

S_3 : Why $\{e, T_{12}\}$ is not a normal subgroup

$$T_{12}(1)=2 \quad T_{12}(2)=1, \quad T_{12}(3)=3$$

$H = \{e, T_{12}\}$ is a subgroup.

Is it normal?

Have to look at $g^{-1}T_{12}g$ $g \in S_3$ and

see if $g^{-1}T_{12}g \in H$.

Try $g = T_{13} \quad (T_{13}(1)=3, T_{13}(2)=2, T_{13}(3)=1)$

Then if $T_{13}^{-1} T_{12} T_{13} \in H$ $T_{13}^{-1} = T_{13}$

either it $= e$ or $= T_{12}$.

Now $T_{13}^{-1} T_{12} T_{13} = e$ would imply

$$T_{12} T_{13} = T_{13} \quad \text{but that is not true}$$

$$(1,2,3) \rightarrow 2,1,3 \rightarrow 2,3,1 \neq e \quad (\text{or } T_{12}T_{13} = T_{13} \Rightarrow T_{12} = e)$$

Can $T_{13}^{-1} T_{12} T_{13} = T_{12}$? ~~no~~

then $T_{12}T_{13} = T_{13}T_{12}$

Is this true

$$(1, 2, 3) \xrightarrow{T_{12}} (2, 1, 3) \xrightarrow{T_{13}} (2, 3, 1)$$

$$(1, 2, 3) \xrightarrow{T_{13}} (3, 2, 1) \xrightarrow{T_{12}} (3, 1, 2)$$

not the same!

So $H = \{e, T_{12}\}$ is not normal.

How about $\langle T_{12}, T_{13} \rangle$

$$(e, (1\ 2\ 3), (1\ 3\ 2)) = A$$

(which is a subgroup.)

$$T_{12}^{-1} (2\ 3\ 1) T_{12} = (3\ 1\ 2) \checkmark \in A$$

$$\begin{aligned} 1 &\rightarrow 2 \rightarrow 3 \rightarrow 3 \\ 2 &\rightarrow 1 \rightarrow 2 \rightarrow 1 \\ 3 &\rightarrow 3 \rightarrow 1 \rightarrow 2 \end{aligned}$$

Other cases similar.

So H is normal.

$$T_{12}^{-1} (3\ 1\ 2) T_{12} = (2\ 3\ 1)$$

What is S_3/A ?

Cosets of A :

$$A \quad T_{12}A = T_{12}, T_{13}, T_{23}$$

and

Since the two cosets are disjoint
(or check directly)

$$T_{12}T_{13} \\ = (131)$$

two cosets A

$$\text{and } B = T_{12}, T_{13}, T_{23}$$

what is A ? It is the identity in

S^3/A . What is B^2

$$= [T_{12}^2] = e$$

$$= [(132)^2]$$

$$(132)(132) = (123)$$

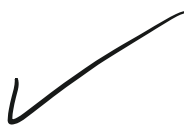
So $S_3/A = \{e, [T_{12}]\}$
two element group.

Note: This works, e.g.,

$$[T_{12}][T_{23}] \text{ should be } [e]$$

while

$$\begin{array}{ccc} 1 & 2 & 3 \\ 6 & 2 & 1 \\ 1 & 3 & 2 \end{array} \in A$$



General (hard) fact

A_4 has a normal subgroup

by A_n $n \geq 5$ has no normal subgroup.

(Later)

So infinite family of "simple groups"
(no normal subgroup ^(except e, whole group) means
by def. group is simple.)

Lots of simple groups. It is known
what they all are (really hard)