Homework 110AH Due October 16, 2024 11pm

1. Let $P(z) = 2 + 3z^2$. Find $\alpha \neq 0$ $\alpha \in \mathbb{C}$ such that $|P(i+\alpha t)| < |P(i)|$ for all sufficiently small positive $t \in \mathbb{R}$.

2. (a) Find $P_1(x)$ and $P_2(x)$ $\ni$

$$(x-2)^2 P_1(x) + (x-3)^2 P_2(x) = 1$$

(P_1, P_2 polynomials).

(b) How did you know part (a) was possible without actually finding P_1, P_2?

3. Suppose G is a finite group and $a \in G$ is an element of order k (i.e. $a^k = e$ but $a^l \neq e$ if $1 \leq l < k$).

(a) Define a relation on G:

$g_1 \sim g_2$ if $\exists$ integer m, $m \geq 0$ $\ni$ $g_1 a^m = g_2$.

Prove $\sim$ is an equivalence relation

(b) Show that the equivalence classes of $\sim$

all have exactly k elements.

(c) Deduce that $k \mid$ order of G

4. Let $F = \mathbb{R}[x]/\sim$ where $p(x) \sim Q(x)$ means $P - Q$ is divisible by $x^2 + 2x + 6$.

(a) Show that F is a field.

(b) Show that $\exists\ \alpha \in F$ such that $\alpha^2 + 1 = 0$.

(c) Deduce that F is really $\mathbb{C}$ in effect. (part of the problem is deciding what this means!)

5. Suppose F is a field and E is another field with $F < E$ (and F has the same operations as E, just restricted to F).

(a) Explain how E becomes a vector space over E

(b) Show that dimension of E over F is $< +\infty$. Deduce that for each $\alpha \in E$, $\exists$ $P(x)$ poly with coefficients in F such that $P(\alpha) = 0$ and degree of P

can be chosen $\leq$ dim of E over F.

6. Prove: If F_1, F_2, F_3 are fields with

$$F_1 \subset F_2 \subset F_3$$ and F_3 finite dimensional over F_1, then

dim (F_3 over F_1)

= dim (F_3 over F_2) $\cdot$ dim (F_2 over F_1)

(this includes that dimensions on the right are finite).

7. Use these ideas to show

$$\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{2}).$$

8. Show $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{2})$ directly

(Suggestion: If $(a+b\sqrt{2})^3 = 2$ then what is $(a-b\sqrt{2})^3$?)