6.4 Asher Christian

2 random sample $x_1, x_2, \ldots, x_n$ from

$N(\mu, \sigma^2)$ $\theta = \sigma^2$ $0 < \theta < \infty$

$\mu \in \mathbb{R}$ $L(\theta)$ is $\hat{\theta} = (1/n) \sum_{i=1}^{n} (x_i - \mu)^2$

unbiased

pdf $(N(\mu, \theta^2)) =$

$$\frac{1}{\sqrt{2\pi\theta^2}} e^{\left(-\frac{(x-\mu)^2}{2\theta^2}\right)}$$

$$L(x_1, \ldots, x_n : \theta) = (2\pi\theta)^{-\frac{n}{2}} e^{\left(-\frac{\sum (x_i - \mu)^2}{2\theta}\right)}$$

$$\ln(L(x_1, \ldots, x_n : \theta)) = -n/2 \ln(2\pi\theta) - \frac{\sum (x_i - \mu)^2}{2\theta}$$

$$\frac{d}{d\theta} L(x_i, \theta) = -\frac{n}{2} \cdot \frac{1}{\theta} + \frac{\sum (x_i - \mu)^2}{2\theta^2}$$

$$0 = -n\theta + \sum (x_i - \mu)^2$$

$$\hat{\theta} = \frac{\sum (x_i - \mu)^2}{n} = \frac{(n-1)S^2}{n}$$

now show unbiased

$$E[\hat{\theta}] = E\left[\frac{\sum (x_i - \mu)^2}{n}\right] = \frac{\sum E[(x_i - \mu)^2]}{n}$$

$$E[(x_i - \mu)^2] = \sum_{i=0}^{n} \frac{(x_i - \mu)^2}{\sqrt{2\pi\theta}} e^{\left(-\frac{(x-\mu)^2}{2\theta}\right)}$$

$$= \sigma^2$$

$$E[\hat{\sigma}] = \frac{\sum \sigma^2}{n} = \frac{n\sigma^2}{n} = \sigma^2$$

6.4

3 $\quad pmf = \frac{\lambda^{x_i} e^{-\lambda}}{x!}$

$$L(\lambda) = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$$

$$\ln(L(\lambda)) = -n\lambda + \sum x_i \ln \lambda - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{d}{d\lambda}\left(\ln(L(\lambda))\right) = -n + \frac{\sum x_i}{\lambda}$$

$$n = \frac{\sum x_i}{\lambda}$$

$$\hat{\lambda} = \frac{\sum x_i}{n} = \overline{X}$$

$$X = \{5.0, 7.1, 12.2, 9.3, 5.4, 1.5, 1.6\}$$

$$\hat{\lambda} = \frac{\sum X}{40} = 2.225$$

6.4-8

pdf $\quad f(x:\theta) = \left(\frac{1}{\theta}\right) x_i^{\frac{(1-\theta)}{\theta}}$

$$L(\theta) = \left(\frac{1}{\theta}\right)^n (\Pi x_i)^{\left(\frac{1-\theta}{\theta}\right)}$$

$$\ln(L(\theta)) = n\ln\left(\frac{1}{\theta}\right) + \left(\frac{1-\theta}{\theta}\right)\left(\sum \ln(x_i)\right)$$

$$\frac{d}{d\theta}\ln(L(\theta)) = -\frac{n}{\theta} - \frac{\sum \ln x_i}{\theta^2}$$

$$0 = -n\theta - \sum \ln x_i$$

$$\hat{\theta} = -\frac{\sum \ln x_i}{n}$$

$$E(\hat{\theta}) = -\frac{\sum E(\ln x_i)}{n}$$

$$E(\ln x_i) = \frac{1}{\theta}\int_0^1 \ln x_i \; x_i^{\left(\frac{1-\theta}{\theta}\right)} dx$$

$$= \frac{1}{\theta}\left(\frac{x_i^{\left(\frac{1-\theta}{\theta}+1\right)}}{\left(\frac{1-\theta}{\theta}+1\right)} \ln x_i - \int \frac{x_i^{\left(\frac{1-\theta}{\theta}\right)}}{\left(\frac{1-\theta}{\theta}+1\right)} dx \right)$$

$$= \frac{x_i^{\left(\frac{1-\theta}{\theta}+1\right)}}{\theta\left(\frac{1-\theta}{\theta}+1\right)} \ln x_i - \frac{x_i^{\frac{(1-\theta}{\theta}+1}}{\theta\left(\frac{1-\theta}{\theta}+1\right)^2}$$

$$= x_i^{(\frac{1}{\theta})} (\ln x_i - \theta) \Big|_0^1$$

$\lim_{x \to 0} x \ln x$

$= \frac{1}{}$

$$-\theta - 0 = -\theta$$

$$E(\hat{\theta}) = \frac{-\sum -\theta}{n} = \theta$$

6.4-12

$x = 0, 1$

each X_i pdf $= p^x (1-p)^{1-x}$

$$\hat{p} = \overline{X} = Y/n = \frac{\sum x_i}{n}$$

a) $E(\hat{p}) = \dfrac{\sum E(x_i)}{n}$

$$E(x_i) = \sum_0^1 x p^x (1-p)^{1-x}$$

$$= p$$

$$E(\hat{p}) = \frac{\sum p}{n} = p$$

b) $Var(\bar{x}) = Var\left(\frac{1}{n} \sum X_i\right)$

$$= \frac{1}{n} Var(\sum x_i)$$

n

$$Var(X_i) = p(1-p)$$

$$Var(\bar{X}) = \frac{1}{n^2}\sum p(1-p)$$

$$= \frac{p(1-p)}{n}$$

C Show that $E(\bar{x}(1-\bar{x})] = (n-1)(p(1-p)/n)$

$$E(\bar{x}(1-\bar{x})] = E\left(\frac{\sum x_i}{n} - \left(\frac{\sum x_i}{n}\right)^2\right)$$

$$= p - \frac{E((\sum x_i)^2)}{n^2}$$

$$= p - \frac{1}{n^2}\left(E(\sum x_i^2) + 2E\left(\sum_{1 \le i < j \le n} x_i x_j\right)\right)$$

$$= p - \frac{1}{n^2}\left(np + n(n-1)\, p^2\right)$$

$$= p - \frac{p}{n} - \frac{(n-1)p^2}{n}$$

$$= \frac{p(n-1)}{n} - \frac{(n-1)p^2}{p}$$

$$= (n-1)\left(\frac{p}{n} - \frac{p^2}{n}\right)$$

$$= (n-1)\frac{(p(1-p))}{n}$$

d find c so $c[\bar{x}(1-\bar{x})]$ is unbiased

$$c = \frac{n}{n-1}$$

6.4-15

$$\mu = \alpha\theta \qquad \sigma^2 = \alpha\theta^2$$

$$0 < \alpha < \infty \qquad 0 < \theta < \infty$$

$$pdf = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{(\alpha-1)} e^{(-x/\theta)}$$

$$L(\alpha,\theta) = \left(\frac{1}{\Gamma(\alpha)\theta^\alpha}\right)^n (\Pi x)^{(\alpha-1)} e^{\left(-\frac{\Sigma x_i}{\theta}\right)}$$

$$\ln(L(\alpha,\theta)) = n\ln\left(\frac{1}{\Gamma(\alpha)\theta^\alpha}\right) + (\alpha-1)\Sigma\ln x_i - \frac{\Sigma x_i}{\theta}$$

$$= -n\ln(\Gamma(\alpha)) - n\alpha\ln(\theta) + (\alpha-1)\Sigma\ln x_i - \frac{\Sigma x_i}{\theta}$$

$$\frac{d}{d\theta}(L(\alpha,\theta)) = \frac{-n\alpha}{\theta} + \frac{\Sigma x_i}{\theta^2}$$

$$n\alpha\theta = \Sigma x_i$$

$$\hat{\theta} = \frac{\Sigma x_i}{n\alpha}$$

$$\frac{d}{d\alpha}(\ln(L(\alpha,\theta))) = -n\psi(\alpha) - n\ln(\theta) + \Sigma\ln x_i$$

$$n\psi(\alpha) = \Sigma\ln x_i - n\ln(\theta)$$

$$\psi(\alpha) = \frac{\Sigma\ln x_i}{n} - \ln\theta$$

$$\alpha\theta = \bar{X}$$

$$\alpha\theta^2 = V$$

$$\alpha = \frac{\bar{X}^2}{V}$$

$$\theta = \frac{V}{\bar{X}}$$

$$\bar{X} = 6.74 \qquad \alpha = 102.498$$

$$V = 0.4432 \qquad \theta = 0.0657$$

6.5 - 3

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

x = midterm

y = final

$\bar{x} = 74.5$

$\bar{y} = 86.5$

$\sum (x_i - \bar{x})(y_i - \bar{y}) = 421$

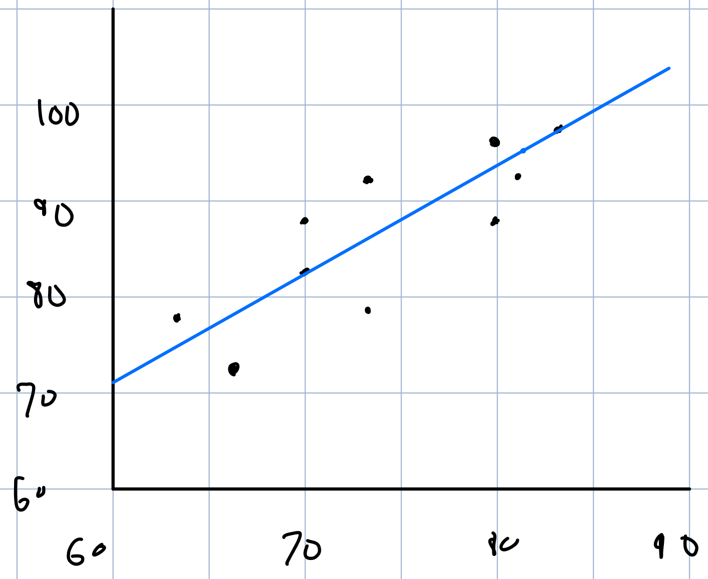
$\sum (x_i - \bar{x})^2 = 414.5$

$\hat{\beta} = 1.015$

$\alpha = 11.172$

$\hat{y} = 11.132 + 1.02(x)$

b)



c)

$$\hat{\sigma}^2 = \frac{1}{n} \sum (y - \hat{\alpha} - \hat{\beta} x_i)^2$$

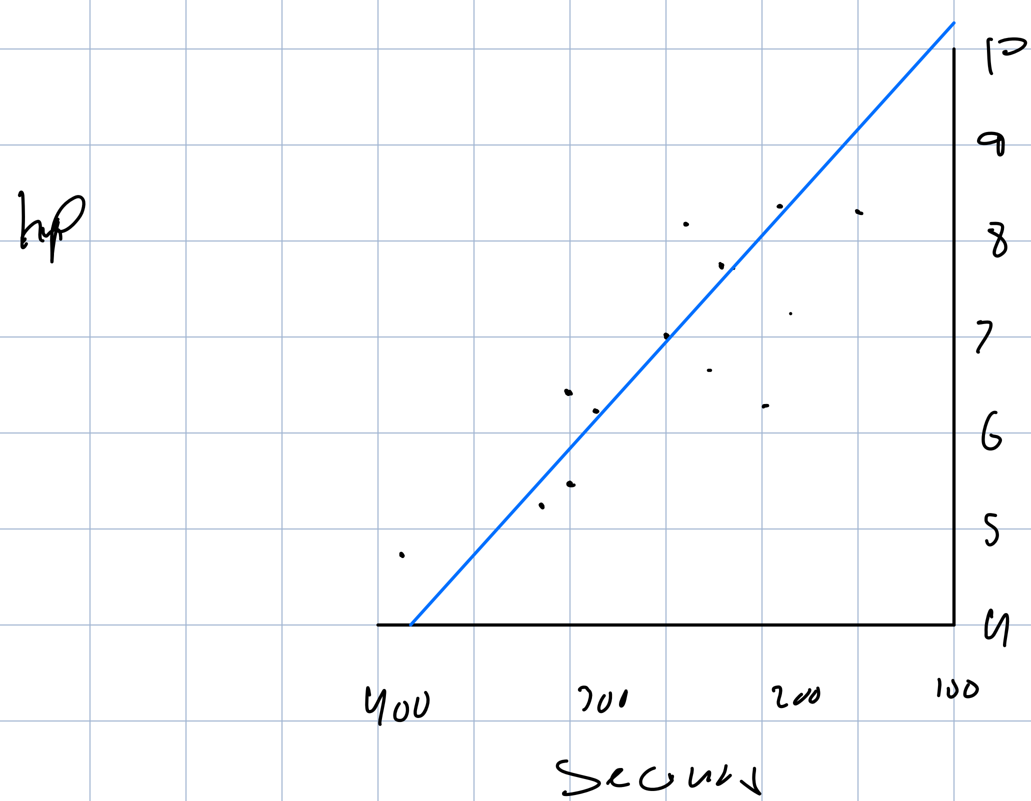
$$= 18$$

6.5-5

a) $\hat{\alpha} = 12.45$

$\hat{\beta} = -0.0222$

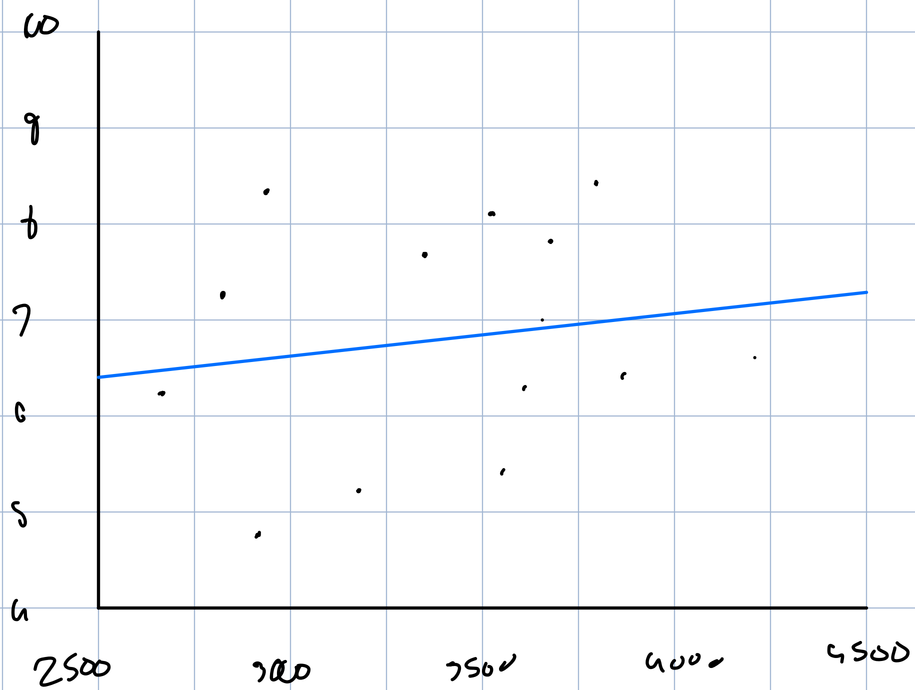
b)



c) $\alpha = 5.47$

$\beta = 0.000395$

d



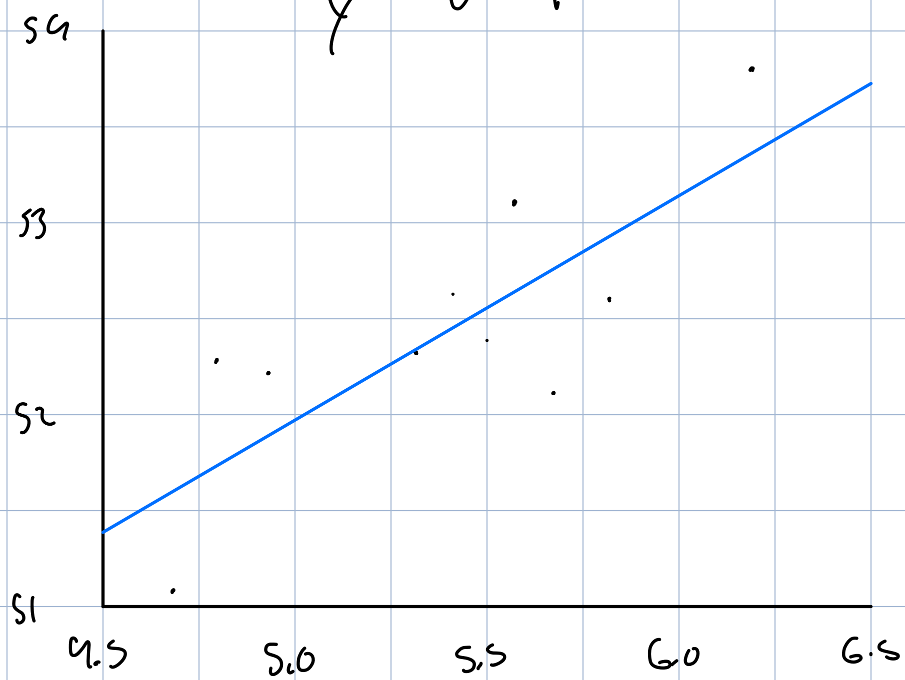
e) horsepower has a stronger correlation

6.5-8

$\hat{\alpha} = 46.587$

$\hat{\beta} = 1.085$

$y = \hat{\alpha} + \hat{\beta} x$



c

Change in length is positivly correlated with cross sectional area