1. Partial Solutions to Homework #2

Exercise 2.5. Prove that any sequence in $\mathbb{R}$ has a monotonic subsequence.

Proof. If $(x_n)_n$ is not bounded from below, we can find $n_1 \in \mathbb{N}$ such that $x_{n_1} < -1$ and then $n_2 \in \{n_1+1, n_1+2, \cdots\}$ such that $x_{n_2} < x_{n_1}$ and then $n_3 \in \{n_2+1, n_2+2, \cdots\}$ such that $x_{n_3} < x_{n_2}$. We repeat the operation to find natural numbers $n_1 < n_2 < \cdots$, such that $x_{n_1} > x_{n_2} > x_{n_3} \cdots$. Thus, $(x_{n_k})_k$ is a monotonic sequence of $(x_n)_n$.

We assume in the sequel that $(x_n)_n$ is bounded from below.

Case 1. Suppose that there exists a subsequence $(x_{n_i})_i$ with no least term and denote by l the greatest lower bound of $(x_{n_i})_i$. To alleviate the notations, we set $y_i = x_{n_i}$. We choose $i_1 \in \mathbb{N}$ such that $l < y_{i_1} < l+1$. Then we choose $i_2 > i_1$ such that $l < y_{i_2} < \min\{l+1/2, y_{i_1}\}$. Inductively, we choose $i_k > i_{k-1}$ such that $l < y_{i_k} < \min\{l+1/k, y_{i_{k-1}}\}$. We have that $(y_{i_k})_k$ is a monotonic subsequence of $(x_n)_n$.

Case 2. Suppose that every subsequence of $(x_n)_n$ has a least term. Choose $n_1 \in \mathbb{N}$ such that

$$x_{n_1} \leq x_n, \qquad \forall n \in \mathbb{N}.$$

Since the sequence $(x_n)_{n>n_1}$ has a least term, we can choose $n_2 \in \{n_1+1, n_1+2, \cdots\}$ such that

$$x_{n_2} \leq x_n, \qquad \forall n \in \{n_1+1, n_1+2, \cdots\}.$$

Note that $x_{n_1} \leq x_{n_2}$. Choose $n_3 \in \{n_2+1, n_2+2, \cdots\}$ such that

$$x_{n_3} \leq x_n, \qquad \forall n \in \{n_2+1, n_2+2, \cdots\}.$$

Note that $x_{n_2} \leq x_{n_3}$. We inductively choose natural numbers $n_1 < n_2 < \cdots$ such that $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots$ to conclude that $(x_{n_k})_k$ is a monotonic subsequence of $(x_n)_n$. □

Exercise 2.15. Let E be a compact metric space, $\{U_i\}_{i\in I}$ a collection of open subsets of E whose union is E. Show that there exists a real number $\epsilon > 0$ such that any closed ball in E of radius ϵ is entirely contained in at least one set U_i.

Proof. Since $\{U_i\}_{i\in I}$ is an open cover of the compact set E, there exists $\{i_1, \cdots, i_n\} \subset I$ such that $E = \cup_{j=1}^n U_{i_j}$. It suffices to show the assertion when I is replaced by $\{i_1, \cdots, i_n\}$. This means that we can assume without loss of generality that $I = \{1, \cdots, n\}$.

Assume on the contrary that for every $\epsilon > 0$ there is $a_\epsilon \in E$ such that for every $i \in I$, we have that $B_\epsilon(a_\epsilon) \not\subset U_i$ (here, $B_\epsilon(a_\epsilon$ is the closed ball of center a_ϵ and radius ϵ). In particular, for every $n \in \mathbb{N}$, we can find $a_n \in E$ such that for every $i \in I$, we have that $B_{1/n}(a_n) \cap U_i^c \neq \emptyset$. Since E is compact, there is an increasing sequence $(n_k)_k$ such that $(a_{n_k})_k$ converges to some a in E. Let $b^i_{n_k} \in B_{1/n}(a_n) \cap U_i^c$. We have that $(b^i_{n_k})_k$ converges to a. Since U_i^c is a closed set and $(b^i_{n_k})_k \subset U_i^c$, we conclude that $a \in U_i^c$ for all i. In other words, $a \in \cap_{i=1}^n U_i^c = \big(\cup_{i=1}^n U_i\big)^c = E^c = \emptyset$, which yields a contradiction. □

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