the characteristic equation matrix of A

$$A-\lambda I = \begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix} - \begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$= \begin{bmatrix}2-\lambda&1&2\\5&3-\lambda&3\\-1&0&-2-\lambda\end{bmatrix}$$

the characteristic equation of A is $|A-\lambda I|=0$

$$\begin{bmatrix}2-\lambda&1&2\\5&3-\lambda&3\\-1&0&-2-\lambda\end{bmatrix}=0$$

$$(2-\lambda)[(3-\lambda)(-2-\lambda)-0]-1[5(-2-\lambda)+3]+2[0+1(3-\lambda)]=0$$

$$(2-\lambda)[-6-3\lambda+2\lambda+\lambda^2)-(-10-5\lambda+3)+6-2\lambda=0$$

$$(2-\lambda)(-6-3\lambda+2\lambda+\lambda^2)-(-10-5\lambda+13)+6-2\lambda=0$$

$$(2-\lambda)[\lambda^2-\lambda-6]+5\lambda+7+6-2\lambda=0$$

$$2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+3\lambda+13=0$$

$$-\lambda^3+3\lambda^2+7\lambda+1=0$$

$$\lambda^3-3\lambda^2-7\lambda-1=0$$

By caley Hamilton theorem

$$A^3-3A^2-7A-I=0$$

$$A^2=A\cdot A=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$$

$$=\begin{bmatrix}4+5-2&2+3+0&4+3-4\\10+15-3&5+9+0&10+9-6\\-2+0+2&-1+0-0&-2+0+4\end{bmatrix}$$

$$=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+25-3 & 7+15+0 & 14+15-6 \\ 44+70-13 & 22+42+0 & 44+42-26 \\ 0-5-2 & 0-3+0 & 0-3+4 \end{bmatrix}$$

142
12552
24
50408
25104
301246

$$= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

$$A^3 - 3A^2 - 7A - I = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} - \begin{bmatrix} 21 & 15 & 9 \\ 66 & 42 & 39 \\ 0 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 14 \\ 35 & 21 & 21 \\ -7 & 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \qquad - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ caley Hamilton theorem is satisfied

Now

$$A^3 - 3A^2 - 7A - I = 0$$

$$I = A^3 - 3A^2 - 7A$$

Multiplying with A^{-1} o.b.s

$$A^{-1} = A^2 - 3A - 7I$$

$$A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 6 \\ 15 & 9 & 9 \\ -3 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

2. Find the inverse of the following matrices by using C-H-T. and also verify C-H-T

i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$

iv) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ v) $\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

solu] i) Given matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

the characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] + 1\left[0-2\right] + 0 = 0$$

$$(1-\lambda)\left[2-2\lambda-\lambda+\lambda^2-1\right] - 2 = 0$$

$$(1-\lambda)\left[\lambda^2-3\lambda+1\right] - 2 = 0$$

$$\lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda-2=0$$

$$-\lambda^3+4\lambda^2-4\lambda-1=0$$

$$\lambda^3-4\lambda^2+4\lambda+1=0$$

By caley hamilton theorem

$$A^3 - 4A^2 + 4A + I = 0$$

$$A^2 = A\cdot A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

wrong
$$= \begin{bmatrix} 1+0+0 & 0+0+2 & 2+0+4 \\ -1-1+0 & 0+1+1 & -2+1+2 \\ 0-1+0 & 0+1+2 & 0+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 6 \\ -2 & 2 & 1 \\ -1 & 3 & 5 \end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0-2 & -1-2-1 & 0-2-2 \\ 2+0+6 & -2+2+3 & 0+2+6 \\ 6+0+10 & -6+1+5 & 0+1+10 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

$$A^3 - 4A^2 + 4A + I = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - \begin{bmatrix} 4 & -8 & -4 \\ 8 & 8 & 12 \\ 24 & 4 & 20 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ caley hamilton theorem is satisfied

$$I = -A^3 + 4A^2 - 4A$$

multiplying with A^{-1}

$$A^{-1} = -A^2 + 4A - 4I$$

$$A^{-1} = -\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4-4 & 2-4+0 & 1+0+0 \\ -2+0-0 & -2+4-4 & 3+4+0 \\ -6+8-0 & -1+4+0 & -5+8-4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 7 \\ 2 & 3 & -1 \end{bmatrix}$$

iv) Given matrix

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

characteristic equation of A is matrix

$$A - \lambda I = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix} = 0$$

$$(7-\lambda)\left[(-1-\lambda)(-1-\lambda) - 4\right] - 2\left(-6(-1-\lambda) - 12\right) - 2\left(-12 - 6(-1-\lambda)\right)$$

$(7-\lambda)[+1+\lambda+\lambda+\lambda^2-4]-2(6+6\lambda-12)-2(-12+6+6\lambda)=0$

$(7-\lambda)[\lambda^2+2\lambda-3]-2(6\lambda-6)-2(6\lambda-6)=0$

$7\lambda^2+14\lambda-21-\lambda^3-2\lambda^2+3\lambda-12\lambda+12-12\lambda+12=0$

$-\lambda^3+5\lambda^2-7\lambda+3=0$

$\lambda^3-5\lambda^2+7\lambda-3=0$

By caley Hamilton theorem

$A^3-5A^2+7A-3I=0$

$$A^2=\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}$$

$$=\begin{bmatrix}49-12-12&14-2-4&-14+4+2\\-42+6+12&-12+1+4&12-2-2\\42-12-6&12-2-2&-12+4+1\end{bmatrix}$$

$$=\begin{bmatrix}25&8&-8\\-24&-7&8\\24&8&-7\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}25&8&-8\\-24&-7&8\\24&8&-7\end{bmatrix}\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}$$

$$=\begin{bmatrix}175-48-48&50-8-16&-50+16+8\\-168+42+48&-48+7+16&+48-14-8\\168-48-42&48-8-14&-48+16+7\end{bmatrix}$$

$$=\begin{bmatrix}79&26&-26\\-78&-25&26\\78&26&-25\end{bmatrix}$$

$$7A=7\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}=\begin{bmatrix}49&14&-14\\-42&-7&14\\42&14&-7\end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - \begin{bmatrix} 125 & 40 & -40 \\ -120 & -35 & 40 \\ 120 & 40 & -35 \end{bmatrix} + \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix}$$

$$- \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 79-125+49-3 & 26-40+14-0 & -26+40-14-0 \\ -78+120-42-0 & -25+35-7-3 & 26-40+14+0 \\ 78-120+42+0 & 26-40+14-0 & -25+35-7+3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ caley Hamilton theorem is ~~not~~ verified

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^3 - 5A^2 + 7A = 3I \rightarrow ①$$

Multiplying ① with A^{-1}

$$A^2 - 5A + 7I = 3A^{-1}$$

$$3A^{-1} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 25-35+7 & 8-10+0 & -8+10+0 \\ -24+30+0 & -7+5+7 & 8-10+0 \\ 24-30+0 & 8-10+0 & -7+5+7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

iii) Given matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(5-\lambda)(5-\lambda)-1\right] - 1\left[-1(5-\lambda)+1\right] + 1\left[1-(5-\lambda)\right] = 0$$

$$(3-\lambda)\left[25-5\lambda-5\lambda+\lambda^2-1\right] - 1\left[-5+\lambda+1\right] + \left[1-5+\lambda\right] = 0$$

$$(3-\lambda)\left[24-10\lambda+\lambda^2\right] - 1(\lambda-4) + 1(\lambda-4) = 0$$

$$(\lambda^2-10\lambda+24)(3-\lambda) \cancel{-(\lambda-4)} \cancel{+(\lambda-4)} = 0$$

$$3\lambda^2 - 30\lambda + 72 - \lambda^3 + 10\lambda^2 - 24\lambda = 0$$

$$-\lambda^3 + 13\lambda^2 - 54\lambda + 72 = 0$$

$$\lambda^3 - 13\lambda^2 + 54\lambda - 72 = 0$$

By caley Hamilton theorem

$$A^3 - 13A^2 + 54A - 72I = 0$$

$$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 9-1+1 & 3+5-1 & 3-1+5 \\ -3-5-1 & -1+25+1 & -1-5-5 \\ 3+1+5 & 1-5-5 & 1+1+25 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 27-7+7 & 9+35-7 & 9-7+35 \\ -27-25-11 & -9+125+11 & -9-25-55 \\ 27+9+27 & 9-45-27 & 9+9+135 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix}$$

$$A^3 - 13A^2 + 54A - 72I$$

$$= \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix} - \begin{bmatrix} 117 & 91 & 91 \\ -117 & 325 & -143 \\ 117 & -117 & 351 \end{bmatrix} + \begin{bmatrix} 162 & 54 & 54 \\ -54 & 270 & -54 \\ 54 & -54 & 270 \end{bmatrix} - \begin{bmatrix} 72 & 0 & 0 \\ 0 & 72 & 0 \\ 0 & 0 & 72 \end{bmatrix}$$

$$= \begin{bmatrix} 27-117+162-72 & 37-91+54-0 & 37-91+54+0 \\ -63+117-54+0 & 127-325+270-72 & -89-325-54+0 \\ 63-117+54+0 & -63+117-54+0 & 153-351+270-72 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Caley Hamilton theorem is verified

$$A^3 - 13A^2 + 54A - 72I = 0$$

$$A^3 - 13A^2 + 54A = 72I$$

$$A^2 - 13A + 54I = 72A^{-1}$$

$$72A^{-1} = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - \begin{bmatrix} 39 & 13 & 13 \\ -13 & 65 & -13 \\ 13 & -13 & 65 \end{bmatrix} + \begin{bmatrix} 54 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 54 \end{bmatrix}$$

$$72A^{-1} = \begin{bmatrix} 9-39+54 & 7-13+0 & 7-13+0 \\ -9+13+0 & 25-65+54 & -11+13+0 \\ 9-13+0 & -9+13+0 & 27-65+54 \end{bmatrix}$$

$$A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

ii) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(-1-\lambda)(-1-\lambda)-4] - 2[2(-1-\lambda)-12] + 3(2-3(-1-\lambda)) = 0$$

$$(1-\lambda)[1+\lambda+\lambda+\lambda^2-4] - 2[-2-2\lambda-12] + 3(2+3+3\lambda) = 0$$

$$(1-\lambda)[\lambda^2+2\lambda-3] - 2(-2\lambda-14) + 3(3\lambda+5) = 0$$

$$\lambda^2+2\lambda-3-\lambda^3-2\lambda^2+3\lambda+4\lambda+28+9\lambda+15 = 0$$

$$-\lambda^3-\lambda^2+18\lambda+40 = 0$$

By Caley Hamilton theorem

$$A^3+A^2-18A-40I=0$$

$$A^2=\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9 & 2-2+3 & 3+8-3\\ 2-2+12 & 4+1+4 & 6-4-4\\ 3+2-3 & 6-1-1 & 9+4+1\end{bmatrix}=\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}\begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}14+6+24 & 28-3+8 & 42+12-8\\ 12+18-6 & 24-9-2 & 36+36+2\\ 2+8+42 & 4-4+14 & 6+16-14\end{bmatrix}=\begin{bmatrix}44&33&22\\24&13&74\\52&14&8\end{bmatrix}$$

$$A^3+A^2-18A-40I$$

$$=\begin{bmatrix}44&33&22\\24&13&74\\52&14&8\end{bmatrix}+\begin{bmatrix}14&3&8\\12&9&-2\\2&4&14\end{bmatrix}-\begin{bmatrix}18&36&54\\36&-18&72\\54&18&-18\end{bmatrix}-\begin{bmatrix}40&0&0\\0&40&0\\0&0&40\end{bmatrix}$$

$$=\begin{bmatrix}44+14-18+40 & 33+3-36-0 & 22+8-54+0\\ 24+12-36-0 & 13+9+18+40 & 74-2-72-0\\ 52+2-54-0 & 14+4-18+0 & 8+14+18-40\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\therefore$ Since Caley Hamilton theorem is verified

$$A^3+A^2-18A-40I=0$$

$$A^3+A^2-18A=40I$$

$$A^2 + A - 18I = 40A^{-1}$$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 14+1-18 & 3+2-0 & 8+3+0 \\ 12+2-0 & 9-1-18 & -2+4+0 \\ 2+3+0 & 4+1+0 & 14-1-18 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

v) Given matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(-3-\lambda)(1-\lambda)-8] + 8[4(1-\lambda)+6] + 2[-16-3(-3-\lambda)] = 0$$

$$(8-\lambda)[-3-\lambda+3\lambda+\lambda^2-8] + 8[4-4\lambda+6] + 2(-16+9+3\lambda) = 0$$

$$(8-\lambda)[\lambda^2+2\lambda-11] + 8(10-4\lambda) + 2(3\lambda-7) = 0$$

$$8\lambda^2+16\lambda-88-\lambda^3-2\lambda^2+11\lambda+80-32\lambda+6\lambda-14 = 0$$

$$-\lambda^3+6\lambda^2+\lambda-22 = 0$$

$$\lambda^3-6\lambda^2-\lambda+22 = 0$$

By caley Hamilton theorem $= 22I - A + 6A$

$$A^3 - 6A^2 - A + 22I = 0$$

$$A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 64-32+6 & -64+24-8 & 16+16+2 \\ 32-12-6 & -32+9+8 & 8+6-2 \\ 24-16+3 & -24+12-4 & 6+8+1 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 304-192+102 & -304+144-136 & 76+96+34 \\ 112-60+36 & 112+45-48 & 28+30+12 \\ 88-64+45 & -88+48-60 & 22+32+15 \end{bmatrix}$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & 109 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$$A^3 - 6A^2 - A + 22I = 0$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & 109 & 70 \\ 69 & -100 & 69 \end{bmatrix} - \begin{bmatrix} 228 & -288 & 204 \\ 84 & -90 & 72 \\ 66 & -96 & 90 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 214-228-8+22 & -296+288+8+0 & 206-204-2+0 \\ 28-24-4+0 & 109+90+3+22 & 70-72+2+0 \\ 69-66-3+0 & -100+76+6+0 & 67-90-1+22 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

caley Hamilton theorem is verified

$$A^3 - 6A^2 - A + 22I = 0$$

$$22I = -A^3 + 6A^2 + A$$

multiplying with A^{-1}

$$22A^{-1} = -A^2 + 6A + I$$

$$22A^{-1} = \begin{bmatrix} -38 & 48 & -34 \\ -14 & 15 & -12 \\ -11 & 16 & -15 \end{bmatrix} + \begin{bmatrix} 48 & -48 & 12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$22A^{-1} = \begin{bmatrix} -38+48+1 & 48-48+0 & -34+12+0 \\ -14+24+0 & 15-18+1 & -12-12+0 \\ -11+18+0 & 16-24+0 & -15+6+1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 11 & 0 & -22 \\ 10 & -2 & -24 \\ 7 & -8 & -8 \end{bmatrix}$$

vi) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I|=0 \rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix}=0$$

$$(1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]+3[10-3(4-\lambda)]=0$$

$$(1-\lambda)[24-6\lambda-4\lambda+\lambda^2-25]-2[12-2\lambda-15]+3[10-12+3\lambda]=0$$

$$(1-\lambda)[\lambda^2-10\lambda-1]-2[-2\lambda-3]+3[3\lambda-2]=0$$

$$\lambda^2-10\lambda-1-\lambda^3+10\lambda^2+\lambda+4\lambda+6+9\lambda-6=0$$

$$-\lambda^3+11\lambda^2+4\lambda-1=0$$

$$\therefore \lambda^3-11\lambda^2-4\lambda+1=0$$

By caley Hamilton theorem

$$A^3-11A^2-4A+I=0$$

$$A^2=A\cdot A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} 1+4+9 & 2+8+15 & 3+10+18 \\ 2+8+15 & 4+16+25 & 6+20+30 \\ 3+10+18 & 6+20+30 & 9+25+36 \end{bmatrix}=\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} 14+50+93 & 28+100+155 & 42+125+186 \\ 25+90+168 & 50+180+280 & \cancel{95}\ 75+225+336 \\ 31+112+210 & 62+224+350 & 93+280+420 \end{bmatrix}$$

$$=\begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

$A^3 - 11A^2 - 4A + I$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 253 & 636 & 793 \end{bmatrix} - \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 770 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 157-154-4+1 & 283-275-8+0 & 353-341-12+0 \\ 283-275-8+0 & 510-495-16+1 & 636-616-20+0 \\ 253-341-12+0 & 636-616-20+0 & 793-770-24+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ caley Hamilton theorem is verified

$$A^3 - 11A^2 - 4A + I = 0$$

$$I = -A^3 + 11A^2 + 4A$$

multiplying with A^{-1}

$$A^{-1} = -A^2 + 11A + 4I$$

$$A^{-1} = \begin{bmatrix} -14 & -25 & -31 \\ -25 & -45 & -56 \\ -31 & -56 & -70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -14+11+4 & -25+22+0 & -31+33+0 \\ -25+22+0 & -45+44+4 & -56+55+0 \\ -31+33+0 & -56+55+0 & -70+66+4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A

note

2017

2018

Solu) Given matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) + 1 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By caley Hamilton theorem

$$A^2 - 5A + 7I = 0$$

$$A^2 = 5A - 7I$$

$$A^3 = 5A^2 - 7A$$

$$A^4 = 5A^3 - 7A^2$$

$$A^5 = 5A^4 - 7A^3$$

$$2A^5 - 3A^4 + A^2 - 4I = 2[5A^4 - 7A^3] - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I$$

$$= 7[5A^3 - 7A^2] - 14A^3 + A^2 - 4I$$

$$= 21A^3 - 48A^2 - 4I$$

$= 21[5A^2 - 7A] - 48A^2 - 4I$

$= 57A^2 - 147A - 4I$

$= 57(5A - 7I) - 147A - 4I$

$= 138A - 403I$

Q. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a polynomial

A. Given matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}$$

The characteristic equation of A is

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) + 2 = 0$

$3 - 3\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

By Cayley Hamilton theorem

$A^2 - 4A + 5I = 0$

$A^2 = 4A - 5I$

$A^3 = 4A^2 - 5A$

$A^4 = 4A^3 - 5A^2$

$A^5 = 4A^4 - 5A^3$

$A^6 = 4A^5 - 5A^4$

Given equation

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$A^6 - 14 = (4A^5 - 5A^4) - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= 3[4A^3 - 5A^2] - 12A^3 + 14A^2$

$= -15A^2 + 14A^2$

$= -A^2 = -4A + 5I$

Quadratic forms

* A homogeneous expression of the second degree in any no. of variables is called a quadratic form.

Ex: 1. $3x^2 + 5xy - 2y^2$ is a quadratic form in x & y

2. $x^2 + 2y^2 - 3z^2 + 2xy - 3yz + 5zx$ is a quadratic form in three variables.

* An expression of the form $Q = X^T AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ where a_{ij} are constants is called a quadratic form in n variables.

Matrix of a Quadratic form

Every quadratic form Q can be expressed as $Q = X^T AX$, the symmetric matrix A is called the matrix of the quadratic form Q and $|A|$ is called the descriminant of the quadratic form

Note:

* If $|A| = 0$ the quadratic form is singlur
* ex: To write the matrix of quadratic form follow the diagram given below:

write the co-efficients of square terms along the diagonal and divide the co-efficients of the product terms, xy, yz, zx by 2 and write them at the appropriate places.

Ex: $Q = 7x^2 + 8xy + 9yz + 2xz + 3y^2 - 5z^2$

$Q = 7xx + 4xy + 4xy + \frac{9}{2}yz + xz + xz + 3yy - 5zz$

$= 7xx + 4xy + xz$
$+ 4yx + 3yy + \frac{9}{2}yz$
$+ zx + \frac{9}{2}zy - 5zz$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & \frac{xy}{2} & \frac{xz}{2} \\ y & \frac{yx}{2} & y^2 & \frac{yz}{2} \\ z & \frac{zx}{2} & \frac{zy}{2} & z^2 \end{array}$$

$$A = \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & 9/2 \\ 1 & 9/2 & -5 \end{bmatrix}$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $X^T = [x \ \ y \ \ z]$

$Q = X^T A X = [x \ y \ z] \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & 9/2 \\ 1 & 9/2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

write the symmetric matrix of the following q.f

1. $x^2 + 2y^2 - 7z^2 - 4xy - 6xz$
2. $2x^2 - 3y^2 + 5z^2 - 6xy - yz + 4zx$
3. $4xy + 6yz + 8zx$
4. $x^2 - y^2 + z^2 + 7xy + 9yz + 11zx$

4. $Q = x^2 - y^2 + z^2 + 7xy + 9yz + 11zx$
$= x^2 - y^2 + z^2 + \frac{7}{2}xy + \frac{9}{2}yz + \frac{11}{2}zx$

$Q = xx - yy + zz + \frac{7}{2}xy + \frac{9}{2}yz + \frac{11}{2}zx$

$= xx + \frac{7}{2}xy + \frac{11}{2}xz$
$+ \frac{7}{2}yx - yy + \frac{9}{2}yz$
$+ \frac{11}{2}zx + \frac{9}{2}zy + zz$

$$A = \begin{bmatrix} 1 & 7/2 & \frac{11}{2} \\ \frac{7}{2} & -1 & \frac{9}{2} \\ \frac{11}{2} & \frac{9}{2} & 1 \end{bmatrix}$$

3. $Q = 4xy + 6yz + 8zx$

$Q = 2xy + 3yz + 4zx$

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

2. $Q = 2x^2 - 3y^2 + 5z^2 - 6xy - yz + 4zx$

$2xx - 3yy + 5zz - 3xy - \frac{1}{2}yz + 2zx$

$$\begin{aligned} Q = {} & 2xx - 3xy + 2zx \\ & -3yx - 3yy - \tfrac{1}{2}yz \\ & +2zx - \tfrac{1}{2}zy + 5zz \end{aligned} \qquad A = \begin{bmatrix} 2 & -3 & 2 \\ -3 & -3 & -1/2 \\ 2 & -1/2 & 5 \end{bmatrix}$$

1. $x^2 + 2y^2 - 7z^2 - 4xy - 6xz$

$Q = xx + 2yy - 7zz - 2xy - 3xz$

$$\begin{aligned} Q = {} & xx - 2xy - 3xz \\ & -2xy + 2yy + 0 \cdot yz \\ & -3xz + 0 \cdot yz - 7zz \end{aligned} \qquad A = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & 0 \\ -3 & 0 & -7 \end{bmatrix}$$

Date 31/12/2018

Write the Quadratic form of corresponding to the matrix

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 5) $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}, X^T = [x \ y \ z] \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^T A X$

$$= [x \ y \ z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x+2y+3z \quad 2x+3z \quad 3x+3y+z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(x+2y+3z) + y(2x+3z) + z(3x+3y+z)$$

$$= x^2+2xy+3zx+2xy+3zx+3zy+z^2$$

$$= x^2+z^2+4xy+6zx+3zy$$

3) Given matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}, X^T = [x \ y \ z] \ ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form

$$Q = X^T A X$$

$$= [x \ y \ z] \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x+2y+5z \quad 2x+3z \quad 5x+3y+4z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(x+2y+5z) + y(2x+3z) + z(5x+3y+4z)$$

$$= x^2+2xy+5zx+2xy+3zy+5zx+3zy+4z^2$$

$$= x^2+4z^2+4xy+10zx+6yz$$

Rank of a Quadratic form

Let X^TAX be a quadratic form the rank $R(A)$ is called the rank of the quadratic form. If 'r' is less than n, $|A|=0$ (or) A is singular then the quadratic form is called "singular" otherwise "non-singular"

Canonical form (or) Normal form of a Quadratic form

Let X^TAX be a quadratic form in n variables then there exist a real non-singular linear transformation $X = PY$ which transforms X^TAX to another quadratic form of type $Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \dots + \lambda_n y_n^2$ then Y^TDY is called the canonical form of quadratic form of X^TAX

Here $D = diag(\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n)$

Index of a Real Quadratic Form

The number of positive terms in canonical form of quadratic form is known as the index of the quadratic form and is denoted by 's'

Signature of a Quadratic form

If 'r' is the rank of the quadratic form and 's' is the index of the quadratic form then $2s-r$ is called the signature of the quadratic form X^TAX.

Nature of Quadratic Forms

⇒ Positive Definite

The Quadratic form X^TAX in 'n' variables is said to be positive Definite if all the Eigen values of A are positive (or) if $r=n$ and $s=n$ i.e., $r=s=n$

⇒ Negative Definite

The Quadratic form X^TAX in n variables is said to be negative definite if $r=n$ and $s=0$ (or) if all the eigen values of A are negative.

⇒ Positive Semi-Definite

The Quadratic form X^TAX in n variables is said to be positive semi definite if $r<n$ & $s=r$ (or) if all the eigen values of $A \geq 0$ and atleast one eigen value is zero

⇒ Negative Semi-Definite

The Quadratic form X^TAX in n variables is said to be negative semidefinite if $r<n$ & $s=0$ (or) if all the eigen values of $A \leq 0$. and atleast one eigen value is zero

⇒ In-Definite

In all other cases, if all the eigen values of A are positive and negative. then the Quadratic form is called in-definite

1. Identify the nature of the Quadratic forms

i) $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$

ii) $x^2+4xy+6xz-y^2+2yz+4z^2$

iii) $x^2+y^2+2z^2-2xy+2yz$

iv) $2x^2+19y^2+16z^2+8xy+8yz+6zx$

Soln i) Given Quadratic form

$$Q = x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$$

$$Q = X^TAX, \quad A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(1-\lambda)-4]+2[-2(1-\lambda)+2]+1(4-(4-\lambda)) = 0$$

$$(1-\lambda)[4-\lambda-4\lambda+\lambda^2\cancel{-4}]+2[\cancel{-2}+2\lambda\cancel{+2}]+\cancel{4}\cancel{-4}+\lambda = 0$$

$$(1-\lambda)[\lambda^2-5\lambda]+4\lambda+\lambda = 0$$

$$\lambda^2\cancel{-5\lambda}-\lambda^3+5\lambda^2\cancel{+5\lambda} = 0$$

$$-\lambda^3+6\lambda^2 = 0$$

$$\lambda^3 = 6\lambda^2$$

$$\lambda = 6 \ ; \ \lambda = 0, 0, 6$$

Eigen values two are zeros and the remaining is positive

Hence given Quadratic form is positive semidefinite

(iii) Given quadratic form

$$Q = 2x^2 + 9y^2 + 6z^2 + 8xy + 8yz + 6zx$$

$$Q = X^T A X \; ; \quad A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 9 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

the characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 & 3 \\ 4 & 9-\lambda & 4 \\ 3 & 4 & 6-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(9-\lambda)(6-\lambda)-16\right] - 4\left[4(6-\lambda)-12\right] + 3\left[16-3(9-\lambda)\right] = 0$$

$$(2-\lambda)\left[54 - 6\lambda - 9\lambda + \lambda^2 - 16\right] - 4\left[24 - 4\lambda - 12\right] + 3\left[16 - 27 + 3\lambda\right] = 0$$

$$(2-\lambda)\left[38 - 15\lambda + \lambda^2\right] - 4\left[12 - 4\lambda\right] + 3\left[3\lambda - 11\right] = 0$$

$$2\lambda^2 - 30\lambda + 76 - \lambda^3 + 15\lambda^2 - 38\lambda - 48 + 16\lambda + 9\lambda - 33 = 0$$

$$-\lambda^3 + 17\lambda^2 - 43\lambda - 5 = 0$$

$$\lambda^3 - 17\lambda^2 + 43\lambda + 5 = 0$$

iv) Given quadratic form

$$Q = x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$$

$$Q = X^T A X, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-1-\lambda)(4-\lambda) - 1\right] - 2\left[2(4-\lambda) - 3\right] + 3\left[2 - 3(-1-\lambda)\right] = 0$$

$$(1-\lambda)\left[-4 - 4\lambda + \lambda + \lambda^2 - 1\right] - 2\left[8 - 2\lambda - 6\right] + 3\left[2 + 3 + 3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2 - 3\lambda - 5\right] - 2\left[-2\lambda + 2\right] + 3\left[3\lambda + 5\right] = 0$$

$$\lambda^2 - 3\lambda - 5 - \lambda^3 + 3\lambda^2 + 5\lambda + 4\lambda - 4 + 9\lambda + 15 = 0$$

$$-\lambda^3 + 4\lambda^2 + 15\lambda + 6 = 0$$

$$\lambda^3 - 4\lambda^2 - 15\lambda - 6 = 0$$

iii) Given Quadratic form

$$Q = x^2+y^2+2z^2-2xy+2xz$$

$$Q = X^TAX \quad A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

the characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-0\right]+1\left[-1(2-\lambda)-0\right]+1\left[0-(1-\lambda)\right]=0$$

$$(1-\lambda)\left[2-2\lambda-\lambda+\lambda^2-0\right]+\left[-2+\lambda\right]+\lambda-1=0$$

$$(1-\lambda)\left[\lambda^2-3\lambda+2\right]-2+\lambda+\lambda-1=0$$

$$\lambda^2-3\lambda+2-\lambda^3+3\lambda^2-2\lambda-3+2\lambda=0$$

$$-\lambda^3+4\lambda^2-3\lambda-1=0$$

$$\lambda^3-4\lambda^2+3\lambda+1=0$$

2. Given matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix} \quad X^T = [x\ y\ z] ;\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^T A X$

$$= [x\ y\ z] \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x+y+5z & x+3y-2z & 5x-2y+4z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x(2x+y+5z) & y(x+3y-2z) & z(5x-2y+4z) \end{bmatrix}$$

$$= 2x^2 + xy + 5zx + xy + 3y^2 - 2zy + 5zx - 2zy + 4z^2$$

$$= 2x^2 + 3y^2 + 4z^2 + 2xy + 10zx - 4zy$$

4. Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \quad X^T = [x\ y\ z] ;\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^T A X$

$$= [x\ y\ z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x+2y+3z & 2x+y+3z & 3x+3y+z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x^2 + 2xy + 3zx + 2xy + y^2 + 3zy + 3zx + 3yz + z^2$$

$$= x^2 + y^2 + z^2 + 4xy + 6zx + 6zy$$

5. Given matrix

$$A=\begin{bmatrix}0 & 5 & -1\\ 5 & 1 & 6\\ -1 & 6 & 2\end{bmatrix} \quad X^T=[x\ y\ z] ; X\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Quadratic form $Q = X^TAX$

$$= [x\ y\ z]\begin{bmatrix}0 & 5 & -1\\ 5 & 1 & 6\\ -1 & 6 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$= [5y-z \quad 5x+y+6z \quad -x+6y+2z]\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$= [x5y-zx+5xy+y^2+6zy-xz+6yz+2z^2]$$

$$y^2+2z^2+10xy+12yz-2xz$$

Date 21/11/19

1. Reduce the matrix $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ to a diagonal part and interupt the result in terms of Quadratic forms also find the rank signature, Index.

Sol) $A = I_3 A I_3$

$$\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}6 & -2 & 2\\ 0 & 7 & -1\\ 0 & -1 & 7\end{bmatrix} \begin{matrix}R_2\to 3R_2+R_1\\ R_3\to 3R_3-R_1\end{matrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ -1 & 0 & 3\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}6 & 0 & 0\\ 0 & 21 & -3\\ 0 & -3 & 21\end{bmatrix} \begin{matrix}C_2\to 3C_2+C_1\\ C_3\to 3C_3-C_1\end{matrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ -1 & 0 & 3\end{bmatrix}A\begin{bmatrix}1 & 1 & -1\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$\begin{bmatrix}6 & 0 & 0\\ 0 & 21 & -3\\ 0 & 0 & 144\end{bmatrix} R_3\to 7R_3+R_2 = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ -6 & 3 & 21\end{bmatrix}A\begin{bmatrix}1 & 1 & -1\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} C_3 \to 7C_3 + C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

$D = P^T A P$

$D = \text{diag}(6, 21, 1008)$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

Quadratic form $= X^T A X$

$$= [x \ y \ z] \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= 6x^2 + 3y^2 + 3z^2 - 4xy + 4xz - 2yz$

Non-singular transformation corresponding to the matrix P is $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= [y_1 + y_2 - 6y_3, \ 3y_2 + 3y_3 \ \ 21y_3]$

$x = y_1 + y_2 - 6y_3 ; \ y = 3y_2 + 3y_3 ; \ z = 21y_3$

Canonical form $= Y^T D Y = 6y_1^2 + 21y_2^2 + 1008y_3^2$

Rank of A is $\rho(A) = 3$ (rank of diagonal matrix non zero row)

Index $= s = 3$ (no. of positive terms)

Signature $= 2s - r = 2(3) - 3 = 3$

2. Find the rank, signature, index of the quadratic form

i) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_1x_3$ by reducing it into canonical form also write the linear transformation which brings about the normal reduction

Sol) Given

Quadratic form

$$= 2x_1^2+x_2^2-3x_3^2+12x_1x_2-8x_2x_3-4x_1x_3$$

Given quadratic form into matrix

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2-3R_1 \\ R_3 \to R_3+R_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} \begin{matrix} C_2 \to C_2-3C_1 \\ C_3 \to C_3+C_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 0 & 81 \end{bmatrix} R_3 \to -17R_3-2R_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 1377 \end{bmatrix} C_3 \to -17C_3-2C_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -2 \\ 0 & 0 & -17 \end{bmatrix}$$

$$D = P^T A P$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 1377 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -2 \\ 0 & 0 & -17 \end{bmatrix}$$

Quadratic form $= X^T A X$

$$= [x\ y\ z] \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 2x^2+6xy-2zx \\ +6xy+y^2-4zy \\ -2xz-4yz-3z^2 \end{matrix}$$

$$= [2x+6y-2z \quad 6x+y-4z \quad -2x-4y-3z]$$

$2x^2+y^2=3z^2+12xy-4zx-8zy$

non singular transformation corresponding to the matrix P is $X=PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -2 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1-3y_2-11y_3 \quad y_2-2y_3 \quad -17y_3]$$

$x=y_1-3y_2-11y_3$; $y=y_2-2y_3$, $z=-17y_3$

canonical form $=Y^TDY = 2y_1^2-17y_2^2+137y_3^2$

Rank of A is $\rho(A)=3$

Index $s=2$

signature $=2s-r=2(2)-3=1$

Date 5/11/2019

2. Reduction to Normal form by orthogonal transformation.

Working Rule:

1. write the co-efficient matrix 'A' associated with the given quadratic form
2. Find the Eigen values of A
3. write the canonical form using $\lambda_1 y_1^2+\lambda_2 y_2^2+\dots+\lambda_n y_n^2$
4. Form the matrix P containing the normalized eigen vectors of A as column vectors. then $X=PY$ gives the required orthogonal transformation which reduces quadratic form to canonical form

1 Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation.

Sol: Given Quadratic form

$$Q = 3x^2+2y^2+3z^2-2xy-2yz$$

The matrix form

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1\left[-(3-\lambda)-0\right]+0=0$$

$$(3-\lambda)\left[6-3\lambda-2\lambda+\lambda^2-1\right]+\left[-3+\lambda\right]=0$$

$$(3-\lambda)\left[\lambda^2-5\lambda+5\right]-3+\lambda=0$$

$$3\lambda^2-15\lambda+15-\lambda^3+5\lambda^2-5\lambda-3+\lambda=0$$

$$-\lambda^3+8\lambda^2-19\lambda+12=0$$

$$\lambda^3-8\lambda^2+19\lambda-12=0$$

$$\begin{array}{r|rrrr} 1 & 1 & -8 & 19 & -12 \\ & 0 & 1 & -7 & 12 \\ \hline & 1 & -7 & 12 & 0 \end{array}$$

$$(\lambda-1)(\lambda^2-7\lambda+12)=0$$

$$(\lambda-1)\left[\lambda^2-4\lambda-3\lambda+12\right]=0$$

$$(\lambda-1)\left[\lambda(\lambda-4)-3(\lambda-4)\right]=0$$

$$(\lambda-1)(\lambda-4)(\lambda-3)=0$$

$$\lambda = 1, 4, 3$$

The are the characteristic roots 1, 4, 3 case (i)

If $\lambda = 1$ $(A-\lambda I)X = 0$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to 2R_2 + R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank $= 2$, $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$2x - y = 0$; $y - 2z = 0$; $z = k$

$2x - 2k = 0$ $y = 2k$

$x = k$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

case (ii)

If $\lambda = 4$ $(A-\lambda I)X = 0$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n - r = 3 - 2 = 1$ L.I.S

$-x-y=0$; $-y-x-y$ $z=k$

$x+k=0$ $-y-k=0$

$x=+k$ $-y=k$

$y=-k$

$$X_3=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-k\\-k\\k\end{bmatrix}=\begin{bmatrix}+1\\-1\\1\end{bmatrix}$$

Eigen

$$x_1=\begin{bmatrix}1\\2\\1\end{bmatrix}\quad x_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}\quad x_3=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

case (iii)

If $\lambda=3$; then $(A-\lambda I)x=0$

$$\begin{bmatrix}0&-1&0\\-1&-1&-1\\0&-1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&-1&0\\-1&-1&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}_{R_3\to R_3-R_1}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\rho(A)=2$; $n=3$

$n-r=3-2$, L.I.S

$-y=0$; $-x-y-z=0$; $z=k$

$-x-0-k=0$

$-x=k$

$x=-k$

$$X_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-k\\0\\k\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}k$$

Here

$\therefore$ we observed that x_1, x_2, x_3 are mutually $\perp$ ler

$x_1x_2=x_2x_3=x_3x_1=0$

The normalized vectors are

$$c_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \quad c_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad c_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P = [c_1 \; c_2 \; c_3] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 1/\sqrt{3} \end{bmatrix}$$

$$D = P^T A P$$

$$D = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 1/\sqrt{6} \\ \frac{-1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

H.W got
u[1]

$$= \begin{bmatrix} 3/\sqrt{6} - 2/\sqrt{6} + 0 & -1/\sqrt{6} + 4/\sqrt{6} - 1/\sqrt{6} & 0 - 2/\sqrt{6} + 3/\sqrt{6} \\ -3/\sqrt{2} + 0 + 0 & 1/\sqrt{2} + 0 - 1/\sqrt{2} & 0 + 0 + 3/\sqrt{2} \\ 3/\sqrt{3} + 1/\sqrt{3} + 0 & -1/\sqrt{3} - 2/\sqrt{3} - 1/\sqrt{3} & 0 + 1/\sqrt{3} + 3/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & \frac{-1}{\sqrt{2}} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

solu] =

$$= \begin{bmatrix} 1/\sqrt{6} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -3/\sqrt{2} & 0 & 3/\sqrt{2} \\ 4/\sqrt{3} & -4/\sqrt{3} & 4/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} + \frac{4}{6} + \frac{1}{6} & -1/\sqrt{12} + 0 + 1/\sqrt{12} & 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18} \\ \frac{-3}{\sqrt{12}} + 0 + 3/\sqrt{12} & 3/2 + 0 + 3/2 & -3/\sqrt{6} + 0 + 3/\sqrt{6} \\ 4/\sqrt{18} - 8/\sqrt{18} + 4/\sqrt{18} & -4/\sqrt{6} + 0 + 4/\sqrt{6} & \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \mathrm{diag}(1, 3, 4)$$

orthogonal transformation

$$X = PY$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x = \frac{y_1}{\sqrt{6}} - \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{3}}$$

$$y = \frac{2y_1}{\sqrt{6}} - y_3/\sqrt{3}$$

$$z = \frac{y_1}{\sqrt{6}} + \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{3}}$$

H.W Date 4/1/19

2. Reduce the Quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal reduction

3. $x_1^2+3x_2^2+3x_3^2-2x_2x_3$

4. $2x^2+2y^2+2z^2-2xy-2yz-2zx$

$\begin{matrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{matrix}$

soln 2. Given Quadratic form

Q.F $= 3x^2+5y^2+3z^2-2yz+2zx-2xy$

The matrix form

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$[(3-\lambda)[3(5-\lambda)-1]+1[-3+1]+1[1-(5-\lambda)]=0$$

$$(3-\lambda)[15-3\lambda-1]+1[-2]+1[1-5+\lambda]=0$$

$$(3-\lambda)[14-3\lambda]-2+\lambda-4=0$$

$$42-14\lambda-9\lambda+3\lambda^2+\lambda-6=0$$

$$3\lambda^2-22\lambda+36=0]$$

$$(3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[-(3-\lambda)+1]+1[1-(5-\lambda)]=0$$

$$(3-\lambda)[15-3\lambda-5\lambda+\lambda^2-1]+[-3+\lambda+1]+1-5+\lambda=0$$

$$(3-\lambda)[\lambda^2-8\lambda+14]+\lambda-2+\lambda-4=0$$

$$3\lambda^2-24\lambda+42-\lambda^3+8\lambda^2-14\lambda+2\lambda-6=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$\begin{array}{r|rrrr} 3 & 1 & -11 & 36 & -36 \\ & 0 & 3 & -24 & 36 \\ \hline & 1 & -8 & 12 & \boxed{0} \end{array}$$

$$\lambda^2-8\lambda+12=0$$

$$(\lambda-3)[\lambda^2-6\lambda-2\lambda+12]=0$$

$$(\lambda-3)[\lambda(\lambda-6)-2(\lambda-6)]=0$$

$$(\lambda-2)(\lambda-3)(\lambda-6)=0$$

$$\lambda=2,3,6$$

case (ii)

If $\lambda=2$ $\quad (A-\lambda I)X=0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2\to R_2+R_1 \\ R_3\to R_3-R_1 \end{matrix} =\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2, \; n = 3$

$n - r = 3 - 2 = 1$, L.I.S

$x - y + z = 0$; $2y = 0$; let $z = k$

$x - 0 + k = 0$ $\quad y = 0$

$x = -k$

$$\therefore X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case (iii)

If $\lambda = 6$ $\quad (A - \lambda I)X = 0$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to 2R_3 - 4R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & +1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to \frac{R_2}{-2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2, \; n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$-3x - y + z = 0$; $y + 2z = 0$; $2k$

$-3x + 2k + k = 0$ $\quad y + 2k = 0$

$-3x = -3k$ $\quad y = -2k$

$x = k$

$$\therefore X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

case(iii)

If $\lambda = 3$ $(A-\lambda I)X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$\rho(A) = 2$
$n = 3$
$n - r = 3 - 2 = 1$
$\therefore$ L.I.S

$x - y = 0$; $y - z = 0$; $z = K$

$x - K = 0$ $y - K = 0$
$x = K$ $y = K$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} ; \; X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; \; X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

we observed that X_1, X_2, X_3 are mutually perpendicular

The normalized vectors

$$e_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad e_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad e_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$D = P^T A P$

$$D = \begin{bmatrix} -\frac{1}{\sqrt2} & 0 & 1/\sqrt2 \\ 1/\sqrt3 & 1/\sqrt3 & 1/\sqrt3 \\ 1/\sqrt6 & \frac{-2}{\sqrt6} & 1/\sqrt6 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt2} & 1/\sqrt3 & 1/\sqrt6 \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ 1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{\sqrt2}+0+\frac{1}{\sqrt2} & \cancel{\frac{1}{\sqrt2}}+0-\cancel{\frac{1}{\sqrt2}} & -\frac{1}{\sqrt2}+0+\frac{3}{\sqrt2} \\ \frac{3}{\sqrt3}-\cancel{\frac{1}{\sqrt3}}+\cancel{\frac{1}{\sqrt3}} & \frac{-1}{\sqrt3}+\frac{5}{\sqrt3}-\cancel{\frac{1}{\sqrt3}} & \cancel{\frac{1}{\sqrt3}}-\cancel{\frac{1}{\sqrt3}}+\frac{3}{\sqrt3} \\ \frac{3}{\sqrt6}+\frac{2}{\sqrt6}+\frac{1}{\sqrt6} & \frac{-1}{\sqrt6}-\frac{10}{\sqrt6}-\frac{1}{\sqrt6} & \frac{1}{\sqrt6}+\frac{2}{\sqrt6}+\frac{3}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ 0 & \frac{1}{\sqrt3} & \frac{-2}{\sqrt6} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt2} & 0 & \frac{2}{\sqrt2} \\ \frac{3}{\sqrt3} & \frac{3}{\sqrt3} & \frac{3}{\sqrt3} \\ \frac{6}{\sqrt6} & \frac{-12}{\sqrt6} & \frac{6}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ \frac{1}{\sqrt2} & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{2}+0+\frac{2}{2} & -\cancel{\frac{2}{\sqrt6}}+0+\cancel{\frac{2}{\sqrt6}} & -\cancel{\frac{2}{\sqrt{12}}}+0+\cancel{\frac{2}{\sqrt{12}}} \\ -\cancel{\frac{3}{\sqrt6}}+0+\cancel{\frac{3}{\sqrt6}} & \frac{3}{3}+\frac{3}{3}+\frac{3}{3} & \cancel{\frac{3}{\sqrt{18}}}-\cancel{\frac{6}{\sqrt{18}}}+\cancel{\frac{3}{\sqrt{18}}} \\ -\cancel{\frac{6}{\sqrt{12}}}+0+\cancel{\frac{6}{\sqrt{12}}} & \cancel{\frac{6}{\sqrt{18}}}-\cancel{\frac{12}{\sqrt{18}}}+\cancel{\frac{6}{\sqrt{18}}} & \frac{6}{36}+\frac{24}{36}+\frac{6}{36} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \operatorname{diag}(2,3,6)$$

orthogonal transformation

$X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ 1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore x = -\frac{y_1}{\sqrt2} + \frac{y_2}{\sqrt3} + \frac{y_3}{\sqrt6}$; $y = \frac{y_2}{\sqrt3} - \frac{2y_3}{\sqrt6}$; $z = \frac{y_1}{\sqrt2} + \frac{y_2}{\sqrt3} + \frac{y_3}{\sqrt6}$

4. Given Quadratic form

$$Q.F = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$$

matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)(2-\lambda)-1] + 1[-(2-\lambda)-1] - 1[1+(2-\lambda)] = 0$$

$$(2-\lambda)[4-2\lambda-2\lambda+\lambda^2-1] + [-2+\lambda-1] - [1+2-\lambda] = 0$$

$$(2-\lambda)[\lambda^2-4\lambda+3] + [\lambda-3] - [3-\lambda] = 0$$

$$(2-\lambda)[\lambda^2-4\lambda+3] + \lambda-3-3+\lambda = 0$$

$$2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 6 = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda[\lambda^2 - 3\lambda - 3\lambda + 9] = 0$$

$$\lambda[\lambda(\lambda-3) - 3(\lambda-3)] = 0$$

$$\lambda(\lambda-3)(\lambda-3) = 0$$

$$\lambda = 0, 3, 3$$

case (i)

If $\lambda = 0$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 + R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 2 \,;\ n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$2x - y - z = 0 \,;\ 3y - 3z = 0 \,;\ z = k$

$2x - k - k = 0 \qquad 3y - 3k = 0$

$x = k \qquad 3y = 3k$

$y = k$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

Case (ii)

If $\lambda = 3 \,;\ (A - \lambda I)x = 0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 1 \,;\ n = 3$

$n - r = 3 - 1 = 2$ L.I.S

$-x - y - z = 0 \,;\ y = k_1 \,;\ z = k_2$

$-x - k_1 - k_2 = 0$

$x + (k_1 + k_2) = 0$

$x = -(k_1 + k_2)$

$$x_2 = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} \text{ (or) } = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_2$$

DIAGONALISATION

if square matrix of A order n has n linearly indipendent eign vectors then a matrix B can be found such that ~~B~~ matrix $B^{-1}AB$ is a diagonal matrix

$$B = [X_1, X_2, X_3]$$

here

X_1, X_2, X_3 are the eign vectors of the given matrix.

Q. Diagonalise the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Sol^n. The CH matrix of A is

$$A - \lambda I$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

Find eign^n of A

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(2-\lambda)(-1-\lambda)-1\right] - 1\left[-1(-1-\lambda)-0\right] - 2\left[-1-0\right] = 0$

$(1-\lambda)\left[-2-2\lambda+\lambda+\lambda^2-1\right] - 1(1+\lambda)+2=0$

$(1-\lambda)\left[\lambda^2-\lambda-3\right] - 1-\lambda+2=0$

$(1-\lambda)(\lambda^2-\lambda-3) - \lambda+1=0 \Rightarrow (1-\lambda)(\lambda^2-\lambda-3)-\lambda+1$

$\Rightarrow (1-\lambda)(\lambda^2-\lambda-3)+(1-\lambda)$

$\lambda^2-\lambda-3-\lambda^3+\lambda^2+3\lambda-\lambda+1=0$

$(1-\lambda)(\lambda^2-\lambda-3)$

$-\lambda^3+2\lambda^2+\lambda-2=0$

$\lambda^3-2\lambda^2-\lambda+2=0$

$D-[x_1,x_2,x_3]$

$\lambda=1 \Rightarrow 1-2-1+2=0$

$$\begin{array}{r|l} & \lambda^2-\lambda-2 \\ \hline (\lambda-1) & \lambda^3-2\lambda^2-\lambda+2 \\ & \lambda^3-\lambda^2 \\ \hline & -\lambda^2-\lambda+2 \\ & -\lambda^2+\lambda \\ \hline & -2\lambda+2 \\ & -2\lambda+2 \\ \hline & 0 \end{array}$$

$(\lambda-1)(\lambda^2-\lambda-2)=0$

$\lambda-1=0 \quad \Big| \quad \lambda^2-\lambda-2=0$

$(\lambda-2)(\lambda+1)=0$

$\lambda = -1, 1, 2.$

The eigen roots of A are = 1, 1, 2

Case I

if $\lambda = 1$ then $|A - 1I| x = 0$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + y - 2z = 0$ —— (1)

$-x + 3y + z = 0$ —— (2)

$y = 0$ —— (3)

Put $y = 0$ in eqn (1) & eqn (2)

$2x - 2z = 0$ —— (4)

$-x + z = 0$ —— (5)

multiplying eqn (5) with (2) + eqn (4)

$2x - 2z = 0$

$-2x + 2z$

$-x + z = 0$

$x = z$

Let $z = k$

$x = k, \quad z = k$

$y = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Case II

If $\lambda = 1$ then $|A - \lambda I| = 0$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y - 2z = 0 \quad \text{— (1)}$$

$$-x + y + z = 0 \quad \text{— (2)}$$

$$y - 2z = 0 \quad \text{— (3)}$$

Let

$z = k$

$$\boxed{\begin{matrix} y - 2z = 0 \\ y - 2z = 0 \end{matrix}}$$

$$y - 2z = 0$$

$$y - 2k = 0$$

$$+2k = +y$$

$$y = 2k.$$

$$-x + 2k + k = 0$$

$$-x + 3k = 0$$

$$x = 3k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Case-III

If $\lambda = 2$ then $|A - \lambda I| = 0$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + y - 2z = 0$

$-x + z = 0$

$y - 3z = 0$

Let $z = k$

$-x + k = 0$

$+x = +k$

$x = k$

$-k + y - 2k = 0$

$y - 3k = 0$

$y = 3k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Hence eign vectors are

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

[illegible]

[illegible]

$$B = [x_1, x_2, x_3] = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$|B| = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2-3) - 3(0-3) + 1(0-2)$$

$$= -1 + 9 - 2$$

$$= 6 \neq 0$$

Cofactor of $1 = (2-3) = -1$

" " $3 = -(0-3) = 3$

" " $1 = (0-2) = -2$

" " $0 = -(3-1) = -2$

" " $2 = (1-1) = 0$

" " $3 = -(1-3) = 2$

" " $1 = (9-2) = 7$

" " $1 = -(3-0) = -3$

" " $1 = 2-0 = 2$

co-factor matrix of $B = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 2 \\ 7 & -3 & 2 \end{bmatrix}$

$$\operatorname{adj} B = \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \operatorname{adj} B = \frac{1}{6} \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} A B = \frac{1}{6} \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -1+2+0 & -1-4+7 & 2-2-7 \\ 3-0+0 & 3+0-3 & -6+0+3 \\ -2-2+0 & -2+4+2 & 4+2-2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2 & -7 \\ 3 & 0 & -3 \\ -4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1+0-7 & 3+4-7 & 1+6-7 \\ 3+0-3 & 9+0-3 & 3+0-3 \\ -4+0+4 & -12+8+4 & -4+12+4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

12/4/18

Diagonalisation by orthogonal transformation

~~suppose~~ suppose A is a real symmetry matrix then a characteristic matrix of A will not be linearly indipendent and also will be orthogonal. If we normalise each characteristic vector or eign vectors (X) we devide each component of X by the square root of the sum of the squares of all elements. write all normalized eign vectors to form normalized ~~mator~~ model matrix B then it can be easily shown that B is an orthogonal matrix and :

B^{-1} equal to B transpose.

therefore the symeterity transform

$B^{-1}AB = D$

Where D is the diagonal matrix.

this transformation B transpose AB is equal to D. is known as orthogonal transformation.

9- Calculation of powers of a matrix

let, A be the given matrix of ~~AB~~ order n.

we know that

proof

$D = B^{-1}AB$

$D^2 = (B^{-1}AB)(B^{-1}AB)$

$= (B^{-1}A)(BB^{-1})(AB)$

$= (B^{-1}A)(I)(AB)$

$D^2 = B^{-1}A^2B$

$D^3 = B^{-1}A^3B$

$\vdots$

$D^n = B^{-1}A^nB$

$(BD^nB^{-1}) = B(B^{-1}A^nB)B^{-1}$

$= A^n$

$A^n = (BD^nB^{-1})$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}$$

Q: $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by O.T.

Sol^n. The ch. matrix of A is

$A - \lambda I$

$$x_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$B = [x_1, x_2, x_3]$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix}$$

The ch. eqn of A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)^2 - 4\right] - 1\left[(1-\lambda) - 2\right] - 1\left[-2-(-1+\lambda)\right]$$

$$(2-\lambda)\left[1-2\lambda+\lambda^2-4\right] - 1\left[-\lambda-1\right] - 1\left[-2+1-\lambda\right] = 0$$

$$(2-\lambda)\left(\lambda^2-2\lambda-3\right) + \lambda+1+1+\lambda = 0$$

$$2\lambda^2 - 4\lambda - 6 - \lambda^3 + 2\lambda^2 + 3\lambda + 2\lambda + 2 = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda - 4\lambda + 4 = 0$$

$$\lambda^2(\lambda-1) - 3\lambda(\lambda-1) - 4(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2 - 3\lambda - 4) = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 = 0$$

$$\lambda(\lambda-4) + 1(\lambda-4) = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\boxed{\lambda = -1, 1, 4}$$

$\lambda = -1, 1, 4.$

The ch roots of the eq^n is -1, 1, 4.

case ①

if $\lambda = -1$, then

$$A - \lambda I = 0$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + y - z = 0$$

$$5y - 5z = 0$$

$$\rho(A) = 2, \quad n = 3, \quad n - r = 3 - 2 = 1 \quad \underline{L.I.S}$$

let,

$$z = k$$

$$5y - 5k = 0$$

$$5y = 5k$$

$$y = k$$

$$3x + k - k = 0$$

$$x = 0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (2)

If $\lambda = 1$ then

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y - z = 0$

$-y - z = 0$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

Let $z = k_1$

$-y - k_1 = 0$

$-y = k_1$

$y = -k_1$

$x - k_1 - k_1 = 0$

$x - 2k_1 = 0$

$x = 2k_1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2K_1 \\ -K_1 \\ K_1 \end{bmatrix} = K_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case 3 if $\lambda = 1$ then

$$A - \lambda I = 0$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y - z = 0$$

$$-5y - 5z = 0$$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

Let $z = K_1$

$$-5y - 5K_1 = 0$$

$$-5y = 5K_1$$

$$y = -K_1$$

$$-2x - K_1 - K_1 = 0$$

$$-2x - 2K_1 = 0$$

$$-2x = 2K_1$$

$$x = -K_1$$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Step: vectors

$B = [X_1\ X_2\ X_3]$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

We observed that eigen vectors are pair wise orthogonal.

$$B = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$B^{-1} = B^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} & 0+\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}} & 0-\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}} \\ \frac{4}{\sqrt{6}}-\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}-\frac{1}{\sqrt{6}}-\frac{2}{\sqrt{6}} & \frac{-2}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{3}}-\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$\theta \cdot m\varrho$

$$\begin{bmatrix} 0-\frac{1}{2}-\frac{1}{2} & 0+\frac{1}{\sqrt2}-\frac{1}{\sqrt2} & 0-\frac{1}{\sqrt6}+\frac{1}{\sqrt6} \\ 0-\frac{1}{\sqrt2}+\frac{1}{\sqrt2} & \frac{4}{6}+\frac{1}{6}+\frac{1}{6} & \frac{2}{\sqrt{18}}-\frac{1}{\sqrt{18}}-\frac{1}{\sqrt{18}} \\ 0+\frac{4}{\sqrt6}-\frac{4}{\sqrt6} & \frac{8}{\sqrt{18}}-\frac{4}{\sqrt{18}}-\frac{4}{\sqrt{18}} & \frac{4}{3}+\frac{4}{3}+\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D(-1, 1, 4)$$

Notes on singular value decomposition for Math 54

Recall that if A is a symmetric $n \times n$ matrix, then A has real eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated), and $\mathbb{R}^n$ has an orthonormal basis $v_1, \ldots, v_n$, where each vector v_i is an eigenvector of A with eigenvalue λ_i. Then

$$A = PDP^{-1}$$

where P is the matrix whose columns are $v_1, \ldots, v_n$, and D is the diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$. Since the vectors $v_1, \ldots, v_n$ are orthonormal, the matrix P is orthogonal, i.e. $P^T P = I$, so we can alternately write the above equation as

$$A = PDP^T. \tag{1}$$

A singular value decomposition (SVD) is a generalization of this where A is an $m \times n$ matrix which does not have to be symmetric or even square.

1 Singular values

Let A be an $m \times n$ matrix. Before explaining what a singular value decomposition is, we first need to define the singular values of A.

Consider the matrix $A^T A$. This is a symmetric $n \times n$ matrix, so its eigenvalues are real.

Lemma 1.1. *If λ is an eigenvalue of $A^T A$, then $\lambda \geq 0$.*

Proof. Let x be an eigenvector of $A^T A$ with eigenvalue λ. We compute that

$$\|Ax\|^2 = (Ax) \cdot (Ax) = (Ax)^T Ax = x^T A^T Ax = x^T (\lambda x) = \lambda x^T x = \lambda \|x\|^2.$$

Since $\|Ax\|^2 \geq 0$, it follows from the above equation that $\lambda \|x\|^2 \geq 0$. Since $\|x\|^2 > 0$ (as our convention is that eigenvectors are nonzero), we deduce that $\lambda \geq 0$. □

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A^T A$, with repetitions. Order these so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $\sigma_i = \sqrt{\lambda_i}$, so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Definition 1.2. The numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ defined above are called the **singular values** of A.

Proposition 1.3. *The number of nonzero singular values of A equals the rank of A.*

Proof. The rank of any square matrix equals the number of nonzero eigenvalues (with repetitions), so the number of nonzero singular values of A equals the rank of A^TA. By a previous homework problem, A^TA and A have the same kernel. It then follows from the "rank-nullity" theorem that A^TA and A have the same rank. □

Remark 1.4. In particular, if A is an $m \times n$ matrix with $m < n$, then A has at most m nonzero singular values, because $\operatorname{rank}(A) \leq m$.

The singular values of A have the following geometric significance.

Proposition 1.5. *Let A be an $m \times n$ matrix. Then the maximum value of $\|Ax\|$, where x ranges over unit vectors in $\mathbf{R}^n$, is the largest singular value σ_1, and this is achieved when x is an eigenvector of A^TA with eigenvalue σ_1^2.*

Proof. Let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors of A^TA with eigenvalues σ_i^2. If $x \in \mathbf{R}^n$, then we can expand x in this basis as

$$x = c_1v_1 + \cdots + c_nv_n, \tag{2}$$

for scalars $c_1, \ldots, c_n$. Since x is a unit vector, $\|x\|^2 = 1$, which (since the vectors $v_1, \ldots, v_n$ are orthonormal) means that

$$c_1^2 + \cdots + c_n^2 = 1.$$

On the other hand,

$$\|Ax\|^2 = (Ax) \cdot (Ax) = (Ax)^T(Ax) = x^TA^TAx = x \cdot (A^TAx).$$

By (2), since v_i is an eigenvalue of A^TA with eigenvalue σ_i^2, we have

$$A^TAx = c_1\sigma_1^2v_1 + \cdots + c_n\sigma_n^2v_n.$$

Taking the dot product with (2), and using the fact that the vectors $v_1, \ldots, v_n$ are orthonormal, we get

$$\|Ax\|^2 = x \cdot (A^TAx) = \sigma_1^2c_1^2 + \cdots + \sigma_n^2c_n^2.$$

Since σ_1 is the largest singular value, we get

$$\|Ax\|^2 \leq \sigma_1^2(c_1^2 + \cdots + c_n^2).$$

Equality holds when $c_1 = 1$ and $c_2 = \cdots = c_n = 0$. Thus the maximum value of $\|Ax\|^2$ for a unit vector x is σ_1^2, which is achieved when $x = v_1$. □

One can similarly show that σ_2 is the maximum of $\|Ax\|$ where x ranges over unit vectors that are orthogonal to v_1 (exercise). Likewise, σ_3 is the maximum of $\|Ax\|$ where x ranges over unit vectors that are orthogonal to v_1 and v_2; and so forth.

2 Definition of singular value decomposition

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let r denote the number of nonzero singular values of A, or equivalently the rank of A.

Definition 2.1. A singular value decomposition of A is a factorization

$$A = U\Sigma V^T$$

where:

- U is an $m \times m$ orthogonal matrix.
- V is an $n \times n$ orthogonal matrix.
- Σ is an $m \times n$ matrix whose i^{th} diagonal entry equals the i^{th} singular value σ_i for $i = 1, \ldots, r$. All other entries of Σ are zero.

Example 2.2. If $m = n$ and A is symmetric, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. The singular values of A are given by $\sigma_i = |\lambda_i|$ (exercise). Let $v_1, \ldots, v_n$ be orthonormal eigenvectors of A with $Av_i = \lambda_i v_i$. We can then take V to be the matrix whose columns are $v_1, \ldots, v_n$. (This is the matrix P in equation (1).) The matrix Σ is the diagonal matrix with diagonal entries $|\lambda_1|, \ldots, |\lambda_n|$. (This is almost the same as the matrix D in equation (1), except for the absolute value signs.) Then U must be the matrix whose columns are $\pm v_1, \ldots, \pm v_n$, where the sign next to v_i is $+$ when $\lambda_i \geq 0$, and $-$ when $\lambda_i < 0$. (This is almost the same as P, except we have changed the signs of some of the columns.)

3 How to find a SVD

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, and let r denote the number of nonzero singular values. We now explain how to find a SVD of A.

Let $v_1, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$, where v_i is an eigenvector of $A^T A$ with eigenvalue σ_i^2.

Lemma 3.1. *(a)* $\|Av_i\| = \sigma_i$.

(b) If $i \neq j$ then Av_i and Av_j are orthogonal.

Proof. We compute

$$(Av_i)\cdot(Av_j) = (Av_i)^T(Av_j) = v_i^T A^T A v_j = v_i^T \sigma_j^2 v_j = \sigma_j^2 (v_i \cdot v_j).$$

If $i = j$, then since $\|v_i\| = 1$, this calculation tells us that $\|Av_i\|^2 = \sigma_j^2$, which proves (a). If $i \neq j$, then since $v_i \cdot v_j = 0$, this calculation shows that $(Av_i)\cdot(Av_j) = 0$. □

Theorem 3.2. *Let A be an $m \times n$ matrix. Then A has a (not unique) singular value decomposition $A = U\Sigma V^T$, where U and V are as follows:*

- *The columns of V are orthonormal eigenvectors $v_1, \ldots, v_n$ of $A^T A$, where $A^T A v_i = \sigma_i^2 v_i$.*
- *If $i \leq r$, so that $\sigma_i \neq 0$, then the i^{th} column of U is $\sigma_i^{-1} A v_i$. By Lemma 3.1, these columns are orthonormal, and the remaining columns of U are obtained by arbitrarily extending to an orthonormal basis for $\mathbb{R}^m$.*

Proof. We just have to check that if U and V are defined as above, then $A = U\Sigma V^T$. If $x \in \mathbb{R}^n$, then the components of $V^T x$ are the dot products of the rows of V^T with x, so

$$V^T x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{pmatrix}.$$

Then

$$\Sigma V^T x = \begin{pmatrix} \sigma_1 v_1 \cdot x \\ \sigma_2 v_2 \cdot x \\ \vdots \\ \sigma_r v_r \cdot x \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

When we multiply on the left by U, we get the sum of the columns of U, weighted by the components of the above vector, so that

$$U\Sigma V^T x = (\sigma_1 v_1 \cdot x)\sigma_1^{-1} A v_1 + \cdots + (\sigma_r v_r \cdot x)\sigma_r^{-1} A v_r$$
$$= [illegible]$$

Since $Av_i = 0$ for $i > r$ by Lemma 3.1(a), we can rewrite the above as

$$\begin{aligned} U\Sigma V^T x &= (v_1 \cdot x)Av_1 + \cdots + (v_n \cdot x)Av_n \\ &= Av_1 v_1^T x + \cdots + Av_n v_n^T x \\ &= A(v_1 v_1^T + \cdots v_n v_n^T)x \\ &= Ax. \end{aligned}$$

In the last line, we have used the fact that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbf{R}^n$, then $v_1 v_1^T + \cdots + v_n v_n^T = I$ (exercise). $\square$

Example 3.3. (from Lay's book) *Find a singular value decomposition of*

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}.$$

Step 1. We first need to find the eigenvalues of $A^T A$. We compute that

$$A^T A = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}.$$

We know that at least one of the eigenvalues is 0, because this matrix can have rank at most 2. In fact, we can compute that the eigenvalues are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Thus the singular values of A are $\sigma_1 = \sqrt{360} = 6\sqrt{10}$, $\sigma_2 = \sqrt{90} = 3\sqrt{10}$, and $\sigma_3 = 0$. The matrix Σ in a singular value decomposition of A has to be a 2×3 matrix, so it must be

$$\Sigma = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}.$$

Step 2. To find a matrix V that we can use, we need to solve for an orthonormal basis of eigenvectors of $A^T A$. One possibility is

$$v_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}.$$

(There are seven other possibilities in which some of the above vectors are multiplied by -1.) Then V is the matrix with v_1, v_2, v_3 as columns, that is,

$$V = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}.$$

Step 3. We now find the matrix U. The first column of U is

$$\sigma_1^{-1}Av_1 = \frac{1}{6\sqrt{10}}\begin{pmatrix}18\\6\end{pmatrix} = \begin{pmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{pmatrix}.$$

The second column of U is

$$\sigma_2^{-1}Av_2 = \frac{1}{3\sqrt{10}}\begin{pmatrix}3\\9\end{pmatrix} = \begin{pmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{pmatrix}.$$

Since U is a 2×2 matrix, we do not need any more columns. (If A had only one nonzero singular value, then we would need to add another column to U to make it an orthogonal matrix.) Thus

$$U = \begin{pmatrix}3/\sqrt{10} & 1/\sqrt{10}\\1/\sqrt{10} & -3/\sqrt{10}\end{pmatrix}.$$

To conclude, we have found the singular value decomposition

$$\begin{pmatrix}4 & 11 & 14\\8 & 7 & -2\end{pmatrix} = \begin{pmatrix}3/\sqrt{10} & 1/\sqrt{10}\\1/\sqrt{10} & -3/\sqrt{10}\end{pmatrix}\begin{pmatrix}6\sqrt{10} & 0 & 0\\0 & 3\sqrt{10} & 0\end{pmatrix}\begin{pmatrix}1/3 & -2/3 & 2/3\\2/3 & -1/3 & -2/3\\2/3 & 2/3 & 1/3\end{pmatrix}^T.$$

4 Applications

Singular values and singular value decompositions are important in analyzing data.

One simple example of this is "rank estimation". Suppose that we have n data points $v_1, \ldots, v_n$, all of which live in $\mathbb{R}^m$, where n is much larger than m. Let A be the $m\times n$ matrix with columns $v_1, \ldots, v_n$. Suppose the data points satisfy some linear relations, so that $v_1, \ldots, v_n$ all lie in an r-dimensional subspace of $\mathbb{R}^m$. Then we would expect the matrix A to have rank r. However if the data points are obtained from measurements with errors, then the matrix A will probably have full rank m. But only r of the singular values of A will be large, and the other singular values will be close to zero. Thus one can compute an "approximate rank" of A by counting the number of singular values which are much larger than the others, and one expects the measured matrix A to be close to a matrix A' such that the rank of A' is the "approximate rank" of A.

For example, consider the matrix

$$A = \begin{pmatrix}[illegible] & [illegible] & [illegible] & [illegible]\\ [illegible] & [illegible] & [illegible] & [illegible]\\ [illegible] & [illegible] & [illegible] & [illegible]\end{pmatrix}$$

The matrix A' has rank 2, because all of its columns are points in the subspace $x_1 + x_2 + x_3 = 0$ (but the columns do not all lie in a 1-dimensional subspace). Now suppose we perturb A' to the matrix

$$A = \begin{pmatrix} 1.01 & 2.01 & -2 & 2.99 \\ -4.01 & 0.01 & 1.01 & 2.02 \\ 3.01 & -1.99 & 1 & -4.98 \end{pmatrix}$$

This matrix now has rank 3. But the eigenvalues of $A^T A$ are

$$\sigma_1^2 \approx 58.604, \quad \sigma_2^2 \approx 19.3973, \quad \sigma_3^2 \approx 0.00029, \quad \sigma_4^2 = 0.$$

Since two of the singular values are much larger than the others, this suggests that A is close to a rank 2 matrix.

For more discussion of how SVD is used to analyze data, see e.g. Lay's book.

5 Exercises (some from Lay's book)

1. (a) Find a singular value decomposition of the matrix $A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find a unit vector x for which $\|Ax\|$ is maximized.

2. Find a singular value decomposition of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

3. (a) Show that if A is an $n \times n$ symmetric matrix, then the singular values of A are the absolute values of the eigenvalues of A.

   (b) Give an example to show that if A is a 2×2 matrix which is not symmetric, then the singular values of A might not equal the absolute values of the eigenvalues of A.

4. Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let v_1 be an eigenvector of $A^T A$ with eigenvalue σ_1^2. Show that σ_2 is the maximum value of $\|Ax\|$ where x ranges over unit vectors in $\mathbb{R}^n$ that are orthogonal to v_1.

5. Show that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then

$$v_1 v_1^T + \cdots + v_n v_n^T = I.$$

6. Let A be an $m \times n$ matrix, and let P be an orthogonal $m \times m$ matrix. Show that PA has the same singular values as A.