

HW 1 - 110AH

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1

For $n \geq 2$, let $B_n : \{1, 2, \dots, n\} \rightarrow \{1, \dots, n\}$ be defined by $B_k(n) = k + 1 - n$ (B_n is the permutation that lists $\{1, \dots, n\}$ in reverse order). Find $\text{sgn}(B_n)$. Starting with $B_n(1)$, it appears at the end of the list of elements so the number it contributes to the sum is 0. For each $B_n(k)$ there are $k - 1$ elements ahead of it with value less than $B_n(k)$. This sum can be written as

$$\sum_{i=1}^n (i - 1) = \frac{n^2 - n}{2}.$$

Therefore $\text{sgn}(B_n) = (-1)^{\frac{n^2 - n}{2}}$

2

Show that $\pi_{ij} : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ with $i, j \in \{1, \dots, n\}$ defined by

$$\pi_{ij}(k) = k, \quad k \neq i, j, i \neq j$$

$$\pi_{ij}(i) = j$$

$$\pi_{ij}(j) = i$$

has $\text{sgn}(\pi_{ij}) = -1$

consider $\pi_{ij}(k)$, $i < j$, $k < i$, or $k > j$. Clearly every element above $\pi_{ij}(k)$ is greater than k so it contributes none to the sum. The same goes for $\pi_{ij}(i)$. Then for each $i < n < j$ the sum of numbers greater than the permutation $\pi_{ij}(n) = 1$. Because the only number less than n above $\pi_{ij}(n)$ is i . There are $j - i - 1$ of these n bringing the total count to $j - i - 1$. for $\pi_{ij}(j)$ There are all of the numbers plus one in between so $j - i$ numbers that are less than j . Adding this to the count we get $2i - 2j - 1$. This number is always odd because it is one less than 2 times an integer and so the $\text{sgn}(\pi_{ij}) = -1$ for all i, j

3

Is every element of S_n (symmetric group on n elements) a product (composition) of interchanges π_{ij} (as in previous problem)?

Starting with the identity I and any bijection S_i . pick j_1 such that $S_i(j_1) = 1$. Then compose $\pi_{1,j_1}I$ to get I_1 . now $I_1(j) = 1$. Proceed inductively on n and for each n first find k_n such that $S_i(k_n) = n$ and the m_n such that $I_{n-1}(m_n) = n$ compose $\pi_{m_n,k_n}I_{n-1}$. All k_i are unique because the permutation is a bijection and all m_n are unique because the previous $m_i, i < n$ point to other elements. Thus with this technique every element of S_n can be constructed as a composition of π_{ij}

4

Explain why $A_n = \{S_\sigma \in S_n : \text{sgn}(\sigma) = 1\}$. Is a normal subgroup by showing that $\pi \rightarrow \text{sgn}(\pi)$ is a homomorphism of S_n onto the group $\{-1, 1\}$, operator multiplication

for any two permutations σ_1, σ_2 . $\text{sgn}(\sigma_1\sigma_2) = \text{sgn}(\sigma_1) * \text{sgn}(\sigma_2)$. This follows from the fact that any adjacent interchange applied to $\sigma = \pi_{i,i+1}\sigma$ flips the sign by -1 and any π_{ij} can be written as

$$\pi_{ij} = \pi_{i,i+1}\pi_{i+1,i+2}\dots\pi_{i+n,j}\pi_{i+n,i+n-1}\dots\pi_{i,i+1}.$$

As a sanity check it is quickly verified that the above expression has an odd number of elements.

So from the previous question any permutation is a product of adjacent interchanges and therefore its sign is known by the amount of adjacent interchanges it has multiplied by the sign of the identity which is 1.

Therefore for any $\pi_1, \pi_2 \in S_n$ $\text{sgn}(\pi_1\pi_2) = \text{sgn}(\pi_1)\text{sgn}(\pi_2)$ and the range of sgn is $-1, 1$ so there is a homomorphism. For any subset A_n of S_n and any $a_1, a_2 \in A_n$ $\text{sgn}(a_1) = \text{sgn}(a_2) = 1$ and $\text{sgn}(a_1a_2) = 1 * 1 = 1$ so the result of any two elements composed with themselves is a member of the subgroup and thus the subgroup is normal.

5

Check explicitly that S_3 has one subgroup of order 3 which is normal and three subgroups of order 2, none of which is normal. S_3 has 6 elements defined

$$\begin{aligned}(1, 2, 3) &= e \\ (2, 1, 3) &= T_{1,2} \\ (1, 3, 2) &= T_{2,3} \\ (3, 2, 1) &= T_{1,3} \\ (3, 1, 2) &= g_3 \\ (2, 3, 1) &= g_2\end{aligned}$$

Notably their inverses and squares are:

$$\begin{aligned}
e^{-1} &= e \\
T_{1,2}^{-1} &= T_{1,2} \\
T_{2,3}^{-1} &= T_{2,3} \\
T_{1,3}^{-1} &= T_{1,3} \\
g_3^{-1} &= g_2 \\
g_2^{-1} &= g_3 \\
g_3^2 &= g_2 \\
g_2^2 &= g_3
\end{aligned}$$

From this the following cyclic subspaces come easily: $\{e, g_3, g_2\}, \{e, T_{1,2}\}, \{e, T_{2,3}\}, \{e, T_{1,3}\}$. Aside from the one listed, any subgroup of order 3 would require 2 distinct T 's. However the combination of $T_i T_j$ with $i \neq j$ is not in the set $\{e, T_i, T_j\}$. Because it is necessarily not equal to the identity. Thus the cyclic group is the only subgroup of order 3. It is easy to check that it is normal because $T_{1,2} g_3 T_{1,2} = g_2$, $T_{1,3} g_3 T_{1,3} = g_2$, $T_{2,3} g_3 T_{2,3} = g_2$. Similarly the same holds in the previous equations if every instance of g_2 is swapped with g_3 and vice versa from the equation $T_i g_3 T_i T_i g_3 T_i = T_i g_2 T_i = g_2^2 = g_3$.

Thus $\{e, g_3, g_2\}$ is a normal subgroup of order 3.

For the subgroups of order 2 there are clearly no others because they would have to be cyclic. They are also not normal as shown early that $T_i T_j \neq T_i$ or e . All subgroups must be order 3 or 2 so we have exhaustively checked all subgroups and determined that the one of order 3 is normal and the 3 of order 2 are not.

6

1. Prove without computing the compositions that π_{12} and π_{23} do not commute by noting that if they did commute then $\{e, \pi_{12}, \pi_{23}, \pi_{12}\pi_{23}\}$ would be a subgroup of S_3 of order 4. we know that $\pi_{12}^{-1} = \pi_{12}$ and $\pi_{23}^{-1} = \pi_{23}$. This information shows that the set $e, \pi_{12}, \pi_{23}, \pi_{12}\pi_{23}$ would be a subgroup for $\pi_{12}\pi_{23} = \pi_{23}\pi_{12}$ and the inverse would be itself for $\pi_{12}\pi_{23}\pi_{23}\pi_{12} = \pi_{12}e\pi_{12} = \pi_{12}\pi_{12} = e$. This is a violation because $4 \nmid 6$.
2. Compare $\pi_{12}\pi_{23}$ and $\pi_{23}\pi_{12}$ to see that they are different. $\pi_{12}\pi_{23} = (2, 3, 1)$ and $\pi_{23}\pi_{12} = (3, 1, 2)$ they are clearly different.

7

If $n \geq 3$ and $k < n$, consider the k -cycle's the permutations of $(1, \dots, n)$ generated by looking at $1, \dots, k$ and moving it to the right $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow k \rightarrow 1$.

1. What is $sgn(\sigma)$?
By counting we have k has $k - 1$ elements ahead of it that are less than k and all other numbers are the less than all numbers ahead of them so $sgn(\sigma) = (-1)^{k-1}$
2. What is the order of σ ? There are k elements corresponding to the possible values of k .

8

Think about k -cycles on every subset of k elements in $\{1, \dots, n\}$

1. is every element of S_3 a product of disjoint cycles? (k arbitrary)
Yes. $e, T_{1,2}, T_{1,3}, T_{2,3}$ are all a singular cycle on a subset of $1, \dots, 3$
 g_2 and g_3 are both cycles on the entire set.
2. S_4, S_n ?
Yes. There are $n!$ elements of S_n . Consider any $\sigma \in S_n$ and the list created by $(1, \sigma(1), \sigma^2(1), \dots, \sigma^{n!}(1))$. This list has $n! + 1$ elements so it must contain one element twice and because σ is a bijection 1 must be repeated. Consider the first occurrence $\sigma^{k+1}(1) = 1$ then $(1, \sigma(1), \sigma^2(1), \dots, \sigma^k(1))$ has only distinct elements. This set also necessarily has at least one element being 1 so $\{1, \dots, n\} \setminus \{1, \sigma(1), \dots, \sigma^k(1)\}$ is of cardinality strictly less than $n!$. Consider any i in this newly created set and the list $(i, \sigma(i), \dots, \sigma^{k_i}(i))$ with $\sigma^{k_i+1}(i) = i$ This list is entirely contained in $\{1, \dots, n\} \setminus \{1, \sigma(1), \dots, \sigma^k(1)\}$ because cycles of disjoint elements are disjoint by the bijectivity of σ . Proceed inductively on i choosing a new i and continually subtracting elements from the starting set $1, \dots, n$. Each iteration generates a distinct cycle and each iteration strictly decreases the number of elements remaining. Therefore it terminates when the last element(s) are put into cycles.