HW 4 - 135

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1.11.24

1 69-1

$$y' = y^2, \quad y(0) = 1.$$

$y_0(x) = 1$

$$y_1(x) = y_0 + \int_{x_0}^{x} y_0(t)^2 dt = 1 + \int_0^x 1 dt = 1 + x.$$

$$\begin{aligned}
y_2(x) &= 1 + \int_0^x (1+t)^2 dt \\
&= 1 + (\frac{1}{3}(1-t)^3\Big|_0^x) \\
&= \frac{1}{3}(1+x)^2 + \frac{2}{3} \\
&= 1 + x + x^2 + \frac{1}{3}x^3 \\
y_3(x) &= 1 + \int_0^x (1 + t + t^2 + \frac{1}{3}t^3)^2 dt \\
&= 1 + \int_0^x (1 + 2t + 3t^2 + \frac{8}{3}t^3 + \frac{5}{3}t^4 + \frac{2}{3}t^5 + \frac{1}{9}t^6) dt \\
&= 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7
\end{aligned}$$

Solving for y directly using separation of variables $\frac{dy}{dx} = y^2 \to \int \frac{dy}{y^2} = \int dx$
$-\frac{1}{y} = x + c \to y = -\frac{1}{x+c}$ with $c = -1$

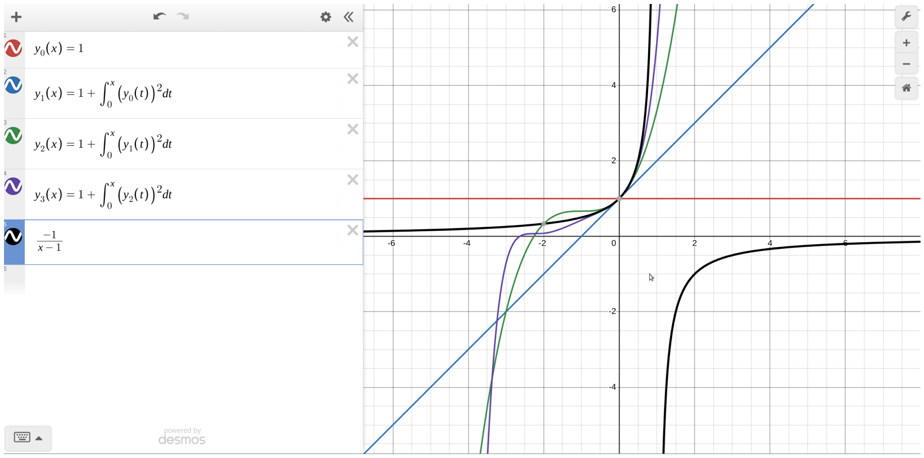

Figure 1: Desmos graph of the piccard iteration and the solution to the ODE

2 69-2

$$y' = 2x(1+y), \quad y(0) = 0.$$

$y_0(x) = 0$

$$\begin{aligned}
y_1(x) &= 0 + \int_0^x (2t)dt \\
&= x^2 \\
y_2(x) &= 0 + \int_0^x (2t(1+t^2))dt \\
&= \int_0^x (2t + 2t^3)dt \\
&= x^2 + \frac{1}{2}x^4 \\
y_3(x) &= \int_0^x (2t(1+t^2+\frac{1}{2}t^4))dt \\
&= x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 \\
y_4(x) &= \int_0^x (2t(1+t^2+\frac{1}{2}t^4+\frac{1}{6}t^6))dt \\
&= x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8
\end{aligned}$$

solving directly

$$
\begin{aligned}
y'(x) &= 2x(1+y) \\
\int \frac{1}{y+1} &= \int 2x \\
ln(y+1) &= x^2 + c \\
y &= De^{x^2} - 1
\end{aligned}
$$

D = 1 to satisfy initial condition.

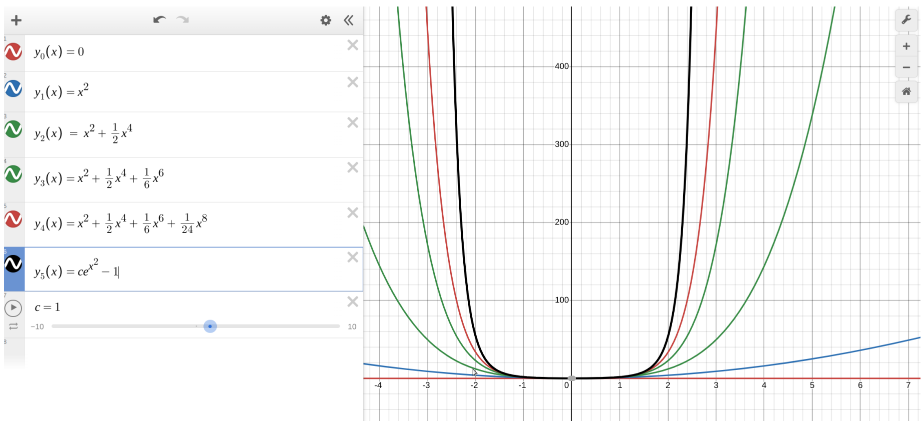

Figure 2: Comparing the first 4 iteratinos with the solution to the ODE

3 69-3

$$
y' = x + y, \quad y(0) = 1.
$$

1. (a)

$$
\begin{aligned}
y_0 &= e^x \\
y_1 &= 1 + \int_0^x (t + e^t) dt \\
&= \frac{1}{2}x^2 + e^x \\
&= \sum_{i=2}^{\infty} \frac{x^i}{i!} - x - 1 + e^x. \\
&= 2e^x - x - 1
\end{aligned}
$$

2. (b)

$$
\begin{aligned}
y_0 &= 1 + x \\
y_1 &= 1 + \int_0^x (2t+1)dt \\
&= 1 + x^2 + x \\
&= \sum_{i=2}^{\infty} \frac{2x^i}{i!} + 1 + x \\
&= 2\sum_{i=0}^{\infty} \frac{x^i}{i!} - x - 1 \\
&= 2e^x - x - 1
\end{aligned}
$$

3. (c)

$$
\begin{aligned}
y_0 &= cos(x) \\
y_1 &= 1 + \int_0^x (t + cos(t))dt \\
&= 1 + \frac{1}{2}x^2 + sin(x) \\
y_2 &= 2 + x + \frac{x^2}{2} + \frac{x^3}{6} - cos(x) \\
y_3 &= 1 + 2x + x^2 + \frac{x^3}{6} + \frac{x^4}{24} - sin(x) \\
y_4 &= x + \frac{3x^2}{2} + \frac{x^3}{3} + \frac{x^4}{24} + \frac{x^5}{120} + cos(x) \\
y^5 &= 1 + x^2 + \frac{x^3}{2} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^6}{720} + sin(x)
\end{aligned}
$$

This one was harder to find a pattern for because the cylce $cos \to sin \to -cos \to -sin$ produces different constant terms and different x terms which propogate and affect the later terms but the graphs clearly approach $2e^x - x - 1$.

4 70-1

1. Theorem A guarantees a solution because y^2 and $2y$ are both continuous on the entirety of $\mathbb{R}^2$ but this is only for some h not the entire line. The solution through $y(0) = 0$ is $y = 0$ and the solution through $y(0) = 1$ is $y = -\frac{1}{x-1}$ which is not continuous on $x = 1$. and the function

$$
f(x) = \begin{cases} -\frac{1}{x-1} & \text{if} x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}.
$$

Satisfies the same requirements

5 70-2

1. (a) $\frac{\partial f}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}$ The partial gets arbitrarily large at values close to zero so bounding the partial is impossible. For example any K that bounds it take $y_1 = \frac{1}{4}$ and $y_0 = (\frac{1}{2K+1})^2$ and the difference is strictly greater than K.

2. (b) For any bound on y greater than 0 we know that the partial of y is strictly decreasing because the second derivative is negative for all positive y. The maximum distance is then taken as the partial evaluated at c and the partial evaluated at the right bound d and their difference is necessarily the largest difference between partial values.