Chapter 20

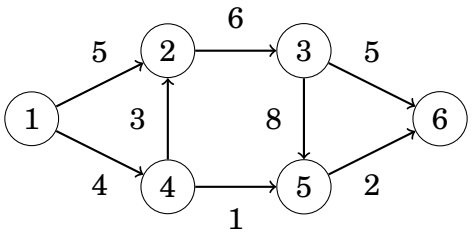
Flows and cuts

In this chapter, we focus on the following two problems:

- **Finding a maximum flow**: What is the maximum amount of flow we can send from a node to another node?
- **Finding a minimum cut**: What is a minimum-weight set of edges that separates two nodes of the graph?

The input for both these problems is a directed, weighted graph that contains two special nodes: the *source* is a node with no incoming edges, and the *sink* is a node with no outgoing edges.

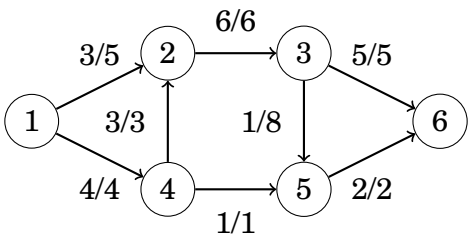
As an example, we will use the following graph where node 1 is the source and node 6 is the sink:



Maximum flow

In the **maximum flow** problem, our task is to send as much flow as possible from the source to the sink. The weight of each edge is a capacity that restricts the flow that can go through the edge. In each intermediate node, the incoming and outgoing flow has to be equal.

For example, the maximum size of a flow in the example graph is 7. The following picture shows how we can route the flow:

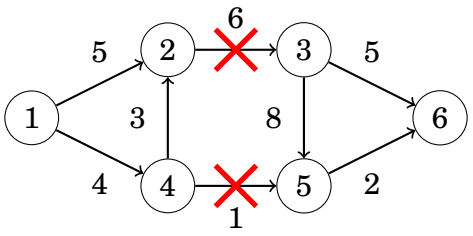


The notation v/k means that a flow of v units is routed through an edge whose capacity is k units. The size of the flow is 7, because the source sends $3+4$ units of flow and the sink receives $5+2$ units of flow. It is easy see that this flow is maximum, because the total capacity of the edges leading to the sink is 7.

Minimum cut

In the **minimum cut** problem, our task is to remove a set of edges from the graph such that there will be no path from the source to the sink after the removal and the total weight of the removed edges is minimum.

The minimum size of a cut in the example graph is 7. It suffices to remove the edges $2 \to 3$ and $4 \to 5$:



After removing the edges, there will be no path from the source to the sink. The size of the cut is 7, because the weights of the removed edges are 6 and 1. The cut is minimum, because there is no valid way to remove edges from the graph such that their total weight would be less than 7.

It is not a coincidence that the maximum size of a flow and the minimum size of a cut are the same in the above example. It turns out that a maximum flow and a minimum cut are *always* equally large, so the concepts are two sides of the same coin.

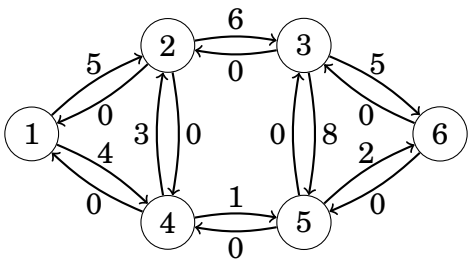
Next we will discuss the Ford–Fulkerson algorithm that can be used to find the maximum flow and minimum cut of a graph. The algorithm also helps us to understand *why* they are equally large.

Ford–Fulkerson algorithm

The **Ford–Fulkerson algorithm** [25] finds the maximum flow in a graph. The algorithm begins with an empty flow, and at each step finds a path from the source to the sink that generates more flow. Finally, when the algorithm cannot increase the flow anymore, the maximum flow has been found.

The algorithm uses a special representation of the graph where each original edge has a reverse edge in another direction. The weight of each edge indicates how much more flow we could route through it. At the beginning of the algorithm, the weight of each original edge equals the capacity of the edge and the weight of each reverse edge is zero.

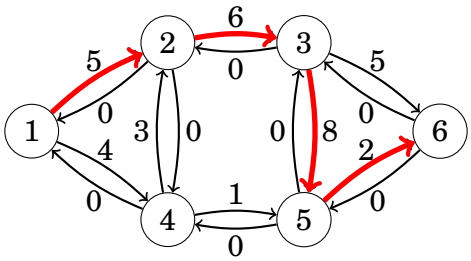
The new representation for the example graph is as follows:



Algorithm description

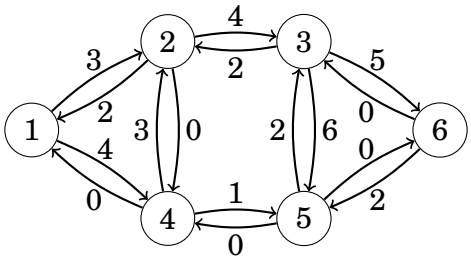
The Ford–Fulkerson algorithm consists of several rounds. On each round, the algorithm finds a path from the source to the sink such that each edge on the path has a positive weight. If there is more than one possible path available, we can choose any of them.

For example, suppose we choose the following path:



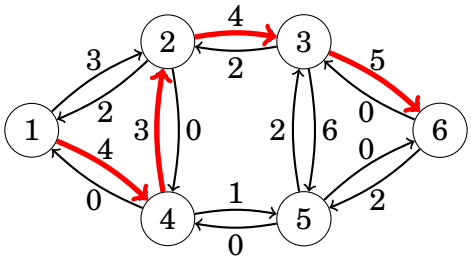
After choosing the path, the flow increases by x units, where x is the smallest edge weight on the path. In addition, the weight of each edge on the path decreases by x and the weight of each reverse edge increases by x.

In the above path, the weights of the edges are 5, 6, 8 and 2. The smallest weight is 2, so the flow increases by 2 and the new graph is as follows:



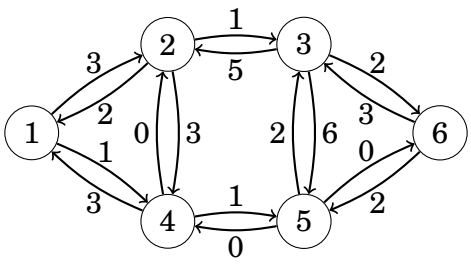
The idea is that increasing the flow decreases the amount of flow that can go through the edges in the future. On the other hand, it is possible to cancel flow later using the reverse edges of the graph if it turns out that it would be beneficial to route the flow in another way.

The algorithm increases the flow as long as there is a path from the source to the sink through positive-weight edges. In the present example, our next path can be as follows:

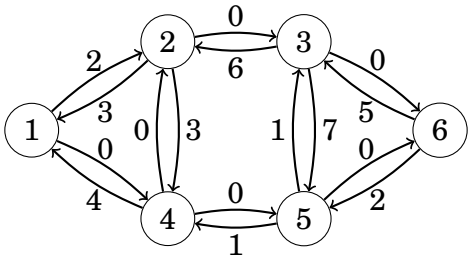


The minimum edge weight on this path is 3, so the path increases the flow by 3, and the total flow after processing the path is 5.

The new graph will be as follows:



We still need two more rounds before reaching the maximum flow. For example, we can choose the paths $1 \to 2 \to 3 \to 6$ and $1 \to 4 \to 5 \to 3 \to 6$. Both paths increase the flow by 1, and the final graph is as follows:



It is not possible to increase the flow anymore, because there is no path from the source to the sink with positive edge weights. Hence, the algorithm terminates and the maximum flow is 7.

Finding paths

The Ford–Fulkerson algorithm does not specify how we should choose the paths that increase the flow. In any case, the algorithm will terminate sooner or later and correctly find the maximum flow. However, the efficiency of the algorithm depends on the way the paths are chosen.

A simple way to find paths is to use depth-first search. Usually, this works well, but in the worst case, each path only increases the flow by 1 and the algorithm is slow. Fortunately, we can avoid this situation by using one of the following techniques:

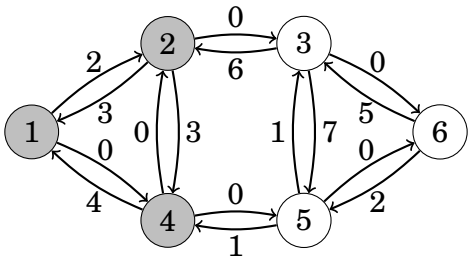
The **Edmonds–Karp algorithm** [18] chooses each path so that the number of edges on the path is as small as possible. This can be done by using breadth-first search instead of depth-first search for finding paths. It can be proven that this guarantees that the flow increases quickly, and the time complexity of the algorithm is $O(m^2 n)$.

The **scaling algorithm** [2] uses depth-first search to find paths where each edge weight is at least a threshold value. Initially, the threshold value is some large number, for example the sum of all edge weights of the graph. Always when a path cannot be found, the threshold value is divided by 2. The time complexity of the algorithm is $O(m^2 \log c)$, where c is the initial threshold value.

In practice, the scaling algorithm is easier to implement, because depth-first search can be used for finding paths. Both algorithms are efficient enough for problems that typically appear in programming contests.

Minimum cuts

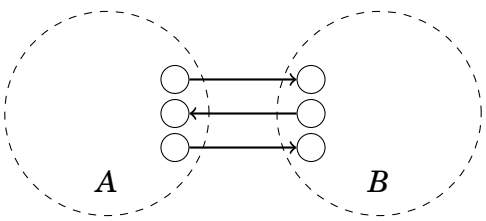
It turns out that once the Ford–Fulkerson algorithm has found a maximum flow, it has also determined a minimum cut. Let A be the set of nodes that can be reached from the source using positive-weight edges. In the example graph, A contains nodes 1, 2 and 4:



Now the minimum cut consists of the edges of the original graph that start at some node in A, end at some node outside A, and whose capacity is fully used in the maximum flow. In the above graph, such edges are $2 \to 3$ and $4 \to 5$, that correspond to the minimum cut $6 + 1 = 7$.

Why is the flow produced by the algorithm maximum and why is the cut minimum? The reason is that a graph cannot contain a flow whose size is larger than the weight of any cut of the graph. Hence, always when a flow and a cut are equally large, they are a maximum flow and a minimum cut.

Let us consider any cut of the graph such that the source belongs to A, the sink belongs to B and there are some edges between the sets:



The size of the cut is the sum of the edges that go from A to B. This is an upper bound for the flow in the graph, because the flow has to proceed from A to B. Thus, the size of a maximum flow is smaller than or equal to the size of any cut in the graph.

On the other hand, the Ford–Fulkerson algorithm produces a flow whose size is *exactly* as large as the size of a cut in the graph. Thus, the flow has to be a maximum flow and the cut has to be a minimum cut.

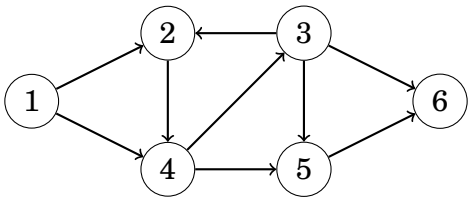
Disjoint paths

Many graph problems can be solved by reducing them to the maximum flow problem. Our first example of such a problem is as follows: we are given a directed graph with a source and a sink, and our task is to find the maximum number of disjoint paths from the source to the sink.

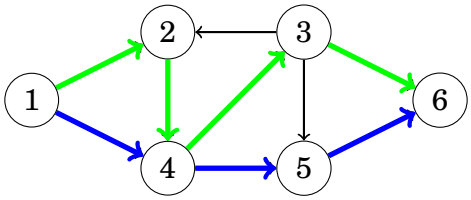
Edge-disjoint paths

We will first focus on the problem of finding the maximum number of **edge-disjoint paths** from the source to the sink. This means that we should construct a set of paths such that each edge appears in at most one path.

For example, consider the following graph:



In this graph, the maximum number of edge-disjoint paths is 2. We can choose the paths $1 \to 2 \to 4 \to 3 \to 6$ and $1 \to 4 \to 5 \to 6$ as follows:



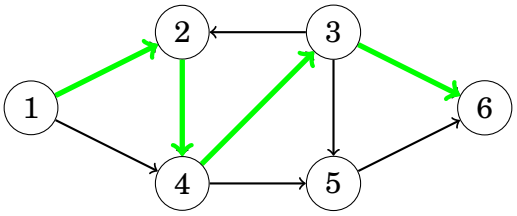
It turns out that the maximum number of edge-disjoint paths equals the maximum flow of the graph, assuming that the capacity of each edge is one. After the maximum flow has been constructed, the edge-disjoint paths can be found greedily by following paths from the source to the sink.

Node-disjoint paths

Let us now consider another problem: finding the maximum number of **node-disjoint paths** from the source to the sink. In this problem, every node, except

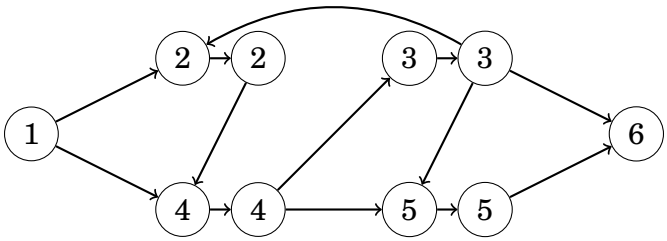
for the source and sink, may appear in at most one path. The number of node-disjoint paths may be smaller than the number of edge-disjoint paths.

For example, in the previous graph, the maximum number of node-disjoint paths is 1:

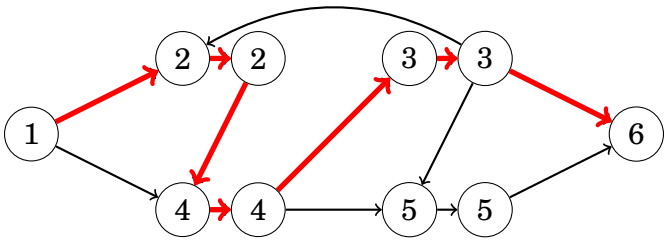


We can reduce also this problem to the maximum flow problem. Since each node can appear in at most one path, we have to limit the flow that goes through the nodes. A standard method for this is to divide each node into two nodes such that the first node has the incoming edges of the original node, the second node has the outgoing edges of the original node, and there is a new edge from the first node to the second node.

In our example, the graph becomes as follows:



The maximum flow for the graph is as follows:



Thus, the maximum number of node-disjoint paths from the source to the sink is 1.

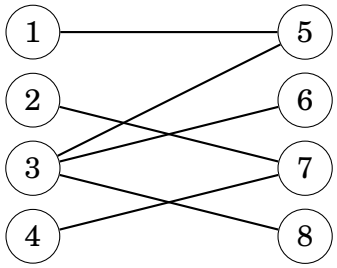
Maximum matchings

The **maximum matching** problem asks to find a maximum-size set of node pairs in an undirected graph such that each pair is connected with an edge and each node belongs to at most one pair.

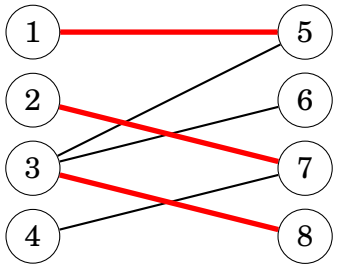
There are polynomial algorithms for finding maximum matchings in general graphs [17], but such algorithms are complex and rarely seen in programming contests. However, in bipartite graphs, the maximum matching problem is much easier to solve, because we can reduce it to the maximum flow problem.

Finding maximum matchings

The nodes of a bipartite graph can be always divided into two groups such that all edges of the graph go from the left group to the right group. For example, in the following bipartite graph, the groups are $\{1,2,3,4\}$ and $\{5,6,7,8\}$.

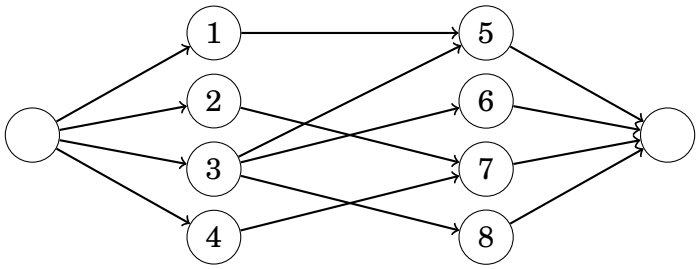


The size of a maximum matching of this graph is 3:

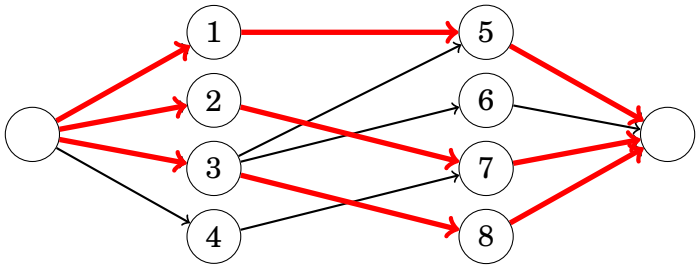


We can reduce the bipartite maximum matching problem to the maximum flow problem by adding two new nodes to the graph: a source and a sink. We also add edges from the source to each left node and from each right node to the sink. After this, the size of a maximum flow in the graph equals the size of a maximum matching in the original graph.

For example, the reduction for the above graph is as follows:



The maximum flow of this graph is as follows:

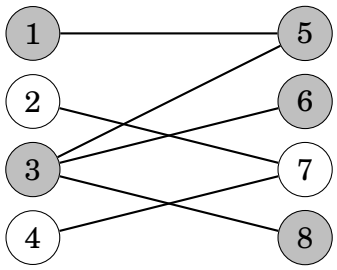


Hall's theorem

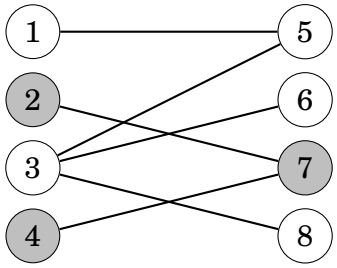
Hall's theorem can be used to find out whether a bipartite graph has a matching that contains all left or right nodes. If the number of left and right nodes is the same, Hall's theorem tells us if it is possible to construct a **perfect matching** that contains all nodes of the graph.

Assume that we want to find a matching that contains all left nodes. Let X be any set of left nodes and let $f(X)$ be the set of their neighbors. According to Hall's theorem, a matching that contains all left nodes exists exactly when for each X, the condition $|X| \leq |f(X)|$ holds.

Let us study Hall's theorem in the example graph. First, let $X = \{1,3\}$ which yields $f(X) = \{5,6,8\}$:



The condition of Hall's theorem holds, because $|X| = 2$ and $|f(X)| = 3$. Next, let $X = \{2,4\}$ which yields $f(X) = \{7\}$:



In this case, $|X| = 2$ and $|f(X)| = 1$, so the condition of Hall's theorem does not hold. This means that it is not possible to form a perfect matching for the graph. This result is not surprising, because we already know that the maximum matching of the graph is 3 and not 4.

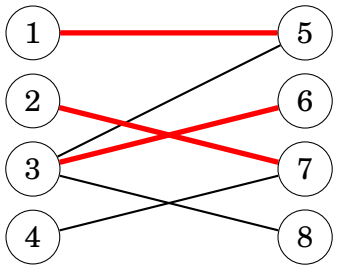
If the condition of Hall's theorem does not hold, the set X provides an explanation *why* we cannot form such a matching. Since X contains more nodes than $f(X)$, there are no pairs for all nodes in X. For example, in the above graph, both nodes 2 and 4 should be connected with node 7 which is not possible.

Kőnig's theorem

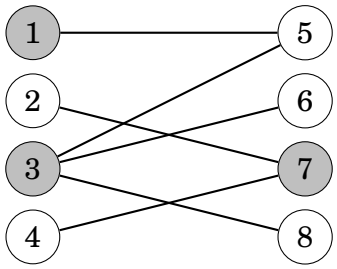
A **minimum node cover** of a graph is a minimum set of nodes such that each edge of the graph has at least one endpoint in the set. In a general graph, finding a minimum node cover is a NP-hard problem. However, if the graph is bipartite, **Kőnig's theorem** tells us that the size of a minimum node cover and the size

of a maximum matching are always equal. Thus, we can calculate the size of a minimum node cover using a maximum flow algorithm.

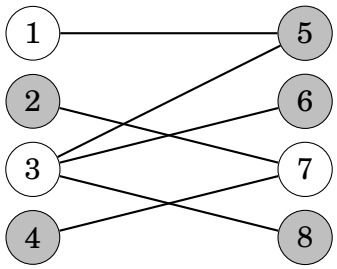
Let us consider the following graph with a maximum matching of size 3:



Now Kőnig's theorem tells us that the size of a minimum node cover is also 3. Such a cover can be constructed as follows:



The nodes that do *not* belong to a minimum node cover form a **maximum independent set**. This is the largest possible set of nodes such that no two nodes in the set are connected with an edge. Once again, finding a maximum independent set in a general graph is a NP-hard problem, but in a bipartite graph we can use Kőnig's theorem to solve the problem efficiently. In the example graph, the maximum independent set is as follows:

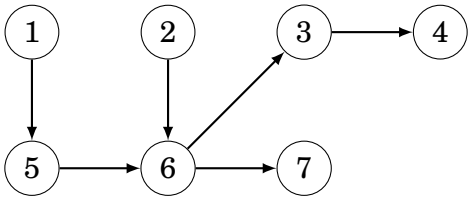


Path covers

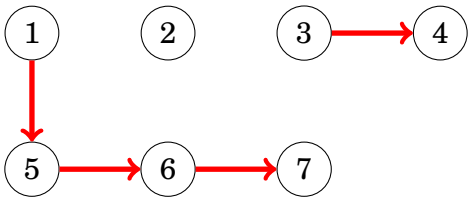
A **path cover** is a set of paths in a graph such that each node of the graph belongs to at least one path. It turns out that in directed, acyclic graphs, we can reduce the problem of finding a minimum path cover to the problem of finding a maximum flow in another graph.

Node-disjoint path cover

In a **node-disjoint path cover**, each node belongs to exactly one path. As an example, consider the following graph:



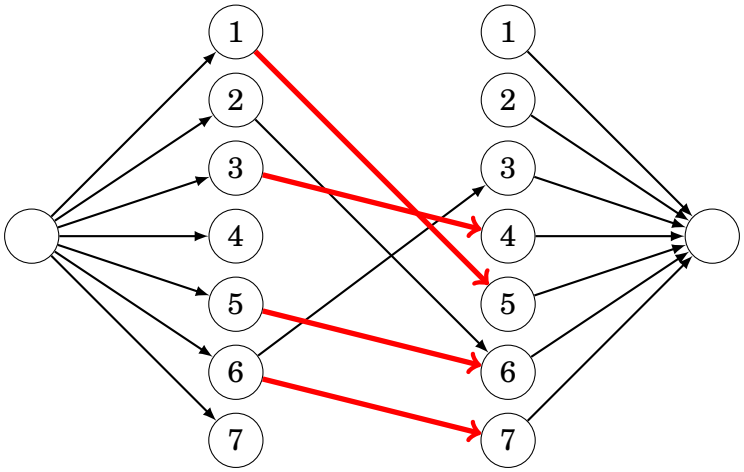
A minimum node-disjoint path cover of this graph consists of three paths. For example, we can choose the following paths:



Note that one of the paths only contains node 2, so it is possible that a path does not contain any edges.

We can find a minimum node-disjoint path cover by constructing a *matching graph* where each node of the original graph is represented by two nodes: a left node and a right node. There is an edge from a left node to a right node if there is such an edge in the original graph. In addition, the matching graph contains a source and a sink, and there are edges from the source to all left nodes and from all right nodes to the sink.

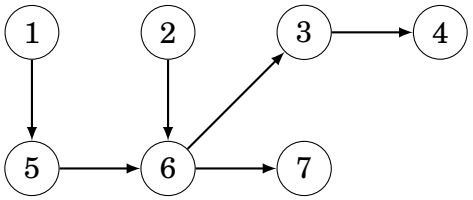
A maximum matching in the resulting graph corresponds to a minimum node-disjoint path cover in the original graph. For example, the following matching graph for the above graph contains a maximum matching of size 4:



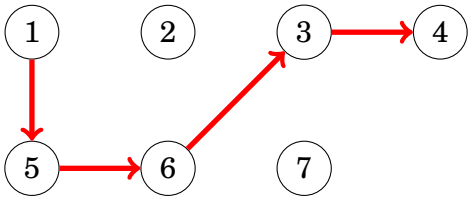
Each edge in the maximum matching of the matching graph corresponds to an edge in the minimum node-disjoint path cover of the original graph. Thus, the size of the minimum node-disjoint path cover is $n - c$, where n is the number of nodes in the original graph and c is the size of the maximum matching.

General path cover

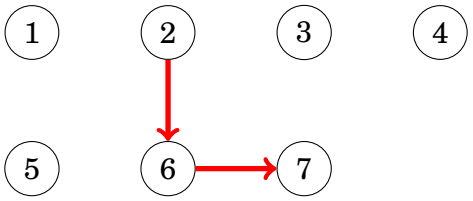
A **general path cover** is a path cover where a node can belong to more than one path. A minimum general path cover may be smaller than a minimum node-disjoint path cover, because a node can be used multiple times in paths. Consider again the following graph:



The minimum general path cover of this graph consists of two paths. For example, the first path may be as follows:

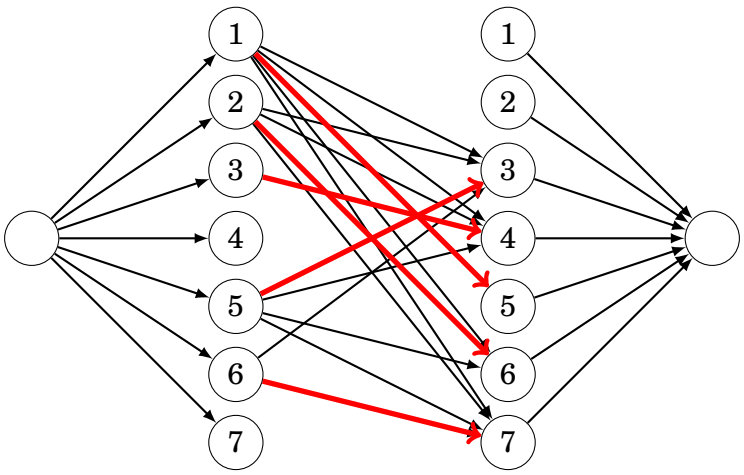


And the second path may be as follows:



A minimum general path cover can be found almost like a minimum node-disjoint path cover. It suffices to add some new edges to the matching graph so that there is an edge $a \rightarrow b$ always when there is a path from a to b in the original graph (possibly through several edges).

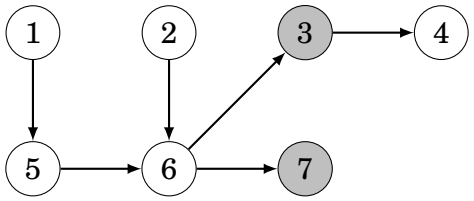
The matching graph for the above graph is as follows:



Dilworth's theorem

An **antichain** is a set of nodes of a graph such that there is no path from any node to another node using the edges of the graph. **Dilworth's theorem** states that in a directed acyclic graph, the size of a minimum general path cover equals the size of a maximum antichain.

For example, nodes 3 and 7 form an antichain in the following graph:



This is a maximum antichain, because it is not possible to construct any antichain that would contain three nodes. We have seen before that the size of a minimum general path cover of this graph consists of two paths.