Homework Due Friday Oct 25, 2024

1. For $n \geq 2$, let $B_n : \{1, 2 \dots n\} \to \{1, \dots n\}$ be defined by

$$B_n(k) = n + 1 - k$$

(B_n is the permutation that lists $\{1, \dots n\}$ in reverse order).

Find $\operatorname{sgn}(B_n)$.

2. Show that $\pi_{ij} : \{1, \dots n\} \to \{1, \dots n\}$

$i, j \in \{1, \dots n\}$ defined by

$$\pi_{ij}(k) = k \qquad k \neq i, j, \; i \neq j$$

$$\pi_{ij}(i) = j$$

$$\pi_{ij}(j) = i$$

has $\operatorname{sgn}(\pi_{ij}) = -1$

3. Is every element of S_n (symmetric group on n elements) is a product (composition) of interchanges π_{ij} (as in previous problem)

Suggestion: Try an interchange to get 1 to go to 1.

4- Explain why A_n $(= \{\sigma \in A_n : sgn(\sigma) = 1\}$ is a normal subgroup by showing that $\pi \to sgn(\pi)$ is a homomorphism of S_n onto the group $\{-1, 1\}$, operation multiplication

5. Check explicitly that S_3 has one subgroup of order 3 which is normal and three subgroups of order 2, none of which is normal.

6(a) Prove without computing the compositions that π_{12} and π_{23} (notation as above) do not commute by noting that if they did commute then

$e, \pi_{12}, \pi_{23}, \pi_{12}\pi_{23}$ would be a subgroup of S_3 of order 4

(b) Compute $\pi_{12}\pi_{23}$ and $\pi_{23}\pi_{12}$ to see

They are different,

7. If $n \geq 3$ and $k < n$, consider the "k-cycle" σ, the permutations of $\{1, \ldots n\}$ generated by looking at $1, \ldots k$ and moving it to the right

$1 \to 2$

$2 \to 3$

$k-1 \to k$

$k \to 1$.

(a) What is $\operatorname{sgn}(\sigma)$?

(b) What is the order of σ?

*8. Think about k-cycles on every subset of k elements in $\{1, \ldots n\}$ (a) Is every element of S_3 a product of disjoint cycles? (k arbitrary)

(b) How about S_4? (c) S_n?