HW 1 - 110AH

Asher Christian 006-150-286

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1 Problem 1

The order of a finite group = the number of elements in the group. Find all the groups of order 4.

1. $\mathbb{Z}/4\mathbb{Z},+ = \{0, 1, 2, 3\}$

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. other options = $\{e, a, b, c\}$ suppose $a \times a \times a = e$. $a \neq a$, $a \neq e$, $a = b$ or $a = c$. Let $a \times a = b$ then we have $b \times a = a \times b = e$ and $b \times b = a \times a \times a \times a = a \times e = a$

	e	a	b	c
e	e	a	b	c
a	a	b	e	
b	b	e	a	
c	c			

since in any group if $ab = ac \Rightarrow b = c$, $ac = c \Rightarrow a = e$ a contradiction. Therefore it is impossible for any element $g \neq e$ with $g \times g \times g = e$
Now we have covered cases where $a \times a \times a = e$ and cases where $a \times b = 0, b \neq a \times a$ The other case is $a \times a = e$ and $b \times b = e$ for $b \neq a$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We have $a \times a = b \times b = c \times c = e$

2 problem 2

Find the greatest common divisor of 2124 and 1024, systematically

$$\begin{aligned} 2124 &= 1024(2) + 76 \\ 1024 &= 76(13) + 36 \\ 76 &= 36(2) + 4 \\ 36 &= 4(9) + 0 \end{aligned}$$

gcd(2124,1024) $= 4$

3 problem 3

Show that the gcd of 111 and 113 is 1 and find $n, m \in \mathbb{Z} \to 111m + 113n = 1$

$$\begin{aligned} 113 &= 111(1) + 2 \\ 111 &= 2(55) + 1 \\ 2 &= 2(1) + 0 \end{aligned}$$

gcd(111,113) $= 1$

$$\begin{aligned} 1 &= 111 - 2(55) \\ &= 111 - (113 - 111)(55) \\ &= 111(56) - 113(55) \end{aligned}$$

$m = 56, n = -55$

4 Problem 4

Assume (as we shall prove later) that if $g \in G$ and $n =$ the smallest positive integer such that $g^n = e$ then $n|$ord(G) where ord(G) = the order of G = number of elements in G. Prove:

1. (a) If G is a finite group, $g \in G$ then $\exists N_g > 0 \to g^{N_g} = e$ so there is smallest such N_g notation ord(g)

2. (b) use the given assumption to show that if ord(G) $= p$, p prime then G is the same group as $\mathbb{Z}/\mathbb{Z}_p$

3. Proof (a)
   assume false for contradiction. Then consider $n =$ ord(G) $= \#$ of elements in G and the corresponding and the list of n+1 elements $g, g^2, ..., g^n, g^{n+1}$. The list must then contain no duplicates for if
   $g^i = g^j$ with $i, j \in [1, n+1] i < j$

Then $g^{j-i}g^i = eg^i \to g^{j-i} = e$ a contradiction. However there are n+1 elements in the list and only n elements in the group so by pigeonhole principle 2 elements must be the same yielding a contradiction and proving our assumption that there is no N_g false. Therefore one of the $g^2, g^3, ..., g^{n+1}$ is g and take the power that is associated with the least satisfying that property to be N_g

4. Proof (b) By the assumption any $g \in G$ with $g^n = e$ take n minimal with this quality n | ord(G) and $n \leq$ ord(G) However ord(G) is prime so n = 1 or n = p. if n = 1 then $g = e$ so consider n = p. then the group is spanned by $\{e, g, g^2, ..., g^{p-1}\}$ Take the power to represent the element so the set becomes $\{0, 1, 2, ..., p-1\}$ and multiplying elements means adding powers just like adding numbers in the $\mathbb{Z}/\mathbb{Z}_p$ Set so they are isomorphic.

5 Problem 5

Let S_3 = the group of 1-1 functions from $1, 2, 3$ to itself.

1. (a) show S_3 is a group when $\times$ = composition of functions (on the right) $(f \times g)(x) = g(f(x))$
2. (b) What is the order of S_3?
3. (c) Is S_3 the same group (except for notation) as $\mathbb{Z}/$ ord$(S_3)\mathbb{Z}$?
4. Proof (a). Assuming all functions are 1-1 onto otherwise inverse functions would be impossible. $e(f(x)) = f(e(x)) = f(x)$ where e is the function that maps by the following ordered pairs $(1, 1), (2, 2), (3, 3)$ for any function (1, a), (2, b), (3,c) $a, b, c \in \{1, 2, 3\} a \neq b \neq c$ the corresponding function (a,1), (b,2), (c,3) is the inverse. for any series of functions a,b,c a(bc) = c(b(a(x))) = (ab)c. Therefor the set is a group under composition
5. (b) there are 6 elements in the group - the first element can be mapped to 3 elements, the second to 2 and the third to 1, multiply together to get 6 total options
6. proof (c). if S_3 is the same as $\mathbb{Z}/\mathbb{Z}_{ord(S_e)}$ then for each element $g \in S_3 \to g^6 = e$ and one element spans the entire set. Let (a,b,c) denote a function that maps 1-a, 2-b, 3-c

$$\begin{cases} (1,2,3) & g = e \\ (2,1,3) & g^2 = e \\ (3,2,1) & g^2 = e \\ (1,3,2) & g^2 = e \\ (2,3,1) & g^3 = e \\ (3,1,2) & g^3 = e \end{cases}.$$

There is no element that spans the set so the set is not isomorphic to $\mathbb{Z}/\mathbb{Z}_{ord(S_3)}$

6 Problem 6

Suppose $N \in \mathbb{Z}^+$. Prove:

1. There exists only finitely many finite groups $G \to ord(G) = N$.(Regarding G_1 and G_2 as the same if they are "isomorphic"

2. Proof
   Consider $g_1, g_2, g_3 \in G$ unique elements. if $g_1g_2 = g_1g_3 \Rightarrow g_1^{-1}g_1g_2 = g_1^{-1}g_1g_2 \Rightarrow g_2 = g_3$. Take the ordering $a_m = (x_1, x_2, x_3, ...x_{n-1}), m < n, x_i \in G, 1 \leq i < n$ For any group a_1 has $(n-1)!$ possible combinations and every subsequent a_i has $(n-i)!$ options. So the total combinations is equal to $\sum_{i=1}^{n-1}(n-i)! \neq \infty$

7 Problem 7

Look at $(N+1)^3 - N^3 = 3N^2 + 3N + 1$ and sum the LHS from $N = 1 to N = n$ use the fact you know $\sum_1^n N$ and $\sum_1^n 1$ to figure out what $\sum_1^n N^2$ is!
We know $\sum_{N=1}^n 1 = N$ and $\sum_{N=1}^n N = \frac{n(n+1)}{2}$
Assume $\sum_{N=1}^n N^3$ follows a polynomial of degree 3

$$a(n+1)^3 + b(n+1)^2 + c(n+1) + d = an^3 + b^2 + cn + d + (n+1)^2$$

$an^3 + 3an^2 + 3an + a + bn^2 + 2bn + b + cn + c + d =$
$an^3 + (3a+b)n^2 + (3a+2b+c)n + (a+b+c+d) = an^3 + (b+1)n^2 + (c+2)n + (d+1)$

$$a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}.$$

$$\sum_{N=1}^{n} N^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

$$\sum_{N=1}^{n} 3N^2 + 3N + 1 = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n + \frac{3}{2}n^2 + \frac{3}{2}n + n = n^3 + 3n^2 + 3n.$$

Note - I was not very smart and I did not realize how plainly it was put. Alow me to try again.

Consider the sequence $\sum_{N=1}^n N^3$, $1, 2^3, 3^3, 4^3, ..., n^3$. and $\sum_{N=1}^n (N+1)^3 = 2^3, 3^3, 4^3, 5^3, ..., n^3, (n+1)^3$. Subtracting the larger sum from the smaller we get.
$(n+1)^3 - 1 = \sum_{N=1}^n (N+1)^3 - N^3$

$$(n+1)^3 - 1 = 3\sum_{N=1}^{n} N^2 + \frac{3n(n+1)}{2} + n$$
$$n^3 + 3n^2 + 3n + 1 - 1 - n - \frac{3}{2}n^2 - \frac{3}{2}n = 3\sum_{N=1}^{n} N^2$$
$$\frac{n^3}{3} + \frac{1}{2}n^2 + \frac{1}{6}n = \sum_{N=1}^{n} N^2$$

The same result obtained before.

8 Problem 8

Can you do this process on prob 4 for higher powers (inductively on the power)? How does it work for 3rd powers?
Yes. Assume the formula is known for $\sum_{N=1}^{n} N^a$ for some $a \in \mathbb{N}$. To determine $\sum_{N=1}^{n} N^{a+1}$ simply consider $\sum_{N=1}^{n}(N+1)^{a+2} - N^{a+2} = (n+1)^{a+2} - 1 =$ sum of powers less $a+2$. Every power less than $a+1$ can be written in temrs of n and the remaining $\sum_{N=1}^{n} N^{a+1}$ can be expressed in terms of powers of n which is the end goal.