Adjoint of the Derivative

1 What was I talking about?

This is what I was trying to say in discussion today as a fun example. Take $V = \mathbb{R}[x]_{\leq n}$ and define inner product on V by

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx$$

Since we're working over $\mathbb{R}$ we can ignore the complex conjugation. Take T as the differentiation operator

$$T(f) = \frac{d}{dx} f \in V$$

What is the adjoint of T? We can use integration by parts to get that for all f, g

$$\begin{aligned}\langle Tf, g \rangle = \langle f, T^*g \rangle &\implies \int_0^1 f'(x)g(x)dx = \int_0^1 f(x)\big(T^*(g)\big)(x)dx \\ &\implies f(x)g(x)\Big|_{t=0}^{t=1} - \int_0^1 f(x)g'(x)dx = \int_0^1 f(x)\big(T^*(g)\big)(x)dx\end{aligned}$$

This is almost what we want to find the adjoint: Pretend S is an operator on V such that $S = (E_1 - E_0) - T$ and E_a is some operator that takes in a function h and spits out a function $E_a h$ where

$$\int (E_a h)(x) f(x) dx = h(a)f(a)$$

However, no such operator exists because it would need to be something resembling multiplication by a function that integrates to 1 but yet is only nonzero at a single point. This is what people call the "Dirac delta function" (which you can google) and it's not actually a function. The rigorous way to define it is actually the way we've done here; it comes about just as some formal linear algebra device (but usually this is all done on an infinite-dimensional function space).

I finished what I said in discussion by saying that the adjoint of differentiation doesn't actually have a nice form because what it's "supposed" to be isn't actually a function. But keep in mind that the adjoint of a linear map on an inner product space always exists and is uniquely defined.

2 What can we compute?

Here's an example that shows you a computation of what the adjoint actually does in one of these cases. Let's take V to be polynomials of degree at most 2 over $\mathbb{R}$. This has a basis

$$\{1, x, x^2\}$$

Gram-Schmidt on this basis gives an orthonormal basis for V of the form

$$\{1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})\}$$

Remember that in an orthonormal basis the adjoint is supposed to be the conjugate transpose of the matrix of the map. So writing the differentation map T in matrix form using our orthonormal basis, we get

$$T = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} \implies T^* = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}$$

What does T^* do in our original basis? Using a change of basis matrix, we can recover

$$T^*(1) = 12x - 6$$
$$T^*(x) = 30x^2 - 24x + 2$$
$$T^*(x^2) = 30x^2 - 26x + 3$$

You can check that, for example

$$\int_0^1 f'(x)x^2 dx = \int_0^1 f(x)(30x^2 - 26x + 3)dx$$

for any polynomial f of degree at most 2. However, this equation is false for higher-degree polynomials (including the example x^3)!