1. Partial Solutions to Homework #1

Exercise 1.6. Find the *g.l.b.* and *l.u.b.* of

$$S := \left\{ \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots, \right\}$$

Solution. We have $\sqrt{2} \in S$ and $\sqrt{2} \leq x$ for all $x \in S$. Hence, g.l.b.$(S) = \sqrt{2}$.

We label the elements of S as $x_1 = \sqrt{2}$, $x_2 = \sqrt{2+\sqrt{2}}$, $x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$ to observe that $x_{n+1} = \sqrt{2+x_n}$. Assume that $x_n \leq 2$. Then $x_{n+1} \leq \sqrt{2+2} = 2$. Since $x_1 \leq 2$, we inductively conclude that $x_n \leq 2$ for all $n \in \mathbb{N}$. Therefore, S is bounded from above by 2, and so, if we set $M := l.u.b.(S)$ then $\sqrt{2} \leq M \leq 2$.

For every $\epsilon \in (0, \sqrt{2})$, we can find n such that $M - \epsilon \leq x_n$. We conclude that $M \geq x_{n+1} = \sqrt{2+x_n} \geq \sqrt{2+M-\epsilon}$ and so, $M^2 \geq 2 + M - \epsilon$, i.e. $(M+1)(M-2) \geq -\epsilon$. Since $\epsilon > 0$ is arbitrary, we have $(M+1)(M-2) \geq 0$. Thus, $M \in (-\infty, -1] \cup [2, +\infty)$. Since we knew that $\sqrt{2} \leq M \leq 2$, this means $M = 2$.

Exercise 1.8. Let X and Y be nonempty subsets of $\mathbb{R}$ whose union is $\mathbb{R}$ and such that each element of X is less than each element of Y. Prove that there exists $a \in \mathbb{R}$ such that X is one of the two sets

$$\{x \in \mathbb{R} : \ x \leq a\} \quad \text{or} \quad \{x \in \mathbb{R} : \ x < a\}.$$

Solution. Since X and Y are nonempty subsets of $\mathbb{R}$, we can choose $y_0 \in Y$ and conclude that y_0 is an upper bound for X. Setting $a = \text{l.u.b}(X)$, we have that $S \subset (-\infty, a]$. If $x \in (-\infty, a)$, we cannot have that $x \in Y$ otherwise x would be an upper bound for X, which is smaller than $a = \text{l.u.b}(X)$. Hence, we must have $x \in X$. In summary, we have proven that

$$(-\infty, a) \subset S \subset (-\infty, a].$$

This proves that if S fails to be equal to $(-\infty, a]$ then it must be equal to $(-\infty, a)$.

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