UNIT-V : NUMERICAL INTEGRATION AND NUMERICAL SOLUTION OF O.D.E

Numerical Integration:

W.K.T a definite integral of the form $\int_a^b f(x)\,dx$ represents the area under the area $y=f(x)$ enclosed between the limits $x=a$ and $x=b$. This integration is possible only if $f(x)$ is explicitly is given and if it is integrable. To evaluate the integral of the type $\int_a^b f(x)\,dx$ we use the following three formulas

1. Trapezoidal rule
2. Simpson's $\frac{1}{3}$ rule (Simpson's rule)
3. Simpson's $\frac{3}{8}$ rule.

Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$= \frac{h}{2}$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

Here $h = \frac{b-a}{n}$, n = no. of sub intervals = even or odd

i.e in Trapezoid rule no of subintervals may be even or odd. h means interval width.

Simpson's $\frac{1}{3}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+\cdots)\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(\text{sum of odd ordinates})+2(\text{sum of even ordinates})\right]$$

Here y_0 = first ordinate, y_n = last ordinate

ordinate means y values.

Here $h = \frac{b-a}{n}$, n = no. of sub intervals.

It should be noted that in simpson's $\frac{1}{3}$ rule, the given interval must be divided into an even no. of sub intervals of width 'h' i.e. n may be 2, 4, 6, 8, 10 ... etc

Simpson's $\frac{3}{8}$ rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+y_7\cdots)+2(y_3+y_6+y_9+\cdots)\right]$$

Here $h = \frac{b-a}{n}$, n = no. of sub intervals.

It should be noted that in simpson's $\frac{3}{8}$ rule, the given interval must be divided into multiple of 3 i.e. n may be 3, 6, 9, 12 ... etc

Note: Suppose if we want to apply these 3 rules for a given problem $\int_a^b f(x)\,dx$, we have to divide $[a,b]$ in to 6 multiple

i.e. n = 6, 12, 18 ... etc. If $n=6$, which is even as well as multiple of 3.

Problems

1. Evaluate $\int_0^1 x^3 dx$ with five sub intervals by trapezoidal rule

Sol: Given integral $\int_0^1 x^3 dx$ compare with $\int_a^b f(x)dx$,

Here $a=0$, $b=1$, $f(x)=x^3$

Given n = five sub intervals $=5$

$\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

The values of x and y are tabulated below

x	0	0.2	0.4	0.6	0.8	1
$y=x^3$	0	0.008	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoid rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$\int_0^1 x^3 dx = 0.26$$

2. Evaluate $\int_0^2 e^{-x^2} dx$ using simpson's rule taking $h=0.25$

Sol: Here, $a=0$, $b=2$, $f(x)=e^{-x^2}$, $h=0.25$

The values of x and y are given below.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x)$	1	0.9394	0.7788	0.5698	0.3679	0.2096	0.1054	0.0468	0.0183

By simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_8)+4(\text{sum of odd coordinates})+2(\text{sum of even ordinates})\right]$$

$$= \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

Note: Here y_0 is the first ordinate, y_8 is the last ordinate. It is not once y_8 is last ordinate it is not considered as even ordinate [In our formula sub all the values exactly once, no value is repeated and no value is missing

$$\int_0^2 e^{-x^2}\,dx = \frac{0.25}{3}\left[(1+0.0183)+4(0.9394+0.5698+0.2076+0.0468)+2(0.7788+0.3679+0.1054)\right]$$

$$\int_0^2 e^{-x^2}\,dx = 0.8821$$

3. Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ using 3 rules and hence find $\log_e 2$ value

Sol: Given $\int_0^1 \frac{1}{1+x}\,dx$ compare $\int_a^b f(x)\,dx$

Here $a=0$, $b=1$, $f(x)=\frac{1}{1+x}$, Taking $n=6$,

$$h=\frac{b-a}{n}=\frac{1}{6}$$

(To apply 3 rules take $n=6$ which is even as well as multiple of 3.)

x	0	1/6	2/6	3/6	4/6	5/6	1
f(x)	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

T-Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1/6}{2}\left[(1+0.5)+2(0.857+0.6667+0.5455+(0.75+0.6))\right]$$

$$= \frac{1/6}{2}\left[(1+0.5)+2(0.857+0.75+0.6667+0.6+0.5455)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.6219$$

($\int$ press $\frac{1}{1+x}$ and then press solve for calc then enter x values ...)

$\frac{1}{3}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{h}{3}\ \frac{1/6}{3}\left[(1+0.5)+4(0.857+0.6667+0.5455)+2(0.75+0.6)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.6932$$

$\frac{3}{8}$ rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{3}{8}\cdot\frac{1}{6}\left[(1+0.5)+3(0.857+0.75+0.6+0.5455)+2(0.6667)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.6932$$

To find $\log_e 2$ value:

W.K.T $\int_0^1 \frac{1}{1+x}\,dx = \log_e(1+x)\Big|_0^1$

$$\int_0^1 \frac{1}{1+x}\,dx = \log_e(2)-\log_e 1$$

$$= \log_e 2$$

$\therefore$ from above

$$\log_e 2 = 0.69327$$

61. Evaluate $\int_0^1 \sqrt{1+x^4}\,dx$ using 3 rules

Sol: In casio fix the function $\sqrt{(1+x^4)}$ To display x operate Alpha) (right bracket) calculate f(x) values for x values

Here $a=0$, $b=1$, $f(x) = \sqrt{1+x^4}$

Take $n=6$ $\therefore h = \frac{b-a}{n} = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	1
f(x)	1	1.0004	1.0062	1.0308	1.0943	1.2175	1.4142

1. T-Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\int_0^1 \sqrt{1+x^4}\,dx = \frac{1}{12}\left[(1+1.41142) + 2(1.0004+1.0062+1.0308+1.0943+1.2175)\right]$$

$$= 1.0927$$

2. $\frac{1}{3}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$\int_0^1 \sqrt{1+x^4}\,dx = \frac{1}{18}\left[(1+1.4142) + 4(1.0004+1.0308+1.2175) + 2(1.0062+1.0943)\right]$$

$$= 1.0894$$

3. $\frac{3}{8}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$\int_0^1 \sqrt{1+x^4}\,dx = \frac{3}{8}\times\frac{1}{6}\left[(1+1.4142)+3(1.0004+1.0062+1.07434+1.2172)+2(1.0308)\right]$$

$$\int_0^1 \sqrt{1+x^4}\,dx = 1.0894$$

from page 6

5 Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using simpson's $\frac{3}{8}$ rule by taking 7 ordinates. Hence obtain an approximate value of π

Sol: Here $a=0$, $b=1$, $f(x)=\frac{1}{1+x^2}$ $\left[f(x) = 1 \div (1+x^2)\right]$

To obtain 7 ordinates take no. of subintervals $n=6$

$\therefore h = \frac{b-a}{n} = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$	1	0.9730	0.9	0.8	0.6923	0.5901	0.5

By simpson's $\frac{3}{8}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3}{8}\times\frac{1}{6}\left[(1+0.5)+3(0.9730+0.9+0.6923+0.5901)+2(0.8)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.7854 \longrightarrow (1)$$

By actual integration $\int_0^1 \frac{1}{1+x^2}\,dx = (\tan^{-1}x)_0^1$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} \longrightarrow (2)$$

from ① & ② L.H.S same so equate R.H.S, we have

$$\frac{\pi}{4} = 0.7854$$

$$\pi = 4(0.7854)$$

$$\pi = 3.1416$$

8. 4.3.P Evaluate $\int_0^{\pi} t\sin t\,dt$ using trapezoidal rule by taking $n=6$

Sol: Here $a=0$, $b=\pi$, $f(t)=t\sin t$, $n=6$

$$\therefore h = \frac{b-a}{n} = \frac{\pi}{6}$$

Here $f(t) = t\sin t$, t is algebraic function, $\sin t$ is trigonometric function. Given limits are from 0 to π (180) i.e in terms of degrees but algebraic function considered only radian values i.e 1, 2, 3 like that. So convert the given limits in terms of radians, in radians $\pi = \frac{22}{7}$, put calcs made in radians.

t (deg)	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
t (rad)	0	0.5238	1.0476	1.5715	2.0953	2.6191	3.1429
f(t)	0	0.2620	0.9075	1.5715	1.8136	1.3096	−0.0041

By T-rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\int_0^{\pi} t\sin t\,dt = \frac{\pi}{12}\left[(0-0.0041)\right] + 2(0.2620+0.9075+1.5715+1.8136+1.3096)]$$

$$= \frac{22/7}{12}\left[-0.0041 + 2(5.8616)\right]$$

$$= 3.0693$$

7 A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and the position of the rocket at $t = 40$ sec. Use Trapezoidal rule.

t	0	5	10	15	20	25	30	35	40
$a(t)$	40.0	45.25	48.50	51.25	54.35	59.48	61.5	64.3	68.7

Sol: If 's' is the distance travelled in time t and 'v' is the velocity at time t, then

acceleration (a) = rate of change of velocity

$$\Rightarrow a(t) = \frac{dv}{dt}$$

$$\int a(t)\,dt = \int \frac{dv}{dt}\,dt$$

$$\int a(t)\,dt = v$$

$$\therefore \text{velocity } (v) = \int_0^{40} a(t)\,dt, \text{ compare } \int_a^b f(t)\,dt$$

Here $a = 0$, $b = 40$, $h = 5$

(or) Here 9 ordinates are given so take no. of subintervals $n = 8$

$$\therefore h = \frac{b-a}{n}$$

$$h = \frac{40-0}{8}$$

$$h = 5$$

by T-Rule

$$\int_a^b f(t)\,dt = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$\int_0^{40} a(t)\,dt = \frac{5}{2}\left[(40+68.7)+2(45.25+48.50+51.25+54.35+59.48+61.5+64.3)\right]$$

$$V = 2194.9$$

position of the rocket at t = 40 seconds

t = velocity x time

$= 2194.9 \times 40$

$= 87796$

8. When a train is moving at 30m/sec, steam is shutoff and brakes are applied. The speed of the train per second after t seconds is given by using simpson's rule. Determine the distance moved by the train in 40 seconds

Time(t)	0	5	10	15	20	25	30	35	40
Speed(v)	30	24	19.5	16	13.6	11.7	10	8.5	7.0

Sol: W.K.T velocity (v) = rate of change of displacement

$$\Rightarrow v = \frac{ds}{dt}$$

$$\int v\,dt = \int \frac{ds}{dt}\,dt$$

$$s = \int v\,dt$$

$$\therefore \text{distance } (s) = \int_0^{40} v\,dt \; ; \quad \text{Here } a=0;\ b=40;\ h=5$$

By simpson's rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$\int_0^{40} v(t)\,dt = \frac{5}{3}\left[(30+7)+4(24+16+11.7+8.5)+2(19.5+13.6+10)\right]$$

$$s = \int_0^{40} v(t)\,dt = 606.6667$$

$$s = 606.6667$$

4. Evaluate $\int_0^{\pi/2} e^{\sin x} dx$, taking $h = \frac{\pi}{6}$

Sol: Here $a = 0$, $b = \pi/2$, $f(x) = e^{\sin x}$, Given $h = \frac{\pi}{6}$

x (degrees)	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$
x (radians)	0	0.5238	1.0476	1.5715
$f(x) = e^{\sin x}$	1	1.6490	2.3779	2.7183

By T-rule

$$\int_a^b f(x) = \frac{h}{2}\left[(y_0+y_3)+2(y_1+y_2)\right]$$

$$\int_0^{\pi/2} e^{\sin x} dx = \frac{\pi}{12}\left[(1+2.7183)+2(1.6490+2.3779)\right]$$

$$= \frac{22}{7\times 12}\left[3.7183+2(4.0269)\right]$$

$$= 3.0832$$

Numerical solution of ordinary differential equations:

Many problems in science and engineering can be formulated into ordinary differential equations. The analytical methods (soln of I order and I degree D.E learn in previous semister) of solving D.E are applicable only to a selected class of differential equations. In that case we use the following Numerical methods to solve first order and first degree ordinary D.E's. In this chapter we solve the first order and first degree differential equations with some conditions.

consider a first order and first degree D.E with initial condition which is

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0$$

To solve this type of D.E's we use the following methods

1. Taylor series expansion
2. Euler's method
3. Modified Eulers method
4. Runga-kutta methods
5. picards method of successive approximation

Taylor series method:

Given D.E $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

To find y values $y_1, y_2, y_3 \ldots$ corresponding to $x = x_1, x_2, x_3 \ldots$ where $x_1 = x_0 + h$, $x_2 = x_1 + h$.

Taylor series expansion of y at $x = x_1$, $x = x_2$ is defined below.

By Taylor series expansion

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$$\therefore y_{(n+1)} = y_{(x_{n+1})} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots$$

Taylor Series problems

Using Taylor Series method, find an approximate value of y at $x=0.1, 0.2$ for the differential equation $y'-2y=3e^x$, $y(0)=0$, compare the numerical solution with exact solution

Given D.E $y'-2y=3e^x$, $y(0)=0$

$$\Rightarrow y'=2y+3e^x, \; y(0)=0$$

compare given D.E with standard form

$$\frac{dy}{dx}=f(x,y) \text{ with } y(x_0)=y_0$$

$$\therefore f(x,y)=2y+3e^x, \; x_0=0, \; y_0=0$$

Now, we have to find the values of y at $x_1=0.1$, $x_2=0.2$ using Taylor series method

$\therefore h=0.1$ $\quad$ ($\because x_1=x_0+h \Rightarrow h=x_1-x_0$
$h=0.1-0$
$h=0.1$)

Given $y'=2y+3e^x$

$$y''=2y'+3e^x$$

$$y'''=2y''+3e^x$$

$$y^{iv}=2y'''+3e^x$$

By Taylors series method

$$y_1=y(x_1)=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{iv}+\cdots$$

$y_0'=2y_0+3e^{x_0}=3$ $\quad$ ($\because y_0'$ means y' at (x_0,y_0) so put $x=x_0, y=y_0$ in R.H.S)

$$y_0''=2y_0'+3e^{x_0}=2\times3+3e^0=9$$

$$y_0'''=2y_0''+3e^{x_0}=2\times9+3\times e^0=21$$

$$y_0^{iv}=2y_0'''+3e^{x_0}=2\times21+3\times e^0=45$$

sub these in the above formula, we have

$$y_1 = y(0.1) = 0 + (0.1)\times 3 + \frac{(0.1)^2}{2}\times 9 + \frac{(0.1)^3}{6}\times 21 + \frac{(0.1)^4}{24}\times 45$$

$$y_1 = 0.3487$$

To find $y_2 = y(0.2)$

$$y_2 = y(x_2) = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$

$$y_1' = 2y_1 + 3e^{x_1} = 2\times 0.3487 + 3\times e^{0.1} = 4.0129$$

$$y_1'' = 2y_1' + 3e^{x_1} = 2\times 4.0129 + 3\times e^{0.1} = 11.3413$$

$$y_1''' = 2y_1'' + 3e^{x_1} = 2\times 11.3413 + 3\times e^{0.1} = 25.9981$$

$$y_1^{iv} = 2y_1''' + 3e^{x_1} = 2\times 25.9981 + 3\times e^{0.1} = 55.3117$$

sub these in the above formula, we have

$$y_2 = y(0.2) = 0.3487 + (0.1)(4.0129) + \frac{(0.1)^2}{2}\times 11.3413 + \frac{(0.1)^3}{6}\times 25.9981 + \frac{(0.1)^4}{24}\times 55.3117$$

$$y_2 = 0.8113$$

Exact soln

Given D.E $y' - 2y = 3e^x$, $y(0) = 0$

clearly this is of the form $\frac{dy}{dx} + P(x)y = Q(x)$

here $P(x) = -2$, $Q(x) = 3e^x$

$$I.F = e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

G.S is $y(I.F) = \int Q(x)(I.F)dx + c$

$$\Rightarrow ye^{-2x} = \int 3e^x e^{-2x}dx + c$$

$$ye^{-2x} = 3\int e^{-x}dx + c$$

$$ye^{-2x} = -3e^{-x} + c$$

$$y = \frac{-3e^{-x}}{e^{-2x}} + \frac{c}{e^{-2x}}$$

$\therefore y = -3e^{x} + ce^{2x}$

But given $y(0)=0$ i.e $y=0$ when $x=0$

$$0 = -3\times e^{0} + c\times e^{0}$$

$$0 = -3 + c$$

$$c = 3$$

$$y = -3e^{x} + 3e^{2x}$$

when $x = 0.1$, $y_1 = y(0.1) = -3\times e^{0.1} + 3\times e^{2(0.1)} = 0.3487$

$x = 0.2$, $y_2 = y(0.2) = -3\times e^{0.2} + 3\times e^{2(0.2)} = 0.8113$

$\therefore$

x	0	0.1	0.2
y (numeric value)	0	0.3487	0.8113
y (exact value)	0	0.3487	0.8113

$\therefore$ Numerical values and exact values are same.

2. Using Taylor series method, solve the D.E $\frac{dy}{dx} = x^2 + y^2$, $y(0)=0$ find y at $x=0.4$ in two steps.

Given D.E $y' = x^2 + y^2$ with $y(0)=0$

Here $x_0 = 0$, $y_0 = 0$

Here initial value $x_0 = 0$ and final value $x = 0.4$

To obtain x from initial value 0 to final value $x = 0.4$. To obtain x from initial value 0 to final value in two steps take $h = 0.2$

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$$

Now our aim is to find the value of y at $x_1 = 0.2$, $x_2 = 0.4$ using Taylor series method.

Given $y' = x^2 + y^2$

$y'' = 2x + 2yy'$

$y''' = 2 + 2[yy'' + y'y'] = 2 + 2yy'' + 2y'^2$

$y^{iv} = 2yy''' + 6y'y''$ (use $\frac{d}{dx}(uv)$ formula $\frac{d}{dx}(y'^2) = 2y'y''$)

To find $y_1 = y(x_1) = y(0.2)$

By Taylor series method

$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$

$y_0' = x_0^2 + y_0^2 = 0$

$y_0'' = 2x_0 + 2y_0y_0' = 2\times 0 + 2\times 0\times 0 = 0$

$y_0''' = 2 + 2y_0y_0'' + 2y_0'^2 = 2$

$y_0^{iv} = 2y_0y_0''' + 6y_0'y_0'' = 0$ $(\because y_0 = 0,\ y_0' = 0)$

sub these in the above formula

$y_1 = 0 + (0.2)(0) + \frac{(0.2)^2}{2}\times 0 + \frac{(0.2)^3}{6}\times 2 + \frac{(0.2)^4}{24}\times 0$

$y_1 = 0.0027$

To find $y_2 = y(x_2) = y(0.4)$

$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$

$y_1' = x_1^2 + y_1^2 = (0.2)^2 + (0.0027)^2 = 0.04$

$y_1'' = 2x_1 + 2y_1y_1' = 2(0.2) + 2(0.0027)(0.04) = 0.4$

$y_1''' = 2 + 2y_1y_1'' + 2y_1'^2 = 2 + 2(0.0027)(0.4002) + 2(0.04)^2$
$= 2.0054$

$y_1^{iv} = 2y_1 y_1''' + 6y_1' y_1'' = 2(0.0027)(2.0054) + 6(0.041)(0.4001)$

$y_1^{iv} = 0.1069$

Sub these values in the above, we have

$$y_2 = 0.0027 + (0.2)(0.041) + \frac{(0.2)^2}{2} \times 0.4001 + \frac{(0.2)^3}{6} \times 2.0054 + \frac{(0.2)^4}{24} \times 0.1069$$

$y_2 = 0.0219$

The values of x and corresponding values of y are tabulated below

x	0	0.2	0.4
y	0	0.0027	0.0219

3. Using Taylor series method find y(1.1), y(1.2) for the D.E, $y' = (xy^{1/3})$, y(1)=1. compare the numerical soln obtained with exact solution.

Ans: Given D.E $y' = xy^{1/3}$ with y(1)=1

here $x_0 = 1$, $y_0 = 1$

Now our aim is to find the values of y at

$x_1 = 1.1$, $x_2 = 1.2$ using Taylor series method

$\therefore h = 0.1$ $(\because x_1 = x_0 + h \Rightarrow h = x_1 - x_0)$

Given $y' = xy^{1/3} \longrightarrow (1)$

$$y'' = x\frac{d}{dx}(y^{1/3}) + y^{1/3}\frac{dx}{dx} \left[\text{using } \frac{d}{dx}(uv)\right]$$

$$y'' = x \cdot \frac{1}{3} y^{1/3 - 1} y' + y^{1/3}$$

$$= \frac{x}{3} y^{-2/3}(xy^{1/3}) + y^{1/3}$$

$$= \frac{x^2}{3} y^{-1/3} + y^{1/3}$$

$$y''' = \frac{1}{3}\left[x^2\frac{d}{dx}\left(y^{-1/3}\right) + y^{-1/3}\frac{d}{dx}\left(x^2\right)\right] + \frac{d}{dx}\left(y^{1/3}\right)$$

$$= \frac{1}{3}\left[-x^2\cdot\frac{1}{3}y^{-4/3}y' + y^{-1/3}(2x)\right] + \frac{1}{3}y^{1/3-1}y'$$

$$= \frac{1}{3}\left[-\frac{x^2}{3}y^{-4/3}\times xy^{1/3} + 2xy^{-1/3}\right] + \frac{1}{3}y^{-2/3}xy^{1/3}$$

$$= -\frac{x^3}{9}y^{-1} + \frac{2x}{3}y^{-1/3} + \frac{x}{3}y^{-1/3}$$

$$= \frac{-x^3}{9y} + xy^{-1/3}$$

To find $y_1 = y(1.1)$

By Taylor series method

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

$$y_0' = x_0 y_0^{1/3} = 1$$

$$y_0'' = \frac{x_0^2}{3}y_0^{-1/3} + y_0^{1/3} = \frac{1}{3} + 1 = \frac{4}{3}$$

$$y_0''' = \frac{-x_0^3}{9y_0} + xy_0^{-1/3} = \frac{-1}{9} + 1 = \frac{8}{9}$$

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}\times\frac{4}{3} + \frac{(0.1)^3}{6}\times\frac{8}{9}$$

$$y_1 = y(1.1) = 1.1068$$

To find $y_2 = y(1.2)$

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots$$

$$y_1' = x_1 y_1^{1/3} = (1.1)(1.1068)^{1/3} = 1.1378$$

$$y_1'' = \frac{x_1^2}{3}y_1^{-1/3} + y_1^{1/3} = \frac{(1.1)^2}{3}(1.1068)^{-1/3} + (1.1068)^{1/3}$$

$$= 1.4243$$

$$y_1''' = \frac{-x_1^3}{9y_1} + x_1 y_1^{-1/3} = \frac{-(1.1)^3}{9(1.1068)} + (1.1)(1.1068)^{-1/3}$$

$$= 0.9298$$

Sub these in the above formula, we have

$$y_2 = 1.1068 + (0.1)(1.1378) + \frac{(0.1)^2}{2}(1.1145) + \frac{(0.1)^3}{6} \times 0.9398$$

$$y_2 = 1.2249$$

Exact soln

Given D.E

$$y' = xy^{1/3}, \; y(1) = 1$$

$$\frac{dy}{dx} = xy^{1/3}$$

$$\frac{1}{y^{1/3}} dy = x\,dx$$

$$\int y^{-1/3} dy = \int x\,dx$$

$$\frac{y^{-1/3+1}}{-1/3+1} = \frac{x^2}{2} + c$$

$$\frac{3}{2} y^{2/3} = \frac{x^2}{2} + c$$

$$\frac{3}{2} y^{2/3} = \frac{2}{3}\left(\frac{x^2}{2}\right) + c$$

$$y^{2/3} = \frac{x^2}{3} + c \times \frac{2}{3}$$

but given $y = 1$, when $x = 1$

$$\therefore \; 1 = \frac{1}{3} + c \times \frac{2}{3}$$

$$\Rightarrow 1 - 1/3 = c \times 2/3$$

$$\left(\frac{2}{3}\right) c = \frac{2}{3}$$

$$c = 1$$

$$y^{2/3} = \frac{x^2}{3} + \frac{2}{3}$$

$$y=\left(\frac{x^2}{3}+\frac{2}{3}\right)^{3/2}$$

put $x=1.1$

$$y_1=\left(\frac{1.1^2}{3}+\frac{2}{3}\right)^{3/2}=1.1068$$

put $x=1.2$

$$y_2=\left(\frac{(1.2)^2}{3}+\frac{2}{3}\right)^{3/2}=1.2279$$

x	1	1.1	1.2
y(N.V)	1	1.1068	1.2279
y(E.V)	1	1.1068	1.2279

4 Tabulate $y(0.1), y(0.2), y(0.3)$ using Taylor series method, given that $y'=y^2+x$ and $y(0)=1$

Given $y'=y^2+x,\ y(0)=1$

Here $x_0=0,\ y_0=1$

Now, we have to find the values of y at $x_1=0.1$, $x_2=0.2$, $x_3=0.3$ using Taylor series method.

$\therefore\ h=0.1$

Given $y'=y^2+x$

$y''=2yy'+1$

$y'''=2[yy''+y'y']$

$y'''=2yy''+2y'^2$

$y^{iv}=2[yy'''+y''y']+2(2y'y'')$

$y^{iv}=2yy'''+6y'y''$

To find $y_1=y(0.1)$

$$y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{iv}+\cdots$$

$y_0' = y_0^2 + x_0 = 1$

$y_0'' = 2y_0y_0' + 1 = 2\times1\times1+1 = 3$

$y_0''' = 2y_0y_0'' + 2y_0'^2 = 2\times1\times3 + 2\times1 = 8$

$y_0^{iv} = 2y_0y_0''' + 6y_0'y_0'' = 2\times1\times8 + 6\times1\times3 = 34$

Sub these in the above formula, we have

$y_1 = 1 + (0.1)\times1 + \frac{(0.1)^2}{2}\times3 + \frac{(0.1)^3}{6}\times8 + \frac{(0.1)^4}{24}\times34$

$y_1 = 1.1165$

To find $y_2 = y(0.2)$

$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$

$y_1' = y_1^2 + x_1 = (1.1165)^2 + 0.1 = 1.3466$

$y_1'' = 2y_1y_1' + 1 = 4.0069$

$y_1''' = 2y_1y_1'' + 2y_1'^2 = 2(1.1165)(4.0069) + 2(1.3466)^2 = 12.5741$

$y_1^{iv} = 2y_1y_1''' + 6y_1'y_1'' = 2(1.1165)(12.5741) + 6(1.3466)(4.0069) = 60.4250$

Sub these in the above formula, we have

$y_2 = 1.1165 + (0.1)\times1.3466 + \frac{(0.1)^2}{2}\times4.0069 + \frac{(0.1)^3}{6}\times12.5741 + \frac{(0.1)^4}{24}\times60.4250$

$y_2 = y(0.2) = 1.2735$

To find $y_3 = y(0.3)$

$y_3 = y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$

$y_2' = y_2^2 + x_2 = (1.2735)^2 + 0.2 = 1.8219$

$y_2'' = 2y_2 y_2' + 1 = 2(1.2735)(1.8217)+1 = 5.6404$

$y_2''' = 2y_2 y_2'' + 2{y_2'}^2 = 2(1.2735)(5.6404) + 2(1.8219)^2$
$= 21.0047$

$y_2^{iv} = 2y_2 y_2''' + 6y_2' y_2'' = 2(1.2735)(21.0047) + 6(1.8219)(5.6404)$
$= 115.1566$

Sub these in the above, we have,

$$y_3 = 1.2735 + (0.1)(1.8219) + \frac{(0.1)^2}{2} \times 5.6404 + \frac{(0.1)^3}{6} \times 21.0047 + \frac{(0.1)^4}{24} \times 115.1566$$

$y_3 = 1.4879$

we can tabulate the values as follows

x	0	0.1	0.2	0.3
y	1	1.1165	1.2735	1.4879

Eulers method

Suppose we wish to solve the eqn $\frac{dy}{dx} = f(x,y)$ subject to the condition that $y(x_0) = y_0$

Now we have to find the value of y at $x_1, x_2, \ldots x_n$

where y_1 at $x_1 = x_0 + h$

y_2 at $x_2 = x_1 + h$

y_3 at $x_3 = x_2 + h, \ldots$

where h is interval difference is taken as very small. If we represents the points (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots$ There it is a curve on xy-plane where slope at $p(x_0, y_0)$ & (x_1, y_1) is

$$m = \frac{dy}{dx} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}$$

but from ① $\frac{dy}{dx} = f(x,y)$ at (x_0,y_0) $\frac{dy}{dx} = f(x_0,y_0)$

$$f(x_0,y_0) = \frac{y_1-y_0}{h}$$

$$y_1 - y_0 = h\,f(x_0,y_0)$$

$$y_1 = y_0 + h\,f(x_0,y_0)$$

$$y_2 = y_1 + h\,f(x_1,y_1) \cdots$$

In general

$y_{n+1} = y_n + h\,f(x_n,y_n)$ $\forall n = 0,1,2,\cdots$ is Eulers formula where h is taken as very small

Problems

1. using Eulers method, solve for y at $x=2$ from $\frac{dy}{dx} = 3x^2+1$, $y(1)=2$, taking step size $h=0.5$

Sol: Given D.E $\frac{dy}{dx} = 3x^2+1$, $y(1)=2$, $h=0.5$

compare this with standard form $\frac{dy}{dx} = f(x,y)$, $y(x_0)=y_0$. Here $f(x,y) = 3x^2+1$, $x_0=1$, $y_0=2$

Now $x_1 = x_0+h = 1+0.5 = 1.5$

$x_2 = x_1+h = 1.5+0.5 = 2$

Now, we have to find the value of y at $x_1=1.5$ & $x_2=2$ by using Eulers method

Eulers formula is $y_{n+1} = y_n + h\,f(x_n,y_n)$ $\forall n=0,1,2\ldots$

put $n=0$

$$y_1 = y_0 + h\,f(x_0,y_0)$$

$$y_1 = y_0 + h(3x_0^2+1)$$

$$y_1 = 2+(0.5)[3(1)^2+1]$$

$$y_1 = 4$$

Put $n=1$ $\therefore y_2 = y_1 + hf(x_1, y_1)$

$y_2 = y_1 + h(5x_1^2 + y_1)$

$y_2 = 4 + (0.5)[5 \times 1.5^2 + 4]$

$y_2 = 7.8750$

The values of x and y are tabulated below

x	1	1.5	2
y	2	4	7.875

2 Given $y' = x^2 - y$, $y(0) = 1$ find $y(0.1)$, $y(0.2)$ using Euler's method

Sol: Given D.E $y' = x^2 - y$, $y(0) = 1$

compare this with standard form $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ here $f(x,y) = x^2 - y$, $x_0 = 0$, $y_0 = 1$

Now we have to find the values of y at $x_1 = 0.1$, $x_2 = 0.2$ using Euler's method. Here $h = x_1 - x_0 = 0.1$

By Euler's method $y_{n+1} = y_n + hf(x_n, y_n)$ $\forall n = 0, 1, 2, \ldots$

$\therefore y_1 = y_0 + h f(x_0, y_0)$

$= y_0 + h(x_0^2 - y_0)$

$y_1 = 1 + (0.1)[0^2 - 1] = 0.9$

$y_2 = y_1 + hf(x_1, y_1)$

$= y_1 + h(x_1^2 - y_1)$

$= 0.9 + (0.1)[(0.1)^2 - 0.9]$

$y_2 = 0.8110$

i.e. $y_1 = 0.9$, $y_2 = 0.8110$

Use Euler's method to find $y(0.1), y(0.2)$, given that $y' = (x^3 + xy^2)e^{-x}$, $y(0) = 1$

Here $f(x,y) = (x^3 + xy^2)e^{-x}$, $x_0 = 0$, $y_0 = 1$.

Now we have to find the values of y at $x_1 = 0.1$, $x_2 = 0.2$ by Euler's method so $h = 0.1$

Now $y_{n+1} = y_n + hf(x_n, y_n)$ $\forall n = 0,1,2,\ldots$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + h(x_0^3 + x_0 y_0^2)e^{-x_0}$$

$$y_1 = 1 + (0.1)[0+0]e^{-0} = 1$$

$$y_1 = 1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = y_1 + h(x_1^3 + x_1 y_1^2)e^{-x_1}$$

$$y_2 = 1 + 0.1[0.1^3 + 0.1 \times 1^2]e^{-0.1}$$

$$y_2 = 0.091$$

<u>Modified Euler's method</u>

Euler's method is very slow if we take h is very small. If h is not taken as small we may not get the accurate sol^n. So Euler's method can be modified as taking the average of the slopes (x_0, y_0) & $(x_1, y_1^{(1)})$

$\therefore$ Modified Euler's formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(1)})\right] \quad \forall n = 0,1,\ldots$$

where $y_1^{(1)} = y_0 + hf(x_0, y_0)$

i.e. initial sol^n is obtained from Euler's method.

Summary of the method

Given $\frac{dy}{dx} = f(x,y)$, $y = y_0$ at $x = x_0$

To find $y_1 = y(x_1)$ at $x_1 = x_0 + h$

$y_1^{(0)} = y_0 + h f(x_0, y_0)$ by euler's method, its modified value is obtained like this

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right] \ldots$$

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(k)})\right]$$

If two successive values of $y_1^{(k)}$ & $y_1^{(k+1)}$ are sufficiently close to one another, we will stop the process and take the common value as y_1

To find $y_2 = y(x_2)$ at $x_2 = x_1 + h$

$y_2^{(0)} = y_1 + h f(x_1, y_1)$ by eulers method, its modified value is

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$$ and so on.

Problems on Modified Eulers method

1. Using Modified Euler method find $y(0.2)$, $y(0.4)$ given that $y' = y + e^x$, $y(0) = 0$

Sol: Given D.E $y' = y + e^x$, $y(0) = 0$

compare with standard form of D.E $\frac{dy}{dx} = f(x,y)$

$y(x_0) = (y_0)$

Here $f(x,y) = y + e^x$, $x_0 = 0$, $y_0 = 0$

Now we have to find the values of y at $x_1 = 0.2$, $x_2 = 0.4$ using modified Eulers method. $h = 0.2$

To find $y_1 = y(x_1)$

By modified Eulers method

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] \quad \forall\, k = 0,1,2,3 \ldots$$

where $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(0)} = y_0 + h(y_0 + e^{x_0})$$

$$y_1^{(0)} = 0 + (0.2)(0 + e^0) = 0.2$$

By modified Eulers method

I approximation of y_1 is

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= y_0 + \frac{h}{2}\left[y_0 + e^{x_0} + y_1^{(0)} + e^{x_1}\right]$$

$$= 0 + \frac{0.2}{2}\left[0 + e^0 + 0.2 + e^{0.2}\right]$$

$$= 0.2421$$

II approximation of y_1 is

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[y_0 + e^{x_0} + y_1^{(1)} + e^{x_1}\right]$$

$$= 0 + \frac{0.2}{2}\left[0 + e^0 + 0.2421 + e^{0.2}\right]$$

$$= 0.24644$$

III approximation of y_1 is

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(2)})\right]$$

$$= y_0 + \frac{h}{2}\left[y_0 + e^{x_0} + y_1^{(2)} + e^{x_1}\right]$$

$$= 0 + \frac{0.2}{2}\left[0 + e^0 + 0.24644 + e^{0.2}\right]$$

$$= 0.2468$$

IV approximation of y_1 is

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(3)})\right]$$

$$= y_0 + \frac{h}{2}\left[y_0 + e^{x_0} + y_1^{(3)} + e^{x_1}\right]$$

$$= 0 + \frac{0.2}{2}\left[0 + e^0 + 0.24684 + e^{0.2}\right]$$

$$= 0.2468$$

Here $y_1^{(3)}$ & $y_1^{(4)}$ values are equal, so, we take

$$y_1 = 0.2468$$

<u>To find $y_2 = y(0.4)$</u>

By modified Eulers method

$$y_2^{(k+1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(k)})\right] \quad \forall k = 0,1,2,\ldots$$

where $y_2^{(0)} = y_1 + h f(x_1, y_1)$

$$y_2^{(0)} = y_1 + h(y_1 + e^{x_1})$$

$$y_2^{(0)} = 0.2468 + (0.2)\left[0.2468 + e^{0.2}\right]$$

$$y_2^{(0)} = 0.5404$$

By modified Eulers theorem

I approximation of y_2 is

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(0)})\right]$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= y_1 + \frac{h}{2}\left[y_1 + e^{x_1} + y_2^{(0)} + e^{x_2}\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[0.2468 + e^{0.2} + 0.5404 + e^{0.4}\right]$$

$$= 0.5968$$

II approximation

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= y_1 + \frac{h}{2}\left[y_1 + e^{x_1} + y_2^{(1)} + e^{x_2}\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[0.2468 + e^{0.2} + 0.5968 + e^{0.4}\right]$$

$$y_2 = 0.6025$$

III approximation

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= y_1 + \frac{h}{2}\left[y_1 + e^{x_1} + y_2^{(2)} + e^{x_2}\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[0.2468 + e^{0.2} + 0.6025 + e^{0.4}\right]$$

$$= 0.6031$$

IV approximation

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[0.2468 + e^{0.2} + 0.6031 + e^{0.4}\right]$$

$$= 0.6031$$

Here $y_2^{(3)}$ & $y_2^{(4)}$ are same so $y_2 = 0.6031$

x	0	0.2	0.4
y	0	0.2468	0.6031

2. Solve the D.E $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by modified Eulers method and compute $y(0.02)$, $y(0.04)$

Sol. Here $f(x,y) = x^2+y$, $x_0=0$, $y_0=1$, $x_1=0.02$, $x_2=0.04$

$\therefore h = 0.02$

To find $y_1 = y(0.02)$

By M.E.T $y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(k)})\right]$

where the initial soln $y_1^{(0)}$ is obtained from Eulers method

Now $y_1^{(0)} = y_0 + hf(x_0,y_0) = y_0 + h(x_0^2+y_0)$

$\Rightarrow y_1^{(0)} = 1+(0.02)(0^2+1) = 1.02$

I approximation

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[x_0^2+y_0+x_1^2+y_1^{(0)}\right]$

$= 1+\frac{0.02}{2}\left[0^2+1+0.02^2+1.02\right]$

$= 1.0202$

II approximation

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right]$

$= y_0 + \frac{h}{2}\left[x_0^2+y_0+x_1^2+y_1^{(1)}\right]$

$= 1+\frac{0.02}{2}\left[0^2+1+0.02^2+1.0202\right]$

$= 1.0202$

Here $y_1^{(1)}$ and $y_1^{(2)}$ are same so

$y_1 = 1.0202$

To find $y_2 = y(0.04)$

By M.E.M $y_2^{(k+1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(k)})\right]$

where initial value $y_2^{(0)} = y_1 + h f(x_1, y_1)$.

$\Rightarrow y_2^{(0)} = y_1 + h\left[x_1^2 + y_1\right]$

$= 1.0202 + 0.02\left[0.04^2 + 1.0202\right]$

$= 1.0406$

I appr

$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2 + y_2^{(0)})\right]$

$y_2^{(1)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(0)}\right]$

$= 1.0202 + \frac{0.02}{2}\left[0.02^2 + 1.0202 + 0.04^2 + 1.0406\right]$

$= 1.0408$

II appr

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= y_1 + \frac{h}{2}\left[x_1^2 + y_1 + x_2^2 + y_2^{(1)}\right]$

$= 1.0202 + \frac{0.02}{2}\left[0.02^2 + 1.0202 + 0.04^2 + 1.0408\right]$

$= 1.0408$

Here $y_2^{(1)}$ and $y_2^{(2)}$ are same so $y_2 = 1.0408$

x	0	0.02	0.04
y	1	1.0202	1.0408

3. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ compute $y(0.02)$, $y(0.04)$ using Eulers modified method

Here $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.02$, $x_2 = 0.04$, $h = 0.02$

To find $y_1 = y(0.02)$

By Modified Eulers theorem

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(k)})\right] \quad \forall k=0,1,2,\dots$$

where initial soln $y_1^{(0)}$ is obtained from Euler method.

$$y_1^{(0)} = y_0 + h\, f(x_0,y_0) = y_0 + h\left(\frac{y_0-x_0}{y_0+x_0}\right)$$

$$y_1^{(0)} = 1+(0.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

I approximation

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[\left(\frac{y_0-x_0}{y_0+x_0}\right) + \left(\frac{y_1^{(0)}-x_1}{y_1^{(0)}+x_1}\right)\right]$$

$$y_1^{(1)} = 1+\frac{0.02}{2}\left[\left(\frac{1-0}{1+0}\right) + \left(\frac{1.02-0.02}{1.02+0.02}\right)\right]$$

$$y_1^{(1)} = 1+0.01\left[1+0.9615\right]$$

$$y_1^{(1)} = 1.0196$$

II approximation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[\frac{y_0-x_0}{y_0+x_0} + \frac{y_1^{(1)}-x_1}{y_1^{(1)}+x_1}\right]$$

$$= 1+\frac{0.02}{2}\left[\left(\frac{1-0}{1+0}\right) + \left(\frac{1.0196-0.02}{1.0196+0.02}\right)\right]$$

$$= 1+0.01\left[1+0.9615\right]$$

$$= 1.0196$$

Here $y_1^{(1)}$ & $y_1^{(2)}$ are same hence $y_1 = 1.0196$

<u>To find $y_2 = y(0.04)$</u>

$$y_2^{(k+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{k})\right] \quad \forall k = 0, 1, 2, \ldots$$

where $y_2^{(0)} = y_1 + h f(x_1, y_1) = y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$

$$= 1.0196 + (0.02)\left(\frac{1.0196 - 0.02}{1.0196 + 0.02}\right) = 1.0388$$

$$y_2^{(0)} = 1.0388$$

<u>I appr</u>

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= y_1 + \frac{h}{2}\left[\left(\frac{y_1 - x_1}{y_1 + x_1}\right) + \left(\frac{y_2^{(0)} - x_2}{y_2^{(0)} + x_2}\right)\right]$$

$$= 1.0196 + \frac{0.02}{2}\left[\left(\frac{1.0196 - 0.02}{1.0196 + 0.02}\right) + \left(\frac{1.0388 - 0.04}{1.0388 + 0.04}\right)\right]$$

$$= 1.0196 + 0.01\left[0.9615 + 0.9258\right]$$

$$= 1.0385$$

<u>II appr</u>

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[\left(\frac{y_1 - x_1}{y_1 + x_1}\right) + \frac{y_2^{(1)} - x_2}{y_2^{(1)} + x_2}\right]$$

$$y_2^{(2)} = 1.0196 + \frac{0.02}{2}\left[\left(\frac{1.0196 - 0.02}{1.0196 + 0.02}\right) + \left(\frac{1.0385 - 0.04}{1.0385 + 0.04}\right)\right]$$

$$= 1.0196 + 0.01\left[0.9615 + 0.9258\right]$$

$$= 1.0385$$

Here $y_2^{(1)}$ & $y_2^{(2)}$ are same so $y_2 = 1.0385$

x	0	0.02	0.04
y	1	1.0196	1.0385

4. Given $y' = x + \sin y$, $y(0) = 1$ compute $y(0.2)$ & $y(0.4)$ by Euler's modified method.

Sol: Here $f(x,y) = x + \sin y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $x_2 = 0.4$, $h = 0.2$

To find $y_1 = y(0.2)$

By M.E.M : $y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(k)})\right]$

where $y_1^{(0)} = y_0 + h f(x_0,y_0)$

$y_1^{(0)} = y_0 + h(x_0 + \sin y_0)$

$y_1^{(0)} = 1 + (0.2)(0 + \sin 1)$

$= 1.1683$

I app:

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$

$= y_0 + \frac{h}{2}\left[x_0 + \sin y_0 + x_1 + \sin y_1^{(0)}\right]$

$= 1 + \frac{0.2}{2}\left[0 + \sin 1 + 0.2 + \sin(1.1683)\right]$

$= 1.1962$

II app:

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right]$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[x_0 + \sin y_0 + x_1 + \sin y_1^{(1)}\right]$

$= 1 + \frac{0.2}{2}\left[0 + \sin 1 + 0.2 + \sin(1.1962)\right]$

$= 1.1972$

III app:

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(2)})\right]$

$= y_0 + \frac{h}{2}\left[x_0 + \sin y_0 + x_1 + \sin y_1^{(2)}\right]$

$= 1 + \frac{0.2}{2}\left[0 + \sin 1 + 0.2 + \sin(1.1972)\right]$

$= 1.1972$

Here $y_1^{(2)} = y_1^{(3)}$

$\therefore y_1 = 1.1972$

To find $y_2 = y(0.4)$

By M.E.M $y_2^{(k+1)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2, y_2^{(k)})]$

where $y_2^{(0)} = y_1 + h f(x_1, y_1)$

$= 1.1972 + 0.2[0.2 + \sin(1.1972)]$

$= 1.4234$

I appr $y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$y_2^{(1)} = y_1 + \frac{h}{2}[x_1 + \sin y_1 + x_2 + \sin y_2^{(0)}]$

$y_2^{(1)} = 1.1972 + \frac{0.2}{2}[0.2 + \sin(1.1972) + 0.4 + \sin(1.4234)]$

$y_2^{(1)} = 1.4492$

II appr

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= y_1 + \frac{h}{2}[x_1 + \sin y_1 + x_2 \sin y_2^{(1)}]$

$= 1.1972$

To find $y_2 = y(0.4)$

By M.E.M $y_2^{(k+1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(k)})]$

where $y_2^{(0)} = y_1 + h f(x_1, y_1)$ obtained from

Eulers method $y_2^{(0)} = y_1 + h(x_1 + \sin y_1)$

$y_2^{(0)} = 1.1972 + 0.2[0.2 + \sin(1.1972)]$

$y_2^{(0)} = 1.4234$

I appr

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[x_1 + \sin y_1 + x_2 + \sin y_2^{(0)}\right]$$

$$= 1.1972 + \frac{0.2}{2}\left[0.2 + \sin(1.1972) + 0.4 + \sin(1.4232)\right]$$

$$= 1.4492$$

II app

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= y_1 + \frac{h}{2}\left[x_1 + \sin y_1 + x_2 + \sin y_2^{(1)}\right]$$

$$= 1.1972 + \frac{0.2}{2}\left[0.2 + \sin(1.1972) + 0.4 + \sin(1.4492)\right]$$

$$= 1.4496$$

III app

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= y_1 + \frac{h}{2}\left[x_1 + \sin y_1 + x_2 + \sin y_2^{(2)}\right]$$

$$= 1.1972 + \frac{0.2}{2}\left[0.2 + \sin(1.1972) + 0.4 + \sin(1.4496)\right]$$

$$= 1.4496$$

Here $y_2^{(2)} = y_2^{(3)}$ so $y_2 = 1.4496$

x	0	0.2	0.4
y	1	1.1972	1.4496

Runge-Kutta Methods (R-K Method):

1. **First Order R-K Method:**

$y_1 = y_0 + hf(x_0, y_0)$. It is nothing but taylors series expansion upto the term in h i.e Eulers method.

2. **Second order R-K Method**

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$, $k_2 = hf(x_0 + h, y_0 + k_1)$

Second order R-K method is nothing but modified Eulers method.

3. **Third order R-K Method**

$$y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + h, y_0 + k_2)$

4. **Fourth Order R-K Method**

This method is most commonly used in practise and is often referred to as "Runge-kutta method" only without any reference to the order.

4^{th} order R-K method or R-K method formula is

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$, $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$k_1 = hf(x_1, y_1)$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$k_4 = hf(x_1 + h, y_1 + k_3)$

1. Obtain the values of y at $x = 0.1, 0.2$ using R-K method of i, second order ii, third order iii, fourth order for the differential eq^n $y' + y = 0$, $y(0) = 1$

Sol: Given D.E $y' = -y$, $y(0) = 1$

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$, $h = 0.1$

i, Second order

To find $y_1 = y(0.1)$

By second order R-K method $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = hf(x_0, y_0) = h(-y_0) = 0.1(-1) = -0.1$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= hf(0 + 0.1, 1 + (-0.1))$ (Sub k_1, k_2 values in the above, we have)

$= hf(0.1, 0.9)$

$= (0.1)(-0.9)$

$= -0.09$

$y_1 = 1 + \frac{1}{2}[-0.1 - 0.09]$

$y_1 = 0.905$

To find $y_2 = y(0.2)$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_1, y_1) = hf(0.1, 0.905)$
$= (0.1)(-0.905)$
$= -0.0905$

$k_2 = hf(x_1 + h, y_1 + k_1) = hf(0.1 + 0.1, 0.905 - 0.0905)$

$\Rightarrow k_2 = hf(0.2, 0.8145)$
$= (0.1)(-0.8145)$
$= -0.08145$

Sub k_1, k_2 values in the above, we have

$$y_2 = 0.905 + \frac{1}{2}[-0.0905 - 0.08145]$$

$$y_2 = 0.8190$$

ii. Third order

To find $y_1 = y(0.1)$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$$

where $k_1 = hf(x_0, y_0) = hf(0, 1)$
$= (0.1)(-1) = -0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2})$
$= (0.1)f(0.05, 0.95)$ $\quad (\because f(x, y) = -y)$
$= (0.1)(-0.95)$
$= -0.095$

$k_3 = hf(x_0 + h, y_0 + k_2) = hf(0 + 0.1, 1 - 0.095)$
$= (0.1)f(0.1, 0.905)$
$= (0.1)(-0.905) = -0.0905$

sub k_1, k_2, k_3 values we have

$$y_1 = 1 + \frac{1}{6}\left[-0.1 + 4(-0.095) - 0.0905\right]$$

$$y_1 = 0.9049$$

To find $y_2 = y(0.2)$

$$y_2 = y_1 + \frac{1}{6}\left[k_1 + 4k_2 + k_3\right]$$

where $k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9049)$

$$= (0.1)(-0.9049)$$

$$= -0.0905$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1) f\left(0.1 + \frac{0.1}{2}, 0.9049 - \frac{0.0905}{2}\right)$$

$$= (0.1) f(0.15, 0.8597)$$

$$= (0.1)(-0.8597)$$

$$= -0.086$$

$$k_3 = hf(x_1 + h, y_1 + k_2) = hf(0.1 + 0.1, 0.9049 - 0.086)$$

$$= (0.1) f(0.2, 0.8189)$$

$$= (0.1)(-0.8189)$$

$$k_3 = -0.0819$$

sub k_1, k_2, k_3 values in the above, we have

$$y_2 = 0.9049 + \frac{1}{6}\left[-0.0905 + 4(-0.086) - 0.0819\right]$$

$$y_2 = 0.8186$$

Fourth order R-K method

To find $y_1 = y(0.1)$

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$k_1 = hf(x_0, y_0) = hf(0, 1)$
$= (0.1)(-1)$
$= -0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$
$= hf(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2})$
$= (0.1) f(0.05, 0.95)$
$= (0.1)(-0.95)$
$k_2 = -0.095$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$
$= hf(0 + \frac{0.1}{2}, 1 - \frac{0.095}{2})$
$= (0.1) f(0.05, 0.9525)$
$= (0.1)(-0.9525)$
$= -0.0953$

$k_4 = hf(x_0 + h, y_0 + k_3)$
$= (0.1) f(0.1, 0.9048)$
$= (0.1)(-0.9048)$
$= -0.0905$

$\therefore y_1 = 1 + \frac{1}{6}[-0.1 + 2(-0.0950) + 2(-0.0953) - 0.0905]$
$y_1 = 0.9048$

To find $y_2 = y(0.2)$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = hf(x_1, y_1) = hf(0.1, 0.9048)$
$= (0.1)(-0.9048)$
$= -0.0905$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$
$= hf(0.1 + \frac{0.1}{2}, 0.9048 - \frac{0.0905}{2})$
$= hf(0.15, 0.8596)$
$= (0.1)(-0.8596)$
$= -0.086$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$
$= hf(0.15, 0.9048 - \frac{0.086}{2})$
$= (0.1) f(0.15, 0.8618)$
$= (0.1)(-0.8618)$
$= -0.0862$

$k_4 = hf(x_1+h, y_1+k_3)$

$= hf(0.1+0.1, 0.9045-0.086)$

$= (0.1) f(0.2, 0.8186)$

$= (0.1)(-0.8186)$

$= -0.0817$

$\therefore y_2 = 0.9045 + \frac{1}{6}[-0.0905 - 2\times0.086 - 2\times0.0862 - 0.0817]$

$y_2 = 0.8127$

2. using R-K method, find $y(0.2)$ for the equation

$\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$

Sol: Given $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = 0.2$

To find $y_1 = y(0.2)$

By 4^{th} order R-K method

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = hf(x_0, y_0) = (0.2) f(0,1) = (0.2)\left(\frac{y-x}{y+x}\right)$

$= (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= (0.2) f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2})$

$= 0.2 f(0.1, 1.1)$

$= (0.2)\left(\frac{1.1-0.1}{1.1+0.1}\right)$

$= 0.1667$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= (0.2) f(0 + \frac{0.2}{2}, 1 + \frac{0.1667}{2})$

$= (0.2) f(0.1, 1.0833)$

$= (0.2)\left(\frac{1.0833-0.1}{1.0833+0.1}\right)$

$= 0.1662$

$k_4 = hf(x_0+h, y_0+k_3)$

$= (0.2)\, f(0+0.2, 1+0.1662)$

$= (0.2)\, f(0.2, 1.1662)$

$= (0.2)\left(\dfrac{1.1662-0.2}{1.662+0.2}\right)$

$k_4 = 0.1414$

Sub k_1, k_2, k_3, k_4 values in ①, we have

$y_1 = 1 + \frac{1}{6}\left[0.2 + 2(0.1667) + 2(0.1662) + 0.1414\right]$

$y_1 = 1.1679$

Find $y(0.1), y(0.2)$ using R-K Method given that

$y' = x^2 - y$ and $y(0) = 1$

To find $y_1 = y(0.1)$

By R.K method $y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

where $k_1 = hf(x_0, y_0)$

$= (0.1)\, f(0,1)$

$= (0.1)(0^2-1)$

$= -0.1$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$

$= (0.1)\, f\left(0+\frac{0.1}{2}, 1-\frac{0.1}{2}\right)$

$= (0.1)\, f(0.05, 0.95)$

$= (0.1)\left[0.05^2 - 0.95\right]$

$= -0.0948$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$

$= (0.1)\, f\left(0+\frac{0.1}{2}, 1-\frac{0.0948}{2}\right)$

$= (0.1)\, f(0.05, 0.9526)$

$= (0.1)\,(0.05^2 - 0.9526)$

$= -0.095$

$k_4 = hf(x_0+h, y_0+k_3)$

$= (0.1)\, f(0+0.1, 1-0.095)$

$= (0.1)\, f(0.1, 0.905)$

$= (0.1)\left[0.1^2 - 0.905\right]$

$= -0.0895$

$y_1 = 1 + \frac{1}{6}\left[-0.1 + 2(-0.09475) + 2(-0.0950) - 0.0895\right]$

$y_1 = 0.9052$

To find $y_2 = y(0.2)$

By R.K Method (4^{th} order) $y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$k_1 = hf(x_1, y_1)$

$= (0.1) f(0.1, 0.9052)$

$= (0.1)\left[0.1^2 - 0.9052\right]$

$= -0.0895$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= (0.1) f\left[0.1 + \frac{0.1}{2}, 0.9052 - \frac{0.0895}{2}\right]$

$= (0.1) f(0.15, 0.8604)$

$= -0.0838$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= (0.1) f(0.15, 0.8633)$

$= (0.1)\left[0.15^2 - 0.8633\right]$

$= -0.0841$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= hf(0.1 + 0.1, 0.9052 - 0.0841)$

$= (0.1) f(0.2, 0.8211)$

$= (0.1)\left[0.2^2 - 0.8211\right]$

$= -0.0781$

$\therefore \Rightarrow y_2 = 0.9052 + \frac{1}{6}\left[-0.0895 + 2(-0.0838) + 2(-0.0841) - 0.0781\right]$

$y_2 = 0.8213$

x	0	0.1	0.2
y	1	0.9052	0.8213

Picard's Method of Successive approximation

Consider the differential eqn $\frac{dy}{dx} = f(x, y)$,

$y(x_0) = y_0 \Rightarrow dy = f(x, y)\,dx$

I.0.B.S

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\left(y\right)_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \longrightarrow ②$$

we find that the R.H.S of eq ② contains the unknown y under the integral sign. An eqn of this kind is called an integral equation and it can be solve by a process of successive approximations.

Picards method gives a sequence of functions $y^{1}(x), y^{(2)}(x), y^{3}(x), \ldots$ which form a sequence of approximation to y converging to $y(x)$

To get the first approximation $y^{1}(x)$, put $y = y_0$ in the integrand of R.H.S of ②, we get

$$y^{1}(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$\text{IIIy} \quad y^{(2)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(1)}x)\,dx$$

$$y^{(3)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(2)}x)\,dx, \ldots$$

In general $\boxed{y^{(n+1)}(x) = y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx \quad \forall\, n = 0,1,2,3,\ldots}$ → (1)

This is the general iterative formula for y. Iterations are repeated until the two successive approximation are sufficiently close, eq(1) is called picards iteration formula.

Note-1 Since this method involves actual integration sometimes it may not be possible to carry out the integration in that case stop the process at that stage.

2. This method is applicable only when integrand of R.H.S exist

Problems

1. Find an approximate value of y for $x=0.1$, $x=0.2$ if $\frac{dy}{dx} = x+y$ and $y=1$ at $x=0$ using picards method. check your answer with exact or analytical soln.

Sol Given D.E $\frac{dy}{dx} = x+y$, $y(0)=1$

compare this with standard form $\frac{dy}{dx} = f(x,y)$, $y(x_0)=y_0$

Here $f(x,y) = x+y$, $x_0=0$, $y_0=1$, $x_1=0.1$, $x_2=0.2$

By picards method, $y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$ $\forall\, n=0,1,2,\ldots$

I approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$\Rightarrow y_1 = y_0 + \int_{x_0}^{x} (x + y_0)\,dx$$

$$= 1 + \int_{0}^{x} (x+1)\,dx$$

$$= 1 + \left(\frac{x^2}{2} + x\right)_0^x$$

$$= 1 + \left(\frac{x^2}{2} + x\right)_0^x$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

$$\therefore y_1(x_1) = y_1(0.1) = 1 + 0.1 + \frac{0.1^2}{2} = 1.105$$

$$y_1(x_2) = y_1(0.2) = 1 + 0.2 + \frac{0.2^2}{2} = 1.22$$

II approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 1 + \int_{0}^{x} \left(x + 1 + x + \frac{x^2}{2}\right) dx$$

$$y_2 = 1 + \int_{0}^{x} \left(1 + 2x + \frac{x^2}{2}\right) dx$$

$$y_2 = \left(1 + x + x^2 + \frac{x^3}{6}\right)_0^x$$

$$\therefore y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

$$\therefore y_2(x_1) = y_2(0.1) = 1 + 0.1 + 0.1^2 + \frac{0.1^3}{6} = 1.1102$$

$$y_2(x_2) = y_2(0.2) = 1 + 0.2 + 0.2^2 + \frac{0.2^3}{6} = 1.2413$$

III approximation

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$y_3 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + \left(x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right)_0^x$$

$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_3(x_1) = y_3(0.1) = 1 + 0.1 + 0.1^2 + \frac{0.1^3}{3} + \frac{0.1^4}{24} = 1.1103$$

$$y_3(x_2) = y_3(0.2) = 1 + 0.2 + 0.2^2 + \frac{0.2^3}{3} + \frac{0.2^4}{24} = 1.2427$$

Here $y_2(x_1)$ and $y_3(x_1)$ are close to each other

but $y_2(x_2)$ and $y_3(x_2)$ are not close to each other

so, we go to next approximation

IV approximation

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx$$

$$y_4 = y_0 + \int_{x_0}^{x} (x + y_3)\,dx$$

$$y_4 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

$$y_4(x_1) = y(0.1) = 1 + 0.1 + 0.1^2 + \frac{0.1^3}{3} + \frac{0.1^4}{12} + \frac{0.1^5}{120} = 1.1103$$

$y_4(x) = y_4(0.2) = 1 + 0.2 + 0.2^2 + \frac{0.2^3}{3} + \frac{0.2^4}{12} + \frac{0.2^5}{120} = 1.2428$

$\therefore$ 3rd and 4th approximations are nearly same

Hence $y_1 = y(0.1) = 1.1103$, $y_2 = y(0.2) = 1.2428$

Exact soln:

Given D.E $\frac{dy}{dx} = x + y$, $y(0) = 1$

$$\frac{dy}{dx} - y = x, \quad y(0) = 1$$

clearly this is L.D.E $\frac{dy}{dx} + p(x)y = Q(x)$

Here $p(x) = -1$, $Q(x) = x$

I.F $e^{\int p(x)dx} = e^{-x}$

G.S is $y(I.F) = \int Q(x)(I.F)dx + c$

$$ye^{-x} = u\int v dx - \int\left(\frac{du}{dx}\int v dx\right)dx + c$$

$$ye^{-x} = \int x e^{-x} dx + c$$

$$ye^{-x} = x\int e^{-x}dx - \int\left(\frac{dx}{dx}\int e^{-x}dx\right)dx + c$$

$$= -xe^{-x} - \int 1(-e^{-x})dx + c$$

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$y = -x - 1 + ce^{x}$$

but given $y(0) = 1$ i.e $y = 1$ at $x = 0$

$$\Rightarrow 1 = -0 - 1 + c$$

$$c = 2$$

Soln is $y = -x - 1 + 2e^{x}$

put $x = 0.1$

$y(0.1) = -0.1 - 1 + 2e^{0} = 1.1103$

put $x = 0.2$

$y(0.2) = -0.2 - 1 + 2e^{0.2} = 1.2428$

∴ Table

x	0	0.1	0.2
$y(N.V)$	1	1.1103	1.2428
$y(E.V)$	1	1.1103	1.2428

2. Find the value of y for $x = 0.4$ by picards method given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$

Sol: Here $f(x,y) = x^2 + y^2$, $y(0) = 0$, $x_0 = 0$, $x = 0.4$

By picards method

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx \quad \forall n = 0, 1, 2, \ldots$$

<u>I approximation:</u>

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_1 = y_0 + \int_{x_0}^{x} (x^2 + y_0^2)\,dx$$

$$y_1 = 0 + \int_{0}^{x} x^2\,dx$$

$$y_1 = \left(\frac{x^3}{3}\right)_0^x \Rightarrow y_1 = \frac{x^3}{3}$$

$$y_{1(0.4)} = 0.0213$$

II approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 0 + \int_0^x (x^2 + y_1^2)\,dx$$

$$= \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$y_2 = \frac{x^3}{3} + \frac{x^7}{63}$$

$$y_2(0.4) = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63}$$

$$\therefore y_2(0.4) = 0.0214$$

Here first and second approximations of y at $x = 0.4$ are close to each other

$$\therefore y(0.4) = 0.0214$$

3. Find the value of y at $x = 0.1$ by picards method given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$

Sol: Here $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $x = 0.1$

By picards method $y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$ for $n = 0, 1, 2, \ldots$

I approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = 1 + \int_0^x \left(\frac{y_0 - x}{y_0 + x} \right) dx$$

$$= 1 + \int_0^x \left(\frac{1 - x}{1 + x} \right) dx$$

$$= 1 + \int_0^x \frac{2 - (x+1)}{1 + x}\, dx$$

$$= 1 + \int_0^x \frac{2}{1+x}\, dx - \int_0^x dx$$

$$= 1 + \left[2 \log (1+x) \right]_0^x - (x)_0^x$$

$$= 1 + \left[2 \log (1+x) - 2 \log (1) \right] - x$$

$$y_1 = 1 + 2 \log (1+x) - x$$

$$y_1 (0.1) = 1 - 0.1 + 2 \log_e (1.1) = 1.0906$$

II approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$y_2 = 1 + \int_0^x \left(\frac{y_1 - x}{y_1 + x}\right) dx$$

$$= 1 + \int_0^x \frac{[1+2\log(1+x) - x] - x}{[1+2\log(1+x) - x] + x}\, dx$$

$$= 1 + \int_0^x \frac{[1+2\log(1+x)] - 2x}{1+2\log(1+x)}\, dx$$

$$= 1 + \int_0^x \frac{1+2\log(1+x)}{1+2\log(1+x)}\, dx - 2\int_0^x \frac{x}{1+2\log(1+x)}\, dx$$

$$= 1 + \int_0^x dx - 2\int_0^x \frac{x}{1+2\log(1+x)}\, dx$$

$$y_2 = 1 + x - 2\int_0^x \frac{x}{1+2\log(1+x)}\, dx$$

which is very difficult to integrate

Hence the first approximation is itself of the value of y. $y(0.1) = 1.0906$.

1. Given that $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$, compute $y(0.1)$ and $y(0.2)$ using picards method.

Sol: Here $f(x,y) = 1 + xy$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$

By picards method $y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx \to n=0,1,\ldots$

I approximation

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = 1+\int_0^x (1+xy_0)\,dx = 1+\int_0^x (1+x)\,dx$$

$$= 1+\left(x+\frac{x^2}{2}\right)_0^x$$

$$y_1 = 1+x+\frac{x^2}{2}$$

$$y_1(0.1) = 1+0.1+0.1^2/2 = 1.1050$$

$$y_1(0.2) = 1+0.2+0.2^2/2 = 1.22$$

II approximation:

$$y_2 = y_0+\int_{x_0}^x f(x, y_1)\,dx$$

$$= 1+\int_0^x \left[1+x\left(1+x+\frac{x^2}{2}\right)\right]dx$$

$$= 1+\int_0^x \left(1+x+x^2+\frac{x^3}{2}\right)dx$$

$$y_2 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$$y_2(0.1) = 1+0.1+0.1^2/2+0.1^3/3+0.1^4/8 = 1.1055$$

$$y_2(0.2) = 1+0.2+0.2^2/2+0.2^3/3+0.2^4/8 = 1.2229$$

III approximation:

$$y_3 = y_0+\int_{x_0}^x f(x, y_2)\,dx$$

$$y_3 = 1+\int_0^x \left(1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8}\right)dx$$

$$y_3 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

$$y_3(0.1) = 1.1053$$

$$y_3(0.2) = 1.2229$$

$\therefore$ 2nd, 3rd approximations are same

$y(0.1) = 1.1053$, $y(0.2) = 1.2227$

5. Obtain the picards second approximate soln of the initial value problem $\frac{dy}{dx} = \frac{x^2}{y^2+1}$, $y(0)=0$

Sol: Here $f(x,y) = \frac{x^2}{y^2+1}$, $x_0 = 0$, $y_0 = 0$

By picards method $y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$, $\forall n = 0, 1 \ldots$

I approximation.

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_1 = 0 + \int_0^x \left(\frac{x^2}{y_0^2+1}\right) dx$$

$$= \int_0^x x^2\,dx = \frac{x^3}{3}$$

II approximation.

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = y_0 + \int_{x_0}^{x} \left(\frac{x^2}{y_1^2+1}\right) dx$$

$$= 0 + \int_0^x \left[\frac{x^2}{\left(\frac{x^3}{3}\right)^2+1}\right] dx$$

$$= \int_0^x \frac{9x^2}{(x^3)^2+9}\,dx$$

put $x^3 = t$

$3x^2\,dx = dt$

as $x \to 0$, $t \to 0$

as $x \to x$, $t \to x^3$

$$= 3 \int_0^{x^3} \frac{dt}{t^2+9}$$

$$\frac{y}{y_2} = 3 \cdot \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right)\Big|_0^{x^3}$$

$$\frac{y}{y_2} = \tan^{-1}\left(\frac{x^3}{3}\right) - \tan^{-1}(0)$$

$$\therefore \frac{y}{y_2} = \tan^{-1}\left(\frac{x^3}{3}\right)$$