Homework Math 110AH Due November 15, 2024

1. Suppose A is an orthogonal $n\times n$ matrix of $\det = 1$, $n \geq 2$. Is there a continuous function $\gamma_A : [0,1] \to SO(n) \ni \gamma_A(0) = I_n$ and $\gamma_A(1) = A$? Prove your answer. (Suggestion: think about $n=2$ first).

2(a) What condition on A guarantees that $\det e^{tA} = 1$ for all t with $|t| < \varepsilon$, some ε.

(b) Does it follow that $\det(e^{tA}) = 1$, for all t if A satisfies the condition you found in part (a)?

3. How is 2(b) related to the fact (which you should prove!) that $\det(e^{tA})$ has a power series in t that converges for all $t \in \mathbb{R}$.

4. (a) Show that $\operatorname{tr}(AB - BA) = 0$

(b) Show that if A, B are skew symmetric ($A^T = -A$, $B^T = -B$) then $AB - BA$ is skew symmetric.

5. With $\|A\|_{op} \overset{\text{def}}{=} \max_{\|\vec{x}\|=1} \|A\vec{x}\|$

A $n\times n$ $\mathbb{R}$-valued matrix

$\|\vec{x}\|$ = usual euclidean length

$\|(x_1, .., x_n)\| = \left(\sum_1^n x_j^2\right)^{1/2}$

Show

(a) $\|AB\| \le \|A\| \cdot \|B\|$

(b) $\|A+B\| \le \|A\| + \|B\|$

6. Use prob 5 to prove (a) the series for e^A converges, all A

and

(b) the serie for $\ln(I+A)$ converges if $\|A\|<1$.

7. Check that formally

$$e^{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots} = 1 + x$$

up to terms of degree ≤ 4 in x.

More problems will be added later.