## Mean Value Theorem:

(1) Rolle's theorem

(2) Lagrange's theorem

(3) Cauchy's theorem

(4) Taylor's theorem

(5) Maclaurin's theorem

### (1) Rolle's Theorem:

* Verify the Rolle's theorem for the following functions.

(1) $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

(2) $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$

(3) $f(x) = x(x+3)\,e^{-x/2}$ in $[-3,0]$

(4) $f(x) = |x|$ in $[-1,1]$

(5) $f(x) = \frac{1}{x^2}$ in $[-1,1]$

(6) $f(x) = \sin x$ in $[-\pi,\pi]$

(7) $f(x) = \tan x$ in $[0,\pi]$

(8) $f(x) = \sec x$ in $[0,2\pi]$

(9) $f(x) = e^x.\sin x$ $[0,\pi]$

(10) $f(x) = (x-a)^m.(x-b)^n$ in $[a,b]$

Rolle's Theorem:

Let $f(x)$ be a function of $x$ defined in $(a,b)$

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is derivable in $(a,b)$

(iii) $f(a) = f(b)$

then $\exists$ a $c \in (a,b) \ni f'(c) = 0$.

(i) $f(x) = \frac{\sin x}{e^x}$ is continuous for all 'x'.

$f(x)$ is continuous in $[0, \pi]$

$\Rightarrow f'(x) = \frac{e^x \cdot \cos x - \sin x \cdot e^x}{(e^x)^2}$

$= \frac{e^x (\cos x - \sin x)}{(e^x)^{\cancel{2}}}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$ is exist $\forall x$.

$\Rightarrow f'(x)$ is exist in the interval $[0, \pi]$

$\therefore f(x)$ is derivable in $(0, \pi)$.

$\Rightarrow$ We have to show that $f(0) = f(\pi)$

$f(0) = \frac{\cancel{\cos 0} + \sin 0}{e^0} = \frac{\cancel{1} \, 0}{1} = 0.$

$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$

Then $\exists$ exist $c \in (a, b) \ni f'(c) = 0$.

$f(x) = \frac{\sin x}{e^x}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$

$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$

$\sin c = \cos c$

$\tan c = 1$

$c = \tan^{-1}(1)$

$\boxed{c = \pi/4} \in [0, \pi]$

② $f(x) = \log\left(\frac{x^2 + ab}{x(a+b)}\right)$ in $[a, b]$ $a > 0, b > 0$.

$f(x) = \log(x^2 + ab) - \log x(a+b)$ is continuous $\forall$ 'x'

except at $x = 0 \notin [a, b]$.

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x) = \log(x^2 + ab) - \log x + \log(a+b)$

$= \frac{1}{\cdot} (2x) - \frac{1}{\cdot} - 0$

$f(x)$ is derivable in $(a,b)$

$f(a) = \log(a^2+ab) - \log a(a+b)$

$= \log(a^2+ab) - \log(a^2+ab)$

$= 0.$

$f(b) = \log(b^2+ab) - \log b(a+b)$

$= \log(b^2+ab) - \log(ab+b^2)$

$= 0$

$f(a) = f(b).$

$\exists$ a $c \in (a,b) \ni f'(c) = 0$

We have $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$

$2c^2 - c^2 - ab = 0$

$c^2 = ab$

$c = \pm\sqrt{ab}$

$c = \sqrt{ab}$ (or) $-\sqrt{ab}$

$\boxed{c = \sqrt{ab}} \in (a,b)$

③ $f(x) = x\cdot(x+3)\,e^{-x/2}$ in $[-3,0]$

sol: $f(x)$ is continuous $\forall x$.

(i) $f(x)$ is continuous in $[-3,0]$

(ii) $f(x) = (x^2+3x)\,e^{-x/2}$

$f'(x) = (x^2+3x)\,e^{-x/2}\cdot\frac{-1}{2} + e^{-x/2}(2x+3)$

$= -\frac{(x^2+3x)}{2}e^{-x/2} + (2x+3)\,e^{-x/2}$

$= e^{-x/2}\left[(2x+3) - \frac{(x^2+3x)}{2}\right]$

$= e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$

$= e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$

$= e^{-x/2}\,(-x^2+x+6)$

We have to show that $f(-3) = f(0)$

$$f(-3) = -3(-3+3)e^{-3/2}$$
$$= -3(0)\,e^{-3/2}$$
$$= 0$$
$$f(0) = 0\,(0+3)e^{-0/2}$$
$$= 0.$$

$$\boxed{\therefore f(-3) = f(0)}$$

Then $\exists$ a $c \in (a,b) \ni f'(c)=0.$

$$f'(x) = \frac{e^{-x/2}}{2}(-x^2+x+6)$$
$$f'(c) = \frac{e^{-c/2}}{2}(-c^2+c+6) = 0$$
$$(-c^2+c+6)\,e^{-c/2} = 0.$$
$$e^{-c/2} = 0 \quad \text{and} \quad -c^2+c+6=0$$
$$c^2-c-6=0$$
$$c^2-3c+2c-6=0$$
$$c(c-3)+2(c-3)=0$$
$$(c-3)(c+2)=0$$
$$c=3,\ \boxed{c=-2} \in (-3,0)$$

(4) $f(x) = |x|$ in $[-1,1]$

Sol: We know that $|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x>0 \end{cases}$

(i) $f(x)=|x|$ is continuous $\forall x$

$\Rightarrow$ $|x|$ is continuous in $[-1,1]$

(ii) The derivative of $|x|$ does not exist.

Because,

L.H.D $\lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to0^-}\left(\frac{-x-0}{x}\right)$$

$= \lim_{x \to 0^-} (-1) = \underline{-1}$

**R.H.D** $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0^+} \frac{x - 0}{x - 0}$

$= \lim_{x \to 0^+} \frac{\cancel{x}}{\cancel{x}}$

$= \lim_{x \to 0^+} (1) = 1$

$$\boxed{\therefore L.H.D \neq R.H.D}$$

Hence Rolle's theorem is not verified.

(10). $f(x) = (x-a)^m . (x-b)^n$ in $[a,b]$.

**sol:** $f(x)$ is exist $\forall x$.

$\rightarrow f(x)$ is continuous in $[a,b]$

$\rightarrow f(x) = (x-a)^m . (x-b)^n$

$f'(x) = (x-a)^m . n(x-b)^{n-1}(1-0) + (x-b)^n m(x-a)^{m-1}(1-0)$

$= n . (x-b)^{n-1} . (x-a)^m + m . (x-a)^{m-1} . (x-b)^n$

$= n . (x-b)^n . (x-b)^{-1} (x-a)^m + m . (x-a)^m . (x-a)^{-1} (x-b)^n$

$= (x-a)^m (x-b)^n [n . (x-b)^{-1} + m . (x-a)^{-1}]$

$= (x-a)^m . (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a}\right)$

$= (x-a)^n . (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)}\right)$

$f(x)$ is exist $\forall x$. except at $x=a$ and $x=b$. $\notin (a,b)$

$\therefore f(x)$ is exist in $(a,b)$.

$\therefore f(x)$ is derivable in $(a,b)$

$f(a) = (a-a)^m . (a-b)^n$

$= (0)^m . (a-b)^n$

$= 0.$

$= (b-a)^m (0)$

$= 0.$

$f(a) = f(b)$

Then $\exists$ a $c \in (a,b) \ni f'(c) = 0$

$f(x) = (x-a)^m \cdot (x-b)^n$

$$f'(x) = (x-a)^m \cdot (x-b)^n \left( \frac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right)$$

$$f'(c) = (c-a)^m \cdot (c-b)^n \left[ \frac{n(c-a) + m(c-b)}{(c-a)(c-b)} \right] = 0.$$

$$= (c-a)^m (c-b)^n \left[ \frac{nc - na + mc - mb}{(c-a)(c-b)} \right] = 0$$

$$= (c-a)^m \cdot (c-b)^n \left[ \frac{(m+n)c - (na+mb)}{(c-a)(c-b)} \right] = 0$$

$(c-a)^m = 0$, $(c-b)^n = 0$ and $(m+n)c - na - mb = 0.$

$\Rightarrow (m+n)c = na + mb.$

$$c = \frac{na+mb}{m+n} \in (a,b)$$

⑤ $f(x) = \frac{1}{x^2}$ in $[-1, 1]$.

sol: $f(x) = \frac{1}{x^2}$

~~f(x) is does not exist at x~~.

$\Rightarrow$ f(x) is does not continuous in $[-1, 1]$ except at $x=0 \in [-1,1]$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$.

$\Rightarrow$ f(x) is does not derivable in $(-1,1)$ except at $t=0$.

But $t=0 \in (-1,1)$

$\therefore$ Rolles theorem can not applied.

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[-\pi,\pi]$

$f'(x) = \cos x$.

$\Rightarrow f'(x)$ is derivable in $(-\pi,\pi)$

We have to show that $f(-\pi) = f(\pi)$

$f(-\pi) = \sin(-\pi) = -\sin\pi = 0$

$f(\pi) = \sin\pi = 0$

$$f(-\pi) = f(\pi)$$

then $\exists$ a $c \in (-\pi,\pi) \ni f'(c) = 0$

$$f(x) = \sin x \Rightarrow f'(x) = \cos x$$

$$f'(c) = 0$$

$$\cos c = 0$$

$$c = \cos^{-1}(0)$$

$$c = \cos^{-1}(\cos \pi/2)$$

$$\boxed{c = \pi/2} \in (-\pi/2, \pi/2)$$

(7) $f(x) = \tan x$ in $[0,\pi]$.

$f(x) = \tan x$

$f(x)$ is exist $\forall x$. except at $x = \pi/2 \in (0,\pi)$.

$\therefore f(x)$ is does not continuous in $[0,\pi]$

$f'(x) = \sec^2 x$

$f'(x)$ is does not exist $\forall x$. Except at $x = 0 \in (0,\pi)$

Rolle's theorem can not be verified.

$f(x) = \sec x.$

$f(x)$ is exist $\forall x$. except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f(x)$ is continuous in $[0, 2\pi]$ except at $x = \pi/2 \in (0, 2\pi)$

$f'(x) = \sec x . \tan x.$

$f'(x)$ is exist $\forall x$. Except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f'(x)$ is derivable in $(0, 2\pi)$ Except at $x = \pi/2$.

$\Rightarrow f(0) = f(2\pi)$ (we have to S.T)

$f(0) = \sec 0° = 1$

$f(2\pi) = \sec 2\pi = 1$

$$\boxed{f(0) = f(2\pi)}$$

Then $\exists$ a $c \in (0, 2\pi) \ni f'(c) = 0$

$\sec c . \tan c = 0$

$\tan c = 0$ and $\sec c = 0$

$c = \tan^{-1}(0)$ $\quad c = \sec^{-1}(0)$

$c = \tan^{-1}(\tan 0)$ $\quad$ ~~sec~~ sec

$\boxed{c = 0}$

(9) $f(x) = e^x . \sin x$ in $[0, \pi]$.

$f(x) = e^x \sin x$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, \pi]$

$f'(x) = e^x \cos x + \sin x \, e^x$

$= e^x (\cos x + \sin x)$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, \pi)$.

we have to show that

$\Rightarrow f(0) = f(\pi)$

$f(0) = e^0 . \sin 0 = 0$

$f(\pi) = e^{\pi} \sin \pi = 0$

$$\boxed{f(0) = f(\pi)}$$

$e^c(\cos c + \sin c) = 0$

$\cos c + \sin c = 0$

$\sin c = -\cos c$

$\frac{\sin c}{\cos c} = -1$

$\tan c = -1$

$c = \tan^{-1}(-1)$

$c = \tan^{-1}(\tan 3\pi/4)$

$\boxed{c = 3\pi/4} \in (0, \pi)$.

Saturday 19/10/19

## Lagrange's Mean Value Theorem:

Let $f(x)$ be a function of $x$. if

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is derivable in $(a,b)$

(iii) Then $\exists$ a $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

① $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

② $f(x) = \log x$ $[1, e]$

③ $f(x) = e^x$ $[0,1]$

④ $f(x) = \frac{1}{x}$ $[1,4]$

⑤ $f(x) = x - x^3$ $[-2,1]$

⑥ if $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

⑦ By using LMVT, show that $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$ and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}(4/3) < \frac{\pi}{4} + \frac{1}{6}$

⑧ By using L.M.V.T, $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}(3/5) > \pi/3 - 1/8$.

⑨ $x \le \sin^{-1}x \le \frac{x}{1-x^2}$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, 1/2]$

$f'(x) = (x^2 - x)(x-2)$

$= x^3 - 2x^2 - x^2 + 2x$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, 1/2)$.

$\Rightarrow$ Then $\exists$ a $c \in (0, 1/2) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f(1/2) = 1/2(1/2 - 1)(1/2 - 2)$

$= \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)$

$= 3/8$

$$3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$$3c^2 - 6c + 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}}{1}$$

$$3c^2 - 6c + 2 - 3/4 = 0$$

$$3c^2 - 6c + 5/4 = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2(3)}$$

$$= \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{6}{6} + \frac{\sqrt{21}}{6}, \quad \frac{6}{6} - \frac{\sqrt{21}}{6}$$

$$= \cancel{1 + \frac{\sqrt{21}}{6}}, \quad 1 - \frac{\sqrt{21}}{6}$$

$$\boxed{c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)}$$

③ $f(x) = \log x$ in $[1, e]$

$f(x)$ is continuous $\forall x$. except at $x=0 \notin (1,e)$

$\Rightarrow f(x)$ is continuous in $[1, e]$

$f'(x) = \frac{1}{x}$.

$f'(x)$ is exist $\forall x$. except at $x=0 \notin (1,e)$

$\Rightarrow f'(x)$ is derivable in $(1, e)$.

$$\frac{1}{c} = \frac{\log e - \log 1}{e-1}$$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$ $e = 2.7 \ldots$

$$\Rightarrow \boxed{c = e-1} \in (1, e)$$ $e-1 = 2.7-1 = 1.7 \ldots$

③ $f(x) = e^x$ in $[0,1]$

$f(x)$ is contincious $\forall x$.

$\Rightarrow f(x)$ is contincious in $[0,1]$

$f'(x) = e^x$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0,1)$.

Then $\exists$ a $c \in (0,1) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$e^c = \frac{e^1 - e^0}{1-0}$$

$$e^c = \frac{e-1}{1}$$

$$e^c = e-1$$

$$\boxed{c = \log(e-1) \in (0,1)}$$

④ $f(x) = \frac{1}{x}$ in $[1,4]$

$f(x)$ is contincuous $\forall x$. except at $x=0 \notin (1,4)$

$\Rightarrow f(x)$ is contincuous in $[1,4]$

$f'(x) = \frac{-1}{x^2}$ is exist $\forall x$. except at $x=0 \notin (1,4)$

$\Rightarrow f(x)$ is derivable in $(1,4)$

Then $\exists$ a $c \in (1,4) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{c^2} = \frac{1/4 - 1}{4-1}$$

$$\frac{-1}{c^2} = \frac{\frac{1-4}{4}}{3}$$

$$\frac{-1}{c^2} = \frac{-\cancel{3}/4}{\cancel{3}}$$

$c^2 = 4$

$c = \sqrt{4} \Rightarrow c = \pm 2$

$\boxed{c = 2 \in (1,4)}$

⑤ $f(x) = x - x^3$ in $[-2,1]$

$f(x) = x - x^3$ is continuous $\forall x$.

$\rightarrow f(x)$ is continuous in $[-2,1]$

$f'(x) = 1 - 3x^2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(-2,1)$

$\exists$ a $c \in (-2,1) \ni f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$1 - 3c^2 = \frac{[1-(1)^3] - [-2-(-2)^3]}{1-(-2)}$$

$$1 - 3c^2 = \frac{(1-1) - [-2-(-8)]}{1+2}$$

$$1 - 3c^2 = \frac{0 - (-2+8)}{3}$$

$3 - 9c^2 = -6$


$\cancel{9}c^2 = \cancel{9}$

$c^2 = 1$

$c = \pm 1$

$\boxed{c = -1 \in (-2,1)}$

⑥ If $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$.

Sol:- Let us take $f(x) = \log(1+x)$

Since $f(x) = \log(1+x)$ is continuous $\forall x > 0$.

and $f(x)$ is derivable $\forall x > 0$.

By using L.M.V.T,

$\exists$ a $c \in (0,x) \ni f'(c) = \dfrac{f(x) - f(0)}{x-0}$

$$1+x$$

$$\frac{1}{1+c} = \frac{\log(1+x) - \log(1+0)}{x-0}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x} \rightarrow ①$$

Given that $0 < c < x$

$$1 < c+1 < x+1$$

$$1 < \frac{1}{c+1} < \frac{1}{x+1}$$

$$1 < \frac{\log(1+x)}{x} < \frac{1}{x+1}$$

$$x < \log(1+x) < \frac{x}{1+x}$$

⑦ Let $f(x) = \tan^{-1}x$ in $[a,b]$

Given, ~~f(x) is continuous ∀ x. except~~

$f(x)$ is continuous in $[a,b]$

and $f(x)$ is derivable in $(a,b)$.

By using L.M.V.T,

then $\exists$ a $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$$

Given that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}(b) - \tan^{-1}(a) > \frac{b-a}{1+b^2}$$

$$\frac{4/3-1}{1+(1)^2} > \tan^{-1}(4/3) - \tan^{-1}(1) > \frac{1/3}{1+(4/3)^2}$$

$$\frac{1/3}{2} > \tan^{-1}(4/3) - \pi/4 > \frac{1/\cancel{3}}{\frac{25}{\cancel{9}3}}$$

$$\frac{1}{6} > \tan^{-1}(4/3) - \pi/4 > \frac{3}{25}$$

$$\frac{1}{6}+\frac{\pi}{4} > \tan^{-1}(4/3) > \frac{3}{25}+\frac{\pi}{4}$$

$$\frac{\pi}{4}+\frac{3}{25} > \tan^{-1}(4/3) > \frac{\pi}{4}+\frac{1}{6}.$$

(8) Let $f(x) = \cos^{-1}x$. in $[a,b]$

Given that, $f(x)$ is continuous in $[a,b]$

and $f(x)$ is derivable in $(a,b)$

By using L.M.V.T,

$$\exists \text{ a } c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(x) = \cos^{-1}x \rightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b) - \cos^{-1}(a)}{b-a}$$

We know that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

Wrong

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\cos^{-1}(a) - \cos^{-1}(b)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} \geq -\{\cos^{-1}(a) - \cos^{-1}(b)\} \geq \frac{-(b-a)}{\sqrt{1-b^2}}$$

Given that $a = 3/5$, $b =$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}(b) - \cos^{-1}(a) > \frac{a-b}{\sqrt{1-b^2}}$$

Given that

Let $f(x), g(x)$ are a functions of $x$.

(i) $f(x), g(x)$ are continuous in $[a,b]$

(ii) $f(x), g(x)$ are derivable in $(a,b)$

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

verify the cauchy's Mean Value Theorem for the following functions.

① $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ $0<a<b$.

② $f(x) = \sin x$, $g(x) = \cos x$ in $[0,\pi/2]$

③ $f(x) = e^{x}$, $g(x) = e^{-x}$ in $[a,b]$

④ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ if $0<a<b$

⑤ $f(x) = x^2+2$, $g(x) = x^3-1$ in $[1,2]$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1,e]$

⑦ $f(x) = x^3$, $g(x) = 2-x$ in $[0,9]$

① $f(x)$ is always continuous $\forall x$.

$g(x)$ is continuous $\forall x$. except at $x=0 \notin (a,b)$ $[0<a<b]$

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2} x^{-3/2}$

$f'(x)$ is exist $\forall x$. except at $x=0 \notin (a,b)$

$f(x)$ is derivable in $(a,b)$.

$g'(x)$ is exist $\forall x$. Except at $x=0 \in (a,b)$

$g(x)$ is derivable in $(a,b)$

$\rightarrow f(x), g(x)$ are derivable in $(a,b)$

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2\sqrt{c}\,c}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\cancel{+}\, c = \frac{\cancel{\sqrt{b}-\sqrt{a}}}{\cancel{+}(\cancel{\sqrt{b}-\sqrt{a}})}\sqrt{ab}$$

$$\boxed{c = \sqrt{ab} \in (a,b)}$$

② $f(x) = \sin x$ , $g(x) = \cos x$ $[0, \pi/2]$

$f(x), g(x)$ are always continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0, \pi/2]$.

$f'(x) = \cos x$, $g'(x) = -\sin x$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in $(0, \pi/2)$

Then $\exists$ a $c \in (0, \pi/2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\cos c}{-\sin c} = \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0}$$

$$\frac{\cos c}{-\sin c} = \frac{1-0}{0-1}$$

$$\frac{\cos c}{\cancel{+}\sin c} = \frac{1}{\cancel{+}1}$$

$$\cos c = \sin c$$

$$\frac{\sin c}{\cos c} = 1$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

$$\boxed{c = \pi/4 \in (0, \pi/2)}$$

③ $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$

$f'(x) = e^{x}$ $\quad g'(x) = -e$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in $(a, b)$

Then $\exists$ a $c \in (a, b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{e^{c}}{-e^{-c}} = \frac{e^{b} - e^{a}}{e^{-b} - e^{-x}}$$

$$-e^{c} \cdot e^{c} = \frac{e^{b} - e^{a}}{\frac{1}{e^{b}} - \frac{1}{e^{a}}}$$

$$-(e^{c})^{2} = \frac{e^{b} - e^{a}}{\frac{e^{a} - e^{b}}{e^{a} \cdot e^{b}}}$$

$$\cancel{-} e^{2c} = \frac{\cancel{e^{b} - e^{a}}}{\cancel{-}(\cancel{e^{b} - e^{a}})} e^{a} e^{b}$$

$$e^{2c} = e^{a} e^{b}$$

$$e^{2c} = e^{a+b}$$

$$2c = a + b$$

$$\boxed{c = \frac{a+b}{2} \in (a, b)}$$

(4) $f(x) = \frac{1}{x^{2}}$, $g(x) = \frac{1}{x}$ in $[a, b]$ $(0 < a < b.)$

$f(x)$ is continuous $\forall x$. Except at $x = 0 \notin (a, b)$

$g(x)$ is continuous $\forall x$. Except at $x = 0 \notin (a, b)$

$\Rightarrow f(x), g(x)$ are continuous in $[a, b]$

$f'(x) = -2x^{-3}$ $\quad g'(x) = \cancel{\log x} \frac{-1}{x^{2}}$

$f'(x)$ is exist $\forall x$. Except at $x = 0$.

$g'(x)$ is exist $\forall x$. Except at $x = 0$

$\Rightarrow f(x), g(x)$ are derivable in $(a, b)$

Then $\exists$ a $c \in (a, b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{\frac{\cancel{-}2}{c^{3}}}{\cancel{-}1/c^{2}} = \frac{1/b^{2} - 1/a^{2}}{1/b - 1/a}$$

$$\frac{2}{c} = \frac{(a+b)(a-b)}{(ab)^2} \times \frac{ab}{a-b}$$

$$\boxed{c = \frac{2ab}{a+b} \in (a,b)}$$

(5) $f(x) = x^2+2$ , $g(x) = x^3-1$. in $[1,2]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow$ $f(x)$, $g(x)$ are continuous in $[1,2]$

$f'(x) = 2x$ , $g'(x) = 3x^2$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow$ $f(x)$, $g(x)$ are derivable in $(1,2)$

Then $\exists$ a $c \in (1,2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{2\cancel{c}}{3c^2} = \frac{[(2)^2+2]-[(1)^2+2]}{[(2)^3-1]-[(1)^3-1]}$$

$$\frac{2}{3c} = \frac{(4+2)-(1+2)}{(8-1)-(1-1)}$$

$$\frac{2}{3c} = \frac{6-3}{7-0}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$\boxed{c = \frac{14}{9} \in (1,2)}$$

(6) $f(x) = \log x$ , $g(x) = \frac{1}{x}$ in $[1,e]$

$f(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$g(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$\Rightarrow$ $f(x)$, $g(x)$ are continuous in $[1,e]$

$f'(x)$ is exist $\forall x$ except at $x=0 \notin (1,e)$

$g'(x)$ is exist $\forall x$. except at $x=0 \notin (1,e)$

$\Rightarrow f(x), g(x)$ is derivable in $(1,e)$.

Then $\exists$ a $c \in (1,e) \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

$$\frac{1/\cancel{c}}{-1/c^{\cancel{2}}} = \frac{\log e - \log 1}{1/e - 1/1}$$

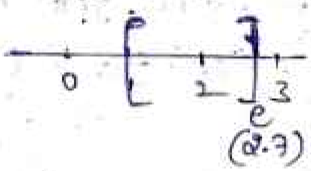
$$-c = \frac{\log_e e - 0}{\frac{1-e}{e}}$$

$$-c = \frac{1-0}{\frac{1-e}{e}}$$

$$-c = \frac{e}{1-e}$$

$$\boxed{c = \frac{e}{e-1} \in (1,e)}$$

$$\boxed{c = 1.58}$$




⑦ $f(x) = x^3$, $g(x) = 2-x$ in $[0,9]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0,9]$

$f'(x) = 3x^2$

$f'(x)$ is exist $\forall x$.

$f(x)$ is derivable in $(0,9)$

$g'(x) = 0-1 = -1$

$g'(x)$ is exist $\forall x$.

$g(x)$ is derivable in $(0,9)$.

$\Rightarrow f(x), g(x)$ are derivable in $(0,9)$

$$\frac{3c^2}{-1} = \frac{(9)^3 - (0)^3(2-0)^3}{(2-9)-(2-0)}$$

$$\cancel{+}\cancel{3}c^2 = \frac{81\times 9/3 - 8}{\cancel{+7}\ \cancel{+/2}}$$

$$c^2 \neq \frac{81\times 9}{7}$$

$$-3c^2 = \frac{721}{-7-2}$$

$$\cancel{+}3c^2 = \frac{721}{+9}$$

$$c^2 = \frac{721}{27}$$

$$c = \sqrt{\frac{721}{27}}$$

$$\boxed{c = 5.1675 \quad \in (0,9)}$$

Wednesday

23/10 Taylor's Expansion And Maclurin's:

Taylor's expansion at $x=0$, $x=1$, $x=\pi/2$.

Taylor's expansion about / at $\}$ $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

This is also called as taylor's expansion in powers of $(x-1)$.

Maclurin's:

at $x=0$, $f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

① $f(x) = \sin x$

② $f(x) = \log(1+x)$

③ $f(x) = \tan^{-1} x$

④ $f(x) = e^x$ at $x=1$

⑤ $f(x) = (1-x)^{5/2}$

⑥ $f(x) = \log x$ in powers of $x-1$ and hence evaluate $\log 1.1$ correct to four decimal process.

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x=2$.

By Taylor's expansion

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \ldots$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \rightarrow f'''(1) = 2$

$f^{iv}(x) = -6x^{-4} \Rightarrow f^{iv}(1) = -6$

wrong

$\therefore f(x) = \log a + (x-a)$

---

(5) $f(x) = \log(x)$

By Taylor's expansion at $x = a$

is $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \ldots$

at $a = 1$

$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{iv}(1) + \ldots \rightarrow (1)$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -6x^{-4} \Rightarrow f^{iv}(1) = -6$

from (1),

$\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!} 2 + \frac{(x-1)^4}{4!}(-6) + \ldots$

$\log x = (x-1) - \frac{(x-1)^2}{2!} + 2 \cdot \frac{(x-1)^3}{3!} - 6 \frac{(x-1)^4}{4!} + \ldots$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \frac{(1.1-1)^4}{4} + \cdots$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.00001}{4} + \cdots$$

$$= 0.1 - 0.005 + 0.0003 + 0.000002$$

No use

$$= 0.105202$$

$$\log(1.1) = 0.0953101799$$

$$\boxed{\log(1.1) \simeq 0.094}$$

(7) $f(x) = 2x^3 - 7x^2 + x + 6$ at $x = 2$.

By Taylor's expansion at $x = 2$

is $f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \frac{(x-2)^4}{4!} f^{iv}(2) + \cdots \rightarrow (1)$

$f(x) = 2x^3 - 7x^2 + x + 6 \rightarrow f(2) = -4$

$f'(x) = 6x^2 - 14x + 1 \rightarrow f'(2) = 24 - 28 + 1 = -3$

$f''(x) = 12x - 14 \rightarrow f''(2) = 24 - 14 = 10$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$\cancel{f^{iv}(x) = 0} \Rightarrow \cancel{f^{iv}(2) = 0}$

from (1),

$$2x^3 - 7x^2 + x + 6 = -4 + (x-2)(-3) + \frac{(x-2)^2}{2!} 10 + \frac{(x-2)^3}{3!} 12$$

$$= -4 + (x-2)(-3) + \frac{(x-2)^2}{2!} 10 + \frac{(x-2)^3}{3!}$$

$$= -4 - 3(x-2) + 10\frac{(x-2)^2}{2!} + 12 \cdot \frac{(x-2)^3}{3!}$$

Now, the Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \rightarrow (1)$$

$f(x) = \log(1+x) \Rightarrow f(0) = \log(1+0) = 0$

$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = -6$

from (1),

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$$

$$= x - \frac{x^2}{2!} + 2.\frac{x^3}{3!} - 6\frac{x^4}{4!} + \dots$$

(5) $f(x) = (1-x)^{5/2}$

By. Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \dots \rightarrow (1)$$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{5}{2}(1-x)^{3/2}(-1) \Rightarrow f'(0) = \frac{-5}{2}$

$f''(x) = \frac{-5}{2}\frac{3}{2}(1-x)^{1/2}(-1) \Rightarrow f''(0) = \frac{15}{4}$

$f'''(x) = \frac{15}{4}\frac{1}{2}(1-x)^{-1/2}(-1) \Rightarrow f'''(0) = -\frac{15}{8}$

$\vdots \qquad \vdots$

from (1),

$$(1-x)^{5/2} = 1 + x.\left(\frac{-5}{2}\right) + \frac{x^2}{2!}\left(\frac{15}{4}\right) + \frac{x^3}{3!}\left(\frac{-15}{8}\right) + \dots$$

$$= 1 - \frac{5}{2}x + \frac{15}{4}\frac{x^2}{2!} + -\frac{15}{8}.\frac{x^3}{3!} + \dots$$

$$= 1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{5}{16}x^3 + \dots$$

Now, the ...

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots \rightarrow ①$$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = -(-\sin x) \Rightarrow f^{iv}(0) = 0.$

from ①,

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

② $f(x) = \tan^{-1} x.$

Now, Maclurin's expansion is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$f(x) = \tan^{-1} x. \Rightarrow f(0) = \tan^{-1}(0) = 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x^2)^2}(2x) \Rightarrow f''(0) = 0.$

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)2(1+x^2)(2x)}{[(1+x^2)^2]^2}$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = \frac{-2(1+0)^2 + 0}{(1+0)^4} = -2$$

from ①,

$$\tan^{-1} x = 0 + x.(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$\tan^{-1} x = x - 2\frac{x^3}{3!} + \cdots$$

Now, Taylor's expansion is

$$f(x) = f(a) + (x-a)f'(x-a) + \frac{x^2}{2!} f''(x-a) + \frac{x^3}{3!} f''' x-a$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$\Rightarrow$ at $a=1$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \longrightarrow (1)$$

$f(x) = e^x \Rightarrow f(1) = e$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

$f'''(x) = e^x \Rightarrow f'''(1) = e$

$f^{IV}(x) = e^x \Rightarrow f^{IV}(1) = e$

$\vdots$

from (1),

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \cdots$$

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right].$$