S_3: Why $\{e, T_{12}\}$ is not a normal subgroup

$T_{12}(1) = 2 \quad T_{12}(2) = 1, \ T_{12}(3) = 3$

$H = \{e, T_{12}\}$ is a subgroup.

Is it normal?

Have to look at $g^{-1} T_{12}\, g \quad g \in S_3$ and

see if $\quad g^{-1} T_{12}\, g \in H$.

Try $\quad g = T_{13} \quad (T_{13}(1) = 3, T_{13}(2) = 2, T_{13}(3) = 1)$

Then if $\quad T_{13}^{-1} T_{12} T_{13} \in H \quad T_{13}^{-1} = T_{13}$

either it $= e$ or $= T_{12}$.

Now $\quad T_{13}^{-1} T_{12}\, T_{13} = e$ would imply

$T_{12} T_{13} = T_{13}$ but that is not true

$(1,2,3) \to 2\,1\,3 \to 2\,3\,1 \neq e \quad$ (or $T_{12}T_{13} = T_{13} \Rightarrow T_{12} = e$)

Can $\quad T_{13}^{-1} T_{12}\, T_{13} = T_{12}$? If so then $T_{12}T_{13} = T_{13}T_{12}$

Is this true

$(1,2,3) \xrightarrow{T_{12}} (2\ 1\ 3) \xrightarrow{T_{13}} (2\ 3\ 1)$

$\ne$

$(1,2,3) \xrightarrow{T_{13}} (3\ 2\ 1) \xrightarrow{T_{12}} (3\ 1\ 2)$

not the same!

So $H = \{e, T_{12}\}$ is not normal.

How about $(1,2,3) \to (2\ 3\ 1)$

$$\left(e,\ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix}\right) = A$$

(which is a subgroup.

$T_{12}^{-1}\ (2\ 3\ 1)\ T_{12} = (3\ 1\ 2)$ ✓ $\in A$

$1 \to 2 \to 3 \to 3$

$2 \to 1 \to 2 \to 1$

$3 \to 3 \to 1 \to 2$

Other cases similar.

So H is normal.

$T_{12}^{-1}\ (3\ 1\ 2)\ T_{12} = (2\ 3\ 1)$

What is S_3/A?

Cosets of A: A

and $T_{12}A = T_{12}, T_{13}, T_{23}$

Since the two cosets are disjoint

$T_{12}T_{13}$

$= (2\,3\,1)$

(or check directly)

two cosets A

and $B = T_{12}, T_{13}, T_{23}$

what is A? It is the identity in

S^3/A. What is B^2

$= [T_{12}^2] = e$

$= [(132)^2]$

$(1\,3\,2)(1\,3\,2) = (1\,2\,3)$

So $S_3/A = \{e, [T_{12}]\}$

two element group.

Note: This works, e.g.

$[T_{12}][T_{23}]$ should be $[e]$

while

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow 2 & 1 & 3 \\ \downarrow 3 & 1 & 2 \end{array} \in A$$

✓

General (hard) fact

A_4 has a normal subgroup

by A_n $n \geq 5$ has no normal subgroup.

(Later)

So infinite family of "simple groups"

(no normal subgroup (except e, whole group) means by def. group is simple.)

Lots of simple groups. It is known what they all are (really hard)