MAE 6225 Homework 1

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1.

$$\frac{\delta\phi}{\delta x} = \frac{-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2}}{12h} :$$

$$\phi_{j-2} = \phi_j - 2h\phi_j{}' + \frac{4h^2}{2!}\phi_j{}'' - \frac{8h^3}{3!}\phi_j{}''' + \frac{16h^4}{4!}\phi_j{}'''' + o\left(h^5\right)$$

$$\phi_{j-1} = \phi_j - h\phi_j{}' + \frac{h^2}{2!}\phi_j{}'' - \frac{h^3}{3!}\phi_j{}''' + \frac{h^4}{4!}\phi_j{}'''' + o\left(h^5\right)$$

$$\phi_j = \phi_j$$

$$\phi_{j+1} = \phi_j + h\phi_j{}' + \frac{h^2}{2!}\phi_j{}'' + \frac{h^3}{3!}\phi_j{}''' + \frac{h^4}{4!}\phi_j{}'''' + o\left(h^5\right)$$

$$\phi_{j+2} = \phi_j + 2h\phi_j{}' + \frac{4h^2}{2!}\phi_j{}'' + \frac{8h^3}{3!}\phi_j{}''' + \frac{16h^4}{4!}\phi_j{}'''' + o\left(h^5\right)$$

$$\frac{\delta\phi}{\delta x} = \alpha_1\phi_{j-2} + \alpha_2\phi_{j-1} + \alpha_3\phi_j + \alpha_4\phi_{j+1} + \alpha_5\phi_{j+2}$$

$$= \left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\right)\phi_j$$

$$+ \left(-2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5\right)h\phi_j{}'$$

$$+ \left(4\alpha_1 + \alpha_2 + \alpha_4 + 4\alpha_5\right)\frac{h^2}{2}\phi_j{}''$$

$$+ \left(-8\alpha_1 - \alpha_2 + \alpha_4 + 8\alpha_5\right)\frac{h^3}{6}\phi_j{}'''$$

$$+ \left(16\alpha_1 + \alpha_2 + \alpha_4 + 16\alpha_5\right)\frac{h^4}{24}\phi_j{}'''' + o\left(h^5\right)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$$

$$(-2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5)\, h = 1$$

$$4\alpha_1 + \alpha_2 + \alpha_4 + 4\alpha_5 = 0$$

$$-8\alpha_1 - \alpha_2 + \alpha_4 + 8\alpha_5 = 0$$

$$16\alpha_1 + \alpha_2 + \alpha_4 + 16\alpha_5 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & 1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & 1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{12h} \\ -\frac{2}{3h} \\ 0 \\ \frac{2}{3h} \\ -\frac{1}{12h} \end{bmatrix}$$

$$\frac{\delta\phi}{\delta x} = \frac{-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2}}{12h} + o\left(h^4\right)$$

This scheme is **4th order accurate**.

$$\frac{\delta\phi}{\delta x} = \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{6h} :$$

$$\phi_{j+1} = \phi_j + h\phi_j' + \frac{h^2}{2!}\phi_j'' + \frac{h^3}{3!}\phi_j''' + o\left(h^4\right)$$

$$\phi_j = \phi_j$$

$$\phi_{j-1} = \phi_j - h\phi_j' + \frac{h^2}{2!}\phi_j'' - \frac{h^3}{3!}\phi_j''' + o\left(h^4\right)$$

$$\phi_{j-2} = \phi_j - 2h\phi_j' + \frac{4h^2}{2!}\phi_j'' - \frac{8h^3}{3!}\phi_j''' + o\left(h^4\right)$$

$$\frac{\delta\phi}{\delta x} = \alpha_1\phi_{j+1} + \alpha_2\phi_j + \alpha_3\phi_{j-1} + \alpha_4\phi_{j-2}$$

$$= \left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\right)\phi_j$$

$$+ \left(\alpha_1 - \alpha_3 - 2\alpha_4\right)h\phi_j'$$

$$+ \left(\alpha_1 + \alpha_3 + 4\alpha_4\right)\frac{h^2}{2}\phi_j''$$

$$+ \left(\alpha_1 - \alpha_3 - 8\alpha_4\right)\frac{h^3}{6}\phi_j''' + o\left(h^4\right)$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

$$\alpha_1 - \alpha_3 - 2\alpha_4 = \frac{1}{h}$$

$$\alpha_1 + \alpha_3 + 4\alpha_4 = 0$$

$$\alpha_1 - \alpha_3 - 8\alpha_4 = 0$$

3

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & -1 & -8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3h} \\ \frac{1}{2h} \\ -\frac{1}{h} \\ \frac{1}{6h} \end{bmatrix}$$

$$\frac{\delta \phi}{\delta x} = \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{6h} + o\left(h^3\right)$$

This scheme is **3rd order accurate**.

2. Construct a one-sided scheme which is **4th order accurate** to approximate $\frac{\partial \phi}{\partial x}$:

$$\frac{\partial \phi}{\partial x} \approx \frac{\delta \phi}{\delta x} = \alpha_1 \phi_{j+4} + \alpha_2 \phi_{j+3} + \alpha_3 \phi_{j+2} + \alpha_4 \phi_{j+1} + \alpha_5 \phi_j$$