# UNIT-1 : Solving Linear system of Equations and Eigen values and Eigen vectors

## Rank of a Matrix :

→ If 'A' is null matrix (or) zero matrix then rank of null matrix is zero

→ If A is a Non-null matrix, we say that 'r' is the rank of given matrix. If it satisfies the following conditions

i. Every $(r+1)^{th}$ order minor of 'A' is zero and

ii. Then there exist [ ∃ ] atleast one $r^{th}$ order minor of 'A' is non-zero. Rank of 'A' is denoted by $\rho(A)$

Note

i. The rank of null-matrix is zero

ii. The rank of a unit matrix half order 'n' is n

iii. If A is non-singular matrix of order 'n' the rank of matrix is n

iv. If A is a singular matrix of order 'n' then the rank of matrix is $< n$

v. The rank of a non-zero row matrix is 1.

vi. The rank of A and $A^T$ are same.

1\. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{3\times 3}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 1[6-1] - 2[4-3] + 3[2-9]$

$= 5 - 2 - 21$

$= -18 \neq 0$

Hence $\rho(A) = 3$

2\. Find the rank of the matrix $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix}$

Given that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 13 & 10 \end{bmatrix}_{3\times 3}$

Now, $|A| = 1(30-26) - 5(0-4) + 4(0-6)$

$= 4 + 20 - 24$

$= 24 - 24$

$= 0$

Hence $\rho(A) = 0$

the rank of A is less than order of matrix

Now the minor of 2x2 matrix is

$|A| = \begin{vmatrix} 3 & 2 \\ 13 & 10 \end{vmatrix}$

$= 30 - 26$

$= 4 \neq 0$

Hence $\rho(A) = 2$

To find the rank of a matrix using elementary transformations (or) operations, we use the following methods:

i. Echelon form
ii. Normal form
iii. PAQ form

Elementary transformations (or) operations:

i. Interchanging of two rows and columns:

If $i^{th}$ row and $j^{th}$ row, $i^{th}$ column and $j^{th}$ column are interchanged. It is denoted by $R_{ij} \longleftrightarrow R_{ji}$ (or) $R_j \leftrightarrow R_i$
$C_i \longleftrightarrow C_j$ (or) $C_j \leftrightarrow C_i$

ii. Multiplying each element of a row with a non-zero scalar:

If $i^{th}$ row is multiplied with a scalar $k$ then it is denoted by $R_i \longleftrightarrow kR_i$ and adding to the corresponding elements of another row. It is denoted by $R_i \rightarrow kR_i + R_j$

iii. The corresponding column operations will be denoted by $C$ instead of $R$. i.e $C_i \longleftrightarrow C_j$, $C_j \longleftrightarrow C_i$
$C_{i} \longrightarrow kC_i + C_j$

iv. The elementary transformations (or) operations is called a row operation (or) a column operation according as it applied to row (or) column.

1. Echelon form:

Working rule:

i. Take the given matrix, let it be 'A'
ii. In this method, we have to apply only row operation
iii. If the value of first element in any row of A matrix is 1 then that row should be changes into

first row of the matrix.

iv. The no of zeros before the 1st non-zero element in a row are increased from top to bottom of the matrix

v. Zero rows if any, must occupy the last row

vi. The no of non-zero rows in the echelon form is called the rank of the matrix.

1. Find the rank of the following matrices using echelon form.

i. $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times4}$ ii. $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}_{4\times5}$

iii. $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}_{3\times4}$ iv. $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}_{3\times4}$

v. $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4}$ vi. $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

i. $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$[A] \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ [illegible] & [illegible] & [illegible] & [illegible] \\ 1 & 5 & 4 & 3 \end{bmatrix}$ [illegible]

$$[A] \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is Echelon form

Hence $\rho(A)$ = the no. of non-zero rows $= 2$

P. $$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix} = A$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + R_1$

$R_4 \to R_4 + 3R_1$

$$[A] \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$[A] \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is Echelon form

Hence $\rho(A)$ = the no. of non-zero rows

$= 2$

iii) G.T $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 8R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -5 & 11 & -4 \\ 0 & -15 & 41 & -33 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$[A] \sim \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -5 & 11 & -4 \\ 0 & 0 & 8 & -21 \end{bmatrix}$$

which is echelon form

Hence $\rho(A)$ = the no of non-zero rows = 3

iv) G.T $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$

$$[A] \sim \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

$R_3 \to 5R_3 - 8R_2$

$$[A] \sim \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 12 & 21 \end{bmatrix}$$

which is echelon form

Hence $\rho(A)$ = the no. of nonzero rows = 3

2. G.T. $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is echelon form

Hence $\rho(A)$ = the no of non-zero rows = 3

vi) $$A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 2 & -4 & 3 & -1 & 0 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 4R_1$

$$[A] \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & 9 & 12 & -3 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$[A] \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 9 & 9 & -4 \end{bmatrix}$$

$R_4 \to 5R_4 - 9R_3$

$$[A] \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 0 & -18 & +16 \end{bmatrix}$$

which is echelon form

$\therefore$ Rank of $A = \rho(A) =$ No. of non-zero rows $= 4$

2. Normal form

Working rule:

i. Take the given matrix, let it be A

ii. In this method, we have to apply both row and column operations

iii. By applying these operations, the matrix A can be reduced into any one of the following normal forms i.e. $N = I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, [I_r, 0]$ where

$I_r$ stands for identity matrix.

iv. The rank of the matrix is the order of the identi matrix

1. Find the rank of the following matrices using normal form, where

i. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ ii. $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

iii. $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix}$ iv. $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

v. $A = \begin{bmatrix} 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 3 \\ 3 & 1 & 4 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ vi. $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

Given that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}_{3\times4}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 4C_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$C_3 \to 3C_3 - 2C_2$

$C_4 \to 3C_4 - 5C_2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -36 \end{bmatrix}$$

$C_2 \to C_2/-3, \quad C_4 \to C_4/-36$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\Rightarrow [A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[A] \sim [I_3 \;\; 0]$$

Hence $\int(A) = r = 3$

ii) Given that $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$[A] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$

$C_4 \to 4C_4 + 3C_2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2/-4, \; C_3 \to C_3/-3$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 8C_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A] \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = r = 3$

Q. GT $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_4$

$$[A] \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 2 & -2 & 0 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 1$

$R_4 \to R_4 - 2R_1$

$$[A] \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 10 & -4 & -10 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$$R_3 \to 10R_3 - R_2$$
$$R_4 \to 5R_4 - R_2$$

$$[A] \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 10 & -4 & -10 \\ 0 & 0 & -6 & 10 \\ 0 & 0 & -6 & 10 \end{bmatrix}$$

$$R_4 \to R_4 - R_3$$

$$[A] \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 10 & -4 & -10 \\ 0 & 0 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to C_2 + 2C_1$$
$$C_3 \to C_3 - C_1$$
$$C_4 \to C_4 - 3C_1$$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & -4 & -10 \\ 0 & 0 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to C_2/10$$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 + 4C_2$

$C_4 \rightarrow C_4 + 10C_2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3/-6$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 + 10C_3$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A] \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = r = 3$

iv. G.T $A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 5 & 1 & 9 & 7 \end{bmatrix}_{3\times 4}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 5R_1$

$A = \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -5 & 11 & -4 \\ 0 & -15 & 41 & -33 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$

$[A] \sim \begin{bmatrix} 1 & 2 & -4 & 5 \\ 0 & -15 & 11 & -4 \\ 0 & 0 & 8 & -21 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 + 4C_1$

$C_4 \to C_4 - 5C_1$

$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 11 & -4 \\ 0 & 0 & 8 & -21 \end{bmatrix}$

$C_4 \to 5C_4 - 4C_2$

$C_3 \to 5C_3 + 11C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \underset{110}{\cancel{88}} & -105 \end{bmatrix}$

$C_2 \to C_2/-5$

$C_3 \to C_3/35$

$C_4 \to C_4/-105$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$[C_4 \to C_4 + C_3]$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

Rank of $A = \rho(A) = r = 3$

3) GT $A = \begin{bmatrix} 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 3 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 2 & 2 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$, $R_4 \leftrightarrow R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 2R_1$

$$[A] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to 2R_4 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$[A] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - 2C_1$

$C_4 \to C_4 - 2C_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_3 \to 2C_3 - 3C_2$

$C_4 \to 2C_4 - 5C_2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$C_2 \to C_2/-2,\ C_3 \to C_3/2,\ C_4 \to C_4/10$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$C_4 \to C_4/2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [I_4]$$

Rank of $A = \rho A = r = 4$

vi) G.T $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

[illegible]

$$[A] \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_2 \to c_2 - 3c_1,\ c_3 \to c_3 - 6c_1,\ c_4 \to c_4 + c_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} c_3 \to c_3 + c_2 \\ c_4 - 2c_2 \end{matrix}$$

$$[A] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of $A = \rho(A) = r = 2$

Normal form through PAQ form

Working rule:

→ Take the given matrix, let it be $A$

→ In this method, we have to apply both row and column operations.

→ Now, we write $A$ as $A_{m \times n} = I_m A I_n$,
where $I_m$ is prefactor and $I_n$ is post factor and $I$ is identy matrix

→ Reduce the matrix $A$ of L.H.S to anyone of the following normal forms.

i.e $N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \end{bmatrix}$

using elimentary row and column operations

→ We apply the elimentary row operation to L.H.S and simultaneously we apply the prefactor

$I_n$ of R.H.S

→ we apply the column operation to A of L.H.S and simultaneously we apply the post factor $I_n$ of R.H.S.

→ The rank of the matrix is the order of identity matrix.

1. Find two non-singular matrices P and Q such that PAQ is in normal form and hence find the rank of A.

i. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$  ii. $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

iii. $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$  iv. $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

v. $A = \begin{bmatrix} 5 & -3 & 1 & [illegible] \\ 3 & 1 & 4 & [illegible] \\ 1 & -4 & 11 & [illegible] \end{bmatrix}$  vi. $A = \begin{bmatrix} 1 & [illegible] & -3 & -2 \\ 2 & [illegible] & [illegible] & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

i. G.T $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{3\times 3}$

Now we write A as $A_{m\times n} = I_m A I_n$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

Hence $\rho(A) = r = 2$, $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$

Now we write A as $A_{3\times 4} = I_3 \, A \, I_4$

i.e. $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_2 \to C_2 - 3C_1$

$C_3 \to C_2 - 6C_2$

$C_4 \to C_4 + C_1$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & [illegible] & [illegible] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_3 \to C_3 + C_1$
$C_4 \to C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -7 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

Here $\rho(A) = r = 2$, $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & -3 & -7 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(iii) Given that $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}_{4 \times 4}$

Now, we write $A$ as $A_{4\times4} = I_4 A I_4$

i.e. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 2 & -2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 10R_3 - R_2$

$R_4 \to 5R_4 - R_2$

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 0 & -6 & 16 \\ 0 & 0 & -1 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & -1 & 0 & -6 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + 2C_1$ , $C_3 \to C_3 - C_1$ , $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & -4 & -6 \\ 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2/10$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1/5 & -1 & -2 \\ 0 & 1/10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + 4C_2$

$C_4 \to C_4 + 6C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1/5 & -1/5 & -4/5 \\ 0 & 1/10 & 2/5 & 3/5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3/-6$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1/5 & 1/30 & -4/5 \\ 0 & 1/10 & -1/15 & 3/5 \\ 0 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 16C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1/5 & 1/30 & -4/3 \\ 0 & 1/10 & -1/15 & 5/3 \\ 0 & 0 & -1/6 & 8/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

Hence $\rho(A) = r = 3$, $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 10 & -6 \\ 5 & 0 & -10 & 0 \end{bmatrix}$

$$Q = \begin{bmatrix} 1 & 1/5 & 1/30 & -4/3 \\ 0 & 1/10 & -1/15 & 5/3 \\ 0 & 0 & -1/6 & 4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iv) Given that $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{3\times 4}$

Now we write

$$A_{3\times 4} = I_3 A I_4$$

$$\begin{bmatrix} 2 & 1 & -3 & 6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ , $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 5R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -5 & -1 & -2 \\ 0 & 0 & -24 & 12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$ , $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -5 & -1 & -2 \\ 0 & 0 & -24 & 12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 5C_3 - C_2$

$C_4 \to 5C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & 60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$C_4 \to 2C_4 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -20 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_2 \to C_2/-5$

$C_3 \to C_3/-120$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & 1/5 & 4/120 & -20 \\ 0 & -1/5 & 1/120 & -5 \\ 0 & 0 & -5/120 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1/5 & 4/120 & -20 \\ 0 & -1/5 & 1/120 & -5 \\ 0 & 0 & -5/120 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

So, rank of $A = \rho(A) = r = 3$

3. Gt $A = \begin{bmatrix} 3 & -2 & 1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -9 \end{bmatrix}_{3\times 4}$

we have $A_{m\times n} = I_m A I_n$

$$A_{3\times 4} = I_3 A I_4$$

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -4 & 11 & -9 \\ 5 & 1 & 4 & -2 \\ 3 & -2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -4 & 11 & -9 \\ 0 & 21 & -51 & 43 \\ 0 & 10 & -32 & 32 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 21R_3 - 10R_2$

$$\begin{bmatrix} 1 & -4 & 11 & -9 \\ 0 & 21 & -51 & 43 \\ 0 & 0 & -162 & 242 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 21 & -10 & -13 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now $C_2 \to C_2 + 4C_1$

$C_3 \to C_3 - 11C_1$

$C_4 \to C_4 + 9C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 21 & -51 & 43 \\ 0 & 0 & -162 & 242 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 21 & -10 & -13 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -11 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 21C_3 + 51C_2$

$C_4 \to 21C_4 - 43C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 \\ 0 & 0 & -3402 & 5082 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 21 & -10 & -13 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -27 & 17 \\ 0 & 1 & 51 & -43 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$C_3 \to \dfrac{C_3}{-3402}$ , $C_4 \to \dfrac{C_4}{5082}$

$C_2 \to \dfrac{C_2}{21}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -5 \\ 21 & -10 & -15 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & 27/3042 & 17/5082 \\ 0 & 1/21 & -51/3042 & -45/5082 \\ 0 & 0 & -21/3042 & 0 \\ 0 & 0 & 0 & 21/5082 \end{bmatrix}$$

$c_4 \to c_4 - c_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -5 \\ 21 & -10 & -15 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & 27/3042 & -55/10000 \\ 0 & 1/21 & -51/3042 & 65/10000 \\ 0 & 0 & -21/3042 & 21/3042 \\ 0 & 0 & 0 & 21/5082 \end{bmatrix}$$

$$[I_3 \; 0] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -5 \\ 21 & -10 & -15 \end{bmatrix} A \begin{bmatrix} 1 & 4/21 & 27/3042 & -55/10000 \\ 0 & 1/21 & -51/3042 & 65/10000 \\ 0 & 0 & -21/3042 & 21/3042 \\ 0 & 0 & 0 & 21/5082 \end{bmatrix}$$

$[I_3 \; 0] = PAQ$

Now Rank of $A = \rho(A) = 3$

where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -5 \\ 21 & -10 & -15 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 4/21 & 27/3042 & 11/2000 \\ 0 & 1/21 & -51/3042 & 13/2000 \\ 0 & 0 & -21/3042 & 21/3042 \\ 0 & 0 & 0 & 21/5082 \end{bmatrix}$$

vi) Given that $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}_{3\times4}$

$A_{3\times 4} = I_3 \, A \, I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 + 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \text{[illegible]} & & \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 6C_3 - 5C_2$

$C_4 \to 6C_4 - 7C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -8 & 26 \\ 0 & 1 & -5 & -7 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$C_2 \to C_2/-6$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -8 & 26 \\ 0 & -1/6 & -5 & -7 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1/3 & -8 & 26 \\ 0 & -1/6 & -5 & -7 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Inverse of a Matrix using elementary row operations or Gauss Jordan Method

Working rule:-

1. Take the given Matrix, let it be $A$
2. In this method, we have to apply only row operations
3. Now, we write $A$ as $A_{m \times n} = I_n A$, where $I_n$ is prefactor and $I$ is identity matrix of order $n$
4. Now we apply the elementary row operation on $A$ of L.H.S and simultaneously we apply the prefactor $I_n$ of R.H.S. We will do this till we get a matrix is of the form $I_n = B \cdot A$. Then $B$ is called the inverse of $A$

1. Find the Inverse of the following matrices using elementary row operations (or) Gauss Jordan method

i) $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

i) G.T $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{3\times 3}$

Now, we write A as $A_{3\times3} = I_3 A$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1 - R_3$

$R_2 \to 3R_2 - 2R_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix} A$$

$R_1 \to R_1/3$

$R_2 \to R_2/3$

$R_3 \to R_3/3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & 1/3 & 1/3 \end{bmatrix} A$$

$$[I_3] = BA$$

where $B = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & 1/3 & 1/3 \end{bmatrix}$

Hence $B$ is called inverse of given matrix

3. $$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}_{4\times4} = A$$

Now, we write that, $A_{n\times n} = I_n A$

$A_{4\times4} = I_4 A$

$$\begin{bmatrix} 1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & -3 & 3 & -1 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & -1 \\ 0 & -7 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} A$$

$R_3 \to 2R_3 - 7R_2$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & -1 \\ 0 & 0 & -6 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_4 \to 6R_4 + R_3$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to R_2 + R_4$

$R_3 \to R_3 - R_4$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 2 & 2 & 6 \\ -6 & -12 & 0 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to 6R_1 - R_3$

$R_2 \to 3R_2 + R_3$

$$\begin{bmatrix} 6 & 6 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 18 & 0 & 6 \\ -6 & -6 & 6 & 12 \\ -6 & -12 & 0 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 6 & 18 \\ -6 & -6 & 6 & 12 \\ -6 & -12 & 0 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to R_1/6,\ R_2 \to R_2/-6,\ R_3 \to R_3/-6$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & -6 \end{bmatrix} A$$

$$\therefore [I_4] = BA$$

where $B = \begin{bmatrix} 8 & 2 & 1 & 8 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

Hence $B$ is called Inverse of the given matrix

(ii) Given that $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}_{3\times 3}$

Now we write $A_{3\times3} = I_3 A$

where $A_{3\times 3} = I_3 A$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} A$$

$R_3 \to R_3/8$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1/8 & 5/8 & 1/4 \end{bmatrix} A$$

$R_2 \to R_2/-1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & 0 \\ 1/8 & 5/8 & 1/4 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & 1/2 \\ 0 & -1 & 0 \\ 1/8 & 5/8 & 1/4 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1/2 \\ 1/8 & -3/8 & 1/4 \\ 1/8 & 5/8 & 1/4 \end{bmatrix} A$$

$[I_3] = BA$

where $B = \begin{bmatrix} -1/4 & 3/4 & 1/2 \\ 1/8 & -3/8 & 1/4 \\ 1/8 & 5/8 & 1/4 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

We can write $A_{3\times3} = I_3 A$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2/2$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3/3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$[I_3] = BA$$

where $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$

v. G.T $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}_{3\times 3}$

we can write $A_{3\times3} = I_3 A$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A$$

$R_3 \to R_3 / -4$

$R_2 \to R_2 / 2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

$R_1 \to R_1 - 6R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

$R_2 \to R_2 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

$$[I_3] = BA$$

where $B = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$

Hence B is called inverse of the given matrix

# Linear system of equations

The General form of m linear equation in n variable is given by :-

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$

The above system can be written into matrix form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$AX = B$$

Here, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

= coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ = variable matrix and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$ = constant matrix

There are two types of system of linear equations. They are i. Homogeneous system of equations.

ii. Non-homogeneous system of equations.

i. Homogeneous system of equations

If all the constants in the given system are zero's then the system is called homogeneous system of equations

Ex: $x_1 + x_2 - x_3 = 0$

$2x_1 + 2x_2 - x_3 = 0$

$3x_1 - x_2 + 2x_3 = 0$

ii. Non-homogeneous system of equations

If all the constants in the given system are nonzero [(i.e $b_n \neq 0$), atleast one $b_n$ is non-zero] then the system is called a non-homogeneous system of equations.

Ex: $2x_1 + x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = -1$

$3x_1 + 3x_2 - 2x_3 = 0$

Augmented Matrix

The matrix obtained by the coefficient matrix together with the constant matrix is called the Augmented matrix.

consistent:

A system is said to be consistent, if it has atleast one solution.

Inconsistent:

A system is said to be inconsistent, if it has no solution.

Working rule for finding consistent and inconsistent

1. Take the given linear system of equations

2. The given linear system of equations can be put into matrix form i.e $AX = B$

3. consider the augmented matrix and reduce it to echelon form by using elementary row operations

i. if $\rho(A) = \rho(AB) = n$, then the given system is consistent and it has a unique solution

ii. if $\rho(A) = \rho(AB) < n$, then the given system is consistent and it has an infinite no. of solutions.

iii. if $\rho(A) \neq \rho(AB)$, then the given system is inconsistent and it has no solutions.

1. Find whether the following equations are consistent if so, solve them

i. $x + y + 2z = 4$,

$2x - y + 3z = 9$,

$3x - y - z = 2$

1 Given equations are $x+y+2z=4$

$2x-y+3z=9$

$3x-y-z=2$

The above equations can be put into matrix form is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

consider augmented matrix $[AB]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$R_3 \to 3R_3 - 4R_2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

which is echelon form

Here $\rho(A)=3$, $\rho(AB)=3=n$

clearly $\rho(A)=\rho(AB)=n=3$, then the given system is consistent and it has a unique solution

By using back substitution, we have

$$-17z = -34$$
$$z = 2$$

$$-3y - z = 1$$
$$-3y = 3$$
$$y = -1$$

$$x + y + 2z = 4$$
$$x = 4 - y - 2z$$
$$x = 4 + 1 - 4$$
$$x = 1$$

Hence $x = 1$, $y = -1$, and $z = 2$ are the values of given system of equations.

3) $2x - y - z = 2$
$x + 2y + z = 2$
$4x - 7y - 5z = 2$

Given equations are
$$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

The above equations can be put in matrix form

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

consider Augmented matrix AB

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$[AB] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(AB) = 2 \quad , n = 3$

clearly $\rho(A) = \rho(AB) < n$

$\therefore$ the given system is consistent and it has infinitely no of solutions

By using back substitution, we have -

$-5y - 3z = -2 \to (1)$

$x + 2y + z = 2 \to (2)$

let $z = k$

from (1), $-5y = -2 + 3z$

$y = \dfrac{-2 + 3k}{-5}$

$y = \dfrac{2 - 3k}{5}$

from (2) $x = 2 - 2y - z$

$= 2 - \dfrac{2(2-3k)}{5} - k$

$$x = \frac{10-4+4k-5k}{5}$$

$$x = \frac{6+k}{5}$$

Hence $x = \frac{6+k}{5}$, $y = \frac{2-3k}{5}$ and $z = k$ are the one of the solution of given system of equations.

2. iii) Find the values of $a$ and $b$ for which the equations $x+y+z=3$, $x+2y+2z=6$, $x+3y+az=b$, have

i, no solution

ii, infinitely no of solution

iii, a unique solution

Given equations are
$$x+y+z=3$$
$$x+2y+2z=6$$
$$x+3y+az=b$$

The above equations can be put into matrix form

i.e.
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$$

consider Augmented matrix AB

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & 3 & a & b \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & a-1 & b-3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & a-3 & b-9 \end{bmatrix}$$

which is echelon form

i. if $a=3$ and $b \neq 9$, then the given system is inconsistence and it has no solutions

ii. if $a=3$ and $b=9$, then the given system is consistent and it has an infinitely no of solutions

iii. if $a \neq 3$ and $b$ is any value, then the given system is consistent and it has unique solution

Gauss - Elimination method:

Working rule:

1. Take the given linear system of equations
2. The above linear system of equations can be put into matrix form i.e $AX=B$.
3. consider the augmented matrix $AB$
4. By using elementary row operations, the augmented matrix $AB$ can be reduced in to echelon form
5. By using back substitution method, solve the equations we get the values of given system of equations.

2\. Solve the following equations using Gauss elimination method

i. $x+y+z=8,\ 2x+3y+2z=19,\ 4x+2y+3z=23$.

ii. $3x+y+2z=3,\ 2x-3y-z=-3,\ x+2y+z=4$

iii. $2x_1+x_2+2x_3+x_4=6,\ x_1-x_2+x_3+x_4=6,\ 4x_1+3x_2+3x_3-3x_4=-1,$

$2x_1+2x_2-x_3+x_4=10$

iv. $x+3y+z=9,\ 2x+y+3z=11,\ 3x+y+4z=16$

i. Given equations are
$$x+y+z=8$$
$$2x+3y+2z=19$$
$$4x+2y+3z=23$$

The above equations can be written into matrix form
$$\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 4 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 19\\ 23\end{bmatrix}$$

consider Augmented matrix [AB]
$$[AB]\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 8\\ 2 & 3 & 2 & 19\\ 4 & 2 & 3 & 23\end{array}\right]$$

$R_2 \to R_2-2R_1$
$R_3 \to R_3-4R_1$

$$[AB]\sim\begin{bmatrix}1 & 1 & 1 & 8\\ 0 & 1 & 0 & 3\\ 0 & -2 & -1 & -9\end{bmatrix}$$

$R_3 \to R_3+2R_2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

which is echelon form

By back substitutions, we have

$$-z = -3$$
$$z = 3$$
$$y = 3$$
$$x+y+z = 8$$
$$x = 8-y-z$$
$$x = 8-3-3$$
$$x = 8-6$$
$$x = 2$$

$\therefore$ $x=2$, $y=3$, $z=3$ are the solutions of given system of equations

6) Given equations are $3x+y+2z = 3$
$$2x-3y-z = -3$$
$$x+2y+z = 4$$

The above equations can be written into matrix form

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

consider Augmented matrix $[AB]$

$$[AB] \sim \left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} \sim [AB]$$

$R_3 \to 7R_3 - 5R_2$

$$[AB] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

which is echelon form

By back substitutions, we have

$8z = -8$ $\quad$ $-7y - 3z = -11$ $\quad$ $x + 2y + z = 4$

$z = -1$ $\quad$ $-7y + 3 = -11$ $\quad$ $x + 4 + (-1) = 4$

$-7y = -14$ $\quad$ $x = 4 - 3$

$y = 2$ $\quad$ $x = 1$

Hence $x=1, y=2, z=-1$ are the solutions of given system of linear equations

(ii) Given equations are $2x_1 + x_2 + 2x_3 + x_4 = 6$

$x_1 - x_2 + x_3 + x_4 = 6$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

The above equations can be written into matrix form

$$\begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -1 \\ 10 \end{bmatrix}$$

consider Augmented matrix [AB]

$$[AB] \sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[AB] \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$

$$[AB] \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 6 \\ 0 & 3 & 0 & -1 & -6 \\ 0 & 7 & -1 & -7 & -25 \\ 0 & 4 & -3 & -1 & -2 \end{bmatrix}$$

$R_3 \to 3R_3 - 7R_2$, $R_4 \to 3R_4 - 4R_2$

$$[AB] \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 6 \\ 0 & 3 & 0 & -1 & -6 \\ 0 & 0 & -3 & -14 & -33 \\ 0 & 0 & -9 & 1 & 18 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$[AB] \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 6 \\ 0 & 3 & 0 & -1 & -6 \\ 0 & 0 & -3 & -14 & -33 \\ 0 & 0 & 0 & 43 & 117 \end{bmatrix}$$

which is echelon form

By back substitutions

$$43x_4 = 117$$

$$x_4 = \frac{117}{43}$$

$$-3x_3 - 11x_4 = -33$$

$$3x_3 + 11x_4 = 33$$

$$3x_3 = 33 - 11x_4$$

$$3x_3 = 33 - 11\left[\frac{117}{43}\right]$$

$$3x_3 = 33 - \frac{1287}{43}$$

$$3x_3 = \frac{1419 - 1638}{43}$$

$$3x_3 = \frac{-219}{43}$$

$$x_3 = \frac{-219}{129}$$

$$x_2 - x_4 = -6$$

$$x_2 = -6 + x_4$$

$$x_2 = -6 + \frac{117}{43}$$

$$x_2 = \frac{-258 + 117}{43}$$

$$x_2 = \frac{141}{43}$$

$$x_1 - x_2 + x_3 + x_4 = 6$$

$$x_1 = 6 + x_2 - x_3 - x_4$$

$$x_1 = 6 + \frac{141}{43} + \frac{219}{129} - \frac{117}{43}$$

$$x_1 = 6 + \frac{219}{129} + \frac{141 - 117}{43}$$

$$x_1 = 6 + \frac{219}{129} + \frac{24}{43}$$

$$x_1 = 6 + \frac{219 + 72}{129}$$

$$x_1 = 6 + \frac{291}{129}$$

$$x_1 = \frac{774 + 291}{129} = \frac{1065}{129}$$

Hence $x_1 = \frac{1065}{129}$, $x_2 = \frac{171}{113}$, $x_3 = -\frac{219}{129}$, $x_4 = \frac{117}{213}$

are the solutions of the given linear system of equations

iv. Given equations are $x+5y+z=9$

$2x+y+3z=12$

$3x+y+4z=16$

The above equations can be written in matrix form

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$$

Augmented matrix $[AB]$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 2 & 1 & 3 & 12 \\ 3 & 1 & 4 & 16 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 3R_1$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 0 & -9 & 1 & -6 \\ 0 & -14 & 1 & -11 \end{bmatrix}$$

$R_3 \to 9R_3 - 14R_2$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 0 & -9 & 1 & -6 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

which is echelon-form

By back substitutions

$-5z = -15$   $-9y + z = -6$   $x + 5y + z = 9$

$z = 3$   $-9y = -6 - z$   $x = 9 - 5y - z$

$-9y = -6 - 3$   $x = 9 - 5 - 3$

$-9y = -9$   $x = 9 - 8$

$y = 1$   $x = 1$

Hence $x=1$, $y=1$, $z=3$ are the solutions of the given system of linear equations.

## Gauss Jordan Method

Working rule:

1. Take the given linear system of equations.
2. The given system of equations can be put into matrix form i.e. $AX = B$.
3. Consider the augmented matrix $AB$.
4. The coefficient matrix $A$ in the augmented matrix can be reduced into identity matrix by using elementary row operations.
5. Solve the equations, we get the values of $x$ and $y$

1. Solve the following system of equations using Gauss Jordan method

i) $x+y+z=8$, $2x+3y+2z=19$, $4x+2y+3z=23$

ii) $x+5y+z=9$, $2x+y+3z=12$, $3x+y+4z=16$

iii) $10x+y+z=12$, $2x+10y+z=13$, $x+2y+20z=7$

iv) $10x+y+z=12$, $x+10y-z=10$, $x-2y+10z=9$

3) Given equations are $x+y+z=8$

$$2x+3y+2z=19$$

$$4x+2y+3z=23$$

The above equations can be put into matrix form as $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ 23 \end{bmatrix}$$

consider Augmented matrix $[AB]$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 2 & 3 & 2 & 19 \\ 4 & 2 & 3 & 23 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & -2 & -1 & -9 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_3' \to R_3' / -1$

$$[AB] \sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_3$

$$[AB] \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence $x=2, y=3, z=3$ are the solutions of given system of linear equations.

ii, Given equations are $x+5y+z=9$

$2x+y+3z=12$

$3x+y+4z=16$

the above equations can be put into matrix form

$$AX = B$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$$

consider Augmented matrix $[AB]$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 2 & 1 & 3 & 12 \\ 3 & 1 & 4 & 16 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 0 & -9 & 1 & -6 \\ 0 & -14 & 1 & -11 \end{bmatrix}$$

$R_3 \to 9R_3 - 14R_2$

$$[AB] \sim \begin{bmatrix} 1 & 5 & 1 & 9 \\ 0 & -9 & 1 & -6 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3/-5$

$$[AB] \sim \begin{bmatrix} 1 & 14 & 0 & 15 \\ 0 & -9 & 1 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$[AB] \sim \begin{bmatrix} 1 & 14 & 0 & 15 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2/-9$

$$[AB] \sim \begin{bmatrix} 1 & 14 & 0 & 15 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 - 14R_2$

$$[AB] \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence $x=1, y=1, z=3$ are the solutions of given system of linear equations.

iii. Given equations are $10x+y+z=12$
$2x+10y+z=13$
$x+y+5z=7$

The above equations can be put into matrix form

$$AX = B$$

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

consider Augmented matrix $[AB]$

$$[AB] \sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad , \quad R_3 \to R_3 - 10R_1$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & -8 & -9 & -1 \\ 0 & -9 & -47 & -38 \end{bmatrix}$$

$R_3 \to 8R_3 + 9R_2$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & -8 & -9 & -1 \\ 0 & 0 & -473 & -473 \end{bmatrix}$$

$R_3 \to R_3 / -473$

$$[AB] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & -8 & -9 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 9R_3$

$$\begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 / 8$

$$\begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 6R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x = 1$, $y = 1$, and $z = 1$ are the solutions

iii) $10x + y + z = 12$

$x + 10y - z = 10$

$x - 2y + 10z = 9$

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

$$[A|B] \sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 10 & 1 & 1 & 12 \\ 1 & -2 & 10 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 10R_1 \qquad R_3 \to R_3 + R_1$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & -99 & 11 & -88 \\ 0 & -12 & 11 & -1 \end{bmatrix}$$

$R_2 \to R_2 / 11$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & -9 & 1 & -8 \\ 0 & -12 & 11 & -1 \end{bmatrix}$$

$R_3 \to 9R_3 - 12R_2$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & -9 & 1 & -8 \\ 0 & 0 & 87 & 87 \end{bmatrix}$$

$R_3 \to R_3 / 87$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & -9 & 1 & -8 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2/-9$

$$[AB] \sim \begin{bmatrix} 1 & 10 & -1 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$[AB] \sim \begin{bmatrix} 1 & 10 & 0 & 11 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 10R_2$

$$[AB] \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[AB] \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x=1, y=1, z=1$ are the solutions

Diagonally dominant system:

The diagonal coefficients are non-zero and are large compared to the other coefficients. Such a system is called a diagonally dominant system.

Gauss-seidal Method:

Working rule:

1. Let us consider the system of equations be

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

2. First we check out the given equations are diagonally dominent system or not.

3. If not, we interchange the given equations, then we get a diagonally domirent system.

4. If yes, then go to step-5

5. Now, we write the given equations we large coefficients of variable in the equations can be expressed in terms of remaining variables and divide that equations by large coefficients i.e

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \longrightarrow ①$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \longrightarrow ②$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \longrightarrow ③$$

4. Take the initial solution as $x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0$

5. put $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$ in ①, we get a new value of $x_1^{(1)}$

6. Now, put $x_1^{(1)}$ and $x_3^{(0)} = 0$ in ②, we get a new

value of $x_3^{(1)}$

Now put $x_1^{(1)}$ and $x_2^{(1)}$ are in ③, we get a new value of $x_3^{(1)}$

These $x_1^{(1)}$, $x_2^{(1)}$ and $x_3^{(1)}$ values are called first iteration or first approximation.

7. We continue like this process up to two decimal places of consecutive iterations are equal or nearly equal

1. Solve the following system of equations using gauss-seidal method

i. $10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$

ii. $x+4y+15z=24$, $x+12y+z=26$, $10x+y-2z=10$

iii. $x+10y+z=6$, $10x+y+z=6$, $x+y+10z=6$

i. Given equations are

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14$$

clearly the given equations are diagonally dominent system. Now, we write the equations as

$$x=\frac{1}{10}[12-y-z] \rightarrow ①$$

$$y=\frac{1}{10}[13-2x-z] \rightarrow ②$$

$$z=\frac{1}{10}[14-2x-2y] \rightarrow ③$$

Take the initial solution as $x^{(0)}=0$, $y^{(0)}=0$ and $z^{(0)}=0$

$1^{st}$ approximation :

put $y^{(0)}=0$, $z^{(0)}=0$ we substitute in (1),

we have $x_1 = \frac{1}{10}[12-0-0] = \frac{12}{10} = 1.2$

put $x_1^{(1)}=1.2$ and $z^{(0)}=0$ we sub in (2), we get

$$y = \frac{1}{10}[13-2(1.2)-0]$$

$$= \frac{1}{10}[13-2.4]$$

$$= 1.06$$

put $x^{(1)}=1.2$, $y^{(1)}=1.06$ we sub in (3), we get

$$z^{(1)} = \frac{1}{10}[14-2(1.2)-2(1.06)]$$

$$= \frac{1}{10}[14-2.4-2.12]$$

$$= 0.948$$

$2^{nd}$ approximation :

put $y^{(1)}=1.06$ and $z^{(1)}=0.948$ we sub in (1)

$$x^{(2)} = \frac{1}{10}[12-1.06-0.948]$$

$$x^{(2)} = 0.9992$$

put $x^{(2)}=0.9992$ and $z^{(1)}=0.948$ we sub in (2)

$$y^{(2)} = \frac{1}{10}[13-2(0.9992)-0.948]$$

$$= 1.0054$$

put $x^{(2)}=0.9992$ and $y^{(2)}=1.0054$ we sub in (3)

$$z = \frac{1}{10}[14-2(0.9992)-2(1.0054)] = 0.9991$$

$3^{rd}$ approximation

put $y^{(2)} = 1.0034$ and $z^{(2)} = 0.9991$ are sub in (1)

$$x = \frac{1}{10}\left[12 - 1.0034 - 0.9991\right]$$

$$x^{(3)} = 0.9996$$

put $x^{(3)} = 0.9996$ and $z^{(2)} = 0.9991$ are sub in (2)

$$y^{(3)} = \frac{1}{10}\left[13 - 2(0.9996) - (0.9991)\right]$$

$$= 0.90036 \quad 1.0001$$

put $x^{(3)} = (0.9996)$ and $y^{(3)} = 0.9003$ are sub in (3)

$$z^{(3)} = \frac{1}{10}\left[14 - 2(0.9996) - 2(0.9003)\right]$$

$$= 1.00008$$

$4^{th}$ approximation

put $y^{(3)} = 1.0000$ and $z^{(3)} = 1.0000$ sub in (1),

we have $x^{(4)} = \frac{1}{10}\left[12 - 1.0001 - 1.0000\right]$

$$x^{(4)} = 0.9999$$

put $x^{(4)} = 0.9999$ and $z^{(3)} = 1.0000$

$$y^{(4)} = \frac{1}{10}\left[13 - 2(0.9999) - 1.0000\right]$$

$$= 1.0000$$

put $x^{(4)} = 0.9999$ and $y^{(4)} = 1.0000$

$$z^{(4)} = \frac{1}{10}\left[14 - 2(0.9999) - 2(1.0000)\right] = 1.0000$$

$5^{th}$ approximation

put $y^{(4)} = 1.0000$ and $z^{(4)} = 1.0000$

we have $x^{(5)} = \frac{1}{10}\left[12 - 1.0000 - 1.0000\right]$

$x^{(5)} = 1.0000$

put $x^{(5)} = 1$ and $z^{(4)} = 1.0000$

we have $y^{(5)} = \frac{1}{10}\left[13 - 2(1) - 1.0000\right]$

$y^{(5)} = 1.0000$

put $x^{(5)} = 1$ and $y^{(5)} = 1$

we have $z^{(5)} = \frac{1}{10}\left[14 - 2(1) - 2(1)\right]$

$= \frac{10}{10} = 1.0000$

$x^{(4)} = 0.9999$ $y^{(4)} = 1.0000$ $z^{(4)} = 1.0000$

$x^{(5)} = 1.0000$ $x^{(5)} = 1.0000$ $z^{(5)} = 1.0000$

clearly $4^{th}$ and $5^{th}$ iterations are equal or nearly equal, so we conclude that $x = x^{(4)} = x^{(5)} = 1.0000$

$y = y^{(4)} = y^{(5)} = 1.0000$

$z = z^{(4)} = z^{(5)} = 1.0000$

Hence $x = 1$, $y = 1$ and $z = 1$ are the solutions of given system of equations.

5) Given equations are $x+4y+15z=24$

$$x+12y+z=26$$

$$10x+y-2z=10$$

clearly the given equations are not diagonally dominent system

so, we interchange first and third equations, we get a diagonally dominent system i.e

$$10x+y-2z=10$$

$$x+12y+z=26$$

$$x+4y+15z=24$$

The above equations can be written as,

$$x=\frac{1}{10}\left[10-y+2z\right]\longrightarrow ①$$

$$y=\frac{1}{12}\left[26-x-z\right]\longrightarrow ②$$

$$z=\frac{1}{15}\left[24-x-4y\right]\longrightarrow ③$$

Take the initial solution $x^{(0)}=0$, $y^{(0)}=0$, $z^{(0)}=0$

$1^{st}$ approximation:

put $y^{(0)}=0$, $z^{(0)}=0$ in ①, we get

$$x^{(1)}=\frac{1}{10}\left[10-0+2(0)\right]=1$$

put $x^{(1)}=1$ and $z^{(0)}=0$

$$y^{(1)}=\frac{1}{12}\left[26-1-0\right]=2.0833$$

put $x^{(1)} = 1$ and $y^{(1)} = 2.0833$

$$z = \frac{1}{15}\left[24 - 1 - 4(2.0833)\right]$$

$$= \frac{14.6664}{15} = 0.9777$$

$2^{nd}$ approximation

put $y^{(1)} = 2.0833$, $z^{(1)} = 0.9777$ sub in ①, we get

$$x^{(2)} = \frac{1}{10}\left[10 - 2.0833 + 2(0.9777)\right]$$

$$= 0.9872$$

put $x^{(2)} = 0.9872$ and $z^{(1)} = 0.9777$ in ①, we get

$$y^{(2)} = \frac{1}{12}\left[26 - 0.9872 - 0.9777\right]$$

$$= 2.0029$$

put $x^{(2)} = 0.9872$ and $y^{(2)} = 2.0029$ in ①, we get

$$z^{(2)} = \frac{1}{15}\left[24 - 0.9872 - 4(2.0029)\right]$$

$$= 1.0000$$

$3^{rd}$ approximation

put $y^{(2)} = 2.0029$, $z^{(2)} = 1.0000$ in ①, we get

$$x^{(3)} = \frac{1}{10}\left[10 - 2.0029 + 2(1.0000)\right]$$

$$= 0.9997$$

put $x^{(3)} = 0.9997$, $z^{(2)} = 1.0000$

$$y^{(3)} = \frac{1}{12}\left[26 - 0.9997 - 1.0000\right]$$

$$= 2.0000$$

put $x^{(3)} = 0.9997$, $y^{(3)} = 2.0000$ in ①, we get

$$z^{(3)} = \frac{1}{15}\left[24 - 0.9997 - 4(2.0000)\right]$$

$$= 1.0000$$

$4^{th}$ approximation

put $y^{(3)} = 2.0000$ and $z^{(3)} = 1.0000$ in ①, we get

$$x^{(4)} = \frac{1}{18}\left[10 - 2.0000 + 2(1.0000)\right]$$

$$= 1$$

put $x^{(4)} = 1$ and $z^{(3)} = 1.0000$ in ①, we get

$$y^{(4)} = \frac{1}{12}\left[26 - 1 - 1.0000\right]$$

$$= 2$$

put $x^{(4)} = 1$ and $y^{(4)} = 2$ in ①, we get

$$z^{(4)} = \frac{1}{15}\left[24 - 1 - 4(2)\right]$$

$$= 1$$

$5^{th}$ approximation

put $y^{(4)} = 2$ and $z^{(4)} = 1$ in ①, we get

$$x^{(5)} = \frac{1}{18}\left[10 - 2 + 2(1)\right] = 1$$

put $x^{(5)} = 1$ and $z^{(4)} = 1$ in ①, we get

$$y^{(5)} = \frac{1}{12}\left[26 - 1 - 1\right] = 2$$

put $x^{(5)}=1$ and $y^{(5)}=2$ in (3), we get

$$z^{(5)} = \frac{1}{15}\left[24-1-4(2)\right] = 1$$

$$x^{(4)}=1, \quad y^{(4)}=2, \quad z^{(4)}=1$$
$$x^{(5)}=1, \quad y^{(5)}=2, \quad z^{(5)}=1$$

clearly $4^{th}$ and $5^{th}$ iterations are equal, so we conclude that

$$x = x^{(4)} = x^{(5)} = 1$$
$$y = y^{(4)} = y^{(5)} = 2$$
$$z = z^{(4)} = z^{(5)} = 1$$

Hence $x=1, y=2, z=1$ are the solutions of given system of equations.

(ii) Given equations are

$$x+10y+z=6$$
$$10x+y+z=6$$
$$x+y+10z=6$$

clearly the given equations are not diagonally dominant system.

So, we interchange first and second equations, we get a diagonally dominant system i.e

$$10x+y+z=6$$
$$x+10y+z=6$$
$$x+y+10z=6$$

the above equations can be written as

$$x = \frac{1}{10}\left[6-y-z\right] \rightarrow (1)$$
$$y = \frac{1}{10}\left[6-x-z\right] \rightarrow (2)$$
$$z = \frac{1}{10}\left[6-x-y\right] \rightarrow (3)$$

take the initial solution $x^{(0)}=0,\ y^{(0)}=0,\ z^{(0)}=0$

$1^{st}$ approximation

put $y^{(0)}=0,\ z^{(0)}=0$ in ①, we get

$$x^{(1)} = \frac{1}{10}\left[6-0-0\right]$$
$$=0.6$$

put $x^{(1)}=0.6$ and $z^{(0)}=0$ in ②, we get

$$y^{(1)} = \frac{1}{10}\left[6-0.6-0\right]$$
$$=0.54$$

put $x^{(1)}=0.6$ and $y^{(1)}=0.54$ in ③, we get

$$z^{(1)} = \frac{1}{10}\left[6-0.6-0.54\right]$$
$$=0.486$$

$2^{nd}$ approximation

put $y^{(1)}=0.54,\ z^{(1)}=0.486$ in ①, we get

$$x^{(2)} = \frac{1}{10}\left[6-0.54-0.486\right]$$
$$=0.4974$$

put $x^{(2)}=0.4974$ and $z^{(1)}=0.486$

$$y^{(2)} = \frac{1}{10}\left[6-0.4974-0.486\right]$$
$$=0.5016$$

put $x^{(2)}=0.4974$ and $y^{(2)}=0.5016$ in ③, we get

$$z^{(2)} = \frac{1}{10}\left[6-0.4974-0.5016\right] = 0.5001$$

$3^{rd}$ approximation

put $y^{(2)} = 0.5016$ and $z^{(2)} = 0.5001$ in (1), we get

$$x^{(3)} = \frac{1}{10}\left[6 - 0.5016 - 0.5001\right]$$

$$= 0.4998$$

put $x^{(3)} = 0.4998$ and $z^{(2)} = 0.5001$ in (2), we get

$$y^{(3)} = \frac{1}{10}\left[6 - 0.4998 - 0.5001\right]$$

$$= 0.5000$$

put $x^{(3)} = 0.4998$ and $y^{(3)} = 0.5000$

$$z^{(3)} = \frac{1}{10}\left[6 - 0.4998 - 0.5000\right]$$

$$= 0.5000$$

put $(x)^{(3)} = 0.4998$ and

$4^{th}$ approximation

put $y^{(3)} = 0.5000$ and $z^{(3)} = 0.5000$ in (1), we get

$$x^{(4)} = \frac{1}{10}\left[6 - 0.5000 - 0.5000\right]$$

$$= 0.5$$

put $x^{(4)} = 0.5$ and $z^{(3)} = 0.5000$ in (2), we get

$$y^{(4)} = \frac{1}{10}\left[6 - 0.5 - 0.50000\right]$$

$$= 0.5$$

put $x^{(4)} = 0.5$ and $y^{(4)} = 0.5$ in (3), we get

$$z^{(4)} = \frac{1}{10}\left[6 - 0.5 - 0.5\right]$$

$$= 0.5$$

$5^{th}$ approximation

put $y^{(4)} = 0.5$ and $z^{(4)} = 0.5$ in ①, we get

$$x^{(5)} = \frac{1}{10}\left[6 - 0.5 - 0.5\right]$$

$$= 0.5$$

put $x^{(5)} = 0.5$ and $z^{(4)} = 0.5$ in ②, we get

$$y^{(5)} = \frac{1}{10}\left[6 - 0.5 - 0.5\right]$$

$$= 0.5$$

put $x^{(5)} = 0.5$ and $y^{(5)} = 0.5$ in ③, we get

$$z^{(5)} = \frac{1}{10}\left[6 - 0.5 - 0.5\right]$$

$$= 0.5$$

clearly $4^{th}$ and $5^{th}$ iterations are equal so, we conclude that

$$x = x^{(4)} = x^{(5)} = 0.5$$
$$y = y^{(4)} = y^{(5)} = 0.5$$
$$z = z^{(4)} = z^{(5)} = 0.5$$

Hence $x = 0.5$, $y = 0.5$, $z = 0.5$ are the solutions of the given linear system of equations

Gauss - Jacobi Method:

Working rule:

1. let us consider the system of equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

2. First we check out whether the given equations are diagonally dominant system or not.

3. If no, then we interchange the given equations, we get a diagonally dominant system.

If yes, then go to next step

4. Now, we write the given equations are as

$$x_1 = \frac{1}{a_{11}}\left[ b_1 - a_{12}x_2 - a_{13}x_3 \right] \longrightarrow ①$$

$$x_2 = \frac{1}{a_{22}}\left[ b_2 - a_{21}x_1 - a_{23}x_3 \right] \longrightarrow ②$$

$$x_3 = \frac{1}{a_{33}}\left[ b_3 - a_{31}x_1 - a_{32}x_2 \right] \longrightarrow ③$$

Let us take the initial approximation as

$x_1^{(0)} = 0$, $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$

5. put $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$ in ①, we get a new value of $x_1^{(1)}$.

put $x_1^{(0)} = 0$, $x_3^{(0)} = 0$ we sub in ②, then we get a new value of $x_2^{(1)}$

put $x_1^{(0)} = 0$, $x_2^{(0)} = 0$ we sub in ③, then we get a new value of $x_3^{(1)}$.

These $x_1^{(1)}$, $x_2^{(1)}$ and $x_3^{(1)}$ values are called first approximation or first iteration.

6. we continue like this process untill upto two consecutive approximation values are equal or nearly equal

1. Solve the following system of equations using Gauss-Jacobi method.

i. $x+10y+z=6$ , $10x+y+z=6$, $x+y+10z=6$

ii. $x+4y+15z=24$, $x+12y+z=26$, $10x+y-2z=10$

iii. $10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$

i. Given equations are $x+10y+z=6$

$$10x+y+z=6$$

$$x+y+10z=6$$

clearly the given equations are not in diagonally dominant system, so, we interchange $1^{st}$ and $2^{nd}$ eqs we get a diagonally dominant system.

i.e

$$10x+y+z=6$$
$$x+10y+z=6$$
$$x+y+10z=6$$

the above equations can be written as

$$x=\frac{1}{10}[6-y-z] \rightarrow (1)$$

$$y=\frac{1}{10}[6-x-z] \rightarrow (2)$$

$$z=\frac{1}{10}[6-x-y] \rightarrow (3)$$

let us take the initial approximation as

$x^{(0)}=0$, $y^{(0)}=0$, $z^{(0)}=0$

$1^{st}$ approximation

put $y^{(0)}=0$, $z^{(0)}=0$ in (1), we get

$$x^{(1)} = \frac{1}{10}\left[6 - 0 - 0\right]$$
$$= 0.6$$

put $x^{(0)} = 0$ and $z^{(0)} = 0$ in ②, we get

$$y = \frac{1}{10}\left[6 - 0 - 0\right]$$
$$y^{(1)} = 0.6$$

put $x^{(0)} = 0$ and $y^{(0)} = 0$ in ③, we get

$$z^{(1)} = \frac{1}{10}\left[6 - 0 - 0\right]$$
$$= 0.6$$

2nd Iteration

put $y^{(1)} = 0.6$ and $z^{(1)} = 0.6$ in ①, we get

$$x^{(2)} = \frac{1}{10}\left[6 - 0.6 - 0.6\right]$$
$$= 0.48$$

put $x^{(1)} = 0.6$ and $z^{(1)} = 0.6$ in ②, we get

$$y^{(2)} = \frac{1}{10}\left[6 - 0.6 - 0.6\right]$$
$$= 0.48$$

put $x^{(1)} = 0.6$ and $y^{(1)} = 0.6$ in ③, we get

$$z^{(2)} = \frac{1}{10}\left[6 - 0.6 - 0.6\right]$$
$$= 0.48$$

3rd Iteration

put $y^{(2)} = 0.48$ and $z^{(2)} = 0.48$ in ①, we get

$$x^{(3)} = \frac{1}{10}\left[6 - 0.48 - 0.48\right]$$
$$= 0.504$$

put $x^{(2)} = 0.48$ and $z^{(2)} = 0.48$ in ②, we get

$$y^{(3)} = \frac{1}{10}\left[6 - 0.48 - 0.48\right]$$
$$= 0.504$$

put $x^{(2)} = 0.48$ and $y^{(2)} = 0.48$ in ③, we get

$$z^{(3)} = \frac{1}{10}\left[6 - 0.48 - 0.48\right]$$
$$= 0.504$$

$4^{th}$ iteration

put $y^{(3)} = 0.504$ and $z^{(3)} = 0.504$ in ①, we get

$$x^{(4)} = \frac{1}{10}\left[6 - 0.504 - 0.504\right]$$
$$= 0.4992$$

put $(x^{3}) = 0.504$ and $z^{(3)} = 0.504$, in ②, we get

$$y^{(4)} = \frac{1}{10}\left[6 - 0.504 - 0.504\right]$$
$$= 0.4992$$

put $x^{(3)} = 0.504$ and $y^{(3)} = 0.504$

$$z^{(4)} = \frac{1}{10}\left[6 - 0.504 - 0.504\right]$$
$$= 0.4992$$

$5^{th}$ iteration

put $y^{(4)} = 0.4992$ and $z^{(4)} = 0.4992$ in ①, we get

$$x^{(5)} = \frac{1}{10}\left[6 - 0.4992 - 0.4992\right]$$
$$= 0.5001$$

put $x^{(4)} = 0.4992$ and $z^{(4)} = 0.4992$ in ②, we get:

$$y^{(5)} = \frac{1}{10}\left[6 - 0.4992 - 0.4992\right]$$

$$= 0.5001$$

put $x^{(4)} = 0.4992$ and $y^{(4)} = 0.4992$ in ③, we get

$$z^{(5)} = \frac{1}{10}\left[6 - 0.4992 - 0.4992\right]$$

$$= 0.5001$$

$6^{th}$ iteration

put $y^{(5)} = 0.5001$ and $z^{(5)} = 0.5001$ in ①, we get

$$x^{(6)} = \frac{1}{10}\left[6 - 0.5001 - 0.5001\right]$$

$$= 0.4999$$

put $x^{(5)} = 0.5001$ and $z^{(5)} = 0.5001$ in ②, we get

$$y^{(6)} = \frac{1}{10}\left[6 - 0.5001 - 0.5001\right]$$

$$= 0.4999$$

put $x^{(5)} = 0.5001$ and $y^{(5)} = 0.5001$ in ③, we get

$$z^{(6)} = \frac{1}{10}\left[6 - 0.5001 - 0.5001\right]$$

$$= 0.4999$$

$7^{th}$ iteration

put $y^{(6)} = 0.4999$ and $z^{(6)} = 0.4999$ in ①, we get

$$x^{(7)} = \frac{1}{10}\left[6 - 0.4999 - 0.4999\right]$$

$$= 0.5000$$

put $x^{(6)} = 0.4999$ and $z^{(6)} = 0.4999$ in ②, we get

$$y^{(7)} = \frac{1}{10}\left[6 - 0.4999 - 0.4999\right]$$

$$= 0.5000$$

put $x^{(6)} = 0.4999$ and $y^{(6)} = 0.4999$ in ③, we get

$$z^{(7)} = \frac{1}{10}\left[6 - 0.4999 - 0.4999\right]$$

$$= 0.5000$$

$8^{th}$ iteration

put $y^{(7)} = 0.5000$ and $z^{(7)} = 0.5000$ in ①, we get

$$x^{(8)} = \frac{1}{10}\left[6 - 0.5000 - 0.5000\right]$$

$$= 0.5$$

put $x^{(7)} = 0.5000$ and $z^{(7)} = 0.5000$ in ②, we get

$$y^{(8)} = \frac{1}{10}\left[6 - 0.5000 - 0.5000\right]$$

$$= 0.5$$

put $x^{(7)} = 0.5000$ and $y^{(7)} = 0.5000$ in ③, we get

$$z^{(8)} = \frac{1}{10}\left[6 - 0.5000 - 0.5000\right]$$

$$= 0.5$$

clearly $7^{th}$ and $8^{th}$ iterations are equal so, we conclude that $x = x^{(7)} = x^{(8)} = 0.5$

$y = y^{(7)} = y^{(8)} = 0.5$

$z = z^{(7)} = z^{(8)} = 0.5$

Hence $x = 0.5$, $y = 0.5$ and $z = 0.5$ are the solutions of the given equations.

3. Given equations are $x + 4y + 15z = 24$

$x + 12y + z = 26$

$10x + y - 2z = 10$

clearly the given equations are not in diagonally dominant system, so we interchange $1^{st}$ and $3^{rd}$

equations, we get a diagonally dominant system

i.e $10x + y - 2z = 10$
$x + 12y + z = 26$
$x + 4y + 15z = 24$

The above equations can be written as

$$x = \frac{1}{10}[10 - y + 2z] \rightarrow (1)$$

$$y = \frac{1}{12}[26 - x - z] \rightarrow (2)$$

$$z = \frac{1}{15}[24 - x - 4y] \rightarrow (3)$$

Let us take the initial approximation as

$x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

1st iteration:

put $y^{(0)} = 0$ and $z^{(0)} = 0$ to (1), we get

$$x^{(1)} = \frac{1}{10}[10 - 0 + 0] = 1$$

put $x^{(0)} = 0$ and $z^{(0)} = 0$ in (2), we get

$$y^{(1)} = \frac{1}{12}[26 - 0 - 0]$$
$$= 2.1666$$

put $x^{(0)} = 0$ and $y^{(0)} = 0$ in (3), we get

$$z^{(1)} = \frac{1}{15}[24 - 0 - 0]$$
$$= 1.6$$

2nd iteration

put $y^{(1)} = 2.1666$ and $z^{(1)} = 1.6$ in (1), we get

$$x^{(2)} = \frac{1}{10}[10 - 2.1666 + 2(1.6)]$$
$$= 1.1033$$

put $x^{(1)} = 1$ and $z^{(1)} = 1.6$ in (2), we get

$$y^{(2)} = \frac{1}{12}[26 - 1 - 1.6]$$
$$= 1.95$$

put $x^{(1)} = 1$ and $y^{(1)} = 2.1666$ in ③, we get

$$z^{(2)} = \frac{1}{15}\left[24 - 1 - 4(2.1666)\right]$$

$= 0.9555$

3rd

put $y^{(2)} = 1.95$ and $z^{(2)} = 0.9555$ in ①, we get

$$x^{(3)} = \frac{1}{10}\left[10 - 1.95 + 2(0.9555)\right]$$

$= 0.9961$

put $x^{(2)} = 1.1033$ and $z^{(2)} = 0.9555$ in ②, we get

$$y^{(3)} = \frac{1}{12}\left[26 - 1.1033 - 0.9555\right]$$

$= 1.9951$

put $x^{(2)} = 1.1033$ and $y^{(2)} = 1.95$ in ③, we get

$$z^{(3)} = \frac{1}{15}\left[24 - 1.1033 - 4(1.95)\right]$$

$= 1.0064$

4th Iteration

put $y^{(3)} = 1.9951$ and $z^{(3)} = 1.0064$ in ①, we get

$$x^{(4)} = \frac{1}{10}\left[10 - 1.9951 + 2(1.0064)\right]$$

$= 1.0017$

put $x^{(3)} = 0.9961$ and $z^{(3)} = 1.0064$ in ②, we get

$$y^{(4)} = \frac{1}{12}\left[26 - 0.9961 - 1.0064\right]$$

$= 1.9997$

put $x^{(3)} = 0.9961$ and $y^{(3)} = 1.9951$

$$z^{(4)} = \frac{1}{15}\left[24 - 0.9981 - 4(1.9951)\right]$$
$$= 1.0015$$

$5^{th}$ iteration:

put $y^{(4)} = 1.9997$ and $z^{(4)} = 1.0015$ in ①, we get

$$x^{(5)} = \frac{1}{10}\left[10 - 1.9997 + 2(1.0015)\right]$$
$$= 1.0003$$

put $x^{(4)} = 1.0017$ and $z^{(4)} = 1.0015$ in ②, we get

$$y^{(5)} = \frac{1}{12}\left[26 - 1.0017 - 1.0015\right]$$
$$= 1.9997$$

put $x^{(4)} = 1.0017$ and $y^{(4)} = 1.9997$ in ③, we get

$$z^{(5)} = \frac{1}{15}\left[24 - 1.0017 - 4(1.9997)\right]$$
$$= 0.9999$$

$6^{th}$ iteration:

put $y^{(5)} = 1.9997$ and $z^{(5)} = 0.9999$ in ①, we get

$$x^{(6)} = \frac{1}{10}\left[10 - 1.9997 + 2(0.9999)\right]$$
$$= 1.0000$$

put $x^{(5)} = 1.0003$ and $z^{(5)} = 0.9999$ in ②, we get

$$y^{(6)} = \frac{1}{12}\left[26 - 1.0003 - 0.9999\right]$$

continue

$$= 1.9999$$

put $x^{(5)} = 1.0003$ and $y^{(5)} = 1.9997$ in ③, we get

$$z^{(6)} = \frac{1}{15}\left[24 - 1.0003 - 4(1.9997)\right]$$
$$= 1.0000$$

$7^{th}$ iteration

put $y^{(6)} = 1.9999$ and $z^{(6)} = 1.0000$ in ①, we get

$$x^{(7)} = \frac{1}{10}\left[10 - 1.9999 + 2(1.0000)\right]$$

$$= 1.0000$$

put $x^{(6)} = 1.0000$ and $z^{(6)} = 1.0000$ in ②, we get

$$y^{(7)} = \frac{1}{12}\left[26 - 1.0000 - 1.0000\right]$$

$$= 2$$

put $x^{(6)} = 1.0000$ and $y^{(6)} = 1.9999$ in ③, we get

$$z^{(7)} = \frac{1}{15}\left[20 - 1.0000 - 4(1.9999)\right]$$

$$= 1.0000$$

clearly $6^{th}$ and $7^{th}$ iterations are nearly equal. so, we conclude that

$$x = x^{(6)} = x^{(7)} = 1.$$
$$y = y^{(6)} = y^{(7)} = 2$$
$$z = z^{(6)} = z^{(7)} = 1$$

Hence $x = 1, y = 2, z = 1$ are the solutions are given equations.

(ii) Given equations are

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

clearly the given equations are in diagonally dominent system. Now, we write the equations as

$$x = \frac{1}{10}\left[12 - y - z\right] \rightarrow (1)$$

$$y = \frac{1}{10}\left[13 - 2x - z\right] \rightarrow (2)$$

$$z = \frac{1}{10}\left[14 - 2x - 2y\right] \rightarrow (3)$$

$1^{st}$ Iteration

Let take initial equations are $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$

put $y^{(0)} = 0$ and $z^{(0)} = 0$ in (1), we get

$$x^{(1)} = \frac{1}{10}\left[12 - 0 - 0\right] = 1.2$$

put $x^{(0)} = 0$ and $z^{(0)} = 0$ in (2), we get

$$y^{(1)} = \frac{1}{10}\left[13 - 2(0) - 0\right]$$

$$= 1.3$$

put $x^{(0)} = 0$ and $y^{(0)} = 0$ in (3), we get

$$z^{(1)} = \frac{1}{10}\left[14 - 2(0) - 2(0)\right]$$

$$= 1.4$$

$2^{nd}$ Iteration

put $y^{(1)} = 1.3$ and $z^{(1)} = 1.4$ in (1), we get

$$x^{(2)} = \frac{1}{10}\left[12 - 1.3 - 1.4\right]$$

$$= 0.93$$

put $x^{(1)} = 1.2$ and $z^{(1)} = 1.4$ in (2), we get

$$y^{(2)} = \frac{1}{10}\left[13 - 2(1.2) - 1.4\right]$$

$$= 0.92$$

put $x^{(1)} = 1.2$ and $y^{(1)} = 1.3$

$$z^{(2)} = \frac{1}{10}\left[14 - 2(1.2) - 2(1.3)\right]$$
$$= 0.9$$

$3^{rd}$ iteration

put $y^{(2)} = 0.92$ and $z^{(2)} = 0.9$ in ①, we get

$$x^{(3)} = \frac{1}{10}\left[12 - 0.92 - 0.9\right]$$
$$= 1.018$$

put $x^{(2)} = 0.93$ and $z^{(2)} = 0.9$ in ②, we get

$$y^{(3)} = \frac{1}{10}\left[13 - 2(0.93) - 0.9\right]$$
$$= 1.024$$

put $x^{(2)} = 0.93$ and $y^{(2)} = 0.92$ in ③, we get

$$z^{(3)} = \frac{1}{10}\left[14 - 2(0.93) - 2(0.92)\right]$$
$$= 1.03$$

$4^{th}$ iteration

put $y^{(3)} = 1.024$ and $z^{(3)} = 1.03$ in ①, we get

$$x^{(4)} = \frac{1}{10}\left[12 - 1.024 - 1.03\right]$$
$$= 0.9946$$

put $x^{(3)} = 1.018$ and $z^{(3)} = 1.03$ in ②, we get

$$y^{(4)} = \frac{1}{10}\left[13 - 2(1.018) - 1.03\right]$$
$$= 0.9934$$

put $x^{(3)}=1.018$ and $y^{(3)}=1.024$ in ③, we get

$$z^{(4)}=\frac{1}{10}\left[14-2(1.018)-2(1.024)\right]$$

$$=0.9916$$

$5^{th}$ iteration

$y^{(4)}=0.9934$ and $z^{(4)}=0.9916$ in ①, we get

$$x^{(5)}=\frac{1}{10}\left[12-0.9934-0.9916\right]$$

$$=1.0015$$

put $x^{(4)}=0.9946$ and $z^{(4)}=0.9916$ in ②, we get

$$y^{(5)}=\frac{1}{10}\left[13-2(0.9946)-0.9916\right]$$

$$=1.0019$$

put $x^{(4)}=0.9946$ and $y^{(4)}=0.9934$ in ③, we get

$$z^{(5)}=\frac{1}{10}\left[14-2(0.9946)-2(0.9934)\right]$$

$$=1.0024$$

$6^{th}$ iteration

put $y^{(5)}=1.0019$ and $z^{(5)}=1.0024$ in ①, we get

$$x^{(6)}=\frac{1}{10}\left[12-1.0019-1.0024\right]$$

$$=0.9995$$

put $x^{(5)}=1.0015$ and $z^{(5)}=1.0024$ in ②, we get

$$y^{(6)}=\frac{1}{10}\left[13-2(1.0015)-1.0024\right]$$

$$=0.9994$$

put $x^{(5)} = 1.0015$ and $y^{(5)} = 1.0017$ in ①, we get

$$z^{(6)} = \frac{1}{10}\left[14 - 2(1.0015) - 2(1.0017)\right]$$

$$= 0.9993$$

$7^{th}$ iterations

put $y^{(6)} = 0.9994$ and $z^{(6)} = 0.9993$ in ①, we get

$$x^{(7)} = \frac{1}{10}\left[12 - 0.9994 - 0.9993\right]$$

$$= 1.0001$$

put $x^{(6)} = 0.9995$ and $z^{(6)} = 0.9993$ in ②, we get

$$y^{(7)} = \frac{1}{10}\left[13 - 2(0.9995) - 0.9993\right]$$

$$= 1.0001$$

put $x^{(6)} = 0.9995$ and $y^{(6)} = 0.9994$ in ③, we get

$$z^{(7)} = \frac{1}{10}\left[14 - 2(0.9995) - 2(0.9994)\right]$$

$$= 1.0002$$

$8^{th}$ iteration

put $y^{(7)} = 1.0001$ and $z^{(7)} = 1.0002$ in ①, we get

$$x^{(8)} = \frac{1}{10}\left[12 - 1.0001 - 1.0002\right]$$

$$= 0.9999$$

put $x^{(7)} = 1.0001$ and $z^{(7)} = 1.0002$ in ②, we get

$$y^{(8)} = \frac{1}{10}\left[13 - 2(1.0001) - 1.0002\right] = 0.9999$$

put $x^{(7)} = 1.0001$ and $y^{(7)} = 1.0001$ in ③, we get

$$x^{(8)} = \frac{1}{10}\left[14 - 2(1.0001) - 2(1.0001)\right]$$
$$= 0.9999$$

$9^{th}$ iteration

put $y^{(8)} = 0.9999$ and $z^{(8)} = 0.9999$ in ①, we get

$$x^{(9)} = \frac{1}{10}\left[12 - 0.9999 - 0.9999\right]$$
$$= 1.0008$$

put $x^{(8)} = 0.9999$ and $z^{(8)} = 0.9999$ in ②, we get

$$y^{(9)} = \frac{1}{10}\left[13 - 2(0.9999) - 0.9999\right]$$
$$= 1.0000$$

put $x^{(8)} = 0.9999$ and $y^{(8)} = 0.9999$ in ③, we get

$$z^{(9)} = \frac{1}{10}\left[14 - 2(0.9999) - 2(0.9999)\right]$$
$$= 1.0000$$

$10^{th}$ iteration

put $y^{(9)} = 1.0000$ and $z^{(9)} = 1.0000$ in ①, we get

$$x^{(10)} = \frac{1}{10}\left[12 - 1.0000 - 1.0000\right]$$
$$= 1$$

put $x^{(9)} = 1.0000$ and $z^{(9)} = 1.0000$ in ②, we get

$$y^{(10)} = \frac{1}{10}\left[13 - 2(1.0000) - 1.0000\right]$$

$y^{(10)} = 1$

put $x^{(9)} = 1.0000$ and $y^{(9)} = 1.0000$ in ③, we get

$$x^{(10)} = \frac{1}{10}\left[14 - 2(1.0000) - 2(1.0000)\right]$$

$$= 1$$

clearly $9^{th}$ and $10^{th}$ iterations are equal. So, we conclude that

$$x = x^{(9)} = x^{(10)} = 1$$
$$y = y^{(9)} = y^{(10)} = 1$$
$$z = z^{(9)} = z^{(10)} = 1$$

Hence $x=1, y=1, z=1$ are the solutions of given system of equations.

<u>Homogenious system of equations</u>

Working rule.

1. Take the given system of equations
2. The given system of equations can be put into matrix form i.e $AX=0$
3. Take the coefficient matrix i.e $A$
4. Now, we apply the elementary row operations on $A$ then it can be reduced into echelon form
5. i. If $\rho(A) = n$, then the given system is consistent and it has a unique solution or zero solution or trivial solution

   ii. If $\rho(A) < n$, then the given system is consistent and it has an infinitely no. of non-trivial solutions.

we shall have $(n-r)$ linearly independent solutions

ii. If $\rho(A) \neq n$, then the given system is inconsistent and it has no solution.

1. Solve the following system of equations.

i. $x+2y+z=0$, $2x+3y+z=0$, $4x+5y+4z=0$, $x+y-2z=0$.

ii. $x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$.

iii. $x_1+2x_3-2x_4=0$, $2x_1-x_2-x_4=0$, $x_1+2x_3-x_4=0$, $4x_1-x_2+3x_3-x_4=0$

iv. $x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$.

i. Given equations are $x+2y+z=0$

$$2x+3y+z=0$$
$$4x+5y+4z=0$$
$$x+y-2z=0$$

The above equations can be put into matrix form is $AX=0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

consider $[A] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -3 & 0 \\ 0 & -1 & -3 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$
$$R_4 \to R_4 - R_2$$

$$[A] \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$R_4 \to 3R_4 + 2R_3$$

$$[A] \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is echelon form

Here $\rho(A) = 3$, $n = 3$

clearly $\rho(A) = n$, then the given system is consistent and it has a trivial solution

Hence $x=0$, $y=0$ and $z=0$ are the solutions of given system of equations.

ii. Given equations are

$$x + 3y + 2z = 0$$
$$2x - y + 3z = 0$$
$$3x - 5y + 4z = 0$$
$$x + 17y + 4z = 0$$

The above equations can be put into matrix form is $AX = 0$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

consider $[A] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 + 2R_2$

$$[A] \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is echelon form

Here $\rho(A) = 2$, $n = 3$

clearly $\rho(A) < n$, then the given system is consistent and it has a ~~trivial~~ non-trivial solutions

~~Hence $x=0, y=0, z=0$ are the solutions of the given~~ $(n-r)$ linearly independent solutions.

i.e $n - r = 3 - 2 = 1$

iii. By using back substitution, we have

$$-7y-z=0 \rightarrow (1)$$

$$x+3y+2z=0 \rightarrow (2)$$

Let $z=k$, then from (1)

$$-y=k/7$$

$$y=-k/7$$

from (2) $x=-2z-3y$

$$=-2k+\frac{3k}{7}$$

$$=\frac{-11k}{7}$$

Hence $x=\frac{-11k}{7}$, $y=-\frac{k}{7}$, $z=k$ are the solutions of given system of equations and $k$ is any real number.

iv. Given equations are $x_1+2x_3-2x_4=0$

$$2x_1-x_2-x_4=0$$

$$x_1+2x_3-x_4=0$$

$$4x_1-x_2+3x_3-x_4=0$$

The above equations can be put into matrix form is $AX=0$

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

consider $[A]$

$$[A] = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 4R_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 7 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$[A] \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A] \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is echelon form.

Here $\rho(A) = 4$, $n = 4$

clearly $\rho(A) = 4$ then the given system is consistent and it has trivial solution.

Hence $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$ are the solutions of given system of equations.

Q. Given equations are $x+y-3z+2w=0$

$2x-y+2z-3w=0$

$3x-2y+z-4w=0$

$-4x+y-3z+w=0$

The above equations can be put into matrix form

is $AX=0$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

consider $[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + 4R_1$

$$[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -5 & 0 \\ 0 & 5 & 0 & 9 \end{bmatrix}$$

Eigen values and Eigen vectors

## Eigen values

Let $A$ be any square matrix of order $n$ and $\lambda$ is any real number then $[A-\lambda I]$ is called the characteristic matrix of $A$, where $I$ is the identity matrix of order $n$.

The equation $|A-\lambda I|=0$ is called the characteristic equation of $A$ and roots of this equation is called eigen values or characteristic roots

## Eigen vectors

If $\lambda$ is an eigen value of the matrix $A$ then there exist a non-zero vector '$x$' such that $[A-\lambda I]x=0$ is called the Eigen vector or characteristic vector of $A$ corresponding to the eigen values

## Working rule for finding Eigen values and vectors

1. Take the given matrix, let it be $A$
2. write the characteristic matrix, i.e $A-\lambda I$
3. write the characteristic equation, i.e $|A-\lambda I|=0$ and simplify it. we get the eigen values $\lambda$.
4. write the non-zero vector is $[A-\lambda I]x=0$ and substute $\lambda$ values and simplify, we get the eigen vectors.

Find the Eigen values of the following matrices

i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$ iii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic matrix of $A$ is $A - \lambda I$

$$[A - \lambda I] = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[ \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right]$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

To find Eigen values

The characteristic equation of $A$ is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$5-\lambda(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - \lambda - 6\lambda + 6 = 0$$

$$\lambda(\lambda - 1) - 6(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, \lambda = 6$$

$\therefore$ The Eigen values of $A$ are $1, 6$

ii. Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$.

The characteristic matrix of $A$ is $[A-\lambda I]$

$$[A-\lambda I] = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{bmatrix}$$

To find Eigen values

The characteristic equation of $A$ is $|A-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda\left[-\lambda(3-\lambda)+2\right] - 1(0) + 0(0) = 0$$

$$-\lambda\left[-3\lambda+\lambda^2+2\right] = 0$$

$$\lambda^2-3\lambda+2 = 0$$

$$\lambda^2-\lambda-2\lambda+2 = 0$$

$$\lambda(\lambda-1)-2(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, 2, 0$$

$\therefore$ The Eigen values are $1, 2, 0$

iii. Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic matrix of $A$ is $[A-\lambda I]$

$$[A-\lambda I] = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

To find Eigen values

The characteristic equation of $A$ is $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda\{(3-\lambda)(2-\lambda)-2\} - 2(2-\lambda-1) + 1(2-(3-\lambda)) = 0$$

$$2-\lambda[6-2\lambda-3\lambda+\lambda^2-2] - 2[-\lambda+1] + (2-3+\lambda) = 0$$

$$2-\lambda[\lambda^2-5\lambda+4] - 2(-\lambda+1) + (\lambda-1) = 0$$

$$2-\lambda(\lambda^2-5\lambda+4) + 2\lambda-2+\lambda-1 = 0$$

$$2\lambda^2-10\lambda+8-\lambda^3+5\lambda^2-4\lambda+2\lambda-2+\lambda-1 = 0$$

$$-\lambda^3+7\lambda^2-11\lambda+5 = 0$$

$$\lambda^3-7\lambda^2+11\lambda-5 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2-6\lambda+5) = 0$$

$\lambda = 1,\quad \lambda^2-6\lambda+5 = 0$

$$\lambda^2-\lambda-5\lambda+5 = 0$$

$$\lambda(\lambda-1)-5(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-5) = 0$$

$$\lambda = 1, 5$$

1. Find Eigen values and Eigen vectors of the following matrices

i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$ , ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} -2 & -2 & -2 \\ -1 & -4 & -3 \\ 0 & 2 & 0 \end{bmatrix}$ iv) $A = \begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & 0 \\ -9 & 0 & 0 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ vi) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

vii) $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ viii) $A = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

i) Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$

the characteristic matrix of $A$ is $[A - \lambda I]$.

$$[A - \lambda I] = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{bmatrix}$$

the characteristic equation of $A$ is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda\left[-\lambda(3-\lambda)+2\right]-1(0)+0(0)=0$$

$$-\lambda\left(-3\lambda+\lambda^2+2\right)=0$$

$\lambda=0$ $\quad \lambda^2-\lambda-2\lambda+2=0$

$$\lambda(\lambda-1)-2(\lambda-1)=0$$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda=1,2$$

$\therefore$ The eigen values are $\lambda=0,1,2$

To find Eigen vectors

Let $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the vector

Now, we consider the system i.e is $[A-\lambda I]X=0$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

case (i): put $\lambda=0$ in (1) we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y \\ z \\ -2y+3z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y=0,\ z=0$

$$-2y+3z=0$$

let $x=k$ (say)

$\therefore\ X_1=\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow k=1$

is the Eigen vector corresponding to the Eigen value $\lambda = 0$

case ii put $\lambda = 1$ in ①, we have

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x+y \\ -y+z \\ -2y+2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x+y=0, \quad -y+z=0, \quad -2y+2z=0$

$x=y \qquad y=z \qquad y=z$

let $y=k$ (say), then $x=k$ and $z=k$

$\therefore X_2 = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, for $k=1$ is the Eigen vector corresponding to the Eigen value $\lambda = 1$

case iii, put $\lambda = 2$ in ①, we have

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x+y \\ -2y+z \\ -2y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x+y=0 \qquad -2y+z=0 \qquad -2y+z=0$

$x=y/2 \qquad z=2y$

Let $y=k$ (say) then $x=k/2$ and $z=2k$

$$\therefore \quad X_3 = \begin{bmatrix} k/2 \\ k \\ 2k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$ for $k=2$ is the eigen value corresponding the eigen value $\lambda=2$

Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic matrix of A is $|A-\lambda I|=0$ i.e

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$6-\lambda\left[(3-\lambda)(3-\lambda)-1\right]+2\left[-2(3-\lambda)+2\right]+2\left[2-2(3-\lambda)\right]=0$

$6-\lambda\left[9-3\lambda-3\lambda+\lambda^2-1\right]+2\left[-6+2\lambda+2\right]+2\left[2-6+2\lambda\right]=0$

$6-\lambda\left[\lambda^2-6\lambda+9-1\right]+2(2\lambda-4)+2(2\lambda-4)=0$

$6-\lambda\left[\lambda^2-6\lambda+8\right]+4\lambda-8+4\lambda-8=0$

$6-\lambda\left[\lambda^2-6\lambda+8\right]+8\lambda-16=0$

$6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-\cancel{8\lambda}+\cancel{8\lambda}-16=0$

$-\lambda^3+12\lambda^2-36\lambda+32=0$

$\lambda^3-12\lambda^2+36\lambda-32=0$

$(\lambda-2)(\lambda^2-10\lambda+16)=0$

$\lambda=2, \quad \lambda^2-2\lambda-8\lambda+16=0$

$$(\lambda-2)(\lambda-2)(\lambda-8)=0$$
$$\lambda = 2, 2, 8$$

To find Eigen vectors

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the vector.

Now, we consider the system is $[A-\lambda I]X=0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (6-\lambda)x-2y+2z \\ -2x+(3-\lambda)y-z \\ 2x-y+(3-\lambda)z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

$$(6-\lambda)x-2y+2z=0$$
$$6x-\lambda x-2y+2z=0.$$

Case i. put $\lambda=2$ in (1), we have

$$4x-2y+2z=0 \Rightarrow 2x-y+z=0$$
$$-2x+y-z=0 \Rightarrow 2x-y+z=0$$
$$2x-y+z=0 \Rightarrow 2x-y+z=0$$

let $y=k_1$ and $z=k_2$ then $x=\dfrac{k_1-k_2}{2}$

$$\therefore x = \begin{bmatrix} \frac{k_1-k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix},$$

ie $x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ for $k_1 = 1$, $k_2 = 2$

is the Eigen vectors corresponding to the Eigen value $\lambda = 2$

case ii put $\lambda = 8$ in ①, we have

$$\begin{bmatrix} -2 & -2 & +2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x - 2y + 2z = 0 \Rightarrow x + y - z = 0 \rightarrow$ ①

$-2x - 5y - z = 0 \quad 2x + 5y + z = 0 \rightarrow$ ②

$2x - y - 5z = 0 \quad 2x - y - 5z = 0 \rightarrow$ ③

Now ① + ②

$\Rightarrow$

$x + y - z = 0$

$2x + 5y + z = 0$

$3x + 6y = 0$

$x + 2y = 0$

$x = -2y$

$y = -x/2$

① + ③

$x + y - z = 0$

$2x - y - 5z = 0$

$3x - 6z = 0$

$x - 2z = 0$

$x = 2z$

$z = x/2$

let $x = k$ (say), then $y = -k/2$ and $z = k/2$

$\therefore x_3 = \begin{bmatrix} +k \\ -k/2 \\ k/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ for $k = 2$, is the

Eigen value corresponding to Eigen vector when case $\lambda = 8$

(ii) Given that $A = \begin{bmatrix} -2 & -2 & -2 \\ -1 & -4 & -3 \\ 0 & 2 & 0 \end{bmatrix}$

The characteristic matrix of $A$ is $[A-\lambda I]$

$$[A-\lambda I] = \begin{bmatrix} -2-\lambda & -2 & -2 \\ -1 & -4-\lambda & -3 \\ 0 & 2 & -\lambda \end{bmatrix}$$

The characteristic equation of $A$ is $|A-\lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & -2 & -2 \\ -1 & -4-\lambda & -3 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$$

$$-2-\lambda\left((-4-\lambda)-\lambda+6\right)+2(\lambda)-2(-2)=0$$

$$-2-\lambda\left[4\lambda+\lambda^2+6\right]+2\lambda+4=0$$

$$-8\lambda-2\lambda^2-12-4\lambda^2-\lambda^3-6\lambda+2\lambda+4=0$$

$$-\lambda^3-6\lambda^2-12\lambda-8=0$$

$$\lambda^3+6\lambda^2+12\lambda+8=0$$

$$\lambda=-2,-2,-2$$

To find Eigen vectors

Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the vector

Now, we consider the system is $[A-\lambda I]X=0$

$$\begin{bmatrix} -2-\lambda & -2 & -2 \\ -1 & -4-\lambda & -3 \\ 0 & 2 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = -2$

$$\begin{bmatrix} -2+2 & -2 & -2 \\ -1 & -4+2 & -3 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -1 & -2 & -3 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2y-2z=0 \Rightarrow y+z=0 \Rightarrow y=-z$

$-x-2y-3z=0, \quad x+2y+3z=0 \Rightarrow x+z=0$

$2y+2z=0 \quad y+z=0 \Rightarrow y=-z$

Let $y=k$ (say) and $z=-k$, $x=k$

$$x = \begin{bmatrix} k \\ k \\ -k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ for } k=1, \text{ is the}$$

Eigen value of corresponding Eigen vectors when $\lambda = -1$

iv) Given that $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 & 3 \\ -1 & -\lambda & 0 \\ -3 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda\left[\lambda^2\right] - 1\left[\lambda\right] + 3\left[-3\lambda\right] = 0$$

$$-\lambda^3 - \lambda - 9\lambda = 0$$

$$-\lambda^3 - 10\lambda = 0$$

$$\lambda^3 + 10\lambda = 0$$

$$\lambda(\lambda^2 + 10) = 0$$

$$\lambda = 0,\ \lambda^2 + 10 = 0$$

$$\lambda^2 = -10$$

$$\lambda = \pm\sqrt{10}$$

$$\lambda = 3.16,\ \lambda = -3.16$$

$$\lambda = 0, 3, -3$$

To find Eigen vectors

Let x be the $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ vector

Now, we consider the system is $[A-\lambda I]x = 0$

$$\begin{bmatrix} -\lambda & 1 & 3 \\ -1 & -\lambda & 0 \\ -3 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = 0$

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + 3z = 0 \Rightarrow y = -3z$

$-x = 0$ $\Rightarrow x = 0$
$-3x = 0$

Let $z = k$ (say), $y = -3k$, $x = 0$

$x = \begin{bmatrix} 0 \\ -3k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$ for $k = 1$ is the Eigen vector corresponding to the Eigen value $\lambda = 0$

put $\lambda = \sqrt{10}i$

$$\Rightarrow \begin{bmatrix} -\sqrt{10}i & 1 & 3 \\ -1 & -\sqrt{10}i & 0 \\ -3 & 0 & -\sqrt{10}i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-\sqrt{10}ix + y + 3z = 0$

$-x - \sqrt{10}iy = 0$ $\quad y = \frac{-x}{\sqrt{10}i}$

$-3x - \sqrt{10}iz = 0 \Rightarrow z = \frac{-3x}{\sqrt{10}i}$

Let $x = k$ then $y = \frac{-k}{\sqrt{10}i}$ and $z = \frac{-3k}{\sqrt{10}i}$

$$x = \begin{bmatrix} k \\ \frac{-k}{\sqrt{10i}} \\ \frac{-3k}{\sqrt{10i}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1}{\sqrt{10i}} \\ \frac{-3}{\sqrt{10i}} \end{bmatrix}$$ is the eigen vector

of the given corresponding eigen value $\lambda = \sqrt{10i}$

put $\lambda = -\sqrt{10i}$

$$\begin{bmatrix} \sqrt{10i} & 1 & 3 \\ -1 & \sqrt{10i} & 0 \\ -3 & 0 & \sqrt{10i} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sqrt{10i}\,x + y + 3z = 0$$
$$-x + \sqrt{10i}\,y = 0 \quad \Rightarrow y = \frac{1}{\sqrt{10i}}$$
$$-3x + \sqrt{10i}\,z = 0 \quad z = \frac{3}{\sqrt{10i}}$$

let $x = k$, $y = \frac{1}{\sqrt{10i}}$, $z = \frac{3}{\sqrt{10i}}$

$$x = \begin{bmatrix} k \\ \frac{1}{\sqrt{10i}} \\ \frac{3}{\sqrt{10i}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{10i}} \\ \frac{3}{\sqrt{10i}} \end{bmatrix}$$ for $k = 1$ is the

Eigen vector of the corresponding eigen value

$\lambda = -10\sqrt{i}$

Given that $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation of $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$1-\lambda\left[(2-\lambda)(3-\lambda)-2\right]+1\left[2-2(2-\lambda)\right]=0$

$1-\lambda\left[6-2\lambda-3\lambda+\lambda^2-2\right]+1\left[2-4+2\lambda\right]=0$

$1-\lambda\left[\lambda^2-5\lambda+4\right]+1\left[2\lambda-2\right]=0$

$\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+2\lambda-2=0$

$-\lambda^3+6\lambda^2-7\lambda+2=0$

$\lambda^3-6\lambda^2+7\lambda-2=0$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 7 & -2 \\ & 0 & 1 & -5 & 2 \\ \hline & 1 & -5 & 2 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2-5\lambda+2)=0$

$\lambda=1,\quad \lambda = \frac{5\pm\sqrt{25-8}}{2}$

$\lambda = \frac{5\pm\sqrt{17}}{2}$

To find Eigen vectors

The characteristic equation of $[A-\lambda I]X=0$

$$\begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = \frac{5+\sqrt{17}}{2}$

$$\left[\frac{-3-\sqrt{17}}{2}\right]x + z = 0 \Rightarrow x = \frac{-2z}{3+\sqrt{17}}$$

$$x + \left(\frac{-1-\sqrt{17}}{2}\right)y + z = 0 \rightarrow (1)$$

$$2x + 2y + \left[\frac{1-\sqrt{17}}{2}\right]z = 0 \rightarrow (2)$$

$$\Rightarrow 2x - 2\left[\frac{1+\sqrt{17}}{2}\right]y + 2z = 0$$

$$2x + 2y + \frac{1-\sqrt{17}}{2} = 0$$

$$(-3-\sqrt{17})y + \left(\frac{3+\sqrt{17}}{2}\right)z = 0$$

$$y = -\frac{(3+\sqrt{17})}{2(-3+\sqrt{17})}z$$

$$y = \frac{+z}{2}$$

Let $z = k$ (say), $y = \frac{-k}{2}$ and $x = \frac{2k}{3+\sqrt{17}}$

$$x_2 = \begin{bmatrix} \frac{2k}{3+\sqrt{17}} \\ -k/2 \\ k \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{17}} \\ -1/2 \\ 1 \end{bmatrix} \text{ for } k = 1 \text{ is}$$

the eigen vector corresponding to the eigen values where $\lambda = \frac{2k}{-3+\sqrt{17}}$

put $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z = 0$, $x + y + z = 0 \Rightarrow x = -y$, $x + y + 2z = 0$
$\Rightarrow z = 0$

$X = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $k=1$ for the Eigen vectors corresponding to the eigen value $\lambda = 1$

put $\lambda = \frac{5-\sqrt{17}}{2}$ in (1), we have

$$\begin{bmatrix} -\frac{3+\sqrt{17}}{2} & 0 & 1 \\ 0 & -\frac{1+\sqrt{17}}{2} & 1 \\ 2 & 2 & \frac{1+\sqrt{17}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{-3+\sqrt{17}}{2}\right)x + z = 0 \Rightarrow x = \frac{2z}{3-\sqrt{17}}$$

$$x + \left(\frac{-1+\sqrt{17}}{2}\right)y + z = 0 \rightarrow (1)$$

$$2x + 2y + \left(\frac{1+\sqrt{17}}{2}\right)z = 0 \rightarrow (2)$$

(1) $\times 2$ − (2) $\times 1$ $\Rightarrow$ $\cancel{2x} + 2\left(\frac{-1+\sqrt{17}}{2}\right)y + 2z = 0$

$$\cancel{2x} + 2y + \left(\frac{1+\sqrt{17}}{2}\right)z = 0$$

$$(-3+\sqrt{17})y + \left(\frac{3\pm\sqrt{17}}{2}\right)z = 0$$

$$(-3+\sqrt{17})y = -\left(\frac{3+\sqrt{17}}{2}\right)z$$

$$y = -\left[\frac{-3+\sqrt{17}}{(-3+\sqrt{17})2}\right]z$$

$$y = \frac{-z}{2}$$

Let $z = k$ (say), $x = \frac{2k}{3-\sqrt{17}}$, $y = \frac{k}{2}$

$$x = \begin{bmatrix} \frac{2k}{3-\sqrt{17}} \\ k/2 \\ k \end{bmatrix} = \begin{bmatrix} \frac{2}{3-\sqrt{17}} \\ 1/2 \\ 1 \end{bmatrix}$$ for $k=1$ is the eigen vector corresponding to the eigen value when

$$\lambda = \frac{5-\sqrt{17}}{2}$$

vii) Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic matrix of $A$ is $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[(1-\lambda)-3\right] + 3\left[1-3(5-\lambda)\right] = 0$$

$$1-\lambda\left[5-5\lambda-\lambda+\lambda^2-1\right] - \left[-2-\lambda\right] + 3\left[1-15+3\lambda\right] = 0$$

$$1-\lambda\left[\lambda^2-6\lambda+4\right] + 2+\lambda + 3\left[3\lambda-14\right] = 0$$

$$1-\lambda\left[\lambda^2-6\lambda+4\right] + 2+\lambda+9\lambda-42 = 0$$

$$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+2+\lambda+9\lambda-42 = 0$$

$$-\lambda^3+7\lambda^2 \cancel{+0\lambda} -36 = 0$$

$$\lambda^3-7\lambda^2 \cancel{+0\lambda} +36 = 0$$

$$\begin{array}{r|rrrr} & 1 & -7 & 0 & 36 \\ -2 & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$(\lambda+2)(\lambda^2-9\lambda+18) = 0$$

$$\lambda = -2, \quad \lambda^2-9\lambda+18 = 0$$

$$(\lambda-3)(\lambda-6) = 0$$

$$\lambda = 3, \lambda = 6$$

case-2 put $\lambda = -2$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x+y+3z=0$ $\quad x+7y+z=0 \rightarrow (1)$ $\quad 3x+y+3z=0 \rightarrow (2)$

$y=-3x-3z$ $\quad \Rightarrow x=-7y-z$ $\quad y=-3x-3z = -3(x+z)$

let $z=(k)$ say (1) and (2) solve

$$\begin{array}{r} 3x+21y+3z=0 \\ 3x+y+3z=0 \\ \hline 20y=0 \\ y=0 \end{array}$$

$\Rightarrow 3x+3z=0$

$3(x+z)=0$

$x+z=0$

$x=-z=-k$

let $z=k$ and we have

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ at } k=1$$

put $\lambda=3$ in (1) we get

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x+y+3z=0 \rightarrow (1)$ $\quad$ solve (1) and (2)

$x+2y+z=0 \rightarrow (2)$

$3x+y-2z=0 \rightarrow (3)$

$$\begin{array}{r} -4x+2y+6z \\ x+2y+z \\ \hline -5x+5z=0 \\ x=z \end{array}$$

(1) and (2) $\Rightarrow$ $-x+y+z=0$

$x+y-2z=0$

$2y-z=0$

$-2y+2z=0$

$y=z$

let $x=k$ (say)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ -k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ at } k=1$$

put $\lambda=6$ in (1)

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x+y+3z=0 \rightarrow (1)$

$x-y+z=0 \rightarrow (2)$

$3x+y-5z=0 \rightarrow (3)$

(1) and (3)

$-5x+y+3z=0$

$3x+y-5z=0$

$-8x+8z=0$

$x=z$

$3x+y-5z=0$

$y-2z=0$

$y=2z$

(2) and (3) $\Rightarrow$ $x-y+z=0$

$3x+y-5z=0$

$4x-4z=0$

$z=x$

let $z=k$ (say), $y=2z$, $x=z$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} k \\ 2k \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ for } k=1$$

vii)

Given $A=\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[-\lambda(1-\lambda)\right] - 1\left[-\lambda\right] = 0$$

$$1-\lambda\left[-\lambda+\lambda^2\right] + \lambda = 0$$

$$1-\lambda(\lambda^2-\lambda) + \lambda = 0$$

$$\lambda^2 - \lambda - \lambda^3 + \lambda^2 = 0$$

$$-\lambda^3 + 2\lambda^2 - \lambda = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda = 0 \quad \lambda^2 - 2\lambda + 1 = 0$$

$$\lambda^2 - \lambda - \lambda + 1 = 0$$

$$\lambda(\lambda-1) - 1(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-1) = 0$$

$$\lambda = 1, 1$$

To find Eigen vectors

case-1, put $\lambda = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y = 0 \quad \Rightarrow x = 0$$
$$y = 0$$

let $z = k$ (say), $x = 0$, $y = 0$

$$x = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ for, $k=1$ is the eigen vector

corresponding to the eigen value $\lambda = 0$

case-ii) put $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0$

$x = 0$

$-z = 0$

$z = 0$

$$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ are the eigen vectors are corresponding eigen values for $\lambda = 1$

Q.iii) Given $A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

the characteristic matrix of $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & -2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$2-\lambda\left[(3-\lambda)^2-1\right] + 2\left[-2(3-\lambda)+2\right] - 2\left[2-2(3-\lambda)\right] = 0$$

$2-\lambda\,[9+\lambda^2-6\lambda]+2[-6+2\lambda+2]-2[2-(6+2\lambda)]=0$

$2-\lambda\,[\lambda^2-6\lambda+9]+2[2\lambda-4]-2[2\lambda-4]=0$

$2\lambda^2-12\lambda+18-\lambda^3+6\lambda^2-9\lambda+4\lambda-8-4\lambda+8=0$

$-\lambda^3+8\lambda^2-21\lambda+18=0$

$\lambda^3-8\lambda^2+21\lambda-18=0$

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 21 & -18 \\ & 0 & 2 & -12 & 18 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

$(\lambda-2)\{\lambda^2-6\lambda+9\}=0$

$(\lambda-2)\{\lambda^2-3\lambda-3\lambda+9\}=0$

$\lambda(\lambda-3)-3(\lambda-3)=0$

$(\lambda-3)(\lambda-3)=0$

$\lambda=2,\quad \lambda=3,3$

case-1, put $\lambda=2$

to find Eigen values

$$\begin{bmatrix} 2-\lambda & -2 & -2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2y-2z=0 \Rightarrow y=-z$

$-2x+y-z=0 \Rightarrow 2x-y+z=0$

$2x-y+z=0 \qquad 2x=0$

$x=0$

let $y=k$ (say), $z=-k$

$$x=\begin{bmatrix} 0 \\ k \\ -k \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$ for $k=1$ is the eigen vector

corresponding to the eigen value $\lambda=2$

case - III : put $\lambda = -3$

$$\begin{bmatrix} -1 & -2 & -2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x - 2y - 2z = 0$.

$-2x - z = 0 \Rightarrow z = -2x$

$2x - y = 0$

$\Rightarrow y = 2x$

let $x = k$, $y = 2k$, $z = -2k$.

$$X = \begin{bmatrix} k \\ 2k \\ -2k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$ for $k = 1$ is the eigen vector for the corresponding eigen value for $\lambda = 3$.