IIR Impulse Invariance Design Project

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Part 1

Discrete-filter design specification:

Passband:

$1-\delta_l \le \left|H_d\,(e^{j\omega})\right| \le 1, \qquad |\omega| \le 0.25\pi$

$\Rightarrow 1-0.2929 \le \left|H_d\,(e^{j\omega})\right| \le 1, \qquad |\omega| \le 0.25\pi$

$\Rightarrow 0.7071 \le \left|H_d\,(e^{j\omega})\right| \le 1, \qquad |\omega| \le 0.25\pi$

Stopband:

$\left|H_d\,(e^{j\omega})\right| \le \delta_h, \quad 0.4\pi \le |\omega| \le \pi$

$\Rightarrow \left|H_d\,(e^{j\omega})\right| \le \sqrt{0.1}, \qquad 0.4\pi \le |\omega| \le \pi$

Corresponding analog-filter design specification with period *T*:

Passband:

$1-\delta_l \le \left|H_d\,(j\Omega)\right| \le 1, \qquad |\Omega| \le \frac{0.25\pi}{T}$

$\Rightarrow 1-0.2929 \le \left|H_d\,(j\Omega)\right| \le 1, \qquad |\Omega| \le \frac{0.25\pi}{T}$

$\Rightarrow 0.7071 \le \left|H_d\,(j\Omega)\right| \le 1, \qquad |\Omega| \le \frac{0.25\pi}{T}$

Stopband:

$\left|H_d\,(j\Omega)\right| \le \delta_h, \quad \frac{0.4\pi}{T} \le |\Omega| \le \frac{\pi}{T}$

$\Rightarrow \left|H_d\,(j\Omega)\right| \le \sqrt{0.1}, \qquad \frac{0.4\pi}{T} \le |\Omega| \le \frac{\pi}{T}$

Part 2

Continuous time butterworth filter formula:

$\left|H_d\,(j\Omega)\right|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$

$\left|H_d\,(j\Omega)\right|^2 = H_d\,(\Omega)\,H_d^*(\Omega) = H_d\,(\Omega)\,H_d\,(-\Omega)$

Passband:

$\left|H_d\,(\frac{\pi}{4T})\right|^2 = \frac{1}{1+(\pi/4T\Omega_c)^{2N}} \ge (0.7071)^2, \qquad |\Omega| \le \frac{0.25\pi}{T}$

Stopband:

$\left|H_d\,(\frac{4\pi}{10T})\right|^2 = \frac{1}{1+(4\pi/10T\Omega_c)^{2N}} \le 0.1, \qquad \frac{0.4\pi}{T} \le |\Omega| \le \frac{\pi}{T}$

Solving for N:

(1) ---------------- $(\frac{0.25\pi}{T\Omega_c})^{2N} = (0.7071)^{-2} - 1 \approx 1$

(2) --------------- $(\frac{0.4\pi}{T\Omega_c})^{2N} = (0.1)^{-1} - 1 = 9$

$(2) \div (1)$ ---------------- $2N log(\frac{0.4}{0.25}) = log(9)$

$\Rightarrow N = \frac{log(9)}{2log(\frac{0.4}{0.25})} = 2.3378 = 3$

$\therefore N \text{ is independent of } T$

Solving for Ω_c with N = 3 from above:

$(\frac{0.25\pi}{T\Omega_c}) \approx 1^{(1/6)} = 1$

$\Rightarrow \Omega_c \approx \frac{0.25\pi}{T}$

$\therefore \Omega_c \text{ is dependent of } T$

Since Ω_c depends on T, then $H_d(\Omega)$ depends on T.

Part 3

Assuming T=1, $\Omega_c \approx \frac{0.25\pi}{1} = 0.7853$

Poles,

$s_k = \pm\Omega_c exp[j(1+N+2k)(\frac{\pi}{2N})]$ Where k=0 to N-1

=> $s_k = \pm 0.7853 exp[j(1+3+2k)(\frac{\pi}{6})]$ Where k=0 to 2

$s_0 = \pm 0.7853 exp[j(1+3)(\frac{\pi}{6})] = \pm 0.7853[cos(4\pi/6) + jsin(4\pi/6)] = -0.393 + j0.68, + 0.393 - j0.68$

$s_1 = \pm 0.7853 exp[j(1+3+2)(\frac{\pi}{6})] = \pm 0.7853[cos(6\pi/6) + jsin(6\pi/6)] = -0.7853 + j0, + 0.7853 - j0$

$s_2 = \pm 0.7853 exp[j(1+3+4)(\frac{\pi}{6})] = \pm 0.7853[cos(8\pi/6) + jsin(8\pi/6)] =$

$-0.393 - j0.68, \ +0.393 + j0.68$

We choose poles on the left half s-place as the poles of $H_c(s)$ for the system to be causal and stable.

The poles of $H_c(s)$ are $-0.393 + j0.68$, $-0.7853 + j0$, $-0.393 - j0.68$

The transfer function of the Butterworth filter is

$$H_c(s) = \prod_{k=1}^{3} \frac{1}{(1-\frac{s}{s_k})} = \frac{\Omega_c^3}{\prod_{k=1}^{3}(s-s_k)} ,$$

$$H_c(s) = \frac{0.7853^3}{(s-0.393+j0.68)(s-0.7853+j0)(s-0.393-j0.68)}$$

Then use partial fraction to find the coefficients in order to apply inverse laplace transform

$$H_c(s) = \frac{0.7853^3}{(s-0.393+j0.68)(s-0.7853+j0)(s-0.393-j0.68)}$$

$= \frac{A_1}{(s-0.393+j0.68)} + \frac{A_2}{(s-0.7853+j0)} + \frac{A_3}{(s-0.393-j0.68)}$

$A_1 = \frac{0.4843}{(0.393-j0.68-0.7853)(0.393-j0.68-0.393-j0.68)} = \frac{0.4843}{(-0.3923-j0.68)(-j1.36)}$

$= \frac{0.4843}{(-0.9248+j0.5335)}$

$A_2 = \frac{0.4843}{(0.7853-0.393+j0.68)(0.7853-0.393-j0.68)} = \frac{0.4843}{(0.3923+j0.68)(0.3923-j0.68)}$

$= \frac{0.4843}{0.1539+04624} = 0.786$

$A_3 = \frac{0.4843}{(0.393+j0.68-0.393+j0.68)(0.393+j0.68-0.7853)} = \frac{0.4843}{(j1.36)(-0.3923+j0.68)}$

$= \frac{0.4843}{(-0.9248-j0.533528)}$

$h_c(t) = \sum_{k=1}^{3} A_k e^{s_k t} \ \ for\ t \geq 0,\ 0\ \ for\ t < 0$

= $\frac{0.4843}{(-0.9248+j0.5335)} e^{(-0.393+j0.68)t} + 0.786e^{-0.7853t} + \frac{0.4843}{(-0.9248-j0.533528)} e^{(-0.393-j0.68)t}$

Part 4

Impulse invariance requires $h[n] = T h_c(nT)$

$h[n] = \sum_{k=1}^{3} A_k e^{s_k nT} \ \ for\ n \geq 0,\ 0\ \ for\ n < 0$

Then,

$h[n] \ = [\ A_1 e^{s_1 nT} \ + A_2 e^{s_2 nT} \ + \ A_3 e^{s_3 nT}]$

$H(z) = \sum_{n=0}^{\infty} T h_c[nT] z^{-n} = \sum_{n=0}^{\infty} (A_1 exp(s_1 nT) z^{-n} + A_2 exp(s_2 nT) z^{-n} + A_3 exp(s_3 nT) z^{-n})$

$= \frac{A_1}{1-exp(s_1 T) z^{-1}} + \frac{A_2}{1-exp(s_2 T) z^{-1}} + \frac{A_3}{1-exp(s_3 T) z^{-1}}$

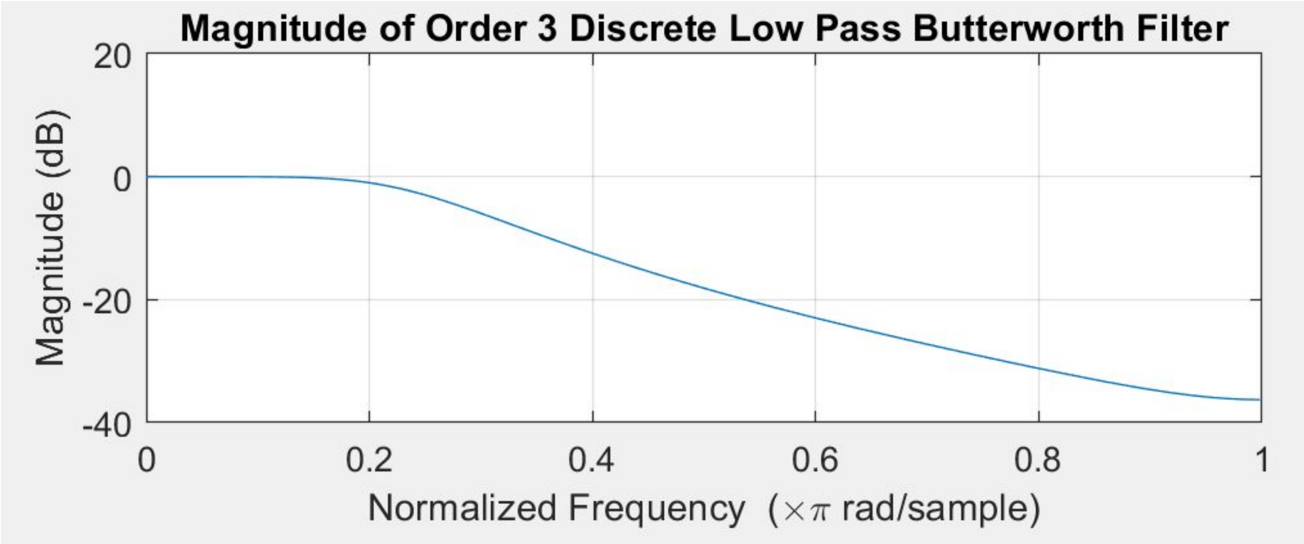
$= \frac{b_1 + b_2 z^{-1} + b_3 z^{-2}}{1-a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$, the coefficients a and b are real valued.

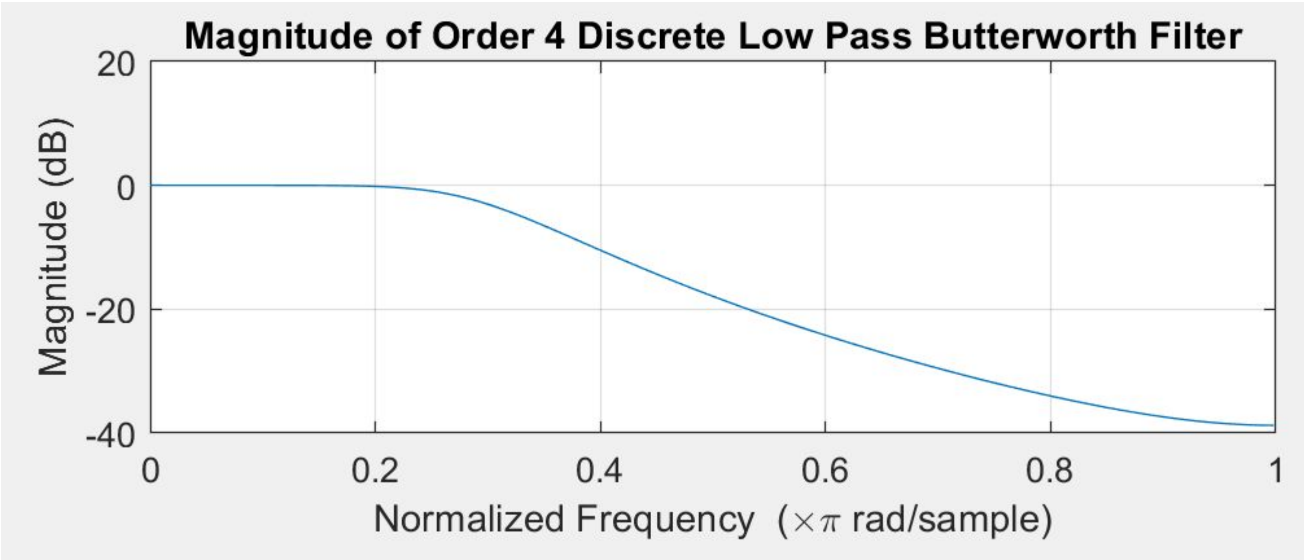
s_k *has radius* ω_c/T => $s_k = j\omega_c q_k/T$ *where* q_k *is on unit circle in complex plane*

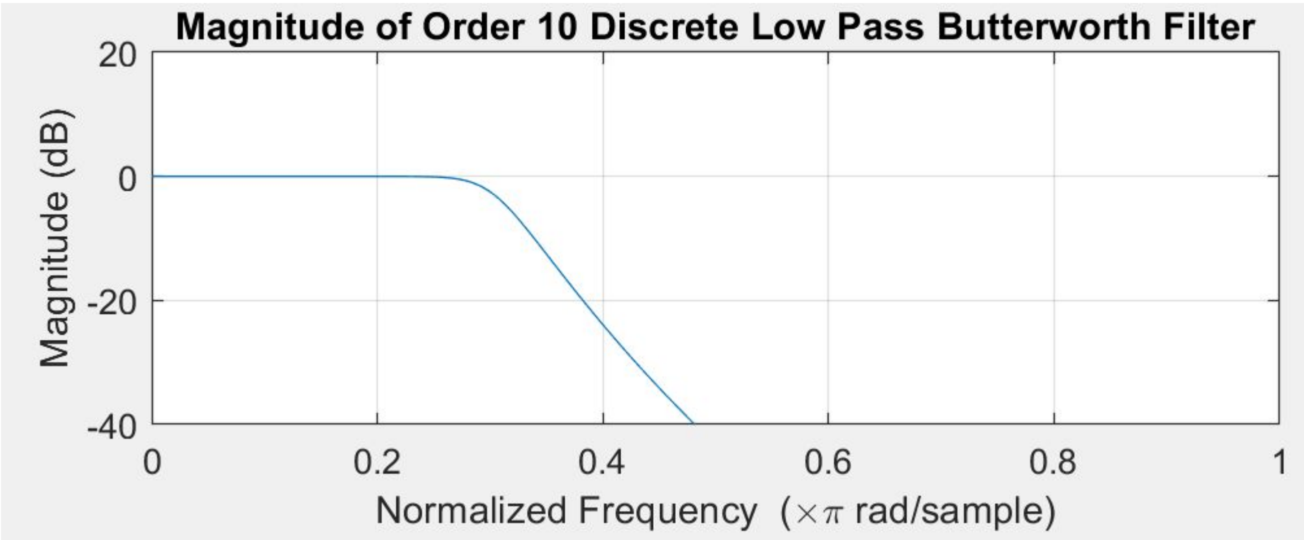
The Z-transform does not depend on T.

It can be seen that a pole at $s = s_k$ in the s-plane transforms to a pole at $z = e^{s_k T}$ in the z-plane. We began our design with discrete-time filter specifications, therefore the parameter T sampling interval has no significance in the design process.

Part 5







For N = 3: 0dB for $|\omega| \leq 0.25\pi$ and -13dB for $0.4\pi \leq |\omega| \leq \pi$

For N = 4: 0dB for $|\omega| \leq 0.25\pi$ and -10dB for $0.4\pi \leq |\omega| \leq \pi$

For N = 10: 0dB for $|\omega| \leq 0.25\pi$ and -23dB for $0.4\pi \leq |\omega| \leq \pi$

Part 6

The transfer function of the digital prototype low pass filter is denoted by $H_{lp}(Z)$, where Z is same as z-transform variable z.

$$Z^{-1} = -(z^{-1}+\alpha)/(1+\alpha z^{-1})$$

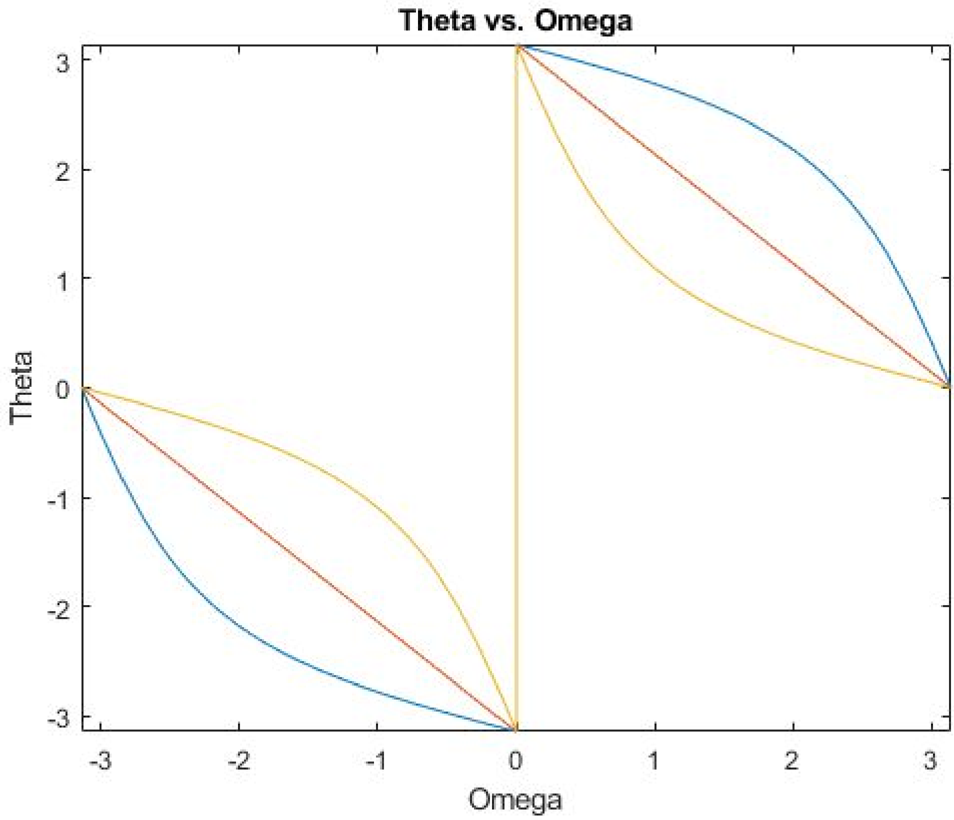
The unit circle in the Z-plane is mapped into the unit circle in the z-plane. If θ and ω are frequency variables in the Z-plane and z-plane, then

$e^{-j\theta} = G(e^{-j\omega})$ and $\theta = -arg\ G(e^{-j\omega})$

$$=> e^{-j\theta} = G(e^{-j\omega}) = \frac{-(e^{-j\omega}+\alpha)}{(1+\alpha e^{-j\omega})} = \frac{(e^{-j\omega}\alpha)}{(1+\alpha e^{-j\omega})}\frac{(1+\alpha e^{j\omega})}{(1+\alpha e^{j\omega})} = \frac{-e^{-j\omega}-\alpha-\alpha-\alpha^2 e^{j\omega}}{1+\alpha e^{-j\omega}+\alpha e^{-j\omega}+\alpha^2} = \frac{-2\alpha-cos(\omega)+jsin(\omega)-\alpha^2 cos(\omega)-\alpha^2 sin(\omega)}{1+cos(\omega)+\alpha\ sin(\omega)+cos(\omega)-\alpha\ sin(\omega)+\alpha^2}$$
$$= \frac{-2\alpha-(1+\alpha^2)cos(\omega)-j(\alpha^2-1)sin(\omega)}{1+2\alpha cos(\omega)+\alpha^2}$$

Then,

$$\theta = -arg\ G(e^{-j\omega}) = -arctan(\frac{(\alpha^2-1)sin(\omega)}{-2\alpha-(1+\alpha^2)cos(\omega)})$$



As we can see from the plot relation between θ and ω, any value of α between -1 and 1 produces varieties of mappings. Setting α to be 0(red), ½(yellow) and -½(blue) on the same plot, it can be observed that the mappings are the same when the domain are swapped.