ization problem is:

$$\max_{Y_{H,t},I_t,N_{H,t},K_t} \; P_{H,t}Y_{H,t} - P_{I,t}I_t - W_{H,t}N_{H,t} - P_{H,t}(R_t^K - 1)K_t \qquad (10)$$
$$\text{s.t.} \quad (9)$$

Finally, H and F competitive firms in the aggregate home and foreign final goods, combine final goods produced in H and F into an aggregate final consumption good using a CD technology. In the case of the H bloc (and F bloc, mutatis mutandis) the resulting specification is given by:

$$C_t = C_{H,t}^{1-\gamma} C_{F,t}^{\gamma} \qquad (11)$$

where $C_t, C_{H,t}, C_{F,t}$ denote the aggregate consumption basket, the aggregate final home good and the aggregate final foreign good. The maximization is then:

$$\max_{C_{H,t},C_{F,t}} P_t C_t - P_{H,t}C_{H,t} - P_{F,t}C_{F,t}$$
$$\text{s.t.} \quad (11) \qquad (12)$$

3.2 Modeling natural capital

This subsection explains in greater detail our modeling strategy for the evolution of natural capital.[12] In line with Dasgupta and Mäler (2004), D'Alessandro (2007) and Kornafel and Telega (2019), in studying our model dynamics we concentrate on two versions of this general specification: one with and one without a critical threshold. These two versions have a well established tradition in the study of fisheries management (Clark (2006), and conservation more in general Clark (2010)), and thus allow us to conceptualize the dynamic resource-harvesting problem that economic agents face when deciding how much of the natural resource to exploit for production and how much to keep in place for (possible) future use.

3.2.1 Natural capital with no critical threshold

In this version the stock of natural capital can always recover to its original, carrying capacity level, no matter what amount of depletion occurs between periods. In particular, in this version, the stock of natural capital depends non-linearly on its "background" or "natural" regeneration rate, which in turn depends on how far the existing stock is from its carrying capacity level CC, as well as on the amount that

[12] We focus on the aggregate economy-wide stock since we rule out idiosyncratic shocks and also assume that the dynamics governing the stock are identical across all its natural capital stock owners.

is exploited for production:

$$K_{N,t+1} = K_{N,t}\left(1 + r_N\left(1 - \frac{K_{N,t}}{CC}\right)\right) - K_{N,t}^b \tag{13}$$

We call the rate at which natural capital accumulates (or decumulates) through the impact of its own regeneration given the beginning of period existing stock k_N's*accumulation rate*:

$$A_{N,t} \equiv r_N K_{N,t}\left(1 - \frac{K_{N,t}}{CC}\right) \tag{14}$$

In other words, the rate of accumulationdepends on Nature'scarrying capacity CC, and it declines as the stock of natural capital approaches CC, as follows:

Proposition 1. *$A_{N,t}$ is monotonously increasing in $K_{N,t}$ for $K_{N,t} < CC/2$ and monotonously decreasing in $K_{N,t}$ for $K_{N,t} > CC/2$; that is, as the stock approaches its carrying capacity, the speed of regeneration diminishes (converging to zero in the limit, as $K_{N,t}$ approaches CC). Moreover, $\frac{\partial^2 A_{N,t}}{\partial K_{N,t}^2} < 0$ for $K_{N,t} \in (0, CC)$.*

Proof. See Appendix A.3. □

3.2.2 Natural capital with a critical threshold

Since the ability of natural capital to recover may change when natural capital exceeds CT, we consider also a second version of the general specification which makes the evolution of natural capital dependent on such threshold. Assuming that the level of CT is known to the agents in the economy, the equation for natural capital under this specification becomes:

$$K_{N,t+1} = K_{N,t}\left(1 + r_N\left(1 - \frac{K_{N,t}}{CC}\right)\left(\frac{K_{N,t}}{CT} - 1\right)\right) - K_{N,t}^b \tag{15}$$

In this case, once $K_{N,t} < CT$, the existing stock of natural capital converges progressively to zero. The accumulation/decumulation rate is then given by:

$$A_{N,t} \equiv r_N K_{N,t}\left(1 - \frac{K_{N,t}}{CC}\right)\left(\frac{K_{N,t}}{CT} - 1\right) \tag{16}$$

In other words, in the presence of a critical threshold, the rate at which natural capital accumulates/decumulates depends not only on CC and r_N but now also on CT. The proposition below analyzes the sign of $A_{N,t}$ in the case in which there is a critical threshold.

Proposition 2. *For values of $K_{N,t} \in (CT, CC)$, $A_{N,t}$ is monotonously increasing if* *and only if*

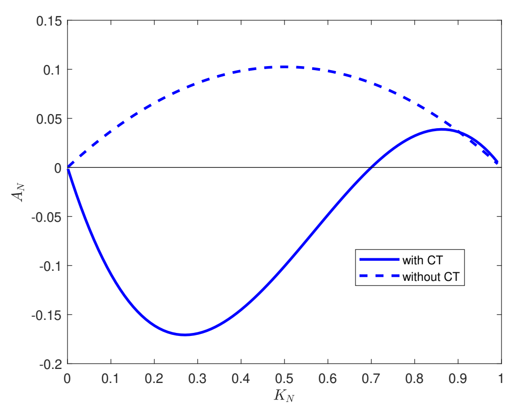
$$1 - 2K_{N,t}/CC + \frac{K_{N,t}}{CC}\left(\frac{CC - K_{N,t}}{K_{N,t} - CT}\right) > 0 \tag{17}$$

and monotonously decreasing otherwise. Moreover, $\frac{\partial^2 A_{N,t}}{\partial K_{N,t}^2} < 0$*. Importantly, the second term in the expression introduces a tension which ultimately determines the sign of* $\frac{\partial A_{N,t}}{\partial K_{N,t}}$*, and which will depend on the distance of* K_N *from* CC *and* CT*; for example, the closer the stock of natural capital is to* CT *(from above) the stronger is the positive pressure coming from being far from* CC*, but also, at the same time, the stronger is the negative pressure coming from being close to* CT*. The maximum in the interval (*CT*,*CC*) reflects the balancing out of this tension.*

Proof. See Appendix A.3. □

To help understand what these two alternative specifications entail for K_N in practice, Figure 1 plots the rate at which natural capital evolves (that is, k_N's accumulation rate A_N) with or without CT, normalizing the value of CC to 1. In line with the above propositions, the figure shows that, in the absence of a critical threshold, the accumulation rate of natural capital is always positive and increases before decreasing in proximity of natural capital's maximum sustainable level, CC (namely, A_N is always above zero in the interval $(0, 1)$, increasing for $K_N < CC/2$ and decreasing for $K_N > CC/2$) . Conversely, in presence of a critical threshold, A_N is negative for values below CT, but positive and increasing for a range of values between CT and CC before converging to zero as K_N approaches CC.

Figure 1: Nature Accumulation Rates (A_N)



Note: $CC = 1, CT = 0.7$. $r_N = 1.4$ when assuming a CT, and $r_N = 0.4$ otherwise.

In practice, under both specifications, the accumulation rate of natural capital remains uncertain because parametric shocks to each specification may affect the evolution of natural capital. To capture this we go one step further in modeling K_N and postulate that there are shocks that affect multiplicatively k_N's accumulation rate. Specifically we define a stationary shock process:

$$ln\ (z_{t+1}) = \rho^N ln\ (z_t) + \sigma_\epsilon \epsilon_{t+1} \tag{18}$$

where $\sigma_t > 0$, $|\rho^N| \leq 1$ and $\epsilon_{t+1} \sim \mathcal{N}(0,1)$. We thus re-write the law of motion of natural capital (in the absence of critical threshold) as:

$$K_{N,t+1} = K_{N,t}\left(1 + z_t r_N\left(1 - \frac{K_{N,t}}{CC}\right)\right) - K^b_{N,t} \tag{19}$$

We adopt the same approach when modeling natural capital in the presence of a critical threshold. The multiplicative assumption implies that the greater A_N, the larger the uncertainty that the agents or the social planner face when making optimal decisions, due to higher possible realizations of the shock.[13] Importantly the (log) formulation of the shock implies that the accumulation rate cannot turn negative following the realization of a bad shock. This is a simplifying assumption, which we adopt to contain the studied equilibria within the economically sustainable region (to the right of CT).

3.3 Aggregation and market clearing

We allow for aggregate shocks in the evolution of natural capital, but rule out idiosyncratic shocks. Then, assuming that each atomistic unit is indexed by i, the aggregate production, stock of natural capital, amount of labor associated with the green and brown technologies, consumption levels, and physical and financial assets are given

[13] In more complex specifications one could assume that σ_ϵ is time-varying. For example, as the stock of nature approaches the CT of the economy, there could be greater variability in the persistence and size of the shocks associated with the regeneration rate, due for example to an increase over time in the frequency and magnitude of extreme natural events. The inclusion of a shock over the regeneration rate could also be interpreted as a parsimonious way to model uncertainty over property rights; for example by setting $\epsilon_{t+1} \sim \mathcal{TN}_{(-\infty,0)}(0,1)$, where $\mathcal{TN}_{(a,b)}(.)$ represents a truncated normal distribution on the interval (a,b), it is possible interpret the (negative) shocks as originating from expropriation of (part) of the newly formed amount of Nature.