2022 Spring - Econometrics - Problem Set 1

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1. From Model A:

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + \epsilon_i \\ \Rightarrow \epsilon_i &= y_i - \beta_0 - \beta_1 x_i \\ \Rightarrow \sum_{i=1}^{n} \epsilon_i^2 &= \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 = Q_a \end{aligned}$$

Obtain first order conditions for minimizing residuals sum of square:

$$\frac{\partial Q_a}{\partial \beta_0} = -2 \sum (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{1}$$

$$\frac{\partial Q_a}{\partial \beta_1} = -2 \sum x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{2}$$

From (1):

$$\sum y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum x_i = 0$$

$$\Rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{3}$$

From (2) $-\ \bar{x}$(1):

$$\begin{aligned} &\sum (x_i - \bar{x})(y_i - \hat{\beta}_0 - \hat{\beta} x_i) = 0 \\ \Rightarrow &\sum (x_i - \bar{x})((y_i - \bar{y}) - \beta_1 (x_i - \bar{x})) = 0 \end{aligned}$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \tag{4}$$

From Model B:

$$\begin{aligned} y_i &= \gamma_0 + \gamma_1 (x_i - \bar{x}) + \epsilon_i \\ \Rightarrow \epsilon_i &= y_i - \gamma_0 - \gamma_1 (x_i - \bar{x}) \\ \Rightarrow \sum_{i=1}^{n} \epsilon_i^2 &= \sum_{i=1}^{n} (y_i - \gamma_0 - \gamma_1 (x_i - \bar{x}))^2 = Q_b \end{aligned}$$

Obtain FOCs:

$$\frac{\partial Q_b}{\partial \gamma_0} = -2 \sum (y_i - \gamma_0 - \gamma_1 (x_i - \bar{x})) = 0 \tag{5}$$

$$\frac{\partial Q_b}{\partial \gamma_1} = -2 \sum (x_i - \bar{x})(y_i - \gamma_0 - \gamma_1 (x_i - \bar{x})) = 0 \tag{6}$$

From (6):

$$\sum y_i - n\hat{\gamma}_0 - \hat{\gamma}_1 \underbrace{\sum(x_i - \bar{x})}_{=0} = 0$$

$$\Rightarrow \hat{\gamma}_0 = \bar{y} \tag{7}$$

With (6) − (5), we have:

$$\begin{aligned}
&\sum(x_i - \bar{x})(y_i - \hat{\gamma}_0 - \hat{\gamma}_1(x_i - \bar{x})) = 0 \\
\Rightarrow &\sum(x_i - \bar{x})(y_i - \bar{y} - \hat{\gamma}_1(x_i - \bar{x})) = 0 \\
\Rightarrow &\sum(x_i - \bar{x})(y_i - \bar{y}) - \hat{\gamma}_1 \sum(x_i - \bar{x})^2 = 0 \\
\Rightarrow &\hat{\gamma}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} \qquad (8)
\end{aligned}$$

As shown above, $\hat{\beta}_1$ and $\hat{\gamma}_1$ are identical and having the same variance, while $\hat{\beta}_0 \neq \hat{\gamma}_0$ with difference in variance:

$$\begin{aligned}
\text{Var}(\hat{\beta}_0) - \text{Var}(\hat{\gamma}_0) &= [\text{Var}(\bar{y}) + \bar{x}^2 \text{Var}(\hat{\beta}_1) - 2\bar{x}\,\text{Cov}(\bar{y}, \hat{\beta}_1)] - \text{Var}(\bar{y}) \\
&= \bar{x}^2 \text{Var}(\hat{\beta}_1) - 2\bar{x}\,\text{Cov}\left(\frac{1}{n}\sum y_i, \frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2}\right) \\
&= \bar{x}^2 \text{Var}(\hat{\beta}_1) - 2\bar{x}\frac{1}{n}\frac{1}{\sum(x_i - \bar{x})^2}\,\text{Cov}\left(\sum y_i, \sum(x_i - \bar{x})y_i\right) \\
&= \bar{x}^2 \text{Var}(\hat{\beta}_1) - \frac{2\bar{x}}{n\sum(x_i - \bar{x})^2}\sum(x_i - \bar{x})\,\text{Var}\,(y_i) \\
&= \bar{x}^2 \text{Var}(\hat{\beta}_1) - \frac{2\bar{x}\sigma_y^2}{n\sum(x_i - \bar{x})^2}\underbrace{\sum(x_i - \bar{x})}_{=0} \\
&= \bar{x}^2 \text{Var}(\hat{\beta}_1) > 0
\end{aligned}$$

Thus, variance of estimator $\hat{\beta}_0$ is larger.

2. For a linear model without intercept, we have FOC for minimizing residual:

$$\frac{\partial Q_a}{\partial \beta_1} = -2\sum x_i(y_i - \beta_1 x_i) = 0 \tag{9}$$

which could be derived into an algebraic property of OLS estimator:

$$\sum_{i=1}^{n} x_i(y_i - \hat{\beta}_1 x_i) = \sum_{i=1}^{n} x_i\hat{\epsilon}_i = 0$$

By definition,

$$\begin{aligned}
\text{SST} &= \sum_{i=1}^{n}(y_i - \bar{y})^2 \\
&= \sum_{i=1}^{n}(\hat{y}_i - \bar{y} + \hat{\epsilon}_i)^2 \\
&= \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(\hat{\epsilon}_i)^2 + 2\sum_{i=1}^{n}(\hat{y}_i - \bar{y})(\hat{\epsilon}_i) \\
&= \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(\hat{\epsilon}_i)^2 + 2\sum_{i=1}^{n}(\hat{\beta}_1 x_i - \bar{y})(\hat{\epsilon}_i) \\
&= \underbrace{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}_{\text{SSE}} + \underbrace{\sum_{i=1}^{n}(\hat{\epsilon}_i)^2}_{\text{SSR}} + 2\hat{\beta}_1 \underbrace{\sum_{i=1}^{n}(x_i\hat{\epsilon}_i)}_{=0} - 2\bar{y}\sum_{i=1}^{n}\hat{\epsilon}_i \\
&= \text{SSE} + \text{SSR} - 2\bar{y}\sum_{i=1}^{n}\hat{\epsilon}_i
\end{aligned}$$

Therefore, for linear model without intercept, SSR isn't necessarily equals to SSE + SSR

3. Construct variables in matrix form:

$$\mathbf{y} = \begin{bmatrix}1\\0\\2\\1\\3\\5\end{bmatrix}, \mathbf{x} = \begin{bmatrix}1&0&0\\1&0&1\\1&0&2\\1&1&0\\1&1&1\\1&1&2\end{bmatrix}, \hat{\beta} = \begin{bmatrix}\hat{\beta}_0\\\hat{\beta}_1\\\hat{\beta}_2\end{bmatrix}$$

(a)

$$\begin{aligned}
\hat{\beta} &= (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y} \\
&= \left(\begin{bmatrix}1&1&1&1&1&1\\0&0&0&1&1&1\\0&1&2&0&1&2\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\1&0&2\\1&1&0\\1&1&1\\1&1&2\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1&1&1&1\\0&0&0&1&1&1\\0&1&2&0&1&2\end{bmatrix}\begin{bmatrix}1\\0\\2\\1\\3\\5\end{bmatrix} \\
&= \begin{bmatrix}6&3&6\\3&3&3\\6&3&10\end{bmatrix}^{-1}\begin{bmatrix}12\\9\\17\end{bmatrix} \\
&= \frac{1}{12}\begin{bmatrix}7&-4&-3\\-4&8&0\\-3&0&3\end{bmatrix}\begin{bmatrix}12\\9\\17\end{bmatrix} \\
&= \frac{1}{12}\begin{bmatrix}-3\\24\\15\end{bmatrix} \\
&= \begin{bmatrix}-\frac{1}{4}\\2\\\frac{5}{4}\end{bmatrix}
\end{aligned}$$

Therefore, $\hat{\beta}_1 = 2$, $\hat{\beta}_2 = 5/4$

(b)

$$\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \hat{\gamma} = \begin{bmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \end{bmatrix}$$

And its coefficients:

$$\begin{aligned} \hat{\gamma} &= (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y} \\ &= \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Therefore, $\hat{\gamma}_1 = 2 = \hat{\beta}_1$

(c)

$$\begin{aligned} \hat{u} = y - x_2\delta = [I - (x_2(x_2'x_2)^{-1}x_2')]y = [I - P_2]y \\ \hat{v} = x_1 - x_2\delta = [I - (x_2(x_2'x_2)^{-1}x_2')]x_1 = [I - P_2]x_1 \end{aligned}$$

where P_2 is an orthogonal projection matrix.

Numerically,

$$\hat{u} = \begin{bmatrix} 0.25 \\ -2.00 \\ -1.25 \\ 0.25 \\ 1.00 \\ 1.75 \end{bmatrix}, \hat{v} = \begin{bmatrix} -0.50 \\ -0.50 \\ -0.50 \\ 0.50 \\ 0.50 \\ 0.50 \end{bmatrix}$$

Regress $\hat{u}$ on $\hat{v}$ with a linear model without intercept:

$$\begin{aligned} \hat{\alpha} &= [\hat{v}'\hat{v}]^{-1}\hat{v}'\hat{u} \\ &= 2 \end{aligned}$$

Therefore, $\hat{\alpha} = 2 = \hat{\beta}_1$

4. (a)

```
X <- matrix(
    c(
        7, 4, 9, 0, 5,
        2, 6, 2, 9, 3,
        3, 7, 0, 0, 5
    ),
    ncol = 3
)
```

```r
Y <- matrix(
    c(6, 2, 4, 2, 1)
)

X_intercept <- cbind(rep(1, 5), X)

beta_hat <- solve(t(X_intercept) %*% X_intercept) %*% t(X_intercept) %*% Y
beta_hat
```

```
##             [,1]
## [1,]  2.38952067
## [2,]  0.29576302
## [3,] -0.05609562
## [4,] -0.20717168
```

(b)
```r
X_new <- c(1, 0, 4, 3)
Y_hat <- X_new %*% beta_hat
Y_hat
```

```
##          [,1]
## [1,] 1.543623
```

5. (a)
```r
data(mtcars)
mtcars
```

```
##                      mpg cyl  disp  hp drat    wt  qsec vs am gear carb
## Mazda RX4           21.0   6 160.0 110 3.90 2.620 16.46  0  1    4    4
## Mazda RX4 Wag       21.0   6 160.0 110 3.90 2.875 17.02  0  1    4    4
## Datsun 710          22.8   4 108.0  93 3.85 2.320 18.61  1  1    4    1
## Hornet 4 Drive      21.4   6 258.0 110 3.08 3.215 19.44  1  0    3    1
## Hornet Sportabout   18.7   8 360.0 175 3.15 3.440 17.02  0  0    3    2
## Valiant             18.1   6 225.0 105 2.76 3.460 20.22  1  0    3    1
## Duster 360          14.3   8 360.0 245 3.21 3.570 15.84  0  0    3    4
## Merc 240D           24.4   4 146.7  62 3.69 3.190 20.00  1  0    4    2
## Merc 230            22.8   4 140.8  95 3.92 3.150 22.90  1  0    4    2
## Merc 280            19.2   6 167.6 123 3.92 3.440 18.30  1  0    4    4
## Merc 280C           17.8   6 167.6 123 3.92 3.440 18.90  1  0    4    4
## Merc 450SE          16.4   8 275.8 180 3.07 4.070 17.40  0  0    3    3
## Merc 450SL          17.3   8 275.8 180 3.07 3.730 17.60  0  0    3    3
## Merc 450SLC         15.2   8 275.8 180 3.07 3.780 18.00  0  0    3    3
## Cadillac Fleetwood  10.4   8 472.0 205 2.93 5.250 17.98  0  0    3    4
## Lincoln Continental 10.4   8 460.0 215 3.00 5.424 17.82  0  0    3    4
## Chrysler Imperial   14.7   8 440.0 230 3.23 5.345 17.42  0  0    3    4
## Fiat 128            32.4   4  78.7  66 4.08 2.200 19.47  1  1    4    1
## Honda Civic         30.4   4  75.7  52 4.93 1.615 18.52  1  1    4    2
## Toyota Corolla      33.9   4  71.1  65 4.22 1.835 19.90  1  1    4    1
## Toyota Corona       21.5   4 120.1  97 3.70 2.465 20.01  1  0    3    1
## Dodge Challenger    15.5   8 318.0 150 2.76 3.520 16.87  0  0    3    2
## AMC Javelin         15.2   8 304.0 150 3.15 3.435 17.30  0  0    3    2
## Camaro Z28          13.3   8 350.0 245 3.73 3.840 15.41  0  0    3    4
## Pontiac Firebird    19.2   8 400.0 175 3.08 3.845 17.05  0  0    3    2
## Fiat X1-9           27.3   4  79.0  66 4.08 1.935 18.90  1  1    4    1
## Porsche 914-2       26.0   4 120.3  91 4.43 2.140 16.70  0  1    5    2
## Lotus Europa        30.4   4  95.1 113 3.77 1.513 16.90  1  1    5    2
## Ford Pantera L      15.8   8 351.0 264 4.22 3.170 14.50  0  1    5    4
## Ferrari Dino        19.7   6 145.0 175 3.62 2.770 15.50  0  1    5    6
## Maserati Bora       15.0   8 301.0 335 3.54 3.570 14.60  0  1    5    8
## Volvo 142E          21.4   4 121.0 109 4.11 2.780 18.60  1  1    4    2
```

```r
X_mtcars <- cbind(
    rep(1, length(mtcars$drat)),
    mtcars$wt,
    mtcars$hp,
    mtcars$qsec,
    mtcars$vs
)

beta_hat_mtcars <- solve(t(X_mtcars) %*% X_mtcars) %*% t(X_mtcars) %*% mtcars$drat
beta_hat_mtcars
```

```
##               [,1]
## [1,]  5.8346396715
## [2,] -0.3138840781
## [3,] -0.0003869982
## [4,] -0.0725544807
## [5,]  0.2823793330
```

(b)
```r
summary(lm(drat ~ wt + hp + qsec + vs, data = mtcars))
```

```
##
## Call:
## lm(formula = drat ~ wt + hp + qsec + vs, data = mtcars)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.76329 -0.24404  0.03211  0.25748  0.68374
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.834640   1.509470   3.865 0.000631 ***
## wt          -0.313884   0.139196  -2.255 0.032449 *
## hp          -0.000387   0.002324  -0.167 0.868992
## qsec        -0.072554   0.091331  -0.794 0.433885
## vs           0.282379   0.279744   1.009 0.321735
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3942 on 27 degrees of freedom
## Multiple R-squared:  0.5266, Adjusted R-squared:  0.4564
## F-statistic: 7.508 on 4 and 27 DF,  p-value: 0.0003348
```