MAE 6225
Homework 2

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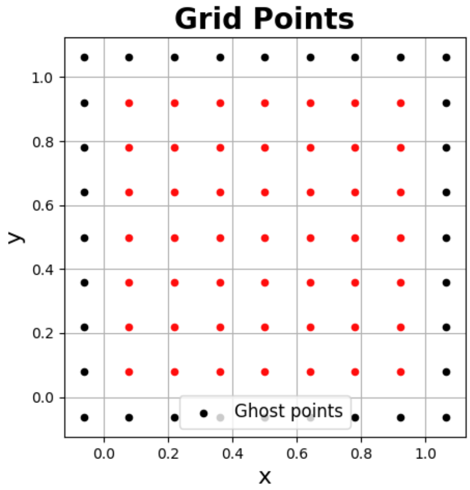

Figure 1: Grid layout

0.1 Discretization of the Poisson Equation

The Poisson equation, $\nabla^2 u = f(x, y)$, was discretized as follows:

$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} \approx \frac{1}{\Delta x^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{yy} \approx \frac{1}{\Delta y^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right)$$

For a uniform grid, $\Delta y = \Delta x = h$:

$$\frac{1}{h^2}\left(u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) = f(x_i, y_j)$$

This formula can be summarized as the linear system $A\mathbf{u} = \mathbf{f}$, where

$$\mathbf{u} = (u_{1,1}, ..., u_{I,1}, ..., u_{i,j}, ..., u_{1,J}, ..., u_{I,J})$$

Matrix A is a sparsely populated diagonal matrix, which can be very resource-intensive to invert if large. Therefore, iterative methods can be used to solve the equation.

In order to evaluate, a grid was constructed to represent the domain in the x and y directions. Figure one illustrates.

0.2 Comparison with Exact Solution, Dirichlet Boundary Conditions

Given the exact solution:

$$u_{ex}(x, y) = \sin(2\pi nx)\sin(2\pi ny)$$

and homogeneous Dirichlet boundary conditions, the Poisson equation was solved for a range of values of n that met the Nyquist criterion. Plots of the exact solution can be seen in figures 2-4.

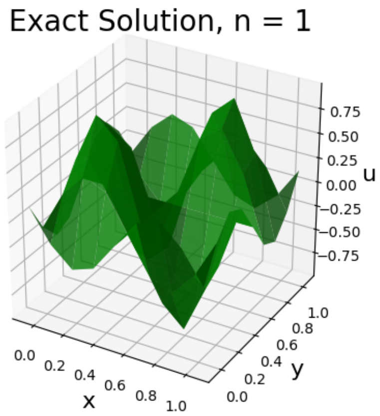

Figure 2: Exact solution with $n = 1$

0.2.1 Jacobi Method

First, a mesh grid was created to represent the x and y domains. A simple 9 by 9 grid was used for the first section, as grid refinement would come later. Plots of the results can be seen in figures 5-7.

0.2.2 Successive Over Relaxation (SOR)

The same mesh grid was created yet again, and a relaxation parameter $\omega = \frac{2}{2-cos(2\pi\Delta x)}$ was assigned. Plots of the results may be seen in figures 8-10.

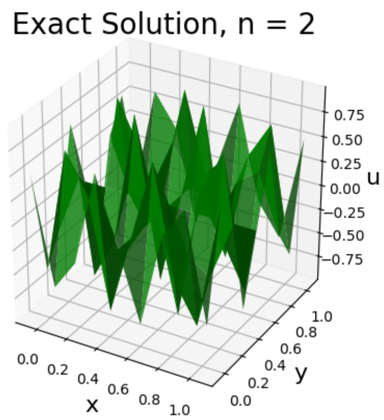

Figure 3: Exact solution with $n = 2$

0.3 Grid Refinement Study

Next, a grid refinement study was performed using a range of grid sizes from 5 by 5 to 513 by 513. A plot of the results can be seen in figure 11.

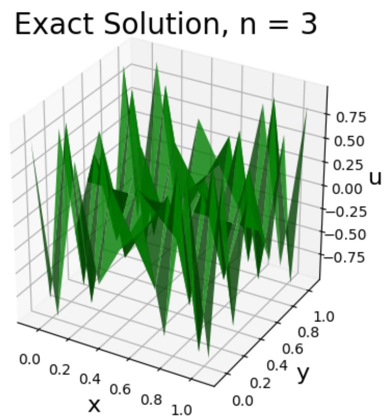

Figure 4: Exact solution with $n = 3$

0.4 Neumann Boundary Conditions

A new exact solution,

$$u_{ex}\left(x, y\right) = cos(2\pi nx)\, cos\left(2\pi nx\right)$$

was given with Neumann boundary conditions. Both methods were used again, and another grid refinement study was performed. Results can be seen in figures 12-21.

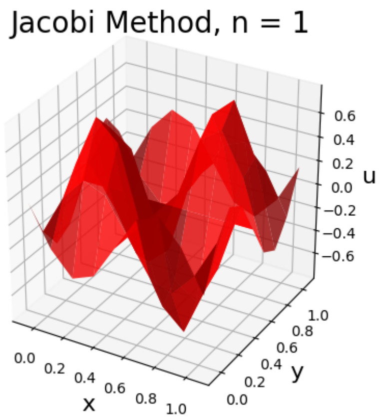

Figure 5: Jacobi method with $n = 1$

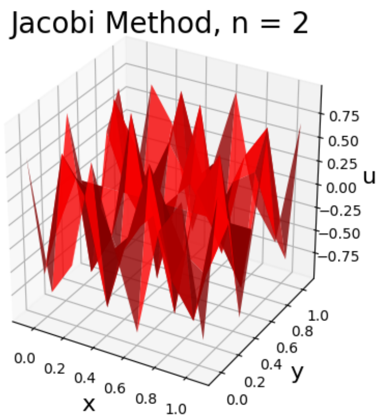

Figure 6: Jacobi method with $n = 2$

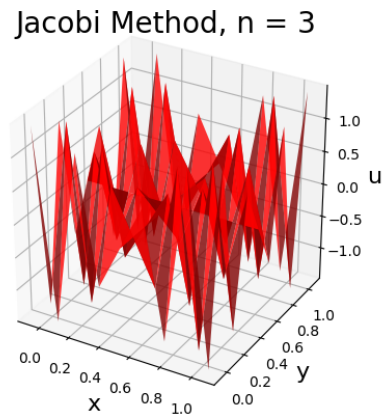

Figure 7: Jacobi method with $n = 3$

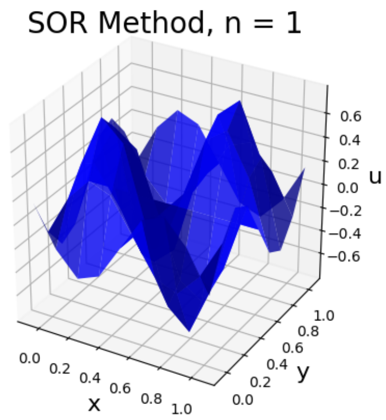

Figure 8: Gauss-seidel method with SOR, $\omega = 1.1$, $n = 1$

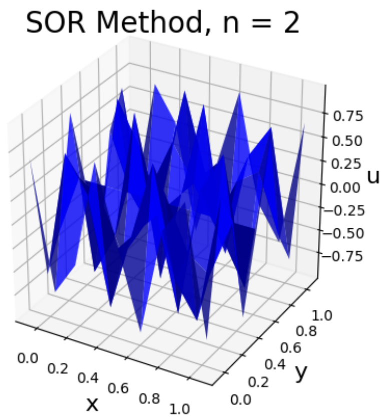

Figure 9: Gauss-seidel method with SOR, ω, $n = 2$

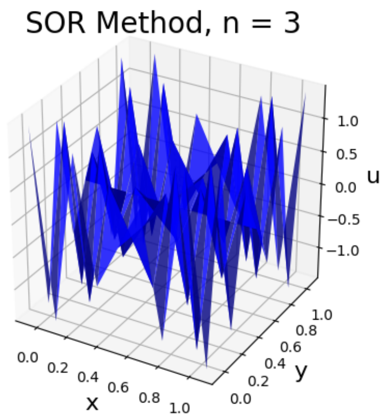

Figure 10: Gauss-seidel method with SOR, ω, $n = 3$

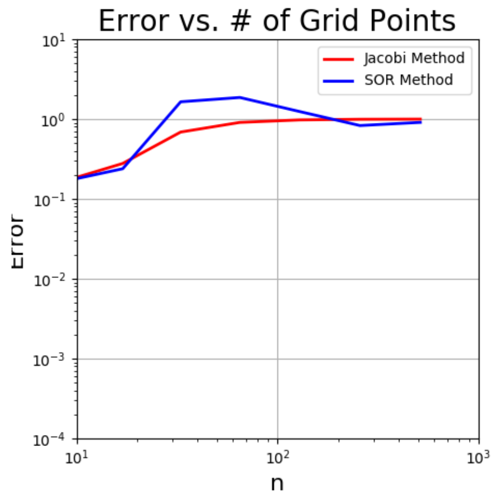

Figure 11: Grid refinement results for both methods used