Assignment 10

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Question 1

Please use the theory of generating functions to determine the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = n$, where the variables are constrained in the following way: $x_1 = 2k$ for some $k \in N$, $x_2 \in 0, 1, 2$ $x_3 = 3q$ for some $q \in N$, and $x_4 \in 0,1$.

Claim: There are n + 1 solutions to $x_1 + x_2 + x_3 + x_4 = n$, where the variables are constrained in the following way: $x_1 = 2k$ for some $k \in N$, $x_2 \in 0, 1, 2$ $x_3 = 3q$ for some $q \in N$, and $x_4 \in 0,1$.

Proof: We can identify x_1, x_2, x_3, x_4 to be

$x_1 = A(x) = \frac{1}{1-x^2}$

$x_2 = B(x) = \frac{1-x^2}{1-x} = 0 + 1 + 2$

$x_3 = C(x) = \frac{1}{1-x^3}$

$x_4 = D(x) = \frac{1-X^2}{1-x} = 0 + 1$

To explore every possibility we multiple A(x)B(x)C(x)D(x) to get $\frac{1}{(1-x)^2}$ and we know that $\frac{1}{(1-x)^2}$ is equal to n + 1 so we can denote that

$\sum_{n>=0} (n + 1)x^2 = \frac{1}{(1-x)^2}$

The coefficient for x^n is equal to n + 1

Question 2

Please revisit and re-solve the pizza problem using generating functions to obtain the closed formula: P (n) = $\frac{1}{2}$

$n^2 + \frac{1}{2}n + 1 = \binom{n}{2} + n + 1.$

Claim: P (n) = $\frac{1}{2}$ n^2 + $\frac{1}{2}$n + 1 is a solution to the pizza problem.

Proof: If we take the recurrence relation formula of $a_n = a_{n-1} + n$ as a generator function we can see that $\sum_{n>=0} a_{n-1}x^n + \sum_{n>=0} nx^n = \sum_{n>=0} a_n x^n$. Using the lemma we were given in class of nx^n is $\frac{x}{(1-x)^2}$. Using this lemma we can get the equation $\frac{x}{(1-x)^2} + x\sum_{n>=0} a_n x^n = \sum_{n>=0} a_n x^n$. With some factorization we can get $\frac{x}{(1-x)^3} + \frac{1}{1-x}$ which is the same as $\sum_{n>=0} P(a_n) = \sum_{n>=0}(\frac{n^2}{2} + \frac{n}{2} + 1$ which outlines P(n) to be $\frac{1}{2}$ n^2 + $\frac{1}{2}$n + 1 and as it is defined as $\binom{n}{2}$ + n + 1.

Question 3

Please solve the following recurrence relations using the theory of generating functions:

(a) $a_n = a_{n-1} + a_{n-2} + n$, n≥2 ,a0 = 0 , a1 = 1

(b) $b_n = 4b_{n-1} - b_{n-2} - 6b_{n-3}$, n≥3, b0 = 0, b1 = 1, b2 = 1