Solution to Problem #1 of Homework #2

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Solution.

We shall give a lower bound of the probability of an ϵ-typical sequence. and hence the number of typical sequences multiply the lower bound should be less then 1.

Lemma 1. *Given x^n, if $y^n \in \mathcal{T}_\epsilon^{(n)}(Y|x^n)$, then $P_{Y|X}(y^n|x^n) \geq 2^{-n(1+\epsilon)H(Y|X)}$*

Proof. Recall that

$$(x^n, y^n) \in \mathcal{T}_\epsilon^{(n)} \iff |\pi(x, y \mid x^n, y^n) - P_{X,Y}(x, y)| \leq \epsilon P_{X,Y}(x, y),\ \forall (x, y) \in \mathcal{X} \times \mathcal{Y} \quad (1)$$

Then

$$\begin{aligned} P(y^n|x^n) &\overset{\text{(a)}}{=} \prod_{i=1}^{n} P(y_i|x_i) \\ &\overset{\text{(b)}}{=} \prod_{(x,y)\in(\mathcal{X},\mathcal{Y})} P(y|x)^{n\pi(x,y|x^n,y^n)} \\ &= \prod_{(x,y)\in(\mathcal{X},\mathcal{Y})} 2^{n\pi(x,y|x^n,y^n)\log P(y|x)} \\ &\overset{\text{(c)}}{\geq} \prod_{(x,y)\in(\mathcal{X},\mathcal{Y})} 2^{n(1+\epsilon)P(x,y)\log P(y|x)} \\ &= 2^{n(1+\epsilon)\sum P(x,y)\log P(y|x)} \\ &\overset{\text{(d)}}{=} 2^{n(1+\epsilon)H(y|x)} \end{aligned}$$

Where

(a) holds since each (x_i, y_i) are independent.

(b) is because $n\pi(x, y|x^n, y^n) = \#\{i : (x_i, y_i) = (x, y)\}$.

(c) is the result of equation (1) since $\log P(y|x) \leq 0$ and

$$|\pi(x, y|x^n, y^n) - P_{X,Y}(x, y)| \leq \epsilon P_{X,Y}(x, y) \implies \pi(x, y|x^n, y^n) \leq (1 + \epsilon)P_{X,Y}(x, y).$$

(d) hold from the definition $H(y|x) = \sum_{x,y} P(x, y) \log P(y|x)$.

□

Now

$$
\begin{aligned}
1 &\geq \sum_{y^n \in \mathcal{T}_\epsilon^{(n)}(Y|x^n)} P_{Y|X}(y^n|x^n) \\
&\geq \left(\min_{y^n \in \mathcal{T}_\epsilon^{(n)}(Y|x^n)} P_{Y|X}(y^n|x^n) \right) \cdot \left| \mathcal{T}_\epsilon^{(n)}(Y|x^n) \right| \\
&\geq 2^{-n(1+\epsilon)H(Y|X)} \cdot \left| \mathcal{T}_\epsilon^{(n)}(Y|x^n) \right|
\end{aligned}
$$

Where the second equality holds by lemma 1, hence

$$
\left| \mathcal{T}_\epsilon^{(n)}(Y|x^n) \right| \leq 1/2^{-n(1+\epsilon)H(Y|X)} = 2^{n(1+\epsilon)H(Y|X)}.
$$

$\square$