requires at least one of A or B to be true.

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Yes!

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Yes! There are four cases:

1. it is sunny, and the moon is made of spaghetti
2. it is not sunny, and the moon is made of spaghetti
3. it is sunny, and the moon is not made of spaghetti
4. it is not sunny, and the moon is not made of spaghetti

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Yes! There are four cases:

1. it is sunny, and the moon is made of spaghetti
2. ~~it is not sunny, and the moon is made of spaghetti~~
3. it is sunny, and the moon is not made of spaghetti
4. ~~it is not sunny, and the moon is not made of spaghetti~~

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Yes! There are four cases:

1. ~~it is sunny, and the moon is made of spaghetti~~
2. ~~it is not sunny, and the moon is made of spaghetti~~
3. ~~it is sunny, and the moon is not made of spaghetti~~
4. ~~it is not sunny, and the moon is not made of spaghetti~~

Entailment

In general, we have n assumptions $P_1, \ldots, P_n$ and conclusion Q.

We are going to say: $P_1, \ldots, P_n \models Q$

Read as $P_1, \ldots, P_n$ *entails* Q

if:

for all "situations" (i.e., valuations)

that make **all** the assumptions P_i true,

the conclusion Q is true.

Entailment

With more symbols

for all valuations v, if, for all i, $[\![P_i]\!]v = \mathsf{T}$, then $[\![Q]\!]v = \mathsf{T}$.

In terms of Semantics

every valuation in all $[\![P_i]\!]$ is also in $[\![Q]\!]$
(in set theory symbols: $([\![P_1]\!] \cap \cdots \cap [\![P_n]\!]) \subseteq [\![Q]\!]$).

Entailment vs Validity

If we have no assumptions, then:

$$\models P$$

exactly when

$$\text{for all } v.\ [\![P]\!]v = \mathsf{T}$$

exactly when

$$P \text{ is valid}$$

Deduction Theorem

$$P_1, \ldots, P_n, P \models Q \qquad \textit{exactly when} \qquad P_1, \ldots, P_n \models P \to Q$$

All these statements are equivalent:

1. $P_1, \ldots, P_n, P \models Q$
2. for all v, if all $[\![P_i]\!]v = T$ and $[\![P]\!]v = T$, then $[\![Q]\!]v = T$
3. for all v, if all $[\![P_i]\!]v = T$, then (if $[\![P]\!]v = T$, then $[\![Q]\!]v = T$)
4. for all v, if all $[\![P_i]\!]v = T$, then $[\![P \to Q]\!]v = T$
5. $P_1, \ldots, P_n \models P \to Q$

Entailment vs satisfiability

So, it is the case that

$$P_1, \ldots, P_n \models Q$$

exactly when

$$\models P_1 \to \cdots \to P_n \to Q$$

exactly when

$$P_1 \to \cdots \to P_n \to Q \text{ is valid}$$

exactly when

$$\neg(P_1 \to \cdots \to P_n \to Q) \text{ is not satisfiable}$$

Summary

- Entailment defines safe deductions.
- Relationship with Validity
- Relationship with "$\rightarrow$" (Deduction Theorem)
- Relationship with Satisfaction.