HW 1 - 115B

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1 Exercise 1

For each of the following vector spaces V and each ordered basis B, find an explicit formula for each vector in the dual basis B^*.

1. $V = k^3$, $B = \{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$
   Let E be the canonical basis then
   $$[I]_E^B = \begin{pmatrix}1 & 1 & 0\\0 & 2 & 0\\1 & 1 & 1\end{pmatrix} [I]_B^E = \begin{pmatrix}1 & -\frac{1}{2} & 0\\0 & \frac{1}{2} & 0\\-1 & 0 & 1\end{pmatrix}.$$
   The canonical dual basis is the column vectors $(1,0,0),(0,1,0),(0,0,1)$ with $f(v)$ corresponding to matrix multiplication Thus
   $$B^* = \{(1,-\frac{1}{2},0),(0,\frac{1}{2},0),(-1,0,1)\}.$$

2. $V = k[x]_{\leq 2}, B = \{1, x, x^2\}$ Let I denote the identity and D denote the derivative operator which was shown previously to be linear. The dual basis can be defined as
   $$B^* = \{I, D, \frac{D}{2}\}.$$
   with function defined as follows.
   For any $f^* \in V^*, f \in V$ $f^*(f) = (f^* \circ f)(0)$

2 Exercise 2

Define some $f \in (\mathbb{R}^2)^*$ $f\begin{pmatrix}x\\y\end{pmatrix} = 2x + y$ and a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ via the formula $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x+2y\\x\end{pmatrix}$

1. *Compute $T^*(f)$.*

$$T^*(f)(v) = f(T(v)) = 2(3x + 2y) + x = 7x + 4y.$$

2. *Compute $[T^*]_{\mathcal{E}^*}$, where $\mathcal{E}$ is the standard ordered basis for $\mathbb{R}^2$ and $\mathcal{E}^* = \{e_1^*, e_2^*\}$ is the dual basis, explicitly by finding scalars a, b, c, d such that $T^*(e_1^*) = ae_1^* + ce_2^*$ etc.*

$$T^* = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

3. *Compute $[T]_{\mathcal{E}}$ and $(T_{\mathcal{E}})^t$*

$$[T]_{\mathcal{E}} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} [T]_{\mathcal{E}}^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

3 Exercise 3

Let V denote a finite dimensional k-vector space. For any subset $S \subset V$ define the annihilator S^0 of S as

$$S^0 := \{f \in V^* : f(x) = 0 \quad \forall x \in S\}.$$

1. *Prove that S^0 is a subspace of V^**
   To show taht S^0 first note that $S^0 \subset V^*$ since every element of S^0 is an element of V^*. Then note that S^0 contains the 0 element $f \in V^* \to f(x) = 0 \forall x \in V$ is clearly an element of S^0. Additionally consider any $f_1, f_2 \in S^0, c \in k$.

$$(f_1 + cf_2)(x) = f_1(x) + cf_2(x) = 0 + c * 0 \;\; \forall x \in S \to (f_1 + cf_2) \in S^0.$$

2. *If W is a subspace of V and $x \notin W$, prove that there exists some $f \in W^0$ such that $f(x) \neq 0$* Since $x \notin W$ $x \neq 0$. Since V is finite dimensional pick a basis with x as an element $B = x, v_1, v_2, ...$ And consider further the linear functional f defined on B s.t.

$$f(v_i) = \begin{cases} 1 & v_i = x \\ 0 & v_i \neq x \end{cases}.$$

   Since $x \notin W$ any vector $w \in W$ written in B coordinates has 0 as the x coefficient and so $f(w) = 0$ satisfying the definition of S^0 and so $f \in S^0$ but $f(x) = 1 \neq 0$.

3. *In class, we constructed an isomorphism $\phi : V \to V^{**}$. Prove that $(S^0)^0 = span(\phi(S))$ where $\phi(S) := \{\phi(s) : s \in S\}$*
   First note that $\phi(v) = \lambda_v$ defined as $\lambda_v(f) = f(v) \quad \forall f \in V^*, v \in V$ To show the equality we must show that for each $\lambda \in \phi(S), \lambda \in (S^0)^0$ and vice-versa Recall the definition of $(S^0)^0$

$$(S^0)^0 = \{\lambda \in V^{**} : \lambda(f) = 0 \quad \forall f \in S^0\}.$$

Consider any $\lambda_v \in \phi(S)$ and the $v \in V$ corresponding to this function. clearly $v \in S$. Consider additionally any arbitrary $f \in S^0$. $\lambda_v(f) = f(v)$ recalling the definition of S^0, since $f \in S^0$ $f(v) = 0$ since $v \in S$ therefore λ_v satisfies the condition of $(S^0)^0$. This proves the first direction.
For the second direction pick a basis $B' = \{s_1, s_2, ..., s_n\}$ for S and since $S \subset V$ pick a basis B for V consisting of all elements of B' and additional elements $B = \{s_1, s_2, ..., s_n, v_1, v_2, ..., v_m\}$. Pick any $\lambda \in (S^0)^0$ and assume for contradiction that $\lambda \notin span(\phi(S))$. Use the isomorphism to get $v = \phi^{-1}(\lambda)$. Using our basis

$$v = \alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n + \beta_1 v_1 + \beta_2 v_2 + ... + \beta_m v_m.$$

and so

$$\lambda(f) = \alpha_1 f(s_1) + ... + \alpha_n f(s_n) + \beta_1 f(v_1) + ... + \beta_m f(v_m).$$

But since $\lambda(f^0) = 0$ whenever $f^0 \in S^0$ each $\beta_i = 0$ by the previous question Thus:

$$\lambda = \alpha_1 \lambda_{s_1} + \alpha_2 \lambda_{s_2} + ... + \alpha_n \lambda_{s_n}.$$

Which is a linear combination of elements of $\phi(S)$ and is thus in the span of $\phi(S)$.

4. *For subspaces W_1 and W_2 of V, prove that $W_1 = W_2$ iff $W_1^0 = W_2^0$.*
To prove the first direction assume first that $W_1^0 = W_2^0$ and assume for the sake of contradiction that $W_1 \neq W_2$ then WLOG there exists some $w \in W_1$ s.t. $w \notin W_2$. But since $w \notin W_2$ there exists some $f \in W_2^0 = W_1^0$ s.t. $f(w) \neq 0$. However this is a contradiction by the definition of W_1^0 since $f(w) = 0$ by definition of $w \in W_1^0$. This proves the first direction. The other direction is trivial since in the definition of the two sets we can interchange W_1 and W_2.

5. *For subspaces W_1 and W_2 prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$* Consider the definition

$$(W_1 + W_2)^0 := \{f \in V^* : f(x) = 0 \ \ \forall x \in (W_1 + W_2)\}.$$

Consider any $f \in (W_1 + W_2)^0$:
$f \in V^*$, $f(w_1) = 0, f(w_2) = 0 \ \ \forall w_1 \in W_1, w_2 \in W_2$. Therefore $f \in W_1^0$ and $f \in W_2^0$ so $f \in W_1^0 \cap W_2^0$:
Consider any $f \in W_1^0 \cap W_2^0$:
$f \in V^*$, $f(w_1) = 0 \ \forall w_1 \in W_1$ since $f \in W_1^0$ and $f(w_2) = 0 \ \forall w_2 \in W_2$ since $f \in W_2^0$ therefor for any $w \in (W_1 + W_2), w = \alpha w_1 + \beta w_2$ for some $\alpha, \beta \in k$, $w_1 \in W_1, w_2 \in W_2$ and $f(w) = f(\alpha w_1 + \beta w_2) = \alpha f(w_1) + \beta f(w_2) = \alpha * 0 + \beta * 0 = 0$ so $f \in (W_1 + W_2)^0$

4 Exercise 4

Prove that if W is a subspace of V, then $dim(W) + dim(W^0) = dim(V)$. There are 2 cases either V is finite dimensional or not. In the first case since V is f.d. so is W. let $dim(V) = n, dim(W) = k$. Consider an ordered basis $B' = \{w_1, ..., w_k\}$ of W and extend it to an ordered basis $B = \{w_1, ..., w_k, ..., w_n\}$ of V. We know that for any $f \in V^*, f = \alpha_1 w_1^* + \alpha_2 w_2^* + ... + \alpha_k w_k^* + ... + \alpha_n w_n^*$ Consider any $f \in W^0$. then

$$f(w_i) = \begin{cases} 0 & i \in \{1, ..., k\} \\ c_i & i \in \{k+1, ..., n\} \end{cases}.$$

For this to hold given the representation above each $\alpha_i = 0, \ \forall i \in \{1, ..., k\}$ Therefore every $f^0 \in W^0$ can be represented as

$$f^0 = \alpha_{k+1} w_{k+1}^* + ... \alpha_n w_n^*.$$

and so $\{w_{k+1}^*, ..., w_n^*\}$ form a basis for W^0 and $dim(W^0) = n-k$ so $k+n-k = n$ is true and we have proved the question.
In the case $dim(V)$ is not finite. we must show that at least one of $dim(W), dim(W^0)$ is not finite. There are two cases $dim(W)$ is finite or not. If it is not finite we are done. Assume $dim(W)$ is finite. Since $dim(V)$ is not finite $dim(V^*)$ is not finite. Assume for contradiction that $dim(W^0)$ is finite then pick a basis for W^0

$$B^0 = \{w_1^*, w_2^*, ..., w_l^*\}.$$

Additionally pick $l+1$ linearly independent vectors in $V \setminus W$ which is possible since W is finite and V infinite.

$$\{v_1, ..., v_{l+1}\}.$$

and the corresponding linear functionals

$$\{v_1^*, ..., v_{l+1}^*\}.$$

defined in the usual way. Clearly each of these linear functionals is in W^0 and each is linearly independent. But this leads to a contradiction because there cannot exist more than l linearly independent linear functionals in W_0 if the basis was defined with l elements. Thus a contradiction showing that indeed W^0 must be infinite.

5 Exercise 5

Suppose that W is a finite dimensional vector space and $T : V \to W$ is a linear transformation. Prove that $ker(T^*) = R(T)^0$
Consider any $f \in ker(T^*)$ Then clearly

$$f(T(v)) = 0.$$

for all $v \in V$ and in particular this implies for any $w \in R(T)$, $f(w) = 0$ since $w \in R(T)$ is equivalent to $w = T(v)$ for some $v \in V$. $f(w) = 0$ for all $w \in R(T)$ satisfies the same requirements that put $f \in R(T)^0$ Thus $ker(T^*) \subset R(T)^0$. Consider any $f \in R(T)^0$ then for any $w \in R(T)$, $f(w) = 0$. In particular for any $v \in V, T(v) \in R(T)$ so $T(v) = w$ for some $w \in R(T)$ and $f(w) = 0$ by definition so $f(T(v)) = 0 \forall v \in V$ and $f \in ker(T^*)$ thus $R(T)^0 \subset ker(T^*)$

6 Exercise 6

Let R denote the 3×3 real matrix $\begin{pmatrix} -3 & -3 & -4 \\ 2 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. *Find all eigenvalues of R. For each eigenvalue, compute the corresponding eigenspace.*
Each eigenvalue is one such that $det(T - \lambda I) = 0$ first

$$det(\begin{pmatrix} -3-\lambda & -3 & -4 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = -(\lambda+1)((\lambda-2)(\lambda+3)+6) = -\lambda(\lambda+1)^2.$$

Clearly $\lambda = -1$ and $\lambda = 0$ are the only eigenvalues of R First to find all eigenvectors of eigenvalue 1 is to find all solutions to $Rv = 0$ These come in the form $\begin{pmatrix} a \\ -a \\ 0 \end{pmatrix}$ for any $a \in \mathbb{R}$ which is found by row reducing R thus the eigenspace of eigenvalue 0 is all vectors of the previous form. To find the eigenspace of eigenvalue -1 solve $(R + I)v = 0$ The solutions to this equation are of the form $\begin{pmatrix} -1.5b - 2c \\ b \\ c \end{pmatrix}$ for any $b, c \in \mathbb{R}$ And so the eigenspace of eigenvalue -1 is spanned by

$$\{\begin{pmatrix} -1.5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\}.$$

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7 Exercise 7

For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by the formula $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix}$, *find a basis B of $\mathbb{R}^2$ s.t. $[T]_B$ is diagnoal* Consider

$$[T]_{\mathcal{E}} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}.$$

First find the eigenvalues of $[T]_{\mathcal{E}}$

$$det(T - \lambda I) = det \begin{pmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) + 2 = (\lambda - 2)(\lambda - 3).$$

So 3 and 2 are eigenvalues. with eigenvectors

$$B := \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}.$$

respectively picking those two vectors as the basis in order we note that

$$[T]_B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Diagonal.

8 Exercise 8

Given some vector space V and a linear endomorphism $T : V \to V$ we define a T-invariant subspace of V to be a subspace $W \subset V$ s.t. $T(W) \subset W$. For the following determine if W is T-invariant subspace of V

1. $V = \mathbb{R}[x], T(f(x)) = f'(x), W = R[x]_{\leq 2}$ Yes since for an arbitrary polynomial ax^n $f'(x) = anx^{n-1}$ so the degree can only decrease and will therefore stay less than or equal to 2.

2. $V = R[x], T(f(x)) = xf(x), W = R[x]_{\leq 2}$ No consider $x^2 \in W$. $T(x^2) = x^3$ which has degree 3 and is not in W

3. $V = k^3, T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}, W = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = x_2 = x_3\}$

   Yes since for any $w \in W, w = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}$ and $T(w) = \begin{pmatrix} 3x_1 \\ 3x_1 \\ 3x_1 \end{pmatrix} \in W$

4. $V = C([0,1])$ $T(f(t)) = (\int_0^1 f(x)dx)t$ and $W = \{f \in V : f(t) = at + b,\ \ a, b \in \mathbb{R}\}$ for every $f \in W$ since f is continuous the integral exists and is a real number and so $T(f) = at$ with $b = 0$ in the definition and so is still in W

5. $V = k^{2\times 2}, T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, W is the subspace of symmetric matrices.

   No for consider an arbitrary $A \in W$

   $$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} b & c \\ a & b \end{pmatrix}.$$

   which is not symmetric unless $a = c$