Sample Problems for Final Math 110AH

1. If in a finite group G, two subgroups H and K with the property that $\gcd(|H|,|K|)=1$ is it necessarily true that $|H\cap K|=1$.

2. Discuss how A_3 (= even permutations of 3 elements) illustrates the Sylow Theorems by finding the Sylow subgroups, checking which ones are conjugate and how the numbers of them are $kp+1$ etc.

3 Write down the formula for $\#X$ for a set X in terms of $\sum_{\text{orbits}} \#\text{orbit}$ and the formula for $\#$ orbit in terms of $|G|$ and $|$some stabilizer$|$ etc.
And prove the formula.

4. If $|G|=p^s$, p prime, $s\geq 1$, is

|center of G| > 1 necessarily?

Prove your answer.

5. (a) If H, K are normal subgroups of a finite group G then is $HK \stackrel{\text{def}}{=} \{hk : h \in H, k \in K\}$ ... a subgroup?

(b) Is HK necessarily normal in G?

(c) Explain why # of elements in

$$HK = |H||K| / |H \cap K|$$

6. Prove (without using general decomposition results) that a finite abelian group G is isomorphic to the direct product of its p-Sylow subgroups where p ranges over the prime divisors of |G|.

7. Discuss why if $P_1(x)$, $P_2(x)$ are polynomials over a field F,

then $\exists$ $Q_1(x)$ and $Q_2(x)$ $\ni$

(1) $Q_1P_1 + Q_2P_2$ has the properties

$$Q_1P_1 + Q_2P_2 \mid P_1 \text{ and } \mid P_2.$$

(2) If $R(x) \mid P_1(x)$ and $R(x) \mid P_2(x)$

then $Q_1P_1 + Q_2P_2 \mid R(x)$.

8. Illustrate prob 7 for $P_1(x) = x^2$, $P_2(x) = (x-1)^2$.

9. Use prob 7 to prove that if the minimal polynomial of $T: \mathbb{C}^n \to \mathbb{C}^n$ is $x^2(x-1)^2$ then

$\mathbb{C}^n \cong V_0^n \oplus V_1^n$ where $V_0 =$ generalized eigenspace of 0 and $V_1 =$ generalized eigenspace of 1.

10. Give an example of a linear transformation $T: \mathbb{C}^2 \to \mathbb{C}^2$ such that 1 is an eigenvalue and the generalized eigenspace of $1 = \mathbb{C}^2$ but $\{v \in \mathbb{C}^2 : Tv = v\}$ has dimension 1.

11. If A is an $n\times n$ $\mathbb{R}$-valued matrix prove $\det e^A = e^{\operatorname{tr} A}$, Prove this.

12. Prove: If A is an $n\times n$ matrix that is close enough to I_n then $\exists B$ $\ni$ $e^B = A$. (part of this is to explain what "close enough" means!)
Hint: Use $\ln(1+x)$ series.

13. Give an example of an $\mathbb{R}$ valued 2×2 matrix A with $\det A = 1$ $\ni$ there is no 2×2 $\mathbb{R}$-valued B with
$$e^B = A$$
and prove your example works

14. Suppose e^{tA} is orthogonal for all t with $|t| < \varepsilon$ some $\varepsilon > 0$.
Prove: A is skew symmetric.

15. Prove from first principles that if $g \in G$, G a finite group, then order of $g \mid |G|$.

16. Explain carefully why if $H \subset G$ $|H| = \frac{1}{2}|G|$ then H is a normal subgroup of G (H, G are finite groups. H a subgroup of G)

17. Outline the proof by induction of the first Sylow theorem (that $p^s \mid |G|$ but $p^{s+1} \nmid |G|$ then $\exists$ a subgroup of order p^s in G).

18. Discuss why two subgroups satisfying prob 17 are always conjugate to each other.

19. Discuss why number of such subgroups $\equiv 1 \mod p$.

20. How many groups of order 35 are there (up to isomorphism). Prove your answer.

21. Prove every group of order 4 is abelian.

22. Consider the group generated by 90° rotations of a square together with reflections in each of the two diagonals) How many elements? Is it abelian?

23(a) Let $SO(3)$ = rotations of $\mathbb{R}^3$ of determinant 1. Show that if $A \in SO(3)$ then the action of A on S^2 $(= \{(x, y, z) : x^2+y^2+z^2 = 1\})$ has a fixed point (i.e. a one point orbit).

(b) Discuss the dimensions of orbits in terms of the dimension of stabilizers after showing dimension of $SO(3) = 3$.

24. Explain why if H is a subgroup of G, the number of conjugates of $G = |G| / |$normalizer of H in $G|$ (G finite group here)

25. (a) Define the sign of a permutation (± 1)

(b) Prove that $\text{sign}(\sigma) = (-1)^{\text{no. of transpositions}}$ when σ is written as a product of transpositions

26. True or false: every permutation of a set of r elements is expressible as a product of no more than $r-1$ transpositions. (Suggestion: Think inductively)

27. Does a group of order $2n$, n odd always have a normal subgroup? Prove

your answer.

28. How many elements in $\mathbb{Z}_{28}$ have order 28? Prove your answer.

29. A group is simple if it has no normal subgroups except $\{e\}$ and G. Are there infinitely many simple groups? Prove your answer.

30. Prove: For each fixed $N \in \mathbb{Z}^+$ there are only finitely many groups with N elements (up to isomorphism)

31. If $N = mn$, $n > 1$, $m > 1$, is there always at least two groups G of order N (up to isomorphism)? Prove your answer.

32. Describe the Euclidean algorithm for polynomials and explain how to find the highest degree common factor by this method.

33. Explain why an irreducible polynomial over $\mathbb{R}$ (irreducible means same as prime, no ~~no~~ nontrivial factorization) is degree ≤ 2.

34. Use your argument for prob 33 to factor $X^4 + 1$ over $\mathbb{R}$

35. Prove that every group G with $p \mid |G|$, p prime, has an element of order p.
(Do not use Sylow Theorems. This is a step in the proof of the Sylow Theorems).

36. (a) State the block decomposition for elements of $SO(n)$ [two-by-two blocks and 1×1 blocks]

(b) Assuming this, explain carefully why for every $A \in SO(n)$ $\exists B$ such that
$e^B = A$

(Suggestion: Do $SO(2)$ first: Look at prob 37 first maybe)

37. Find
$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}}, \quad t \in \mathbb{R}.$$