Robot Autonomy

Lecture 18: Gaussian Processes

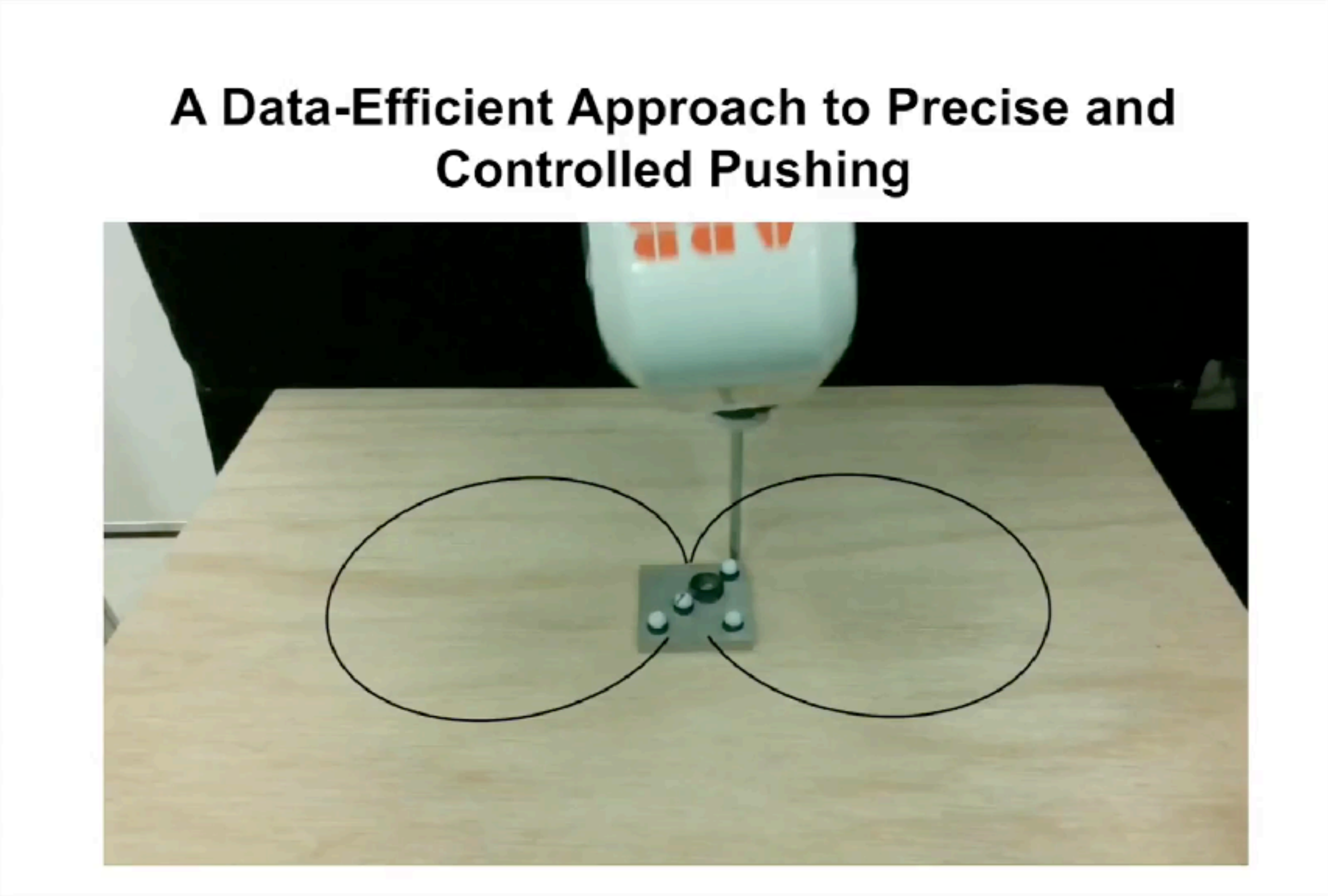
Oliver Kroemer

Motivation

- Robotics involves a wide range of mapping functions
  - State and action to next state
  - Controller parameters to reward
- Learn functions from data
- May only have a few samples
- Want to use a flexible representation
- Want to model uncertainty of learned model

- Model functions with Gaussian process regression

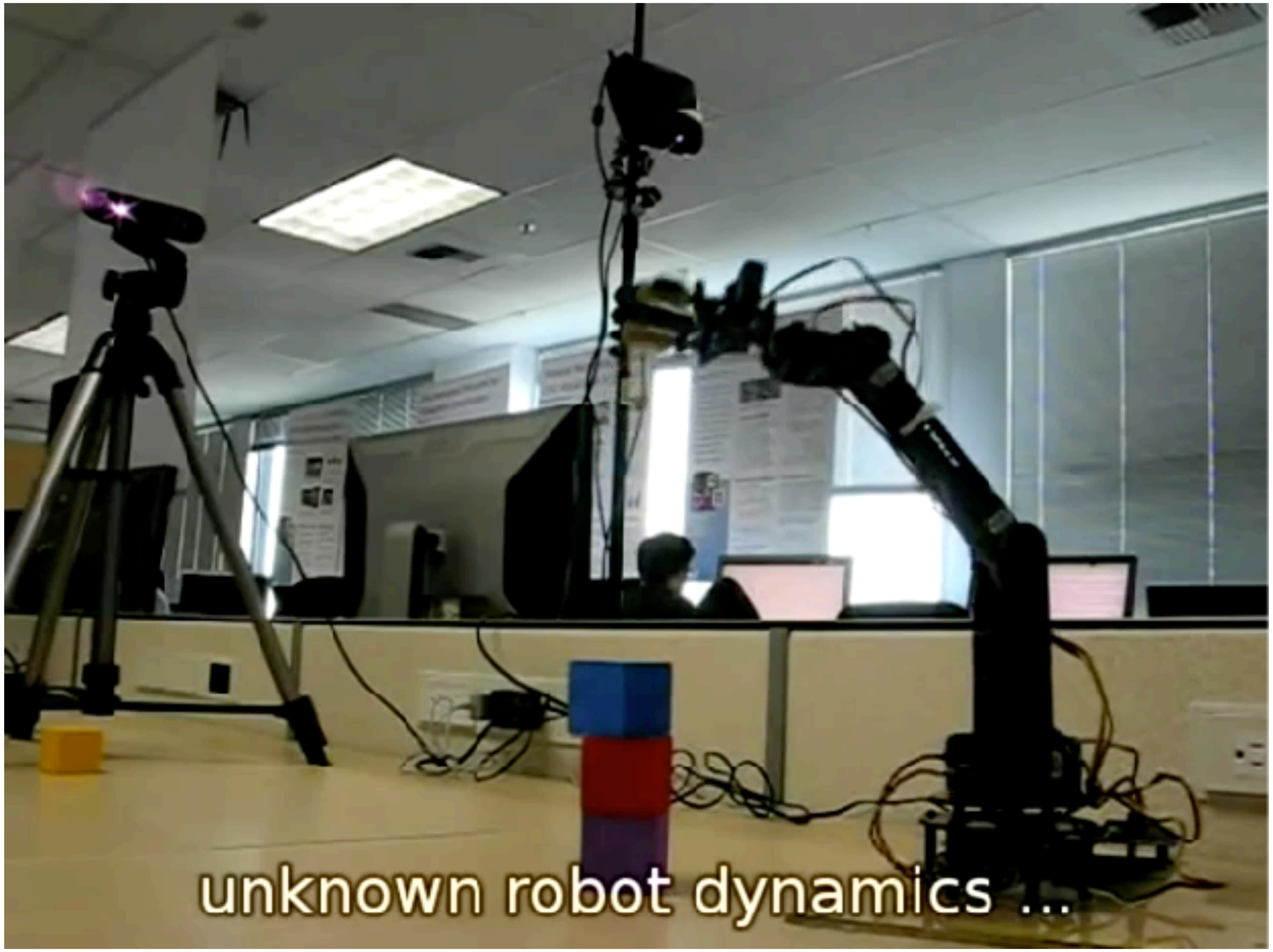
GP Forward Model



State and Action -> Next State

GP Model-Based RL

- Learn a Gaussian Process model of the task for RL

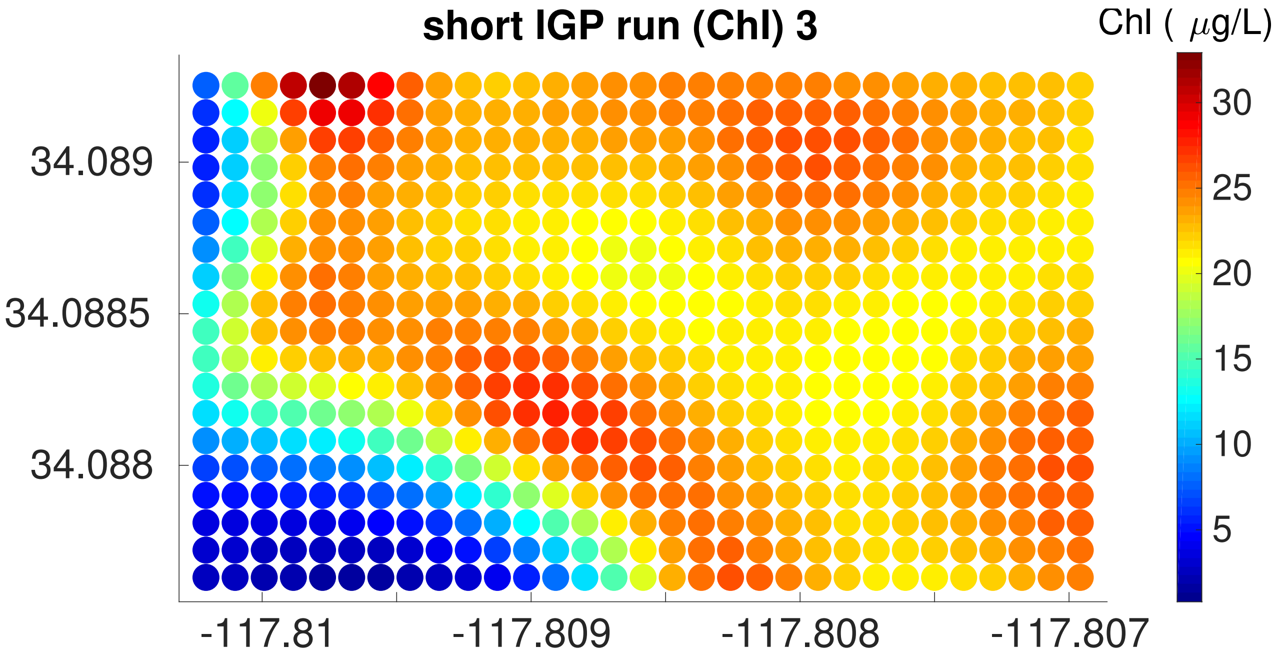


State and Action -> Next State

Deisenroth et al. Learning to Control a Low-Cost Manipulator using Data-Efficient Reinforcement Learning RSS 2011

GP Sample Selection

Looking for Chlorophyll

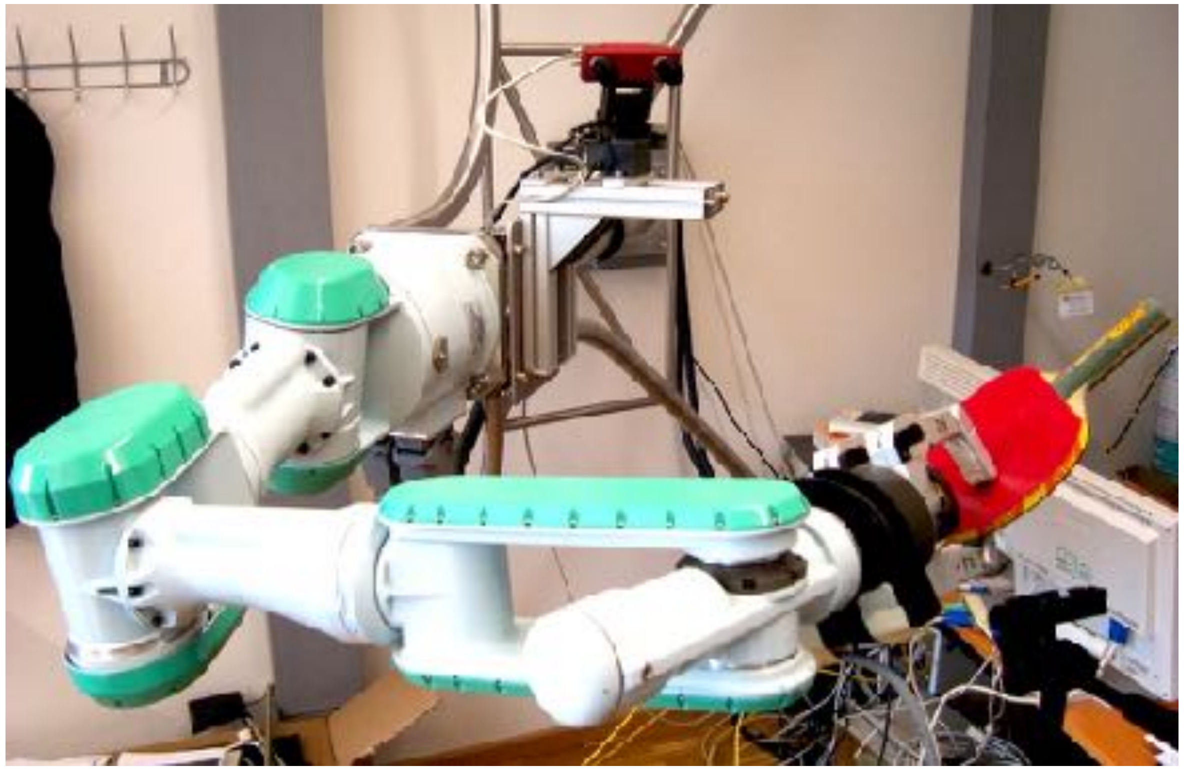


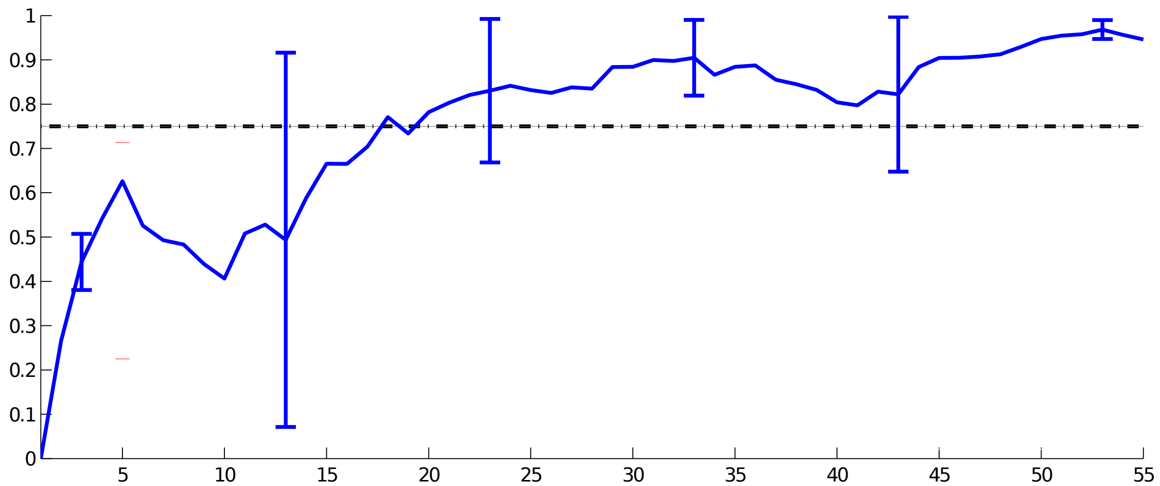
State -> Measurement Value

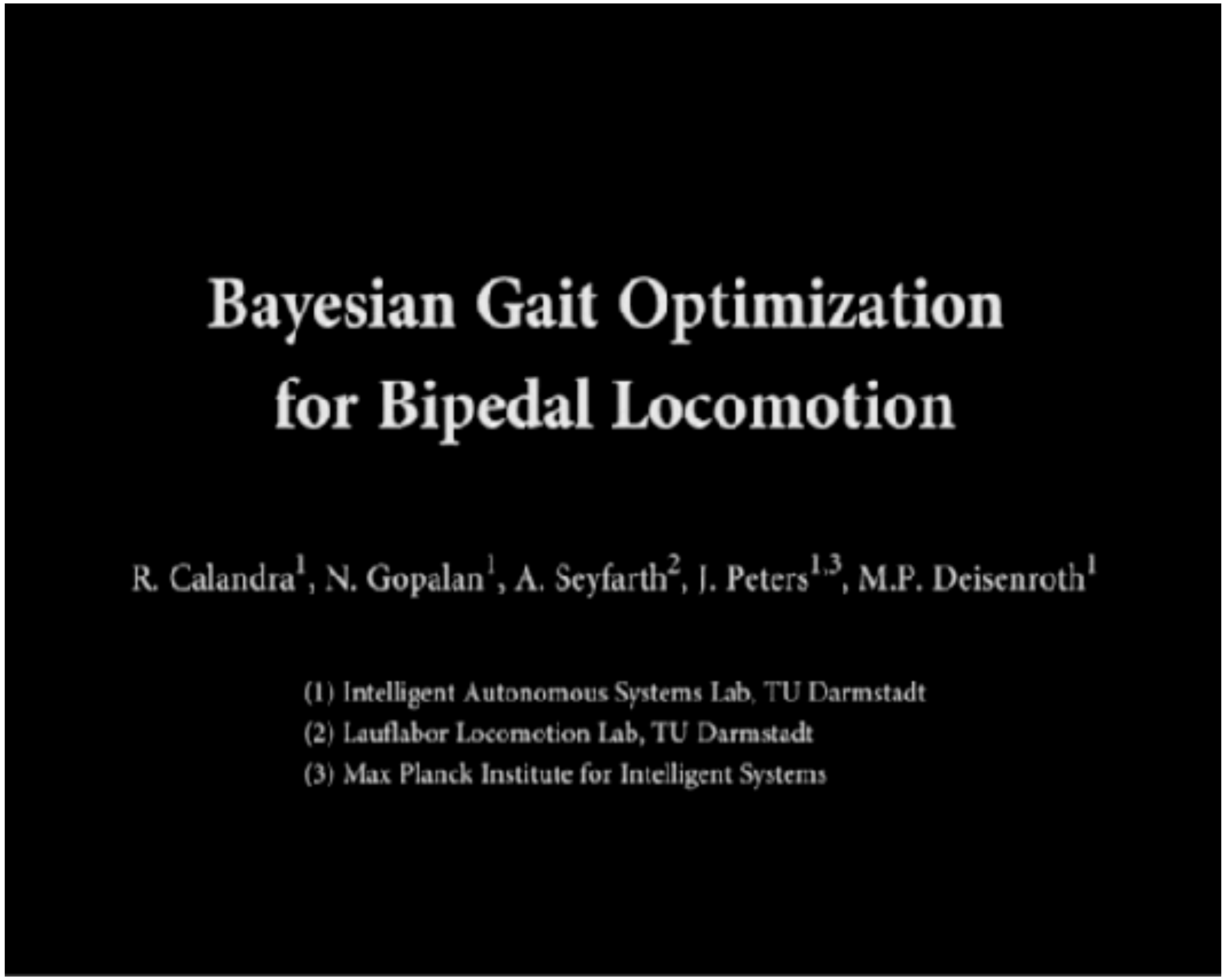
Kemna et al. On-board Adaptive Informative Sampling for AUVs: a Feasibility Study, Oceans 2018

Bayesian Optimization

- Learn optimal grasp frames for specific objects

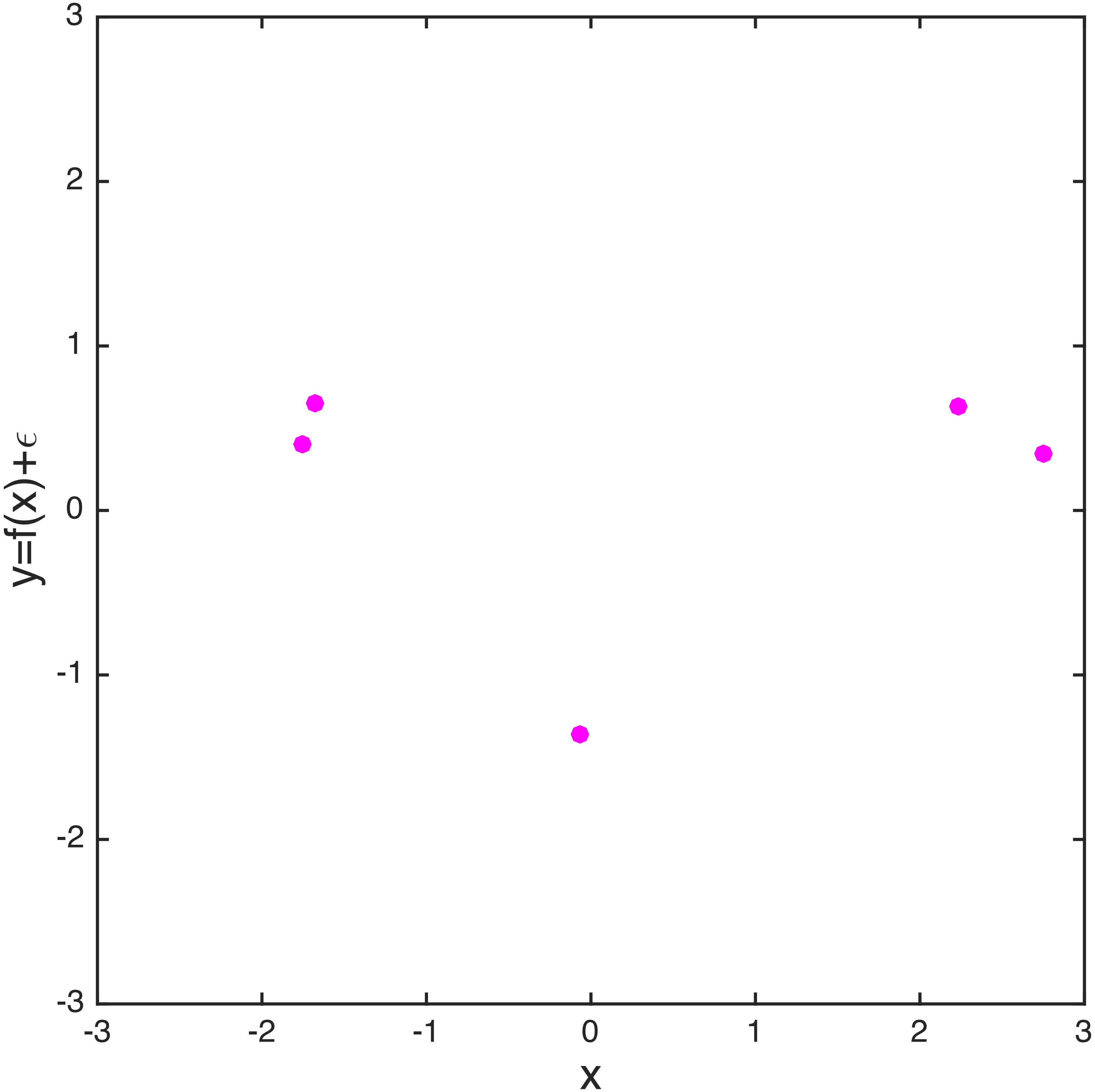






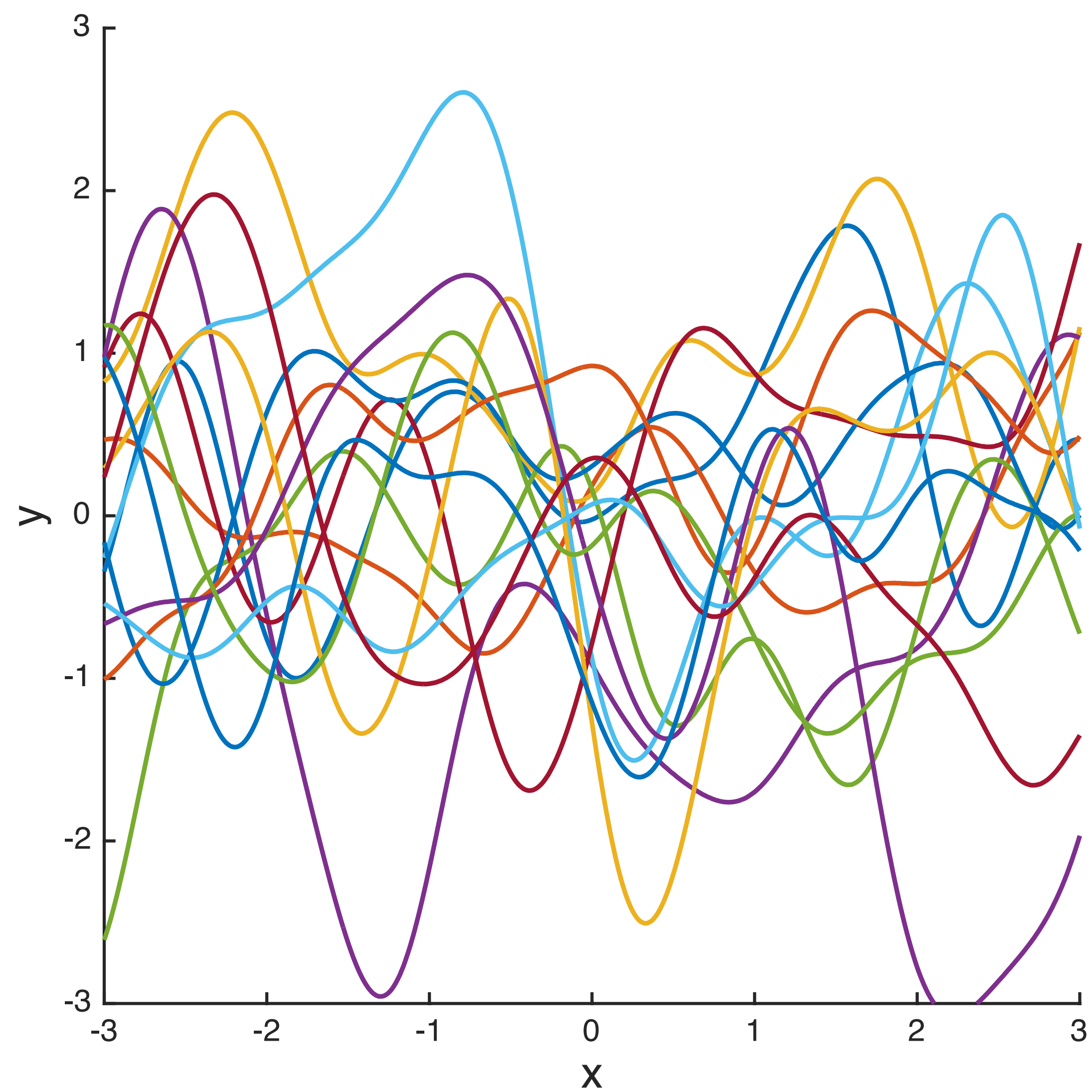
Action Parameters -> Reward

Problem



Function Distributions

- The function f(x) is unknown



- May assume a prior on functions (e.g.,smooth, near zero)

Function Distributions

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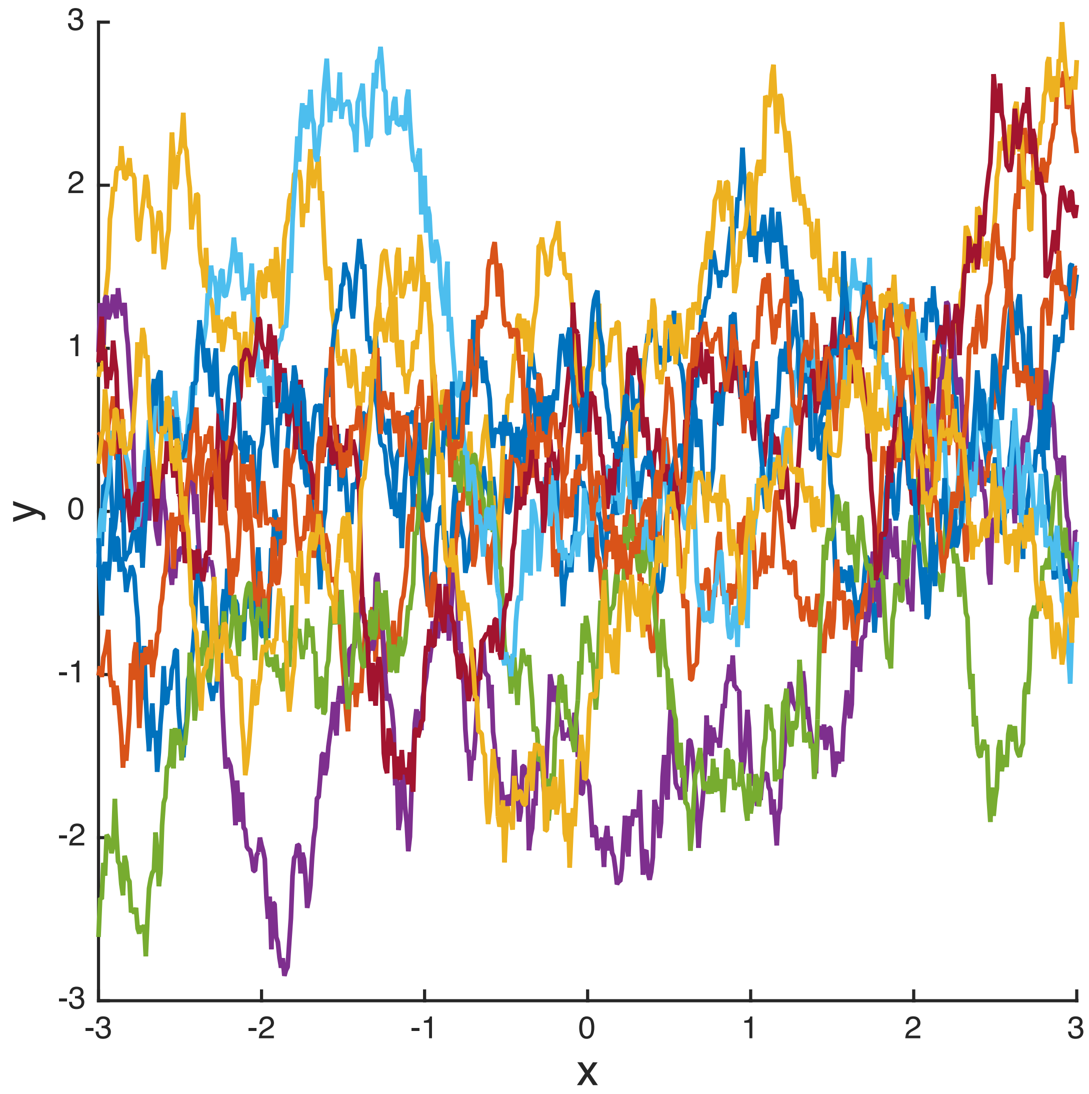
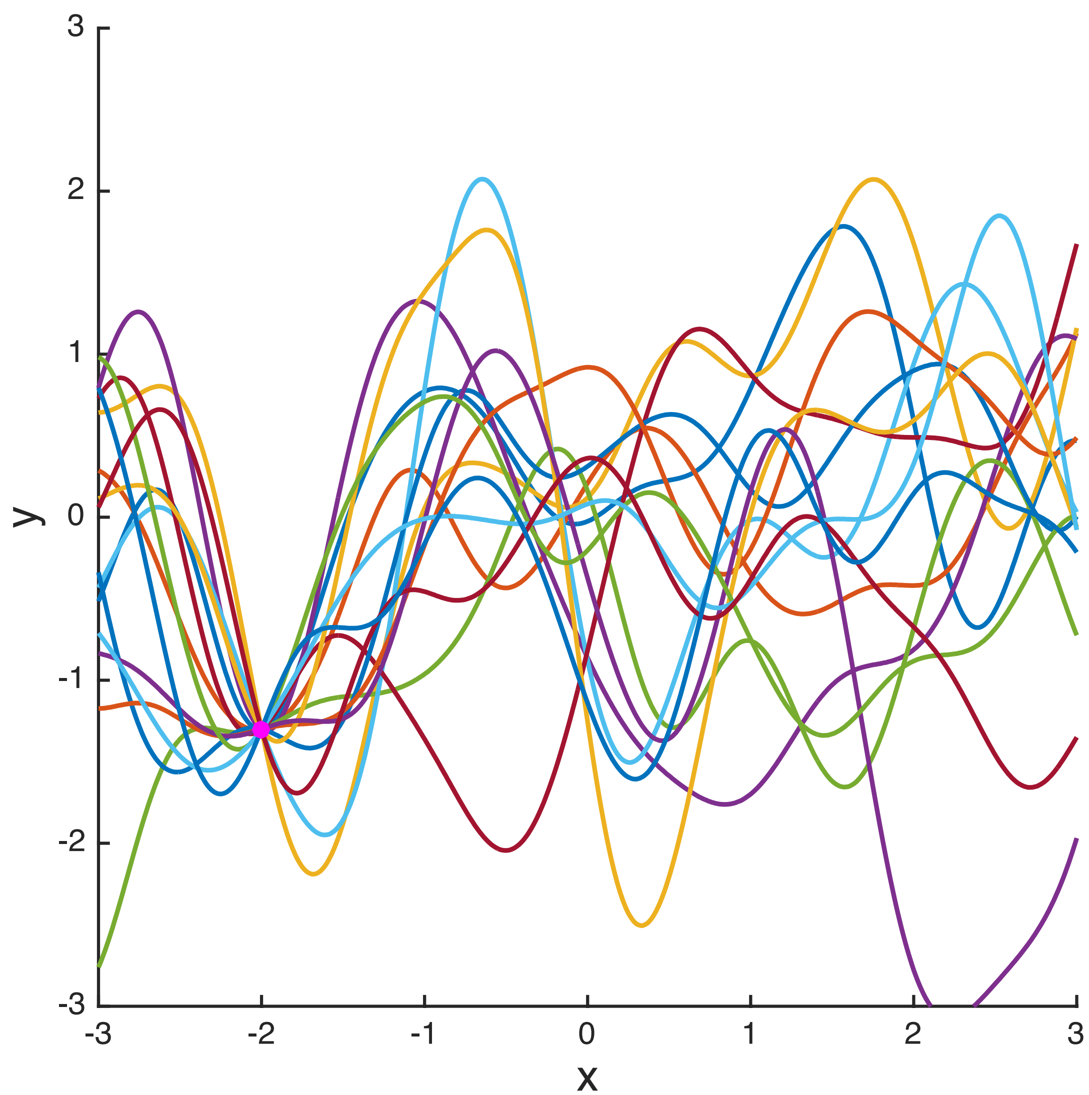


- May assume a prior on functions (e.g.,smooth, near zero)

Function Distributions

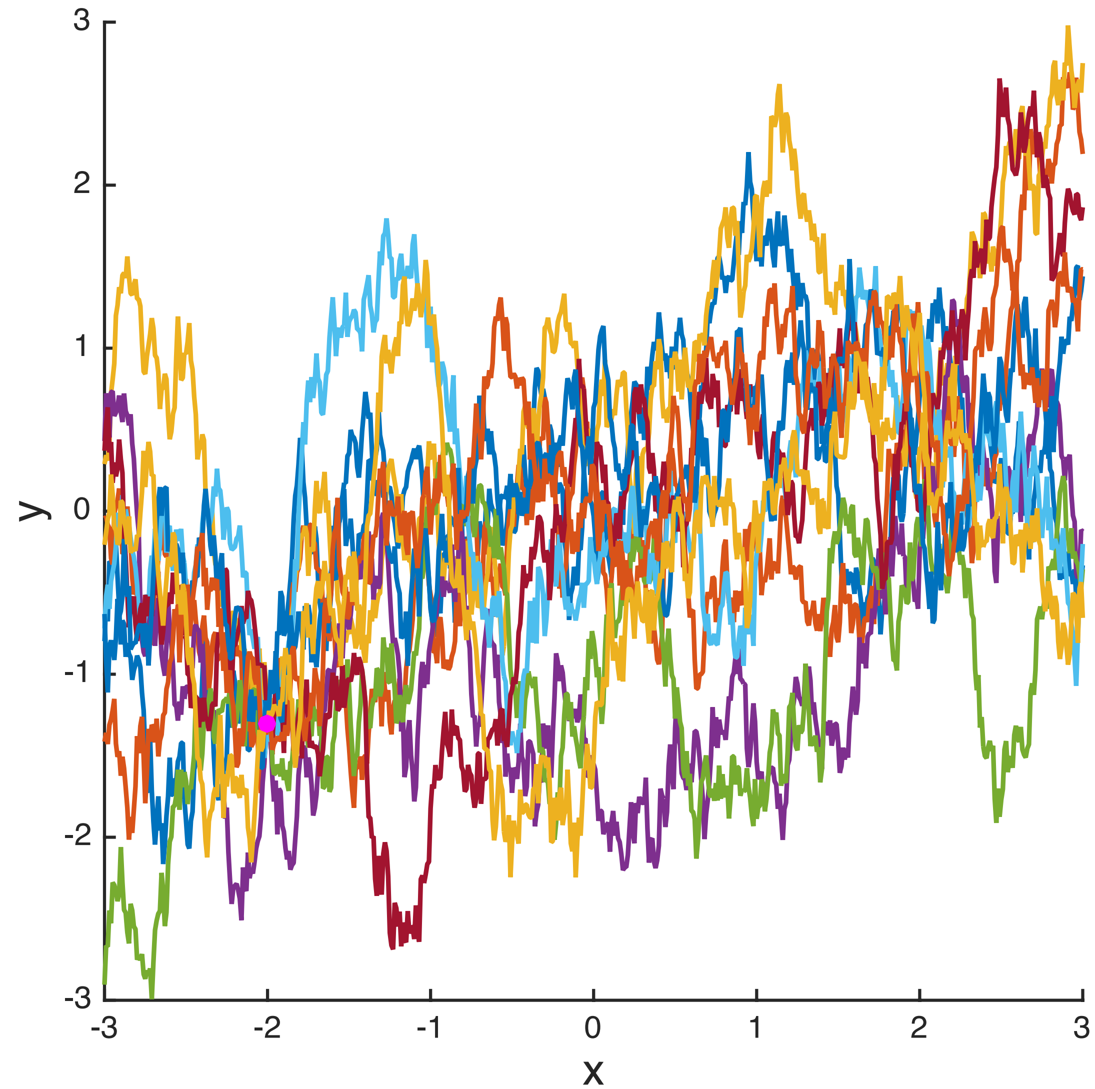
- Nearby values tend to be more correlated to each other



- Easier to predict values close to sample at $x = -2$

Function Distributions

- Nearby values tend to be more correlated to each other



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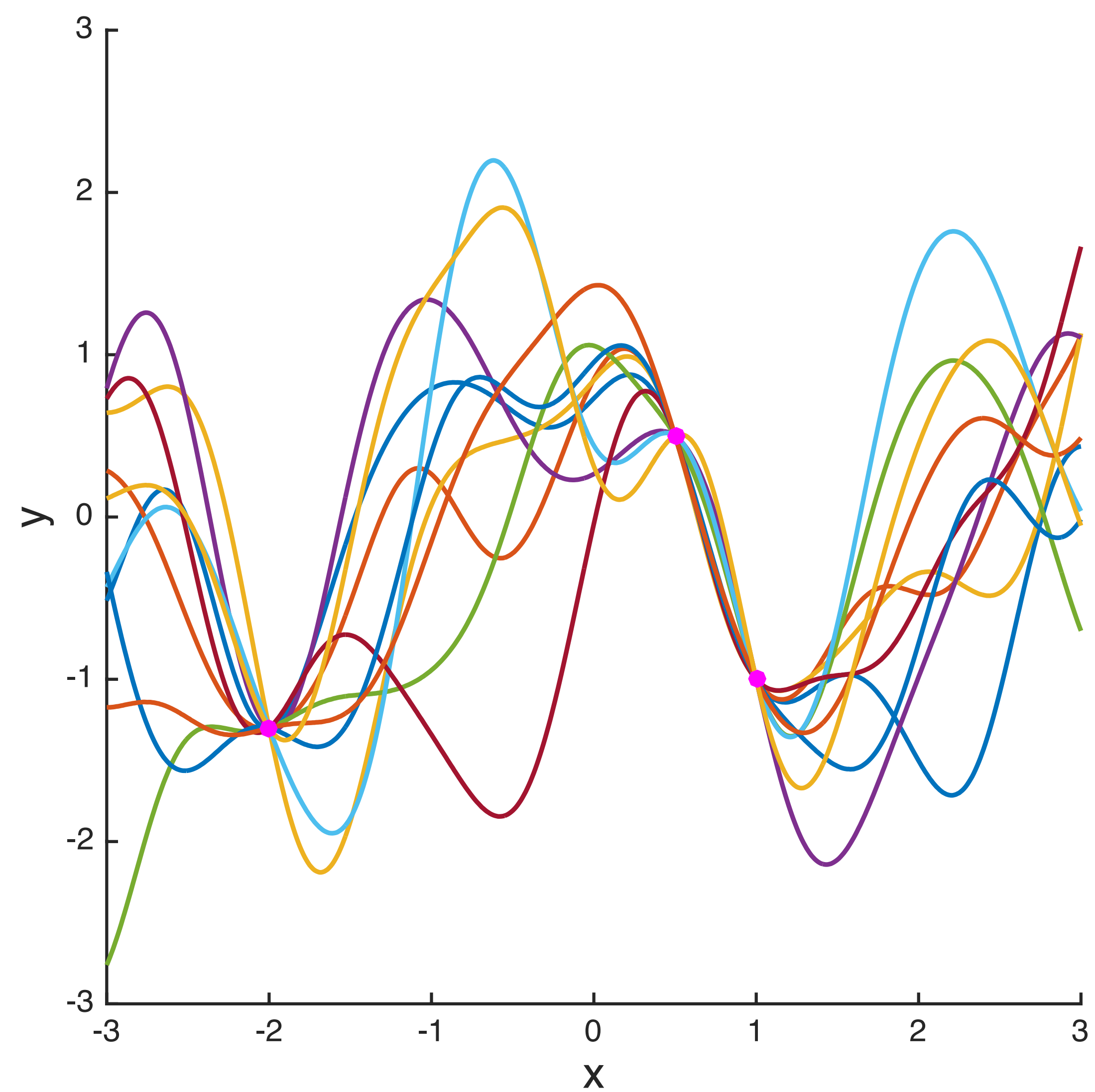


- Capture more of the shape with additional samples

Function Distributions

- Do not know the exact structure of the function
- Assume values are more correlated when Δx is smaller
  - If there is no smoothness/correlation, optimization is intractable
- Correlations provide local information about value
  - x is more likely to give high value if near a high-value sample
- Want to exploit these correlations for making prediction
- How to capture the correlation between the y values?

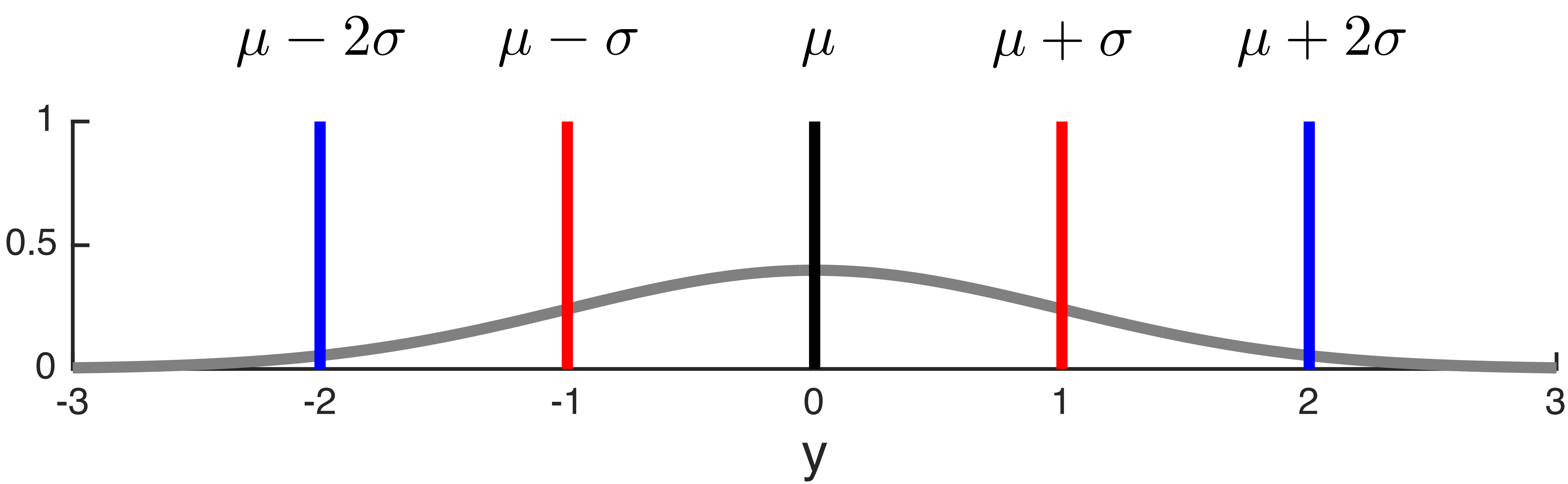
multivariate Gaussian distributions!

Gaussian Distribution

- Normal distribution

$$p(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{0.5(y-\mu)^2}{\sigma^2}\right)$$

- Standard normal distribution (zero mean, stddev of one)



Multivariate Gaussian Distributions

- For two or more variables we have the joint distribution

$$\begin{bmatrix} y_A \\ y_B \end{bmatrix} \sim p\left(y_A, y_B\right) = \mathcal{N}\left(\mu_{A,B}, \Sigma_{A,B}\right)$$

$$p\left(y_A, y_B\right) = \frac{1}{(2\pi)^{d/2} \det\left(\Sigma_{A,B}\right)^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} y_A \\ y_B \end{bmatrix} - \mu_{A,B}\right)^T \Sigma_{A,B}^{-1}\left(\begin{bmatrix} y_A \\ y_B \end{bmatrix} - \mu_{A,B}\right)\right)$$

  - Mean:

$$\mu_{A,B} = \begin{bmatrix} \mu_a \\ \mu_b \end{bmatrix}$$

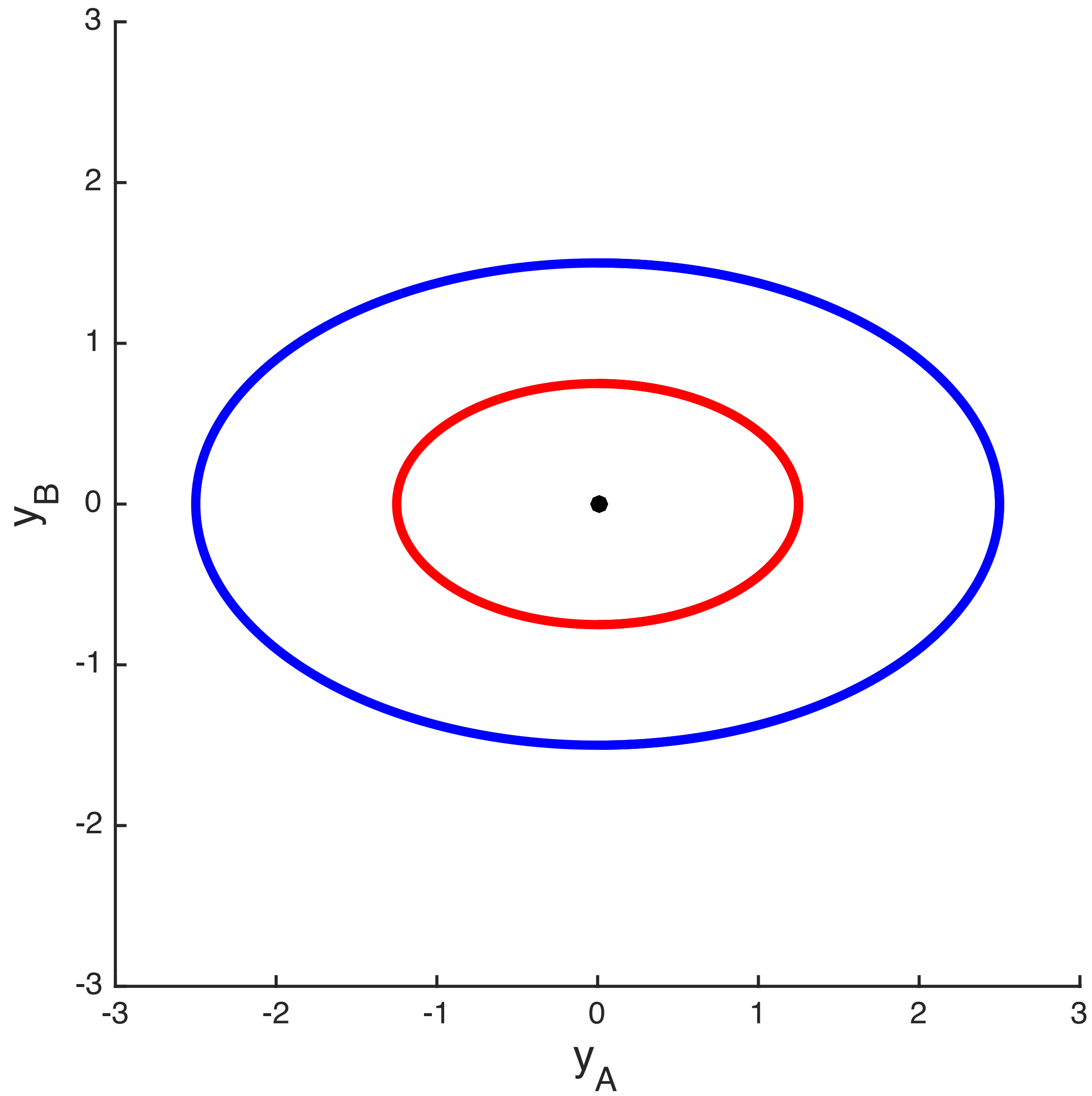
  - Covariance:

$$\Sigma_{A,B} = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix} \qquad \Sigma_{ba} = \Sigma_{ab}^T$$

- Focus on 2 samples, use notation for arbitrary number

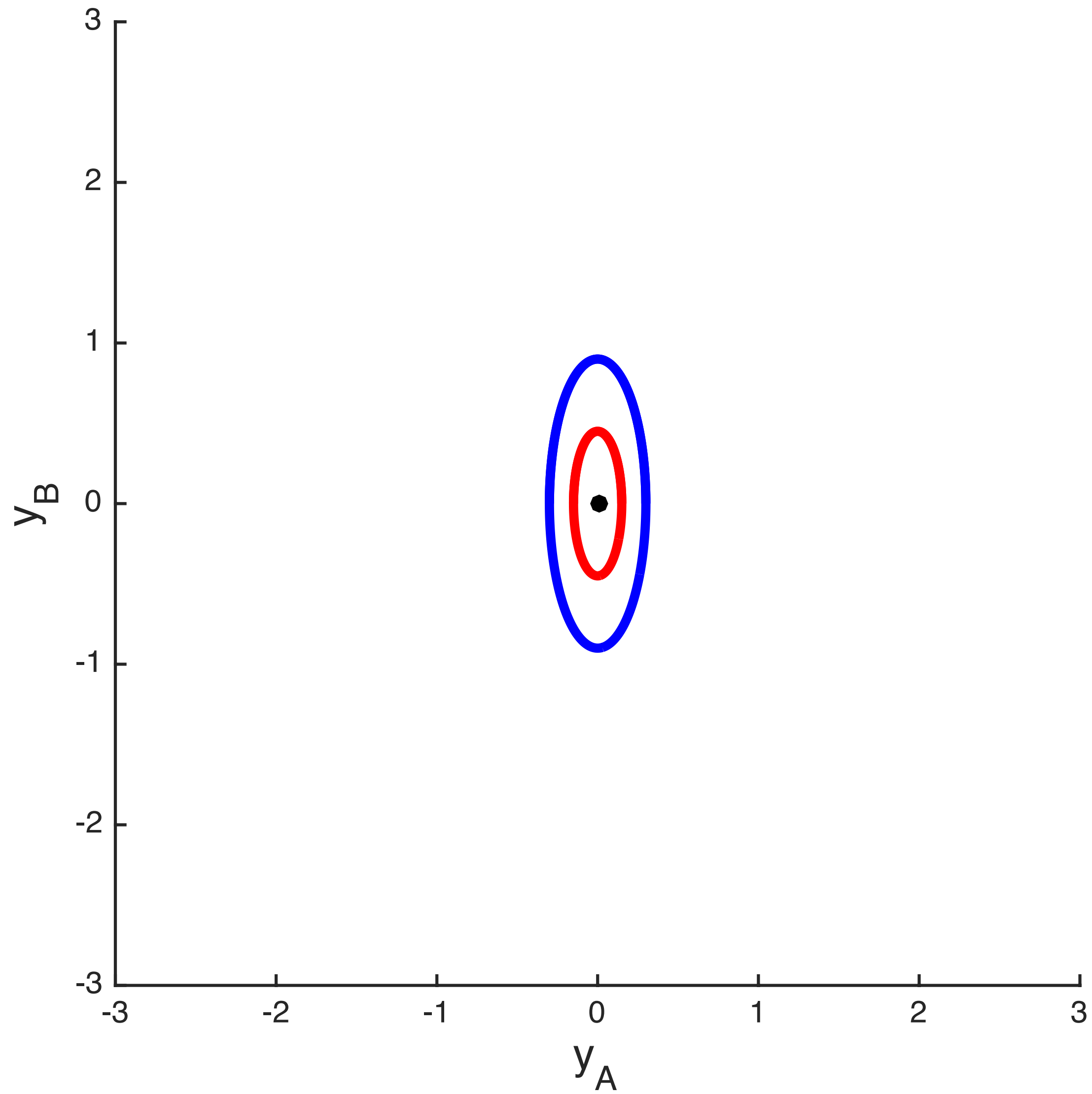
Joint Gaussian Distribution

$$\mu_{A,B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Sigma_{A,B} = \begin{bmatrix} 1.25^2 & 0 \\ 0 & 0.75^2 \end{bmatrix}$$



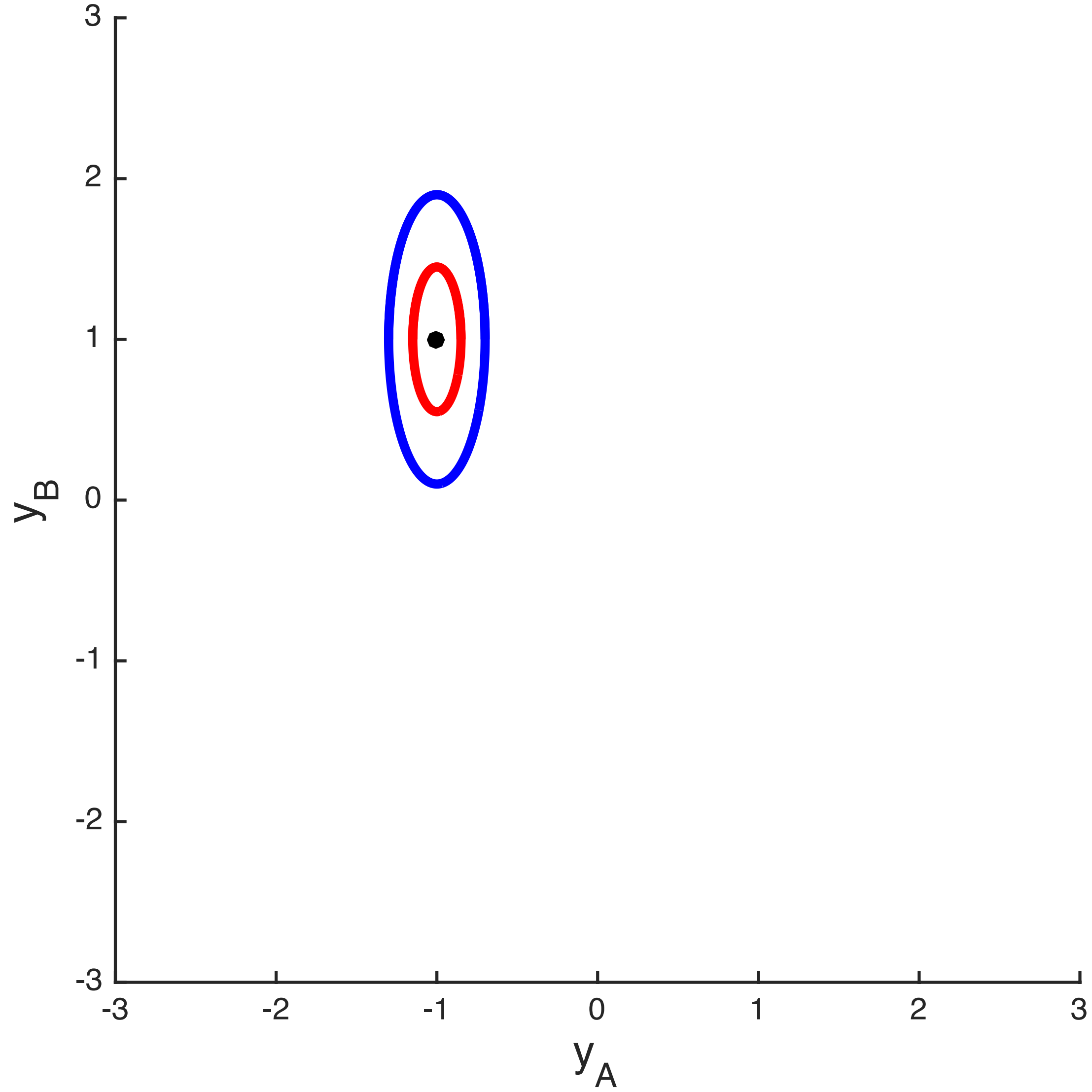
Joint Gaussian Distribution

$$\mu_{A,B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Sigma_{A,B} = \begin{bmatrix} 0.15^2 & 0 \\ 0 & 0.45^2 \end{bmatrix}$$



Joint Gaussian Distribution

$$\mu_{A,B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \Sigma_{A,B} = \begin{bmatrix} 0.15^2 & 0 \\ 0 & 0.45^2 \end{bmatrix}$$



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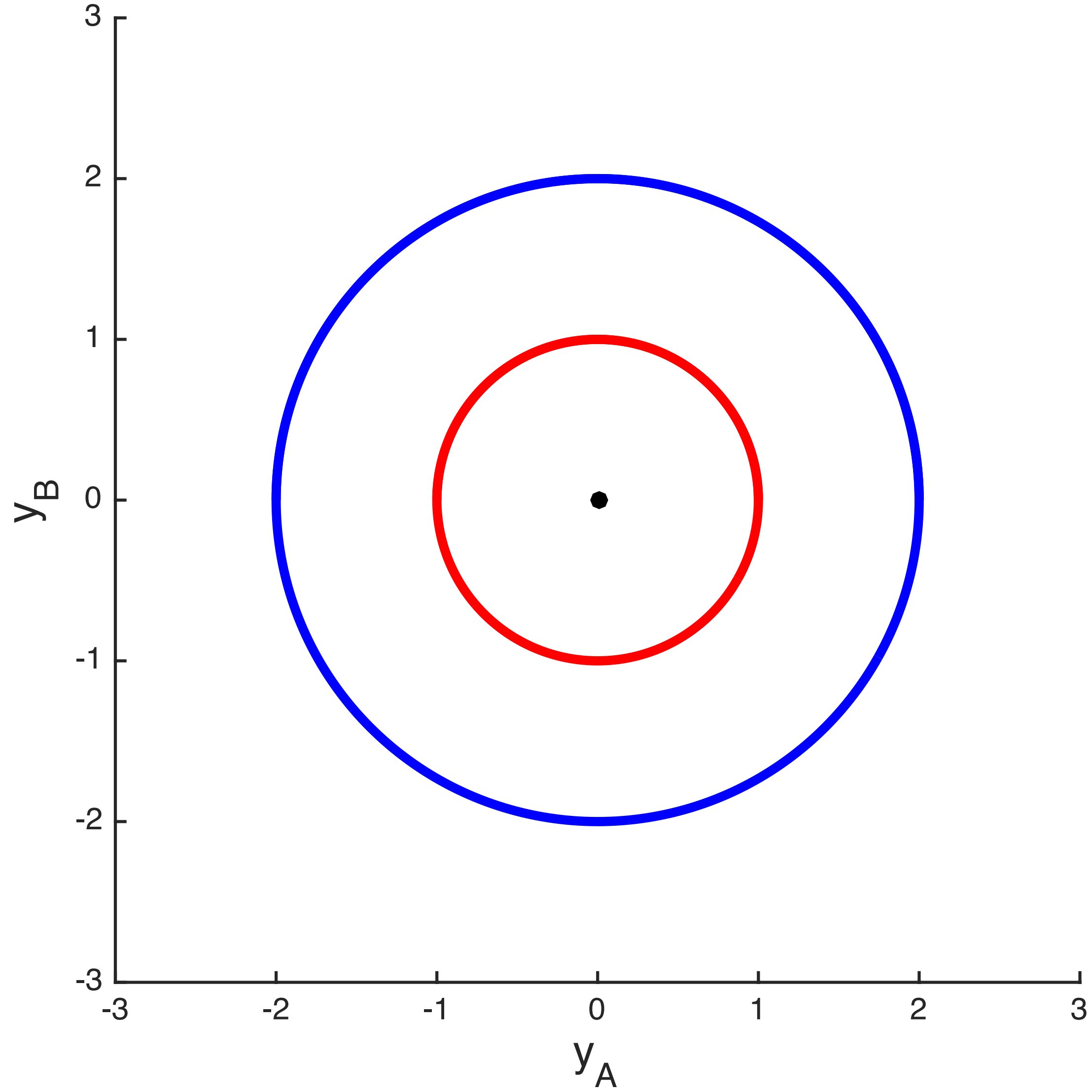
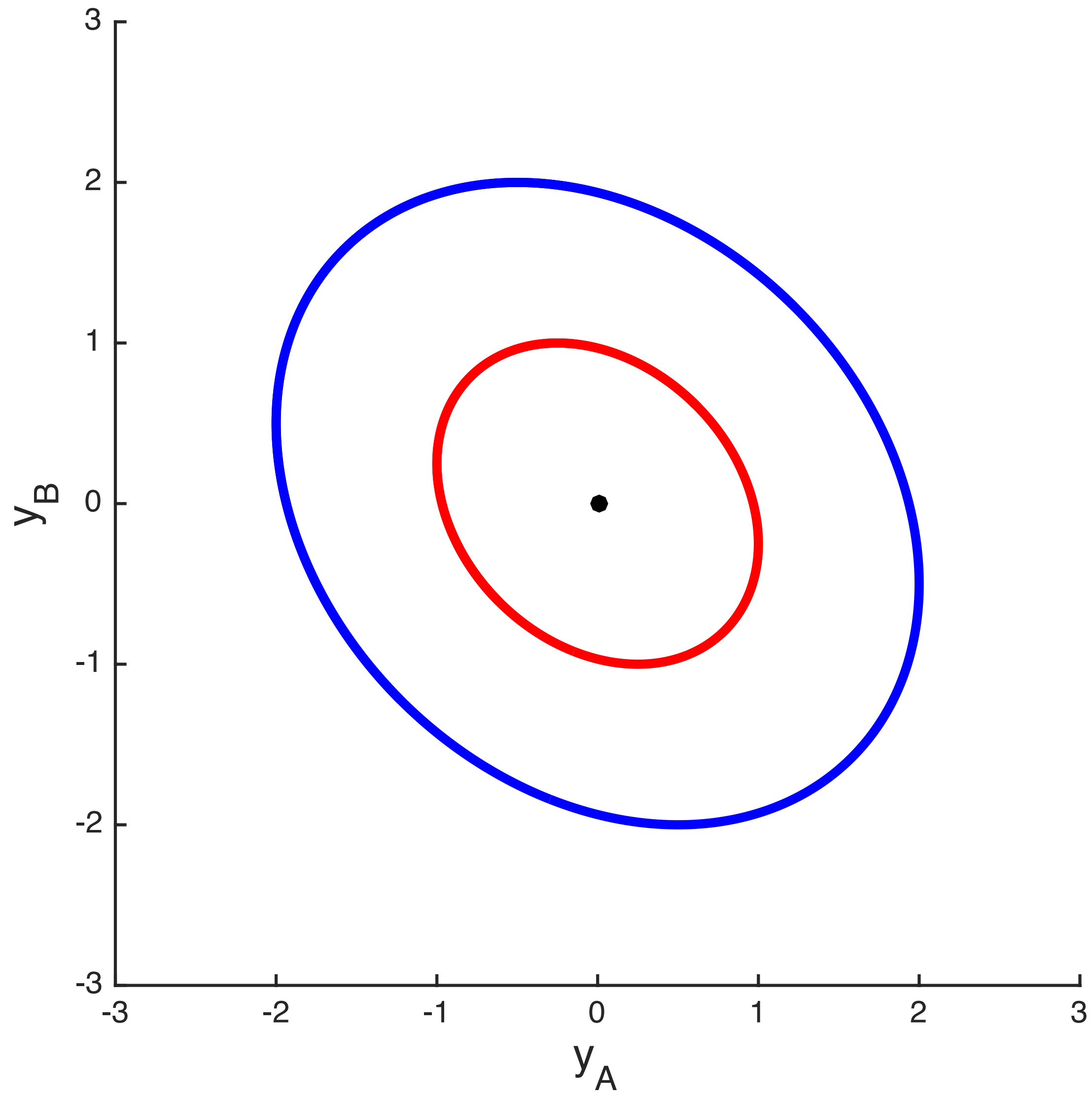


Joint Gaussian Distribution

$$\mu_{A,B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Sigma_{A,B} = \begin{bmatrix} 1 & -(0.5^2) \\ -(0.5^2) & 1 \end{bmatrix}$$



negative correlation

Joint Gaussian Distribution

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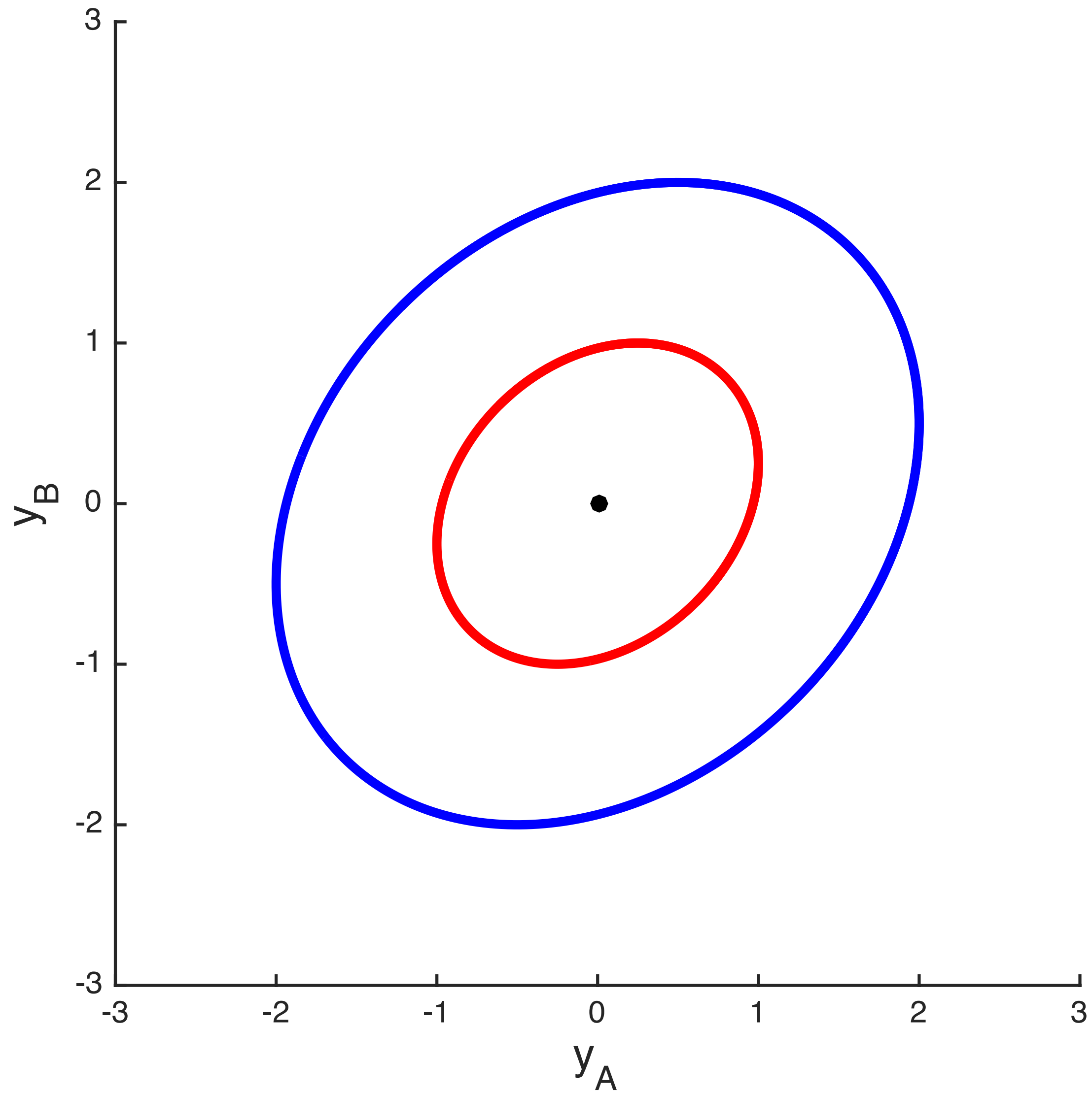

positive
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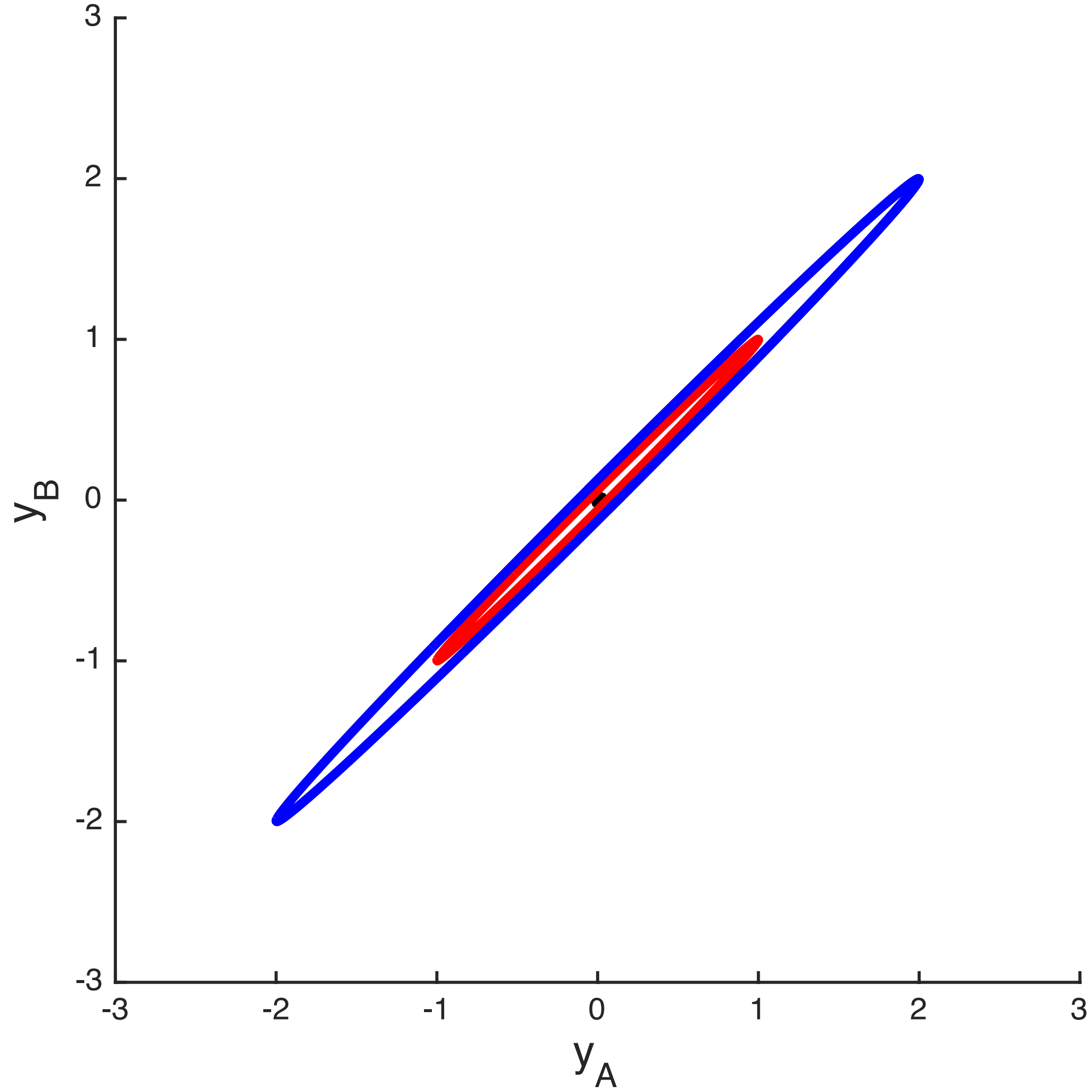

Conditional Gaussian Distribution

- Given the joint distribution:

$$\begin{bmatrix} y_A \\ y_B \end{bmatrix} \sim p(y_A, y_B) = \mathcal{N}(\mu_{A,B}, \Sigma_{A,B})$$

$$\mu_{A,B} = \begin{bmatrix} \mu_a \\ \mu_b \end{bmatrix} \qquad \Sigma_{A,B} = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}$$

- The conditional distribution of y_B given y_A is:

$$[y_B] \sim p(y_B | y_A) = \mathcal{N}\left(\mu_{B|A}, \Sigma_{B|A}\right)$$

$$\mu_{B|A} = \mu_b + \Sigma_{ba}\Sigma_{aa}^{-1}(y_a - \mu_a)$$

can redefine y with zero mean

$$\Sigma_{B|A} = \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}$$

  - Gives distribution over y_B after observing y_A

Conditional Gaussian Distribution

- Given the joint distribution:

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$$\mu_{B|A} = \underline{\Sigma_{ba}\Sigma_{aa}^{-1} y_a}$$ linear relationship

$$\Sigma_{B|A} = \Sigma_{bb} - \underline{\Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}}$$ covariance reduction

  - Gives distribution over y_B after observing y_A

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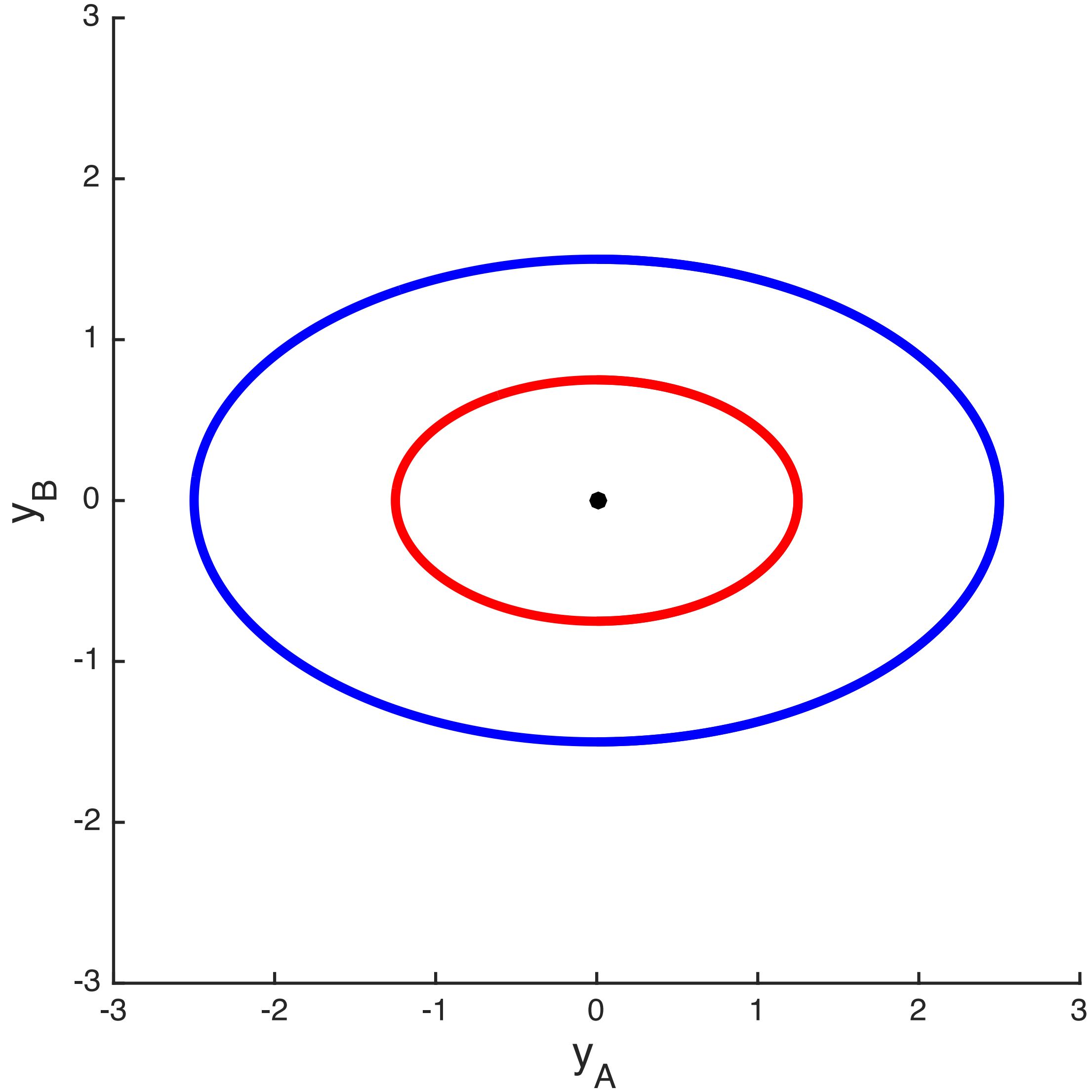
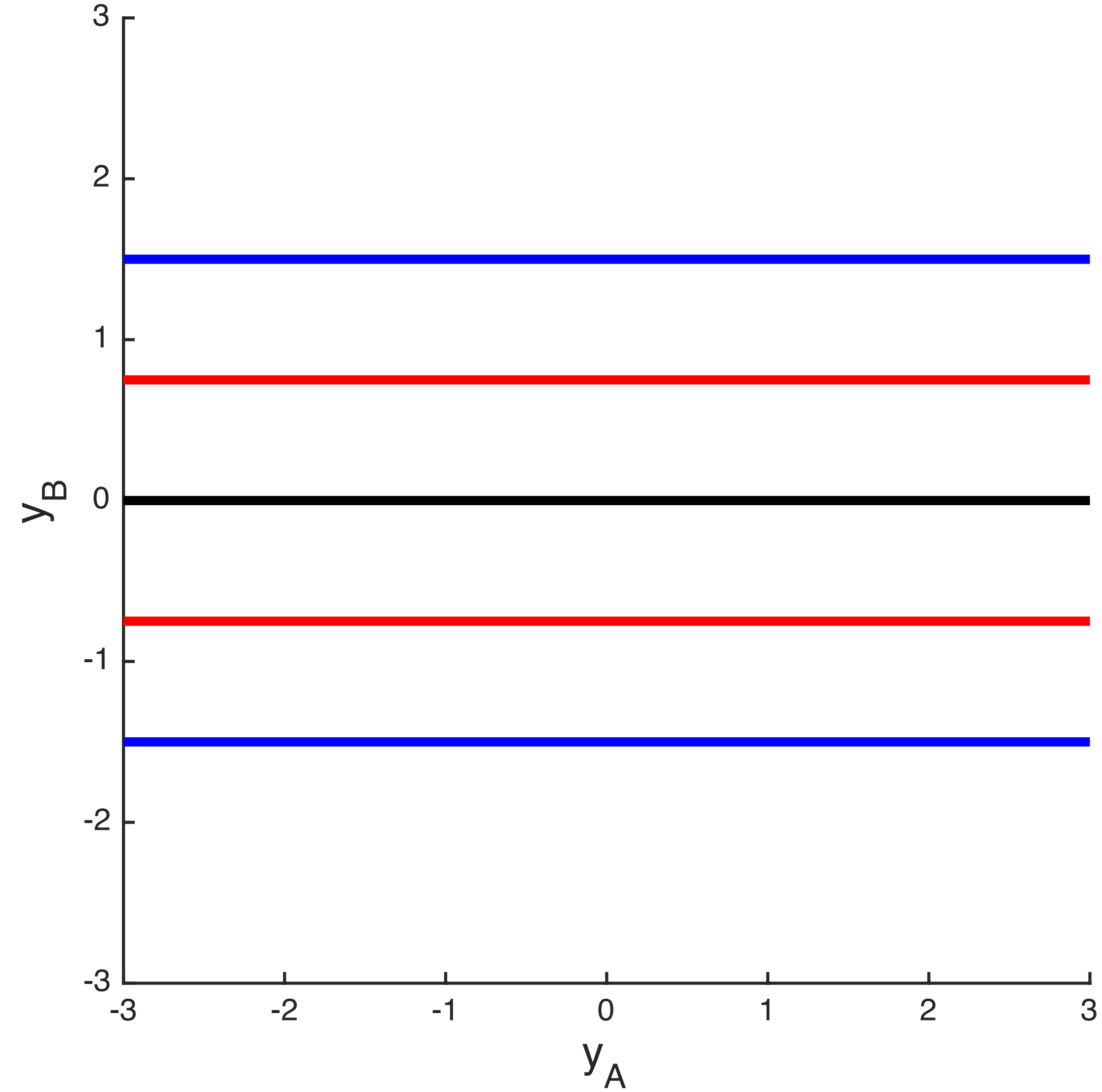


$$p(y_A, y_B)$$



$$p(y_B | y_A)$$

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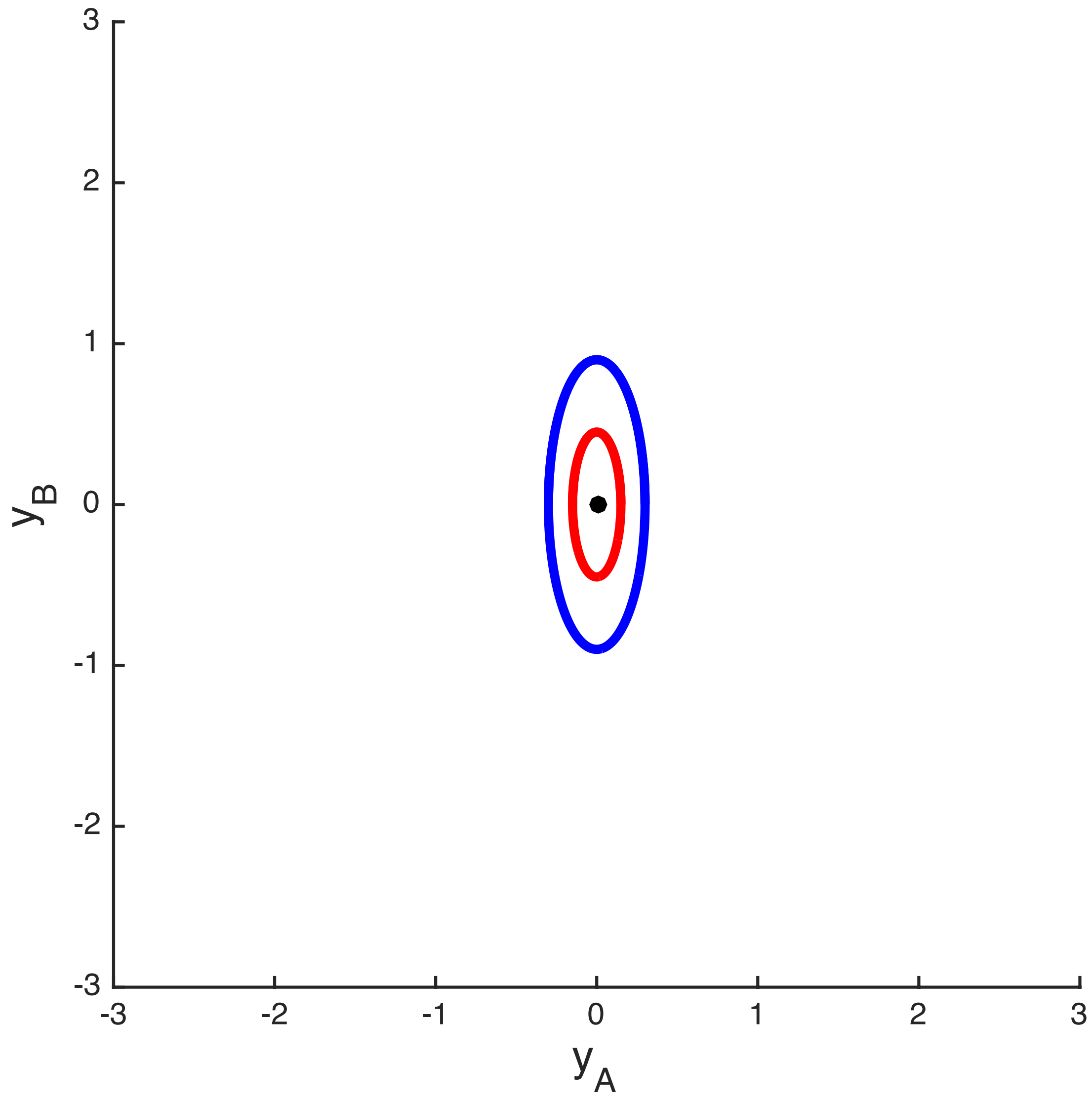

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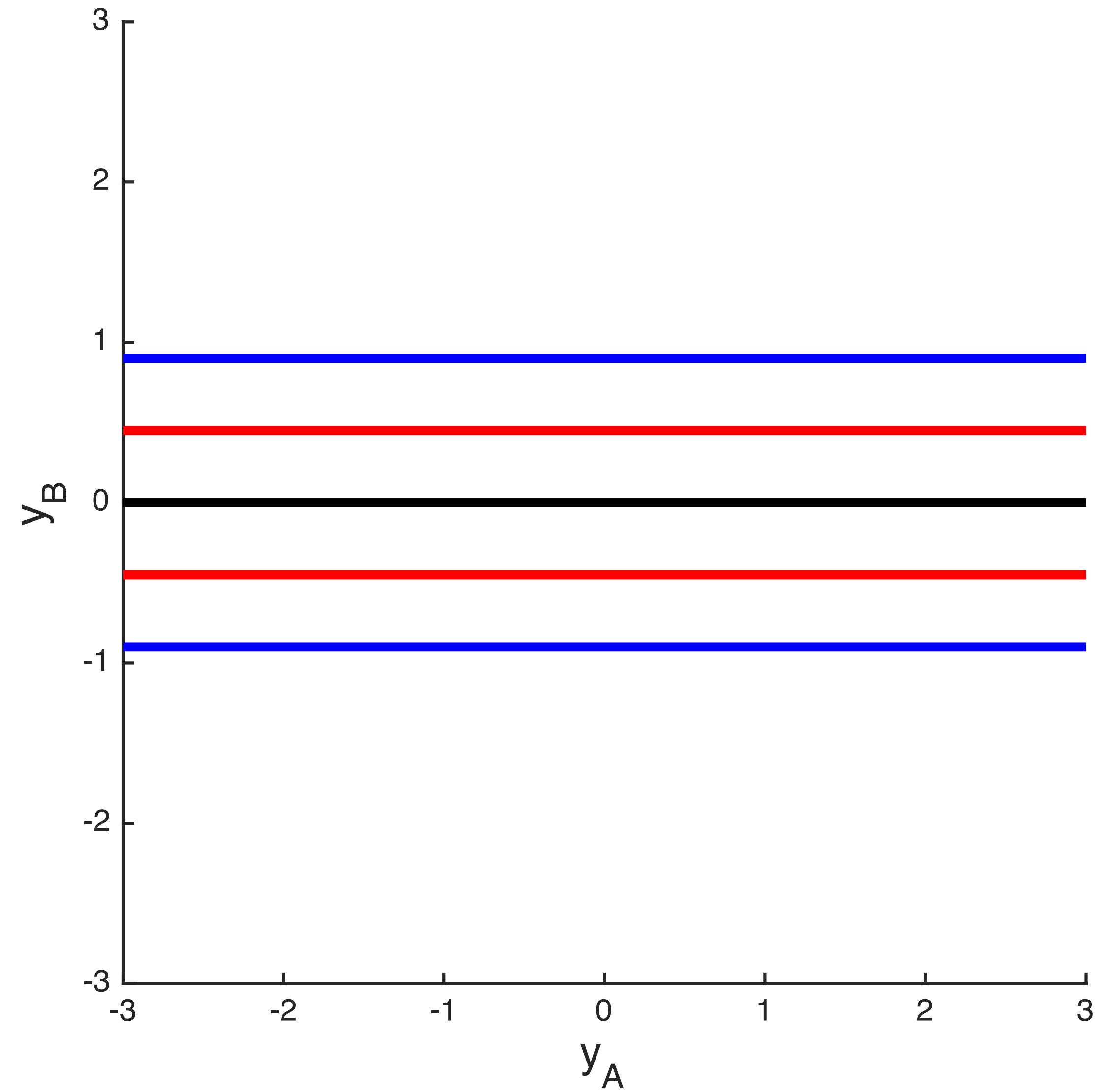

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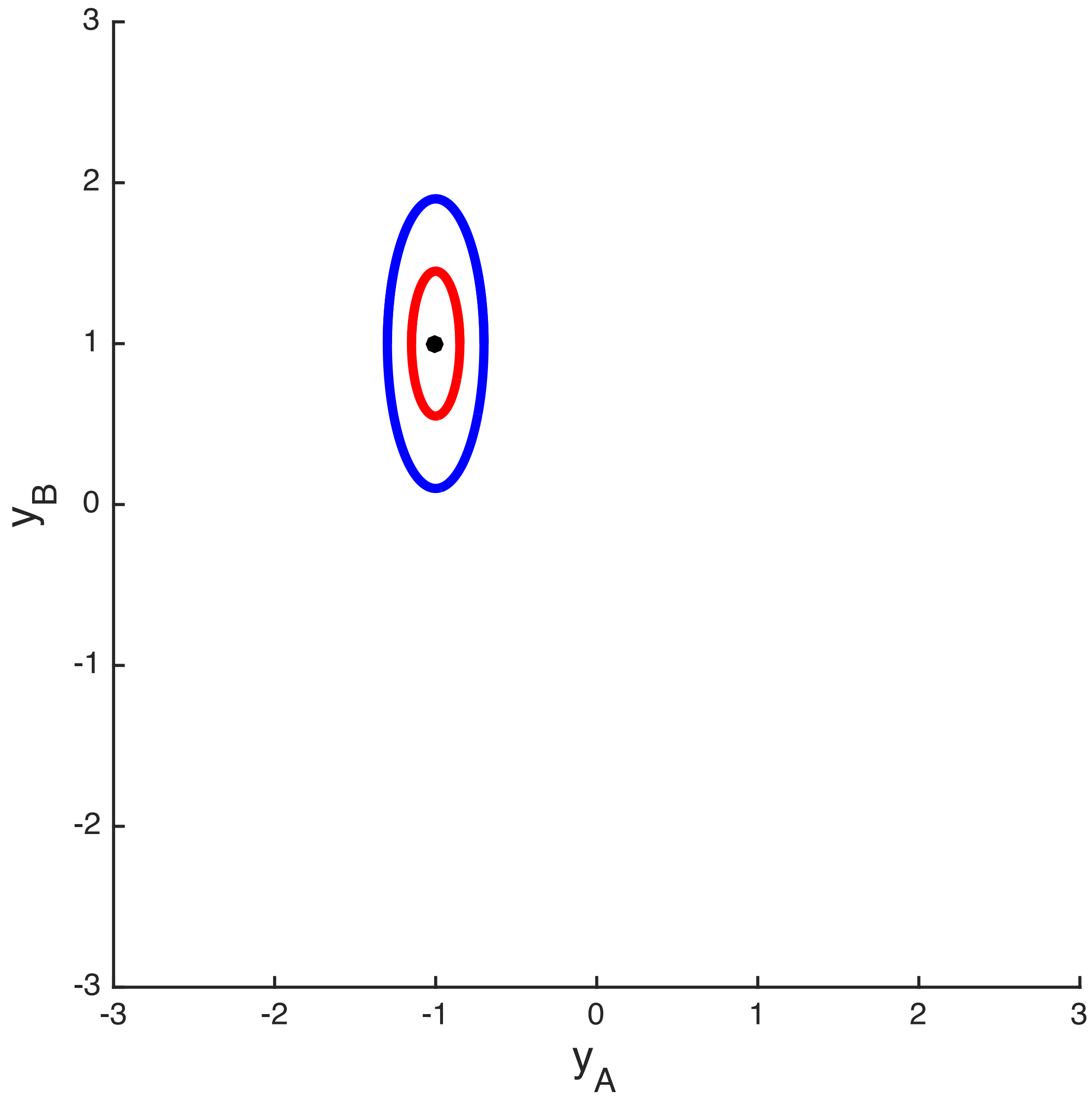

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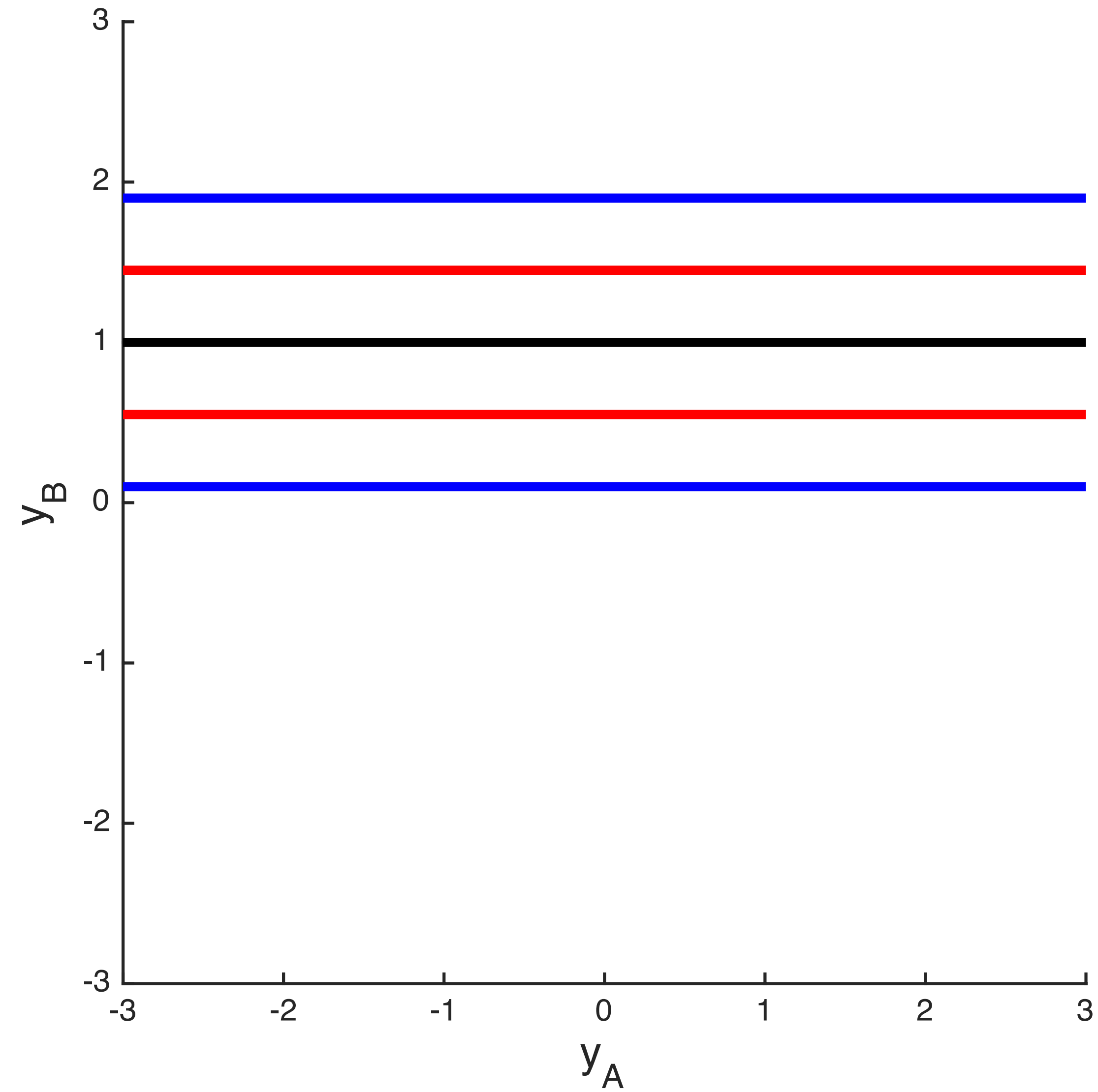

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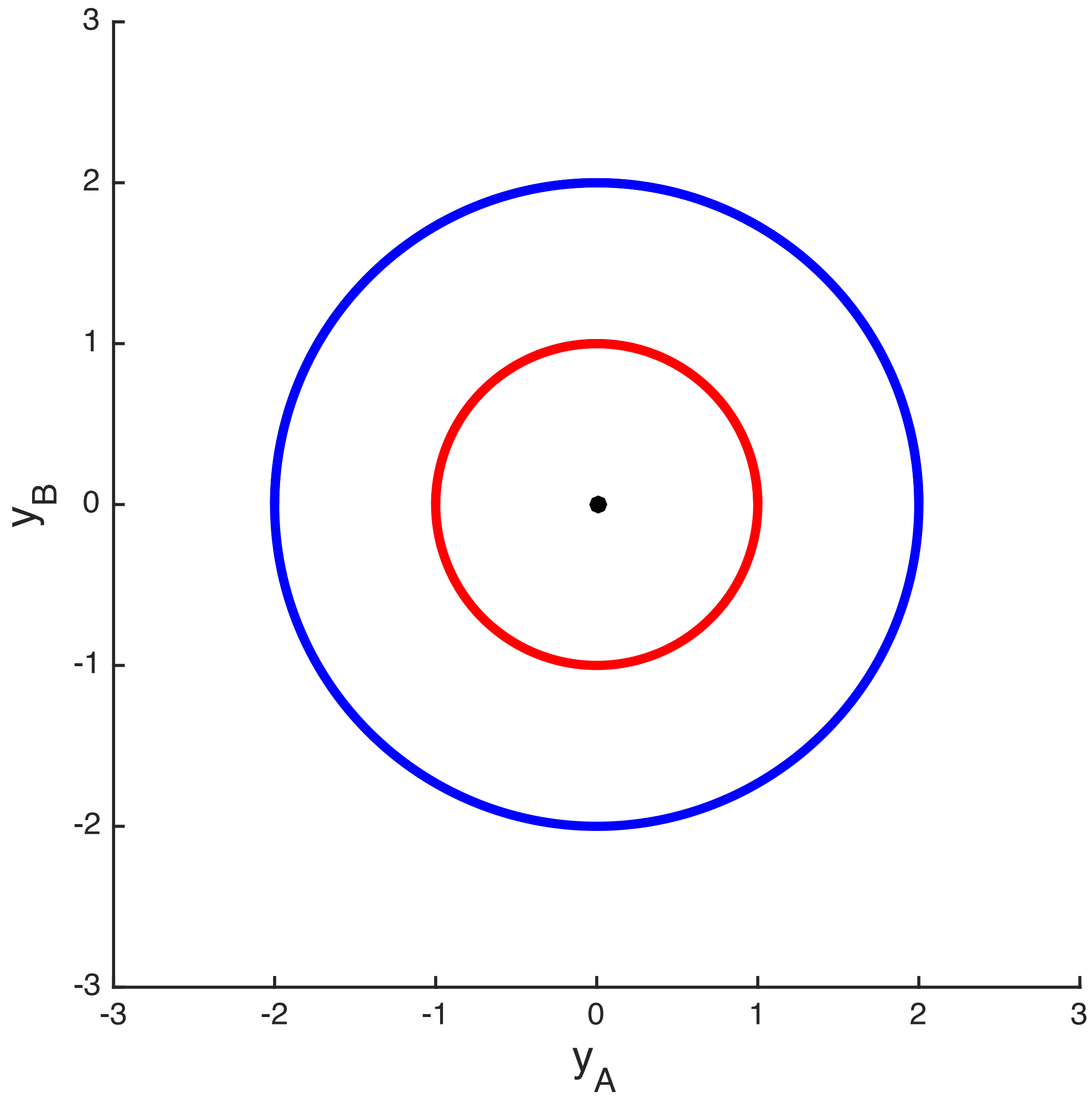

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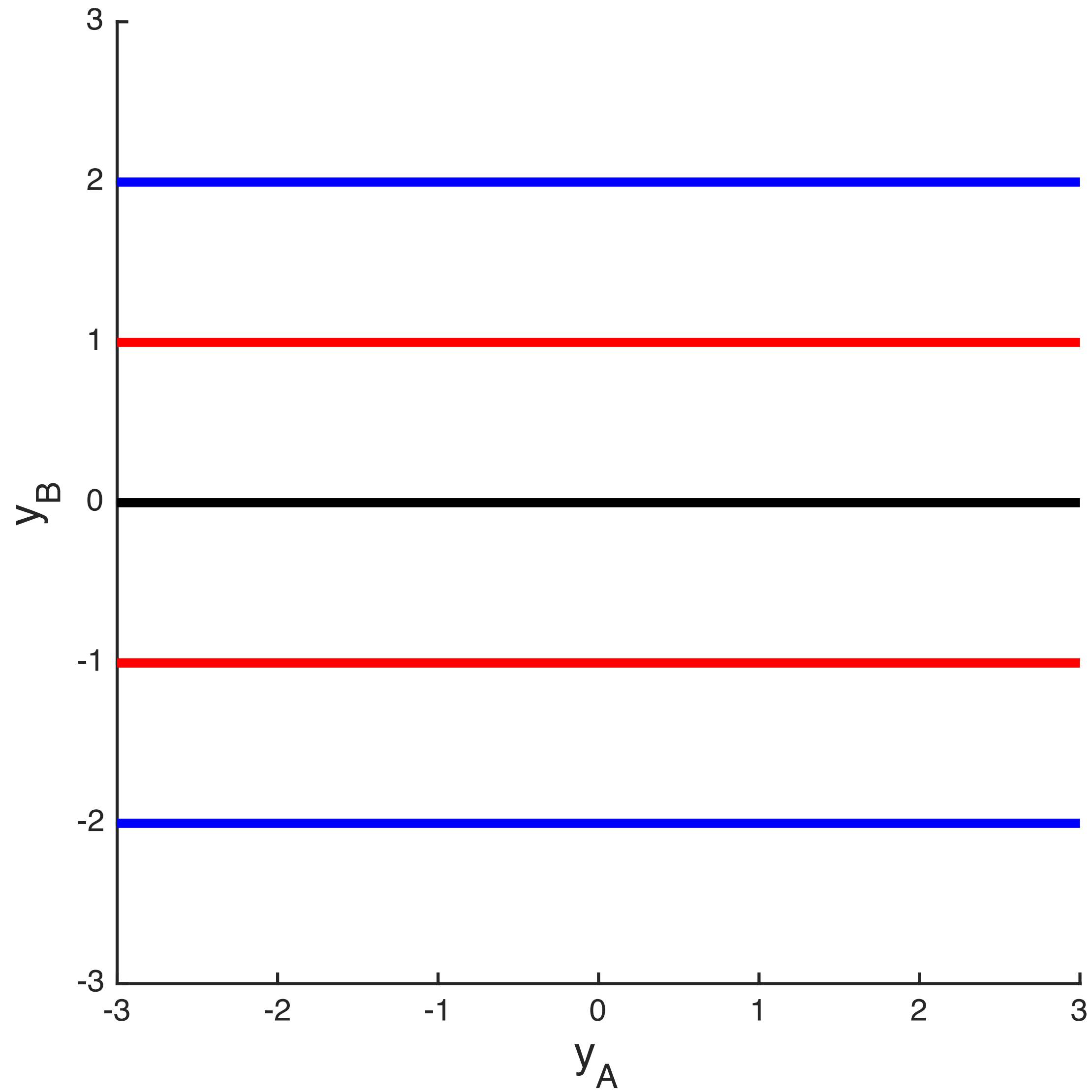

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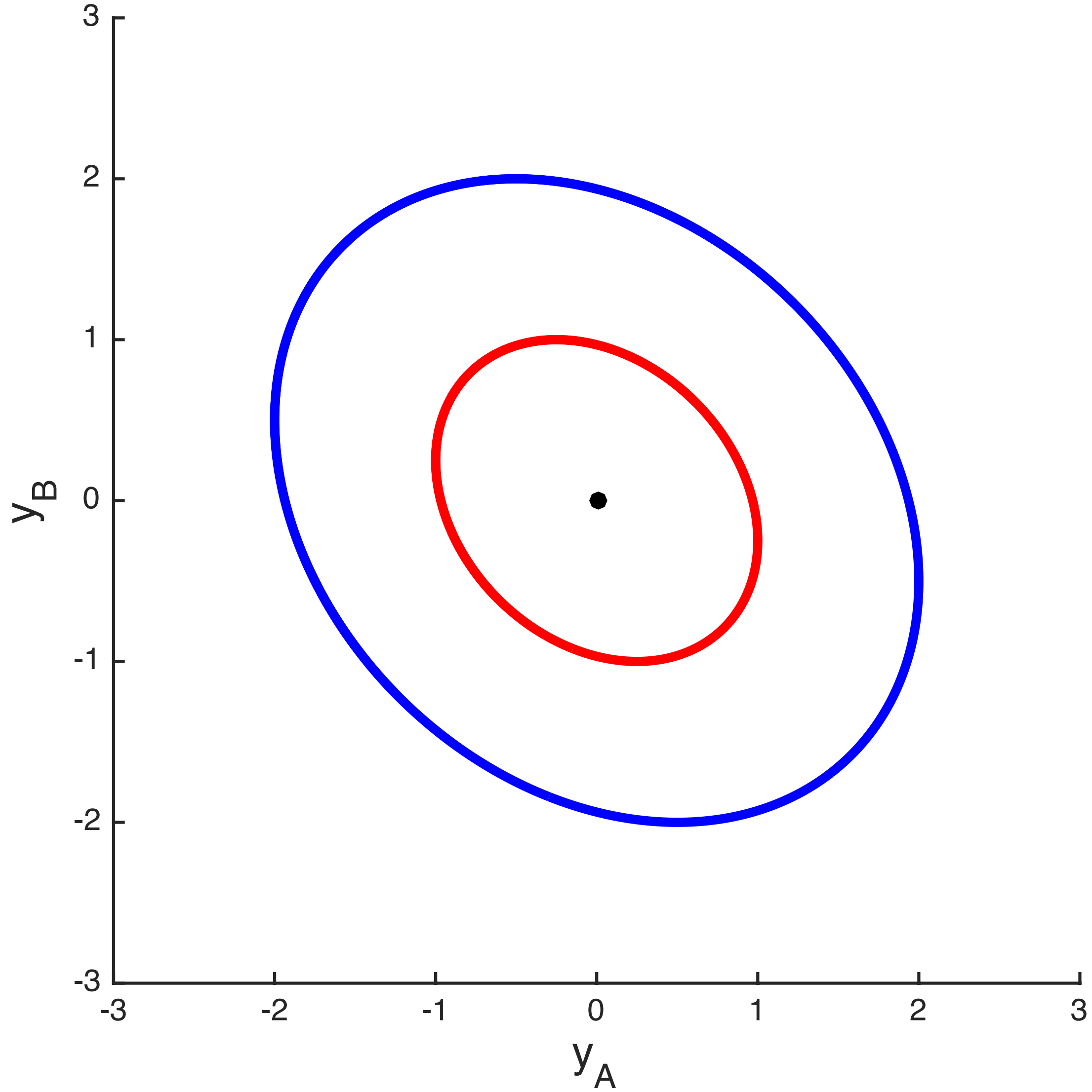

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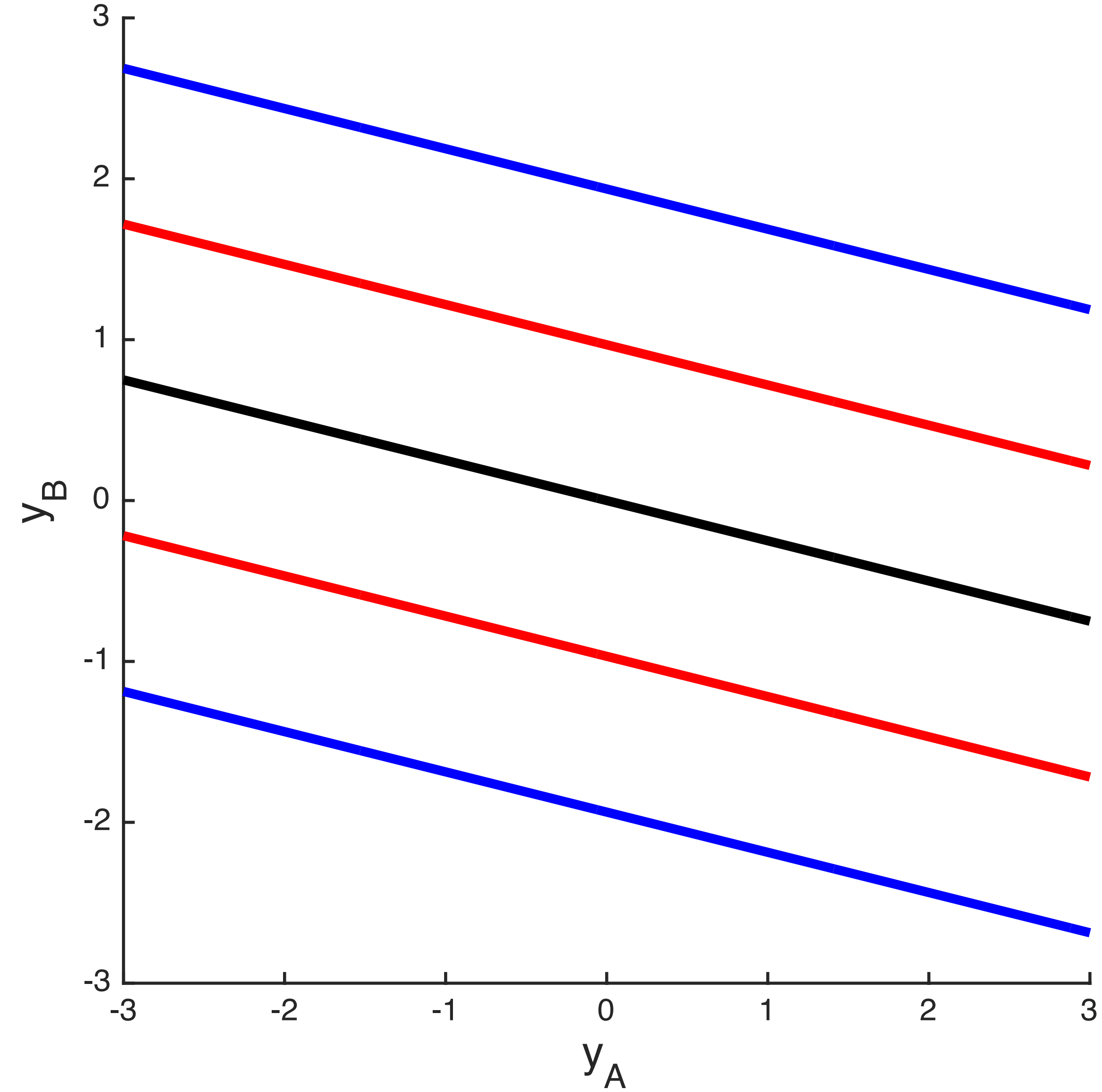

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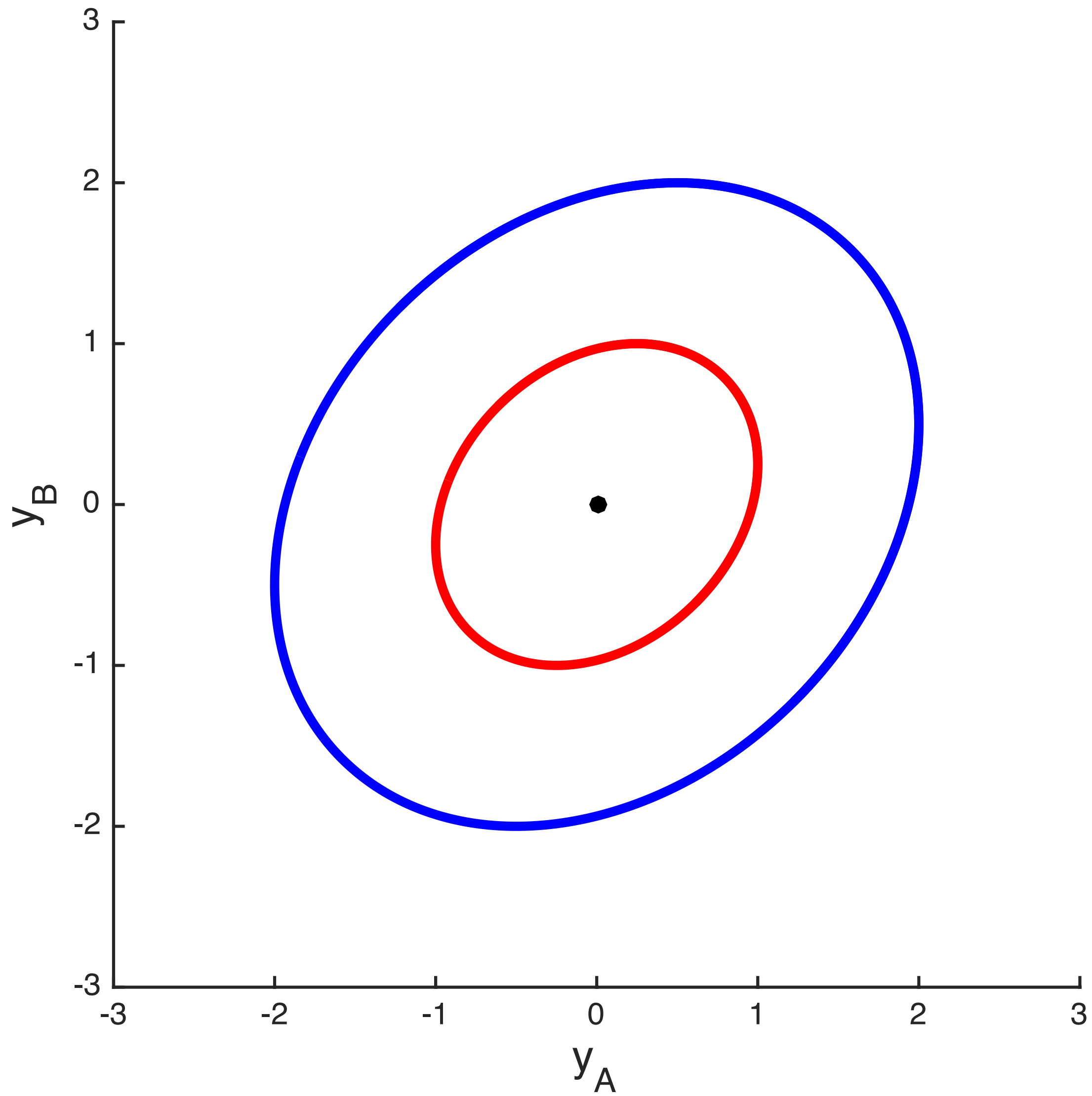

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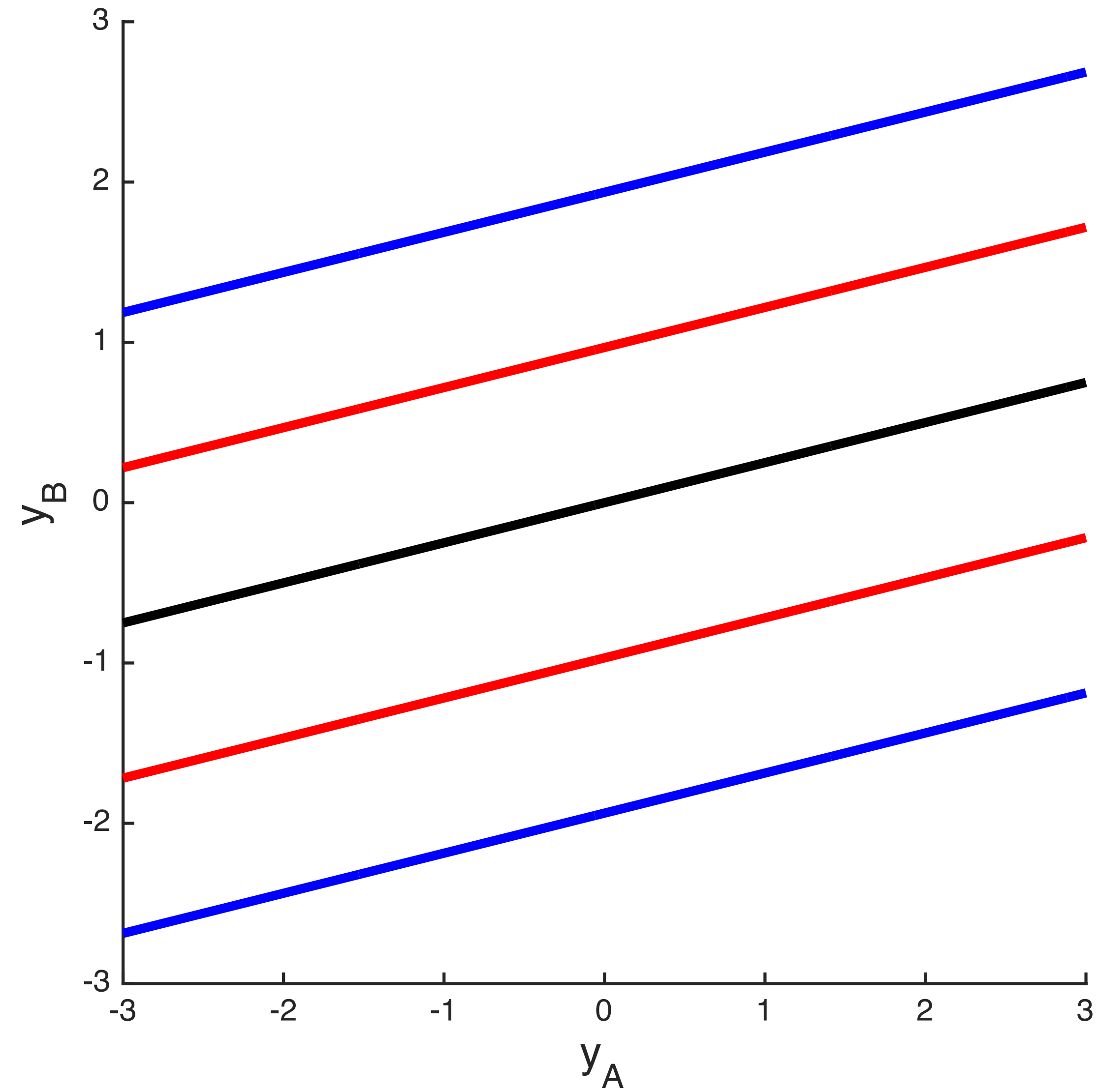

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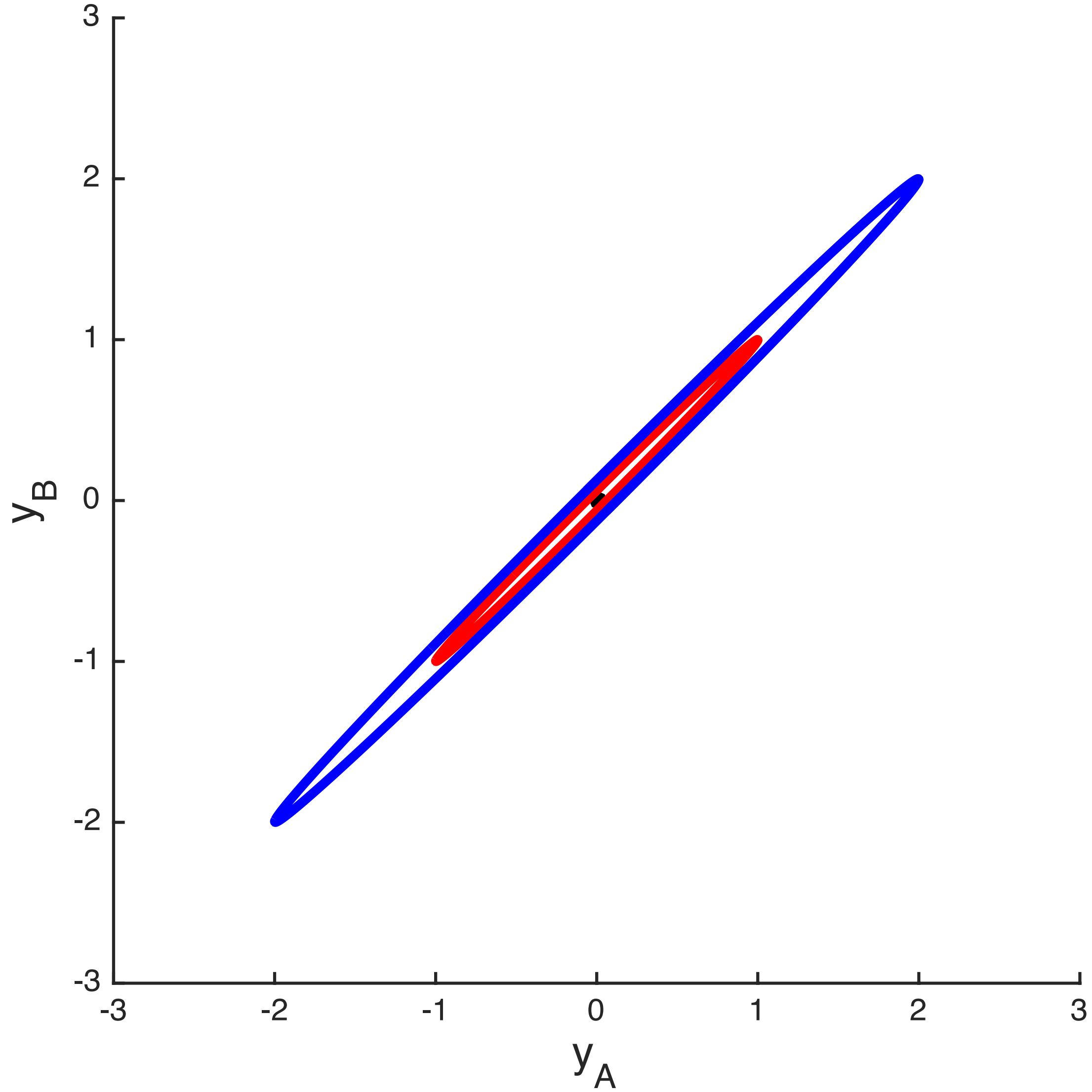

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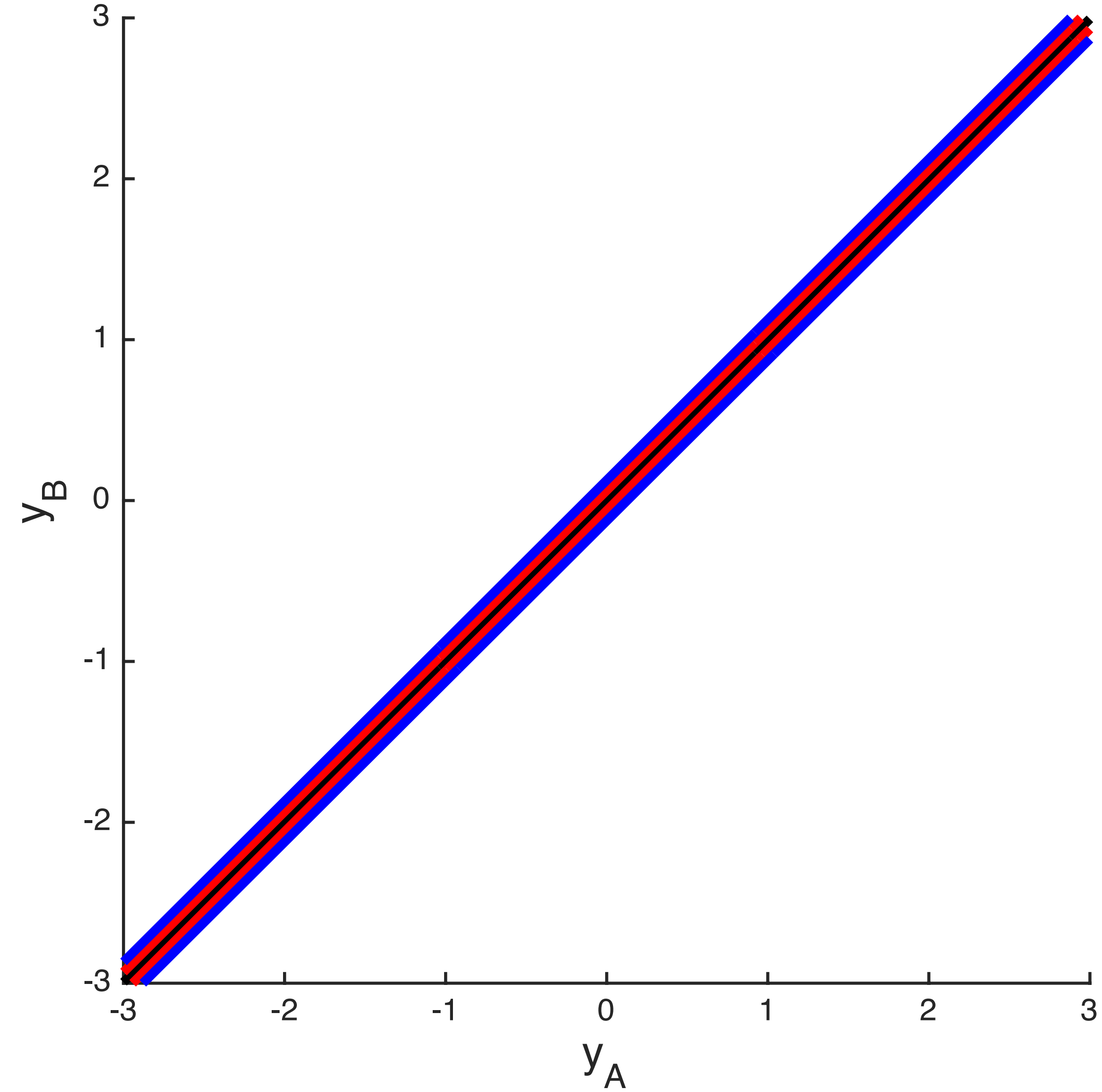
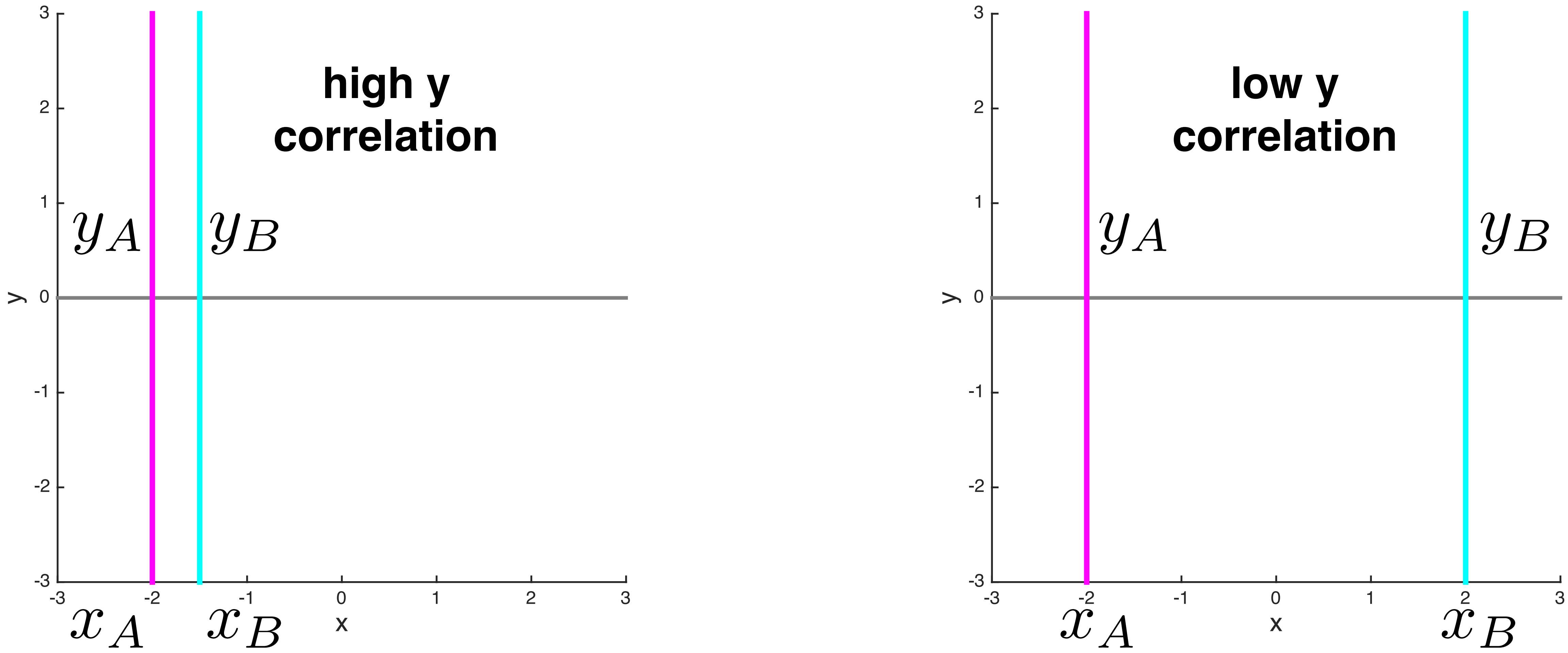


$p(y_B|y_A)$

$$\mu_{B|A} = \Sigma_{ba}\Sigma_{aa}^{-1}y_a$$
$$\Sigma_{B|A} = \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}$$

Regression

- Want to capture correlations between function samples



- Use Gaussian distributions to capture value correlations

$$\begin{bmatrix} y_A(x_A) \\ y_B(x_B) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix} \right)$$

Kernel Functions

- Need to specify covariance matrix as functions of x_A, x_B

$$\begin{bmatrix} y_A(x_A) \\ y_B(x_B) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix} \right)$$

- Valid covariance matrix must be positive definite
- Use kernel functions to specify covariances

$$\begin{bmatrix} y_A(x_A) \\ y_B(x_B) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} K(x_A, x_A) & K(x_A, x_B) \\ K(x_B, x_A) & K(x_B, x_B) \end{bmatrix} \right)$$

- For multiple samples in x_A and x_B the K are matrices:

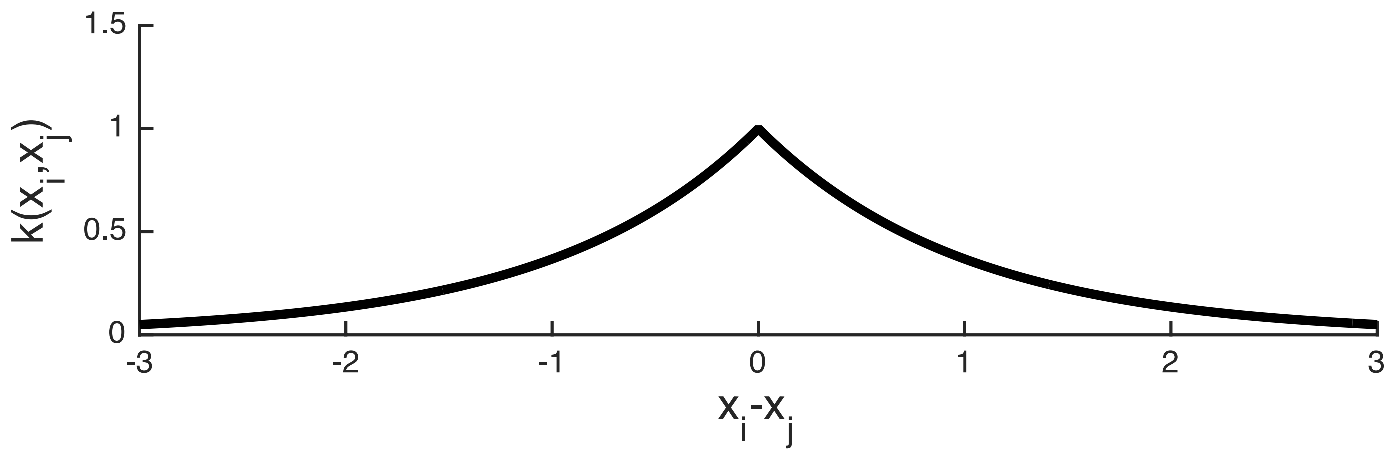
$$[K(x_C, x_D)]_{i,j} = k(x_{Ci}, x_{Dj})$$

  - i.e., apply kernel function to each pair of samples

Example Kernel Functions

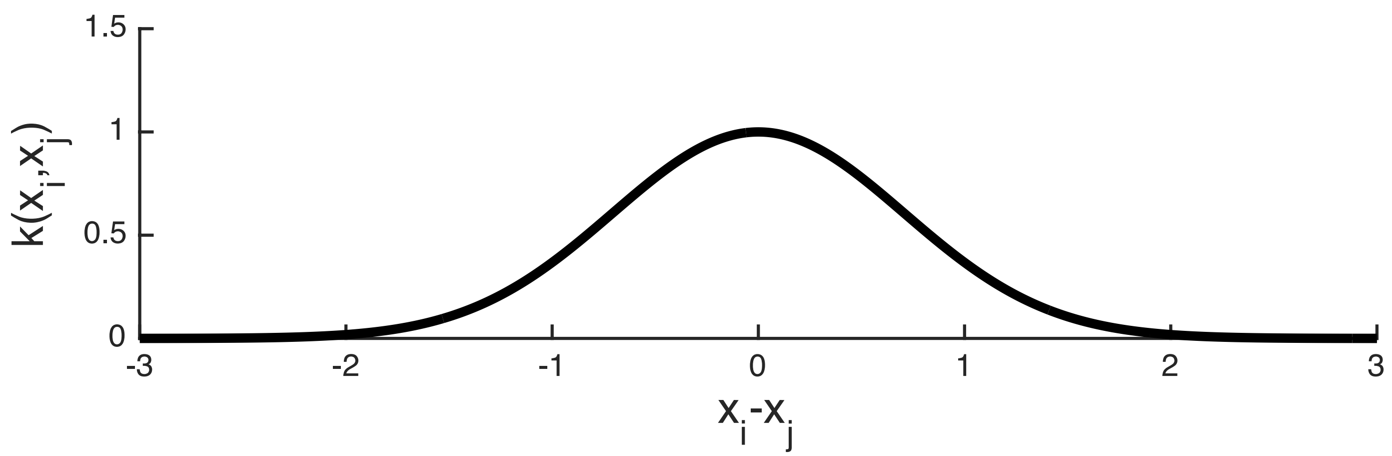
- Exponential kernel function

$$k(x_i, x_j) = \sigma_a^2 \exp(-\left\| x_i - x_j \right\| / \sigma_l)$$



- Squared exponential kernel function

$$k(x_i, x_j) = \sigma_a^2 \exp(-(x_i - x_j)^2 / \sigma_l^2)$$



- Hyperparameters: amplitude σ_a^2 and length scale σ_l

Noise

- Are the y values identical for samples of the same x ???

Noise

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no, noisy output

$$y = f(x) + \epsilon \qquad\qquad \epsilon \sim \mathcal{N}(0, \sigma_n^2)$$

- Need to add sample noise to the model

$$\begin{bmatrix} y_A(x_A) \\ y_B(x_B) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} K(x_A, x_A) & K(x_A, x_B) \\ K(x_B, x_A) & K(x_B, x_B) \end{bmatrix} + \sigma_n^2 I \right)$$

  - Additional noise variance hyperparameter

- Correlations between samples rather than function values

Gaussian Process Regression

$$\begin{bmatrix} y_A(x_A) \\ y_B(x_B) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} K(x_A, x_A) & K(x_A, x_B) \\ K(x_B, x_A) & K(x_B, x_B) \end{bmatrix} + \sigma_n^2 I \right)$$

- Using our conditional Gaussian equations we get

$$\mu_{B|A} = K(x_B, x_A)(K(x_A, x_A) + \sigma_n^2 I)^{-1} y_A$$

$$\Sigma_{B|A} = K(x_B, x_B) + \sigma_n^2 I - K(x_B, x_A)(K(x_A, x_A) + \sigma_n^2 I)^{-1} K(x_A, x_B)$$

- In practice, the A has m samples and B usually has one

$$\mu_{B|A} = k(x_A, x_B)^T (K(x_A, x_A) + \sigma_n^2 I)^{-1} y_A$$

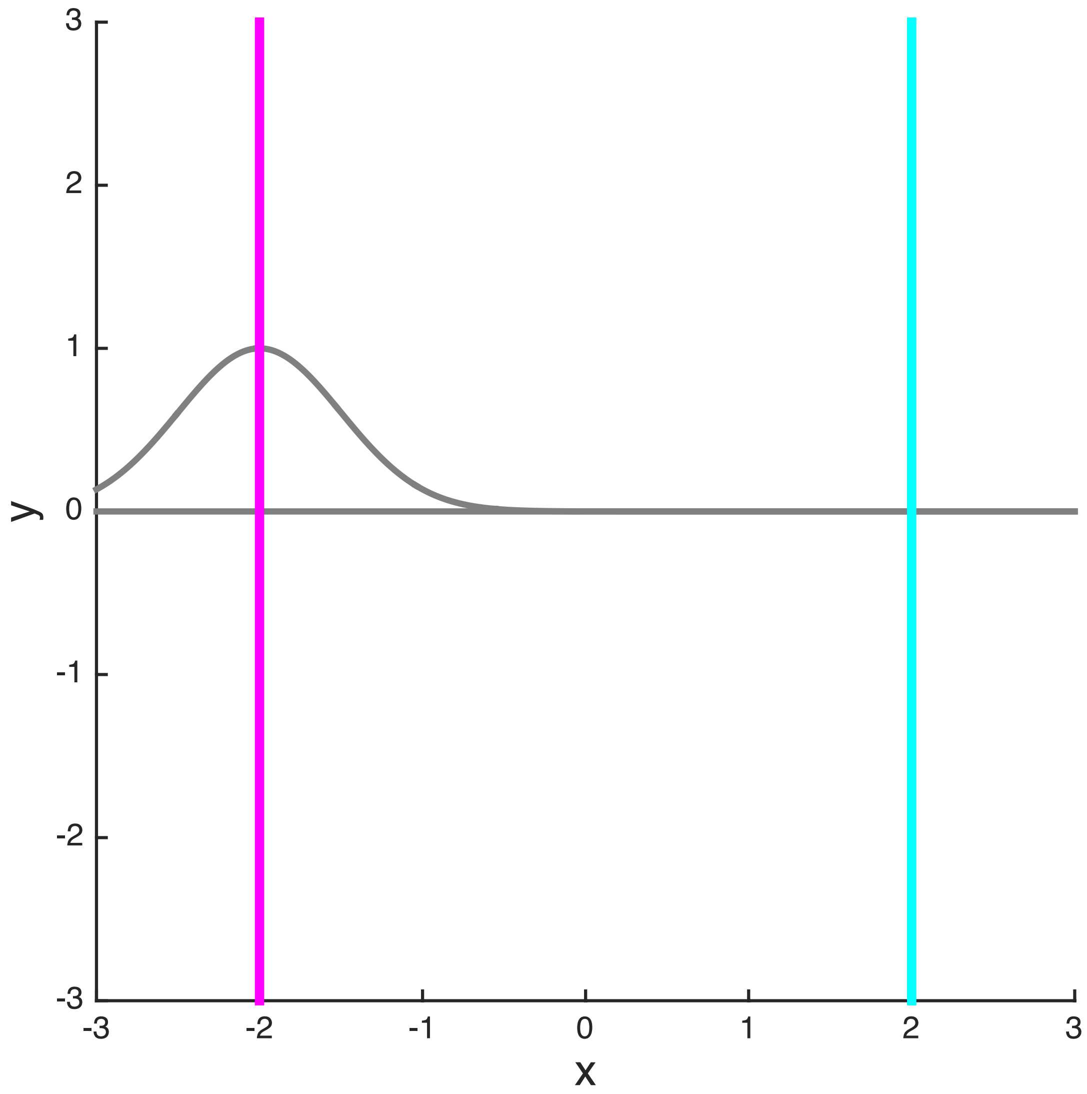
$1 \times m \qquad m \times m \qquad m \times m \qquad m \times 1$

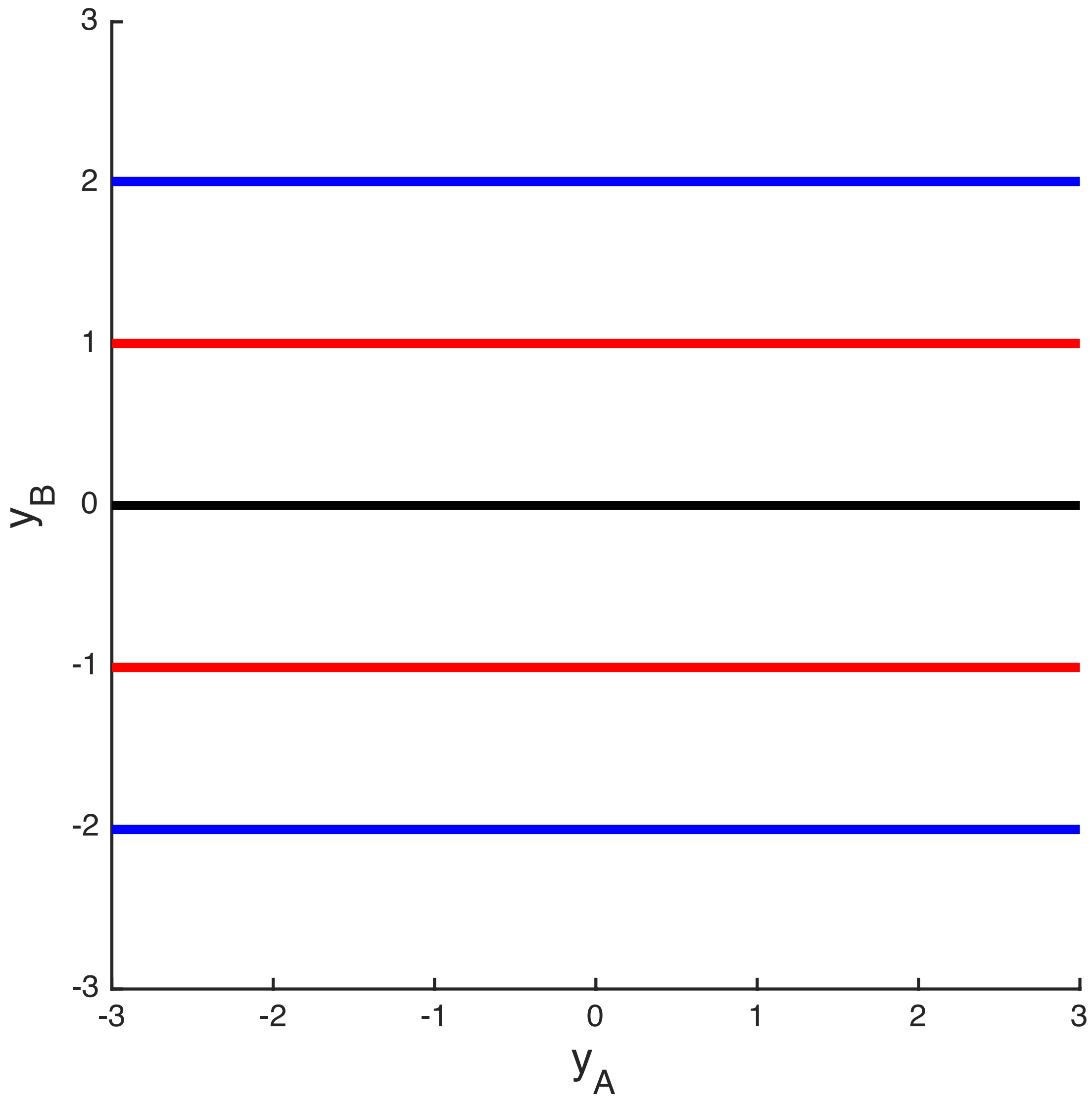
$$\Sigma_{B|A} = k(x_B, x_B) + \sigma_n^2 - k(x_A, x_B)^T (K(x_A, x_A) + \sigma_n^2 I)^{-1} k(x_A, x_B)$$

$1 \times 1 \qquad 1 \times 1 \qquad 1 \times m \qquad m \times m \qquad m \times m \qquad m \times 1$

Gaussian Process Regression

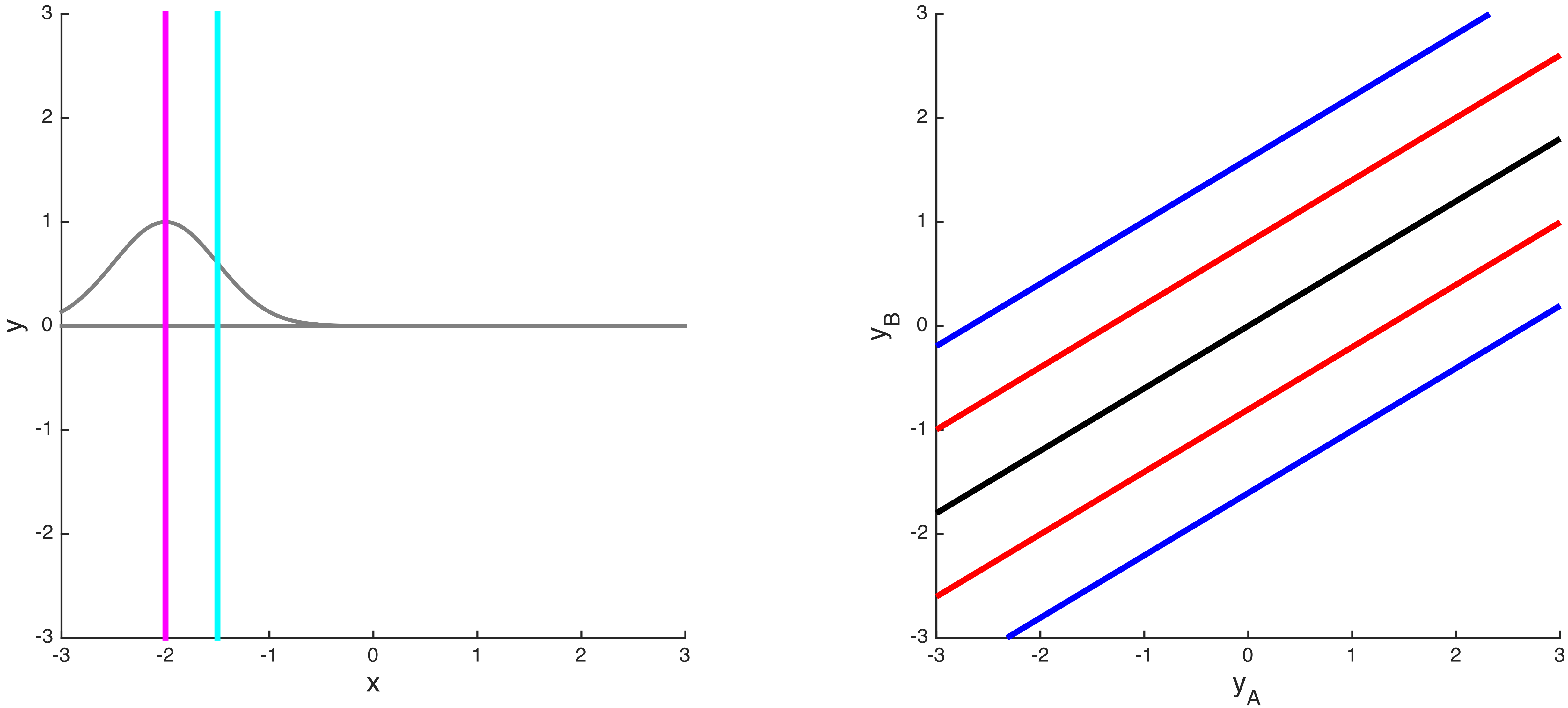
- Consider two points far away





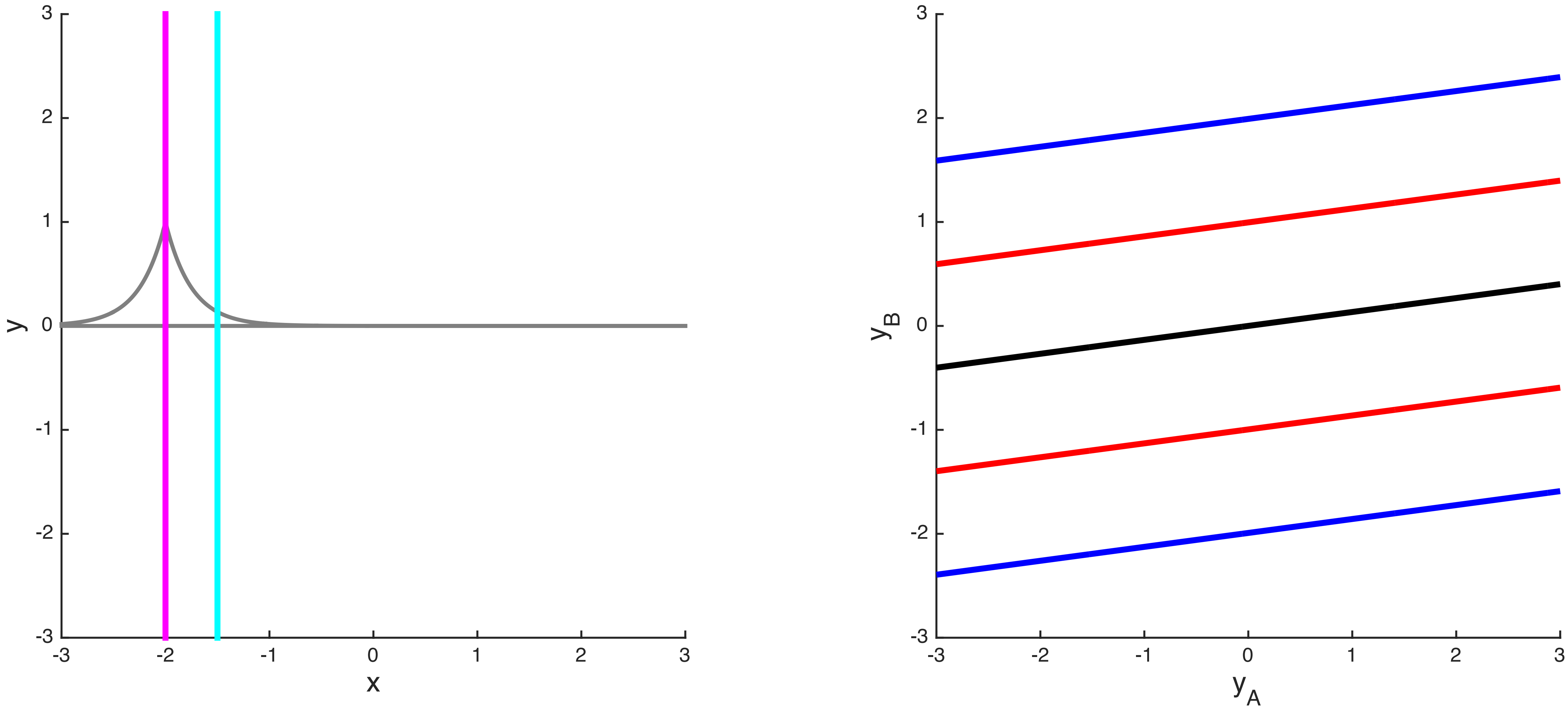
Gaussian Process Regression

- Consider two points close together



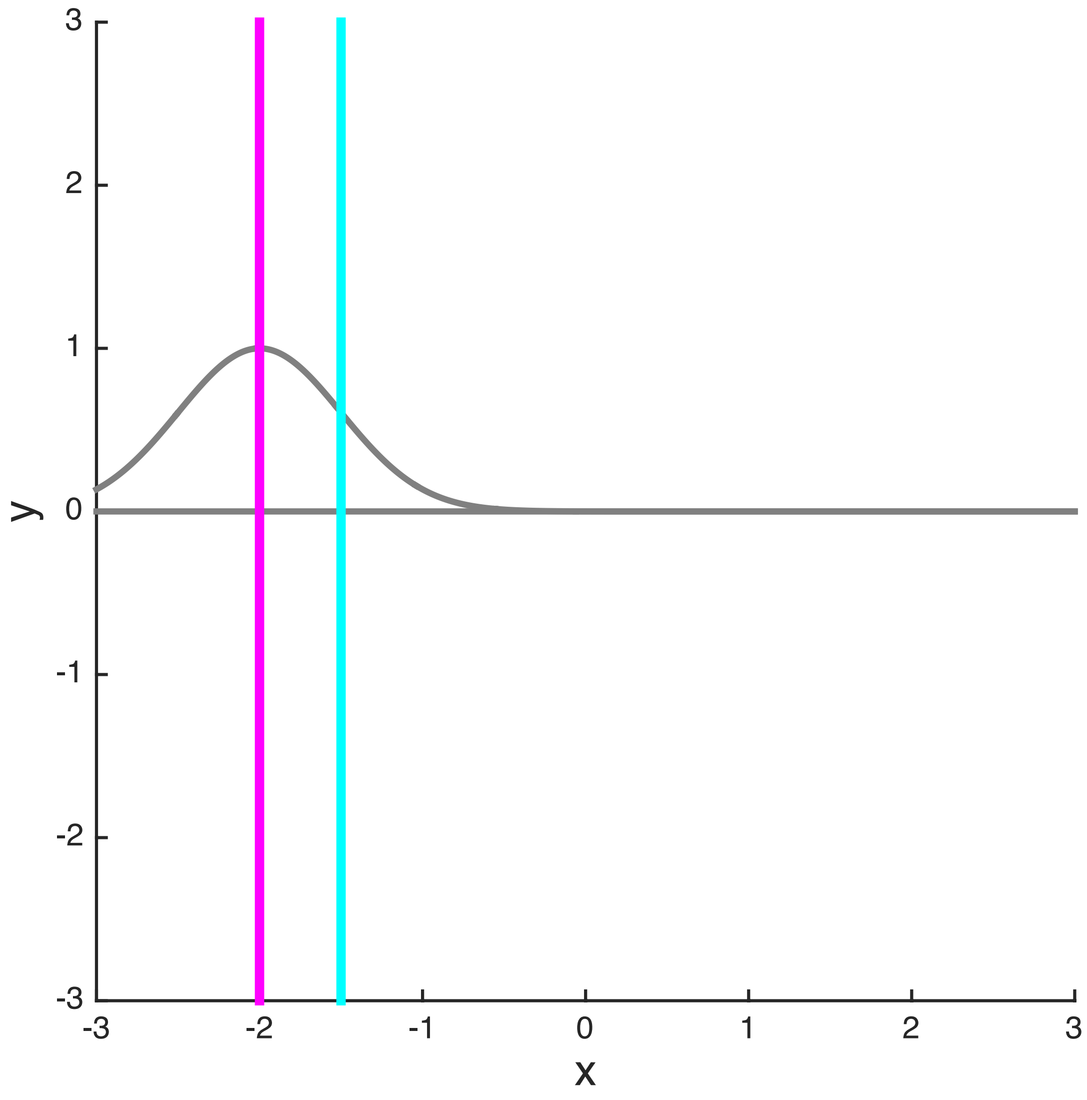
Gaussian Process Regression

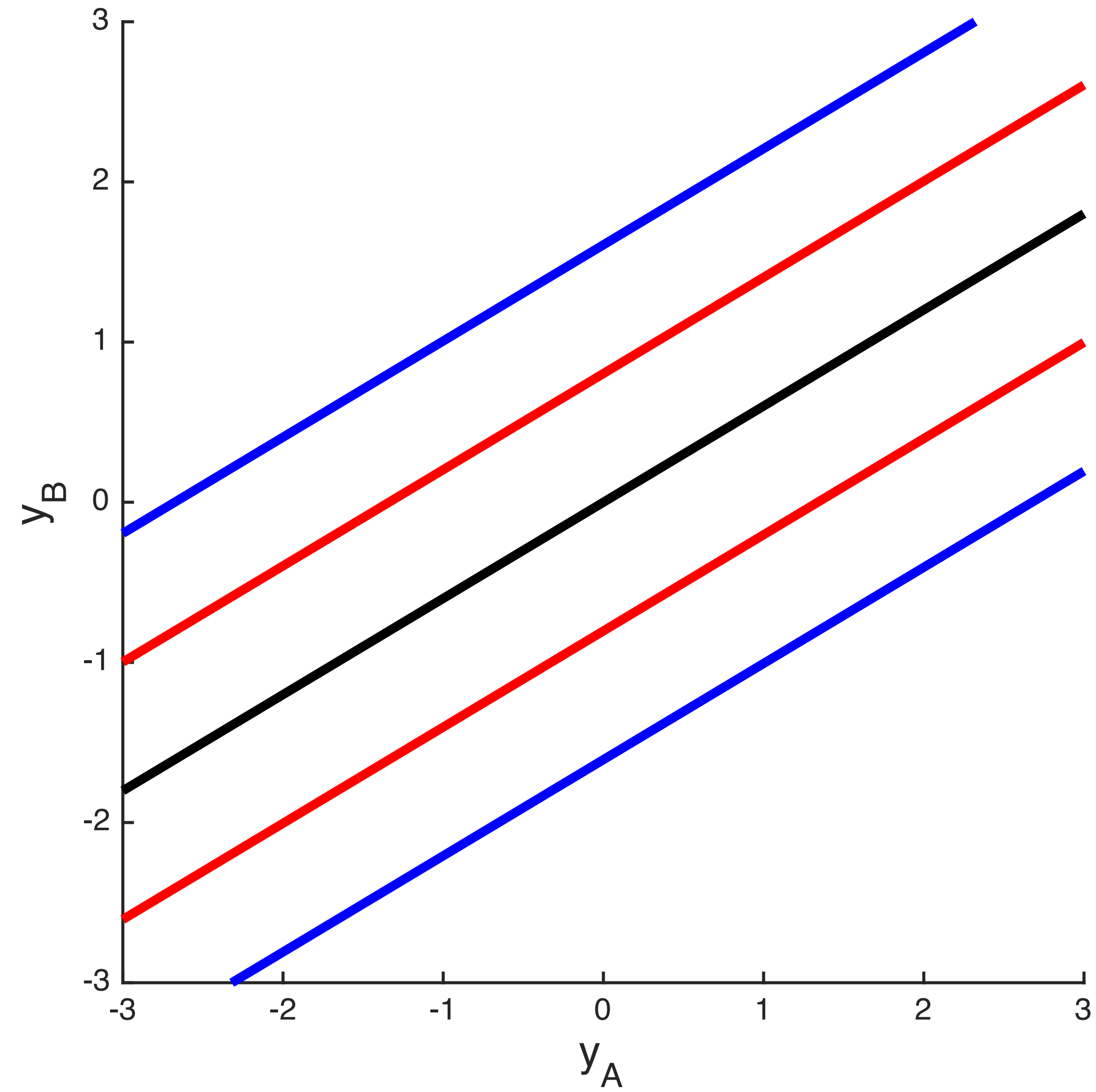
- Different kernel will represent different correlations



Gaussian Process Regression

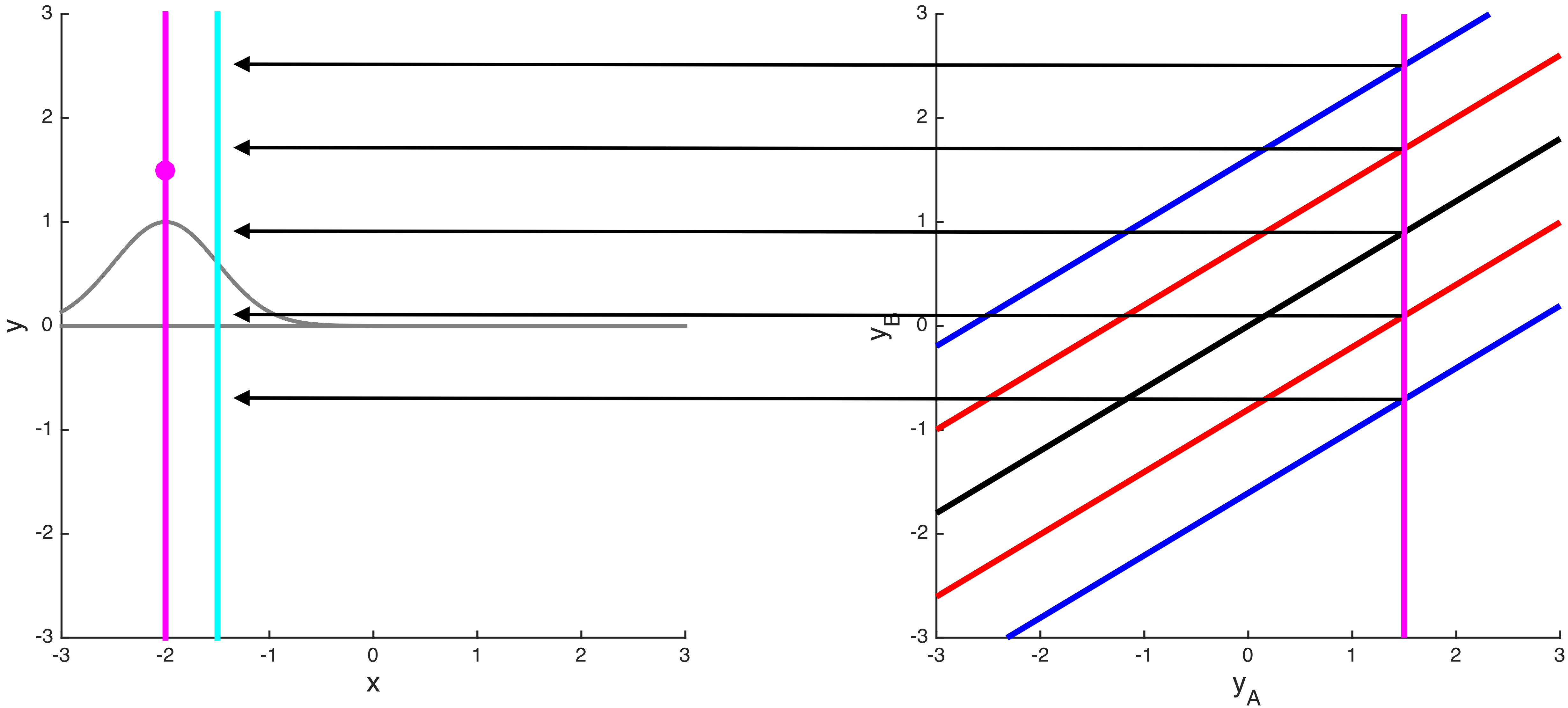
- What happens when we observe a sample $y_A(x_A)$?





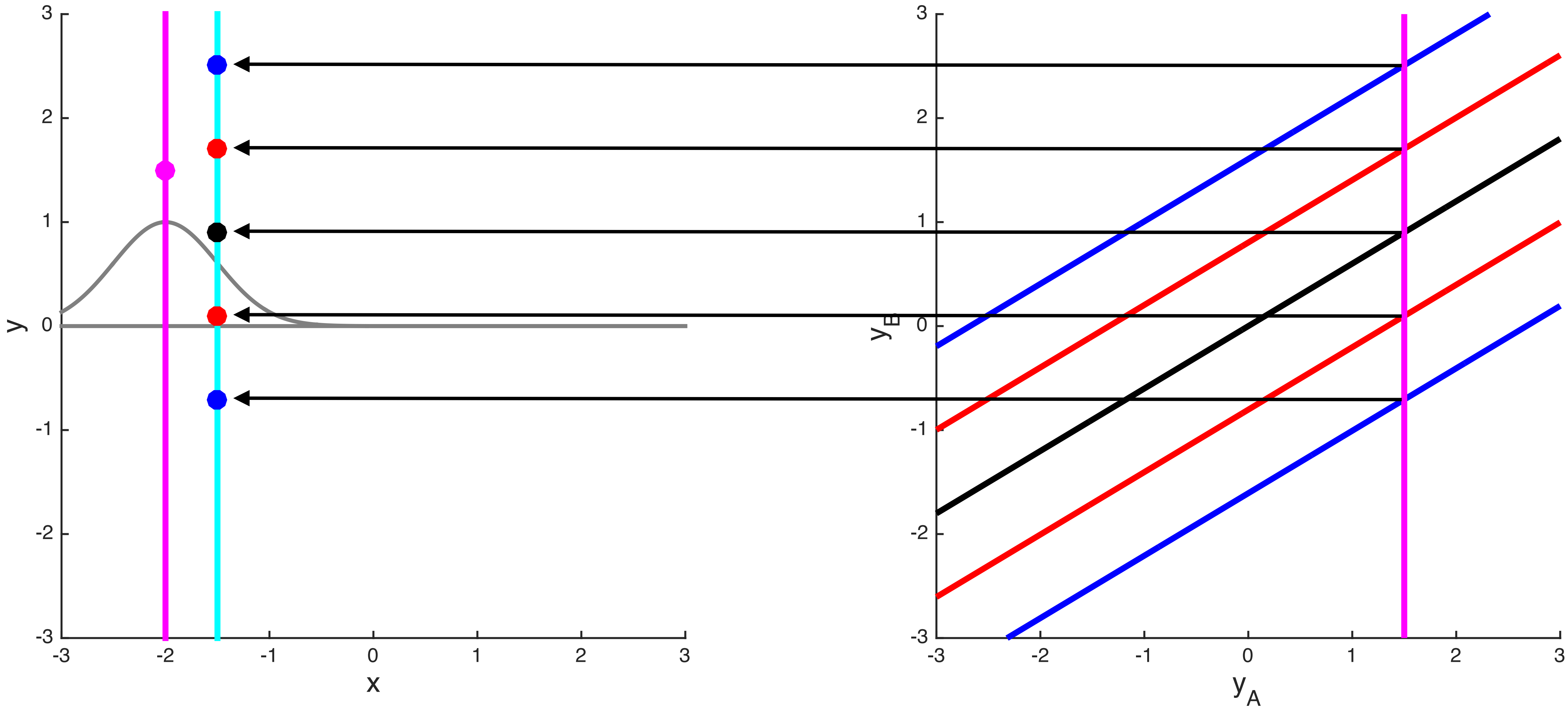
Gaussian Process Regression

- What happens when we observe a sample $y_A(x_A)$?



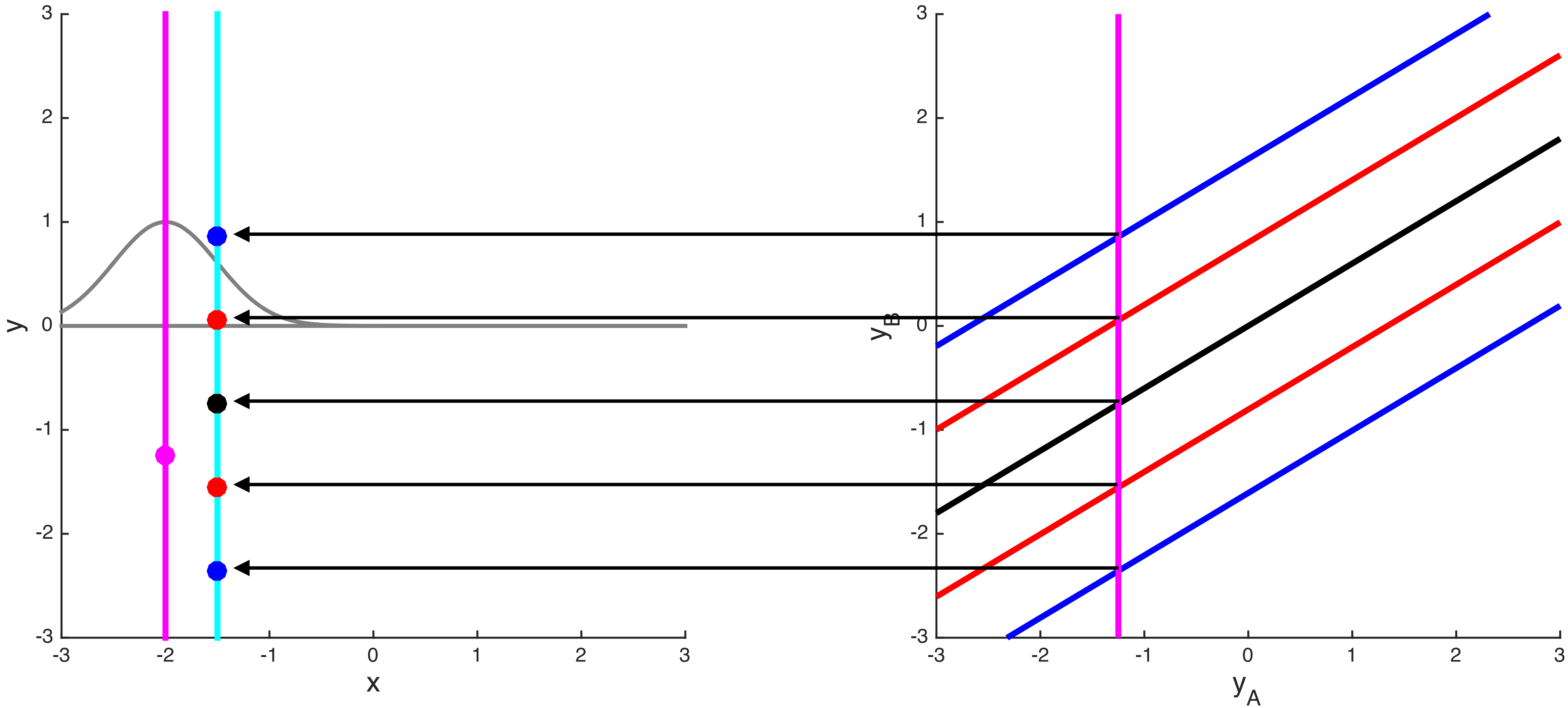
Gaussian Process Regression

- Get the corresponding conditional distribution



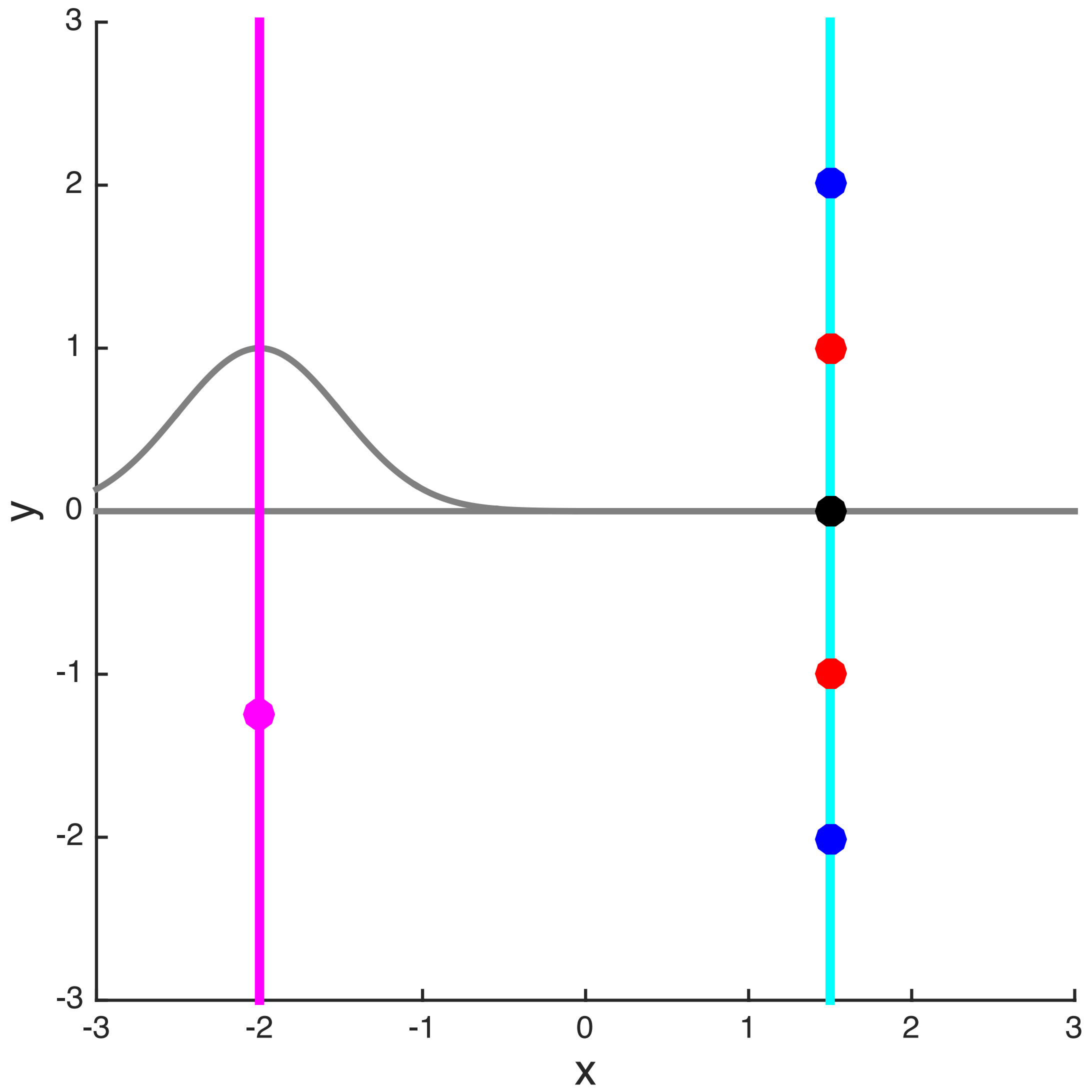
Gaussian Process Regression

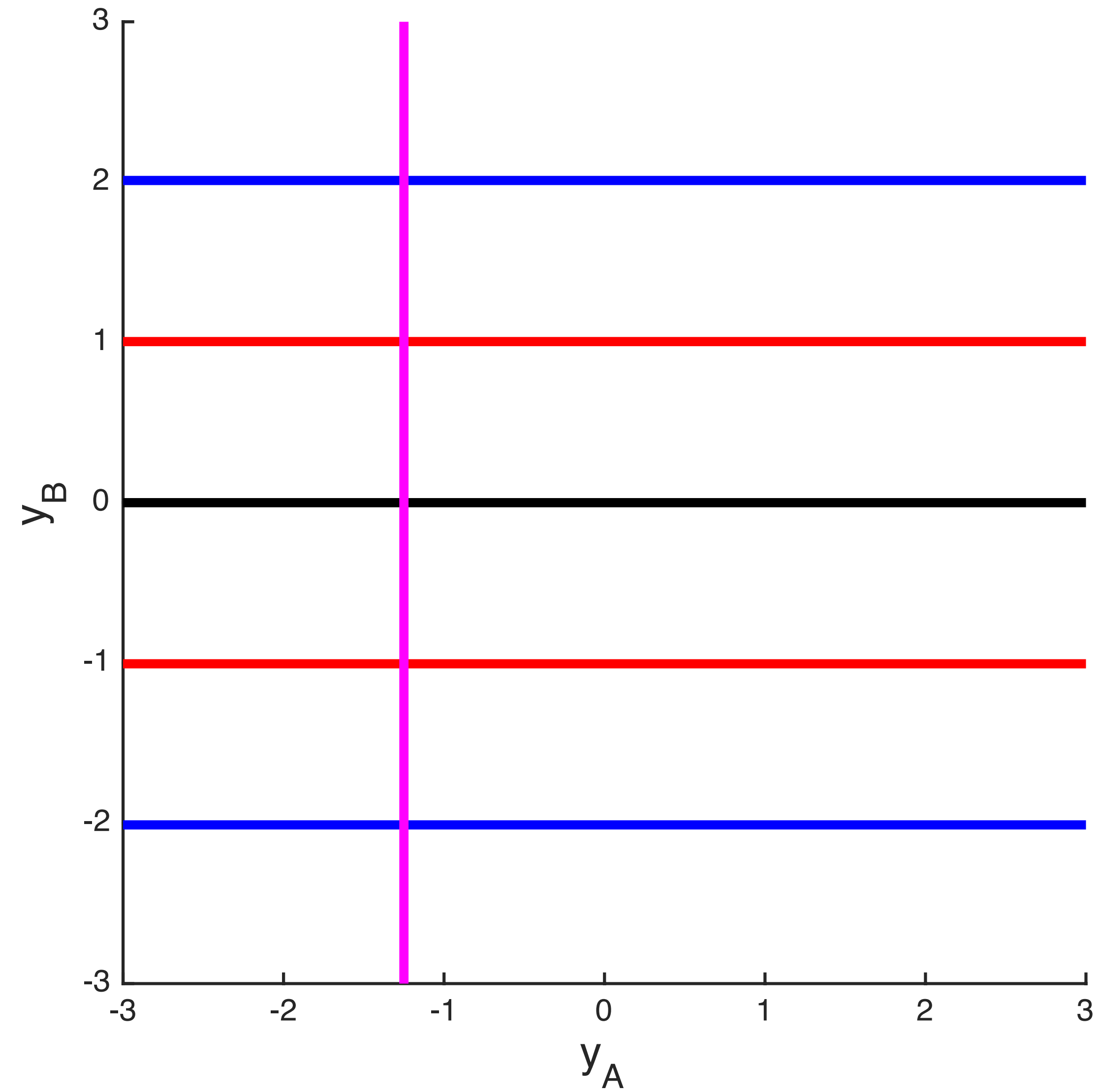
- A different value gives a different distribution



Gaussian Process Regression

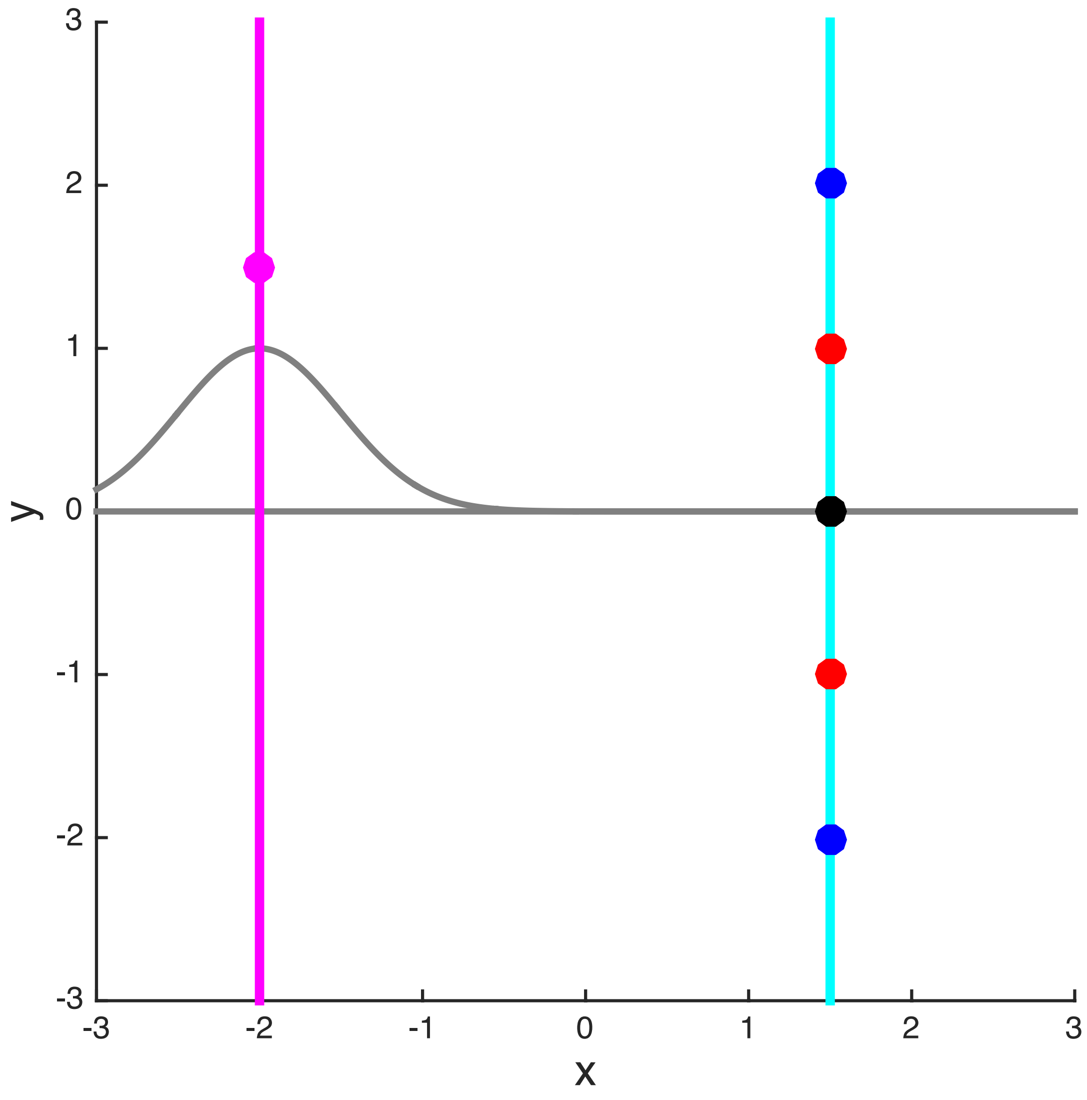
- What if we are further away?

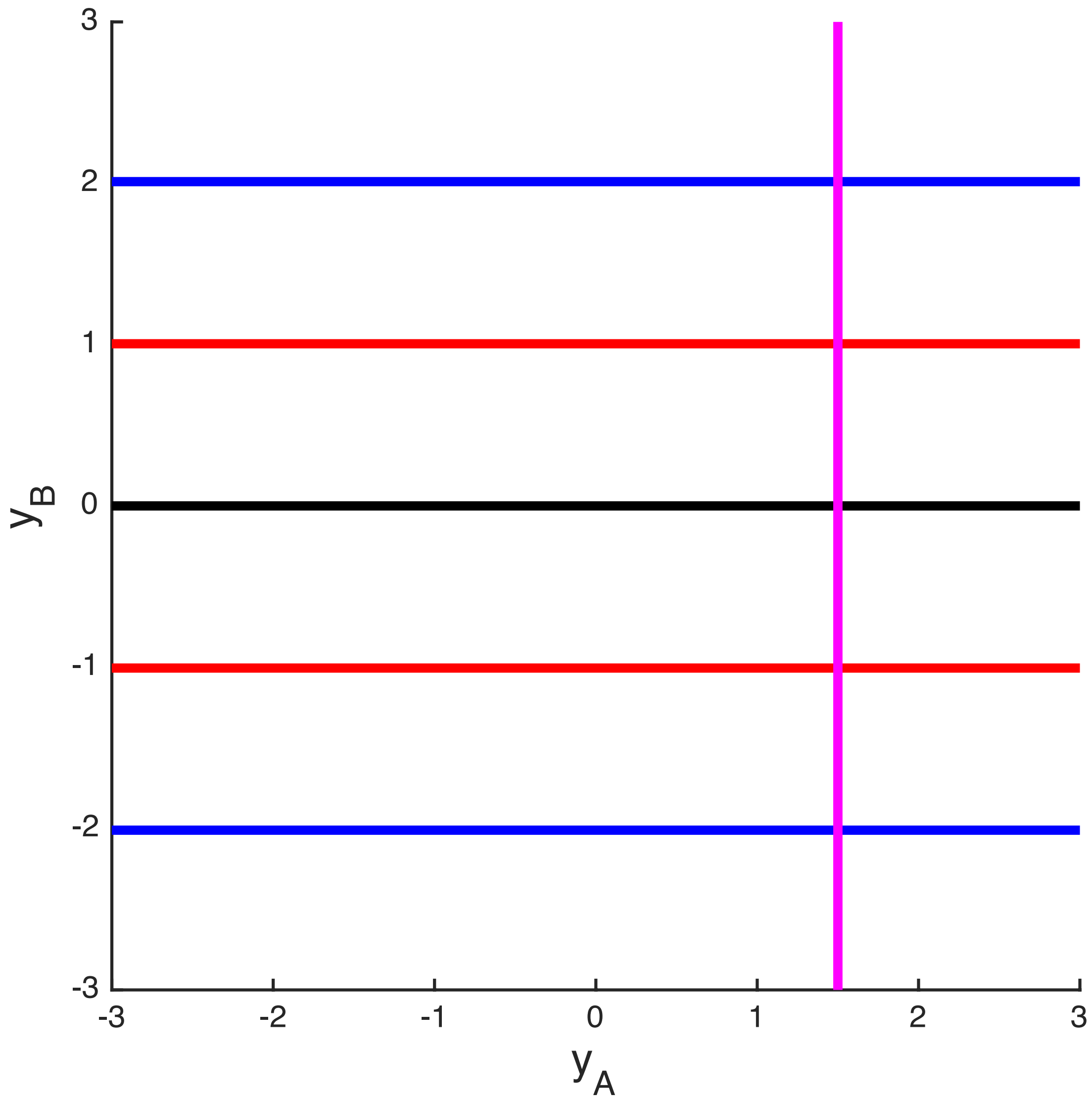




Gaussian Process Regression

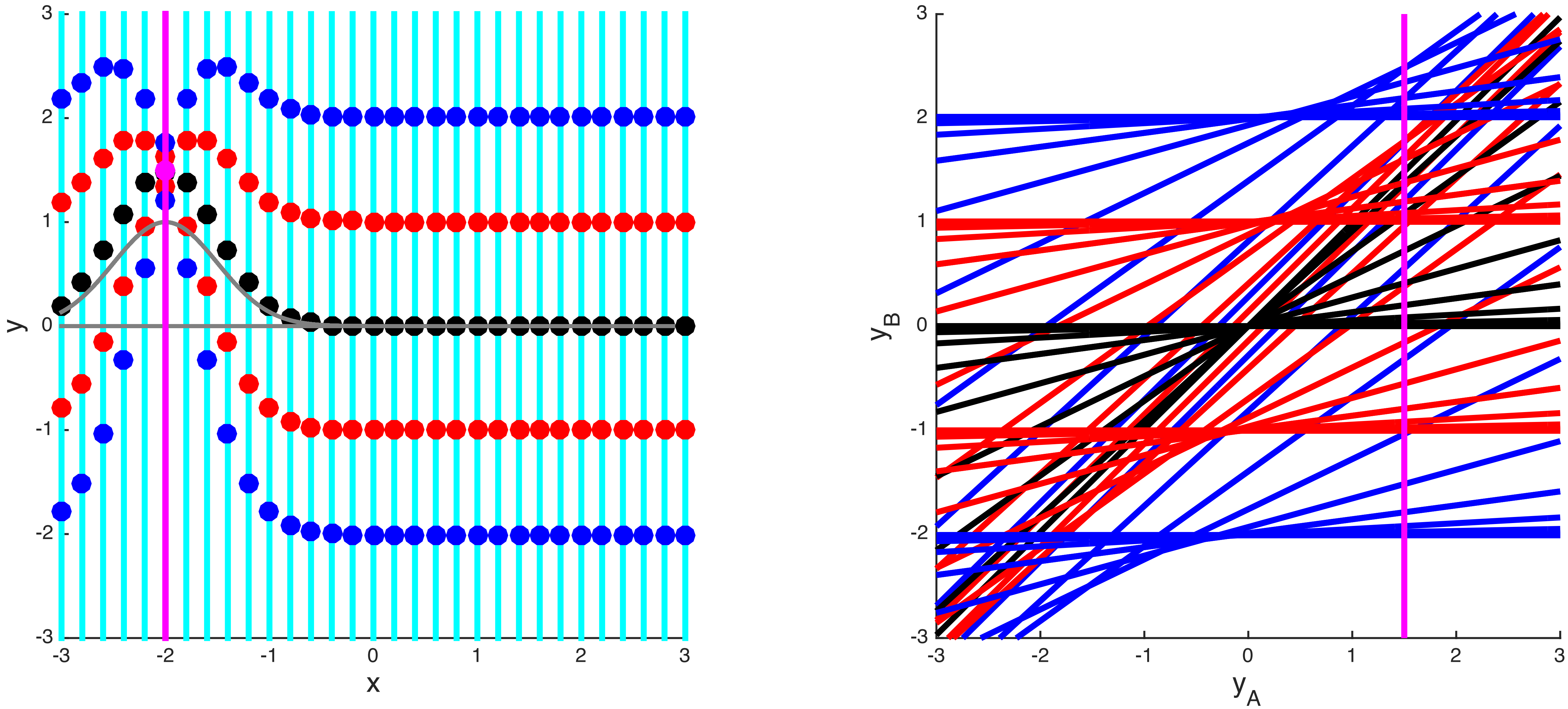
- Almost no change in distribution to to low correlation





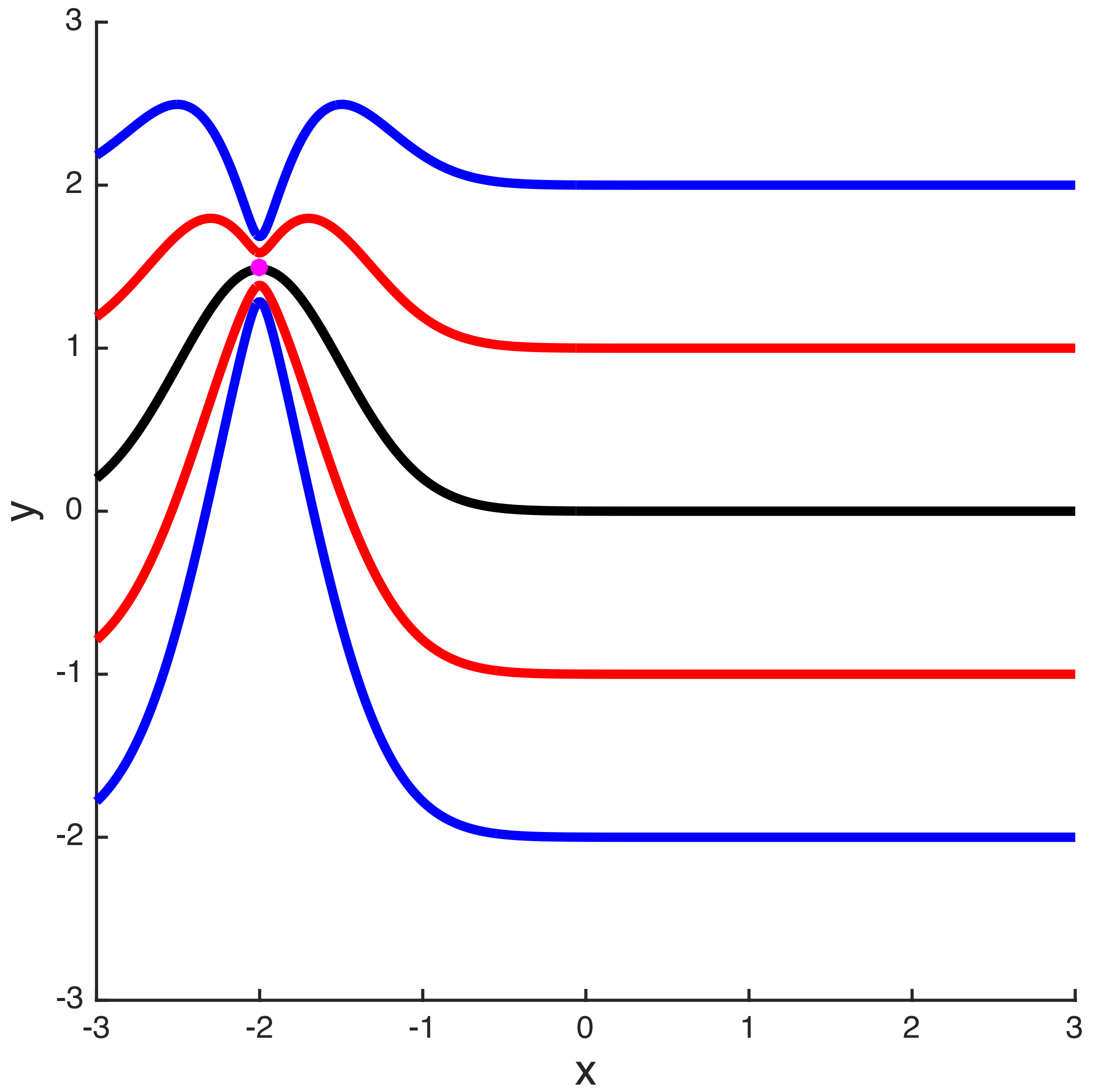
Gaussian Process Regression

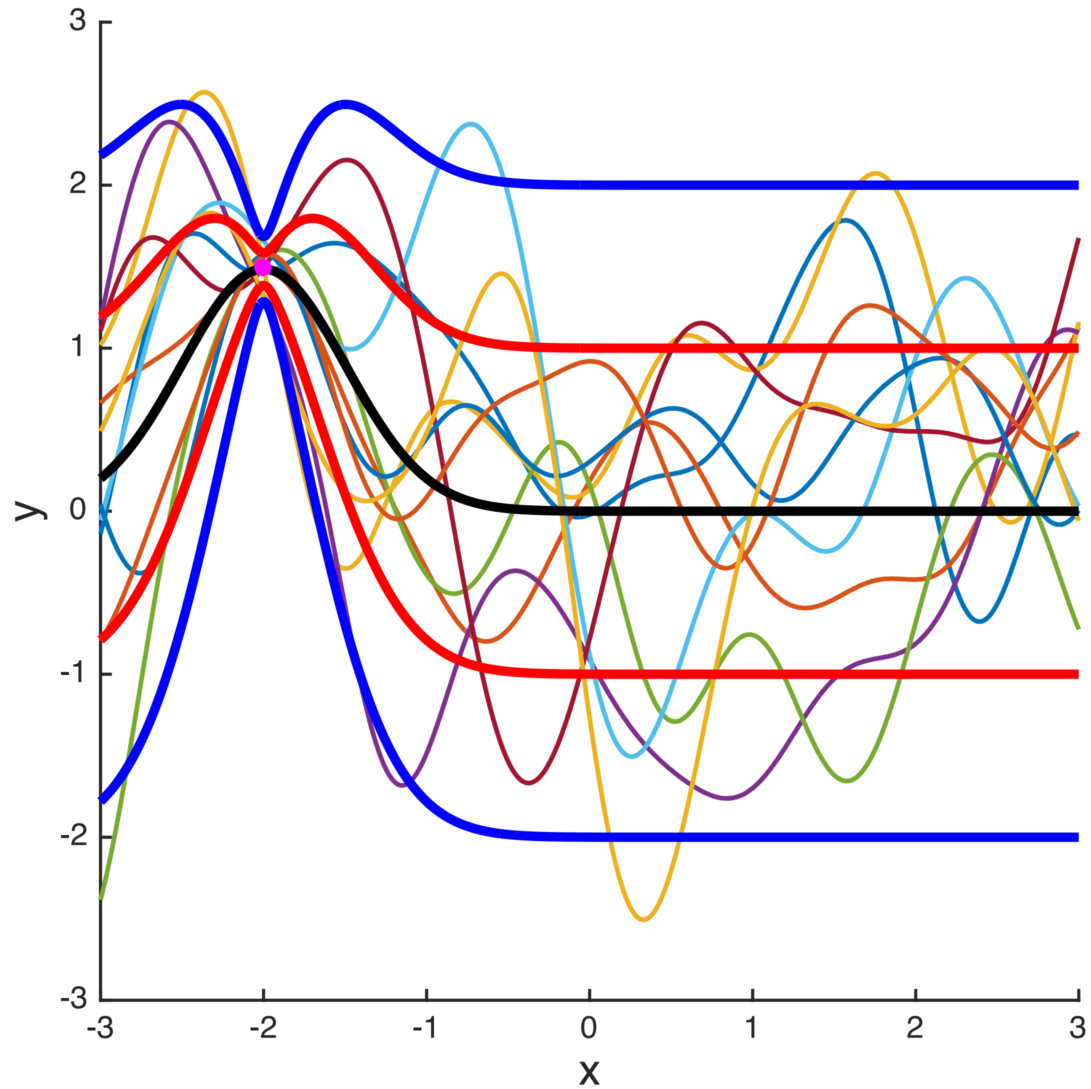
- Sweeping through values gives us…



Gaussian Process Regression

- Sweeping through values gives us…

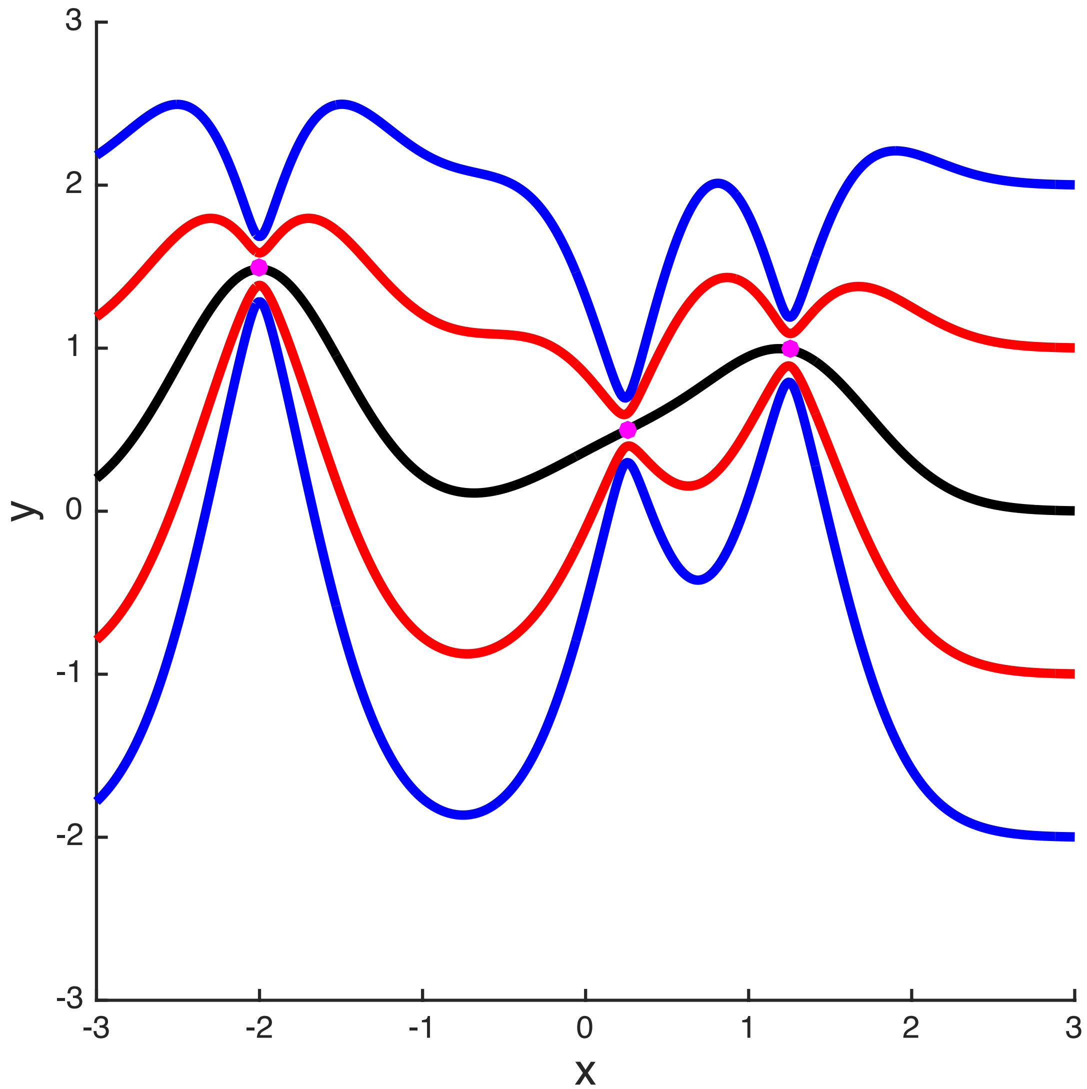


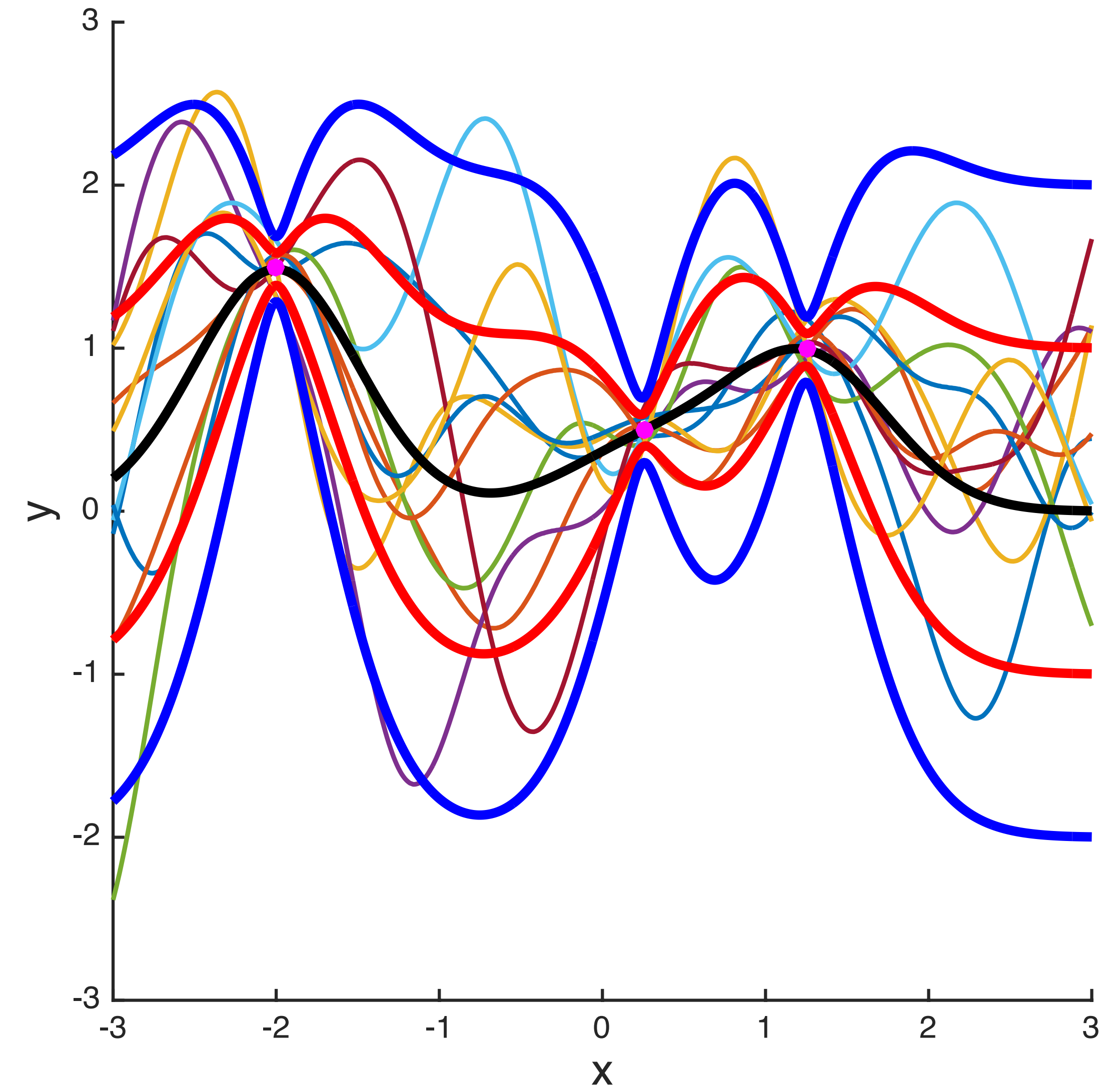


- Distribution over functions!

Gaussian Process Regression

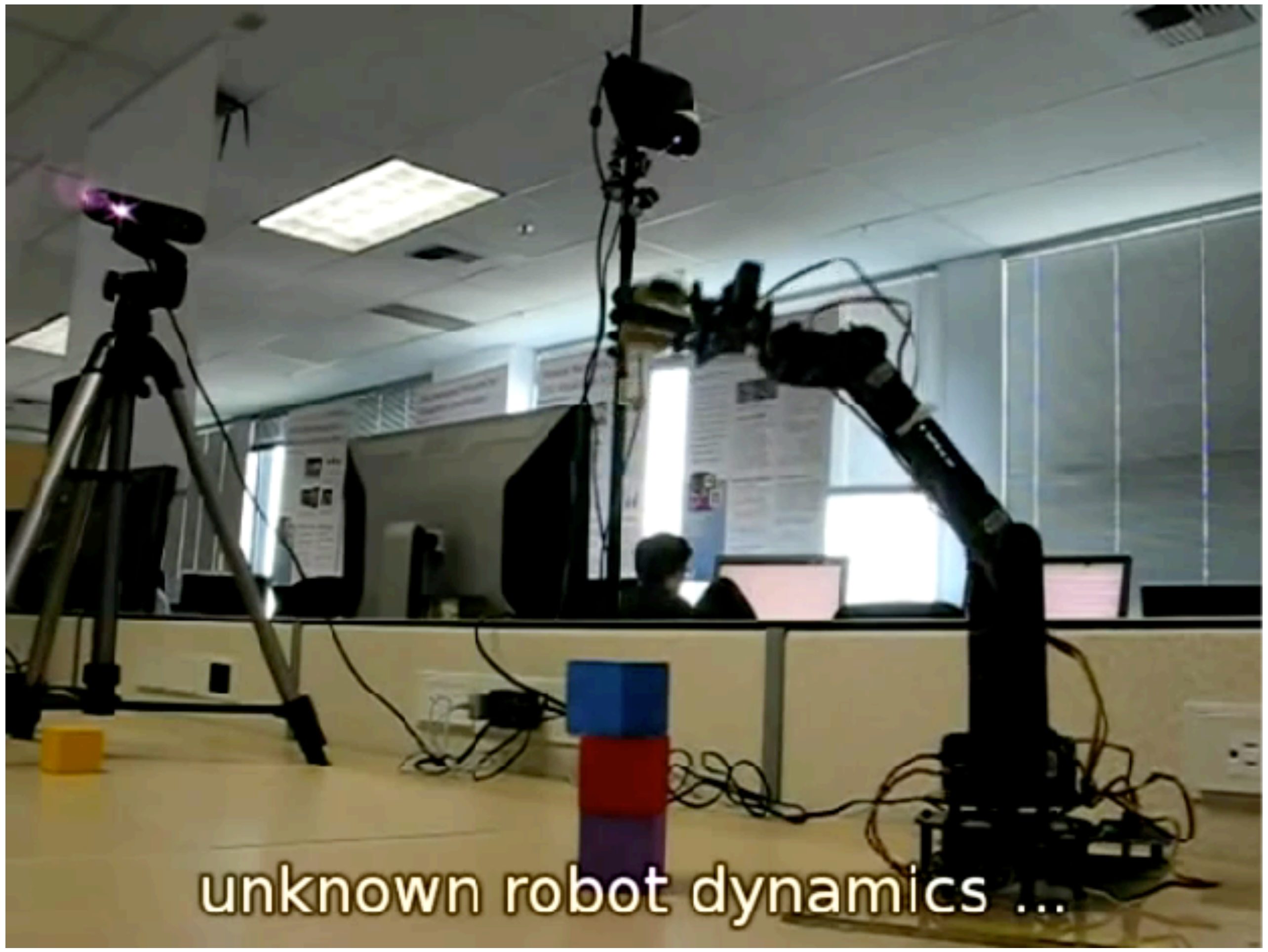
- Conditioning on more samples…

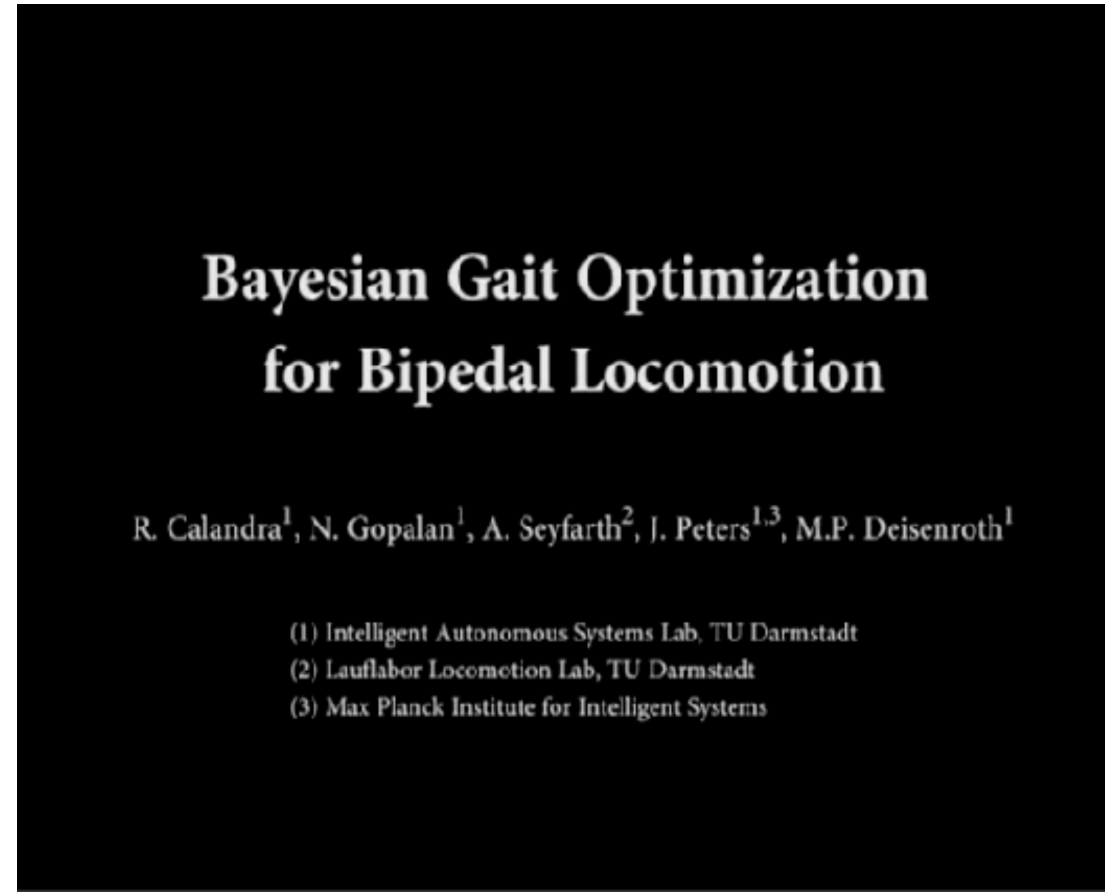


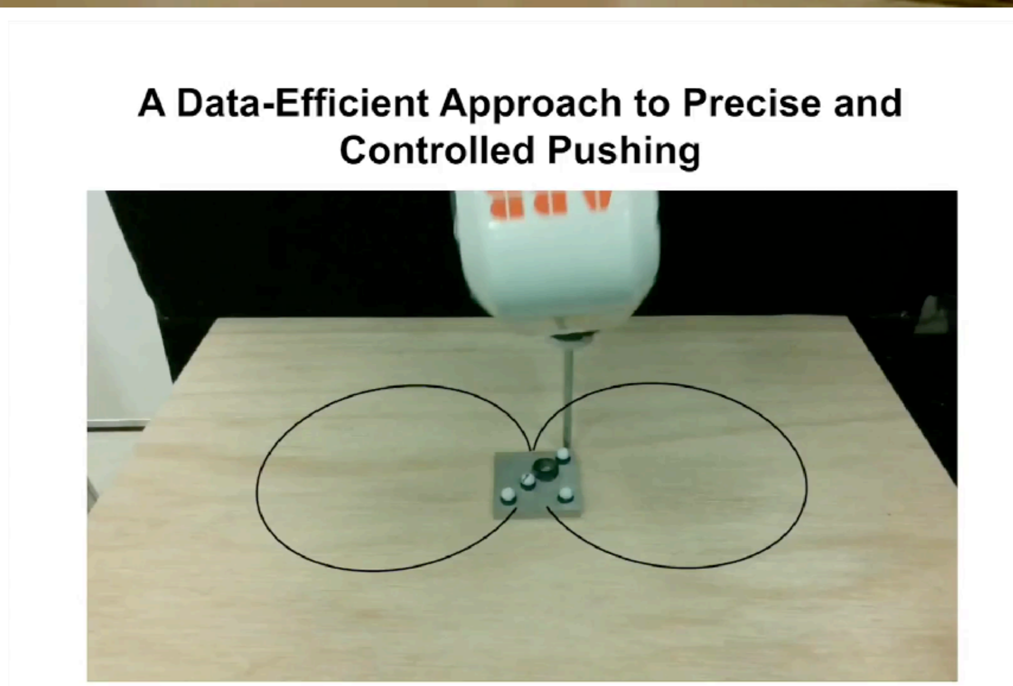


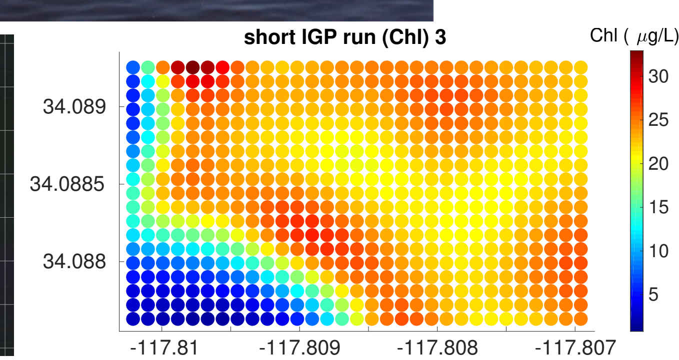
- Refines posterior distribution over functions

Applications









Questions?

(Thank you!)