Homework (second part)

8. If $e^A = I_n$, does A have to be symmetric (I_n is of course orthogonal so this is a natural question).

Suggestion: $P^{-1} I_n P = I_n$ whereas $P^{-1} e^A P = e^{P^{-1} A P}$ and $P^t A P$ does not have to be symmetric even if A is?)

9. If $A \in Gl(n, \mathbb{R})$ with $\det A > 0$, does A have to have a square root – i.e. does there have to be a $B \in Gl(n, \mathbb{R}) \Rightarrow B^2 = A$?

[This is related to A have a $\ln$ since if $A = e^C$ then $(e^{C/2})^2 = A$].

10. How about prob 9 for $A \in SL(n, \mathbb{R})$ ($\det A = 1$)?

11. Work out a proof that $\det e^A = e^{\operatorname{tr} A}$ by doing the following sequence of steps:

(a) Using $\det e^{(t+\Delta t)A} = \det e^{tA} \det e^{(\Delta t)A}$

show that if $F(t) = \det e^{tA}$

then $$F'(t) = \det e^{tA} \cdot F'(0)$$

(b) Show $$F'(0) = \operatorname{tr} A$$

by writing out

$$e^{(\Delta t)A} - I \quad \text{and}$$

throwing away all higher degree than 1 in Δt terms.

(c) Deduce that

$$F'(t) = F(t) \operatorname{tr} A$$

and apply what you know about differential equations to get $\det(tA) = e^{t \operatorname{tr} A}$ (by putting $t = 1$).

12. Is every element $A \in SL(n, \mathbb{R})$ $= e^B$ for some trace 0 B?

Prove your answer.