HW 5 - 115B

Asher Christian 006-150-286

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1 Exercise 1

Give an example of an inner product space V and a linear operator $T : V \to V$ such that $\ker(T)$ *and* $\ker(T^*)$ *are not equal.* Consider $V = \mathbb{R}^2$ and $T = L_A$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

and standard inner product. then

$$\ker(T) = \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}.$$

and

$$[T^*]_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with

$$\ker(T^*) = \text{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}.$$

thus the kernels are not equal

2 Exercise 2

Let V be a finite dimensional inner product space, and let W be a subspace

1. *Prove $V = W \oplus W^\perp$*
   Pick an orthogonal basis B' for W, $B' = \{v_1, v_2, ..., v_n\}$. Extend this basis to an orthogonal basis B for V, $B = \{v_1, v_2, ..., v_n, v_{n+1}, ..., v_m\}$ This can be done by first picking a basis then using the graham schmidt process. Additionally $\{v_{n+1}, ..., v_m\}$ is a basis for $W^\perp$ this is because consider any element $w \in W^\perp$ then $w = \sum_{i=1}^m \alpha_i v_i$ but since $w \in W^\perp$, w must be orthogonal to every element in W that is $\langle w, v_j \rangle = \langle \sum_{i=1}^m \alpha_i v_i, v_j \rangle = \sum_{i=1}^m \alpha_i \langle v_i, v_j \rangle = 0$ if $j \leq n$ thus $\alpha_i = 0$ for all $i \leq n$ Then for any $v \in V$, $v = \sum_{i=1}^m \alpha_i v_i$. Let $w_1 = \sum_{i=1}^n \alpha_i v_i$ and $w_2 = \sum_{i=n+1}^m \alpha_i v_i$ Then clearly $w_1 \in W$, $w_2 \in W^\perp$ because $\langle w_1, w_2 \rangle = \sum_{i=1}^n \beta_i (\sum_{j=n+1}^m \langle v_i, \gamma_j v_j \rangle) = 0$

holds for any $v_1 \in W, v_2 \in W^\perp$. Additionally since B is a basis this is the unique representation. Thus since $\text{Span}\{v_{n+1}, ..., v_m\}$ is $W^\perp$ every element is a unique sum of elements in W and $W^\perp$

2. *Show that if T is a projection on W along $W^\perp$, then $T = T^*$*
Pick an orthonormal basis $B' = \{v_1, v_2, .., v_n\}$ of W and extend to $B = \{v_{n+1}, ..., v_m\}$ of V orthonormal, then if $v = \sum_{i=1}^m \alpha_i v_i$ and $Tv = \sum_{i=1}^n \alpha_i v_i$. if $w = \sum_{i=1}^m \beta_i v_i$ then

$$\langle Tv, w\rangle = \sum_{i=1}^n \langle \alpha_i v_i, w\rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \alpha_i v_i, \beta_j v_j\rangle = \sum_{i=1}^n \alpha_i \overline{\beta_i}.$$

And similarly

$$\langle v, Tw\rangle = \sum_{i=1}^m \langle \alpha_i v_i, \sum_{j=1}^n \beta_i v_i\rangle = \sum_{i=1}^m \sum_{j=1}^n \langle \alpha_i v_i, \beta_j v_j\rangle = \sum_{j=1}^n \alpha_j \overline{\beta_j}.$$

The key point to note is that the inner product of distinct basis elements is 0 whereas the inner product of similar elements is 1 by normality. and so

$$\langle Tv, w\rangle = \langle v, Tw\rangle \implies T = T^*.$$

3 Exercise 3

By Hw 4 Q 8 Part (b) this question is proven.

4 Exercise 4

For each lin op T on an inner product space V, determine whether T is normal, self-adjoint, or neither.

(a) $V = \mathbb{R}^2$ standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 2y \\ -2x + 5y \end{pmatrix}$
In the standard basis (which is orthonormal under standard inner product) we have

$$[T]_\mathcal{E} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \implies [T^*]_\mathcal{E} = \overline{[T]_\mathcal{E}^t} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} = [T]_\mathcal{E}.$$

and since the basis is orthonormal we can freely go in between representations so $T = T^*$ and thus T is self-adjoint and therefore normal.

(b) $V = \mathbb{C}^2$ with standard inner product, $T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2x + iy \\ x + 2y \end{pmatrix}$
In the standardard basis (again orthonormal under complex standard inner product) we have

$$[T]_\mathcal{E} = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \implies [T^*]_\mathcal{E} = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix}.$$

and

$$[TT^*]_{\mathcal{E}} = \begin{pmatrix} 5 & 2+2i \\ 2-2i & 5 \end{pmatrix} \quad [T^*T]_{\mathcal{E}} = \begin{pmatrix} 5 & 2+2i \\ -2i+2 & 5 \end{pmatrix}.$$

Thus T is normal but is not self-adjoint.

(c) $V = \mathbb{R}[x]_{\leq 2}, T(f) = f'$ and $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$
Short computation yields an orthonormal basis using the gram-schmidt process on $\{1, x, x^2\}$ to get the orthonormal basis

$$B = \{1, \sqrt{12}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6}).$$

and

$$[I]_B^{\mathcal{E}} = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & \sqrt{12} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{pmatrix} \quad [I]_{\mathcal{E}}^B = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{5}}{30} \end{pmatrix} [D]_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$[D]_B = [I]_{\mathcal{E}}^B [D]_{\mathcal{E}} [I]_B^{\mathcal{E}} = \begin{pmatrix} 0 & \sqrt{3}/6 & -\frac{\sqrt{15}}{30} + \frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{15}}{15} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$[D]_B[D^*]_B = \begin{pmatrix} a^2 + b^2 & \cdots \\ \vdots & \ddots \end{pmatrix}.$$

but

$$[D^*]_B[D]_b = \begin{pmatrix} 0 & \cdots \\ \vdots & \ddots \end{pmatrix}.$$

so the operators are not the same and therefore the operator is not normal

(d) $V = \mathbb{R}^{2\times 2}$ and $T(M) = M^t$ where $\langle A, B \rangle = tr(B^*A)$
consider first comparing

$$\langle A^t, B \rangle = tr(\overline{B^t}A^t) = tr(A\overline{B}).$$

and

$$\langle A, B^t \rangle = tr(\overline{B}A).$$

but for any $A, B \in \mathbb{R}^{2\times 2}$

$$tr(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ji} = \sum_{j=1}^n \sum_{i=1}^n B_{ji}A_{ij} = \sum_{j=1}^n (BA)_{jj} = tr(BA).$$

and so the two inner products are the same and thus $T = T^*$

5 Exercise 5

Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint iff $TU = UT$.

$$\langle TUv, w\rangle = \langle Uv, T^*w\rangle = \langle v, U^*T^*w\rangle.$$

similarly

$$\langle UTv, w\rangle = \langle v, T^*U^*w\rangle.$$

since this holds for any v, w it must be the case that $TU = UT$ if and only if $U^*T^* = T^*U^*$

6 Exercise 6

Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \quad T_2 = \frac{1}{2i}(T - T^*).$$

(a) *Prove that T_1 and T_2 are self adjoint and that $T = T_1 + iT_2$*

$$\begin{aligned}
\langle \frac{1}{2}(T + T^*)v, w\rangle &= \frac{1}{2}(\langle Tv, w\rangle + \langle T^*v, w\rangle) \\
&= \frac{1}{2}(\langle v, T^*w\rangle + \langle v, Tw\rangle) \\
&= \langle v, \frac{1}{2}(T^* + T)w\rangle
\end{aligned}$$

similarly

$$\begin{aligned}
\langle \frac{1}{2i}(T - T^*)v, w\rangle &= \frac{1}{2i}(\langle Tv, w\rangle - \langle T^*v, w\rangle) \\
&= \frac{1}{2i}(\overline{\langle T^*w, v\rangle - \langle Tw, v\rangle}) \\
&= \overline{\frac{i}{2}\langle (T^* - T)w, v\rangle} \\
&= \langle v, \frac{i}{2}(T^* - T)w\rangle \\
&= \langle v, \frac{1}{2i}(T - T^*)w\rangle
\end{aligned}$$

And so both operators are self adjoint by direct computation

$$T_1 + iT_2 = \frac{1}{2}(T + T^*) + \frac{i}{2i}(T - T^*) = \frac{1}{2}(T + T^* + T - T^*) = \frac{1}{2}(2T) = T.$$

by linearity.

(b) *Suppose also that $T = U_1 + iU_2$ where U_1 and U_2 are self-adjoint prove that $U_1 = T_1$ and $U_2 = T_2$* note that

$$\langle (U_1 + iU_2)v, w\rangle = \overline{\langle U_1 w, v\rangle} + \overline{-i\langle U_2 w, v\rangle} = \langle v, (U_1 - iU_2)w\rangle$$

Thus

$$T^* = U_1 - iU_2.$$

and so

$$T_1 = \frac{1}{2}(T + T^*) = \frac{1}{2}(U_1 + iU_2 + U_1 - iU_2) = \frac{1}{2}(2U_1) = U_1.$$

and therefore $T_2 = U_2$

(c) *Prove that T is normal iff $T_1T_2 = T_2T_1$*

$$\begin{aligned} T_1T_2 &= \frac{1}{2}(TT_2 + T^*T_2) \\ &= \frac{1}{4i}(T^2 - TT^* + T^*T - (T^*)^2) \\ T_2T_1 &= \frac{1}{2i}(TT_1 - T^*T_1) \\ &= \frac{1}{4i}(T^2 + TT^* - T^*T - (T^*)^2) \end{aligned}$$

By direct computation if T is normal then $TT^* = T^*T$ and so both equations are equal likewise if the two equations are equal then

$$TT^* - T^*T = -TT^* + T^*T \iff T^*T = TT^*.$$

7 Exercise 7

Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following

(a) *If T is self-adjoint, then $T|_W$ is self adjoint.*
Let $v, w \in W$ $Tv = T|_W v$ and $Tw = T|_W w$ and

$$\langle T|_W v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, T|_W w\rangle.$$

this holds for arbitrary v, w and so $T|_W$ is self-adjoint.

(b) *$W^\perp$ is T^*-invariant*
let $w \in W^\perp$ and $v \in W$ then

$$\langle v, T^*w\rangle = \langle Tv, w\rangle = \langle u, w\rangle.$$

for some $u \in W$ since W is T-invariant, thus $\langle v, T^*w\rangle = 0$ for all $v \in W$ and so $T^*w \in W^\perp$ thus $W^\perp$ is T^*-invariant

(c) *If W is both T- and T^*-invariant, then $(T|_W)^* = (T^*)|_W$* If W is both T and T^* invariant then for any $v, w \in W$ $Tv = T|_W v$ and $T^*w = T^*|_w$ and likewise

$$\langle T|_W v, w\rangle = \langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, T^*|_W w\rangle.$$

(d) *If W is both T and T^* invariant and T is normal then $T|_W$ is normal* If W is T and T^* invariant, then by previous questions $(T|_W)^* = (T^*)|_W$ and so for any $w \in W$, $T|_W(T^*)|_W w = TT^*w = T^*Tw = (T^*)|_W T|_W w$ and since this holds for all elements of w the functions are equivalent and thus $T|_W$ is normal.

8 Exercise 8

Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V, prove that if W is T-invariant, then W is also T^*-invariant. First pick an orthonormal basis $B' = \{v_1, v_2, ..., v_n\}$ for W and extend it to $B = \{v_1, ..., v_n, v_{n+1}, ..., v_m\}$ orthonormal basis of V possible by gram schmitdt process. Then

$$[T]_B = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \quad [T^*]_B = [T]_B^* = \begin{pmatrix} A_{1,1}^* & 0 \\ A_{1,2}^* & A_{2,2}^* \end{pmatrix}.$$

By T-invariance of W and likewise

$$[TT^*]_B = [T]_B[T^*]_B = \begin{pmatrix} A_{1,1}A_{1,1}^* + A_{1,2}A_{1,2}^* & ... \\ \vdots & A_{2,2}A_{2,2}^{*}` \end{pmatrix}.$$

and

$$[T^*T]_B = [T^*]_B[T]_B = \begin{pmatrix} A_{1,1}^*A_{1,1} & ... \\ \vdots & \ddots \end{pmatrix}.$$

Thus

$$A_{1,1}A_{1,1}^* + A_{1,2}A_{1,2}^* = A_{1,1}^*A_{1,1}.$$

and likewise

$$\mathrm{Tr}(A_{1,1}A_{1,1}^*) + \mathrm{Tr}(A_{1,2}A_{1,2}^*) = \mathrm{Tr}(A_{1,1}^*A_{1,1}).$$

but $\mathrm{Tr}(AA^*) = \mathrm{Tr}(A^*A)$ by the following

$$\mathrm{Tr}(AA^*) = \sum_{i=1}^{n}(AA^*)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}A_{ji}^* = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\overline{A_{ij}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{A_{ij}}A_{ij}.$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ji}^*A_{ij} = \sum_{j=1}^{n}(A^*A)_{jj} = \mathrm{Tr}(A^*A).$$

since

$$(a + bi)(a - bi) = a^2 + b^2 = (a - bi)(a + bi).$$

so

$$\mathrm{Tr}(A_{1,2}A_{1,2}^*) = 0.$$

but since

$$A_{ij}\overline{A_{ij}} \geq 0.$$

this implies that each $(A_{1,2})_{ij} = 0$ for all i, j in its dimension and thus $A_{1,2} = 0$ and

$$[T]_B = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix} [T^*]_B = \begin{pmatrix} A_{1,1}^* & 0 \\ 0 & A_{2,2}^* \end{pmatrix}.$$

from which it is evident that W is T^*-invariant.