The discrete charm of the fundamental group

January 31, 2021

This worksheet covers discrete analogs of symmetric products and braid groups of graphs. Let G be a simple locally finite graph. The *discrete fundamental group* (*DFG*) of G is the initial term of a sequence $A_n(G)$ of combinatorially-defined groups called the *discrete homotopy groups* of G. There are accompanying discrete homology groups $\mathcal{H}_n(G)$ and a Hurewicz theorem in dimension 1, as well as a natural map from $A_n(G)$ to $H_n(G)$ for all other n. For background and references, see my paper "Discrete homotopy of token configurations."

[1]:
```
load('dfg.sage')
```

0.1 Computing the DFG

While the DFG is defined combinatorially, it can be shown that $A_1(G)$ is isomorphic to the fundamental group of the 2-complex obtained by attaching a 2-cell to each 3- and 4-cycle of G. To compute the DFG using `dfg.sage`, simply call `discrete_fundamental_group(G)`.

[2]:
```
for i in range(3,8):
    G = graphs.CycleGraph(i)
    print("DFG of " + str(i) + "-cycle: "
          + str(discrete_fundamental_group(G)))
```

```
DFG of 3-cycle: Finitely presented group <  |  >
DFG of 4-cycle: Finitely presented group <  |  >
DFG of 5-cycle: Finitely presented group < e |  >
DFG of 6-cycle: Finitely presented group < e |  >
DFG of 7-cycle: Finitely presented group < e |  >
```

The DFG of an n-cycle is trivial if $n = 3$ or 4, and infinite cyclic otherwise.

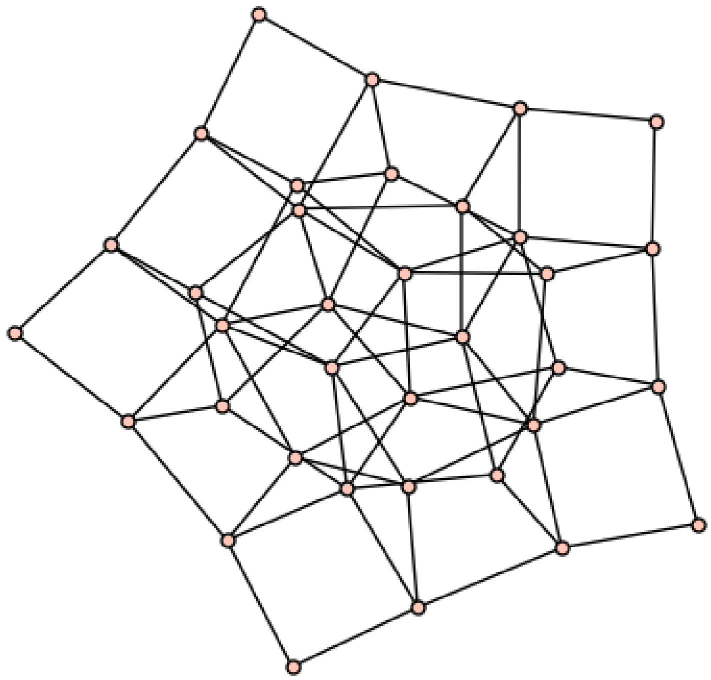
0.2 Reduced powers

The reduced powers of a graph are graphical analogs of the symmetric powers of a topological space. An *n-token configuration* on G is an assignment of n tokens to the vertices of G with multiple tokens allowed on each vertex. Two n-token configurations are *adjacent* if they differ by moving exactly one token to an adjacent vertex. Consider the graph whose vertices correspond to the n-token configurations on G, with two vertices adjacent if and only if the corresponding n-token configurations are adjacent. This is called the *nth reduced power* of G and is denoted by $G^{(n)}$.

To construct $G^{(n)}$, call `reduced_power(G, n)`.

[3]:
```
G1 = graphs.CycleGraph(5)
H1 = reduced_power(G1, 3)
H1.graphplot(vertex_labels=False, vertex_size=25).show()
```

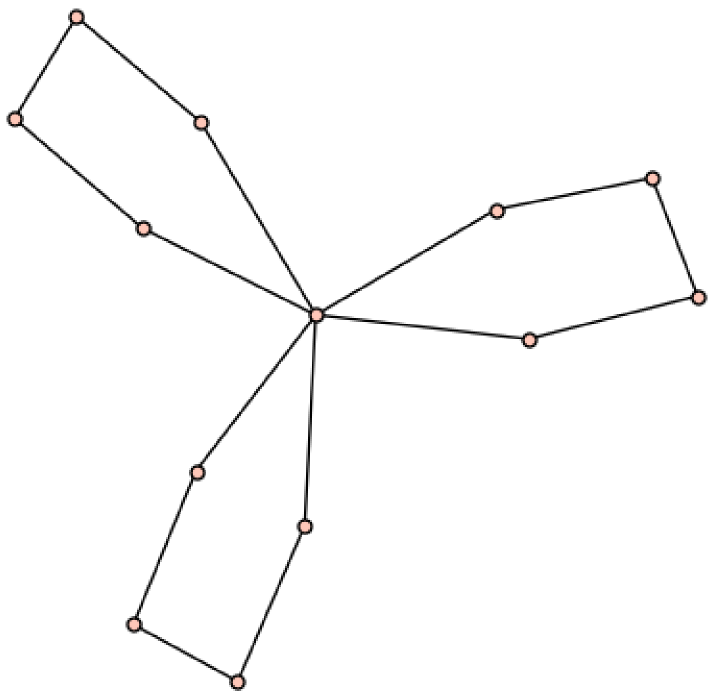
[3]:



It can be shown for all $n \geq 2$ that $A_1(G^{(n)})$ is isomorphic to the nth discrete cubical homology group $\mathcal{H}_1(G)$, i.e. the abelianization of $A_1(G)$. This mirrors a classical theorem in topology due to P. A. Smith. For example, the DFG of the 2nd reduced power of a k-fold wedge sum of m-cycles is a free abelian group on k generators, as long as $m \geq 5$:

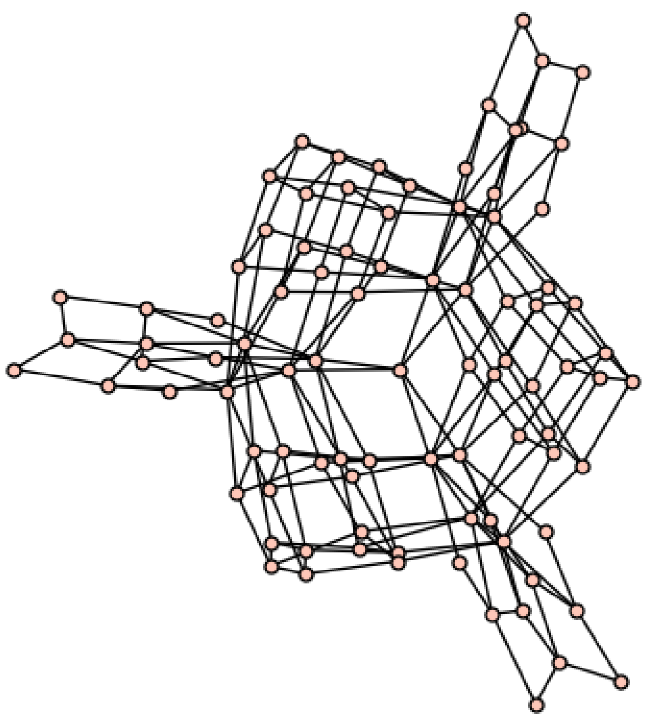
[4]:
```
G2 = polygon_wedge(5, 3)
H2 = reduced_power(G2, 2)
G2.graphplot(vertex_labels=False, vertex_size=25).show()
print("DFG of 2nd reduced power: "
      + str(discrete_fundamental_group(H2)))
H2.graphplot(vertex_labels=False, vertex_size=25).show()
```

[4]:



```
DFG of 2nd reduced power: Finitely presented group < e0, e13, e63 |
e0^-1*e63^-1*e0*e63, e13*e0^-1*e13^-1*e0, e13*e63^-1*e13^-1*e63 >
```

[4]:

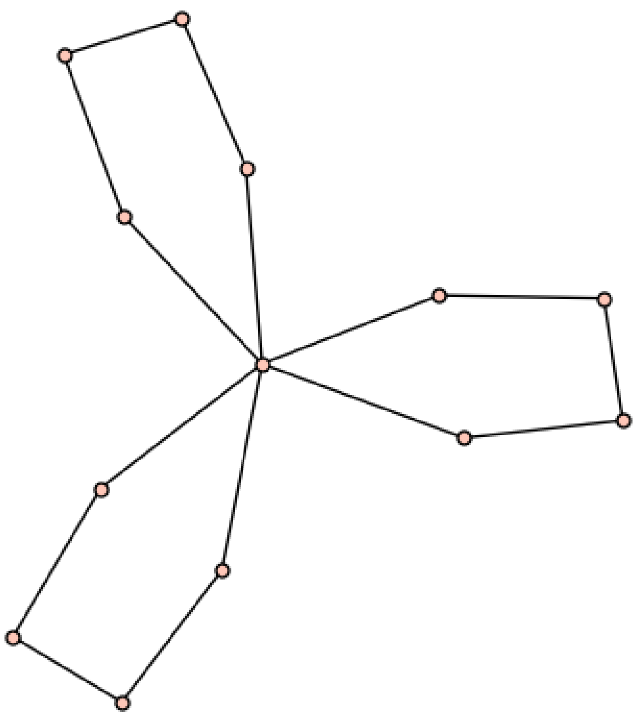


0.3 Token graphs

The *nth token graph* of G, denoted $G^{[n]}$, is the subgraph of $G^{(n)}$ induced by all n-token configurations with at most one token on each vertex of G. Token graphs are discrete analogs of unordered configuration spaces. It can be shown that if n is sufficiently small, then $A_1(G^{[n]})$ is isomorphic to the n-strand braid group of G, of interest in motion planning problems from topological robotics.

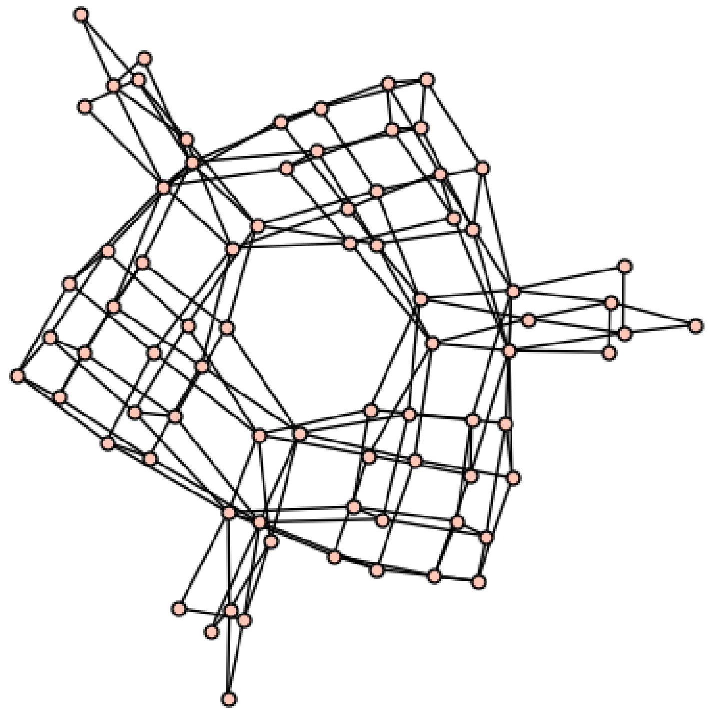
[5]:
```
G3 = polygon_wedge(5, 3)
H3 = token_graph(G2, 2)
G3.graphplot(vertex_labels=False, vertex_size=25).show()
print("DFG of 2nd token graph: "
      + str(discrete_fundamental_group(H3)))
H3.graphplot(vertex_labels=False, vertex_size=25).show()
```

[5]:



```
DFG of 2nd token graph: Finitely presented group < e10, e11, e14, e26, e30, e31,
e59, e61, e63, e73 |  >
```

[5]:



Here, $A_1(G^{[2]})$ is a free group on 10 generators. In general, if G is a k-fold wedge sum of m-cycles and $m \geq 5$, then $A_1(G^{[2]})$ is isomorphic to the 2-strand braid group of a bouquet of k circles, i.e. a free group on $3\binom{n}{2} + 1$ generators.