3.2.1 (b), (d), (e)

$$\int_0^\pi x\cos kx\,dx = \frac{x\sin kx}{k} - \int \cos kx\,dx$$

$$= \frac{x\sin kx}{k}\Big|_0^\pi - \int_0^\pi \frac{\sin kx}{x}dx$$

$$\underbrace{\qquad}_{0}$$

$$= \frac{\cos kx}{k^2}\Big|_0^\pi$$

$$= \frac{\cos \pi k}{k^2} - \frac{1}{k^2}$$

(b) $|x|$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos kx\,dx = \frac{\cos \pi k}{k^2} - \frac{1}{k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin kx\,dx = 0$$

$$|x| \sim \frac{\pi}{2} + \sum_{i=1}^{\infty} \frac{2}{\pi}\left(\frac{\cos i\pi}{i^2} - \frac{1}{i^2}\right)$$

(d) x^2

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\,dx = \frac{1}{\pi}\left[\underbrace{\frac{x^2\sin kx}{k}\Big|_{-\pi}^{\pi}}_{0} - \frac{2}{k}\int_{-\pi}^{\pi} x\sin kx\,dx\right]$$

$$= +\frac{1}{\pi}\left[\frac{2}{k}\int_{-\pi} -x\sin kx\,dx\right]$$

$$\int_{-\pi}^{\pi} x \sin kx \, dx = \frac{x \cos kx}{k}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos kx \, dx}{k}$$

$u = x \qquad dv = -\sin kx$

$du = 1 \qquad v = \dfrac{\cos kx}{k}$

$$= \frac{\pi \cos \pi k}{k} - \frac{-\pi \cos(-\pi k)}{k} - \frac{\sin kx}{k^2}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi \cos \pi k}{k}\right)$$

$$a_k = \frac{4 \cos \pi k}{k^2} = \frac{4}{k^2}(-1)^k$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin kx \, dx = 0$$

$$x^2 \sim \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx$$

e) $f(x) = \sin^3 x = \dfrac{3 \sin x - \sin 3x}{4}$

fourier sum

$$f(x) = \begin{cases} \cos x, & |x| < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2/\pi$$

$$a_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cos kx \, dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos((k+1)x) + \cos((k-1)x)}{2} dx$$

$$\frac{1}{\pi}\left(\frac{\sin(k+1)x}{2(k+1)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin(k-1)x}{2(k-1)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{\pi}\left(\frac{\sin(k+1)\pi/2}{k+1} + \frac{\sin(k-1)\pi/2}{k-1}\right) \text{ if } k \neq 1$$

$$= \frac{1}{2} \quad k = 1$$

if $k \equiv 1 \mod 2 \quad k \neq 1$

$a_k = 0$

if $k \equiv 2 \mod 4 \qquad k \equiv 3 \mod 4 \ni 0$

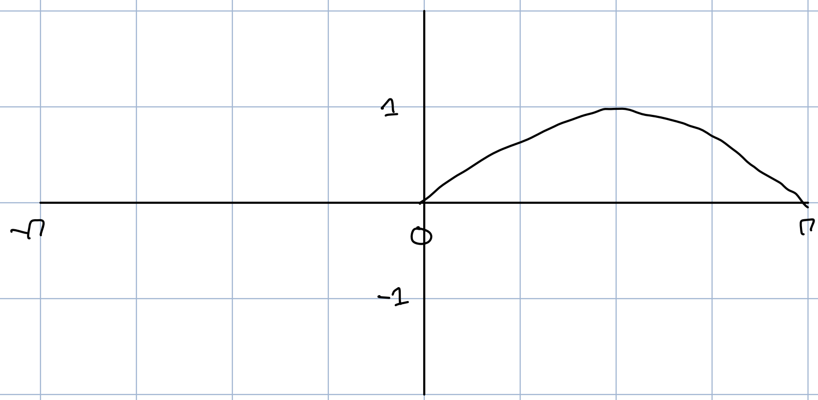
$a_k = \frac{1}{k-1} - \frac{1}{k+1} \qquad k \equiv 0 \mod 4$

$a_k = \frac{1}{k+1} - \frac{1}{k-1}$

$$f(x) \sim \frac{1}{\pi} + \sum_{k=1}^{\infty}\left(\frac{1}{\pi}\left(\frac{\sin\frac{(k+1)\pi}{2}}{k+1} + \frac{\sin\frac{(k-1)\pi}{2}}{k-1}\right)\right)$$

3.2.25 $\quad a, b, c, d$

a) $f(x) = \begin{cases} \sin x & 0 < x \le \pi \\ 0 & -\pi \le x < 0 \end{cases}$



b

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x \, dx = \frac{2}{\pi}$$

$$a_k = \frac{1}{\pi}\int_0^\pi \sin x \cos kx \, dx = \frac{\sin x \sin kx}{k} - \frac{1}{k}\int \sin kx \cos x \, dx$$

$u = \sin x \quad dv = \cos kx$

$du = \cos x \quad v = \frac{\sin kx}{k}$

$$\int \sin kx \cos x \, dx = \frac{-\cos x \cos kx}{k} - \frac{1}{k}\int \sin x \cos kx \, dx$$

$u = \cos x \quad dv = \sin kx$

$du = -\sin x \quad v = \frac{-\cos kx}{k}$

$$a = \frac{\sin x \sin kx}{k} + \frac{\cos x \cos kx}{k^2} + \frac{a}{k^2}$$

$$k^2 a = k \sin x \sin kx + \cos x \cos kx + a$$

$$a(k^2-1) = \underline{\qquad\qquad}$$

$$\int \sin x \cos kx = \frac{k \sin x \sin kx + \cos x \cos kx}{k^2-1}$$

$$\frac{1}{\pi}\int_0^\pi \sin x \cos kx = \frac{1}{\pi}\left(\frac{-\cos k\pi - 1}{k^2-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{k+1}-1}{k^2-1}\right)$$

$k \neq 1$

if $k = 1$

$$\int_0^\pi \sin x \cos x \, dx = \frac{\sin^2 x}{2}\Big|_0^\pi$$

$$= 0$$

$$b_k = \frac{1}{\pi}\int_0^\pi \sin x \sin kx \, dx = \int_0^\pi -\frac{\cos(k+1)x - \cos(k-1)x}{2} \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(k-1)x}{2(k-1)} - \frac{\sin(k+1)x}{2(k+1)}\right]_0^\pi$$

$= 0$ if $k \neq 1$

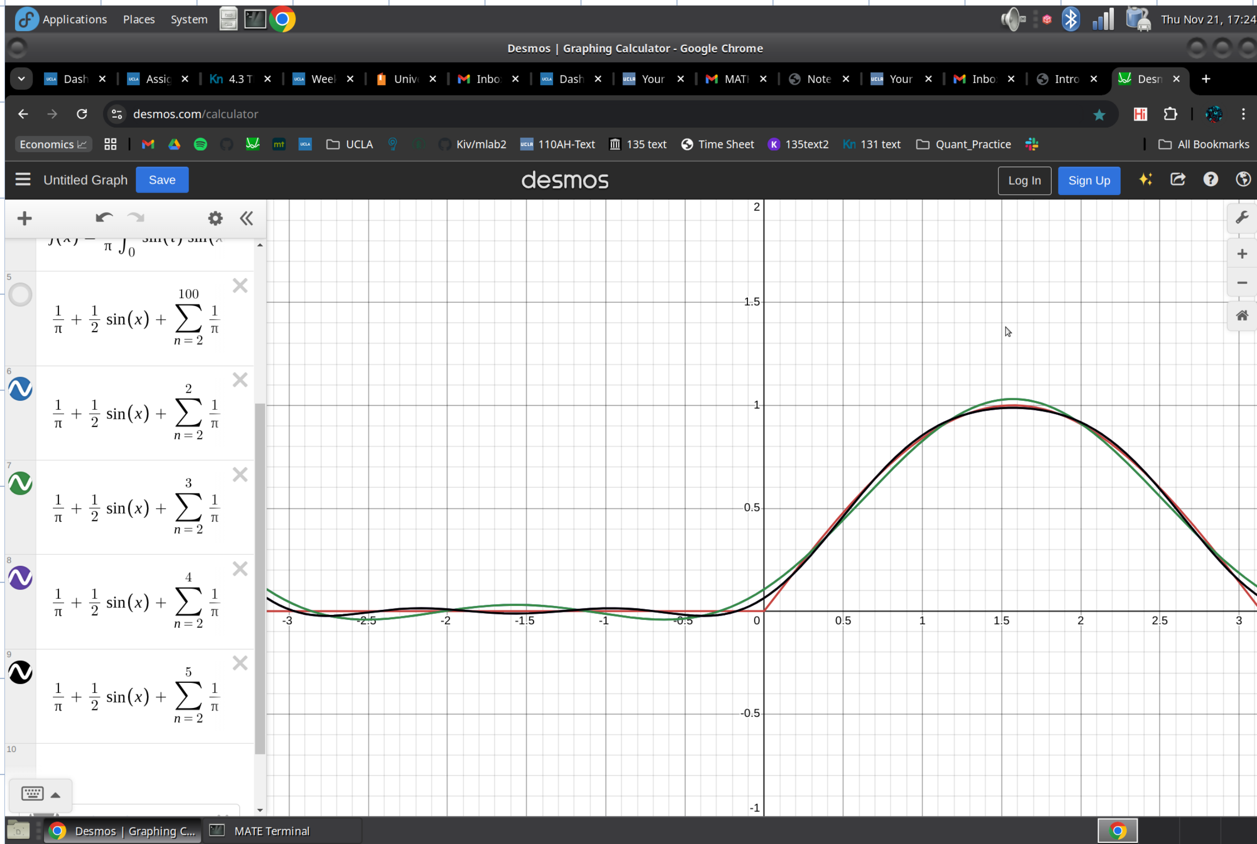
if $k = 1$

$$b_k = \frac{1}{\pi}\int_0^\pi \sin^2(x) = \frac{1}{\pi} \cdot -\frac{\sin(2x) - 2x}{4}\Big|_0^\pi$$

$$= \frac{1}{2}$$

So $f(x) \sim \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{k=2}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^{k+1}-1}{k^2-1}\right)\cos kx$

c)



d) Because f is continuous f converges to f.

2.2.51 a, b, c

$$(\sin x)^3 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3$$

a) $f(x) = \sin x$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \, e^{-ikx} \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-ikx} \, dx$$

$$= \frac{1}{4i\pi}\left[\int_{-\pi}^{\pi} e^{x(i-ik)} \, dx - \int_{-\pi}^{\pi} e^{x(-i-ik)} \, dx\right]$$

$$= \frac{1}{4i\pi}\left[\frac{e^{i(1-k)x}}{i(1-k)}\Bigg|_{-\pi}^{\pi} - \frac{e^{-i(1+k)x}}{-i(1+k)}\Bigg|_{-\pi}^{\pi}\right]$$

$= 0$ if $k \neq 1$ or $k \neq -1$

if $k = 1, -1 \Rightarrow \int = \pm 2\pi$

$$c_k = \frac{1}{2i} \quad k \in \{-1, 1\}$$

$$f(x) \sim -\frac{1}{2i} e^{-ix} + \frac{1}{2i} e^{ix} = \sin x$$

b) $$\sin^3 x = c_k \frac{1}{2\pi}\int \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 e^{-ikx} \, dx$$

$$= \frac{1}{-16i\pi}\int \left(e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}\right) e^{-ikx} \, dx$$

$$= \frac{1}{-16i\pi}\int e^{i(3-k)x} - 3e^{i(1-k)x} + 3e^{-i(k+1)x} - e^{-i(k+3)x} \, dx$$

$= 0$ if $k \notin \{3, 1, -1, -3\}$

$$C_3 = -\frac{1}{8i} \quad C_1 = \frac{3}{8i} \quad C_{-1} = \frac{-3}{8i} \quad C_{-3} = \frac{1}{8i}$$

$$\sin^3 x \sim \frac{e^{-3ix} - e^{3ix}}{8i} + \frac{3(e^{ix} - e^{-ix})}{8i}$$

d) $f(x) = |x|$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x e^{-ikx}\,dx - \int_{-\pi}^{0} x e^{-ikx}\,dx\right)$$

$$\int x e^{-ikx} = \frac{x e^{-ikx}}{-ik} + \frac{1}{ik}\int e^{-ikx}\,dx$$

$u = x \quad dv = e^{-ikx}$

$du = 1 \quad v = \frac{e^{-ikx}}{-ik}$

$$= -\frac{1}{ik}\left(x e^{-ikx} + \frac{e^{-ikx}}{ik}\right)$$

$$\int_0^{\pi} x e^{-ikx} = -\frac{1}{ik}\left(\pi e^{-ik\pi} + \frac{e^{-ik\pi}}{ik} - \frac{1}{ik}\right)$$

$+ \qquad +$

$$-\int_{-\pi}^{0} x e^{-ikx} = \frac{1}{ik}\left(\frac{1}{ik} + \pi e^{ik\pi} - \frac{e^{ik\pi}}{ik}\right)$$

$$\frac{1}{ik}\left(\frac{2}{ik} - \pi(e^{ik\pi} - e^{-ik\pi}) - \frac{(e^{ik\pi} + e^{-ik\pi})}{ik}\right)$$

$$= \frac{1}{ik}\left(\frac{2}{ik} + 2\pi i \sin(k\pi) - \frac{2\cos(k\pi)}{ik}\right)$$

$$= -\frac{2}{k^2} + \frac{2\pi}{k}\sin(k\pi) + \frac{2\cos k\pi}{k^2}$$

$$\Rightarrow \frac{1}{2\pi} \frac{2}{k^2}\left(\cos \pi k - 1\right)$$

$$= \begin{cases} \frac{1}{\pi k^2} & k \text{ odd} \\ 0 & k \text{ even} \neq 0 \end{cases}$$

$$\begin{cases} \frac{\pi}{2} & k = 0 \end{cases}$$

$$|x| \sim \frac{\pi}{2} + \sum_{-\infty}^{\infty} \begin{cases} \frac{1}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

3.2.55 2)3

$$f(x) = xe^{ix}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{ix}e^{-ikx}\,dx$$

$$\int xe^{i(1-k)x}\,dx = \frac{-ixe^{i(1-k)x}}{(1-k)} + \frac{e^{i(1-k)x}}{(1-k)^2}$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-k}\left(\frac{1}{1-k}\left(e^{i(1-k)\pi} - e^{-i(1-k)\pi}\right) - i\pi\left(e^{i(1-k)\pi} + e^{-i(1-k)\pi}\right)\right)\right]$$

$$= \frac{1}{2\pi(1-k)}\left(\frac{2i}{(1-k)}\sin((1-k)\pi) - 2i\pi\left(\cos((1-k)\pi)\right)\right)$$

$$= \left(\frac{i}{1-k}\right)(-1)^k \quad \text{if} \quad k \neq 1$$

$$\text{if} \quad k = 1 \qquad c_k = 0$$

$$f(x) = xe^{ix} \sim \sum_{\substack{-\infty \\ n\neq 1}}^{\infty} \left(\frac{i}{1-n}\right)(-1)^n e^{inx}$$

$$x e^{ix} = x\cos x + i x \sin x$$

$$\frac{f(x)-f(\cancel{-x})}{2} = \frac{xe^{ix}+xe^{-ix}}{2} = x\cos x \qquad f(x)-f(-x)$$

$$\Downarrow$$

$$x\cos x \sim \frac{1}{2}\left(\sum_{\substack{-\infty \\ n\neq 1}}^{\infty} \left(\frac{i}{1-n}\right)(-1)^n \left(e^{inx}-e^{-inx}\right)\right)$$

$$\frac{f(x)+f(-x)}{2i} = \frac{xe^{ix}-xe^{-ix}}{2i} = x\sin x$$

$$x\sin x \sim \frac{1}{2i}\left(\sum_{\substack{-\infty \\ n\neq 1}}^{\infty} \left(\frac{i}{1-n}\right)(-1)^n \left(e^{inx}+e^{-inx}\right)\right)$$

3.2.58

$$f(x) \sim \sum_{-\infty}^{\infty} c_k e^{ikx}$$

$$f(x-a) \sim \sum_{-\infty}^{\infty} c_k e^{ik(x-a)} = \sum_{-\infty}^{\infty} \frac{c_k e^{ikx}}{e^{ika}}$$

just take each

$$c_k \to \frac{c_k}{e^{ika}}$$