COMMENTS ON HOMEWORK ASSIGNMENT #1: MATH 131AH

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1. Preamble

Definition 1 (Useful in Exercises 1.2 and 1.3)**.** *Let $f : X \to Y$ be a function, let $A \subset X$ and $C \subset Y$. We define*

$$f(A) := \{f(x) \,:\, x \in A\}, \quad \textit{and} \quad f^{-1}(C) = \{x \in A \,:\, f(x) \in C\}.$$

RULES TO BE TAKEN FOR GRANTED. You are allowed to use the following rules which you can find in the book by Rosenlicht, since we have already covered Chapter II, Sections 1, 2. In fact, you could also decide to check them as exercises.

(i) If $a > b$ and $b > c$ then $a > c$.
(ii) If $a > b$ and $b \geq c$ then $a > c$.
(iii) If $a > b > 0$ and $c \geq d > 0$ then $ac > bd$.
(iv) We assume to be familiar with the rules of sign: $(positive) \cdot (positive) = (positive)$, etc...
(v) By the rules of sign, for any $a \in \mathbb{R}$, we have $a^2 \geq 0$ and equality holds only if $a = 0$.
(vi) If $a > 0$ then $1/a > 0$.
(vii) If $a > b > 0$ then $1/a < 1/b$.

Since $1^2 = 1$ then by (v), $1 > 0$.

Let us prove (vi). Assume $a > 0$ and $1/a < 0$. Then By the rules of sign, $1 = aa^{-1} < 0$, which yields a contradiction.

Let us prove (vii). Assume that $a > b > 0$. Then $ab > 0$ and so, $(ab)^{-1} > 0$. We use (iii) to conclude that $(ab)^{-1}a > (ab)^{-1}b$, which concludes the proof.

2. An example of solution

One of the goals of Math 131AH is to learn how to write rigorous and elegant proofs, without losing sight of the fact that your proof must be complete. For example, we are going to propose a solution to Exercise 1.5.

Solution to Exercise 1.2 (i). Let $f : X \to Y$ be a function, let A and B be subsets of X. Since $A, B \subset A \cup B$, we infer $f(A), f(B) \subset f(A \cup B)$ and so, $f(A) \cup f(B) \subset f(A \cup B)$. Conversely, if $y \in f(A \cup B)$ then $y = f(x)$ for some $x \in A \cup B$. Either $x \in A$ in which case $y = f(x) \in f(A)$ or $x \in B$ in which case $y = f(x) \in f(B)$. Either way $y \in f(A) \cup f(B)$.

Since $A \cap B \subset A$ we conclude that $f(A \cap B) \subset f(A)$. By symmetry $f(A \cap B) \subset f(B)$ and so, $f(A \cap B) \subset f(A) \cap f(B)$.

Solution to Exercise 1.5. We first observe the following fact for any arbitrary $\alpha, \beta \in \mathbb{R}$: assume that $\alpha > \beta$ and assume on the contrary that we also have $-\alpha \geq -\beta$. Then, we would have that $\alpha - \beta > 0$ and $\beta - \alpha \geq 0$. Adding up, we conclude that $(\alpha - \beta) + (\beta - \alpha) > 0$, which yields the contradiction that $0 > 0$.

(i) Suppose that $a, b \in \mathbb{R}$ and $a < b < 0$. Then $-a > -b > 0$ and so, $1/(-a) < 1/(-b)$. By the above argument then $1/a > 1/b$.

(ii) Suppose that $a, b, x, y \in \mathbb{R}$ and $a < x < b$, $a < y < b$. Exchanging the role of x and y if necessary, we can assume without loss of generality that $x \leq y$. We have $a < x \leq y < b$ and so,

$$y < b, \ -x < -a \qquad \Longrightarrow \qquad y - x < b - a$$

which is the desired result.

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