HW 4 - 131AH

ASHER CHRISTIAN 006-150-286

1. Exercise 1.2

If $f : \mathbb{R}^2 \setminus \{0,0)\} \to \mathbb{R}$*, three limits we can consider are*

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y), \quad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \quad \lim_{(x,y)\to(0,0)} f(x,y).$$

Compute these limits if they exist for

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad f(x,y) = \frac{x^2-y^2}{x^2+y^2}.$$

2. Exercise 1.3

Find a sequence of functions $f_n : [0,1] \to \mathbb{R}$ *that converges to the zero function and such that the sequence* $(\int_0^1 f_n(x)dx)_n$*, increases without bound.*

Let $(f_n)_n$ be defined such that

$$f_n : [0,1] \to \mathbb{R} := \begin{cases} -e^n(x)(x-\frac{1}{n}) & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1 \end{cases}.$$

$$\lim_{n\to\infty} f_n(x) = 0.$$

because for each $x \in [0,1]$ if $x = 0$ it is always zero, and if $x \neq 0$ then pick $n > \frac{1}{x}$ which makes $f_n(x) = 0$ Thus the function converges to zero for each point in $[0,1]$. The integral:

$$\int_0^1 f_n(x)dx = \int_0^{\frac{1}{n}} -e^n(x^2 - \frac{x}{n}) = -e^n \int_0^{\frac{1}{n}} x^2 - \frac{x}{n}.$$

since $x^2 - \frac{x}{n}$ is continuous, by the fundamental theorem of calculus we have

$$\int_0^{\frac{1}{n}} x^2 - \frac{x}{n} = \frac{(\frac{1}{n})^3}{3} - \frac{(\frac{1}{n})^2}{2n} - 0 + 0 = n^{-3}(\frac{1}{3} - \frac{1}{2}) = -\frac{1}{6}n^{-3}.$$

and

$$\int_0^1 f_n(x)dx = \frac{1}{6}e^n n^{-3} = \frac{e^n}{6n^3}.$$

$$\frac{e^1}{1^3} > 1.$$

$$\frac{e^{n+1}}{(n+1)^3} = \frac{e^n}{n^3}\frac{e}{\frac{(n+1)^3}{n^3}} = \frac{e^n}{n^3}(\frac{n}{n+1})^3 e > \frac{e^n}{n^3}(\frac{4}{5})^3 e = \frac{e^n}{n^3}\frac{64}{125}e > \frac{e^n}{n^3}\frac{128}{125}.$$

for $n > 4$ since $\frac{n}{n+1} = 1$

and so the integral is unbounded and its limit does not exist.

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3. Exercise 1.5

Show that if $\sum_{n=1}^{\infty} a_n$ *is a convergent series of real numbers and* $(v_k)_k$ *is a subsequence of* $(n)_{n=1}^{\infty}$ *then*

$$(a_1 + ... + a_{v_1}) + (a_{v_1+1} + ... + a_{v_2}) + (a_{v_2+1} + ... + a_{v_3}) + ... = \sum_{n=1}^{\infty} a_n.$$

Let $(s_n)_n$ be a sequence with each s_n defined by $(s_n) = (a_{v_{n-1}} + a_{v_{n-1}+1} + ... + a_{v_n})$ taking v_0 to be 1 We aim to show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} s_n$
First we show that $\sum_{n=1}^{\infty} s_n$ exists. For this let $N \in \mathbb{N}$ be such that the partial sums $\sum_{i=1}^{n} a_i, \sum_{i=1}^{m} a_i$ are less than ϵ apart when $n, m > N$ Pick N' s.t. $v_{N'} > N$ then any two partial sums

$$\sum_{i=1}^{n} s_i - \sum_{i=1}^{m} s_i = s_{n+1} + s_{n+2} + ... + s_m = a_{v_n} + ... + a_{v_m}.$$

assuming $n < m$ and this is the same form as a cauchy sequence for the original sum and thus is less than epsilon and the sum exists.
To show that the two sums are equal, for any $\epsilon > 0$ pick N_1 such that the partial sums of N_1 or more terms of a_i are within $\frac{\epsilon}{3}$ of eachother and pick N_2 such that s_i sums are the same and set $N = max\{N_1, N_2\}$

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{N} a_i + \sum_{i=N}^{\infty} a_i.$$

$$\sum_{i=1}^{\infty} s_n = \sum_{i=1}^{N} s_i + \sum_{i=N}^{\infty} s_i.$$

$$|\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} s_i| \leq |\sum_{i=1}^{N} s_i - \sum_{i=1}^{N} a_i| + |\sum_{i=N}^{\infty} a_i| + |\sum_{i=N}^{\infty} s_i| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $(v_n)_n$ is a subsequence of $\mathbb{N}$ and by the definition of s_n, $\sum_{i=1}^{N} s_i$ is equivalent to $\sum_{i=1}^{v_N} a_i$ and $v_N \geq N$ so the first term is equivalent to the cauchy statement made previously and thus less than $\frac{\epsilon}{3}$. Consider additionally the partial sums of the last two terms. Each partial sum is equivalent to $\sum_{i=1}^{m} x_i - \sum_{i=1}^{N} x_i$ with $x_i \in \{a_i, s_i\}$ and thus is also a cauchy difference since $m, N \geq N$ and so those terms too are less than $\frac{\epsilon}{3}$ justifying the last step. Therefore the two sums are within ϵ of eachother for any ϵ and are therefore the same.

4. Exercise 1.6

Let $(a_n)_n \subset [0, +\infty)$ *be a sequence of positive numbers which is monotone non increasing. Show that the following hold.*
(i) If $\sum_{n=1}^{\infty} a_n$ *is convergent then* $\lim_{n\to+\infty} na_n = 0$.
(ii) $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if* $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *is convergent.*
Since the series is convergent it is also cauchy in particular for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ s.t.

$$|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i| < \epsilon.$$

whenever $n, m \geq N_\epsilon$ To show the limit converges to zero, for any epsilon pick $N = 2(N_\epsilon + 1)$ for $\frac{\epsilon}{2}$ as before and for any $n > N$ consider $m = \frac{1}{2}n$ flooring m if odd and $n-1$

$$\frac{\epsilon}{2} \geq |a_{\frac{n}{2}} + a_{\frac{n}{2}+1} + ... + a_n| = \sum_{i=1}^{n} a_i - \sum_{i=1}^{\frac{n}{2}-1} a_i \geq |a_n + a_n + ... + a_n| \geq \frac{1}{2}n|a_n|.$$

The last part is an inequality because of the case where $\frac{n}{2}$ is floored and an extra term is included thus overcounting by $\frac{1}{2}a_n$ multiplying through by 2 on both sides we get

$$\epsilon \geq n|a_n|.$$

for any $n > N$ and thus the limit of $n|a_n|$ is equal to zero proving (i)
For (ii) first note that

$$\sum_{n=1}^{\infty} 2^n a_{2^n} \leq a_1 + \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

which also converges since the two differ by a constant a_1 Expanding this out we see every partial sum

$$\sum_{i=0}^{n} 2^i a_{2^i} = a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + ... + a_{2^n} \geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + ... + a_{2^{n+1}-1} = \sum_{i=0}^{2^{n+1}-1} a_i.$$

This is due to the fact that a_n is monotonic non-increasing. By the comparison test this series converges proving the first direction that the convergence of $2^n a_{2^n}$ implies the convergence of a_n. to show the second direction that a_n convergence implies $2^n a_{2^n}$ convergence.
To show this first we note that since $\sum_{i=1}^{\infty} a_i$ converges, for all $n > N$ $\frac{|a_{n+1}|}{|a_n|} < \xi$ with $\xi \in [0, 1)$ This implies then for all n s.t. $2^n > N$

$$\frac{|a_{2^{n+1}}|}{|a_{2^n}|} < \xi^{2n}.$$

Since there are 2^n elements between the two values and so

$$\frac{|2^{n+1} a_{2^{n+1}}|}{|2^n a_{2^n}|} < 2\xi^{2n}.$$

Increasing N so $\xi^{2N} < \frac{1}{2}$ (which is posisble because the value converges to zero as n goes to infinity) we see that for this new (possibly larger) N the series defined by $\sum_{i=1}^{\infty} 2^i a_{2^i}$ satisfies the ratio test and therefore converges proving the second direction.

5. Exercise 1.7

Integral Test) Let $f : [1, +\infty) \to \mathbb{R}$ be a monotone non increasing function. Prove that the following are equivalent.

1. *(i) $\sum_{n=1}^{\infty} f(n)$ is convergent*
2. *(ii) $\lim_{n\to+\infty} \int_1^n f$ exists*

First to prove (i) implies (ii) consider the two functions $f_1, f_2 : [1, N] \to \mathbb{R}$

$$f_1(x) = f(\lfloor x \rfloor), f_2(x) = f(\lceil x \rceil).$$

For arbitrary N And note that since N is finite, f_1 and f_2 are step functions. additionall

$$\forall x \in [0, N] \quad f_2(x) \leq f(x) \leq f_1(x).$$

By the monotonicity of f. The integrals of f_1 and f_2 are well defined and in particular

$$\int_1^N f_1(x)dx = \sum_{n=1}^{N-1} f(n).$$

$$\int_1^N f_2(x)dx = \sum_2^N f(n).$$

And

$$\int_1^N f_2(x)dx \leq \int_1^N f(x)dx \leq \int_1^N f_1(x)dx.$$

But the integrals of f_1 and f_2 are bounded since the sums are bounded and since we can assume each $f(x) \geq 0$ since by the sum converging $f(x)$ must approach 0. So the integrals are bounded above by the infinite sum $\sum_{n=1}^{\infty} f(n)$, below by 0 and are monotone increasing. Additionally since f is montone and bounded it is integrable. so the integral exists for each N and since the partial sums below and above are cauchy, the integral itself must be cauchy and so it converges and the limit exists. Seen below for any $\epsilon > 0$ pick N such that the lower and upper sums are cauchy within ϵ. for all $m, n > N$ Then

$$\int_m^n f_2(x)dx \leq \int_m^n f(x)dx \leq \int_m^n f_1(x)dx.$$

$$-\epsilon \leq \int_m^n f(x)dx \leq \epsilon.$$

Additionally since the limit is over all real numbers note that since the integral is strictly increasing a bound that works for all integer differences greater than N will work for all real numbers greater than N

Now to prove (ii) implies (i). Assume that $\lim_{n\to\infty} \int_1^n f$ exists. Let $I_n = \int_{n-1}^n f$ Then $\int_1^n f = \sum_2^n I_n$. Since f is monotone nonincreasing it achieves its maximum on the interval $[n-1, n]$ at $n-1$ and its minimum at n. In particular $|I_n| \geq |f(n)|$ since $f(n)$ is nonnegative. Thus by the comparison test if $\lim_{n\to\infty} \int_1^n f = \lim_{n\to\infty} \sum_2^n I_n$ exists, by the comparison test $\sum_1^{\infty} f(n)$ exists and converges absolutely

6. Exercise 1.8

For which $p > 0$ the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \sum_{n=2}^{\infty} \frac{1}{n(logn)^p}, \sum_{n=3}^{\infty} \frac{1}{nlog(n)(loglog(n))^p}.$$

All elements of these series are strictly posiive for large N so if they converge the converge absolutely. So it suffices to check if they converge absolutely.

For any $p > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

converges by the integral test for if $p \neq 1$ then

$$\lim_{n\to\infty} \int_1^n \frac{1}{x^p} dx = \lim_{n\to\infty} \frac{n^{1-p}}{1-p} - \frac{1}{1-p}.$$

The non-constant term of this limit converges to 0 if $p > 1$ and diverges towards infinity if $p < 1$. In the case $p = 1$ we have shown in class that this is the harmonic series and it diverges. Using the result from Exercise 1.6 we see that the second sum converges if and only if

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{2^n (log(2^n))^p} = \sum_{n=1}^{\infty} \frac{1}{n^p log(2)^p} = \frac{1}{log(2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

converges We have shown this in the previous question to only converge when $p > 1$ Similarly for the last series using the same rule

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n log(2^n)(log(log(2^n)))^p} = \frac{1}{log(2)} \sum_{n=1}^{\infty} \frac{1}{n(log(nlog(2)))^p}.$$

This series also only converges if the following converges

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n (log(2^n log(2)))^p} = \sum_{n=1}^{\infty} \frac{1}{(nlog(2) + loglog(2))^p} = \frac{1}{(log(2))^p} \sum_{n=1}^{\infty} \frac{1}{(n + \frac{loglog(2)}{log(2)})^p}.$$

This series only converges when $p > 1$ since by the integral test and u substitution as before this is simply a shifted version of the first example.

7. Exercise 1.9

On the set $\mathbb{R} \setminus \{-1, -2, ...\}$ *show the convergence of the series*

$$\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+x}) = \sum_{n=1}^{\infty} \frac{n+x-n}{n(n+x)} = x \sum_{n=1}^{\infty} \frac{1}{n^2 + nx}.$$

If $x \geq 0$ then by the comparison test this series converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. If $x < 0$ for any $\epsilon > 0$ pick N such that if $n_2 > n_2 > N$ the series $\frac{1}{n^2}$ is cauchy within ϵ. i.e.

$$\sum_{n=n_1}^{n_2} \frac{1}{n^2} < \epsilon.$$

then pick $M = N + |x|$ The for the same n_2, n_1

$$\sum_{n=n_1+|x|}^{n_2+|x|} \frac{1}{n^2 + nx} = \frac{1}{(n_1 + |x|)^2 - (n_1 + |x|)|x|} + ... + \frac{1}{(n_2 + |x|)^2 - (n + |x|)|x|}.$$

and for any $n > 0$

$$\frac{1}{(n + |x|)^2 - (n + |x|)|x|} = \frac{1}{n^2 + 2n|x| + |x|^2 - n|x| - |x|^2} = \frac{1}{n^2 + n|x|} < \frac{1}{n^2}.$$

so within this new M, the series is cauchy. Thus for any x and any $\epsilon > 0$ there exists some M such that for any $n, m > M$ the partial sums up to n and m are within ϵ apart and the series converges. The series does not converge uniformly however since picking $\epsilon = \frac{1}{2}$ for any N such that if $n_1, n_2 > N$

$$\sum_{n=n_1}^{n_2} \frac{1}{n} - \frac{1}{n+x} < \epsilon.$$

pick $x = -N + \frac{1}{2}$ Then the term
$$\frac{1}{N} - \frac{1}{N - N + \frac{1}{2}} = \frac{1}{N} + 2.$$
Is clearly greater than $\frac{1}{2}$ and so the series does not converge uniformly.

8. Exercise 1.10

Root test: let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers such that there exists $r \in (0,1)$ such that $\sqrt[n]{|a_n|} \leq r$ for all sufficiently large n. Show that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$|a_n|^{\frac{1}{n}} \leq r \to |a_n| \leq r^n.$$
chopping off the first terms until the inequality holds
$$\sum_{n=n_1}^{\infty} |a_n| \leq \sum_{n=n_1}^{\infty} r^n.$$
In particular the second series converges since it is the geometric series with each partial sum equal to
$$\frac{1-r^n}{1-r}.$$
The limit of which is well defined. And so a_n in series converges absolutely by comparison test.

9. Exercise 1.11

Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series of real numbers then the series $\sum_{m,n=1}^{\infty} a_n b_m$ is also absolutely convergent and
$$\sum_{m,n=1}^{\infty} a_n b_m = (\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n).$$

10. Exercise 1.12

Let $(c_n)_{n=0}^{\infty} \subset \mathbb{R}$. Prove that the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^n$ is $\frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$
if $\limsup_{n\to\infty} c_n = \infty$ then clearly $R = 0$ because we can always find a c_n arbitrarily large so for this proof assume $R \neq 0$ First note that a power series converges absolutely within its radius of convergence. Now to show that if $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$ for any $-R < x < R$ considering the absolute sum. And noting that $|c_n|$ is bounded let $M = sup_n |c_n|$
$$\sum_{n=1}^{\infty} |c_n x^n| \leq \sum_{n=1}^{\infty} |c_n| (\frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} + \epsilon})^n \leq \sum_{n=1}^{\infty} (\frac{|c_n|^{\frac{1}{n}}}{|c_n|^{\frac{1}{n}} + \epsilon})^n$$
$$= \sum_{n=1}^{\infty} (\frac{1}{1 + \frac{\epsilon}{|c_n|^{\frac{1}{n}}}})^n \leq \sum_{n=1}^{\infty} (\frac{1}{1 + \frac{\epsilon}{M^{\frac{1}{n}}}})^n \leq \sum_{n=1}^{\infty} (\frac{1}{1 + \frac{\epsilon}{M}})^n.$$

Assuming $M \geq 1$ otherwise, replace $\frac{\epsilon}{M}$ with ϵ in the last statement. Either way this is clearly a power series and so converges absolutely. Thus proving that the radius of convergence holds. Now to prove that R is a the maximum such bound for the radius of convergence. Assume for contradiction that there exists some $R' > R$ that can serve as a radius of convergence. Pick $R < x < R'$ and consider the subsequence $(c_{n_k})_k$ such that $\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \lim_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}}$

$$\sum_{n=1}^{\infty} c_{n_k} x^{n_k} \leq \sum_{n=1}^{\infty} c_n x^n.$$

$$\sum_{k=1}^{\infty} |c_{n_k} x^{n_k}| = \sum_{k=1}^{\infty} |c_{n_k}|| \frac{1}{\lim_{k\to\infty} |c_{n_k}^{\frac{1}{n_k}}| - \epsilon} |^{n_k}.$$

Consider the individual terms and $N > 0$ such that $|c_{n_k}^{\frac{1}{n_k}} - \lim_{k\to\infty} c_{n_k}^{\frac{1}{n_k}}| < \frac{\epsilon}{2}$

$$|\frac{c_{n_k}^{\frac{1}{n_k}}}{\lim_{k\to\infty} |c_{n_k}|^{\frac{1}{n_k}} - \epsilon}|^{n_k} \geq |\frac{c_{n_k}^{\frac{1}{n_k}}}{c_{n_k}^{\frac{1}{n_k}} - \frac{\epsilon}{2}}|^{n_k} > 1.$$

Thus every element after N terms is greater than 1 so the partial sums of this new series do not converge. Additionally there are infinitely many elements corresponding to these c_{n_k} in the original series so for arbitrary partial sums greater than N there are some that have an element of c_{n_k} in them and thus are in absolute value larger than 1. Therefore the original series does not converge this is a contradiction which shows that R is the least upper bound on the radius of convergence.