Normalizers, Index, and Number of Conjugates

$H < G$ $\quad N_G(H) = \{g \in G;\ g^{-1} H g = H\}$

$N_G(H)$ is a subgroup:

$g_1^{-1} H g_1 = H$, $g_2^{-1} H g_2 = H$

$\Rightarrow \quad g_1^{-1} g_2^{-1} H g_2 g_1 = H$

$\Rightarrow \quad (g_2 g_1)^{-1} H g_2 g_1 = H$

$\Rightarrow \quad g_2 g_1 \in N_G(H)$. So

$N_G(H)$ is closed under products.

$e \in N_G(H)$ obvious, $g \in N_G(H)$

$\Rightarrow g^{-1} \in N_G(H)$ clear

$g^{-1} H g = H \Rightarrow g H g^{-1} = H$

$(g^{-1})^{-1} H g^{-1} = H$

(left multiply by g, right multiply by g^{-1}) So $g \in N_G(H) \Rightarrow g^{-1} \in N$. ☐

Theorem: $H \subset G$ subgroup.

$$\text{no of conjugates of } H = \frac{\text{ord}(G)}{\text{ord}\, N_G(H)}$$

(RHS is an integer by Lagrange's Theorem)

Example: There three two element subgroups of S_3 which are all conjugate to each other.

Normalizer of each one = the 2 element group itself.

E.g. $N_{S_3}(\{\epsilon, T_{12}\}) = \{\epsilon, T_{12}\}$

So no of conjugates of $\{\epsilon, T_{12}\}$ namely

$3 = \operatorname{ord}(G) / \operatorname{ord} N_G(H)$

$= 6/2$ So theorem works in this case.

Proof of Theorem: A conjugate

$g_1^{-1} H g_1$ = another conjugate

$g_2^{-1} H g_2 \iff g_2 g_1^{-1} H g_1 g_2^{-1} = H$

$\iff g_1 g_2^{-1} \in N_G(H) \iff g_1 \in N_G(H) g_2$

$\iff g_1, g_2$ are in the same coset (right coset) of $N_G(H)$.

The right cosets are disjoint and their union = G and each contains $N_G(H)$ elements.

So number of right cosets $= \operatorname{ord}(G)/\operatorname{ord} N_G(H)$

But as shown there is a 1-1 correspondence between the right cosets of $N_G(H)$ and the conjugates of H in G $\square$

This is a special case of a general idea that turns up often:

If G is a group then a group action Σ of G on a set X is a homomorphism of G into the set of bijective functions from X to X. We write

$g \rightsquigarrow \Sigma(g)$ where $x \mapsto \Sigma(g)(x)$ notation: $x\,\Sigma(g)$ action on right

bijection of X to X,

The stabilizer of $x_0 \in X \stackrel{\text{def}}{=} \{g \in G : x_0\Sigma_g = x_0\}$.

The orbit of $x_0 \stackrel{\text{def}}{=} \{x_0\,\Sigma(g) : g \in G\}$

Lemma: no of elements in orbit of x_0

= no of elements in G / no of elements in stabilizer of x_0

(if the numbers are finite, the only case we are generally interested in)

Proof: $x_0 \Sigma(g_1) = x_0 \Sigma(g_2) \Leftrightarrow$

$x_0 = x_0 \Sigma(g_1 g_2^{-1})$ (since $x_0 \Sigma(g_1) \Sigma(g_2^{-1}) = x_0 \Sigma(g_1 g_2^{-1})$.

So $x_0 \Sigma(g_1) = x_0 \Sigma(g_2)$

$\Leftrightarrow$ $g_1 g_2^{-1} \in$ stabilizer of x_0.

$\Leftrightarrow$ $g_1 =$ (stabilizer of x_0) g_2.

$\Leftrightarrow$ g_1 & g_2 are in the same (right) coset of stabilizer of x_0.

So no of elements in orbit of x_0

= no of elements in G / number of elements in a right coset of stabilizer of x_0

= order of G / order of stabilizer of x_0 $\square$

Note that different orbits may have different sizes and different size of stabilizer. But if $x_1, x_2 \in$ same orbit then stabilizer of x_1 and stabilizer of x_2 have same order

(and in fact stabilizer of x_1 and stabilizer of x_2 are conjugate subgroups of G, conjugation by $g \in G$ such that $x_1 \Sigma(g) = x_2$

(Exercise!)

Confusing at first but you'll get used to it.