Hw 3 - 170S - Asher Christian

6.7 (1, 2, 7)

(1) $X_1, X_2, \dots, X_n$ from $N(0, \sigma^2)$ $\quad pdf = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x_i^2}{2\sigma^2}}$

(a) find sufficient statistic Y for σ^2

$$\text{Joint pdf} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{\frac{-\Sigma x_i^2}{2\sigma^2}}$$

Let $Y = \Sigma x_i^2$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{\frac{-Y}{2\sigma^2}}$$

then Joint pdf $= (\text{Joint}) \cdot 1$

where $h(x_1, \dots, x_n) = 1$

So $Y = \Sigma x_i^2$ is a sufficient statistic

(b)

$$\ln(\text{Joint}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\Sigma x_i^2}{2\sigma^2}$$

$$\frac{d}{d\sigma^2} \ln(J) = \frac{-n}{2\sigma^2} + \frac{\Sigma x_i^2}{2(\sigma^2)^2}$$

$$0 = -\frac{n}{2}\sigma^2 + \frac{\Sigma x_i^2}{2}$$

$$\sigma^2 = \frac{\Sigma x_i^2}{n}$$

$$= \frac{Y}{n}$$

(C) $$E[\tfrac{Y}{n}] = E\left[\frac{\Sigma x_i^2}{n}\right]$$

$$= \frac{\sum E[x_i^2]}{n}$$

$$E[x_i^2] = \int_{-\infty}^{\infty} \frac{x_i^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int x^2 e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{\cancel{2\pi}\cancel{\sigma^2}}} \cdot \frac{\sqrt{\pi}}{\cancel{2}} \cdot \cancel{2}\sqrt{2}\, \sigma^{\cancel{3}\,2} = \sigma^2$$

$$E[\sigma^2] = \frac{n\sigma^2}{n} = \sigma^2$$

unbiased

(2) $X_1, X_2, \dots, X_n$ from poisson

$\lambda > 0$ pdf $\frac{\lambda^{x} e^{-\lambda}}{x!}$

$P(X_1 = x_1, \dots, X_n = x_n \mid Y = y)$

$Y = X_1 + \dots + X_n$ and $\Sigma x_i = y$

$Y = n\bar{X}$

$P(X_1 = x_1, \dots, X_n = x_n) = \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\Pi x_i!}$ $P(Y=y) = \frac{(n\lambda)^y e^{-n\lambda}}{y!}$

$P(X_1 = x_1, \dots, X_n = x_n \mid Y = y) = \frac{\cancel{\lambda^y} \cancel{e^{-n\lambda}}}{\Pi x_i} \cdot \frac{y!}{n^y \cancel{\lambda^y} \cancel{e^{-n\lambda}}} = \frac{y!}{(\Pi x_i) n^y}$

(7) Let $X_1, X_2, \ldots$ be random sample

with pmf : $p(1-p)^{x-1}$ $x = 1, 2, 3, \ldots$

$0 < p < 1$

a) Show that $Y = \sum x_i$ is a sufficient statistic

$$f(x_1, \ldots, x_n \mid p) = p^n (1-p)^{\sum x_i - n}$$

let $Y = \sum x_i$

$$= p^n (1-p)^{Y-n}$$

$$h(x_1, \ldots, x_n) = 1$$

Y is sufficient

b) find a function of Y unbiased estimator of p

$$E[Y] = \sum E[x_i]$$

$$E[x] = \sum_{x=1}^{\infty} x p (1-p)^{x-1} = \sum_{x=1}^{\infty} x (1-q)(q)^{x-1}$$

$S = 1, q, q^2, \dots, q^n$

$q^S = q$

$(q-1)S = 1$

$S = \frac{1}{q-1}$

$= p \sum x q^{x-1}$

$= (1-q) \cdot \frac{d}{dq} \sum_0^\infty q^x$

$(1-q) \cdot \frac{d}{dq} \frac{1}{1-q}$

$(1-q) \cdot \frac{1}{(1-q)^2} = \frac{1}{1-q} = \frac{1}{p}$

$f(x) = \frac{1}{x}$

$E(f(x)) = \theta$

Unbiased

6.8 (1, 4, 5, 6)

(1) Y is sum of poisson distribution with mean θ

$$f(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$f(Y) = \frac{n^{Y} \theta^{Y} e^{-n\theta}}{Y!}$$

$$f(\theta|Y) \propto \theta^{\alpha+Y-1} e^{-(n+\beta)\theta}$$

$$= \text{Gamma}(\alpha+Y, n+\beta)$$

$$= \frac{(n+\beta)^{\alpha+Y}}{\Gamma(\alpha+Y)} \theta^{\alpha+Y-1} e^{-(\beta+n)\theta}$$

b) point estimate = sample mean

to minimize $[w(y) - \theta]^2 = \frac{\alpha}{\beta}$

$$= \frac{\alpha+Y}{n+\beta}$$

$$\frac{Y}{n}\left(\frac{n}{n+\frac{1}{\beta}}\right) + \alpha\beta\left(\frac{}{\beta(n+\frac{1}{\beta})}\right)$$

$$= \frac{Y}{n+\frac{1}{\beta}} + \frac{\alpha}{n+\frac{1}{\beta}} = \frac{Y+\alpha}{n+\frac{1}{\beta}} \text{ if } \bar{\beta} = \frac{1}{\beta}$$

then it match

(4)

$$f(x|\theta) = 3\theta x^2 e^{-\theta x^3}$$

$$p(\theta) = G(4, \tfrac{1}{4}) = \frac{1}{\Gamma(4)\,(\frac{1}{4})^4}\,\theta^3\, e^{-4\theta}$$

$$f(x_1, \dots, x_n | \theta) = (3\theta)^n \left(\prod x_i\right)^2 e^{-\theta \sum x_i^3}$$

$$pf(\theta | x_1, \dots, x_n) \propto \theta^{n+3}\, e^{-\theta(\sum x_i^3 + 4)}$$

$$\text{Joint} = \text{Gamma}\left(n+4, \frac{1}{4+\sum x_i^3}\right)$$

$$\text{mean} = \frac{n+4}{4+\sum x_i^3}$$

(5) estimatic $w(Y) \quad [\theta - w(Y)]$

for ex 6.8.3

The Sample is normal so mean works and is equal to median as given in problem

$$\frac{y\,\sigma_0^2 + \theta_0\,\sigma^2/n}{\sigma_0^2 + \sigma^2/n}$$

$$cdf = \frac{1}{\theta}x$$

(6) $$pdf = \frac{1}{\theta} \qquad \text{Size } n$$

a) $$h(\theta) = \frac{\beta \alpha^{\beta}}{\theta^{\beta+1}}$$

$$g_n(y;\theta) = \frac{n!}{(n-1)!}\left(\frac{1}{\theta}y\right)^{n-1} \cdot \frac{1}{\theta}$$

$$= n\left(\frac{1}{\theta}\right)^{n} y^{n-1}$$

$$\text{joint} \propto \left(\frac{1}{\theta}\right)^{n} \cdot \left(\frac{1}{\theta}\right)^{\beta+1}$$

$$\propto (\theta)^{-(n+\beta+1)}$$

$$= \frac{(n+\beta)\alpha^{(n+\beta)}}{\theta^{(n+\beta+1)}}$$

$$\int_{\alpha}^{\infty} \theta \frac{(n+\beta)\alpha^{(n+\beta)}}{\theta^{(n+\beta+1)}}$$

$$= (n+\beta)\alpha^{n+\beta} \int_{\alpha}^{\infty} \frac{1}{\theta^{n+\beta}}$$

$$= (n+\beta)\alpha^{n+\beta}\left[\frac{-1}{1-n-\beta} \cdot \frac{1}{\alpha^{n+\beta-1}}\right]$$

$$= \frac{(n+\beta)\alpha}{(n+\beta-1)}$$

b) $n = 4 \quad \beta = 2$

$\alpha = 1$

find median of

$$\frac{6}{\theta^7}$$

$$1 < \theta < \infty$$

$$\int_1^m 6\theta^{-7} \, d\theta = 0.5$$

$$\left[-\theta^{-6} \Big|_1^m \right] = 0.5$$

$$1 - m^{-6} = 0.5$$

$$m^{-6} = 0.5$$

$$m = 0.5^{-\frac{1}{6}} \approx 1.1222$$

1.122462 is median

and minimizes absolute

value loss

7.1 (3, 7, 15)

3 $X_1, \dots, X_7$ $X_i \sim N(\mu, \sigma^2)$

a) point estimate of $\mu = \bar{X} = 15.757143$

b) point estimate of $\sigma = S = \sqrt{3.2095} = 1.792$

find 95.%

c) $$\varepsilon = Z_{0.025} \frac{S}{\sqrt{7}}$$

$$= (1.96) \frac{(1.792)}{\sqrt{7}}$$

$$= 1.32717$$

$$P(14.43 < \mu < 17.084) = 0.95$$

7

$$X_1, \dots, X_9 \sim N(\mu, \sigma^2)$$

$$\bar{X} = 20{,}4$$

$$\varepsilon = t_{0{,}025} \frac{S}{\sqrt{9}}$$

$$S = 1{,}858$$

$$t_{0{,}025} = 2{,}306$$

$$P(19{,}472 < \mu < 22{,}328) = 0{,}95$$

(15) $X_1, \dots, X_{20} \sim N(\mu, \sigma^2)$

$\bar{x} = 25.475$

$s = 2.493544$

b) $\varepsilon = t_{0.01} \frac{s}{\sqrt{20}}$

$t_{0.01}(19) = 2.539$

$\varepsilon = 1.4156$

$P(24.05932 < \mu < \infty) = 0.99$