16-811 Homework 3 Re-submit

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Q1:

(a):

$$f(x) = \frac{1}{3} + 2sinh(x) = \frac{1}{3} + e^x - e^{-x}$$

$$f(0) = \frac{1}{3}$$

$$f'(x) = e^x + e^{-x}$$

$$f''(x) = e^x - e^{-x}$$

Then we can generalize that:

$$f^{(2n)}(x) = e^x - e^{-x}$$

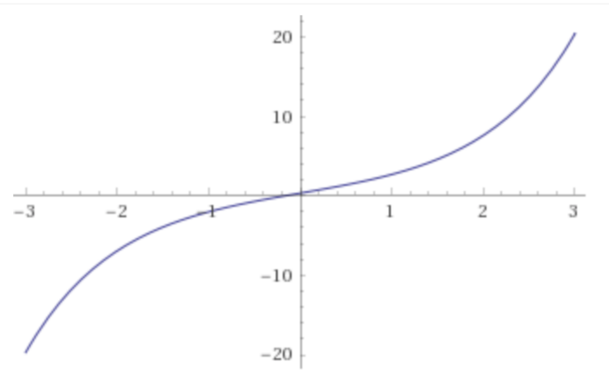
$$f^{(2n-1)}(x) = e^x + e^{-x}$$

where $n = 1, 2, 3, ...$

So the Taylor expansion of $f(x)$ around $x = 0$ is:

$$\begin{aligned} f(x) &= \frac{1}{3} + \frac{e^0 + e^0}{1!}x + \frac{e^0 - e^0}{2!}x^2 + \frac{e^0 + e^0}{3!}x^3 + ... \\ &= \frac{1}{3} + 2(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...) \end{aligned} \tag{1}$$

(b):



(c):

Let's define the best uniform approximation by a quadratic function:

$$g(x) = ax^2 + bx + c$$

Then the error function is given by:

$$e(x) = f(x) - g(x) = \frac{1}{3} + 2sinh(x) - ax^2 - bx - c$$

We need to find $2+2=4$ points x_0, x_1, x_2, x_3 to do the best uniform approximation, at which the error $e(x)$ is smallest, with equal magnitude and alternating sign.

Since $f(x) > 0$ and any order of its derivative $f^{(n)}(x) > 0$, so we have:

$$x_0 = -3, x_3 = 3$$

$$e(-3) = -e(x_1) = e(x_2) = -e(3)$$

Since $f(x) =$ is odd symmetric with respect to $(0, \frac{1}{3})$, we can say that the middle two points are also symmetric to $x = 0$, therefore $x_1 = -x_2$

Let's analyze the two equations below:

$$e(x_0) = -e(x_3)$$

$$e(x_1) = -e(x_2) = -e(-x_1)$$

Putting $x_0 = -3, x_3 = 3$ into the first equation and simplifies it, we get:

$$\frac{1}{3} - 9a - c = 0 \Rightarrow c = \frac{1}{3} - 9a$$

Expanding the second equation gives:

$$\frac{2}{3} - 2ax_1^2 - 2c = 0$$

which gives:

$$c = \frac{1}{3} - ax_1^2$$

Combining the two resulting equations:

$$\frac{1}{3} - 9a = \frac{1}{3} - ax_1^2 \Rightarrow 9a = x_1^2 a$$

and we know that $x_1 \neq \pm 3$, therefore:

$$a = 0,\ c = \frac{1}{3}$$

Putting this back into $e(x_0)$ and $e(x_1)$:

$$e(x_0) = e^{-3} - e^3 + 3b$$

$$e(x_1) = e^{x_1} - e^{-x_1} - bx_1$$

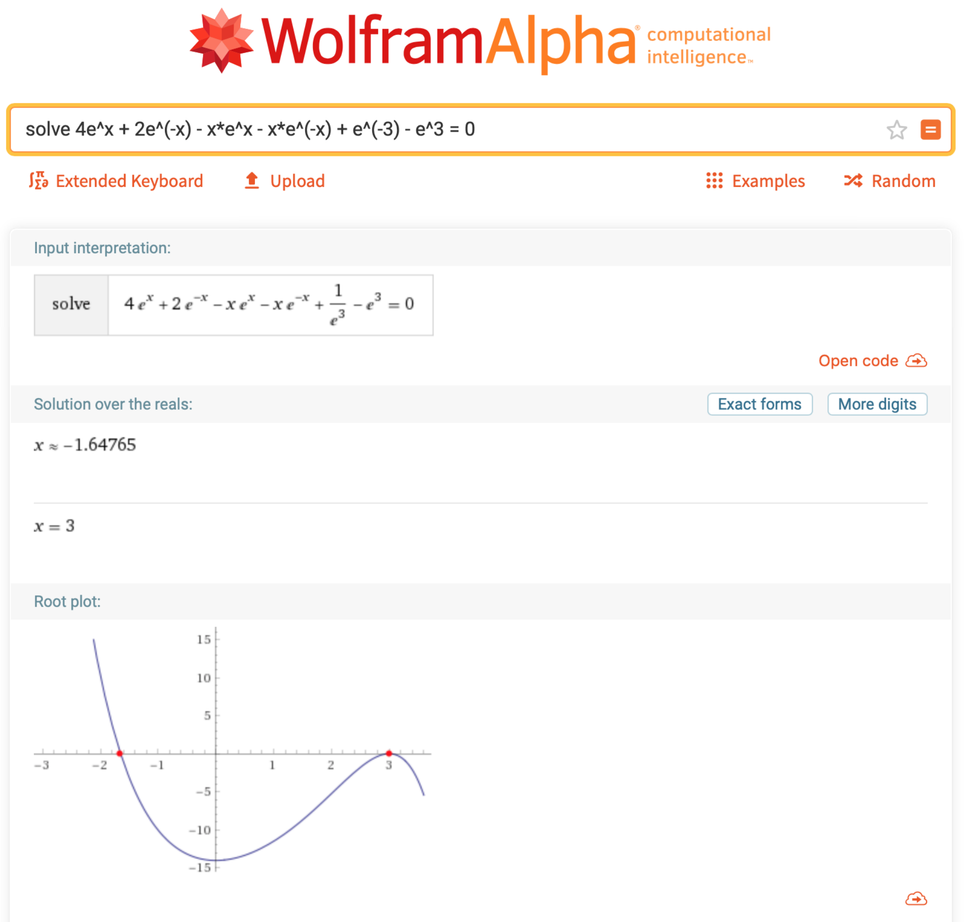
Since the error function is of its local minima at x_0, x_1, x_2, x_3, therefore:

$$e'(x_1) = e^{x_1} + e^{-x_1} - b = 0 \Rightarrow b = e^{x_1} + e^{-x_1}$$

Substitute this back into the equation $e(x_0) = -e(x_1)$ and simplifies it:

$$(4 - x_1)e^{x_1} + (2 - x_1)e^{-x_1} + e^{-3} - e^3 = 0$$

We can use numerical methods (i.e. Muller's method) to solve this. I used WolframAlpha to solve this:



Since we know that $x_1 \neq \pm 3$, then $x_1 = -1.64765$

$$b = e^{x_1} + e^{-x_1} = 5.38726$$

Putting everything together, we have the best uniform approximation by a quadratic function:

$$g(x) = 5.38726x + \frac{1}{3}$$

$$e(x) = f(x) - g(x) = e^x - e^{-x} - 5.38726x$$

Putting any of $x = x_0, x_1, x_2$ or x_3 into the error function and take its absolute value to get the L_∞ error:

$$L_\infty = 3.874$$

In terms of the L_2 error:

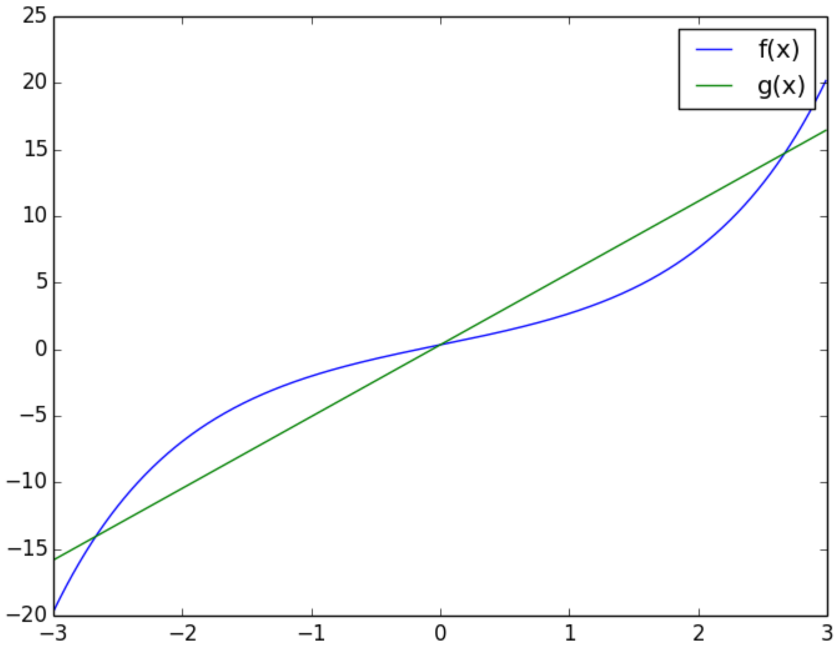
$$L_2 = \sqrt{\int_{-3}^{3} |e(x)|^2 dx}$$

using WolframAlpha:

Computation result:

$$\sqrt{\int_{-3}^{3} |2\sinh(x) - 5.38726\, x|^2 \, dx} = 6.62518$$

Visualization of the uniform approximation:



(d): To compute the Legendre polynomials for least squares approximation, Use the basis functions:

$$\phi_0(x) = 1,\ \phi_1(x) = x,\ \phi_2(x) = x^2$$

Then we get:

$$p_0(x) = 1$$

$$p_1(x) = (x - \frac{< x\phi_0, \phi_0 >}{< \phi_0, \phi_0 >})\phi_0 = x - \frac{\int_{-3}^{3} x dx}{\int_{-3}^{3} 1 dx} = x$$

$$p_2(x) = (x - \frac{< x\phi_1, \phi_1 >}{< \phi_1, \phi_1 >})\phi_1 - \frac{< \phi_1, \phi_1 >}{< \phi_0, \phi_0 >}\phi_0 = (x-0)x - \frac{18}{6} = x^2 - 3$$

Then calculate the coefficients of the Legendre polynomials:

$$\frac{< f(x), p_0(x) >}{< p_0(x), p_0(x) >} = \frac{\int_{-3}^{3} (\frac{1}{3} + 2sinh(x))dx}{\int_{-3}^{3} 1dx} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{< f(x), p_1(x) >}{< p_1(x), p_1(x) >} = \frac{\int_{-3}^{3} x(\frac{1}{3} + 2sinh(x))dx}{\int_{-3}^{3} x^2 dx} = \frac{\int_{-3}^{3} (xe^x - xe^{-x})dx}{18} = \frac{4e^3 + 8e^{-3}}{18} = 4.4856$$

$$\frac{< f(x), p_2(x) >}{< p_2(x), p_2(x) >} = \frac{\int_{-3}^{3} (\frac{1}{3} + 2sinh(x))(x^2 - 3)dx}{\int_{-3}^{3} (x^2 - 3)^2 dx} = \frac{\int_{-3}^{3} (2x^2 sinh(x))dx}{\frac{216}{5}} = 0$$

So the best least squares approximation of $f(x)$ by a quadratic function is given by:

$$p(x) = \frac{1}{3} + 4.4856x$$

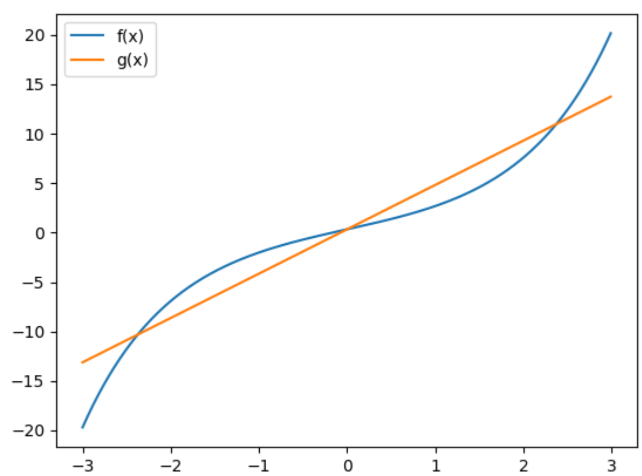
Define the error function $e(x)$:

$$e(x) = f(x) - p(x) = e^x - e^{-x} - 4.4856x$$

$$L_\infty = |e(3)| = 6.57895$$

$$L_2 = \sqrt{\int_{-3}^{3} |e(x)|^2 dx} = 5.40912$$

Visualization of the least squares approximation:



Q4:

(a).

Denote:

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Suppose the least squares approximation plane that fits the data is represented by:

$$ax + by + c = z$$

Then

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is the solution to

$$Ax = b$$

Since we have many points, for an over-determined system, we can use the pseudo-inverse:

$$x = (A^T A)^{-1} A^T b$$

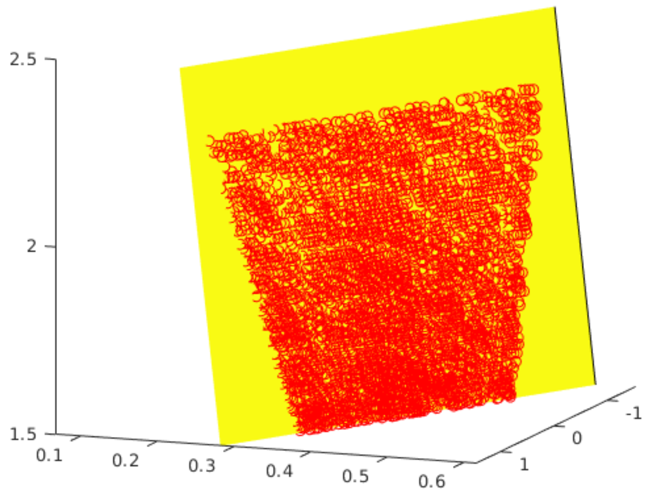
Then we get the coefficients:

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1.79305733 \\ -18.74523538 \\ 9.43910208 \end{bmatrix}$$

So the fitted plane is given by:

$$-1.79305733x + -18.74523538y + 9.43910208 = z$$

Visualization of the fitted plane, red are the original points, yellow is the fitted plane:



Average distance = 0.0029

See q4a.m for more details.

(b).

The fitted plane is given by:

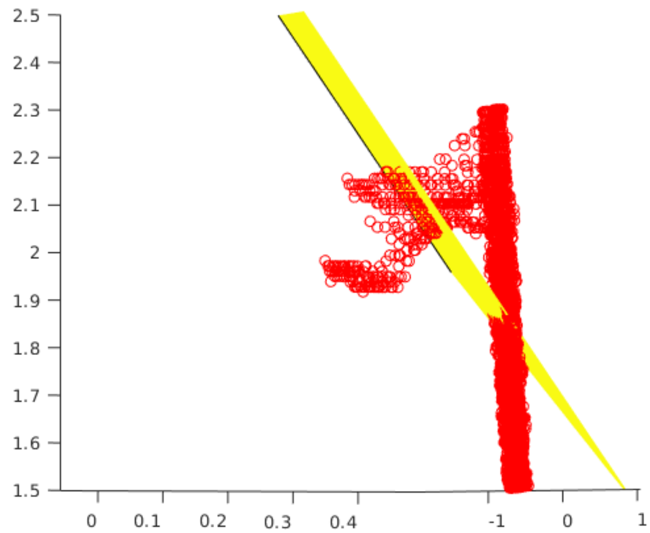
$$-0.2001x + -2.0224y + 2.6567 = z$$

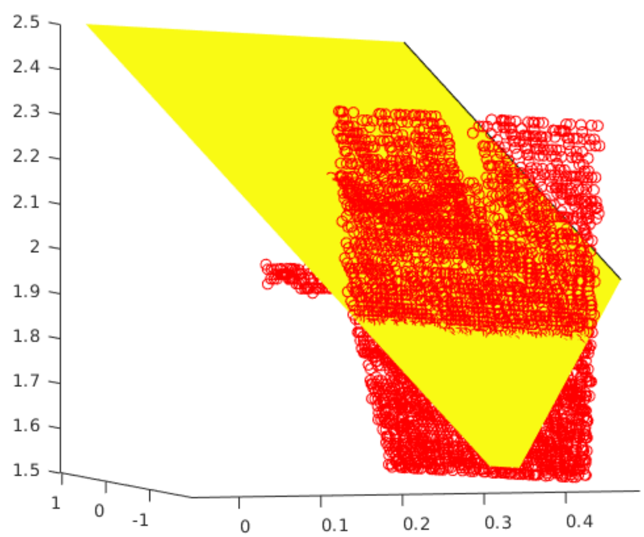
It looks off since it is significantly affected by outliers in the data.

Average distance = 0.0771

See q4b.m for more details.

Visualization of the fitted plane, red are the original points, yellow is the fitted plane:





(c).

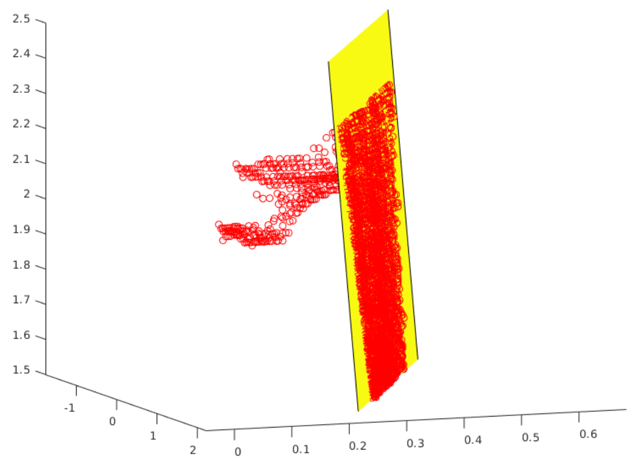
We can use RANSAC to fit a best plane that is robust to outliers.

See q4c.m and ransac.m for more details.

The fitted plane is given by:

$$-2.0122x + -19.1974y + 9.6383 = z$$

Average distance = 0.0156



(d).

By observing the points of the hallway, we can separate the points that belong to different sides of the wall with its x and y coordinate, into 4 clusters that correspond to the 4 different planes. Then we apply RANSAC to each of the cluster to find the best fitted plane.

See q4d.m for more details.

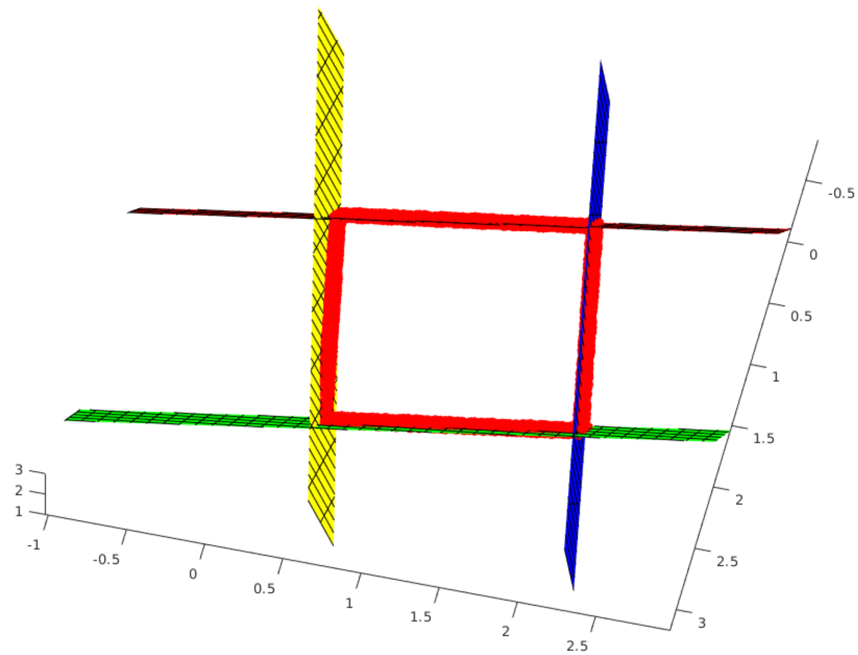
The fitted planes are given by:

$$5.3301x_1 + 1.0340y_1 + -1.5209 = z_1$$

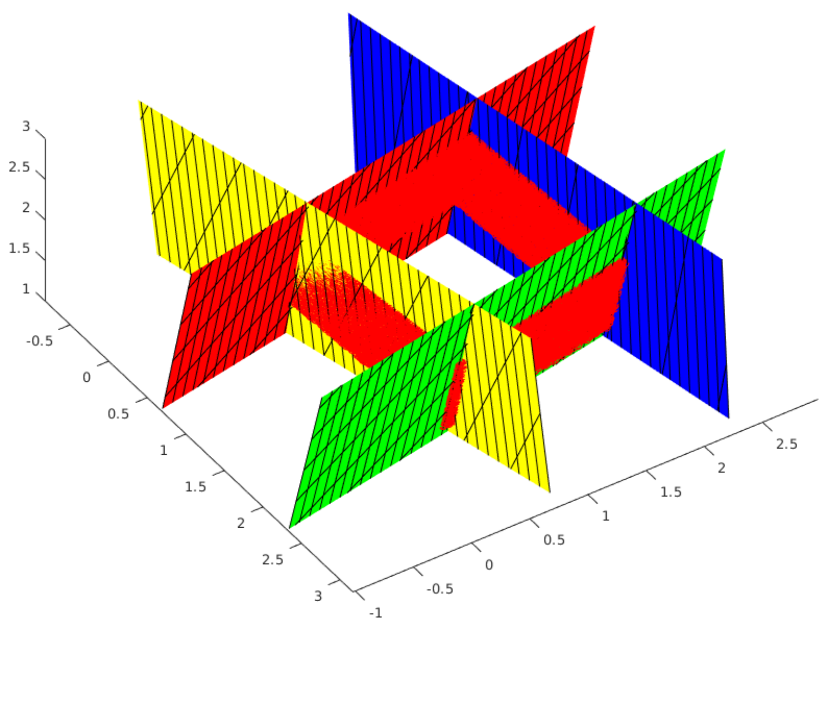
$$4.7020x_2 + 0.9144y_2 + -8.9542 = z_2$$

$$2.6936x_3 + -12.2470y_3 + 2.5234 = z_3$$

$$5.8897x_4 + -33.3014y_4 + 60.6147 = z_4$$



-0.5
0
0.5
1
1.5
2
2.5
3
-1
-0.5
0
0.5
1
1.5
2
2.5
3
2
1



3
2.5
2
1.5
1
-0.5
0
0.5
1
1.5
2
2.5
3
-1
-0.5
0
0.5
1
1.5
2
2.5

(e).

Similar to part (d), we can find the planes with RANSAC and clustering based on x and y coordinates. Then we can determine the smoothness of each surface by calculating the average distance of all the points that belong to that corrsponding plane.

See q4e.m for more details.

The smoothness calculated are:

$s_{left} = 0.0985$

$s_{right} = 0.2451$

$s_{upper} = 0.0894$

$s_{floor} = 0.0663$

So the floor is the safest to traverse (the blue plane shown below).