Spacecraft and Aircraft Dynamics

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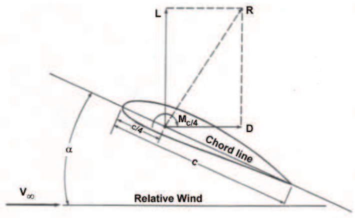
Lecture 4: Contributions to Longitudinal Stability

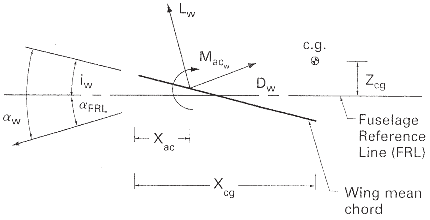
In this lecture, we will discuss

Airfoils:

- The wing Contribution to pitching moment
  - Rotate Lift and Drag vectors
  - Calculate moment about CG ($\vec{M} = \vec{r} \times \vec{F}$)
  - Use approximations to simplify expression.

Two Confusing Figures





Confusion: For an airfoil, angle of attack is measured to the zero-lift-line.

- Thus $C_{M0} = 0$ for an un-inclined *airfoil*.

Confusion: We assume that the aerodynamic center is on the FRL.

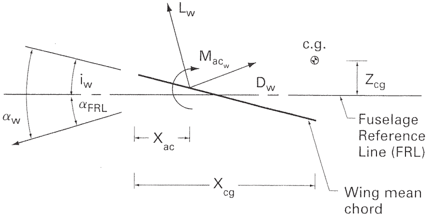
- Thus as measure from the CG,

$$\vec{r}_{ac} = \begin{bmatrix} X_{cg} - X_{ac} \\ 0 \\ Z_c g \end{bmatrix}.$$

- If there is any confusion on a problem, ask me to clarify.

Next Subject: Wing Contribution

Lift and Drag



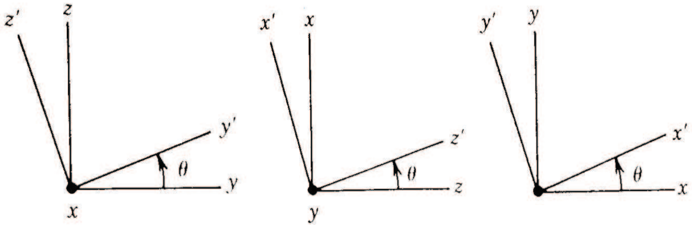
The wing is a lifting surface which produces both **Lift** and **Drag**.

- Lift is perpendicular to free-stream velocity.
  - If $\alpha_{FRL} > 0$, then lift vector pitches forward (nose-down direction) in the body-fixed frame.
- Drag is parallel to free-stream velocity.
  - If $\alpha_{FRL} > 0$, then drag also rotates in the nose-down direction.

To determine the contributions of Lift and drag in the body-fixed frame, these forces must be *rotated* by the angle of attack and any additional wing inclination.

Rotating the Lift and Drag

Rotation Matrices

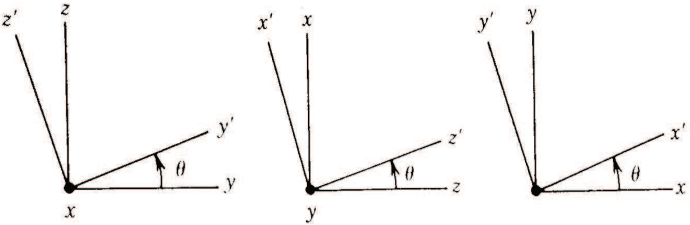


Lift and Drag vectors rotate with angle of attack and wing inclination.

Example: Given a vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a pitch-up rotation, θ,

$$\vec{v}' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \end{bmatrix}$$

The matrix is called a *rotation matrix*.



Rotation matrices can be used to calculate the effect of **ANY** rotation.

Roll Angle ϕ **:**

$$\vec{v}' = R_1(\phi)\vec{v}$$

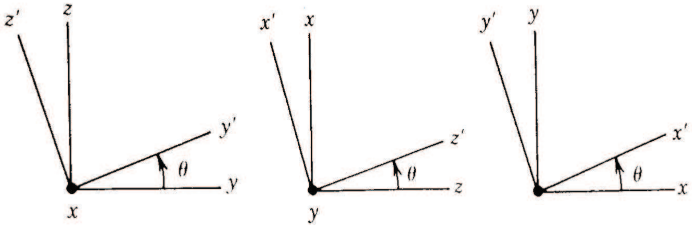
Pitch Angle θ **:**

$$\vec{v}' = R_2(\theta)\vec{v}$$

Yaw Angle ψ **:**

$$\vec{v}' = R_3(\psi)\vec{v}$$

Remember to use the right-hand rule to determine what is a positive rotation.



The rotation matrices are (for reference):

Roll Rotation (ϕ) :

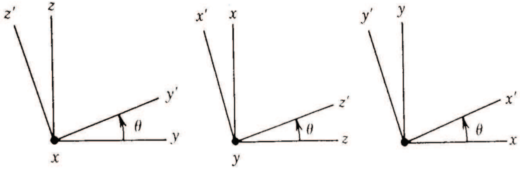
$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Pitch Rotation (θ):

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Yaw Rotation (ψ):

$$R_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation matrices, can be used to calculate a sequence of rotations:

Roll-Pitch-Yaw:

$$\vec{v}_{RPY} = R_3(\psi)R_2(\theta)R_1(\phi)\vec{v}$$

Note the *order* of multiplication is critical.

$$\vec{v}_{RPY} = \left(R_3(\psi)\left(R_2(\theta)\left(R_1(\phi)\vec{v}\right)_1\right)_2\right)_3$$

Consider a pure lift force of $10MN$ after a pitch up of $10\deg$ and a yaw of $20\deg$.

$$\vec{L} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

$$\begin{aligned}
\vec{L}_{PY} &= \begin{bmatrix} \cos 20 & -\sin 20 & 0 \\ \sin 20 & \cos 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 10 & 0 & \sin 10 \\ 0 & 1 & 0 \\ -\sin 10 & 0 & \cos 10 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \\
&= \begin{bmatrix} 0.9397 & -0.3420 & 0 \\ 0.3420 & 0.9397 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix} \begin{bmatrix} 0.9848 & 0 & 0.1736 \\ 0 & 1.0000 & 0 \\ -0.1736 & 0 & 0.9848 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \\
&= \begin{bmatrix} 0.9254 & -0.3420 & 0.1632 \\ 0.3368 & 0.9397 & 0.0594 \\ -0.1736 & 0 & 0.9848 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} -1.6318 \\ -0.5939 \\ -9.8481 \end{bmatrix}
\end{aligned}$$

Compare this to the same rotations in the reverse order (Yaw, then Pitch)

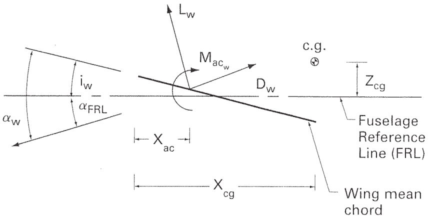
$$\vec{L}_{YP} = \begin{bmatrix} \cos 10 & 0 & \sin 10 \\ 0 & 1 & 0 \\ -\sin 10 & 0 & \cos 10 \end{bmatrix} \begin{bmatrix} \cos 20 & -\sin 20 & 0 \\ \sin 20 & \cos 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

$$= \begin{bmatrix} -1.7365 \\ 0 \\ -9.8481 \end{bmatrix}$$

Which is still in the $x - z$ plane!!! **Why?**

Compare to Pitch, Yaw value:

$$\vec{L}_{PY} = \begin{bmatrix} -1.6318 \\ -0.5939 \\ -9.8481 \end{bmatrix}$$



The **magnitude** of lift and drag are given by:

$$L_{wing} = C_{L,w}QS \qquad \text{and} \qquad D_{wing} = C_{D,w}QS$$

The two important angles are angle of attack of the aircraft, α_{FRL} and

- inclination i_w of the wing. Most wings are rotated up - this increases the angle of attack of the wing.

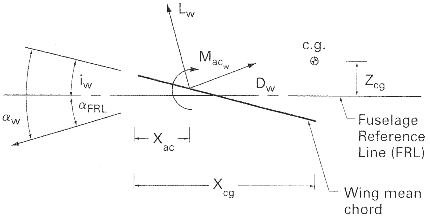
C_L can be determined as

$$C_L = C_{L\alpha}\alpha_w = C_{L\alpha}(\alpha_{FRL} + i_w)$$

where $\alpha_w = \alpha_{FRL} + i_w$ is the angle of the wing with the free-stream.

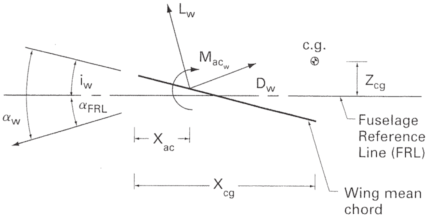
Next Subject: Wing Contribution

Lift and Drag



C_D is usually determined using $C_{D\alpha}$ and C_{D0} from the *tables*.

$$C_D = C_{D\alpha}\alpha_w + C_{D0}$$



The wing makes three contributions to the moment about the Center of Gravity (CG) of the aircraft.

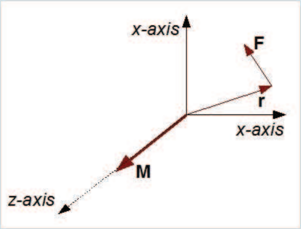
- The lift force, multiplied by "lever arm"
- The drag force, multiplied by a shorter "lever arm"
- The moment exerted on the wing itself (due to C_{M0})

As a reference point, we choose the leading edge of the wing.

- X_{ac} is the distance from the leading edge to the aerodynamic center of the wing
- X_{cg} is the distance from the leading edge to the center of mass of the aircraft
- Z_{cg} is the height of the CG above the Fuselage Reference Line (FRL)

The moment produced by a force about point is

$$\vec{M} = \vec{r} \times \vec{F}$$

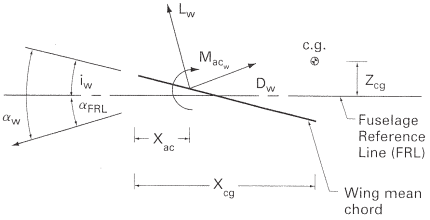


For example, $\vec{F} = \begin{bmatrix} F_x \\ 0 \\ F_z \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} r_x \\ 0 \\ r_z \end{bmatrix}$, which implies

$$\vec{M} = \vec{r} \times \vec{F} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & 0 & r_z \\ F_x & 0 & F_z \end{bmatrix} = -(r_x F_z - F_x r_z)\hat{y}.$$

Wing Contribution: Simple Case

Zero Angle of Attack



Consider when $\alpha = 0$. Then the force in the body-fixed frame, the lift and drag forces are

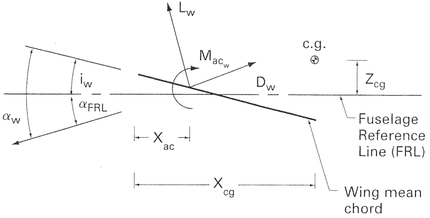
$$F_{w,\alpha=0} = \begin{bmatrix} -D_w \\ 0 \\ -L_w \end{bmatrix}, \qquad \text{applied at} \qquad \vec{r}_w = \begin{bmatrix} X_{cg} - X_{ac} \\ 0 \\ Z_{cg} \end{bmatrix}$$

Note that we assume the aerodynamic center is on the FRL. Now we have

$$\begin{aligned} \vec{M}_{w,\alpha=0} &= M_{0,w}\hat{y} + \vec{r} \times \vec{F} = M_w\hat{y} + \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ X_{cg} - X_{ac} & 0 & Z_{cg} \\ -D_w & 0 & -L_w \end{bmatrix} \\ &= \left(M_{0,w} + L_w(X_{cg} - X_{ac}) - D_w Z_{cg}\right)\hat{y}. \end{aligned}$$

Wing Contribution: Simple Case

Zero Angle of Attack

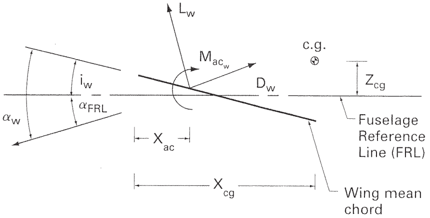


Thus the pitching moment produced by the lift and Drag *at zero angle of attack* ($\alpha_{FRL} = 0$) is

$$M_{w,\alpha=0} = M_{0,w} + L_w(X_{cg} - X_{ac}) - D_w Z_{cg}.$$

Converting to non-dimensional form, the pitching moment is

$$C_{Mw,\alpha_{FRL}=0} = C_{M,w} + C_{L,w}(X_{cg} - X_{ac}) - C_{D,w} Z_{cg}.$$



- Because Lift is perpendicular to free-stream and Drag is parallel, a positive angle of attack will rotate $\vec{F}$ in the negative pitch direction (nose down-direction).
- In the body-fixed frame, these forces become

$$\vec{F}_w = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} \begin{bmatrix} -D_w \\ 0 \\ -L_w \end{bmatrix} = \begin{bmatrix} -D_w \cos\alpha + L_w \sin\alpha \\ 0 \\ -D_w \sin\alpha - L_w \cos\alpha \end{bmatrix}$$

  where $\alpha = \alpha_{FRL}$ and recall $\sin(-\alpha) = -\sin\alpha$, $\cos(-\alpha) = \cos\alpha$

Now we can calculate the moment contribution as before from

$$\vec{M}_w = M_{0,w} + \vec{r}_w \times \vec{F}_w$$

Wing Contribution

Moment Contribution: Angle of Attack

This contribution is readily calculated as

$$\vec{M}_w = \vec{M}_{0,w} + \vec{r}_w \times \vec{F}_w$$

That is,

$$\begin{aligned} M_w &= M_{0,w} + F_x Z_{cg} - F_z (X_{cg} - X_{ac}) \\ &= M_{0,w} + (L_w \sin\alpha - D_w \cos\alpha)\, Z_{cg} \\ &\quad + (D_w \sin\alpha + L_w \cos\alpha)\,(X_{cg} - X_{ac}) \end{aligned}$$

or, in *non-dimensional* form, divide by $QS\bar{c}$ to get

$$\begin{aligned} C_{M,w} = C_{M,0,w} &+ (C_{L,w} \sin\alpha - C_{D,w} \cos\alpha)\, \frac{Z_{cg}}{\bar{c}} \\ &+ (C_{D,w} \sin\alpha + C_{L,w} \cos\alpha)\,(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}) \end{aligned}$$

Recall that

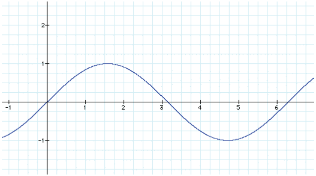
$$C_{L,w} = C_{L\alpha}\alpha_w = C_{L\alpha}(\alpha_{FRL} + i_w) \qquad \text{and} \qquad C_D = C_{D\alpha}\alpha_w + C_{D0}$$

At this point we decide the equation are **too complicated**. We need to make some Approximations.

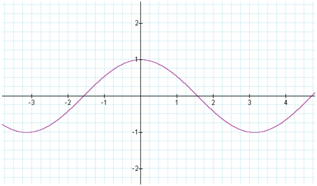
Our moment equation is now

$$C_{M,w} = C_{M,0,w} + \left(C_{L,w}\sin\alpha - C_{D,w}\cos\alpha\right)\frac{Z_{cg}}{\bar{c}} + \left(C_{D,w}\sin\alpha + C_{L,w}\cos\alpha\right)\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right)$$

But this is nonlinear and I promised linear equations. Therefore we use the **Small Angle Approximations** to linearize. If $\alpha \cong 0$, then



$$\sin\alpha \cong \alpha$$



$$\cos\alpha \cong 1$$

Wing Contribution: Simplification

Other Approximations

Replacing $\sin\alpha$ with α and $\cos\alpha$ with 1, we get

$$C_{M,w} \cong C_{M,0,w} + \left(C_{L,w}\alpha - C_{D,w}\right)\frac{Z_{cg}}{\bar{c}} + \left(C_{D,w}\alpha + C_{L,w}\right)\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right).$$

The equation is **linear**, but still complicated.

We now make 2 more assumptions

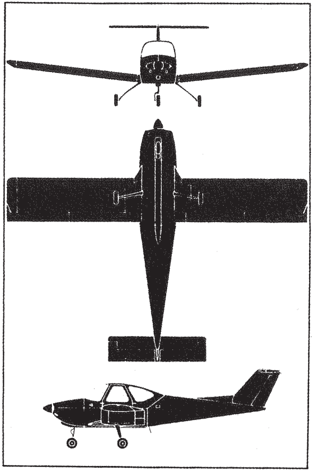
1. Z_{cg} is small compared to $X_{cg} - X_{ac}$.
   - $C_{D,w}Z_{cg} + C_{L,w}\left(X_{cg} - X_{ac}\right) \cong C_{L,w}\left(X_{cg} - X_{ac}\right)$
2. α is small.
   - $C_{D,w}\alpha + C_{L,w} \cong C_{L,w}$

What we have left is

$$C_{M,w} \cong C_{M,0,w} + C_{L,w}\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right).$$

Beechcraft Skipper USA

Type: light training aircraft   Accommodation: two pilots, one passenger



Dimensions:
Length: 23 ft 10 in (7.3 m)
Wingspan: 30 ft (9.1 m)
Height: 7 ft 6 in (2.3 m)

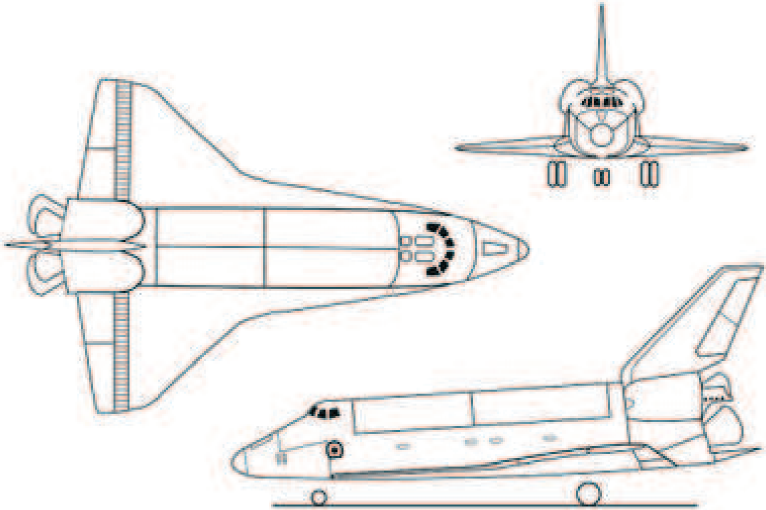
Weights:
Empty: 1103 lb (500 kg)
Max T/O: 1650 lb (748 kg)
Payload: n\a

Performance:
Max speed: 120 mph (196 kmh)
Range: 413 nm (764 km)
Power plant: one O-235 piston engine
Thrust: 115 hp (85 kw)

Variants:
none

Notes: Built in small numbers, the Skipper is mainly used for training purposes.

Wing Contribution: Approximations
Exceptions



Wing Contribution: Continued

We now include the expression for Lift:

$$C_{L,w} = C_{L\alpha,w}\alpha_w = C_{L\alpha,w}\left(\alpha_{FRL} + i_w\right)$$

this yields

$$\begin{aligned}
C_{M,w} &= C_{M,0,w} + C_{L\alpha,w}\left(\alpha_{FRL} + i_w\right)\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right) \\
&= C_{M,0,w} + C_{L\alpha,w} i_w \left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right) + C_{L\alpha,w}\alpha_{FRL}\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right)\alpha_{FRL}
\end{aligned}$$

Thus we have

$$C_{M,total} = C_{M,total,0} + C_{M,total,\alpha}\alpha_{FRL}$$

where

$$C_{M,total,0} = C_{M,0,w} + C_{L\alpha,w} i_w \left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right)$$

$$C_{M,total,\alpha} = C_{L\alpha,w}\left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right)$$

Wing Contribution: Stability

Recall for nose-up steady flight, we need

$$C_{M,total,0} = C_{M,0,w} + C_{L\alpha,w} i_w \left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}} \right) \geq 0$$

and

$$C_{M,total,\alpha} = C_{L\alpha,w} \left(\frac{X_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}} \right) \leq 0$$

Since $C_{L\alpha,w} > 0$, this can only be achieved when we have *both*

CG ahead of wing:

$$X_{cg} - X_{ac} < 0$$

which is unusual.

Negative Camber:

$$C_{M,0,w} > 0$$

which would be odd.

Thus we need a tail contribution.

Conclusion

In this lecture, we have

- Calculated the moment about the CG produced by Lift and Drag on a Wing.
- Made small-angle approximations to get a linear expression.
- Neglected some Moment contributions.
- Developed impractical conditions for stability of a wing-only aircraft

Next Lecture

- Do the same thing for the tail.
- Discuss our new design variables.
- Discuss equilibrium and stability conditions.