Simple Harmonic Motion

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April 15, 2020

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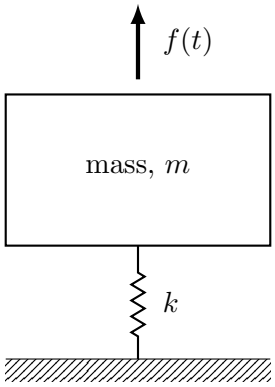
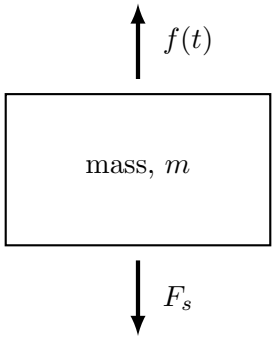
1 Overview

Simple harmonic motion (SHM) is the term used to describe the motion of a mass connected to a spring, as shown below.



Let's take a look at the free body diagram for this one-dimensional system:



From here we can write the equation of motion:

$$\Sigma F_x = m\ddot{x} = f(t) - F_s$$

Hook's Law tells us that for a spring, the force required to strech it is proportional to the length it is being stretched, or $F_s = kx$, yielding

$$m\ddot{x} = f(t) - kx$$
$$m\ddot{x} + kx = f(t)$$

Let's assume the simplest $f(t)$ possible, $f = 0$:

$$m\ddot{x} = -kx$$
$$\Rightarrow \ddot{x} = \frac{k}{m}x$$
$$\Rightarrow x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Figuring out the coefficients:

$$x(0) = x_0 = A\sin(0) + B\cos(0)$$
$$\Rightarrow x_0 = B$$
$$\Rightarrow x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$
$$\dot{x} = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right) - x_0\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$
$$\Rightarrow \dot{x}(0) = \dot{x_0} = A\sqrt{\frac{k}{m}}\cos(0) - x_0\sqrt{\frac{k}{m}}\sin(0)$$
$$\Rightarrow \dot{x_0}\sqrt{\frac{m}{k}} = A$$
$$\Rightarrow x(t) = \dot{x_0}\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$