HW 1 - 115B

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1 Exercise 1

Assume V and W are vector spaces over a field k, and let $T : V \to W$ denote a linear transformation between them. Prove that if T has an inverse, then that inverse is a linear function.
To prove this statement we first assume that T has an inverse $T' : W \to V$ such that $T' \circ T = id_V$ and $T \circ T' = id_W$ and then show that it must be a linear function.
For T' to be a linear function it must satisfy

$$T'(u + v) = T'(u) + T'(v), u, v \in W$$
$$T'(cu) = cT(u), c \in k, u \in W$$

By invertibility of T for any $u', v' \in W$ there exist *unique* $u, v \in V$ s.t. $u' = T(u), v' = T(v)$
By linearity of T

$$T'(u'+v') = T'(T(u)+T(v)) = T'(T(u+v)) = id_V(u+v) = u+v = T'(T(u))+T'(T(v)) = T'(u')+T'(v').$$

Showing that the first property of linearity holds. Likewise

$$T'(cu') = T'(cT(u)) = T'(T(cu)) = cu = cT'(T(u)) = cT'(u').$$

Thus we have shown that both properties of a linear function hold and thus T' $= T$ inverse is a linear function.

2 Exercise 2

Assume that v_1, v_2, v_3 are vectors in a vector space V over some field k. Is it possible that the vectors v_1, v_2, v_3 are linearly independent but the vectors $w_1 = v_1 + v_2$, $w_2 = v_1 + v_3$, $w_3 = v_2 + v_3$ are linearly dependent
Assume for contradiction that w_1, w_2, w_3 are linearly dependent. Then there exists some $x_1, x_2, x_3 \in k$ not all 0 s.t. $x_1w_1 + x_2w_2 + x_3w_3 = 0$ where 0 is the zero vector in V. expanding this equation gives

$$x_1(v_1 + v_2) + x_2(v_1 + v_3) + x_3(v_2 + v_3) = 0.$$

$$(x_1 + x_2)v_1 + (x_1 + x_3)v_2 + (x_2 + x_3)v_3 = 0.$$

By the linear independence of the $v_i's$ the previous equation implies further that

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

and solving shows

$$x_3 = x_2 = x_1 = 0$$

This is a contradiction since each x_i is 0 even when it was explicitly said that was not the case. Thus the vectors w_1, w_2, w_3 are linearly *independent*

3 Exercise 3

Assume V is a vector space consisting of 49 vectors over a field k.

1. *Prove that the set of elements in k is finite.*
   Assume for the sake of contradiction that k has infinite elements. Since there can only be one zero element of V pick a non-zero element v Consider the set $\{k_1 v, k_2 v, ...k_{50} v\}$ with each k_i different which is possible since there are infinitely many different k. No two elements in this set can be the same for if $k_i v = k_j v$ this implies $(k_i - k_j)v = 0$ which futher implies $k_i = k_j$ since $v \neq 0$. This creates a contradiction since there are only 49 elements in V. Therefore k must be finite and in particular must have no more than 49 elements.

2. *Prove that V is finite dimensionsal and that $dim(V) = 1$ or $dim(V) = 2$*
   Since V has 49 vectors any basis has at most 49 vectors and thus is of finite dimension. Let K be the number of elements in k since k is finite and D be the dimension or number of basis vectors of V. There must be then K^D elements in V since K^D represents counting all possible combinations of basis vectors. and so $K^D = 49^1 = 7^2$ Since this is the unique prime factorizeation there are no other $p_1^{e_1}...p_k^{e_k}$ $p_i, e_i \in \mathbb{N}$ that can represent it by the fundamental theorem of arithmetic. and so either $K = 49, D = 1$ or $K = 7, D = 2$ showing that the dimension is either 1 or 2.

4 Exercise 4

Assume V is a vector space over some field k, and let W_1, W_2 denote subspaces of V. Define

$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

1. *Prove that $W_1 \subset W_1 + W_2$ and $W_2 \subset W_2 + W_1$*
   Firstly by the commutative property of vector addition, $W_1 + W_2 = W_2 + W_1$ by commuting the addition within the set definition. To show $W_1 \subset W_1 + W_2$ for any $w \in W_1$ let $w_2 = 0$ then $w + w_2 = w \in W_1 + W_2$. Similarly picking $w_1 = 0$ and $w_2 = w$ we can show the same for the second relation.

2. *Prove that $W_1 + W_2$ is a subspace of V.*
   To show this we only need to show that $W_1 + W_2$ is closed under addition and scalar multiplication and includes the zero vector.
   since W_1 and W_2 are subspaces they both include the zero vector and $0 + 0 = 0$ so $W_1 + W_2$ contains the zero vector.
   consider arbitrary elements $v, u \in W_1 + w_2$ by definition $v = w_1^1 + w_2^1, u = w_1^2 + w_2^2$

   $$v + u = w_1^1 + w_2^1 + w_1^2 + w_2^2 = (w_1^1 + w_1^2) \in W_1 + (w_2^1 + w_2^2) \in W_2.$$

   and so $u + v \in W_1 + W_2$. Let $c \in k$

   $$cv = c(w_1^1 + w_2^1) = cw_1^1 \in W_1 + cw_2^1 \in W_2.$$

   The last property holding by the fact that W_1 and W_2 are subspaces. Thus $W_1 + W_2$ is a subspace.

3. *Assume that $W \subset V$ is a subspace such that $W_1 \subset W$ and $W_2 \subset W$. Prove $W_1 + W_2 \subset W$*
   Any element $v \in W_1 + W_2$ is of the form $w_1 \in W_1 + w_2 \in W_2$ but by the containment of $W_1 \subset W$ and $W_2 \subset W$ $w_1, w_2 \in W$ and so $w_1 + w_2 \in W$ but subspace definition. Therefore $w_1 + w_2 \in W$ for all w_1, w_2 and so $W_1 + W_2 \in W$

5 Exercise 5

Let k denote some field. For each function f below, determine if f is al inear functional and prove your answer is correct. You may assume standard results from calculus.

1. $V = \mathbb{R}[x], f(p(x)) := 4p'(0) + p''(1)$
   For this and all subsequent questions I will verify that $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(cv) = cf(v)$ with $v_1, v_2, v \in V$ and $c \in k$.

   $$f(p_1(x) + p_2(x)) = 4(p_1 + p_2)'(0) + (p_1 + p_2)''(1) = 4(p_1' + p_2')(0) + (p_1'' + p_2'')(1).$$

   This follows from the linearity of the derivative

   $$= 4p_1'(0) + p_1''(1) + 4p_2'(0) + p_2''(1) = f(p_1) + f(p_2).$$

   $$f(cp) = 4(cp)'(0) + (cp)''(1) = 4cp'(0) + cp''(1) = cf(p).$$

   Again by linearity of derivative. Thus f is a linear functional.

2. $V = k^2$, $f(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$
This is not a linear functional because although the function is linear its codomain is not the field k it is still k^2

3. $V = k^{2\times 2}, f(A) = tr(A)$
This function has a proper codomain $tr(A) = a_{11} + a_{22}$

$$tr(A+B) = (a_{11}+b_{11})+(a_{22}+b_{22}) = (a_{11}+a_{22})+(b_{11}+b_{22}) = tr(A)+tr(B).$$
$$tr(cA) = (ca_{11} + ca_{22}) = c(a_{11} + a_{22}) = ctr(A).$$

Yes this function is a linear functional

4. $V = \mathbb{R}[x], f(p(x) = \int_0^1 p(x)dx$

$$f(p_1 + p_2) = \int_0^1 (p_1 + p_2)dx = \int_0^1 p_1 dx + \int_0^1 p_2 dx = f(p_1) + f(p_2).$$
$$f(cp) = \int_0^1 cpdx = c\int_0^1 pdx = cf(p).$$

Yes this function is a linear functional since evaluation this integral will give values in $\mathbb{R}$.

5. $V = \mathbb{Q}^3, f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x^2 + y^2 + z^2$ No because

$$f(c\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = c^2x^2+c^2y^2+c^2z^2 = c^2(x^2+y^2+z^2) = c^2 f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) \neq cf(\begin{pmatrix} x \\ y \\ z \end{pmatrix}).$$

for any $c \neq 0, 1$ and $x, y, z \neq 0$

6 Exercise 6

Assume that m, n are positive integers, and fix $x_1, ..., x_m \in \mathbb{R}$ s.t. $x_i \neq x_j$ if $i, j \in \{1, ..., m\}$ s.t. $i \neq j$. Let

$$W := \{f \in \mathbb{R}[x]_{\leq n} : f(x_1) = f(x_2) = ... = f(x_m) = 0\}.$$

Where $\mathbb{R}[x]_{\leq n}$, denotes degree less than or equal to n.

1. *Prove that W is a vector space over $\mathbb{R}$*
Since $\mathbb{R}[x]_{\leq n}$ is a vector space and W is a subset of this space it suffices to show that W is closed under vector addition, scalar multiplication and has the zero element. clearly the zero element is part of this set because $0(x) = 0$ for all x if $f_1, f_2 \in W$ then

$$(f_1 + f_2)(x_i) = f_1(x_i) + f_2(x_i) = 0 + 0 = 0.$$

additionally $deg(f_1 + f_2) = \max\{deg(f_1), deg(f_2)\}$ so $f_1 + f_2 \in W$ also if $c \in \mathbb{R}$

$$cf(x_i) = c * 0 = 0.$$

and so $cf \in W$

2. *Compute the dimension of W.*
The rank of $\mathbb{R}_{\leq n}$ is $n + 1$ for the $n + 1$ choices of coefficients of a degree n polynomial. Consider the linear transformation $T : \mathbb{R}_{\leq n} \to \mathbb{R}^m$ defined by $T(f) = (f(x_1), f(x_2), ..., f(x_m))$ the kernel of T is all functions s.t. $f(x_i) = 0$ for all $i \in \{1, ..., m\}$ which is the definition of W. This function is also clearly linear. Applying Rank-Nullity we get that $dim(\mathbb{R}_{\leq n}) = dim(Im(T)) + dim(Ker(T))$ the image of T is the entirety of $\mathbb{R}^m$ because we can construct a polynomial to correspond to any vector in $\mathbb{R}^m$ and so has dimension m Thus:

$$n + 1 = m + dim(W).$$

$$dim(W) = n + 1 - m.$$