Lecture Nov, 27

Sylow Theorems:

General thought: X finite set. Want to know $\# X$ ← number of elements in X

$\equiv 1 \pmod p$ p prime.

$\iff$

$\# X = kp+1$

Approach: Suppose G finite group that acts on X.

$X = \bigcup$ orbits of group action
disjoint union

$\text{orbit}(x) = \{xg : g \in G\}$ ← sets are either identical or disjoint

orbit(x_1) orbit(x_2)

$x_1 g_1 = x_2 g_2 \Rightarrow x_1 = x_2 g_2 g_1^{-1}$

$x_2 = x_1 g_1 g_2^{-1}$ x_1, x_2 are same orbit

orbit(x_1) = orbit(x_2)

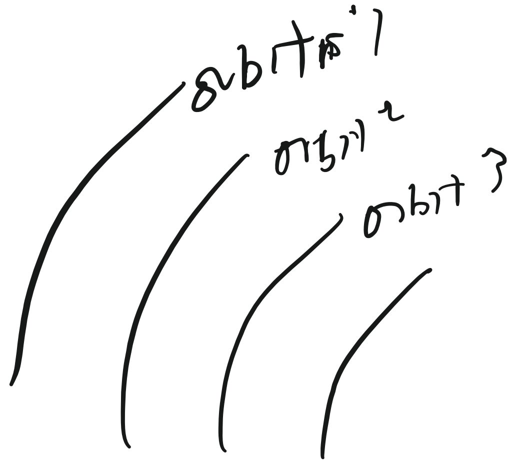
general element of orbit(x_1)

$x_1 g = x_2 (g_2 g_1^{-1} g) \in$ orbit(x_2)

every element of x_1 orbit $\in$ orbit of x_2

Vice versa works. orbit(x_1) = orbit(x_2)

X



orbit(x_1) doesn't have to orbit(x_2)

if the two orbits are different.

Examples: Rotation group on $\mathbb{R}^2$ origin

orbit($\vec{0}$) = $\{\vec{0}\}$

orbit($\vec{x}$) = circle of radius $|\vec{x}|$ around origin

$\vec{x} \neq \vec{0}$

Basic principle:

$$\# \text{orbit}(x_1) = |G| \Big/ \underbrace{|\text{stabilizer of } x_1|}$$

has order independent of which point in orbit you pick.

Stabilizers of pts in same orbit are conjugate subgroup, so have same order

(Review this if not clear)

$\# X =$ sum over orbits $|G| \Big/$ "stabilizer" = size of orbit

= |stabilizer of $x_1 \in$ orbit|

independent of x_1

$$\# X = \#\text{fixed point orbits} + \sum_{\substack{\text{non-single} \\ \text{pt orbits}}} |G| / |\text{stabilizer}|$$

$|\text{st.}| < |G|$

if orbit not a single pt,

|Stabilizer size| = |G| $\quad$ # orbit = $|G|/|G| = 1$

$\iff$ orbit = {single fixed point}

$\# X \equiv 1 \pmod{p}$: will follow if

there is one fixed point and

all non-point orbits have size

divisible by p.

then get

$\# X \equiv 1 \ (p)$ $\quad$ How to be sure of this

Would follow if $\quad |G| = p^s \quad$ some $s \geq 1$.

~~still~~ $|G| = p^s r \quad p \nmid r$

$$\frac{|G|}{|Stab| < |G|} \implies \frac{p^s}{\text{some smaller than power of } p}$$

$r > 1$ case.

|Stabilizer| $\quad p^k \cdot \binom{\text{not my}}{?p} \quad k<s$ | divisible by p!

X = set of all p Sylow subgroups of G

($X \neq \phi$ by previous work) $\quad |G| = p^s r$

Two things of interest

G acting on X = set of p Syl. sub.

by conjugation. Clear that $S_0 \in X$

$g^{-1} S_0 g \in X$, p Sy $\quad$ order $|g^{-1} S_0 g| = |S_0|$

Special trick; $\quad = p^s$ by definition

G background - Action not by G but by

$\underline{\underline{S_0}} < G \quad$ G acts on $X \Rightarrow S_0$ acts on X

$xg \in X \quad x \in X \quad g \in G$

Induces action of subgroup.

X = union of orbits $\quad |S_0| = p^s$

= fix pts of S_0 acting + orbit size div by p.

g

$$p \,\Big|\, \frac{|S_0|}{|\mathrm{Stab}|} \quad \text{(= proper subgroup of } S_0\text{)}$$

S_0 p Sylow subgroups

$\exists\, S_0 \qquad p^k$

$\# X \equiv$ no of fixed pts (mod p).

Goal: $\# X \equiv 1 \pmod p$

Need: One and only one fixed point.

Translate: Fixed point is an $S_1 \in X$ p-Sylow subgroup $\ni$

for every $g \in S_0 \qquad g^{-1} S_1 g = S_1$

Clearly true if $S_1 = S_0$: S_0 is a fixed pt.

(Conj of H by $h \in H \quad h^{-1} H h = H$ ✓)

Only fix pt? If $S_1 \neq S_0$ could be a fixed pt?

Would mean that $S_0 \subset$ normalizer of S_1.

$\Leftrightarrow$ S_1 is a fixed of the action of S_0 on X (all p Sylow subgroups) by Conj.

Last time: p Sylow subgroup S_1 then

$N(S_1) \supset$ p Sylow subgroup

contains not other element of order p

and in particular doesn't contain any nonidentity element of S_0 p-sylow if $S_0 \neq S_1$,

So $N(S_1) \supsetneq S_0$ $\quad S_0 \neq S_1$

Shows: Only fixed pt of S_0 acting on X = all p Syl subg is S_0 itself which is a fixed pt.

Conclusion: $\#X \equiv 1 \pmod p$

Next stage: Want to know that if act on X by G (whole group) then ~~(x)~~ $S_0 \in X$

orbit $(S_0) = X$

All p-Sylow subgroups are conjugate.

(in printed notes (two pages) earlier)

Look at action of G its by conj on X:

= union of orbits.

Pick any one p-Sylow sub S_0 or S_1, . .

then get action of S_0 on X, S_1 on X_1 . .

$\text{orbit}_G(S_0) \leftarrow$ invariant under action by S_0

invariant under action by S_1

(a subset is "invariant" under a group action if it goes itself when any group element is applied).

Example: Look at rotations of

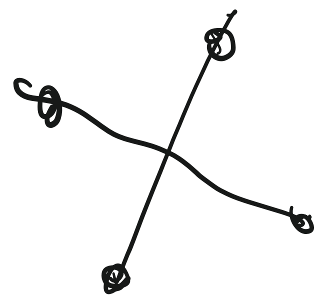
$\mathbb{R}^2$ around origin, Group action.

orbit of this group action, $\{\vec{0}\}$, circle around origin. ←

Orbits are invariant under the action of any subgroup

e.g. $0°$, $90°$, $180°$, $270°$ rotations

$\{\vec{0}\}$, 4 elements in a circle



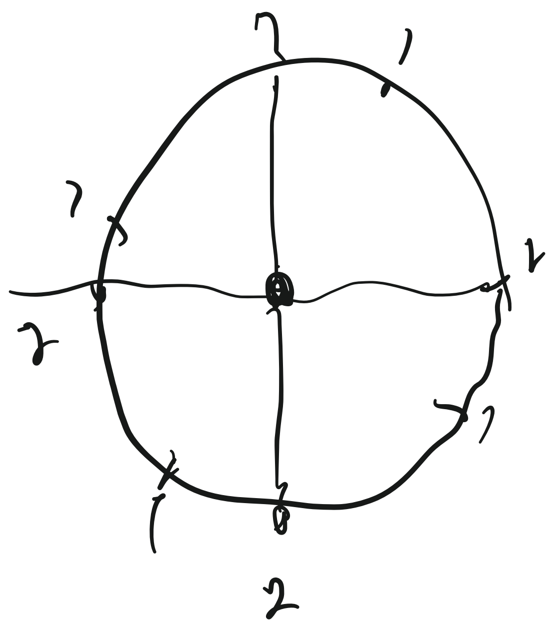
orbit of whole grp ⊃ orbits of subgrp.

circle is invariant under $90°$

or

$\{0\}$ each orbit of whole grp

= union of orbits of subgroup



Back to Sylow:
Now we want to look

S_0 $(\subset G)$
action by conj on X
subgroup of G action on X

orbit of S_0 in G, (subset of X
hope: $\stackrel{?}{=} X$)
invariant under the S_0 action,

pts in orbit $\equiv 1$ (mod p).

Suppose $S_1 \notin$ orbit of S_0 under G
(hope does not happen)
orbit(S_1) under G action is inv under S_1 action
also inv under S_0 action

orbit of $S_0 \equiv 1$ (mod p).

orbit of S_1 under G action ← S_0 inv,

__no__ fixed pts for S_0 action,

$\Rightarrow$ no. of elements in orbit of S_1 under G

$\equiv 0 \pmod{p}$,

Take grp of order p^s act without fixed pts on set Then set has

$\# \equiv 0 \pmod{p}$ $|\text{stabilizer}| < p^s$

stabilizer $| p^s$

orbit sizes are all div by p.

Contradiction:

orbit of S_1 size $\equiv 1 \pmod{p}$

orbit of S_1 size $\equiv 0 \pmod{p}$.

Read carefully in printed

p^s subgr inv, orbit then p^s order acting

Then fixed pt $\# \equiv 1$

or no fixed pt $\# \equiv 0$

Both things apply. X

Check this one carefully.