HW 4 - 115B

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1 Exercise 1

V f.d.v.s. $W_1 \subset V$ $W_2 \subset V$ *subspaces. prove* $V = W_1 \oplus W_2$ *iff* $V = W_1 + W_2$ *and* $W_1 \cap W_2 = \{0\}$ To prove the first direction it suffices to show that if $v = w_1 + w_2 = u_1 + u_2$, $w_1, u_1 \in W_1$ $w_2, u_2 \in W_2$ that $w_1 = u_1$ and $w_2 = u_2$ Indeed the above implies that

$$0 = (w_1 - u_1) + (w_2 - u_2).$$

If one of the terms is 0 then it implies that the other 2 elements are equal so we will only consider the case where both terms are not equal zero. Then the set $\{(w_1 - u_1), (w_2 - u_2)\}$ is linearly dependent but since $W_1 \cap W_2 = \{0\}$ $(w_1 - u_1) \in W_1$ is not in W_2 and thus not in the span of any vectors in W_2 so it must be linearly independent with respect to any set of vectors in W_2 a contradiction thus the first direction has been proved. To prove the other direction assume that $V = W_1 \oplus W_2$ This means that any $v \in V$ can be written as $w_1 + w_2$ with $w_1 \in W_1$ $w_2 \in W_2$ unique. This directly shows that $V = W_1 + W_2$. Assume for contradiction that $W_1 \cap W_2 \neq \{0\}$ then there exists some $v \in W_1 \cap W_2$ $v \neq 0$ But then

$$v = v + 0 = 0 + v.$$

a contradiction since v can be represented as two ways with one element from each subspace.

2 Exercise 2

V f.d.v.s. with basis B with $B_1, B_2, ..., B_m$ *a partition of B. Prove that* $V = \oplus_{i=1}^{m} \mathrm{span}(B_i)$

First note that $\mathrm{span}(B_i)$ is a subspace of V and that each $\mathrm{span}(B_i)$ is disjoint by the disjoint nature of the B_i. And note that

$$B_i \cap (B_{j_1} \cup B_{j_2} \cup ... \cup B_{j_n}) = \{0\}.$$

Provided $j_k \neq i$. Thus inducting on the number n of partitions of B. In the case of only one element in B, B_1 clearly $\mathrm{span}(B_1) = V$ since it is the span of the basis for the vector space. Inducting on n assume it holds for $n-1$ for n partitions note that B_1 is disjoint from $B_2 \oplus B_3 \oplus ... \oplus B_n$ since it is disjoint from the spans of the bases. and any element of V can be written as

$$v = x_{1,1} + x_{1,2} + ... + x_{1,n_1} + x_{2,1} + ... + x_{n,n_n}.$$

with $x_{i,j} \in \text{span}(B_i)$ so clearly each v can be written as a sum of elements in B_1 and in $B_2 \oplus B_3 \oplus ... \oplus B_n$ and since they are disjoint as previously discussed, $V = \oplus_{i=1}^{n} \text{span}(B_i)$

3 Exercise 3

T operaton on fdvs V. Prove that T is diagnoalizable iff V is direct sum of one-dimensional T-inv subspaces

If V is a direct sum of one-dimensional T-invariant subspaces. each subspace is an eigenspace since for any $v \in W$ a one-dimensional T-invariant subspace $Tv = \alpha v$ since otherwise v, Tv would be lin dep and W would not be one-dim T inv. So V is a direct sum of eigenspaces and thus T diagonalizable. For the second direction if instead T is diagonalizable pick the basis $B = \{v_1, v_2, ..., v_n\}$ such that $Tv_i = \alpha_i v_i$ The basis such that $[T]_B$ is diagonal and consider the subspace

$$W_i = \text{span}(v_i).$$

The subspaces are disjoint, T invariant, and their sum is all of V b y construction and so their direct sum is V by the previous question.

4 Exercise 4

Let V be a fdvs

Assume T is a projection onto W_1 along W_2. Show that the range of T is W_1 and the kernel of T is W_2 since $V = W_1 \oplus W_2$ each $v = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$ then for all v $Tv = w_1$ and so for any $w_1 \in W_1$ there exists some $v \in V$ such that $w_1 = Tv$ additionally if $w_i \notin W_1$ assume $w_i = Tv$ for some v and $w_i = w_1 + w_2$ for some $w_2 \in W_2 \neq 0$ then Then $v = v_1 + v_2$ for some $v_1 \in W_1, v_2 \in W_2$ and $Tv = v_1$ so $v_1 = w_1 + w_2$ so $(v_1 - w_1) = w_2$ but $w_2 \in W_2$ and the other side is in W_1 and the two are disjoint a contradiction thus the range is precisely W_1. The kernel of T is W_2 for take any element of W_2 then if $v = 0 + w_2$, $Tv = 0$ so $w_2 \in \ker(T)$ assume there is an element $v \notin W_2$ such that $Tv = 0$ then $v = w_1 + w_2$ as before and $Tv = w_1 = 0$ implies that $w_1 = 0$ implies that $v \in W_2$ so the kernel is the entirety of W_2.

Prove that a linear endomorphism $T : V \to V$ is a projection iff $T = T^2$

If $T = T^2$ then for any $v \in \text{Im}(T)$ $v = Tw = T^2 w = T(v)$ which implies that $T|_{\text{Im}(T)} = \text{Id}|_{\text{Im}(T)}$. Since $V = \text{Im}(T) \oplus \ker(T)$ and whenever $v = v_1 + v_2$ with $v_1 \in \text{Im}(T)$ and $v_2 \in ker(T)$, then $Tv = v_1$ Thus T is a projection. If T is a projection then there exist W_1, W_2 such that $V = W_1 \oplus W_2$ and every

$v = w_1 + w_2$ with $w_1 \in W_1$ $w_2 \in W_2$ then $Tv = w_1$ indeed

$$T^2 v = T(Tv) = T(w_1) = w_1 = Tv.$$

and so $T^2 = T$ proving the second direction.

5 Exercise 5

Let $n \in \mathbb{Z}^{>0}$ and let $A \in k^{n \times n}$ be the matrix

$$A = \begin{pmatrix} 1 & 2 & ... & n \\ n+1 & n+2 & ... & 2n \\ \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & ... & n^2 \end{pmatrix}.$$

Compute the characteristic polynomial of A We start by using the fact that collumn rank is equal to row rank is equal to the rank of a transformation. The column rank is 2 because row 2 - row 1 is

$$\begin{pmatrix} n+1 \\ n+2 \\ \vdots \\ 2n \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

there fore the column of all 1s is in the span of the column vectors as is the first column. if $\vec{v_1}$ is the first column and $\vec{v_2}$ is the column of all ones, then every column i is

$$\vec{v_1} + in\vec{v_2}.$$

Thus every other column is in the span of $\vec{v_1}$ and $\vec{v_2}$, additionally these vectors are clearly linearly independent so the column space is spanned by two linearly independent vectors and thus the rank of the transformation is 2. since $L_A : V \to V$ acts on an n dimensional vector space the kernel must be $\dim(n-2\ )$. Thus there must be $n-2$ lin indep eigenvectors of eigenvalue 0. So It suffices to find two more eigenvectors and their eigenvalues to uniquely determine the degree n characteristic polynomial. Additionally if $W = \text{span}(\vec{v_1}, \vec{v_2})$ then W is $T-$invariant becausde

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n i \\ n^2 + \sum_{i=1}^n i \\ 2n^2 + \sum_{i=1}^n i \\ \vdots \\ (n-1)n^2 + \sum_{i=1}^n i \end{pmatrix}.$$

$$= \sum_{i=1}^n i \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + n^2 (\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}).$$

A linear combination of the two vectors. Likewise

$$A\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^n i^2\\ n\sum_{i=1}^n i+\sum_{i=1}^n i^2\\ 2n\sum_{i=1}^n i+\sum_{i=1}^n i^2\\ \vdots\\ (n-1)n\sum_{i=1}^n i+\sum_{i=1}^n i^2\end{pmatrix}=\sum_{i=1}^n i^2\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}+(n\sum_{i=1}^n i)(\begin{pmatrix}1\\2\\\vdots\\n\end{pmatrix}-\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}).$$

Thus if $B=\{\vec{v_1},\vec{v_2}\}$ is a basis for A then

$$A|_W=\begin{pmatrix}\sum_{i=1}^n i-n^2 & \sum_{i=1}^n i^2-n\sum_{i=1}^n i\\ n^2 & n\sum_{i=1}^n i\end{pmatrix}=\begin{pmatrix}\frac{n-n^2}{2} & \frac{(n+1)(n-n^2)}{6}\\ n^2 & \frac{n^2(n+1)}{2}\end{pmatrix}.$$

Computing the characteristic polynomial we get

$$\lambda=\frac{n}{2}\left(\frac{1}{2}+\frac{n^2}{2}\pm\sqrt{\frac{n^4}{4}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{n^1}{3}+\frac{1}{4}}\right).$$

Thus these are the remaining two roots that are non-zero and so the characteristic polynomial is

$$x^2(x-\lambda_1)(x-\lambda_2).$$

6 Exercise 6

Assume $T:V\to V$ is some invertible linear endomorphism of an inner product space. Show that T^* is invertible and moreover $(T^*)^{-1}=(T^{-1})^*$ T^* satisfies

$$\langle T(v),w\rangle = \langle v,T^*(w)\rangle.$$

for all $v,w\in V$. Furthermore every T has an adjoint in particular there exists $J:V\to V$ such that

$$\langle T^{-1}(v),w\rangle=\langle v,J(w)\rangle.$$

$$\langle v,w\rangle=\langle T^{-1}(T(v)),w\rangle=\langle T(v),J(w)\rangle=\langle v,T^*(J(w))\rangle.$$

Therefore $T^*\circ J=I$ and in particular $J=(T^*)^{-1}$ this last statement verifiable by switching the order from $T^{-1}(T(v))$ to $T(T^{-1}(v))$

7 Exercise 7

For each of the following inner product spaces V and linear operators T on V, evaluate the adjoint of T at the given vector in V

1. $V=\mathbb{R}^2, T(\begin{pmatrix}a\\b\end{pmatrix})=\begin{pmatrix}2a+2b\\a-3b\end{pmatrix}, \vec{v}=\begin{pmatrix}3\\5\end{pmatrix}$

$$T^*(\vec{v})=\begin{pmatrix}11\\-9\end{pmatrix}.$$

2. $V = \mathbb{C}^2, T(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \begin{pmatrix} 2z_1 + iz_2 \\ (1-i)z_1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix}$

$$T^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}.$$

$$T^*(\vec{v}) = \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix}.$$

3. $V = \mathbb{R}[x]_{\leq 1}$ $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx, T(f) = f' + 3f, \vec{v} = 4 - 2x$ We want to find $T^*(\vec{v})$ such that

$$\langle T(w), 4-2x \rangle = \langle w, T^*(4-2x) \rangle.$$

for all w for general $w = a + bx$, $T(w) = (b+3a) + 3bx$

$$\int_{-1}^{1} ((b+3a)+3bx)(4-2x)dx = 4b+24a = \int_{-1}^{1} (a+bx)(c+dx)dx = 2ac + \frac{2bd}{3}.$$

Which means that

$$T^*(\vec{v}) = 12 + 6x.$$

8 Exercise 8

Let T be a linear endomorphism on an inner product space V. we say any linear operator $U : V \to V$ is self adjoint if V is its own conjugate transpose.

1. *Let $U_1 = T + T^*$ and $U_2 = TT^*$ Prove that U_1 and U_2 are self adjoint*
Indeed since $(T^*)^* = T$

$$\langle (T+T^*)(v), w \rangle = \langle T(v), w \rangle + \langle T^*(v), w \rangle = \langle v, T^*(w) \rangle + \langle v, T(w) \rangle = \langle v, (T^*+T)(w) \rangle.$$

likewise

$$\langle (TT^*)(v), w \rangle = \langle T^*(v), T^*(w) \rangle = \langle v, (TT^*)(w) \rangle.$$

2. *Assume T is a linear endomorphism of an inner product space V. Prove that T preserves lengths of all vectors if and only if it preserves all inner products. More precisely, prove that $||T(\vec{v})|| = ||\vec{v}||$ for all $\vec{v} \in V$ if and only if $\langle T(\vec{v}), T(\vec{w}) \rangle = \langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$* To prove the first diretion if $\langle T(\vec{v}), T(\vec{w}) \rangle = \langle \vec{v}, \vec{w} \rangle$ then

$$||T(\vec{v})|| = \sqrt{\langle T(\vec{v}), T(\vec{v}) \rangle} = \sqrt{\langle \vec{v}, \vec{v} \rangle} = ||\vec{v}||.$$

For the second direction we show first that the real part and then the imaginary part of the inner products are preserved given that T preserves lengths. First for the real parts

$$\begin{aligned} \langle T(v) + T(w), T(v) + T(w) \rangle &= \langle v + w, v + w \rangle \\ \langle T(v), T(v) \rangle + \langle T(w), T(v) \rangle + \langle T(v), T(w) \rangle + \langle T(w), T(w) &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ \langle T(v), T(w) \rangle + \overline{\langle T(v), T(w) \rangle} &= \langle v, w \rangle + \overline{\langle v, w \rangle} \end{aligned}$$

Thus the real parts are aligned, similarly

$$\langle T(v) + iT(w), T(v) + iT(w)\rangle = \langle v + iw, v + iw\rangle$$
$$\langle T(v), T(v)\rangle + i\langle T(w), T(v)\rangle - i\langle T(v), T(w)\rangle + \langle T(w), T(w)\rangle = \langle v, v\rangle + i\langle w, v\rangle - i\langle v, w\rangle + \langle w, w\rangle$$
$$\langle T(w), T(v)\rangle - \overline{\langle T(w), T(v)\rangle} = \langle w, v\rangle + \overline{\langle w, v\rangle}$$

And thus the imaginary parts are aligned and so the inner products are equal for all $w, v \in V$