Sample problems for midterm M

1. Suppose G is a group in which every element other than the identity has order 2. Show G is abelian.

2. Does every infinite group have infinitely many distinct subgroups? Prove your answer (Suggestion: Consider the case separately that there is an element of infinite order and that every element has finite order)

3. Let $\mathbb{C}^*$ be $\mathbb{C} - \{0\}$ with multiplication as an operator. Is this isomorphic to $\mathbb{R} - \{0\}$ with multiplication as operation? Why or why not?

4 Show that if H is a subgroup of order n in a group G of order $2n$, then H is normal in G.

5. Does prob 4 work if G is of order $3n$, H of order n. Prove or give a counterexample

6. Suppose H and K are normal in a group G with $H \cap K = \{e\}$.

(a) Show that $hK = KH$ if $h \in H$, $k \in K$.

(b) Show that the subgroup of G generated by H and K together is isomorphic to $H \otimes K$ ($\overset{\text{def}}{=} \{(h, k) : h \in H, k \in K\}$ with operation $(h_1, k_1) \times (h_2, k_2) = (h_1 h_2, k_1 k_2)$)

7. Suppose $f: G_1 \to G_2$ is a homomorphism. Prove: $\ker F$ ($\overset{\text{def}}{=} \{g \in G_1 : F(g) = e \in G_2\}$) is a normal subgroup of G_1.

8. In the situation of problem 7, is the image of F isomorphic $G_1 / \ker F$? Prove your answer.

9. Find $Q_1(x)$ and $Q_2(x)$ such that

$$1 = Q_1(x)(x-1)^3 + Q_2(x)(x-2)^3.$$

10. Explain why there exists a polynomial of pos. deg. $P(x)$ such that $P(A) = 0$, for A an $n \times n$ matrix with $\mathbb{R}$ coefficients.

11. Suppose $P(x)$ is an irreducible polynomial over $\mathbb{R}$. Prove: $\deg P \leq 2$.

12. Find $\alpha \in \mathbb{C}, \alpha \neq 0$, such that $|P(i+t\alpha)| < |P(i)|$ for all sufficiently small $t > 0$

where $P(x) = x^3 + 5x^2 + 7$

13. If $\lambda \in \mathbb{C}$, A an $n \times n$ $\mathbb{C}$-valued matrix, λ an eigenvalue of A, then by definition the generalized eigenspace of λ is

$$G_\lambda(A) = \{v : (A - \lambda I)^k = 0 \text{ for some } k > 0\}.$$

Prove: If λ_1, λ_2 are two eigenvalues of A with $\lambda_1 \neq \lambda_2$ then $G_{\lambda_1}(A) \cap G_{\lambda_2}(A) = \{\vec{0}\}$

14. Explain why the special orthogonal group $SO(n)$ has dimension $n(n-1)/2$, $n \geq 2$

($SO(n)$ = set of $n \times n$ $\mathbb{R}$-valued matrices with $\det = 1$)

15. Prove: the number of conjugates of a subgroup $H \subset G$ = index of $N_G(H)$ in G

(where $N_G(H)$ = the normalizer of H in G $\stackrel{\text{def}}{=} \{g \in G : g^{-1} H g\} = H$).

16. Prove $N_G(H)$ is a subgroup of G.

17. Illustrate part 15 for subgroups of S_3. after finding all the subgroups and determining which ones are normal and which not normal.

18. Explain why (without using the theorem about subgroups of index 2) that

A_n is normal in S_n where

A_n = group of even permutations

("even" means sgn $= +1$).

19. Is every permutation in S_n a product of $n-1$ "transpositions" (interchanges)? (for all $n \geq 2$). Prove your answer.

20. Discuss why each permutation in S_n is a product of disjoint "cycles".

21. What is $\text{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ \downarrow & \downarrow & \cdots & \downarrow \\ n & n-1 & & 1 \end{pmatrix}$ (reversing order permutation)