MAE6225
Project: Part I
Due March 01 (at noon via email)

Part A:
Consider the Poisson equation

$$\nabla^2 u = f(x, y) \tag{1}$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

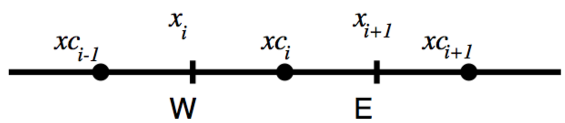

Figure 1: Discretization of the second derivative on a uniform mesh.

1. Discretize (1) on a uniform mesh using the following approximation for the second derivatives (see Fig. 1):

$$\begin{aligned}
u_{xx}(xc_i) &= \frac{u_x(x_{i+1}) - u_x(x_i)}{x_{i+1} - x_i} \\
&= \frac{1}{x_{i+1} - x_i}\left[\frac{u(xc_{i+1}) - u(xc_i)}{xc_{i+1} - xc_i} - \frac{u(xc_i) - u(xc_{i-1})}{xc_i - xc_{i-1}}\right] \\
&= \frac{u(xc_{i+1}) - u(xc_i)}{(xc_{i+1} - xc_i)(x_{i+1} - x_i)} - \frac{u(xc_i) - u(xc_{i-1})}{(xc_i - xc_{i-1})(x_{i+1} - x_i)}.
\end{aligned} \tag{2}$$

   This expression is second-order accurate on a uniform grid. Use an analogous expression for the $y-$derivatives. Use the cell-centered grid arrangement shown in Fig. 2. The ghost points (shaded) can be used to evaluate the boundary condition.

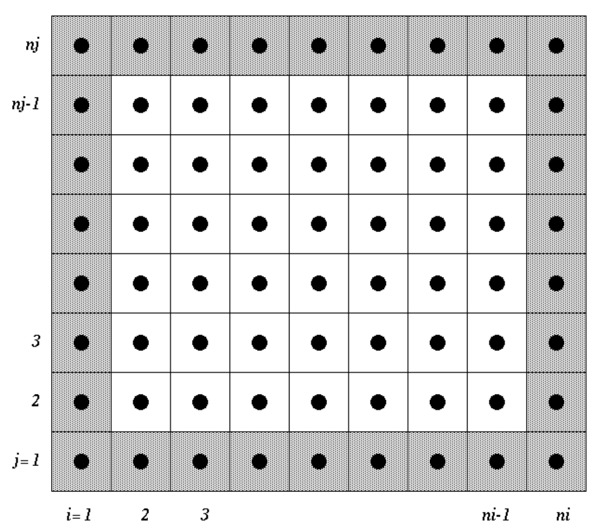

Figure 2: Grid system for the solution of the Poisson equation. The shaded points are ghost points.

2. Consider the exact solution to the Poisson equation:

$$u_{ex}(x,y) = \sin(2\pi n x)\sin(2\pi n y) \tag{3}$$

   with homogeneous Dirichlet conditions on the boundaries; solve equation (1) for a range of values of $1 \leq n \leq [\min(ni, nj) - 1]/2$. Use the following methods:

   (a) Jacobi

   (b) SOR

3. Perform a grid refinement study and show how the error changes as the number of points is increased. You can use a different number of points in x and y.

4. Repeat the steps for

$$u_{ex}(x,y) = \cos(2\pi n x)\cos(2\pi n y) \tag{4}$$

   and homogeneous Neumann conditions on the boundaries. Here, the BC for one boundary point will have to be set to Dirichlet to avoid the singularity given by all Neuman conditions.

5. Discuss the choice of the relaxation parameter ω, and how it is affected by the choice of n, the number of grid points and the type of boundary conditions.

All results above should be summarized in a brief report. Email a pdf file to me together with a zip file with your source code.