Normal Subgroups

G a group, H a subgroup, $H \leq G$

then H is normal in G (notation $H \lhd G$)

if $g^{-1}Hg = H$ for all $g \in G$.

Equivalent: $gH = Hg \quad \forall g \in G$

Note that whether H is normal or not

then $g^{-1}Hg$ is itself a subgroup

of G. It is a "conjugate" of H.

Exercises: (1) $\operatorname{ord}(g^{-1}Hg) = \operatorname{ord} H$

(2) $h \to g^{-1}hg$ is an isomorphism

of H onto $g^{-1}Hg$

The <u>normalizer</u> of H in $G \overset{\text{def}}{=}$

$\{g \in G : g^{-1}Hg = H\}$. Notation $N_G(H)$

Then H is normal in G if and only

if $N_G(H) = G$.

Lemma: $N_G(H)$ is a subgroup of G.

Proof (sketch): If $g^{-1}Hg = H$ then

$gHg^{-1} = H$ (multiply on left by

g_1 or right by g^{-1}). So H is "closed under inverses": $g \in N_G(H) \Rightarrow g^{-1} \in N_G(H)$

If $g_1, g_2 \in N_G(H)$ then

$$(g_1 g_2)^{-1} H g_1 g_2 = (g_2^{-1} g_1^{-1}) H (g_1 g_2)$$

$$= g_2^{-1} \underbrace{(g_1^{-1} H g_1)}_{= H} g_2 = g_2^{-1} H g_2 = H,$$

So $N_G(H)$ is "closed under products" □

Lemma: The number of conjugates of H

= ord (G) / order $N_G(H)$

(number of conjugates of H = index of normalizer of H)

Proof: $g_1^{-1} H g_1 = g_2^{-1} H g_2$

$\Leftrightarrow g_2 g_1^{-1} H g_1 g_2^{-1} = H$

$\Leftrightarrow g_1 g_2^{-1} \in N_G(H) \Leftrightarrow g_1, g_2$

$\in$ same coset of $N_G(H)$.

No of conjugates of $H = \dfrac{\text{no of } g \in G}{\text{size of cosets of } N_G(H)}$

$= \text{ord}(G) / \text{ord}\, N_G(H)$ $\square$

Example: $H = \{e, T_{12}\} < S_3$.

H is not normal. So

$H \subseteq N_G(H) \subsetneq S_3$. So order $N_G(H) < 6$

Since H is a subgroup of $N_G(H)$

order of $N_G(H) =$ order $H \cdot$ some integer ≥ 1

But order $H = 2$ while order $N_G(H) \mid 6$

So order $N_G(H)$ cannot be 4 (or 6 since $4 \nmid 6$)

since H is not normal. So

order $N_G(H) = 2$ and $N_G(H) = H$.

Exercise: Check directly that

$g^{-1} H g \neq H$ if $g \notin \{e, T_{12}\}$.

Theorem (Sylow): If order $(G) = p^k r$

$p \nmid r$, p prime then $\exists$ a subgroup S

of G of order p^k and all such S

are conjugate and the number of

order p^k subgroups $= pl + 1$

for some l.

The order p^k subgroups are called Sylow subgroups for p.

This theorem is not easy to prove,
Proof later.

But it is very powerful.

Example: The only group of order 35 is $\mathbb{Z}/_{35}$.

Reason: The number of 5-Sylow subgroup is $5l+1$ some l, but this no. divides 35 since it = index of normalizer of $N_G(H)$; since {all 5-Sylow subgroups} = {conjugates of S}.

But this $\Rightarrow l=0$ since 6, 11, .. do not divide 7. So S_5 (the 5-Sylow subgroup) is normal.

Similarly there is only one 7-Sylow subgroup S_7 and it is normal.

Note that $S_5 \cap S_7 = \{e\}$

Since an element $\neq e$ cannot have order 5 and order 7 at once.

Also $s_1 \in S_5$, $s_2 \in S_7 \Rightarrow s_1 s_2 = s_2 s_1$

Reason $s_1 s_2 s_1^{-1} s_2^{-1} = s_1 (s_2 s_1^{-1} s_2^{-1})$

$$\underbrace{s_1 (\underbrace{s_2 s_1^{-1} s_2^{-1}}_{\in S_5})}_{\in S_5}$$

but $s_1 s_2 s_1^{-1} s_2^{-1} = (s_1 s_2 s_1^{-1}) s_2^{-1}$

$$\underbrace{(\underbrace{s_1 s_2 s_1^{-1}}_{\in S_7}) s_2^{-1}}_{\in S_7}$$

So $s_1 s_2 s_1^{-1} s_2^{-1} = e$

and so $s_1 s_2 = s_2 s_1$

Thus S_5 and S_7 generate a subgroup of the group of order 35 isomorphic to direct product of S_5 and S_7. But this direct product has 35 elements. So whole group of order 35 $\cong$ direct product of S_5 & S_7

$\simeq \mathbb{Z}_{35}$.

Direct product of two groups (def)

$G_1 \otimes G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$

with operation $(g_1, g_2)(g_1', g_2')$
$= (g_1 g_1', g_2 g_2')$

Part of Sylow business:

$\cancel{ord}(G) = p^k r$ $\quad p \nmid r \quad p$ prime

then for each $l < k$ $\quad (l \geq 1)$

$\exists$ a subgroup of order p^l and

each subgroup is $\subset$ some p-Sylow subgrp (of order p^k by def).