

MAE 6225 Homework 1

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1.

$$\frac{\delta\phi}{\delta x} = \frac{-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2}}{12h} :$$

$$\phi_{j-2} = \phi_j - 2h\phi_j' + \frac{4h^2}{2!}\phi_j'' - \frac{8h^3}{3!}\phi_j''' + \frac{16h^4}{4!}\phi_j'''' + o(h^5)$$

$$\phi_{j-1} = \phi_j - h\phi_j' + \frac{h^2}{2!}\phi_j'' - \frac{h^3}{3!}\phi_j''' + \frac{h^4}{4!}\phi_j'''' + o(h^5)$$

$$\phi_j = \phi_j$$

$$\phi_{j+1} = \phi_j + h\phi_j' + \frac{h^2}{2!}\phi_j'' + \frac{h^3}{3!}\phi_j''' + \frac{h^4}{4!}\phi_j'''' + o(h^5)$$

$$\phi_{j+2} = \phi_j + 2h\phi_j' + \frac{4h^2}{2!}\phi_j'' + \frac{8h^3}{3!}\phi_j''' + \frac{16h^4}{4!}\phi_j'''' + o(h^5)$$

$$\frac{\delta\phi}{\delta x} = \alpha_1\phi_{j-2} + \alpha_2\phi_{j-1} + \alpha_3\phi_j + \alpha_4\phi_{j+1} + \alpha_5\phi_{j+2}$$

$$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \phi_j$$

$$+ (-2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5) h\phi_j'$$

$$+ (4\alpha_1 + \alpha_2 + \alpha_4 + 4\alpha_5) \frac{h^2}{2} \phi_j''$$

$$+ (-8\alpha_1 - \alpha_2 + \alpha_4 + 8\alpha_5) \frac{h^3}{6} \phi_j'''$$

$$+ (16\alpha_1 + \alpha_2 + \alpha_4 + 16\alpha_5) \frac{h^4}{24} \phi_j'''' + o(h^5)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$$

$$(-2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5) h = 1$$

$$4\alpha_1 + \alpha_2 + \alpha_4 + 4\alpha_5 = 0$$

$$-8\alpha_1 - \alpha_2 + \alpha_4 + 8\alpha_5 = 0$$

$$16\alpha_1 + \alpha_2 + \alpha_4 + 16\alpha_5 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & 1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & 1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{12h} \\ \frac{2}{-3h} \\ 0 \\ \frac{2}{3h} \\ -\frac{1}{12h} \end{bmatrix}$$

$$\frac{\delta\phi}{\delta x} = \frac{-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2}}{12h} + o(h^4)$$

This scheme is **4th order accurate**.

$$\frac{\delta\phi}{\delta x} = \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{6h} ;$$

$$\phi_{j+1} = \phi_j + h\phi_j' + \frac{h^2}{2!}\phi_j'' + \frac{h^3}{3!}\phi_j''' + o(h^4)$$

$$\phi_j = \phi_j$$

$$\phi_{j-1} = \phi_j - h\phi_j' + \frac{h^2}{2!}\phi_j'' - \frac{h^3}{3!}\phi_j''' + o(h^4)$$

$$\phi_{j-2} = \phi_j - 2h\phi_j' + \frac{4h^2}{2!}\phi_j'' - \frac{8h^3}{3!}\phi_j''' + o(h^4)$$

$$\frac{\delta\phi}{\delta x} = \alpha_1\phi_{j+1} + \alpha_2\phi_j + \alpha_3\phi_{j-1} + \alpha_4\phi_{j-2}$$

$$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\phi_j$$

$$+ (\alpha_1 - \alpha_3 - 2\alpha_4)h\phi_j'$$

$$+ (\alpha_1 + \alpha_3 + 4\alpha_4)\frac{h^2}{2}\phi_j''$$

$$+ (\alpha_1 - \alpha_3 - 8\alpha_4)\frac{h^3}{6}\phi_j''' + o(h^4)$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

$$\alpha_1 - \alpha_3 - 2\alpha_4 = \frac{1}{h}$$

$$\alpha_1 + \alpha_3 + 4\alpha_4 = 0$$

$$\alpha_1 - \alpha_3 - 8\alpha_4 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & -1 & -8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3h} \\ \frac{1}{2h} \\ -\frac{1}{h} \\ \frac{1}{6h} \end{bmatrix}$$

$$\frac{\delta\phi}{\delta x} = \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{6h} + o(h^3)$$

This scheme is **3rd order accurate**.

2. Construct a one-sided scheme which is 3rd **order accurate** to approximate $\frac{\partial \phi}{\partial x}$:

$$\frac{\partial \phi}{\partial x} \approx \frac{\delta \phi}{\delta x} = \alpha_1 \phi_{j+4} + \alpha_2 \phi_{j+3} + \alpha_3 \phi_{j+2} + \alpha_4 \phi_{j+1} + \alpha_5 \phi_j + o(h^4)$$

$$\begin{aligned} \phi_{j+4} = & \phi_j + 4h_1 \phi_j' + \frac{16h_1(h_1 + h_2)}{2!} \phi_j'' + \frac{64h_1(h_1 + h_2)(h_1 + h_2 + h_3)}{3!} \phi_j''' \\ & + \frac{256h_1(h_1 + h_2)(h_1 + h_2 + h_3)(h_1 + h_2 + h_3 + h_4)}{4!} \phi_j'''' \end{aligned}$$

$$\begin{aligned} \phi_{j+3} = & \phi_j + 3h_1 \phi_j' + \frac{9h_1(h_1 + h_2)}{2!} \phi_j'' + \frac{27h_1(h_1 + h_2)(h_1 + h_2 + h_3)}{3!} \phi_j''' \\ & + \frac{81h_1(h_1 + h_2)(h_1 + h_2 + h_3)(h_1 + h_2 + h_3 + h_4)}{4!} \phi_j'''' \end{aligned}$$

$$\begin{aligned} \phi_{j+2} = & \phi_j + 2h_1 \phi_j' + \frac{4h_1(h_1 + h_2)}{2!} \phi_j'' + \frac{8h_1(h_1 + h_2)(h_1 + h_2 + h_3)}{3!} \phi_j''' \\ & + \frac{16h_1(h_1 + h_2)(h_1 + h_2 + h_3)(h_1 + h_2 + h_3 + h_4)}{4!} \phi_j'''' \end{aligned}$$

$$\begin{aligned} \phi_{j+1} = & \phi_j + h_1 \phi_j' + \frac{h_1(h_1 + h_2)}{2!} \phi_j'' + \frac{h_1(h_1 + h_2)(h_1 + h_2 + h_3)}{3!} \phi_j''' \\ & + \frac{h_1(h_1 + h_2)(h_1 + h_2 + h_3)(h_1 + h_2 + h_3 + h_4)}{4!} \phi_j'''' \end{aligned}$$

$$\phi_j = \phi_j$$

$$\begin{aligned}
\frac{\delta\phi}{\delta x} &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \phi_j \\
&+ (4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4) h_1 \phi_j' \\
&+ (16\alpha_1 + 9\alpha_2 + 4\alpha_3 + \alpha_4) \frac{h_1 (h_1 + h_2)}{2} \phi_j'' \\
&+ (64\alpha_1 + 27\alpha_2 + 8\alpha_3 + \alpha_4) \frac{h_1 (h_1 + h_2) (h_1 + h_2 + h_3)}{6} \phi_j''' \\
&+ (256\alpha_1 + 81\alpha_2 + 16\alpha_3 + \alpha_4) \frac{h_1 (h_1 + h_2) (h_1 + h_2 + h_3) (h_1 + h_2 + h_3 + h_4)}{24} \phi_j''''
\end{aligned}$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$$

$$(4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4) h_1 = 1$$

$$16\alpha_1 + 9\alpha_2 + 4\alpha_3 + \alpha_4 = 0$$

$$64\alpha_1 + 27\alpha_2 + 8\alpha_3 + \alpha_4 = 0$$

$$256\alpha_1 + 81\alpha_2 + 16\alpha_3 + \alpha_4 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 16 & 9 & 4 & 1 & 0 \\ 64 & 27 & 8 & 1 & 0 \\ 256 & 81 & 16 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{h_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} \\ 0 & \frac{4}{3} & -\frac{7}{3} & \frac{7}{6} & -\frac{1}{6} \\ 0 & -3 & \frac{19}{4} & -2 & \frac{1}{4} \\ 0 & 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ 1 & -\frac{25}{12} & \frac{35}{24} & -\frac{5}{12} & \frac{1}{24} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{h_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4h_1} \\ \frac{3h_1}{4} \\ -\frac{h_1}{4} \\ h_1 \\ -\frac{25}{12h_1} \end{bmatrix}$$

$$\frac{\delta\phi}{\delta x} = \frac{-3\phi_{j+4} + 16\phi_{j+3} - 36\phi_{j+2} + 48\phi_{j+1} - 25\phi_j}{12h_1} + o(h^4)$$