HOMEWORK ASSIGNMENTS: MATH 131AH

WILFRID GANGBO

Only the Exercises marked $(*)$ will be collected and either two or three of them will be graded from each set of homework assignment. However, we suggest that you work all exercises of the assignment.

3. Homework #3: Due on Monday 11 November

Exercise 3.1 $(*)$**.** *Let E be a compact metric space, $\{U_i\}_{i \in I}$ a collection of open subsets of E whose union is E. Show that there exists a real number $\epsilon > 0$ such that any closed ball in E of radius ϵ is entirely contained in at least one set U_i.*

Hint. *If not, take bad balls of radius $1, 1/2, 1/3, \cdots$ and a cluster point of their centers.*

Exercise 3.2 $(*)$**.** *We call a metric space sequentially compact if every sequence has a convergent subsequence. Prove that a metric space E is sequentially compact if and only if every infinite subset has a cluster point.*

Exercise 3.3 $(*)$**.** *Let E be a sequentially compact metric space, $\{U_\alpha\}_{\alpha \in I}$ a cover of open subsets of E. Show that there exists a real number $\epsilon > 0$ such that for any $x \in E$, the closed ball $B_\epsilon(x)$ of center x and radius ϵ is entirely contained in at least one set U_α.*

Hint. *Prove this by contradiction.*

Exercise 3.4 $(*)$**.** *We call a metric space totally bounded if, for every $\epsilon > 0$, the metric space is the union of a finite numbers of closed balls of radius ϵ. Prove that a metric space totally bounded if and only if every sequence has a Cauchy subsequence.*

Exercise 3.5 $(*)$**.** *Let E be a metric space. Consider the following relations:*

(i) E is compact. (ii) E is sequentially compact. (iii) E is totally bounded and complete.

1. *Prove first that (ii) and (iii) are equivalent.*

2. *Prove first that (i) and (ii) are equivalent.*

Exercise 3.6 $(*)$**.** *Discuss the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ if*

(i)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(ii)

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z} \text{ have no common divisors, } q > 0. \end{cases}$$

Exercise 3.7. *Let E, E' be metric spaces, $f : E \to E'$ a continuous function.*

(i) Show that if S is a closed subset of E' then $f^{-1}(S)$ is a closed subset of E.

(ii) Show that if $E' = \mathbb{R}$ then $\{p \in E : \ f(p) \geq 0\}$, $\{p \in E : \ f(p) = 0\}$ are closed.

Exercise 3.8 (∗)**.** *Let U, V be non empty open intervals in $\mathbb{R}$ and let $f : U \to V$ be a function onto V which is strictly increasing.*

(i) Prove that f is continuous and $f^{-1} : V \to U$ is continuous.

(ii) Prove that $f(U')$ is an open interval for every open interval $U' \subset U$.

Exercise 3.9 (∗)**.** *Let E be a metric space, S a nonempty subset of E, and let $f : E \to \mathbb{R}$ be the function which takes the value 1 at each point of S and the value 0 at each point of S^c. Prove that the set of points of E at which f is not continuous is precisely the boundary of S.*

Exercise 3.10 (∗)**.** *(i) Prove that if S is a nonempty compact subset of a metric space E and $p_0 \in E$ then $\min\{d(p_0, p) : p \in S\}$ exists.*

(ii) Prove that if S is a nonempty closed subset of $\mathbb{R}^n$ and $p_0 \in \mathbb{R}^n$ then $\min\{d(p_0, p) : p \in S\}$ exists.

Exercise 3.11 (∗)**.** *Let E, E' be metric spaces, $f : E \to E'$ a continuous function. Prove that if E is compact and f is one-to-one onto then $f^{-1} : E' \to E$ is continuous.*

Exercise 3.12. *Prove that for any metric space E and any $p_0 \in E$, the real–valued function sending any p to $d(p_0, p)$ is uniformly continuous.*

Exercise 3.13 (∗)**.** *Let S be a subset of the metric space E with the property that each point of S^c is a cluster point of S (one then calls S dense in E). Let E' be a complete metric space and $f : S \to E'$ a uniformly continuous function. Prove that f can be extended to a continuous function from E to E' in one and only one way, and that this extended function is also uniformly continuous.*

Exercise 3.14. *Show that if $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree then $f(\mathbb{R}) = \mathbb{R}$.*

Exercise 3.15 (∗)**.** *A metric space E is said to be arcwise connected if, given any $p, q \in E$, there exists a continuous function $f : [0, 1] \to E$ such that $f(0) = p$, $f(1) = q$. Show that*

(i) an arcwise connected metric space is connected

(iii) any connected open subset of $\mathbb{R}^n$ is arcwise connected.

Department of Mathematics, UCLA, Los Angeles, CA 90095

Email address: wgangbo@math.ucla.edu