Hw 3 135 Asher Christian

58 - 6

Solve $y' + 4y + 5\int_0^x y\,dx = e^{-x}, \quad y(0) = 0$

$$pL[y] - y(0) + 4L[y] + \frac{5L[y]}{p} = \frac{1}{p+1}$$

$$L[\int_0^x y\,dx]$$

$$g(x) = \int_0^x y\,dx$$

$$L[g'(x)] = pL[g(x)] - \overset{0}{g(0)} = L[y(x)]$$

$$L[g(x)] = \frac{L[y(x)]}{p}$$

$$L[y]\left(p + 4 + \frac{5}{p}\right) = \frac{1}{p+1}$$

$$L[y] = \frac{p}{(p^2+4p+5)(p+1)}$$

$$p = (Ap+B)(p+1) + C(p^2+4p+5)$$

$$= Ap^2 + Ap + Bp + B + Cp^2 + C4p + C5$$

$$= (A+C)p^2 + (4C+A+B)p + (B+5C)$$

$A = -C$

$B = -5C$

$$4C - C - 5C = 1$$

$$C = -\frac{1}{2}$$

$$L[y] = \frac{1}{2}\left(\frac{p+5}{p^2+4p+5} - \frac{1}{p+1}\right)$$

$$\left(\frac{p+5}{p^2+4p+5} = \frac{p+2+3}{(p+2)^2+1} = \frac{p+2}{(p+2)^2+1} + \frac{3}{(p+2)^2+1}\right.$$

$$y = \frac{1}{2}\left(e^{-2x}(\cos x + 3\sin x) - e^{-x}\right)$$

51

3

b $\quad xy'' + (2x+3)y' + (x+3)y = 3e^{-x} \quad y(0)=0$

$$= -\frac{d}{dp}[p^2Y] - 2\frac{d}{dp}[pY] + 3pY - \frac{d}{dp}Y + 3Y = \frac{3}{p+1}$$

$$= -(2pY) - \left(p^2\frac{dY}{dp}\right) - 2Y - 2p\frac{dY}{dp} + 3pY - \frac{dY}{dp} + 3Y = \frac{3}{p+1}$$

$$-\frac{dY}{dp}(p^2+2p+1) - Y(2p+2-3p-3) = \frac{3}{p+1}$$

$$-\frac{dY}{dp}(p^2+2p+1) + Y(p+1) = \frac{3}{p+1}$$

$$-\frac{dY}{dp}(p+1)^2 + Y(p+1) = \frac{3}{p+1}$$

$$Y' - \frac{Y}{p+1} = -\frac{3}{(p+1)^3}$$

$$\mu(p) = e^{\int^p -\frac{1}{t+1}\,dt} = e^{-\ln(p+1)} = \frac{1}{p+1}$$

$$\left(\frac{Y}{p+1}\right)' = -\frac{3}{(p+1)^4}$$

$$\int_p \frac{3}{(p+1)^4}\,dp = \frac{1}{(p+1)^3}$$

$$\frac{Y}{p+1} = \frac{1}{(p+1)^3}$$

$$Y = \frac{1}{(p+1)^2}$$

$$y = xe^{-x}$$

5a

$$L\left[\frac{f(x)}{x}\right] = \int_0^\infty e^{-px}\frac{f(x)}{x}\,dx = \int_p^\infty F(p)\,dp \qquad p=0$$

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_0^\infty \frac{1}{p+a} - \frac{1}{p+b}\,dp$$

$$= \ln(p+a) - \ln(p+b)\Big|_0^\infty$$

$$= \ln\left(\frac{p+a}{p+b}\right)\Big|_0^\infty = -\ln\left(\frac{a}{b}\right)$$

8 a, b

f(x) periodic w/ period a

$f(x+a) = f(x)$

$$F(p) = \frac{1}{1-e^{-ap}} \int_0^a e^{-px} f(x)\, dx$$

$x = y + a$

$y = x - a$

$dx = dy$

$$\int_0^\infty e^{-px} f(x)\, dx = \int_0^a e^{-px} f(x)\, dx + \int_a^{2a} e^{-px} f(x)\, dx + \ldots$$

$$= \int_0^a e^{-px} f(x) + \int_0^a e^{-p(y+a)} f(x)\, dy \ldots$$

$$= \int_0^a e^{-px} f(x)\, dx + e^{-pa} \int_0^a e^{-px} f(x)\, dx + e^{-2pa} \int$$

$$= \sum_{i=0}^{\infty} e^{-pia} \int_0^a e^{-px} f(x)\, dx$$

$$S = \sum_{i=0}^{\infty} \left(e^{-pa}\right)^i$$

$$S e^{-pa} = \sum_{i=1}^{\infty} (e^{-pa})^i$$

$$S(1 - e^{-pa}) = 1$$

$$S = \frac{1}{1 - e^{-pa}}$$

$$= \frac{1}{1 - e^{-pa}} \int_0^a e^{-px} f(x)\, dx$$

b) $$F(p) = \frac{1}{1-e^{-2p}} \int_0^2 e^{-px} f(x)\, dx$$

$$= \frac{1}{1-e^{-2p}} \left(-\frac{e^{-px}}{p} \Big|_0^1 \right)$$

$$= \frac{1}{1-e^{-2p}} \left(\frac{-e^{-p}}{p} + \frac{1}{p} \right)$$

$$= \frac{1-e^{-p}}{(1-e^{-2p})\, p}$$

52

1 Find $L^{-1}\left[\frac{1}{p^2+a^2}\cdot\frac{1}{p^2+a^2}\right]$ by convolution

$$L^{-1}\left[\frac{1}{a^2}L[\sin ax]\,L[\sin ax]\right]$$

$$= \frac{1}{a^2}\int_0^x \sin(at)\,\sin(a(x-t))\,dt$$

$$= -\frac{1}{a^2}\int_0^x \sin(at-ax)\,\sin(at)\,dt$$

$$= -\frac{1}{a^2}\int_0^x \frac{\cos(ax)}{2}-\frac{\cos(2at-ax)}{2}\,dt$$

$$= -\frac{1}{a^2}\left(\frac{\cos(ax)x}{2}-\frac{\sin(2at-ax)}{4a}\Big|_0^x\right)$$

$$= \frac{1}{a^2}\left(\frac{\sin(ax)}{2a}-\frac{\cos(ax)x}{2}\right)$$

52

2 (a, c)

a $$y(x) = 1 - \int_0^x (x-t)\,y(t)\,dt$$

$$L[y] = \frac{1}{p} - \frac{1}{p^2}L[y]$$

$$L[y]\left(1+\frac{1}{p^2}\right) = \frac{1}{p}$$

$$L[y] = \frac{1}{p\left(1+\frac{1}{p^2}\right)}$$

$$= \frac{p^2}{p(p^2+1)}$$

$$= \frac{p}{p^2+1}$$

$$= \cos x$$

c $$e^{-x} = y(x) + 2\int_0^x \cos(x-t)\, y(t)\, dt$$

$$\frac{1}{p+1} = Y + 2\left(\frac{p}{p^2+1}\right)(Y)$$

$$= Y\left(1 + 2\left(\frac{p}{p^2+1}\right)\right)$$

$$Y = \frac{1}{(p+1)\left(1+\frac{2p}{p^2+1}\right)}$$

$$= \frac{p^2+1}{(p+1)(p^2+1+2p)} = \frac{p^2+1}{(p+1)^3}$$

$$= \frac{(p+1)^2}{(p+1)^3} - \frac{2(p+1)}{(p+1)^3} + \frac{2}{(p+1)^3}$$

$$= \frac{1}{(p+1)} - \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3}$$

$$y = e^{-x} - 2e^{-x}x + e^{-x}(x^2)$$

$$= e^{-x}(x-1)^2$$

52-5

Show that $y = \frac{1}{a}\int_0^x f(t)\sin(a(x-t))\,dt$

solves $y'' + a^2 y = f(x)$

$$p^2 \mathcal{L}[y] + a^2 \mathcal{L}[y] = \mathcal{L}[f(x)]$$

$$\mathcal{L}[y] = \frac{1}{a}\mathcal{L}[f(x)]\frac{a}{p^2+a^2}$$

$$\cancel{(p^2+a^2)}\left(\frac{1}{\cancel{a}}\mathcal{L}[f(x)\frac{\cancel{a}}{\cancel{p^2+a^2}}\right)$$

$$\mathcal{L}[f(x)] = \mathcal{L}[f(x)]$$

true

53 - 2 (b)

$$e^{at} * e^{bt}$$

$$\int_0^t e^{a\tau} e^{b(t-\tau)} \, d\tau$$

$$= e^{bt} \int_0^t e^{a\tau} e^{-b\tau} \, d\tau$$

$$= e^{bt} \int_0^t e^{\tau(a-b)} \, d\tau$$

$$= e^{bt} \left. \frac{e^{\tau(a-b)}}{a-b} \right|_0^t$$

$$= \frac{e^{(a-b)t}}{a-b} e^{bt} - \frac{e^{bt}}{a-b}$$

$$= \frac{e^{at} - e^{bt}}{}$$

a-b

4c

$$y'' - y' = t^2 \qquad y(0) = y'(0) = 0$$

$$L[A(t)] = \frac{1}{p} \cdot \frac{1}{p^2 - p} = \frac{1}{p^2(p-1)}$$

$$1 = A(p-1) + Bp^2 + Cp(p-1)$$

$$= p^2(B+C) + p(A-C) - A$$

$C = A$

$A = -1$

$B = 1$

$$= \frac{1}{p-1} - \frac{1}{p} - \frac{1}{p^2}$$

$$A(t) = e^x - 1 - x$$

$f(t) = t^2$

$f'(t) = 2t$

$f(0) = 0$

$$y(t) = \int_0^t \left[e^{t-\tau} - 1 - (t-\tau)\right] 2\tau \, d\tau$$

$$= 2e^t \int_0^t \tau e^{-\tau} - 2\int_0^t \tau - 2t\int_0^t \tau + 2\int_0^t \tau^2$$

$$= 2e^t \left[-te^{-t} - e^{-t} + 1\right] - t^2 - t^3 + \frac{2}{3}t^3$$

$$= -\frac{1}{3}x^3 - x^2 - 2x - 2 + 2e^x$$

$$h(t) = L^{-1}\left(\frac{1}{p(p-1)}\right) = -1 + e^x$$

$$y(t) = \int_0^t (-1 + e^{(t-\tau)}) \tau^2 \, d\tau$$

$$= -\frac{1}{3}t^3 - t^2 - 2t - 2 + 2e^t$$

S3-8 ccs

$$L\frac{dI}{dt} + RI = E(t) \quad I(0) = 0$$

$$E(t) = E_0 \sin wt$$

$$LI' + RI = E_0 \sin \omega t$$

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{Lp+R}\right]$$

$$= \frac{1}{L} e^{-\frac{R}{L}t}$$

$$y(t) = \int_0^t \left(\frac{1}{L} e^{-\frac{R}{L}(t-\tau)}\right)\left(E_0 \sin \omega \tau\right) d\tau$$

$$= \frac{E_0}{L} e^{-\frac{R}{L}t} \int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau \, d\tau$$

$$\int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau \, d\tau = \frac{L}{R} e^{\frac{R}{L}t} \sin \omega t - \frac{L\omega}{R}\int_0^t e^{\frac{R}{L}t} \cos \omega t \, dt$$

$U = \sin \omega t$

$du = \omega \cos \omega t$

$dv = e^{\frac{R}{L}t}$

$v = \frac{L}{R} e^{\frac{R}{L}t}$

$$\int_0^t e^{\frac{R}{L}t} \cos \omega t \, dt = \frac{L}{R} e^{\frac{R}{L}t} \cos \omega t + \frac{L\omega}{R} \int_0^t e^{\frac{R}{L}t} \sin \omega t \, dt$$

$$\int_0^t e^{\frac{R}{L}\tau} \sin w\tau \, d\tau = \frac{L}{R} e^{\frac{R}{L}t} \sin wt - \frac{Lw}{R}\left(\frac{L}{R} e^{\frac{R}{L}t} \cos wt + \frac{Lw}{R} \int_0^t e^{\frac{R}{L}t} \sin wt \, dt \right)$$

$$J = \frac{L}{R} e^{\frac{R}{L}t} \sin wt + \frac{L^2 w}{R^2} e^{\frac{R}{L}t} \cos wt + \frac{L^2 w^2}{R^2} J$$

$$J\left(1 + \frac{L^2 w^2}{R^2}\right) =$$

$$J = \frac{\frac{L}{R} e^{\frac{R}{L}t} \sin wt - \frac{L^2 w}{R^2} e^{\frac{R}{L}t} \cos wt}{1 + \frac{L^2 w^2}{R^2}} + \frac{\frac{L^2 w}{R^2}}{1 + \frac{L^2 w^2}{R^2}}$$

$$I(t) = \frac{E_0}{R} \left(\frac{\left(\sin wx - \frac{Lw}{R} \cos wx + \frac{Lw}{R} e^{-\frac{R}{L}x} \right.}{1 + \frac{L^2 w^2}{R^2}} \right)$$