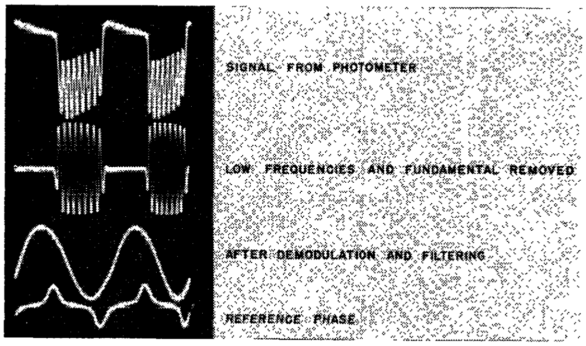

Fig. 5—Typical oscilloscope presentation from simulator.

graphs of several background distributions and superimpose on them a more brilliant spot to simulate our function J_S. Let us choose a rotational scanning operation where we rotate the filter shown in Fig. 3.

The signal from this system is converted to an electrical signal by the photometer and we may observe the output on an oscilloscope (Fig. 5) and indeed be able to obtain a frequency spectrum by using a wave analyzer. We may, in this way, examine many sizes and intensities of spots and the effect of the different backgrounds on our output signal.

It is interesting to note a very special case where our spot is made very small and the background removed; this condition approaches the unidimensional case. A plot of the transform is shown in Fig. 6.

Conclusions

The analogy between the two-dimensional space filter problem and the one-dimensional electrical filter problem is seen to be formally complete. The necessity to obtain time dependent space filtering, for various practical applications, modifies the formal symmetry of the analogy without changing the basic two-dimensional character. In many cases of electrical filter design, the analytic attack, though straightforward, proves to be less direct than an analog computation. Similarly, for the two-dimensional filter, an appropriate analog, together with our analytic understanding of the filter process, provides the designer with an effective solution for his problem.

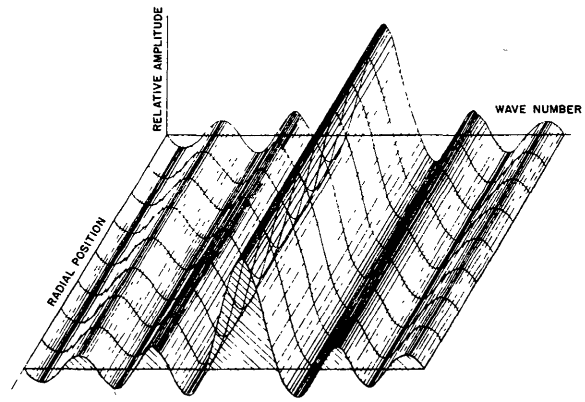

Fig. 6—Transform of n-wedged filter.

Ideal Transformers in the Synthesis of Analog Computer Circuits

R. H. MacNEAL† and G. D. McCANN†

Introduction

THE NEED FOR ideal transformers in the solution of network synthesis problems, especially in multi-terminal problems, has long been recognized.[1–3] Nevertheless, there has been a general feeling that solutions to synthesis problems containing ideal transformers are of little more than academic interest because of the impracticability of constructing transformers good enough to be called ideal.[4] Whenever possible, inductances in the network are associated with the transformer so that it can be replaced by coils having self and mutual inductance.[2] This attitude toward the ideal transformer is inappropriate in at least one branch of electrical engineering where network synthesis techniques are employed, namely in analog computing of the direct analogy (or network analyzer) type.[5] Such computers customarily operate in the audio frequency range and contain high quality passive circuit elements which are adjustable in small steps. In the design of such computers the choice between "ideal" transformers and mutual inductance coils is an easy one to make. The availability of "supermalloy,"[6] which has an initial permeability of 70,000 or more makes

† Department of Electrical Engineering, California Institute of Technology, Pasadena, Calif.

[1] W. Cauer, "Ideale transformatoren und lineare transformationen," *Elec. Nach. Tech.*, vol. 9, p. 157; May, 1932.

[2] W. Cauer, Ein Reacktanztheorem; Sitz. d Preuss Akad. der Wissen; Phys. Math., pp. 673–681; 1931.

[3] C. M. Gewertz, "Network Synthesis," The Williams and Wilkins Co., Baltimore, Md.; 1933.

[4] R. Bott and R. J. Duffin, "Impedance synthesis without use of transformers," *Jour. Appl. Phys.*, vol. 20, p. 816; August, 1949.

[5] H. E. Criner, G. D. McCann, and C. E. Warren, "A new device for the solution of transient vibration problems by the method of electrical-mechanical analogy," *Jour. Appl. Mech.*, vol. 12, pp. 135–141; September, 1945

E. L. Harder and G. D. McCann, "A large scale general purpose electric analog computer," *Trans. AIEE*, Part I, vol. 67, pp. 664; 1948.

[6] R. M. Bozorth, "Ferromagnetism," D. Van Nostrand Co., New York, p. 143; 1951.

possible the design of ideal transformers which have parasite effects no more serious in practice than those of an ordinary inductor. The ideal transformer has in its favor a much greater versatility than the mutual inductance (making possible mutual resistance and capacitance) and probably a cost advantage as well, when the adjustability requirement is considered.

The use of ideal transformers in the design of analog computer circuits has been illustrated in a large number of papers, some of which are more than twenty years old.[7,8] Most of these papers are concerned with the derivation of an electrical analogy for a specific type of physical system (e.g., bending of a beam,[9] pin-ended truss,[7] airplane fuselage shell[10]). The approach of the present paper is somewhat more general than this, but still somewhat more specific than that of the network-synthesis theoretician. The field of interest is limited to analog computing, but the results are intended to apply to the synthesis of analogies for all kinds of mechanical systems.

It is recognized that information concerning a complex system may be presented in a variety of different ways. The information may come as a detailed description of the inner workings of the system, or this information may already have been combined with the aid of general physical principles to form equations of state. Another possibility is that the information may come in the form of measurements taken in tests made on the system, or in tests made on parts of the system. In the latter instance the parts may be installed, or they may be removed from the system when tested. In some cases the information may be presented in all of these ways at once.

In the papers cited above which are concerned with the derivation of electrical analogies for specific systems, it is assumed that the information is presented as a detailed description of the inner workings of the system. The usual (or best) method employed for deriving analogies when information is presented in this fashion is first to subdivide the system to its ultimate elements–springs, masses, joints, beam bending elements, etc.; then to write down analogies for these elements by inspection; and finally to interconnect the elements. When all or part of the information is presented in the form of equations of state or as the result of tests, this method is obviously impracticable for the parts of the system affected. Under such circumstances the system, or its affected parts, must be regarded as mechanical black boxes into which we are not permitted to look but which must be replaced with equivalent electrical black boxes which behave in an identical (or rigorously analogous) manner at their terminals. Unfortunately from the point of view of the computer engineer, the number of such terminals is large and the methods for deriving the networks are not altogether straightforward.

The scope of this paper is restricted to the synthesis of systems characterized by ordinary algebraic equations with constant, real coefficients. For such systems general methods of synthesis are well known.[1]

The All-Resistor Network

The need for ideal transformers in network synthesis problems can be illustrated by considering the limitations placed on the form of the equations of a network not containing transformers, or, in view of the restriction to equations with real coefficients, a network of resistors. We shall consider that the unknowns appearing in the equations of a network with $n+1$ nodes are the voltages from each of n nodes to a single reference node. The form of the equations written in matrix form is

$$[Y][E] = [I] \tag{1a}$$

or

$$\begin{bmatrix} Y_{11} & Y_{12} \cdots Y_{1n} \\ Y_{21} & Y_{22} \cdots Y_{2n} \\ \cdot \ \cdot \ \cdot & \cdot \ \cdot \ \cdot \ \cdot \\ Y_{n1} & Y_{n2} \cdots Y_{nn} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \cdot \\ E_n \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ I_n \end{bmatrix}. \tag{1b}$$

A network satisfying these equations for the case $n=3$ is shown in Fig. 1. Values of conductance of the

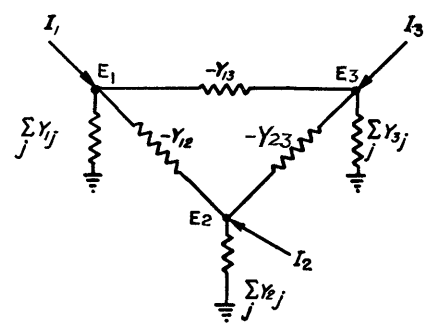

Fig. 1—All-resistor network.

resistors calculated by elementary network analysis are shown in the figure. The network in Fig. 1 is the most general all-resistor network with three independent node pairs. If all of the resistors in this network are to be *positive* and *bilateral*, then the following restrictions must be placed on the coefficients in (1b).

$$\begin{aligned} Y_{ij} = Y_{ji} &\leqq 0 \qquad i \neq j \ \ i, j = 1 \cdots n \\ \sum_{j=1}^{n} Y_{ij} &\geqq 0 \qquad i = 1, 2 \cdots n. \end{aligned} \tag{2}$$

These conditions are far more restrictive than the general conditions for synthesis with passive elements (including transformers), as will be seen.

[7] V. Bush, "Structural analysis by electric circuit analogies," *Jour. Franklin Inst.*, vol. 217, pp. 289–329; March, 1934.
[8] R. R. M. Mallock, "An electrical calculating machine," *Proc. Roy. Soc.*, ser. A, vol. 140, pp. 457–483; 1933.
[9] G. D. McCann and R. H. MacNeal, "Beam vibration analysis with the electric analog computer, *Jour. Appl. Mech.* vol. 72, pp. 13–26; March, 1950.
[10] R. H. MacNeal, "Electrical Analogies for Stiffened Shells with Flexible Rings," NACA TN 3280; December, 1954.

One way of easing the above restrictions is to transform the unknown variables by a linear co-ordinate transformation. Such a transformation might be a simple scale factor multiplication (a diagonal transformation) or it might involve the reduction of $[Y]$ to the unit matrix, i.e., a solution of the problem giving $[E]$ explicitly in terms of $[I]$. Such transformations are not generally permissible in theoretical network synthesis, nor are they permissible from the point of view of analog computing. The terminal voltages $[E]$ and the input currents $[I]$ must be represented in the network. The practical reason for this is that the system represented by (1) may well be only a part of a larger system, in which case $[E]$ and $[I]$ represent also the currents and voltages in other parts of the system.

The All-Transformer Network

The equations of a three-winding ideal transformer, shown in Fig. 2, are

$$\frac{E_1}{N_1} = \frac{E_2}{N_2} = \frac{E_3}{N_3}, \tag{3}$$

$$N_1 I_1 + N_2 I_2 + N_3 I_3 = 0. \tag{4}$$

Stated in words, the general conditions for an ideal, multiwinding, single-core transformer are that the voltages across the various windings are directly proportional to their turns ratios and that the sum of the ampere turns (magnetomotive force) is zero.

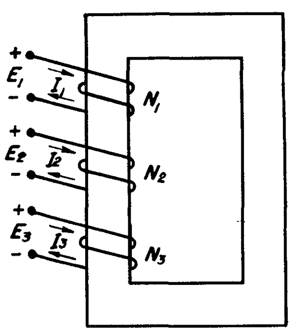

Fig. 2—Ideal transformer.

Consider now the network of transformer windings shown in Fig. 3. All the coils in one horizontal row are wound on the same core of an ideal transformer. Coils in different rows are not magnetically coupled. The relative number of turns in each winding is indicated by the symbol T_{ij}. From the manner of connection and (3), the voltage E_1 is given by

$$E_1 = T_{11}\bar{E}_1 + T_{12}\bar{E}_2 + T_{13}\bar{E}_3 \tag{5}$$

or, in general,

$$[E] = [T][\bar{E}]. \tag{6}$$

Also, from (4)

$$\bar{I}_1 = T_{11}I_1 + T_{21}I_2 + T_{31}I_3 \tag{7}$$

or, in general,

$$[\bar{I}] = [T]^T[I]. \tag{8}$$

(In this equation, the superscript t indicates that the matrix $[T]$ has been transposed.)

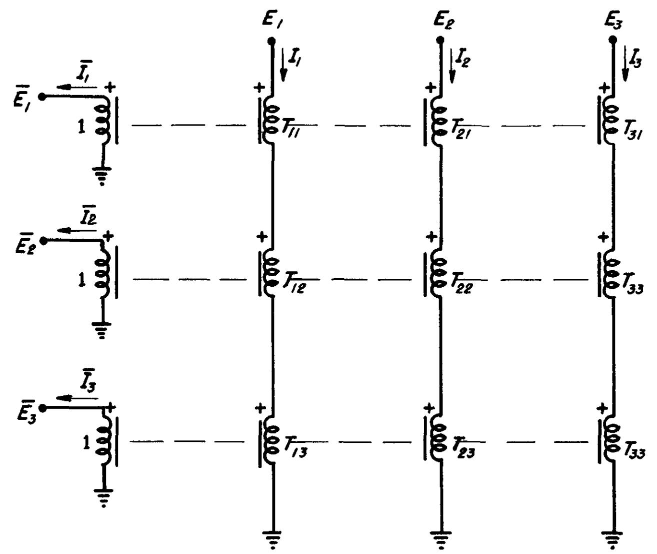

Fig. 3—All-transformer network.

Eqs. (6) and (8) can be regarded as a *workless* transformation from variables $[\bar{I}]$, $[\bar{E}]$ to variables $[I]$, $[E]$, since

$$[I]^T[E] = [I]^T[T][\bar{E}] = \{[T]^T[I]\}^T[\bar{E}]$$
$$= [\bar{I}]^T[\bar{E}]. \tag{9}$$

Hildebrand gives an explanation of the matrix identity employed in (9).[11] Note that from the point of view of physical realizability the only restriction on the elements of $[T]$ is that they be real numbers. Hence the network of Fig. 3 is capable of solving (6) with arbitrary real coefficients (if $[T]$ is nonsingular). Voltages $[E]$ are established by external generators and the voltages $[\bar{E}]$ read off. A practical version of this network computer was invented by Mallock[8] in 1933 and sold commercially. Considering the improvement in transformer iron in the last twenty years this network has inviting characteristics as a computer even today.

From the point of view of network synthesis a more important characteristic of this network is that it provides an *electrical* means for carrying out a co-ordinate transformation. Thus the objection to co-ordinate transformation mentioned in the previous section is overcome because *both* the original and the transformed co-ordinates appear in the network. Let us apply the transformation given by (6) and (8) to (1a):

$$[Y][T][\bar{E}] = [I]$$
$$[T]^T[Y][T][\bar{E}] = [\bar{I}]. \tag{10}$$

[11] F. B. Hildebrand, "Methods of Applied Mathematics," Prentice-Hall, Inc., New York, p. 13; 1952.

Define $[\bar{Y}]=[T]^T[Y][T]$. Eq. (10) can be synthesized by inspection if a transformation can be found which reduces $[\bar{Y}]$ to a diagonal matrix with either zero or positive elements on the principal diagonal and zero elsewhere. If such a transformation exists, the original admittance matrix $[Y]$ is said to be *positive semi-definite.* Hence a sufficient condition for the passive synthesis of (1a) is that the matrix $[Y]$ be positive semi-definite. An efficient transformer network for carrying out the synthesis is described in the next section.

The co-ordinate changing properties of transformer networks have been used extensively in connection with the direct analogy type of analog computer.[12,13] For example, they provide the means for interconnecting analogies for parts of a mechanical system which are oriented in different space directions (e.g., the interconnection of an airplane's fuselage and its swept-back wings). Fig. 4 shows a very simple transformer circuit for synthesizing a negative coupling element.

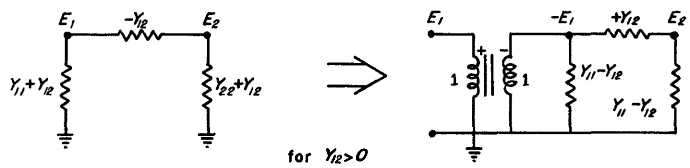

Fig. 4—Synthesis of negative coupling element.

Cauer Networks

Cauer's network, as originally presented,[1] synthesizes a system with equations presented in the form

$$[Z][I] = [E]. \tag{11}$$

$$[T][R][T]^T = \left[\begin{array}{c|c|c|c} R_{11} & R_{11}T_{21} & R_{11}T_{31} & \cdots \\ \hline R_{11}T_{21} & R_{22}+R_{11}T_{21}^2 & R_{11}T_{21}T_{31}+R_{22}T_{32} & \cdots \\ \hline R_{11}T_{31} & R_{11}T_{21}T_{31}+R_{22}T_{32} & R_{33}+R_{11}T_{31}^2+R_{22}T_{32}^2 & \cdots \\ \hline \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \end{array}\right]. \tag{18}$$

In this section Cauer's network will first be derived in its original form and then in a form suitable for the synthesis of (1a). It is assumed that the matrices $[Z]$ and $[Y]$ are positive semi-definite. This does not imply that they are necessarily nonsingular.

As stated in the previous section, the network can be synthesized if a co-ordinate transformation is found which transforms $[Z]$ into a diagonal matrix.

Let this transformation be given by

$$\begin{aligned} [E] &= [T][\bar{E}] \\ [\bar{I}] &= [T]^T[I]. \end{aligned} \tag{12}$$

[12] W. T. Russell, "Lumped Parameter Analogies for Continuous Mechanical Systems," Calif. Inst. Tech., Ph.D. thesis; 1950.

[13] R. H. MacNeal, G. D. McCann, and C. H. Wilts, "The solution of aeroelastic problems by means of electrical analogies," *Jour. Aero. Sci.*, vol. 18, pp. 777–789; December, 1951.

The resulting equation in terms of transformed variables is

$$[\bar{E}] = [R][\bar{I}], \tag{13}$$

where it is assumed that

$$[R] = \begin{bmatrix} R_{11} & 0 & 0 & \cdots & 0 \\ 0 & R_{22} & 0 & \cdots & 0 \\ 0 & 0 & R_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & R_{nn} \end{bmatrix} \tag{14}$$

and $R_{ii} \geqq 0$.

(If any R_{ii} equal zero it is assumed that they occupy the last positions in the matrix. This condition can be satisfied by rearranging rows and columns in the T matrix.)

Premultiply (13) by $[T]$:

$$[E] = [T][\bar{E}] = [T][R][\bar{I}] = [T][R][T]^T[I]. \tag{15}$$

Hence

$$[Z] = [T][R][T]^T. \tag{16}$$

The problem is to find matrices $[T]$ and $[R]$ which satisfy this equation. Let $[T]$ be a *triangular* matrix of the form

$$[T] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ T_{21} & 1 & 0 & \cdots & 0 \\ T_{31} & T_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & \cdots & 1 \end{bmatrix}. \tag{17}$$

Then

Note that, given the elements of $[Z]$, the elements of $[T]$ and of $[R]$ can be evaluated *one at a time.* Explicitly,

$$\begin{gathered} R_{11} = Z_{11}; \quad T_{21} = Z_{12}/R_{11}; \quad T_{31} = Z_{13}/R_{11} \\ R_{22} = Z_{22} - R_{11}T_{21}^2, \text{ etc.} \end{gathered} \tag{19}$$

The triangular transformation matrix is efficient from the point of view of electrical synthesis since fewer windings are required than in the case of a full transformation matrix. Cauer's "Z-network" is shown in Fig. 5(a). The barred co-ordinates can be eliminated by placing the R's on the other side of the windings, as shown in Fig. 5(b). The total number of windings required in the synthesis of n simultaneous equations by Cauer's network is $n\cdot(n+1)/2-1$. If multiwinding transformers are not available, they may be replaced by $n\cdot(n-1)/2$ 2-winding transformers.

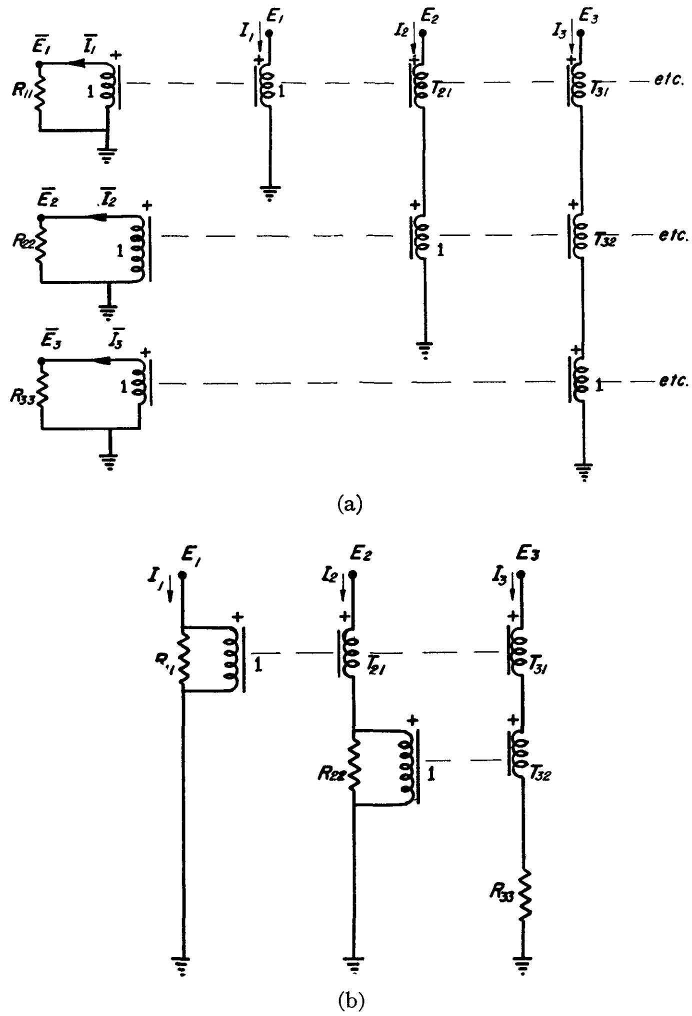

Fig. 5—Cauer's Z-network, (a) with barred co-ordinates represented, (b) with barred co-ordinates eliminated.

The necessary and sufficient conditions for the synthesis of (11) by Cauer's network may be stated in a way that does not involve matrix theory. If (11) is to represent a passive electrical network then it is evidently necessary that the input impedance at any given terminal must be positive (or at least zero) whether some, none, or all of the other terminals are short-circuited. It is easy to demonstrate that this condition is also a sufficient one for the synthesis of Cauer's network. Every element in Cauer's network can be computed by the formulas in (19) so that the only real question is whether or not the resistors are positive. From inspection of Fig. 5(b) it is evident that R_{11} is the input impedance at terminal 1 with all other terminals open-circuited; R_{22} is the input impedance at terminal 2 with terminal 1 short-circuited and all other terminals open-circuited; R_{33} is the input impedance at terminal 3 with terminals 1 and 2 short-circuited, and all others open-circuited; etc. Hence all resistors are positive (or zero).

We will next construct Cauer's "Y-network" for a system specified by equations of the form of (1a). This network, shown in Fig. 6, is the "dual" of the "Z-network" shown in Fig. 5. The analysis proceeds by analogy with the derivation of the Z-network. Let the transformation be

$$
\begin{aligned}
[I] &= [S][\bar{I}] \\
[\bar{E}] &= [S]^T[E].
\end{aligned} \tag{20}
$$

Then if

$$
[\bar{I}] = [G][\bar{E}] \tag{21}
$$

where $[G]$ is a diagonal matrix, it is evident that

$$
[Y] = [S][G][S]^T. \tag{22}
$$

$[S]$ is chosen as a triangular matrix identical in form to $[T]$ so that the elements of $[S]$ and $[G]$ are determined by formulas rigorously analogous to (19).

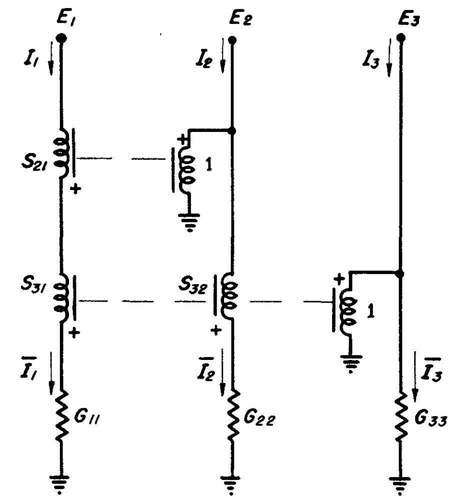

Fig. 6—Cauer's Y-network.

Cauer's networks are important because of their generality. Unfortunately for practical applications, the number of transformer coils required increases approximately as the square of the number of terminals. On the other hand, under special conditions no transformer at all may be required (the all-resistor network). An important unsolved problem is the determination of the *minimum* number of transformers required to synthesize a given system of equations and the construction of a network exhibiting this minimum number.[14]

Synthesis of Systems with "Restrained" Terminals

Many of the problems that arise in analog computing can be translated into problems in abstract network synthesis, but sometimes the problem is one that would not independently occur to a network theoretician as an interesting and useful problem on which to work. Such is the nature of the problem considered in this section. As has frequently been demonstrated, the abstract mathematical sciences depend for their growth on an influx of practical problems, however much these prob-

[14] J. K. Delson, "Networks Involving Ideal Transformers," Calif. Inst. Tech., Ph.D. thesis; 1953.

lems may have been predigested and stripped of their practical aspects.

We may assume, without loss of generality, that electrical analogies for mechanical systems are always set up with force analogous to current and displacement analogous to voltage. Equations written in the form of (1a) are then force equations for the mechanical system and are equations of state (Newton's equations, Lagrange's equations) in the usual sense. Mechanical equations written in the form of (11) ordinarily describe the results of tests, i.e., the displacements that are produced by application of a set of external forces on a body. Hence both forms of Cauer's network are useful in analog computing. The Y-network is useful when information concerning the system is presented as equations of state, and the Z-network is useful when information concerning the system is presented as a matrix of "influence coefficients." In the latter case the information so presented never describes the entire system, for if it did it would constitute a *solution* of the problem. Hence, the influence coefficients describe tests made on a subsystem while it is disconnected from the rest of the system. We can regard the subsystem as a black box electrical network with $n+1$ terminals (one ground terminal). The test the results of which are described by (11) consists of the insertion of a current at each node in turn with all other nodes free (open-circuited), and the measurement of the resulting terminal voltages. It should be recognized that, from the mechanical point of view, this might be a *meaningless* experiment. As an elementary example consider a straight segment of rod capable of resisting tension and compression. In the analogous electrical circuit the terminal voltages are the axial displacements at the ends. (See Fig. 7.) Such a

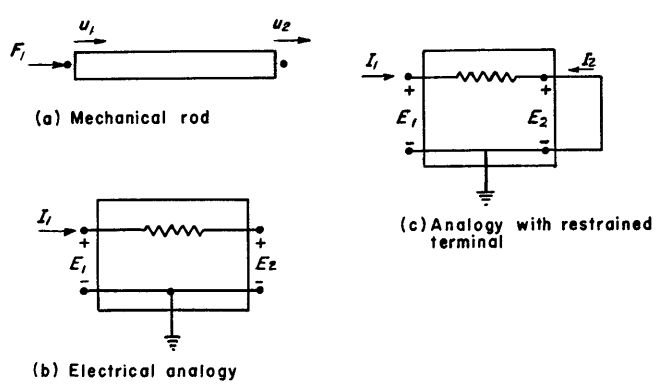

Fig. 7—Analogy for tension member.

system cannot be in equilibrium under the action of a force exerted at one end of the rod. In this example the $[Z]$ matrix does not exist, a fact which may variously be regarded as due to the singularity of the corresponding $[Y]$ matrix; as due to the existence of a rigid body translational degree of freedom; or, electrically, as due to the fact that no connection with ground is provided. The general elastic body has 6 elastic degrees of freedom (3 translations and 3 rotations) corresponding to a $[Y]$ matrix whose determinant and all of whose minor determinants of the first 5 orders vanish (defect = 6). An obvious way to remedy the situation is to *restrain* enough terminals to prevent rigid body motion. If this is done the $[Z]$ matrix in (11) will not describe the entire subsystem but only the influence which the remaining *free* terminals have on each other. A method for obtaining a complete description of the partially restrained system that is suitable for network synthesis is the subject of this section.

A black box is shown in Fig. 8 with any number of terminals (five shown). These are divided into two

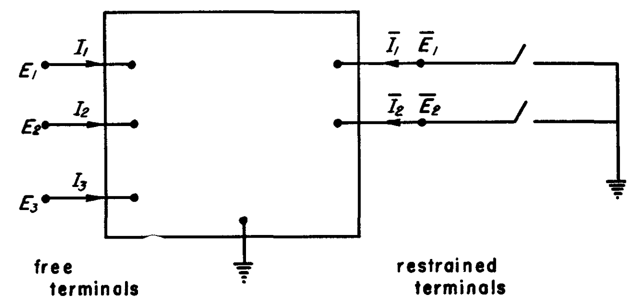

Fig. 8—System with some free terminals and some restrained terminals.

groups; free terminals (unbarred quantities), and restrained terminals (barred quantities). The restrained terminals are so chosen that rigid body motion is impossible when they are grounded. Their number may, however, exceed the minimum number required to prevent such motion. The $[Y]$ matrix for the black box exists and may be partitioned according to the free and restrained terminals.

$$\left[\begin{array}{c|c} Y_{ii} & Y_{ji} \\ \hline Y_{ij}^T & Y_{jj} \end{array}\right]\left[\begin{array}{c} E_i \\ \hline \bar{E}_j \end{array}\right] = \left[\begin{array}{c} I_i \\ \hline \bar{I}_j \end{array}\right], \tag{23}$$

where

$$i = 1, 2 \cdots p \qquad j = p + 1, \cdots n.$$

In accordance with the above discussion $[Y_{ii}]$ is nonsingular. Its inverse is defined as $[Z]$, i.e.:

$$[Y_{ii}]^{-1} = [Z]. \tag{24}$$

We wish to rewrite (23) so that $[E_i]$, voltage at free terminals, and $[\bar{I}_j]$, current at restrained terminals, both appear on the right.

Write the top half of (23) separately and solve for $[E_i]$:

$$[Y_{ii}][E_i] + [Y_{ij}][\bar{E}_j] = [I_i] \tag{25}$$

$$[E_i] = [Z][I_i] - [Z][Y_{ij}][\bar{E}_j]. \tag{26}$$

Write the bottom half of (23) separately:

$$[\bar{I}_j] = [Y_{ij}]^T[E_i] + [Y_{jj}][\bar{E}_j]$$
$$= [Y_{ij}]^T[Z][I_i] + \{[Y_{jj}] - [Y_{ij}]^T[Z][Y_{ij}]\}[\bar{E}_j]. \tag{27}$$

Define

$$[T] = -[Z][Y_{ij}]$$
$$[\bar{Y}] = [Y_{jj}] - [Y_{ij}]^T[Z][Y_{ij}]. \qquad (28)$$

Then, combining the results of the last three equations:

$$\left[\begin{array}{c|c} Z & T \\ \hline -T^T & \bar{Y} \end{array}\right]\left[\begin{array}{c} I_i \\ \hline \bar{E}_j \end{array}\right] = \left[\begin{array}{c} E_i \\ \hline \bar{I}_j \end{array}\right]. \qquad (29)$$

Each of the matrices appearing in (29) has a definite significance in terms of physical measurements on a mechanical system. $[Z]$ is the matrix of influence coefficients with the restrained terminals restrained. $[T]$ is a matrix giving the displacements at the free terminals due to displacements at the restrained terminals. If $[\bar{Y}]=0$, these displacements correspond to rigid body motion. $[\bar{Y}]$ is the matrix of force coefficients for the restrained terminals with the free terminals unrestrained. Hence if the number of restrained terminals is just equal to the minimum number required to prevent rigid body motion, $[\bar{Y}]$ will equal zero.

A network satisfying (29) is shown in block form in Fig. 9. This network employs a Cauer Z-network, a Cauer Y-network, and an all-transformer network. The Cauer Z-network differs from that shown in Fig. 5(b) in that the terminals at the bottom of Fig. 5(b) are connected to the input of the T-network instead of being grounded. An explicit representation of the network for the case of three free and two restrained terminals is shown in Fig. 10. It is interesting to observe that the number of transformer windings and the number of resistors required in this network are respectively equal to the number that would be required in an all-Y or all-Z representation.

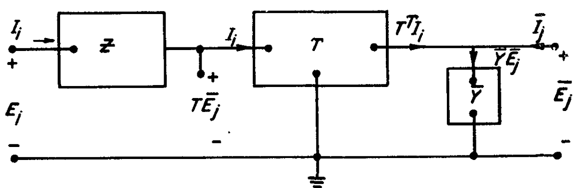

Fig. 9—General synthesis of system with restrained terminals.

Practical Applications

The analog computer solution of problems in structural analysis furnishes many illustrations of the use of the networks described in this paper.

Consider an airplane wing, which may be represented as a cantilever beam with vertical bending and torsional displacements, connected to the airplane's fuselage. Information concerning the wing is presented as a detailed description of its structural properties (EI and GJ vs spanwise station). Information concerning the fuselage is contained in the statement that it may be considered to be rigidly restrained at its center of gravity (for the particular problem at hand) and in a series of measurements giving the vertical displacement and pitching and rolling rotations at the side of the fuselage as a result of load applied at this point. Electrically the airplane wing can be represented by a standard beam analogy,[9] while the fuselage can be represented by a Cauer Z-network, such as that shown in Fig. 5(b). If the statement that the fuselage is fixed at its center of gravity is now amended to read that the inertia loads of the fuselage may be considered to be concentrated at the center of gravity, the fuselage can be represented by a circuit similar to Fig. 10. In this case the Y-network is a set of

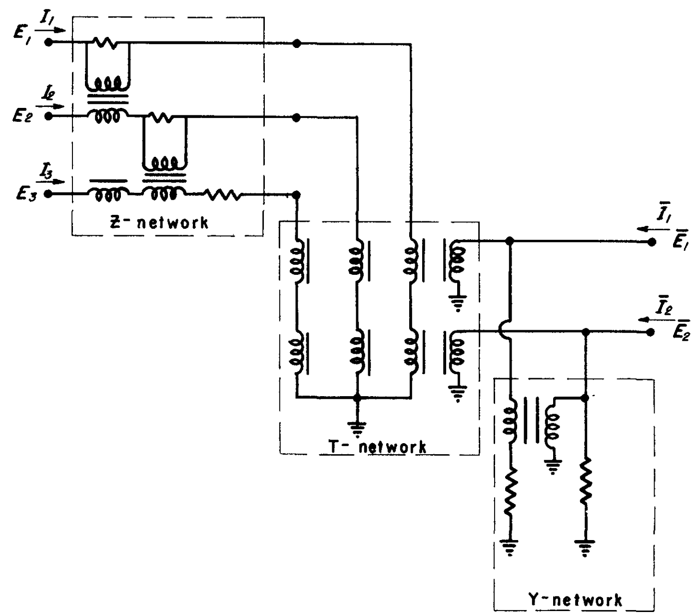

Fig. 10—Synthesis of system shown in Fig. 8.

three uncoupled capacitors representing the mass and the rolling and pitching moments of inertia of the fuselage.

An important type of analog computer circuit is the analogy for what may be termed the *one-dimensional element.* This is a structural element of narrow cross section for which the internal strains can be derived with sufficient accuracy from knowledge of the (approximately) rigid motions of its cross sections. Examples are straight, curved, and twisted beams with bending, torsional, and extensional degrees of freedom. Analogies can be synthesized for one-dimensional elements by first replacing distributed loads (if any) by concentrated loads and then deriving analogies for the segments between load points by methods described in this paper. Since the restraint of six degrees of freedom at one end of a one-dimensional element is just sufficient to prevent rigid body motion, the circuit will not contain a $\bar{y}$-network and the T-network can be derived from rigid body considerations.

As a subcase consider the straight segment of uniform beam shown in Fig. 11 for which a bending analogy is desired. In this case the $[T]$ matrix is given by

$$\begin{bmatrix} W \\ \theta \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \overline{W} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{W} \\ \bar{\theta} \end{bmatrix}. \quad (30)$$

The $[Z]$ matrix (obtained most easily from strain energy and Castigliano's theorem), is

$$\begin{bmatrix} W \\ \\ \theta \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} V \\ \\ M \end{bmatrix} = \begin{bmatrix} \dfrac{l^3}{3EI} & \dfrac{l^2}{2EI} \\ \\ \dfrac{l^2}{2EI} & \dfrac{l}{EI} \end{bmatrix} \begin{bmatrix} V \\ \\ M \end{bmatrix}. \quad (31)$$

The $[\overline{Y}]$ matrix is zero.

The resulting network is shown in Fig. 12. This form of the electrical beam analogy was first derived by Russell.[12]

In another paper the application of ideal transformers to the synthesis of purely reactive networks will be described and their importance for analog computer methods of vibration analysis will be pointed out.

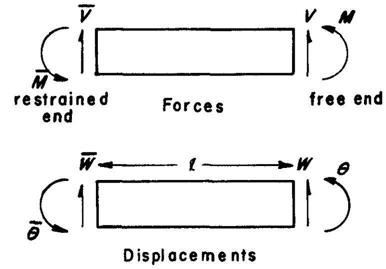

Fig. 11—Beam bending element.

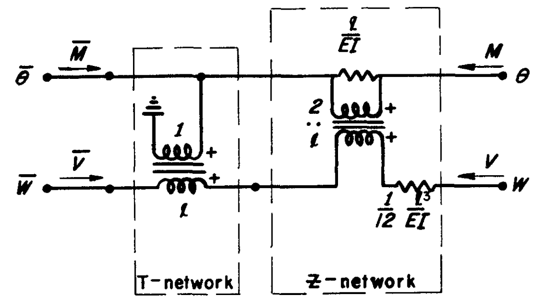

Fig. 12—Synthesis of analogy for beam bending element.

A New Approach to Grounding in DC Analog Computers*

C. M. EDWARDS†

Summary—**The difficulty of minimizing ground current effects and the problems caused by offset voltages in the development of dc analog computers is presented. A conventional grounding system is discussed and a ground isolation system is disclosed as a means of greatly reducing the existing differences in chopper references.**

Introduction

IN THE PAST, considerable attention has been given to two major problems in the development of dc electronic analog computers. These problems are the difficulty of minimizing drift in associated electronic circuits and the difficulty of minimizing ground current effects.

Drift in the electronic circuits is, in general, caused by grid current and cathode potential changes in the low level stages of the dc amplifiers. Unless extreme care is taken in the design of a conventional dc amplifier, over a short time interval the drift referred to the input stage will be in the range of 10 to 100 millivolts. Thus a dc computing amplifier, operating as an integrator with a full scale output of 100 volts and a gain factor of one second, might drift 0.1 per cent of full scale in one second if the drift referred to the input is 100 millivolts.

Since a drift of this magnitude can hardly be tolerated in a complex computer, considerable effort has been expended to minimize dc amplifier drift. The most successful technique developed to date[1] involves the use of a stabilizing circuit which provides low-frequency, driftless gain ahead of the drift point. A dc computing amplifier, so stabilized, can be readily maintained to have less than 0.5 millivolt drift referred to the input.

Ground current effects in a large scale dc analog computer can be troublesome from a number of standpoints. The most significant effects are those associated with the generation of offset voltages and the determination of system stability. Although the problem of stability should not be minimized, it does not present as great a problem in dc analog computers restricted to the audio frequency range as does the problem of offset voltages.

Offset voltages cause computing errors in much the same way as amplifier drift causes errors. The ground current offset problem has been greatly accentuated by the advent of the stabilized amplifier with its inherent low drift. If it is important to maintain amplifier drift under 0.5 millivolt, then it is equally important to reduce offset voltages to under 0.5 millivolt. Such a re-

* The system described in this article was developed in connection with a program sponsored by Special Devices Center, Office of Naval Research, Department of the Navy through the Bureau of Aeronautics.

† Res. Labs., Bendix Aviation Corp., Detroit, Mich.

[1] E. A. Goldberg, "Stabilization of wide-band direct current amplifiers for zero and gain," *RCA Rev.*, vol. II, pp. 296–300; June, 1950.