HW 3 - 115B

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1 Exercise 1

Let A be a 2×2 diagonalizable matrix. Prove the statement of Cayley-Hamilton theorem directly, using the fact that $A = QDQ^{-1}$ for some invertible $Q \in k^{2\times 2}$ and some diagonal $D \in k^{2\times 2}$ First we show that it is true for D

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$P_D(t) = (\lambda_1 - t)(\lambda_2 - t).$$

$$P_D(D) = (\begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 - \lambda_2 \end{pmatrix})(\begin{pmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This holds for any $\lambda_1, \lambda_2 \in k$ Now we use determinant rules

$$\det(D-tI) = \det(Q(D-tI)Q^{-1}) = \det((QD-Q(tI))Q^{-1}) = \det(QDQ^{-1}-t(QIQ^{-1})).$$

$$= \det(A - tI).$$

So the two matrices have the same characteristic polynomial. Additionally

$$P_D(A) = P_D(QDQ^{-1}) = QP_D(D)Q^{-1} = Q0Q^{-1} = 0.$$

The fact used here is that for each term $a_i t^i$

$$a_i(QDQ^{-1})^i = a_i(QD^iQ^{-1}) = Q(a_iD^i)Q^{-1}.$$

This holds for every element of the polynomial and the Q and Q^{-1} can be factored out by the distributive property. So we have shown that the characteristic polynomial annihilates A

2 Exercise 2 / 3

For each linear endomorphism T on the vector space V find an ordered basis for the T-cyclic subspace generated by the vector $\vec{v}$.
Additionally Compute the characteristic polynomial of $T|_W$

1. $V = \mathbb{R}^4$, $T(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} w+x \\ x-y \\ w+y \\ w+z \end{pmatrix}, \vec{v} = \vec{e_1}$
   Listing out elements generated
$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 0 \\ -3 \\ 3 \\ 3 \end{pmatrix}.$$
   But $\vec{v_4} = -3\vec{v_2} + 3\vec{v_3}$ so only the first three are linearly independent and from the theorem presented in lecture we may stop with the first three vectors that are linearly independent and they form a basis
   Using the basis as described before
$$[T|_W]_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}.$$
   With
$$\det(T_W - \lambda I) = \det(\begin{pmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & -3 \\ 0 & 1 & 3-\lambda \end{pmatrix}.$$
$$= -\lambda(-\lambda(3-\lambda) + 3) = \lambda(-\lambda^2 + 3\lambda - 3).$$

2. $V = \mathbb{R}[x]_{\leq 3},\ T(f(x)) = f''(x), \vec{v} = x^2$ Listing elements
$$\vec{v_1} = x^2, \vec{v_2} = 2, \vec{v_3} = 0.$$
   And so a basis is $\vec{v_1}$ and $\vec{v_2}$
   Using the basis
$$[T|_W]_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
   We get the characteristic polynomial
$$\det(T - \lambda I) = \lambda^2.$$

3. $V = k^{2\times 2},\ T(A) = A^T,\ \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Listing elements
$$\vec{v_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
   This is a linearly dependent set so only the first vector is considered and by itself it forms a basis of the T-cyclic subspace.
   The characteristic polynomial associated with this matrix is
$$P_T(\lambda) = (1 - \lambda).$$

4. $V = k^{2\times 2},\ T(A) = L_{\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}}(A),\ \vec{v} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

   Listing elements

$$\vec{v_1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \vec{v_2} = \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}, \vec{v_3} = \begin{pmatrix}3 & 3\\ 6 & 6\end{pmatrix}.$$

   This set is linearly dependent since $\vec{v_3} = 3\vec{v_2}$ so the first two vectors form a basis Using this basis a matrix representation is as follows

$$[T|_W]_B = \begin{pmatrix}0 & 0\\ 1 & 3\end{pmatrix}.$$

   With characteristic polynomial

$$P_T(\lambda) = (-\lambda)(3-\lambda).$$

3 Exercise 4

Let V and W be non-zero finite dimensional k-vector spaces and let

$$T : V \to W.$$

be a linear transformation

1. *Prove that T is onto if and only if T^* is one-one*
   For the first direction if T^* is one-one assume for contradiction that T is not onto. Then there exists some $w \in W$ s.t. $Tv \neq w_0$ for all $v \in V$. Since W is finite dimensional pick a basis for W $B = \{w_0, w_1, w_2, ..., w_n\}$ with w_0 as before. Consider the element $f \in W^*$ defined on the basis as

$$f(w_i) = \begin{cases}1 & i = 0\\ 0 & i \neq 0\end{cases}.$$

   Then for any $v \in V$,
$$T^*(f)(v) = f(Tv) = 0.$$

   This implies that $T^*(f) = 0 \in V^*$ but this is a contradiction since we assumed that T^* is one-one since $T^*(0) = 0$ as well.
   To prove the second direction, if T is onto assume for contradiction that T^* is not one-one Then there exists some $f_1, f_2 \in W^*,\ \ f_1 \neq f_2$ such that

$$T^*(f_1) = T^*(f_2)$$
$$f_1(Tv) = f_2(Tv)$$

   For all $v \in V$. But since $f_1 \neq f_2$ there exists some $w \in W$ such that $f_1(w) \neq f_2(w)$ and by T onto we have there exists some $v \in V$ such that $T(v) = w$. This is a contradiction because we have two v that agree

2. *Prove that T^* is onto iff T is one-one*
   To prove the first direction, if T is one-one pick a basis $\{v_1, v_2, ..., v_n\}$ for V. Then since T is one-one the set $\{Tv_1, Tv_2, ..., Tv_n\}$ is linearly independent and extend it to a basis $\{Tv_1, ..., Tv_n, w_1, ..., w_m\}$ for W Let $f \in V^*$. THen
   $$f = \alpha_1 v_1^* + ... + a_n v_n^*.$$
   with each $\alpha_i \in k$ where v_i^* is the linearly functional defined on the basis taking v_i to one and all other basis vectors to 0. Then pick $g \in W^*$ such that
   $$g = \alpha_1 (Tv_1)^* + ... + \alpha_n (Tv_n)^*.$$
   Then
   $$T^* g(v_i) = g(Tv_i) = \alpha_i = f(v_i).$$
   and so the linear functionals agree on a basis and thus agree so we have shown that every f is in the form T^*g for some $g \in W^*$.
   For the second direction if T^* is onto assume for contradiction that T is not one-one Then there exists
   $$v_1, v_2 \in V \;\; v_1 \neq v_2 \;\; Tv_1 = Tv_2.$$
   since $v_1 \neq v_2$ there exists some $f \in V^*$ such that $f(v_1) \neq f(v_2)$ but since T^* is onto there exists some $g \in W^*$ such that $T^*g = f$
   $$g(Tv_1) = f(v_1) = \alpha = g(w).$$
   $$g(Tv_2) = f(v_2) = \beta = g(w).$$
   so $g(w) = \alpha$ and $g(w) = \beta$ but $\beta \neq \alpha$ a contradiction thus T is one-one

4 Exercise 5

Fix some $d \in \mathbb{Z}^{\geq 1}$ and some scalars $a_0, ..., a_{d-1} \in k$ let A denote the $d \times d$ matrix

$$\begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 0 & -a_{d-2} \\ 0 & 0 & ... & 1 & -a_{d-1} \end{pmatrix}.$$

Prove that the characteristic polynomial of A is $(-1)^d(a_0 + a_1 t + ... + a_{k-1} t^{d-1} + t^d$
First proving the base case $d = 2$

$$\det(\begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} - tI) = \det(\begin{pmatrix} -t & -a_0 \\ 1 & -a_1 - t \end{pmatrix}) = (-1)^2(a_0 + a_1 t + t^2).$$

Now assume it holds for $d < n$, then for the $n \times n$ matrix

$$\det(\begin{pmatrix} -t & 0 & ... & 0 & -a_0 \\ 1 & -t & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & -t & -a_{n-2} \\ 0 & 0 & ... & 1 & -a_{n-1} - t \end{pmatrix}$$

$$= (-t)(\det(\begin{pmatrix} -t & ... & 0 & -a_1 \\ 1 & ... & 0 & -a_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & ... & -t & -a_{n-2} \\ 0 & ... & 1 & -a_{n-1} - t \end{pmatrix}) + (-1)^{n-1}(-a_0)\det(\begin{pmatrix} 1 & -t & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & -t \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

$$= (-t)((-1)^{n-1})(a_1 + a_2 t + ... + a_{n-1}t^{n-2} + t^{n-1})) + (-1)^n(a_0).$$

$$= (-1)^n(a_0 + a_1 t + ... + a_{n-1}t^{n-1} + t^n.$$

Proving the iterative step, so by induction we have shown the relationship holds.

5 Exercise 6

Let T be a linear endomorphism of a finite dimensional vector space V.

1. *Prove that if the characteristic polynomial of T spl,its, then so does the characteristic polynomial fo the restriction of T to any T-invariant subspace of V.*
   Using the definition that splitting refers to being the product of linear factors it suffices to show that $f_{T|_W} | f_T$ when W is T-invariant. Suppose that W is T invariant and consider any basis $B_1 = \{w_1, w_2, ..., w_n\}$ of W since W is finite dimensional. Extend that basis to $B_2 = \{w_1, w_2, ..., w_n, v_1, ..., v_d\}$ of V again since finite dimensional. Consider the representation

   $$[T]_{B_2} = \begin{pmatrix} [T|_W]_{B_1} & A_1 \\ 0 & A_2 \end{pmatrix}.$$

   where A_1 and A_2 are block matrices. and so

   $$[T - tI]_{B_2} = \begin{pmatrix} [T|_W - tI]_{B_1} & A_1 \\ 0 & A_2 - tI \end{pmatrix}.$$

   And By the rules of taking determinants of block matrics

   $$f_T = f_{T|_W} \det(A_2 - tI).$$

   and so $f_{T|_W}$ divides the characteristic polynomial and in particular if the characteristic polynomial splits so does the restriction to any T invariant subspace.

2. *Deduce that if the characteristic polynomial of T splits, then any nonzero T-invariant subspace of V contains an eigenvector of T*
   Consider the characteristic polynomial of the restriction $f_{T|_W}$ with W being the $T-$invariant subspace. By the previous part of the question, $f_{T|_W}$ splilts and is of positive degree. In particular there exists a factor

$$(\alpha - t).$$

   for some α in the base field. We use the fact that since α is a root of the characteristic polynomial it implies that the transformation $T|_W - \alpha I$ is not invertible and so there exists some $v \in W$ with $v \neq 0$ such that $(T|_W - \alpha I)v = 0$ which is equivalent to $T|_W v = \alpha v$ thus W has an eigenvector of $T|_W$ but by definition $T|_W v = Tv$ since $v \in W$ and so $Tv = \alpha v$ and v is an eigenvector of T

Let T be a linear operator on a finite dimensional vector space V, and let W be a T-invariant subspace of V

1. *suppose that $v_1, v_2, ..., v_d$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $\sum_{i=1}^{d} v_i$ is in W, then $v_i \in W$ for all $i \in \{1, 2, ..., d\}$*
   We prove inductively a stronger argument that if $\sum_{i=1}^{d} \beta_i v_i \in W$ then each $v_i \in W$ with $\beta_i \neq 0$ for all i. For the base case if $d = 1$ the statement is trivially true after dividing the single term by β_1. Assume now that the statement holds for $d = n$ then for $d = n + 1$ we have

$$v = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n + \beta_{n+1} v_{n+1}$$
$$Tv = \alpha_1 \beta_1 v_1 + \alpha_2 \beta_2 v_2 + ... + \alpha_n \beta_n v_n + \alpha_{n+1} \beta_{n+1} v_{n+1}$$

   Consider the element

$$\alpha_{n+1} v - Tv = \beta_1(\alpha_{n+1} - \alpha_1)v_1 + \beta_2(\alpha_{n+1} - \alpha_2)v_2 + ... + \beta_n(\alpha_{n+1} - \alpha_n)v_n + \beta_{n+1}(\alpha_{n+1} - \alpha_{n+1})v_{n+1}.$$

   the coefficient for element v_i is

$$\beta_i(\alpha_{n+1} - \alpha_i).$$

   which is not zero if $i \neq n + 1$ since all eigenvalues are distinct and so this vector

$$w = a_{n+1} v - Tv \in W.$$

   and by the inductive hypothesis this implies that each $v_1, v_2, ..., v_n$ is in W. But since each of these vectors is in W we can isolate the v_{n+1} term in the original vector by the equation

$$v_{n+1} = \frac{1}{\beta_{n+1}}(v - \sum_{i=1}^{n} \beta_i v_i).$$

   and so v_{n+1} is in W proving the inductive step. in particular picking each $\beta_i = 1$ we get the statement of the original question.

2. *Suppose that* $\dim(V) = n$ *and* T *has* n *distinct eigenvalues. Prove that* V *itself is a* T*-cyclic subspace.*
Consider the vector $v = \sum_{i=1}^{n} v_i$ with each v_i and eigenvector which in a previous problem we showed could be found for each distinct eigenvalue and let W be the T-cyclic subspace generated by v. By the previous question each $v_i \in W$. The set $\{v_1, v_2, ..., v_n\}$ is linearly independent because assume for contradiction they were not then without loss of generality let v_1 be a linear combination of the other vectors

$$v_1 = \beta_2 v_2 + \beta_3 v_3 + ... + \beta_n v_n.$$

$$Tv_1 = \beta_2 \alpha_2 v_2 + \beta_3 \alpha_3 v_3 + ... + \beta_n \alpha_n v_n \neq \alpha_1 v_1.$$

since the eigenvalues are distinct. And since W is a subspace of V its dimension is less than or equal to n, however since W has n linearly independent vectors it must have dimension at least n, and so its dimension will be exactly n and since $v_1, v_2, ..., v_n$ are n linearly independent vectors in W they form a basis. and since $\dim(W) = \dim(V)$ $W = T$. So V is a T-cyclic subspace generated by v.

6 Exercise 8

Prove that the restriction of a diagnoalizable linear operator T *to any non-trivial* T*-invariant subspace is also diagonalizable*
Let T be a diagnoalizable linear operator, then there exists some basis where every element is an eigenvector. Assume for contradiction that W is a T-invariant subspace but $T|_W$ is not diagnoalizable, then there is some vector $v \in W$ such that v can not be represented as the sum of linearly independent eigenvectors. Consider the representation $v = \sum_{i \in I} \alpha_i v_i$ of v in V where each v_i is an eigenvector under the transformation T. This is possible because T is diagonalizable. This sum must be finite and so by the previous question, grouping elements of equal eigenvalue there exist vectors corresponding to each eigenvalue in W, the span of which includes v a contradiction and so W is spanned by eigenvectors under T and $T|_W$ is diagonalizable.
In particular if each eigenvalue is distinct we are done and if not for any eigenvalue let $\{w_1, w_2, ..., w_d\}$ be the eigenvectors corresponding to that eigenvalue, if W has any linear combination of such vectors they can be grouped into sums of vectors with the same eigenvalue and whichever subset is in W can be represented as a unique basis element.

7 Exercise 9

Let $A \in k^{n \times n}$ *for some* $n \in \mathbb{Z}^{\geq 0}$*. Prove that* $\dim(\mathit{span}\{I_n, A, A^2, A^3, ...\}) \leq n$
consider the characteristic polynomial $f_A(t)$. $\deg(f_A) \leq n$ in particular if

$$f_A(t) = a_0 + a_1 t + ... + (-1)^n t^n.$$

then

$$f_A(A) = a_0 I_n + a_1 A + a_2 A^2 + ... + (-1)^n A^n = 0.$$

so A^n is in the span of $\{I_n, A, A^2, ..., A^{n-1}\}$. And so if $T : k^{n\times n} \to k^{n\times n}$ defined by left multiplication by A the set described is a T-cyclic subspace generated by I_n and so by identical argument as described in class it contains no more than n elements since A^n is a linear combination of the other elements. Thus the dimension of the span is less than or equal to n.