build error-correcting equipment into the machine. One of the necessary consequences of any error-correcting scheme is that it interprets many kinds of errors which are beyond its capabilities as being simple errors which it can correct and in attempting to correct an apparent simple error it will compound the real error. By programming, one can interrupt the solution in a way to minimize the damage done to other data in the machine and one can cause the computer to return to a verified point earlier in the solution.

"There is evidence that the reliability of at least one computer is rapidly becoming such that discussions of complicated redundancy checking systems seem almost superfluous. This still does not mean that we will not continue to use redundancy, but it does mean that we will look very carefully at such checking systems and insist that their costs in terms of equipment be as low as the costs in terms of time for programmed checks giving the same over-all reliability."

The third speaker, Dr. W. C. Carter, presented a paper by himself and John E. Makota.

In this paper, which is highly technical, the authors said:

"We propose to summarize the gain obtained by redundancy and will inquire as to whether this gain can best be obtained by using the extra equipment for checking or by using this extra equipment with other operating features. This discussion will be slanted toward small computers used for scientific work, and some conclusions that are reached will not hold under other hypotheses.

"The amount of results obtained for a fixed expenditure from a digital computer depends upon the amount of meaningful information obtained per unit time and the amount of time spent in problem preparation.

"The amount of correct information obtained per unit of scheduled operating time is a function of the reliability of the computer components, the amount and type of periodic testing of the system, and the amount of time necessary to repair the machine.

"Costs can be reduced not only by increasing the machine duty factor, but also by reducing problem preparation costs—especially for scientific problems.

"Thus there are many alternatives to adding checking equipment to increase the machine duty factor. The following additional equipment will provide increased speed or reduced programming costs and so must be considered as possible alternatives to checking equipment:

1. Floating-point operations.
2. Double precision operations.
3. *B*-lines.
4. Fast access lines.
5. Fast multiplication equipment.

These last alternatives may decrease programming time and thus may reduce costs.

6. Compiling routines.
7. Interpretive routines."

After introducing a hypothetical, though perfectly reasonable, machine to use for comparing these various factors, the authors conclude with the following statements:

"These considerations support the conclusion that all possible steps to increase machine speed should be taken first.

"Where programming costs are an important factor, as in varied scientific applications, the addition of programming aids should be made before checking.

"They also support the conclusion that there is a genuine economical and operational advantage in building checking into a machine under any circumstances."

These papers were followed by almost an hour of lively discussion of various pros and cons. No essentially new information seems to have come out of this, but a good many opinions were forcibly expressed. A vote at the request of Dr. V. M. Wolontis (of Bell Telephone Laboratories) on the question of how many people wanted complete self-checking gave about 60 per cent for and 40 per cent against it.

Small Digital Computers to Assist Large Digital Computers

PANEL DISCUSSION

THE participants in the panel agreed that there had been very few examples of installations where both large computers and small computers of the type discussed at the conference were available.

However, the installations at Douglas Aircraft and United Aircraft had contained, or did still contain, small-scale IBM card-programmed calculators (CPC) in addition to the large-scale IBM *701* computers. Mr. Lowe stated that the smaller computers had been abandoned at his installation because of the proved greater efficiency of the larger machine. Mr. Ramshaw stated that the United Aircraft installation still utilizes six CPC's because of their stand-by capacity for data reduction from the large number of test stations there. Use of these smaller (although not stored-program) calculators for these data-reduction problems prevented the necessity for stopping the larger machine to perform data reduction. It was later suggested from the floor by Dr. Grosch of the General Electric Company, that development now underway would allow such remote tests stands to "cut in" automatically on a large-scale computer for small-job computations, without causing more than a momentary delay of the major problem being computed.

It was the general consensus of the speakers that large computers performed more efficiently than small computers. However, Dr. Carr pointed out that, among the possibilities, small computers with the same instruction code as a large computer could be used to "check out" problems on the large machines, that small machines could perhaps be used to handle "exceptions" in a fashion parallel to the main computation performed by a large machine, and that small computers, with enough imagination on the part of the users, might be effectively applied to act as test equipment for larger computers, serve as "buffers" or "inertia storage" for input-output and computer control, and generally serve as accessory devices.

It was pointed out from the floor that

PANEL MEMBERS: J. W. Carr, III (chairman), University of Michigan, Ypsilanti, Mich.; John Lowe, Douglas Aircraft Corporation, Santa Monica, Calif.; Walter Ramshaw, United Aircraft Corporation, East Hartford, Conn.

computers of the Burroughs *E 101* type, with emphasis on low cost and ease of coding, would probably prove extremely useful in many laboratories where large machines were already available, in serving as a local calculator for groups removed physically from the major equipment.

Other discussion from the floor centered around the use of plugboards on small machines to provide competition with large machines, the code-checking possibilities of small computers for large computers, and relative merits of supplementing large machines by small computers at remote locations.

In general, the absence of small computer users, designers, and builders left the discussion mainly to the adherents of larger computers. Nevertheless, the interest engendered by some of the novel proposals introduced indicated that the use of small computers as assistants to the larger machines is not a dead issue.

Numerical Solution of Differential Equations

H. M. GURK

MORRIS RUBINOFF
ASSOCIATE MEMBER AIEE

THE problem of finding solutions to various types of differential equations has intrigued mathematicians from a theoretical point of view for many years. It has also plagued many applied scientists for an equally long period of time. Except for the relatively few equations whose solutions are available in closed form, the best one can do is to obtain approximate solutions.

These approximate or numerical solutions of differential equations are usually obtained in a stepwise fashion. Thus, an approximate solution of the equation

$$\frac{dx}{dt}=f(x, t)$$

is obtained as a sequence of values $x_n, n=0,1,\ldots$. x_n is taken as the numerical solution of the equation at the times $t = nh(n=0,1,\ldots)$ where h is a fixed positive interval called the integration step. If we let $\dot{x}_i=f(x_i,ih)$, then numerical solutions are usually obtained by the use of formulas of the following types:

1. The open formulas

$$x_n=\sum_{j=1}^{M} a_{jo}x_{n-j}+h\sum_{j=1}^{N} b_{jo}\dot{x}_{n-j} \quad M\geq 1,\ N\geq 0$$

where the a_{jo}'s and the b_{jo}'s are real.

Such a formula is denoted as an O_{MN} formula.

2. The closed or repeated closure formulas

$$x_n=\sum_{j=1}^{P}a_{jc}x_{n-j}+h\sum_{j=0}^{Q-1}b_{jc}\dot{x}_{(n-j)} \qquad P,Q\geq 1$$

where for each n this is applied in an iterative procedure with the value of $\dot{x}_n$ at any iterative step being obtained using the approximate available; initially we would have to make an educated guess, perhaps by using an open formula. This formula is denoted rC_{PQ}—the r standing for repeated.

3. The mixed formula $[O_{MN},C_{PQ}]$ consisting of the open method O_{MN} followed by a single application of the rC_{PQ} formula, which we denote just as C_{PQ}, the result of this being the accepted value of x_n. All the ordinates, that is, x_{n-j}'s, in both the open and closed parts of this mixed formula are the final values computed from the closed formula at previous steps, and all derivatives ($\dot{x}_{n-j}$'s) are computed using the values computed from the open formula.

In the following discussion we will call any open formula O_{MN} with given coefficients, a quadrature formula of type O_{MN}. Likewise, we will talk about quadrature formulas of type rC_{PQ}, and of type $[O_{MN},C_{PQ}]$.

Classically there has been one method used in choosing the quadrature coefficients which appear in any open or closed formula. This is the so-called polynomial method. For a given type of formula this is equivalent to choosing the coefficients so that the positive integer R is a maximum where the equations $\dot{x}=0$, $\dot{x}=1, \dot{x}=t, \dot{x}=t^2,\ldots,\dot{x}=t^R$ are solved correctly by the quadrature formula. For these open and closed types of formulas, such sets of coefficients are easily found since the procedure amounts only to the solution of a set of simultaneous linear algebraic equations. The mixed quadrature formulas commonly used are the combinations of classical open and classical closed formulas.

Using the classical open or closed formulas, one can easily compute that the truncation error per step, that is, the error one obtains at the nth step assuming infinite precision and all previous values correct, is of the order of $Ch^{R+2}\left|f^{(R+1)}(x(\tau),\tau)\right|$ where τ is some value of t between the greatest and least values used in that step, and C is a constant. Thus, for a given formula, one can make the truncation error per step as small as desired by decreasing h, the size of the step. If the total interval over which the equation is to be solved, and an upper bound for the $(R+2)$th derivative in this interval are known, then the total truncation error can be reduced below any desired bound by making the step length sufficiently small. The classical open or closed formula of a given type is the formula of that type which generally makes the truncation error per step tend to zero fastest as h tends to zero.

It is often the case that the solution of the differential equation is desired over an essentially infinite interval and the knowledge of a bound for the error per step is not of much use. In these cases the asymptotic behavior of the numerical solution may be very important and a close approximation of the true solution's asymptotic behavior is desired. With this in mind, we come to the notion of the

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