

HW 4 - 131AH

ASHER CHRISTIAN 006-150-286

1. EXERCISE 4.1

Prove that a continuous real-valued function on a closed interval in \mathbb{R}^2 cannot be one-to-one

Let $f : [a_1, b_1] \times [a_2, b_2] \rightarrow \mathbb{R}$ continuous. $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$

Let $f_1 : [a_1, b_1] \rightarrow \mathbb{R}, g_1 : [a_2, b_2] \rightarrow \mathbb{R}, f_2 : [a_2, b_2] \rightarrow \mathbb{R}, g_2 : [a_1, b_1] \rightarrow \mathbb{R}$

with $f_1(x) = f(x, a_2), g_1(x) = f(b_1, x), f_2(x) = f(a_1, x), g_2(x) = f(x, b_2)$

If $f(a_1, a_2) \neq f(b_1, b_2)$ we are done. assume the values are not equal. Then construct two continuous functions

$$F_1(x) : [0, (b_1 - a_1) + (b_2 - a_2)] \rightarrow \mathbb{R} = \begin{cases} f_1(x + a_1) & x < (b_1 - a_1) \\ g_1(x - (b_1 - a_1) + a_2) & x \geq (b_1 - a_1) \end{cases}.$$

$$F_2(x) : [0, (b_1 - a_1) + (b_2 - a_2)] \rightarrow \mathbb{R} = \begin{cases} f_2(x + a_2) & x < (b_2 - a_2) \\ g_2(x - (b_2 - a_2) + a_1) & x \geq (b_2 - a_2) \end{cases}.$$

Let $m = (b_1 - a_1) + (b_2 - a_2)$

$$F_1(0) = F_2(0) = f(a_1, a_2), F_1(m) = F_2(m) = f(b_1, b_2).$$

and both F_1 and F_2 are continuous on the entire interval because their components are continuous and the points of their piecewise connection are continuous. So for any c between $f(a_1, a_2), f(b_1, b_2)$ by the intermediate value theorem There exists some x_1, x_2 s.t. $F_1(x_1) = F_2(x_2) = c$. We also have that if $f(x, y) = F_1(x_0), (x, y) \notin \{(a_1, a_2), (b_1, b_2)\}$ then F_2 does not correspond directly to any x, y in our starting interval. This goes to show that the two paths created by F_1, F_2 are disjoint except for their start and endpoints. Therefore x_1, x_2 correspond to two unique points in the original closed interval that have the same value therefore in this first case f is not one-one

In general terms this proof considers the paths taken by the boundary of our initial interval and notes that either they must have some point in common or taking the boundary from the intermediate corners creates a sufficient path.

2. EXERCISE 4.2

Prove that the sequence of functions

$$\sqrt{x}, \sqrt{x + \sqrt{x}}, \sqrt{x + \sqrt{x + \sqrt{x}}}.$$

on $(0, +\infty)$ is convergent and find the limit function.

Let $(f_n)_n$ be the sequence of functions defined by $f_0 = \sqrt{x}$ and $f_{n+1} = \sqrt{x + f_n}$

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and Set $x \in [0, +\infty)$ and we have that $(f_n(x))_n$ is a monotone increasing sequence since $x > 0$.

$$\sqrt{x} < \sqrt{x+c}.$$

for all $c > 0$ which implies that $f_1(x) > f_0(x)$

$$f_{n+1}(x) = \sqrt{x + f_n(x)} > \sqrt{x + f_{n-1}(x)} = f_n(x).$$

And so the sequence $(f_n(x))_n$ is strictly increasing. We also have that

$$\sqrt{x} < \frac{1}{2} + \sqrt{\frac{1}{4} + x}.$$

let $a(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}$ Inductively if since $f_0(x) < a(x)$ assume $f_n(x) < a(x)$ then

$$f_{n+1} = \sqrt{x + f_n(x)} < \sqrt{x + a(x)} = \sqrt{a(x)^2} = a(x).$$

since

$$a(x)^2 = \frac{1}{4} + \frac{1}{4} + x + \frac{2}{2}\sqrt{\frac{1}{4} + x} = x + a(x).$$

Thus the sequence $(f_n(x))_n$ is both strictly increasing and bounded from above by $a(x)$. And so define $g(l) = \sqrt{l+x}$ with our x fixed. $g(l)$ is continuous for

$$\lim_{h \rightarrow 0} (g(l+h) - g(l)) = \lim_{h \rightarrow 0} \sqrt{l+h+x} - \sqrt{l+x} = \lim_{h \rightarrow 0} \frac{(l+h+x) - (l+x)}{\sqrt{l+h+x} + \sqrt{l+x}} = \lim_{h \rightarrow 0} \frac{h}{\text{term} > 0 \forall h} = 0.$$

continuity of g implies that

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} g(f_n(x)) = g(\lim_{n \rightarrow \infty} f_n(x)) = \sqrt{x + \lim_{n \rightarrow \infty} f_n(x)}.$$

Basic algebra setting the limit of $f_n(x)$ constant shows that

$$\lim_{n \rightarrow \infty} f_n(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}.$$

Thus the limit exists for all values $x \in (0, \infty)$. and so the functions converge at all points. For $x = 0$ $\sqrt{0} = 0 = \sqrt{0 + \sqrt{0}}$ and so the limit is equal to 0.

3. EXERCISE 4.4

Let $a, b \in \mathbb{R}, a < b$, and for $n = 1, 2, 3, \dots$ let $f_n : [a, b] \rightarrow \mathbb{R}$ be a monotone nondecreasing function. Prove that if the sequence $(f_n)_n$ converges to f then f is a monotone nondecreasing function. Prove that if we further assume that f is continuous then the convergence is uniform.

(i) We can assume that the sequence $(f_n)_n$ converges then for each $x \in [a, b]$ then $\lim_{n \rightarrow \infty} f_n(x)$ exists and it is $f(x)$. pick any $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ and consider for all $n \in \mathbb{N}$ $f_n(x_2) - f_n(x_1) \geq 0$ thus $\lim_{n \rightarrow \infty} f_n(x_2) - f_n(x_1) \geq 0$ and thus $f(x_2) - f(x_1) \geq 0$. so f is monotone nondecreasing.

(ii) Suppose for contradiction that $(f_n)_n$ does not converge uniformly then $\exists \epsilon > 0$ such that $\forall N > 0 \exists x \in [a, b]$ such that $|f_n(x) - f(x)| > \epsilon$ for some $n > N$ First fix this bad $\epsilon_0 > 0$ and set $\epsilon < \frac{\epsilon_0}{3}$. Then set $A = f(a), B = f(b)$ partition $[A, B]$ into ϵ length segments $[A, A + \epsilon], [A + \epsilon, A + 2\epsilon], \dots, [A + r\epsilon, B]$ also consider the unique set of points that map to the end points of this partition

$$S := \{a, \inf(f^{-1}(A + \epsilon)), \inf(f^{-1}(A + 2\epsilon)), \dots, \inf(f^{-1}(A + r\epsilon)), b\}.$$

S can be constructed because f is continuous and monotone non decreasing so every point between start and end has nonempty preimage. The infimum is used because we will later pick a point above an element and show that it converges uniformly. For each $s \in S$ by convergence there exists some $N_s \in \mathbb{N} > 0$ s.t. $|f_n(s) - f(s)| < \epsilon$ for all $n > N_s$. Call $N(S)$ the set of all such N_s . Now set $N = \max\{N_s\}$. Which satisfies point wise convergence for all points in S . Now let $x \in [a, b]$ be any point such that $|f_n(x) - f(x)| > \epsilon$ for some $n > N$. x must not be in S so pick $s_1, s_2 \in S$ such that $s_1 < x < s_2$ and $f(s_2) < f(s_1) + \epsilon$ (in other words the two "closest" s in S). From the monotone non-decreasing property of each f_n we have that $f_n(s_1) \leq f_n(x) \leq f_n(s_2)$. From construction By the pointwise convergence of each s_i we have

$$f(s_1) - \epsilon \leq f_n(x) \leq f(s_2) + \epsilon.$$

and

$$f(s_1) - \epsilon \leq f_n(x) \leq f(s_1) + 2\epsilon.$$

and since

$$|f(x) - f(s_1)| < \epsilon.$$

by the construction of the set S . we have

$$-2\epsilon \leq (f_n(x) - f(x)) \leq 3\epsilon.$$

and so

$$|f_n(x) - f(x)| \leq 3\epsilon < \epsilon_0.$$

a contradiction for all x which proves that the functions converge uniformly.

4. EXERCISE 4.5

(i) Let E and G be a compact metric spaces and $\phi : E \rightarrow G$ is continuous. Show that the map $F : C(G) \rightarrow C(E)$ defined by $F(g) = g \circ \phi$ for $g \in C(G)$ is uniformly continuous.

Using the metric $d(f, g) = \max\{d(f(p) - g(p)) : p \in E\}$ which is possible because E, G are both compact so the maximum exists. We have that $g \circ \phi$ is continuous. Fix $\epsilon > 0$

$$d(F(g), F(f)) = \max\{d(g(\phi(p)), f(\phi(p))) : p \in E\}.$$

Goal find δ s.t. $d(f, g) = \max\{d(f(p), g(p)) : p \in G\} < \delta$ then $d(F(g), F(f)) < \epsilon$ picking $\delta = \epsilon$ we have that if $d(f, g) < \delta$ then $\max\{d(f(p), g(p)) : p \in G\} < \delta$ and since $\phi : E \rightarrow G$ has only outputs in G we have

$$\max\{d(f(p), g(p)) : p \in G\} \geq \max\{d(g(\phi(p)), f(\phi(p))) : p \in E\} = d(F(g), F(f)).$$

and so

$$d(F(g), F(f)) < \epsilon.$$

5. EXERCISE 4.6

Let E be a compact metric space. Show that $C(E)$ is a complete normed vector space if we add its elements in the usual way, multiply them by real numbers in the usual way, and take $\|f\| = \max\{|f(p)| : p \in E\}$ for all $f \in C(E)$. We must verify all axioms to show that it is a complete normed vector space.

1. associativity

$$f + (g + h) = (f + g) + h \text{ follows by definition of adding functions}$$

2. commutativity

$$f + g = g + f \text{ also follows from definition of functions}$$

3. existence of 0
set $0 = f : E \rightarrow \mathbb{R}$ s.t. $f(p) = 0 \forall p \in G$. clearly $0 + f = f$.
4. inverse elements
set $-f : G \rightarrow \mathbb{R}$ by $-f(p) = -1 * f(p)$ clearly $-f + f = 0$ by adding elements pointwise.
5. scalar multiplication
 $a(bf) = (ab)f$ this is clear from the associativity of multiplication pointwise. This also includes the identity scalar multiplication $1f = f$.
6. distributivity
 $a(f + g) = af + ag$ by defining $af : G \rightarrow \mathbb{R} := af(p) = a \cdot f(p)$ distributivity follows $(a(f + g)) : G \rightarrow \mathbb{R} := a(f(p) + g(p)) = af(p) + ag(p)$ Distributivity of scalar multiplication is similarly trivial.
7. non-negativity of norm
absolute value is non-negative
8. $\|f\| = 0 \Leftrightarrow f = 0$.
This is true for if $\|f\| = 0$ then $f(p) = 0$ for all $p \in E$ and thus $f = 0$.
9. $\|\lambda f\| = |\lambda| \|f\|$ this is true because $|\lambda f(p)| = |\lambda| |f(p)|$ and thus we can factor out the factor of λ from the norm.
10. $\|f + g\| \leq \|f\| + \|g\|$
by compactness first pick $p_0 \in E$ such that

$$\begin{aligned} \|f + g\| &= |(f + g)(p_0)| = |f(p_0) + g(p_0)| \leq |f(p_0)| + |g(p_0)| \\ &\leq \max\{|f(p)| : p \in E\} + \max\{|g(p)| : p \in E\} = \|f\| + \|g\| \end{aligned}$$

We can pick this p_0 because E is compact and f is continuous so $|(f + g)(p)|$ attains its maximum at a point p_0 .

11. Completeness
The vector space is complete if every Cauchy sequence $(f_n)_n$ converges to some $f \in C(E)$. $\|f_n - f_m\| < \epsilon$ for all $n, m < N$ since $(f_n)_n$ is Cauchy. Thus

$$\max\{|(f_n - f_m)(p)| : p \in E\} < \epsilon.$$

this implies that for any $p_0 \in E$ $|f_n(p_0) - f_m(p_0)| < \epsilon$ and so the sequence of points is Cauchy and since \mathbb{R} is complete it converges to a point $s_0 \in \mathbb{R}$ and we can set $f : E \rightarrow \mathbb{R}$ so that $f(p) = s$. The sequence $(f_n)_n$ converges to f and it does so uniformly because for any $\epsilon > 0$ the maximum distance between any f_n with $n > N$ (N dependent on epsilon) the maximum distance between the functions and the convergent function is bounded and so every point is within ϵ from our final function. And from this uniform convergence we have that f is continuous and thus in $C(E)$.

6. EXERCISE 4.7

Which one of the following sequence of functions converge uniformly on $[0, 1]$?

$$(i)f_n(x) = \frac{x}{1 + nx^2}, \quad (ii)g_n(x) = \frac{nx}{1 + nx^2}, \quad (iii)h_n(x) = \frac{nx}{1 + n^2x^2}.$$

$(f_n)_n$ is the only sequence that converges uniformly for any $\epsilon > 0$ set $N > \frac{1}{\epsilon^2}$ then for all $x \in [0, 1]$

$$|f_n(x) - 0| = \frac{x}{1 + nx^2} < \frac{x}{1 + \frac{x^2}{\epsilon^2}} < \epsilon.$$

The final inequality holds because

$$\frac{x}{1 + \frac{x^2}{\epsilon^2}} = \frac{x\epsilon^2}{\epsilon^2 + x^2}.$$

suppose $\frac{x\epsilon^2}{\epsilon^2 + x^2} \geq \epsilon$ this implies that $\frac{x\epsilon}{\epsilon^2 + x^2} \geq 1$ This is a contradiction because $a > 0, b \geq 0 \rightarrow ab < a^2 + b^2$ And so the sequence converges to $f(x) = 0$ within ϵ for all $n > N$ and and all $x \in [0, 1]$ and so it converges uniformly. Additionally since the bounds of x were of no importance other than the equality $x \geq 0$ $(f_n)_n$ converges to 0 on the entirety of $[0, \infty)$

For $g_n(x)$ pick $\epsilon = 1$ and then for any N pick $n > N$ such that $n > 4$ $m = n^2$ and then

$$|g_n(\frac{1}{n}) - g_m(\frac{1}{n})| = |\frac{n(\frac{1}{n})}{1 + n(\frac{1}{n})^2} - \frac{m\frac{1}{n}}{1 + m(\frac{1}{n})^2}| = |\frac{n}{n+1} - \frac{\frac{m}{n}}{1 + \frac{m}{n^2}}|$$

and since

$$1 > \frac{n}{n+1}.$$

and

$$\frac{\frac{m}{n}}{1 + \frac{m}{n^2}} = \frac{n}{2} > 2.$$

we have

$$|\frac{n}{n+1} - \frac{\frac{m}{n}}{1 + \frac{m}{n^2}}| > 1 = \epsilon.$$

And thus there exists some $x \in [0, 1]$ namely $\frac{1}{n}$ that violates the fact that for every $\epsilon > 0$ there must be a N such that the maximum distance between two sequential functions is less than ϵ .

If instead we hold x fixed and $\epsilon > 0$

$$g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

if $x \neq 0$ choose $N > \frac{1-\epsilon x}{\epsilon x^3} \in \mathbb{N}$

$$|g(x) - g_n(x)| = |\frac{1}{x} - \frac{nx}{1 + nx^2}| = \frac{1}{x + nx^3} < \frac{1}{x + \frac{x^3 - x^4\epsilon}{x^3\epsilon}} = \frac{1}{x + \frac{1}{\epsilon} - x} = \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

if $x = 0$ then $g_n(x) = 0 \forall n$ and convergence is trivial. And so $g_n(x)$ converges to $g(x)$

Finally to show that $(h_n)_n$ does not converges uniformly for pick $\epsilon < \frac{1}{10}$ then for all $N > 0$ pick any n and $m = 2n$ then the points

$$|h_n(\frac{1}{n}) - h_m(\frac{1}{n})| = |\frac{n\frac{1}{n}}{1 + (n\frac{1}{n})^2} - \frac{m\frac{1}{n}}{1 + (m\frac{1}{n})^2}| = |\frac{1}{2} - \frac{2}{1+4}| = \frac{1}{10} > \epsilon.$$

By the argument before this sequence does not converge uniformly. Additionally $(h_n)_n$ converges to 0. for given $\epsilon > 0$ set $N > \frac{\epsilon}{x}$ and then

$$|h_n(x)| = \frac{nx}{1 + (nx)^2} < \frac{\epsilon}{1 + (nx)^2} < \epsilon.$$

because the denominator is strictly positive. Therefore every point $x \in [0, \infty)$ converges and thus the function itself converges to 0.

7. EXERCISE 4.8

Assuming you know the general properties of the sin function, discuss the differentiability of (i)

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

(ii)

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

By composition of functions we know that at $x \neq 0$ both (i) and (ii) are differentiable since polynomials are differentiable and $\frac{1}{x}$ is differentiable at $x \neq 0$.

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin(\frac{1}{h})}{h} = \lim_{h \rightarrow 0} \sin(\frac{1}{h}).$$

This limit does not exist because we can take two separate sequences that converge to zero

$$(x_n)_n : x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad \lim_{n \rightarrow \infty} x_n = 0.$$

$$(y_n)_n : y_n = \frac{1}{2\pi n} \quad \lim_{n \rightarrow \infty} y_n = 0.$$

but

$$\sin(\frac{1}{x_n}) = \sin(\frac{\pi}{2} + 2\pi n) = 1.$$

and

$$\sin(\frac{1}{y_n}) = \sin(2\pi n) = 0.$$

and so we constructed two sequences that converge to zero but whose limits when applied to the sine function do not converge to the same value. Thus $f(x)$ is not differentiable at $x = 0$ For $g(x)$ we show

$$\lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h \rightarrow 0} h \sin(\frac{1}{h}) = 0.$$

This limit exists because we have for all h

$$-|h| \leq h \sin(\frac{1}{h}) \leq |h|.$$

since this works for all h we have

$$\lim_{h \rightarrow 0} -|h| \leq \lim_{h \rightarrow 0} h \sin(\frac{1}{h}) \leq \lim_{h \rightarrow 0} |h|.$$

$$0 \leq \lim_{h \rightarrow 0} h \sin(\frac{1}{h}) \leq 0.$$

and so the limit exists and is 0 which is the derivative.

8. EXERCISE 4.9

Suppose that a, b are two real numbers such that $a < b$ and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable at $c \in (a, b)$. for any $r, s \in \mathbb{R}$, compute

$$\lim_{h \rightarrow 0} \frac{f(c + rh) - f(c + sh)}{h}.$$

we have

$$\lim_{h \rightarrow 0} \frac{f(c + rh) - f(c + sh)}{h} = \lim_{h \rightarrow 0} \frac{f(rh + c) - f(c)}{h} - \lim_{h \rightarrow 0} \frac{f(sh + c) - f(c)}{h}.$$

these limits exist because given that r and s are constant and that

$$\lim_{h \rightarrow 0} \frac{f(ah + c) - f(c)}{h} = a \lim_{h \rightarrow 0} \frac{f(ah + c) - f(c)}{ah} = af'(c).$$

the above simplifies then to

$$\lim_{h \rightarrow 0} \frac{f(rh + c) - f(c)}{h} - \lim_{h \rightarrow 0} \frac{f(sh + c) - f(c)}{h} = rf'(c) - sf'(c) = (r - s)f'(c).$$

9. EXERCISE 4.10

Suppose that a, b are real numbers such that $a < b$ and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Prove that the following are equivalent

(i) $f' \geq 0$ on (a, b)

(ii) f is monotone nondecreasing on (a, b) To prove (i) \rightarrow (ii) Consider for contradiction assume that f is not monotone nondecreasing then there exist two points α, β with $a \leq \alpha < \beta \leq b$ s.t. $f(\alpha) > f(\beta)$. By mvt $f(\beta) - f(\alpha) = f'(c)(\beta - \alpha)$ The left hand side is strictly less than zero but the right hand side is greater than or equal to zero and thus we have a contradiction.

To prove (ii) \rightarrow (i) assume monotone nondecreasing and pick any point $c \in (a, b)$

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h}.$$

limits of points approaching from the left and the right must agree since the limit exists. From the left we have

$$\lim_{h \rightarrow 0^-} \frac{f(c + h) - f(c)}{h} \leq 0 \rightarrow \geq 0.$$

$$\lim_{h \rightarrow 0^+} \frac{f(c + h) - f(c)}{h} \geq 0 \rightarrow \geq 0.$$

and since all other possible limit forms must agree with these the final limit must be greater than or equal to zero and thus the derivative is greater than or equal to zero on all of (a, b)

10. EXERCISE 4.12

Let $a, b \in \mathbb{R}$ be such that $a < b$ and let f be a differentiable real valued function on an open subset of \mathbb{R} that contains $[a, b]$. Show that if γ is any real number between $f'(a)$ and $f'(b)$ then there exists a number $c \in (a, b)$ such that $\gamma = f'(c)$ Since γ is strictly between $f'(a)$ and $f'(b)$ we can break the question into two cases the first is $f'(b) < f'(a)$ and $f'(b) < \gamma < f'(a)$. Consider

$$g : [a, b] \rightarrow \mathbb{R} := f(x) - \gamma x.$$

Since $[a, b]$ is a closed set and g is continuous g achieves its maximum on $[a, b]$ by extreme value theorem.

$g'(x) = f'(x) - \gamma$ then clearly g does not achieve a maximum at a or b because

$$g'(a) = f'(a) - \gamma > 0 \quad g'(b) = f'(b) - \gamma < 0.$$

and if either was a maximum then for all $t \in (a, b)$

$$\frac{f(a) - f(t)}{a - t} \leq 0 \quad \frac{f(b) - f(t)}{b - t} \geq 0.$$

which contradicts our statement about the derivative. Thus g achieves its maximum on some $x \in (a, b)$ We proved in class that at a maximum the derivative is equal to zero so $g'(x) = 0$ and

$$0 = f'(x) - \gamma \rightarrow \gamma = f'(x).$$

so $x = c$