16-811 Homework 3

Changsheng Shen (Bobby)
changshs@andrew.cmu.edu

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Q1:

(a):

$$f(x) = \frac{1}{3} + 2sinh(x) = \frac{1}{3} + e^x - e^{-x}$$

$$f(0) = \frac{1}{3}$$

$$f'(x) = e^x + e^{-x}$$

$$f''(x) = e^x - e^{-x}$$

Then we can generalize that:

$$f^{(2n)}(x) = e^x - e^{-x}$$

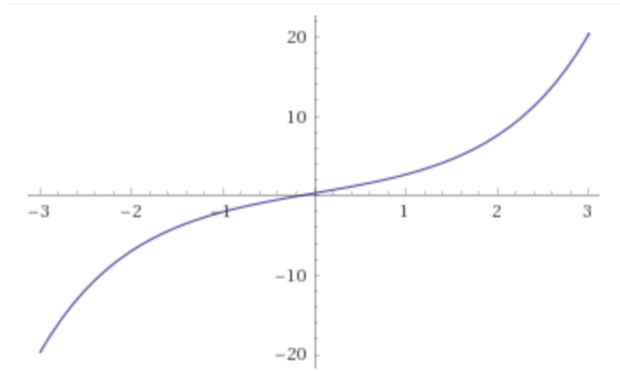
$$f^{(2n-1)}(x) = e^x + e^{-x}$$

where $n = 1, 2, 3, ...$

So the Taylor expansion of $f(x)$ around $x = 0$ is:

$$\begin{aligned} f(x) &= \frac{1}{3} + \frac{e^0 + e^0}{1!}x + \frac{e^0 - e^0}{2!}x^2 + \frac{e^0 + e^0}{3!}x^3 + ... \\ &= \frac{1}{3} + 2(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...) \end{aligned} \tag{1}$$

(b):



(c):

Let's define the best uniform approximation by a quadratic function:

$$g(x) = ax^2 + bx + c$$

Then the error function is given by:

$$e(x) = f(x) - g(x) = \frac{1}{3} + 2sinh(x) - ax^2 - bx - c$$

We need to find $2+2=4$ points x_0, x_1, x_2, x_3 to do the best uniform approximation, at which the error $e(x)$ is smallest, with equal magnitude and alternating sign.

Since $f(x) > 0$ and any order of its derivative $f^{(n)}(x) > 0$, so we have:

$$x_0 = -3, x_3 = 3$$

$$e(-3) = -e(x_1) = e(x_2) = -e(3)$$

Since $f(x) =$ is odd symmetric with respect to $(0, \frac{1}{3})$, we can say that the middle two points are also symmetric to $x = 0$, therefore $x_1 = -x_2$

Let's analyze the two equations below:

$$e(x_0) = -e(x_3)$$

$$e(x_1) = -e(x_2) = -e(-x_1)$$

Putting $x_0 = -3, x_3 = 3$ into the first equation and simplifies it, we get:

$$\frac{1}{3} - 9a - c = 0 \Rightarrow c = \frac{1}{3} - 9a$$

Expanding the second equation gives:

$$\frac{2}{3} - 2ax_1^2 - 2c = 0$$

which gives:

$$c = \frac{1}{3} - ax_1^2$$

Combining the two resulting equations:

$$\frac{1}{3} - 9a = \frac{1}{3} - ax_1^2 \Rightarrow 9a = x_1^2 a$$

and we know that $x_1 \neq \pm 3$, therefore:

$$a = 0,\ c = \frac{1}{3}$$

Putting this back into $e(x_0)$ and $e(x_1)$:

$$e(x_0) = e^{-3} - e^3 + 3b$$

$$e(x_1) = e^{x_1} - e^{-x_1} - bx_1$$

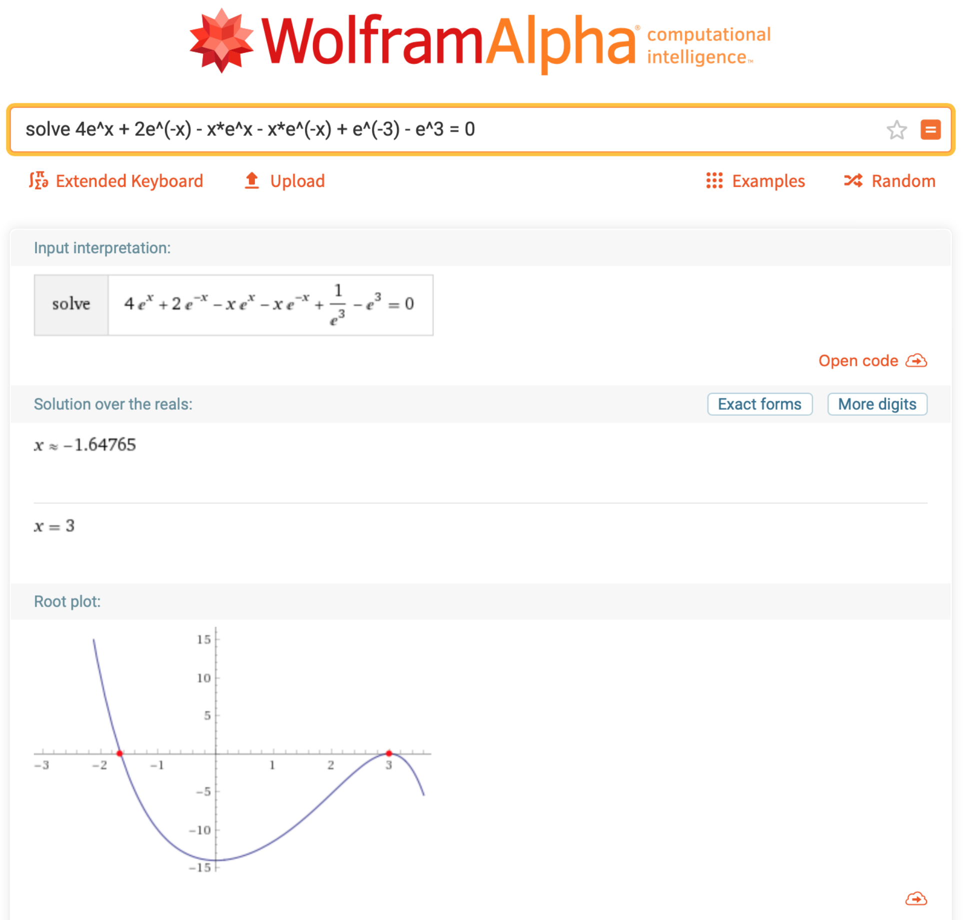
Since the error function is of its local minima at x_0, x_1, x_2, x_3, therefore:

$$e'(x_1) = e^{x_1} + e^{-x_1} - b = 0 \Rightarrow b = e^{x_1} + e^{-x_1}$$

Substitute this back into the equation $e(x_0) = -e(x_1)$ and simplifies it:

$$(4 - x_1)e^{x_1} + (2 - x_1)e^{-x_1} + e^{-3} - e^3 = 0$$

We can use numerical methods (i.e. Muller's method) to solve this. I used WolframAlpha to solve this:



Since we know that $x_1 \neq \pm 3$, then $x_1 = -1.64765$

$$b = e^{x_1} + e^{-x_1} = 5.38726$$

Putting everything together, we have the best uniform approximation by a quadratic function:

$$g(x) = 5.38726x + \frac{1}{3}$$

$$e(x) = f(x) - g(x) = e^x - e^{-x} - 5.38726x$$

Putting any of $x = x_0, x_1, x_2$ or x_3 into the error function and take its absolute value to get the L_∞ error:

$$L_\infty = 3.874$$

In terms of the L_2 error:

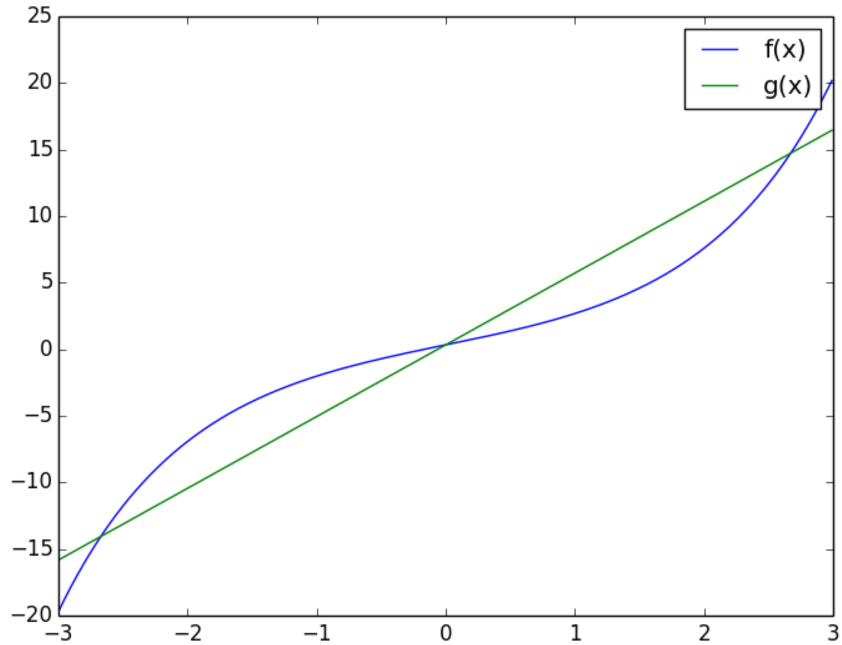
$$L_2 = \sqrt{\int_{-3}^{3} |e(x)|^2 dx}$$

using WolframAlpha:

Computation result:

$$\sqrt{\int_{-3}^{3} |2\sinh(x) - 5.38726\, x|^2 \, dx} = 6.62518$$

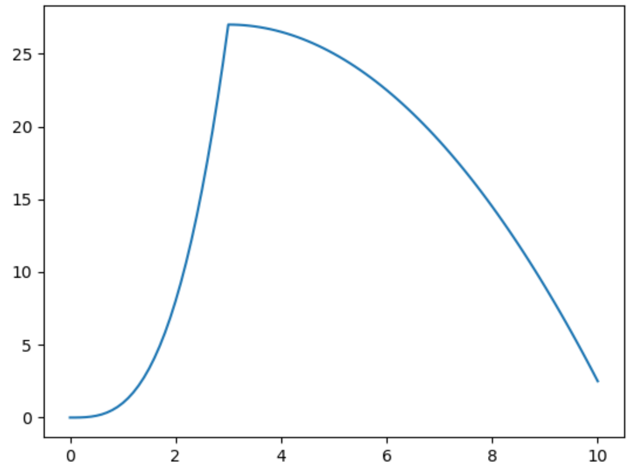
Visualization of the uniform approximation:



(d):

Q2:

Plotting the original function:



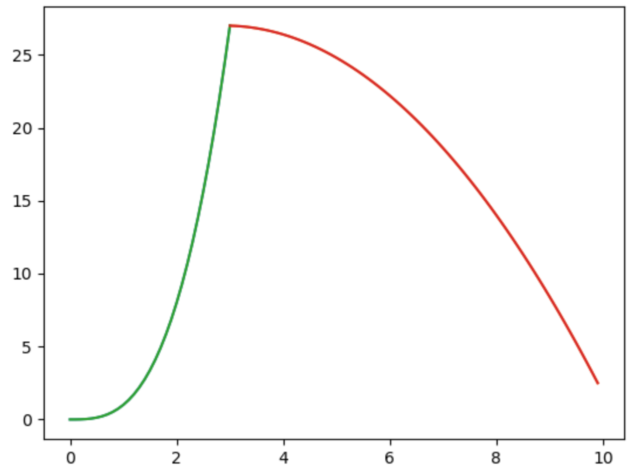
We can observe that there is a discontinuity at $x = 3$, so we fit the function separately.

Using basis functions $\phi_1 = 1, \phi_2 = x, \phi_3 = x^2, \phi_4 = x^3$, and least squares approximation (see q2.py), we fit $f(x)$ as:

$$f(x) = x^3 \quad (0 \leq x \leq 3)$$
$$f(x) = -0.5x^2 + 2.9x + 22.795 \quad (3 \leq x \leq 10)$$

Approximation error is very small (mean error 1e-15)

Plotting the fitted function:



Q3:

(a).

$$T_0(cos\theta) = cos(0) = 1$$

$$T_1(cos\theta) = cos(\theta) \Rightarrow T_1(x) = x$$

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - 2x^2 + 1 = 8x^4 - 8x^2 + 1$$

(b).

$$< T_3, T_4 >= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}(4x^3 - 3x)(8x^4 - 8x^2 + 1)dx$$

Since the integral is evaluated on $[-1, 1]$, substitute x with $sin\theta$:

$$\begin{aligned} < T_3, T_4 > &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1-sin^2\theta}}(4sin^3\theta - 3sin\theta)(8sin^4\theta - 8sin^2\theta + 1)dsin\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{cos\theta}(4sin^3\theta - 3sin\theta)(8sin^4\theta - 8sin^2\theta + 1)cos\theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (32sin^7\theta - 56sin^5\theta + 28sin^3\theta - 3sin\theta)d\theta \end{aligned} \tag{2}$$

Notice that this is an odd function, trivially the integral of this function from $[\frac{-\pi}{2}, \frac{\pi}{2}]$ is 0. Therefore, $< T_3, T_4 >= 0$, so they are orthogonal polynomials relative to their inner product.

(c).

Evaluate the inner product of any T_n:

Let $x = cos\theta$, then:

$$T_n(x) = T_n(cos\theta) = cos(n\theta)$$

$$
\begin{aligned}
< T_n, T_n > &= \int_{-\pi}^{0} \frac{1}{\sqrt{1 - cos^2\theta}} cos^2(n\theta) dcos\theta \\
&= \int_{-\pi}^{0} \frac{1}{sin\theta} cos^2(n\theta)(-sin\theta) d\theta \\
&= -\int_{-\pi}^{0} cos^2(n\theta) d\theta \\
&= -\frac{1}{2}\int_{-\pi}^{0} (cos(2n\theta) + 1) d\theta \\
&= \frac{\pi}{2}
\end{aligned} \tag{3}
$$

Therefore, the length of T_n is $\sqrt{< T_n, T_n >} = \sqrt{\frac{\pi}{2}}$ for all $n \geq 0$

(d).

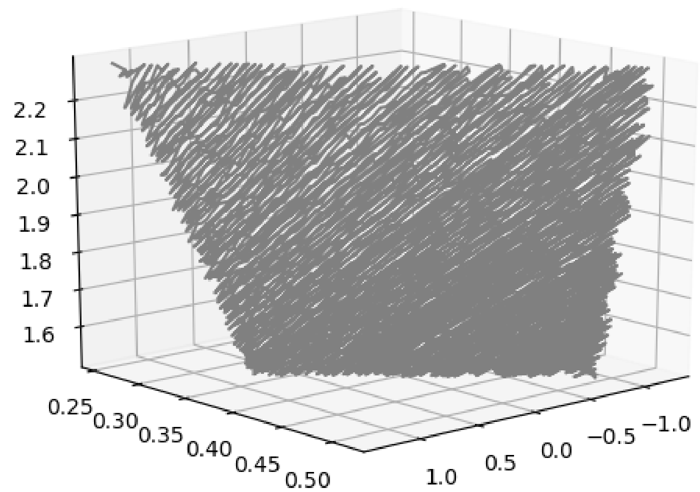
For any i, j such that $i \geq 0$, $j \geq 0$, $i \neq j$, similar to the derivation above:

$$
\begin{aligned}
< T_i, T_j > &= \int_{0}^{\pi} cos(i\theta) \cdot cos(j\theta) d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} [cos(i\theta + j\theta) + cos(i\theta - j\theta)] d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} [cos((i + j)\theta) + cos((i - j)\theta] d\theta \\
&= 0
\end{aligned} \tag{4}
$$

With the trivial fact that the integral of $cos\theta$ of a non-zero integer multiple of θ from 0 to π is 0, based on its symmetry.

Q4:

(a).



(b).

It looks off since it is significantly affected by outliers in the data.

(c).

We can use RANSAC to fit a best plane that is robust to outliers.