Additional Sample Problems

(to help with decomposition into products of Sylow subgroups etc.)

1. Prove: If H, K are normal subgroups of a finite group G then

$HK \overset{def}{=} \{hk : h \in H, k \in K\}$ is a normal subgroup of G.

Steps: (1) HK is closed under products

$h_1k_1h_2k_2 = h_1(k_1h_2)k_2$

$k_1h_2 \in k_1H = Hk_1$ since H is normal

so $k_1h_2 = h_3k_4$ and

$h_1k_1h_2k_2 = h_1h_3k_4k_2 \in HK$

(2) Subset $\neq \emptyset$ of G closed under products is a subgroup.

(3) $g^{-1}HKg = g^{-1}Hg\,g^{-1}Kg = HK$ since H and K are normal.

Prop 2: If H, K are normal in G and $\gcd(|H|, |K|) = 1$, then $|HK| = |H||K|$.

Proof: $h_1 k_1 = h_2 k_2 \Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1}$

So $h_2^{-1} h_1 = k_2 k_1^{-1} \in H$ and $\in K$.

If $g \in H \cap K$ then $|g| \mid H$ and $|g| \mid K$ and $\gcd(|H|, |K|) = 1 \Rightarrow$ $g = e$. So $h_1 = h_2$ and $k_1 = k_2$.

Conclusion follows $|HK| = |H||K|$.

Prop 3: If H, K are normal and $|H \cap K| = 1$ then $hk = kh$ for all $k \in K$ $h \in H$

Proof: $hkh^{-1}k^{-1} \in H$ since

$= h\underbrace{(kh^{-1}k^{-1})}_{\in H}$ since H is normal so $\cancel{k} H k^{-1} = H$]

Similarly $\underbrace{(hkh^{-1})}_{\in K} k^{-1} \in K$ so

$hkh^{-1}k^{-1} \in H \cap K = \{e\}$ so

$hk = kh$

Prop 4 If H, K normal subgroups, $\gcd(|H|, |K|) = 1$. Then

$$HK \cong H \otimes K$$ (internal) direct product

(follows from previous.)

Prop 5: If $|G| = p_1^{s_1} \cdots p_n^{s_n}$ ← G not necessarily abelian here!

and all p_j-Sylow subgroups are normal, then G = direct product of $S_1, \ldots S_n$

where $S_j = p_j$-Sylow subgroup (unique since normal ⟹ only one since all are conj)

Proof by induction:

$$S_1 S_2 = S_1 \otimes S_2 \text{ by previous.}$$

$|S_1 S_2| = p_1^{s_1} p_2^{s_2}$ and $S_1 S_2$ is normal.

So $(S_1 S_2) S_3 = (S_1 S_2) \otimes S_3$

$= S_1 \otimes S_2 \otimes S_3$.

Continue to get $S_1 S_2 \cdots S_n \cong S_1 \otimes \cdots \otimes S_n$

But $\# S_1 \otimes \dots \otimes S_k = p_1^{s_1} \cdots p_k^{s_k} = |G|$

So $S_1 \otimes \dots \otimes S_k = G$.

This of course works for G abelian automatically since all subgroups are normal.

G abelian = direct product of cyclics has been shown (in class and in handout)

i.e.

$$G = \mathbb{Z}_{n_1} \otimes \dots \otimes \mathbb{Z}_{n_k}$$

So to finish abelian group structure

we need

Prop. 6: If $n = p_1^{k_1} \cdots p_l^{k_l}$

then

$$\mathbb{Z}_n = \mathbb{Z}_{p_1^{k_1}} \cdots \mathbb{Z}_{p_l^{k_l}}$$ ← direct product

To prove this we use Lemma on p. 39 of handout which gives

$$\mathbb{Z}_{p_1^{n_1}} \otimes \mathbb{Z}_{p_2^{n_2}} \cong \mathbb{Z}_{p_1^{n_1} p_2^{n_2}}$$

Inductively

$$\mathbb{Z}_{p_1^{k_1} p_2^{k_2}} \otimes \mathbb{Z}_{p_3^{k_3}}$$

$$\cong \mathbb{Z}_{p_1^{k_1} p_2^{k_2} p_3^{k_3}}$$

The p 3s item shows: $\mathbb{Z}_{p_1} \otimes \mathbb{Z}_{p_2^{k_2}} \cong$

$$\mathbb{Z}_{p_1^{k_1}} \mathbb{Z}_{p_2^{k_2}} \cong \mathbb{Z}_{p_1^{k_1} p_2^{k_2}}$$

since $\gcd(p_1^{k_1}, p_2^{k_2}) = 1$.

So we get G abelian

$= \bigotimes$ $\mathbb{Z}_{\text{prime powers of some sort}}$ ← come from $\mathbb{Z}_{n_1}$ etc.

direct

What is the p_j-Sylow subgroup of G

it $= \bigotimes \mathbb{Z}_{p_j \text{ powers}}$ ← all p_j power items produced together.

Example: $\mathbb{Z}_{36}$ $\quad 36 = 2^2 \cdot 3^2$

$$= \mathbb{Z}_4 \otimes \mathbb{Z}_9$$

But note: abelian G $\quad |G| = 36$

need not be $\mathbb{Z}_{36}$, could be $\mathbb{Z}_{36} \cong \mathbb{Z}_4 \otimes \mathbb{Z}_9$

or $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_9$ or $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_3$

or $\mathbb{Z}_4 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_3$.

So order does not tell whole story.

(Familiar from way back on account of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ & $\mathbb{Z}_4$ both of order 4!)

2-Sylow of $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_9$ is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

3 Sylow of $\mathbb{Z}_4 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_3$ is $\mathbb{Z}_3 \otimes \mathbb{Z}_3$

etc.

Get 2's ylow by lumping $\mathbb{Z}_{\text{power of 2}}$ items

3 Sylow by lumping $\mathbb{Z}_{\text{power of 3}}$ items

G = direct product of p Sylows.

All fits together!