10-3 $\quad f(x,y) = x^2|y|$

both x^2 & $|y| \leq 1$ on $|x| \leq 1$ $|y| \leq 1$

and x^2 & $|y| \geq -1$ the fn

$$\frac{\partial f}{\partial y} = \begin{cases} -x^2 & y<0 \\ x^2 & y>0 \end{cases}$$ undefined at $y=0$

$$|f(x,y_1) - f(x,y_2)| = |x^2|y_1| - x^2|y_2|| = x^2||y_1| - |y_2||$$

$$\leq x^2|y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$$

by triangle inequality $\quad$ by $\sup_{x \in [-1,1]} x^2 = 1$

So it satisfies lipshitz with $K=1$

even though all points with $y=0$ have no derivative

4

a $\quad f(x,y) = xy^2 \qquad \frac{\partial f}{\partial y} = 2xy$

$a \leq x \leq b \qquad c \leq y \leq d$

$$|f(x,y_1) - f(x,y_2)| = |y_1 - y_2||f_y(x, p_0)|$$

$y_1 \leq p_0 \leq y_2$ if WLOG $y_1 < y_2$

on any interval $|f_y|$ achieves its maximum

at $x^* = \max(|a|, |b|)$ $\; y^* = \max(|c|, |d|)$

and $f_y(x^*, y^*) = 2x^* y^* < \infty$

bc the previous argument breaks down

as $c \to -\infty$, $d \to +\infty$ in which

case $y^* \to \infty$ and $f(x^*, y^*) = 2x^* y^* \to \infty$

So $\nexists$ K satisfying lipshitz

1. $f(y) = (1+y)^{1/3}$ $\frac{\partial f}{\partial y} = \frac{1}{3}(1+y)^{-2/3}$

$\frac{\partial f}{\partial y} \to \infty$ as $y \to -1$ so

the problem is not well posed for $y(0)=0$

A solution exists

$$y = \left(\tfrac{2}{3}x\right)^{3/2} - 1$$

It is not unique

$y = -1$ is also

a solution

any combination

$$f(x) = \begin{cases} -1 & x < c \\ \tfrac{2}{3}(x-c)^{3/2} - 1 & x \geq c \end{cases}$$

is a solution

2

$$f(y) = |\sin y|$$

$$\frac{df}{dy} = \begin{cases} \cos x & \sin x > 0 \\ -\cos x & \sin x < 0 \end{cases}$$

$f(y)$ is periodic

$$|f(x, y_1) - f(x, y_2)| = \big|\, |\sin y_1| - |\sin y_2| \,\big| \leq |\sin y_1 - \sin y_2|$$

by MVT $\quad |\sin y_1 - \sin y_2| \leq \cos(c) \, |y_1 - y_2| \leq |y_1 - y_2|$

$\max |\cos c| = 1$

So $f(x, y)$ is Lipschitz. coefficient $K = 1$

Since K is irrespective of x

so taking $a \leq x \leq b$ with

$a \to -\infty$ and $b \to \infty$ Lipschitz is satisfied

$\forall x, y$. and all y init values are valid

3 $f: [a,b] \to [a,b]$ $\exists$ $c \in (0,1)$

a by definition of contractive mapping

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le c|x_n - x_{n-1}|$$

b So $|x_2 - x_1| \le c|x_1 - x_0| \le c|b-a|$

inductively $|x_{n+1} - x_n| \le c|x_n - x_{n-1}| = c \cdot c^{n-1}|x_1 - x_0| \le c^n|b-a|$

c

$$x^* = \sum (x_{n+1} - x_n) \le \sum_{n=0}^{\infty} c^n |b-a| = |b-a| \sum_{n=0}^{\infty} c^n$$

$$= \frac{|b-a|}{1-c}.$$

because $x^* = \lim_{n\to\infty} x_n$ converges

and $f(x^*)$ continuous

$f(x^*) = \lim_{n\to\infty} f(x_n)$ So $f(x^*) = x^*$

d if $y^* \in [a,b]$ $f(y^*) = y^*$

$$|x^* - y^*| = |f(x^*) - f(y^*)| = c|x^* - y^*|$$

this implies $c = 0$ but $c \in (0,1)$

So $c \neq 0$ a contradiction thus $|x^* - y^*| = 0$

4.

$f(x) = \sqrt{x+1}$ $\quad f' = \frac{1}{2}(x+1)^{-1/2}$

$|f(x_1) - f(x_2)| = f'(c)|x_1 - x_2|$

$x \in [0,2] \Rightarrow f'(x^*) \leq 0.5$

So f is a contraction mapping

$f(x) = x \Rightarrow x = \sqrt{x+1}$

$x^2 = x+1$

$x^2 - x - 1 = 0 \Rightarrow x = \frac{1}{2} + \frac{\sqrt{5}}{2} \approx 1.618$

Starting with $x_0 = 0$

$x_1 = 1$

$x_2 = \sqrt{2} \approx 1.414$

$x_3 = \sqrt{\sqrt{2}+1} \approx 1.553$

$\left| x_3 - \frac{\sqrt{5}+1}{2} \right| = 0.0642$