HW 4 - 110AH

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12.11.24

1

Suppose A is an orthogonal $n \times n$ matrix of det = 1. $n \geq 2$. Is there a continuous function $\gamma_A : [0,1] \to SO(n)$ such that $\gamma_A(0) = I_n$ and $\gamma_A(1) = A$? Prove your answer. (Suggestion: think about $n = 2$ first). Starting with $n = 2$ we recall that all $A \in SO(2)$ are of the form

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

With the case of $\theta = 0$ being the identity. Thus the function of t that is the matrix

$$\begin{pmatrix} cos(t\theta) & -sin(t\theta) \\ sin(t\theta) & cos(t\theta) \end{pmatrix}.$$

is equal to the original matrix at $t = 1$ and is equal to identity at $t = 0$ and each $t \in [0,1]$ results in a transformation in $SO(2)$ thus the function satisfies the above properties. If $A \in SO(n)$ then A splits into block form given some basis in a way that looks like

$$A = \begin{pmatrix} \begin{pmatrix} cos(\theta_1) & -sin(\theta_1) \\ sin(\theta_1) & cos(\theta_1) \end{pmatrix} & 0 & \dots & 0 \\ 0 & \begin{pmatrix} cos(\theta_2) & -sin(\theta_2) \\ sin(\theta_2) & cos(\theta_2) \end{pmatrix} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since we have proven the existence of the continous function for each block it follows that one can create the similar function $\gamma_A : [0,1] \to SO(n)$ defined by

$$\gamma_A(t) = \begin{pmatrix} \begin{pmatrix} cos(t\theta_1) & -sin(t\theta_1) \\ sin(t\theta_1) & cos(t\theta_1) \end{pmatrix} & 0 & \dots & 0 \\ 0 & \begin{pmatrix} cos(t\theta_2) & -sin(t\theta_2) \\ sin(t\theta_2) & cos(t\theta_2) \end{pmatrix} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This function satisfies all of the conditions outlined in the question.

2

1. (a)
   What condition on A guarantees that $\det e^{tA} = 1$ *for all t with* $|t| < \epsilon$ *for some* ϵ.

   We know that $\det(e^{tA}) = e^{t\,\mathrm{Tr}(A)}$ thus if $t\,\mathrm{Tr}(A) = 0$ then the determinant is 1. Thus The condition that the trace of A is zero guarantees that the determinant of the exponentiation is equal to 1 for all t.

2. (b)
   Does it follow that $\det(e^{tA}) = 1$ *for all t if A satisfies the condition you found in part (a)?* Yes it follows directly because if the trace is 0 then for all t $t\,\mathrm{Tr}(A) = 0$ for all t.

3

How is 2(b) related to the fact which you should prove that $\det(e^{tA})$ *has a power series in t that converges for all* $t \in \mathbb{R}$

We know that

$$\det(e^{tA}) = \det(I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + ...).$$

fixing t we have $||tA||_{\mathrm{op}} = c$ and by the property of operator norm $(I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + ...)$ converges if the operator norm converges $||(I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + ...)||_{\mathrm{op}}$ which is less than or equal to

$$(1 + ||tA|| + \frac{||tA||^2}{2!} + \frac{||tA||^3}{3!} + ...).$$

This is important because if $\mathrm{Tr}\, B = 0$ then the function $A = e^{tB}$ always has determinant 1.

4

1. (a) *Show that* $\mathrm{Tr}(AB - BA) = 0$ We know that $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$ So

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}B_{kj} = \sum_{k=1}^{n}\sum_{j=1}^{n} B_{kj}A_{jk} = \sum_{i=1}^{n}(BA)_{ii} = \mathrm{Tr}(BA).$$

   We also have that

$$\mathrm{Tr}(A+B) = \sum_{i=1}^{n}(A+B)_{ii} = \sum_{i=1}^{n}(A_{ii}+B_{ii}) = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(B).$$

   Putting them together we have $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0$

2. (b) *Show that if A, B are skew symmetric ($A^T = -A, B^T = -B$ then $AB - BA$ is skew symmetric.*
$(AB - BA)^T = (B^T A^T - A^T B^T) = (BA - AB) = -(AB - BA)$

5

With $||A||_{op} = \max ||A\vec{x}||, ||\vec{x}|| = 1$ $n \times n$ $\mathbb{R}$valued matrix and $||\vec{x}||$ usual euclidean length $||x_1, ..., x_n|| = (\sum_1^n x_j^2)^{\frac{1}{2}}$ Show

1. (a) $||AB|| \leq ||A|| \times ||B||$
   Let $\vec{x}$ be s.t. $||\vec{x}|| = 1$ then $||B\vec{x}|| \leq ||B||$ and let $\vec{w} = B\vec{x}$.
   Then $||A\frac{\vec{w}}{||\vec{w}||}|| \leq ||A||$.
   Therefore $||A\vec{w}|| \leq ||A|| \cdot ||\vec{w}||$
   replace $\vec{w}$ with $B\vec{x}$ we have
   $||AB\vec{x}|| \leq ||A|| \cdot ||B||$
   Since this holds for all $\vec{x}$ satsifying $||\vec{x}|| = 1$ we have
   $||AB|| \leq ||A|| \cdot ||B||$

2. (b) $||A + B|| \leq ||A|| + ||B||$ Let $\vec{x}$ again be s.t. $||\vec{x}|| = 1$ then
   $||A\vec{x}|| \leq ||A||$ and $||B\vec{x}|| \leq ||B||$
   Thus $||(A + B)\vec{x}|| = ||A\vec{x} + B\vec{x}|| \leq ||A\vec{x}|| + ||B\vec{x}|| \leq ||A|| + ||B||$ and so
   $||(A + B)\vec{x}|| \leq ||A|| + ||B||$ for all $\vec{x}$ satisfying $||\vec{x}|| = 1$ thus it holds that
   $||A + B|| \leq ||A|| + ||B||$

6

Use prob 5 to prove

1. (a) *the series for e^A converges for all A*
   $e^A = \sum_{i=1}^{\infty} \frac{A^i}{i!}$ let $(a_n)_n = \sum_{i=1}^{n} \frac{A^i}{i}$ for any $\epsilon > 0$ $\exists N$ s.t $\frac{x}{N+1} < 1$ and
   $\frac{||A||^N}{N!} < \epsilon(1 - \frac{||A||}{N+1})$. WLOG $n < m$

   $$||A_m - A_n|| = ||\sum_{i=n}^{m} \frac{A^i}{i!}|| \leq \sum_{i=n}^{m} \frac{||A||^i}{i!} \leq \sum_{i=n}^{\infty} \frac{||A||^i}{i!} = \frac{||A||^n}{n!}(1 + \frac{||A||}{n+1} + \frac{||A||^2}{(n+1)(n+2)} + ...).$$

   $$\leq \frac{||A||^n}{n!} \sum_{i=1}^{\infty} (\frac{||A||}{(n+1)})^i = \frac{||A||^n}{n!} \frac{1}{1 - \frac{||A||}{n+1}} < \epsilon.$$

   Thus $(a_n)_n$ is cauchy and since R^{n^2} is complete, it converges to some element.

2. (b) *The same for $\ln(I + A)$ converges if $||A|| < 1$.* Assuming the Taylor series for $\ln(I + A) =$

   $$A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + ... = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}A^i}{i}.$$

By the same logic

$$\| \ln(I + A) \| \leq \| \sum_{i=1}^{\infty} \frac{(-1)^{i-1} A^i}{i} \| \leq \sum_{i=1}^{\infty} \frac{(-1)^{i-1} \|A\|^i}{i}.$$

This converges for all $\|A\| < 1$ by the convergence of the taylor series for $ln(1 + x)$ This implies by the previous part that the sequence of partial sums is cauchy which shows that after a certain point and given an $\epsilon > 0$ the partial sums do not allow the sum to deviate more than ϵ from the final solution.

7

Check that formally $e^{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots} = 1 + x$ up to terms of degree ≤ 4 in x.

$$= 1 + \sum_{i=1}^{\infty} \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots)^i}{i!} = 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^2}{2} - \frac{x^3}{3} + \frac{11x^4}{24} + \frac{x^3}{6} - \frac{3x^4}{12} + \frac{x^4}{24} + \ldots$$

$$= 1 + x + O(5+).$$

8

if $e^A = I_n$, does A have to be skew symmetric (I_n is of course orthogonal so this is a natural question). Suggestion $P^{-1} I_n P = I_n$ whereas $P^{-1} e^A P = e^{P^{-1} A P}$ and $P^{-1} A P$ does not have to be skew symmetric even if A is?

$$e^{P^{-1} A P} = \sum_{i=0}^{\infty} \frac{(P^{-1} A P)^i}{i!} = \sum_{i=1}^{\infty} \frac{P^{-1} A^i P}{i!} = P^{-1} \sum_{i=0}^{\infty} \frac{A^i}{i!} P = P^{-1} e^A P.$$

And so since

$$A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} e^A = I.$$

But taking

$$P = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} P A P^{-1} = \begin{pmatrix} 6\pi & 4\pi \\ -10\pi & -6\pi \end{pmatrix}.$$

This matrix is not skew symmetric but

$$e^{P A P^{-1}} = I.$$

So It is possible for the exponentiated matrix not to be skew symmetric

9

If $A \in Gl(n, \mathbb{R})$ with $\det(A) > 0$, does A have to be a square root - i.e. does there have to be a $B \in Gl(n, \mathbb{R}) \to B^2 = A$? [This is related to A have a ln since if $A = e^C$ then $(e^{\frac{c}{2}})^2 = A$].
let

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad \det(A) = 1.$$

Then for any

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + cd & cb + d^2 \end{pmatrix}.$$

giving the system of equations

$$a^2 + bc = -2$$
$$b(a + d) = 0$$
$$c(a + d) = 0$$
$$cb + d^2 = -\frac{1}{2}$$

and $b \neq 0, c \neq 0$ so $a = -d$ which implies

$$bc = -2 - a^2$$
$$= -2 - d^2$$
$$cb = -\frac{1}{2} - d^2$$
$$-2 = -\frac{1}{2}$$

Thus no square root exists for some $A \in GL(n, \mathbb{R})$

10

How about prob 9 if $A \in SL(n, \mathbb{R})$ $(\det(A) = 1)$? The example used in the last question has determinant 1 so the same statement holds that there are some matrices with no square root.

11

Work out a proof that $\det(e^A) = e^{\operatorname{Tr} A}$ by doing the following sequence of steps:
(a) using $\det(e^{(t+\Delta t)A} = \det(e^{tA}) \det(e^{(\Delta t)A})$ Show that if $F(t) = \det(e^{tA})$ then $F'(t) = \det(e^{tA}) \cdot F'(0)$
(b) Show $F'(0) = \operatorname{Tr} A$ by writing only $e^{(\Delta t)A} - 1$ and throwing away all higher degree than 1 in Δt terms.
(c) Deduce that $F'(t) = F(t) \operatorname{Tr}(A)$ and apply what you know about differential

equations to get $\det(A) = e^{\operatorname{Tr} A}$ *by putting* $t = 1$

if $F(t) = \det(e^{tA})$ then

$$F'(t) = \lim_{\Delta t \to 0} \frac{F(\Delta t + t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\det(e^{(\Delta t + t)A}) - \det(e^{tA})}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\det(e^{tA})(\det(e^{\Delta t A}) - 1)}{\Delta t} = \det(e^{tA}) \lim_{\Delta t \to 0} \frac{\det(e^{\Delta t A}) - 1}{\Delta t} = \det(e^{tA})F'(0) = F(t)F'(0).$$

$$F'(0) = \lim_{\Delta t \to 0} \frac{\det(e^{\Delta t A}) - 1}{\Delta t}.$$

$$e^{\Delta t A} = I + \Delta t A + O(2+) = \begin{pmatrix} 1 + \Delta t a_{1,1} + O(2+) & \Delta t a_{1,2} + O(2+) & ... & ... \\ \Delta t a_{2,1} + O(2+) & 1 + \Delta t a_{2,2} + O(2+) & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & 1 + \Delta t a_{n,n} + O(2+) \end{pmatrix}.$$

for the determinant when divided by Δt not to go to zero we can consider only terms with order less than 2 in the determinant. It is clear that when computing the determinant taking the product of any element not on the diagonal will result in at least two elements not on the diagonal and thus order of at least 2 for all elements of the product. It then suffices to only compute the first order terms computed along the diagonal.

$$\det(e^{\Delta t A}) = 1 + \sum_{i=1}^{n} \Delta t a_{i,i} + O(2+).$$

And so

$$F'(0) = \lim_{\Delta t \to 0} \frac{\det(e^{\Delta t A}) - 1}{\Delta t} = \lim_{\Delta t \to 0} \sum_{i=1}^{n} a_{i,i} + O(1+) = \sum_{i=1}^{n} a_{i,i} = \operatorname{Tr} A.$$

Therefore since $F : \mathbb{R} \to \mathbb{R}$, $F'(0) = \operatorname{Tr} A$ and

$$F'(t) = F(t) \operatorname{Tr} A.$$

we have

$$F(t) = e^{t \operatorname{Tr} A}.$$

and

$$\det(e^{tA}) = e^{t \operatorname{Tr} A}.$$

more specifically

$$F(1) = \det(e^{A}) = e^{\operatorname{Tr} A}.$$

12

Is every element $A \in Sl(n, \mathbb{R}) = e^B$ *for some Trace 0 B? Prove your answer.*
By question 10. There exist some elements of $A \in SL(n, \mathbb{R})$ s.t. for all $G, G^2 \neq A$. Therefore if $e^B = A$ for some B then $(e^{\frac{B}{2}})^2 = e^B = A$ and thus $e^{\frac{B}{2}}$ would be a square root of A which we showed does not necessarily exist therefore not all elements of $SL(n, \mathbb{R})$ arise from an exponential map.