Algorithm HW#1

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1 Problem 1.

1. False. Or else for all c, there exists $n_0(c) > 0$ s.t. $n/2 + 5 \le cn$, $\forall n \ge n_0(c)$. Choose $c = 1/2$, then $n/2 + 5 \le n/2 \iff 5 \le 0$, $\forall n \ge n_0(1/2)$, which leads to a contradiction, hence $n/2 + 5 \notin o(n)$.

2. False, let $f(n) = n^{1/2}, g(n) = n^{1/3}$, then

   - $f(n) \in o(n)$, since for all $c > 0$, let $N = 4/c^2 > 0$, then if $n \ge N$,

   $$\frac{n^{1/2}}{n} = \frac{1}{n^{1/2}} \le \frac{c}{2} \le c \implies n^{1/2} \le cn$$

   - $g(n) \in O(n)$ by choosing $c = 1$ and $N = 1$, then $n^{1/3} \le cn = n$ for all n.

   - $f(n) \notin O(g(n))$, since for all $c > 0$, choose $N = c^6$, then for $n \ge N$,

   $$\frac{n^{1/2}}{n^{1/3}} = n^{1/6} \ge \left(c^6\right)^{1/6} \ge c \iff n^{1/2} \ge cn^{1/3},$$

   so $f(n) \notin O(g(n))$.

   Hence we disprove the statement by giving a counter example.

2 Problem 2.

$$n^n > n! > 2^{3n} > 2^n > (\log_2(n))! > 3n^3 + 1 > \sqrt{n} + 3 = 2^{log_4(n)} > n^{0.01} > \log_2 n = \ln n$$

Here $f(n) > g(n)$ means that $g(n) \in o(f(n))$ and $f(n) = g(n)$ means that $f(n) = \Theta(g(n))$. Notice that in both case $f(n) = \Omega(g(n))$.

3 Problem 3.

1. We shall assume that $\inf_{1 \le n \le 3} T(n) > 0$, or else there exists a function $T(n)$ such that $T(n) \notin \Theta(f(n)), \forall f$.

   We shall prove that $T(n) \in \Theta\left(n(\log n)^2\right)$ by induction.
   Let

   $$c_1 = \min\left(\frac{1}{9(\log 9)^2} \inf_{3 \le n \le 9} T(n),\ \frac{1}{2\log 3}\right), \quad c_2 = \max\left(\frac{1}{3(\log 3)^2} \sup_{3 \le n \le 9} T(n),\ \frac{1}{\log 3}\right).$$

   Then $c_1 > 0$, since $T(n) = 3T(n/3) + n\log n > T(n/3)$, so
   $\inf_{3 \le n \le 9} T(n) > \inf_{1 \le n \le 3} T(n) > 0$. Now for $3 \le n \le 9$

   $$T(n) \ge \frac{n(\log n)^2}{9(\log 9)^2} T(n) \ge n(\log n)^2 \frac{1}{9(\log 9)^2} \inf_{3 \le n \le 9} T(n) \ge c_1 n(\log n)^2$$
   $$T(n) \le \frac{n(\log n)^2}{3(\log 3)^2} T(n) \le n(\log n)^2 \left(\frac{1}{3(\log 3)^2} \sup_{3 \le x \le 9} T(n)\right) \le c_2 n(\log n)^2$$

   So $c_1 n(\log n)^2 \le T(n) \le c_2 n(\log n)^2$ for $3 \le n \le 9$.

   Assume now $n > 9$ and for all $3 \le k < n$, $T(k) \le ck$. Then since $n/3 \ge 3$, we have

   $$\begin{aligned} T(n) &= 3T(n/3) + n\log n \ge c_1 n(\log n/3)^2 + n\log n \\ &= c_1 n(\log n - \log 3)^2 + n\log n \\ &= c_1 n(\log n)^2 + (c_1 n(\log 3)^2 + n\log n - 2c_1 n(\log n)(\log 3)) \\ &\ge c_1 n(\log n)^2 + (n\log n - 2c_1 n(\log n)(\log 3)) \\ &\ge c_1 n(\log n)^2 + \left(n\log n - \frac{2\log 3}{2\log 3} n(\log n)\right) \\ &= c_1 n(\log n)^2 \end{aligned}$$

   and

   $$\begin{aligned} T(n) &= 3T(n/3) + n\log n \le c_2 n(\log n/3)^2 + n\log n \\ &= c_2 n(\log n - \log 3)^2 + n\log n \\ &= c_2 n(\log n)^2 - (2c_2 n(\log n)(\log 3) - c_2 n(\log 3)^2 - n\log n) \\ &= c_2 n(\log n)^2 - (c_2 n(\log n)(\log 3) - c_2 n(\log 3)^2) - (c_2(\log 3)n(\log n) - n\log n) \\ &\le c_2 n(\log n)^2 \end{aligned}$$

   because $\log n \ge \log 3$ and $c_2 \ge 1/\log 3 \implies c_2 \log 3 \ge 1$.

   So $c_1 n(\log n)^2 \le T(n) \le c_2 n(\log n)^2$. Hence by induction, we prove that
   $T(n) \in \Theta(n(\log n)^2)$.

2. We shall prove that $T(n) \in \Theta(n^3)$ by induction.

Assume for all $1 \le k < n$, $T(k) \le ck^3$. Then

$$\begin{aligned} T(n) &= 4T(n/2) + n^3 \\ &\le 4c(n/2)^3 + n^3 \\ &= cn^3/2 + n^3 \\ &= n^3 + n^3 \\ &= 2n^3 = cn^3 \end{aligned}$$

Hence by induction, we prove that $T(n) \in O(n^3)$.
Notice also that $T(n) = 4T(n/2) + n^3 \ge n^3$, so $T(n) \in \Omega(n^3)$, hence $T(n) \in \Theta(n^3)$.

4 Problem 4

1. True. For all c, let $c' = \frac{1}{2c}$. Since $f(n) \in o(n^2)$, exists N such that $n > N \implies f(n) \le c'n^2$. Then $n^2 - cf(n) \ge n^2 - n^2/2 = n^2/2$. So for $c_0 = 1/2$ and $N_0 = N$, $n \ge N_0 \implies c_0 n^2 \le n^2 - cf(n)$, hence $n^2 - cf(n) \in o(n^2)$.

2. False. Let $f(n) = n^2/2$ and $c = 2$, then

   - $2^{f(n)} \in o(2^{n^2})$, since for any $c' > 0$, if $c' \ge 1$ then $2^{f(n)} \le 2^{n^2}$ for all $n \ge 0$, so assume that $0 < c' < 1$, then choose $N = \sqrt{2\log_2(1/c)}$, for $n \ge N$,

   $$2^{n^2/2} \ge 2^{\log_2(1/c)} \ge \frac{1}{c} \implies 2^{f(n)} = 2^{n^2/2} \le c2^{n^2}$$

     hence $2^{f(n)} \in o(2^{n^2})$.
   - If $c = 2$, then $n^2 - cf(n) = n^2 - 2n^2/2 = 0$, and obviously $0 \notin \Omega(n^2)$.

   Hence we disprove the statement by giving a counter example.

5 Problem 5

1. Let $f(n) = n^{2n}$, Then

   $$T(n) = 2^n n^n T(n/2) \implies T(n) = \frac{n^{2n}}{(n/2)^n} T(n/2) \implies \frac{T(n)}{f(n)} = \frac{T(n/2)}{f(n/2)} = c$$

   where c is a constant. Let c' be such constant that $c' = c\frac{T(1)}{f(1)}$,
   so $T(n) = c\frac{T(1)}{f(1)} f(n) = c'f(n)$, hence $T(n) \in \Theta(f(n)) = \Theta(n^{2n})$.

2. The original recursive formula is equivalent to

   $$T(n) = \frac{n^2}{(n/2)^2} T(n/2) + \frac{n^2}{2\log_2 n} \implies \frac{T(n)}{n^2} = \frac{T(n/2)}{(n/2)^2} + \frac{1}{2\log_2 n}$$

   Let $k = \log_2 n$, so

   $$\frac{T(2^k)}{4^k} = \frac{T(2^{k-1})}{4^{k-1}} + \frac{1}{2k}$$

If we let $f(k) = T(2^k)/4^k$, then $f(k) = f(k-1) + 1/2k$, with initial condition $f(0) = T(1) = 1$. (or $C_1 \le f(k) \le C_2$ if $1 \le k < 2$.) Hence

$$f(k) = C \sum_{m=1}^{k} \frac{1}{2m} = 2C \sum_{m=1}^{k} \frac{1}{m}$$

and notice that since $1/x$ monotonic decrease,

$$\log(k+1) - \log 2 \le \int_2^{k+1} \frac{1}{x}\,\mathrm{d}x \le \sum_{m=2}^{k} \frac{1}{m} \le \int_1^k \frac{1}{x}\,\mathrm{d}x = \log(k)$$

and by the fact that $\log(k+1) \le \log(2k) = \log(k) + \log 2$, so $\sum\limits_{m=1}^{k} 1/m \in \Theta(\log m)$.

Hence $f(k) \in \Theta(\log k)$ and $T(2^k) = 4^k f(k) \in \Theta(4^k \log k)$. Thus

$$T(n) = 4^{\log_2 n} \log(\log_2 n) = n^2 \log\log_2 n \in \Theta(n^2 \log\log n).$$

6 Problem 5

1. True, $T(n) = \Theta(f(n)) \iff T(n) = \Theta(n^2)$ since $f(n) = \Theta(n^2)$.

   Again since $f(n) = \Theta(n^2)$, then exists N_1, c_1, N_2, c_2, $n \ge N_1 \implies f(n) \ge c_1 n^2$ and $n \ge N_2 \implies f(n) \le c_2 n^2$, let $N = \max(N_1, N_2)$, and let

   $$d_1 = \min\left(\frac{1}{(2N)^2} \inf_{N \le n < 2N} f(n),\ 2c_1\right), \quad d_2 = \max\left(\frac{1}{N^2} \sup_{N \le n < 2N} f(n),\ 2c_2\right)$$

   Then for $N \le n < 2N, d_1 n^2 \le f(n) \le d_2 n^2$.

   Now we shall prove that the inequality also holds for any $n \ge 2N$. If for $k = n/2$, $d_1 k^2 \le f(k) \le d_2 k^2$ holds. Then

   $$f(n) \ge 2d_1 k^2 + c_1 n^2 \ge \frac{1}{2} d_1 n^2 + \frac{1}{2} d_1 n^2 = d_1 n^2$$

   and

   $$f(n) \le 2d_2 k^2 + c_2 n^2 \ge \frac{1}{2} d_2 n^2 + \frac{1}{2} d_2 n^2 = d_2 n^2$$

   So $d_1 n^2 \le f(n) \le d_2 n^2$. Hence by induction, $f(n) \in \Theta(n^2)$ and the proof is complete.

2. False, Let $f(n)$ be the function such that $f(2^t) = 4^{k^2}$ if $(k-1)^2 < t \le k^2$ for all $k \in \mathbb{N}$. Then $f(n) \in \Omega(n^2)$, since $f(2^t) = 4^{k^2} > 4^t > (2^t)^2$ and hence $f(n) > n^2$. Then for any $k \in \mathbb{N}$, we have

   $$\begin{aligned} T\left(2^{k^2}\right) &= 2T\left(2^{k^2}/2\right) + f\left(2^{k^2}\right) = 2T\left(2^{k^2-1}\right) + f\left(2^{k^2}\right) \\ &= 4T\left(2^{k^2-2}\right) + 2f\left(2^{k^2-1}\right) + f\left(2^{k^2}\right) \\ &\vdots \\ &= 2^k T\left(2^{k^2-k}\right) + \sum_{t=0}^{k} 2^t f\left(2^{k^2-t}\right) \end{aligned}$$

Notice that $(k-1)^2 \le (k-1)k = k^2-k \le k^2$ so $f\left(2^{k^2-t}\right) = 4^{k^2}$ when $t=k$. Thus

$$T\left(2^{k^2}\right) = 2^k T\left(2^{k^2-k}\right) + \sum_{t=0}^{k} 2^t f\left(2^{k^2-t}\right) \ge 2^k 4^{k^2}$$

So $T(n) = 2^{\sqrt{\log n}} n^2$ for all $n = 2^{k^2}$. Obviously $2^{\sqrt{\log n}} n^2 \notin O(n^2)$ since $2^{\sqrt{\log n}} \to \infty$ as $n \to \infty$. Hence $T(n) \notin O(n^2)$, and we disprove the statement by giving a counter example.

7 Problem 7

1. We shall proof that $T(n) \in \Theta(n \log n)$ by induction. Choose c_1', c_2' so that $c_1' n \log n \le T(n) \le c_2' n \log n$ for $n \le 4$, and let

   $$c_1 = \min\left(c_1', \frac{2}{\log 4}\right), \quad c_2 = \max\left(c_2', \frac{2}{\log 4/3}\right)$$

   So $c_1 n \log n \le T(n) \le c_2 n \log n$ for all $n \le 4$.

   Now if $n > 4$, Assume that for $k = n/2$, $c_1 k \log k \le T(k) \le c_2 k \log k$. Then

   $$\begin{aligned} T(n) &= T(a) + T(b) + 2n \\ &\ge c_1 a \log a + c_1 b \log b + 2n = c_1 a \log a + c_1 b \log b + 2(a+b) \\ &= c_1 a\left(\log(a/n) + \log n + \frac{2}{c_1}\right) + c_1 b\left(\log(b/n) + \log n + \frac{2}{c_2}\right) \\ &= c_1(a+b)\log n + \left(\frac{2}{c_1} - \log(n/a)\right) + \left(\frac{2}{c_1} - \log(n/b)\right) \\ &\ge c_1 n \log n \end{aligned}$$

   since $\log(n/a), \log(n/b) \le \log 4 \le 2/c_2$.

   Similarly,

   $$\begin{aligned} T(n) &= T(a) + T(b) + 2n \\ &\le c_2 a \log a + c_2 b \log b + 2n \\ &= c_2(a+b)\log n - \left(\log(n/a) - \frac{2}{c_2}\right) - \left(\log(n/b) - \frac{2}{c_2}\right) \\ &\le c_2 n \log n \end{aligned}$$

   since $\log(n/a), \log(n/b) \le \log(4/3) \le 2/c_2$. Hence by induction, $c_1 n \log n \le T(n) \le c_2 n \log n$ and thus $T(n) \in \Theta(n \log n)$.

2. We shall proof that $T(n) \in \Theta(n^2)$.

   First we show that $a^2 + b^2 \le 8n^2/9$. WLOG, assume $a \ge b$. If $a \le 2n/3$, then $a^2 + b^2 \le 2a^2 \le 8n^2/9$. If $a > 2n/3$, then $b < n/3$, hence $a^2 + b^2 \le (3n/4)^2 + (1/3)^2 = 97n/144 \le 8n^2/9$.

Now Choose c' so that $T(n) \leq c'n^2$ for $n \leq 4$, and let $c = \max(c', 9)$

If $n > 4$, Assume that for $k = n/2$, $T(k) \leq ck^2$. Then

$$
\begin{aligned}
T(n) &= T(a) + T(b) + n^2 \\
&\leq ca^2 + cb^2 + n^2 = c(a^2 + b^2) + n^2 \\
&\leq c\left(8n^2/9\right) + n^2 \\
&= cn^2 \left(\frac{8 + 9/c}{9}\right) \\
&\leq cn^2
\end{aligned}
$$

since $c \geq 9$, so $T(n) \in O(n^2)$. Notice that $T(n) = T(a) + T(b) + n^2 \geq n^2$, so $T(n) \in \Omega(n^2)$ hence $T(n) \in \Theta(n^2)$.