# Free Product

**Definition 1.** Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty set of groups $G_\alpha$ (We shall assume that these groups are disjoint). The **free product** of $\mathcal{S}$ is a pair $(G, \{i_\alpha\})$ where $G$ is a group and $i_\alpha :: G_\alpha \to G$ are homomorphism which has the following universal property: For any group $H$ and homomorphisms $\varphi_\alpha :: G_\alpha \to H$, there exists a unique group homomorphism $\varphi :: G \to H$ such that $\varphi \circ i_\alpha = \varphi_\alpha$ for all $\alpha \in \mathcal{A}$ (e.g., the diagram below commutes). We write $G = \mathop{\ast}_\alpha G_\alpha$ (some also write $G = \prod^* G_\alpha$ or $G = \coprod G_\alpha$).

$$
\begin{tikzcd}
G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "\varphi_\alpha"'] & \mathop{\ast}_\alpha G_\alpha \arrow[d, "\exists\varphi"] \\
 & H
\end{tikzcd}
$$

Figure 1: The diagram of the free product

*Remark.* The free product could also be defined as the **coproduct** of $\{G_\alpha\}$ if we consider the morphisms as group homomorphisms in the category of groups.

**Definition 2.** A **word** (with respect to a free product) is a finite sequence $g_1 g_2 \cdots g_n$ such that every $g_i$ is in one of $G_\alpha$. Each $g_i$ in it is called a **factor**. We shall denote the empty word by 1. The **length** of a word is the number of factors in it. Define $W(\{G_\alpha\})$ or simply $W$ to be the set of all of the words. For each $g$, let $\mathfrak{G}(g)$ be the group which $g$ belongs, and $g^{-1}$ be the inverse of $g$ in that group.

Write $xy$ as the **concatenate** of two words $x, y$.

We define two **reducing operations** on words

1. Drop a factor $g_i$ which is the identity element.
2. Collapse two factors $g_i g_{i+1}$ to a single one $g^*$ if $\mathfrak{G}(g_i) = \mathfrak{G}(g_{i+1})$ and their product in that group is $g^*$.

Together with the inverse of these two reducing operations, we have four **operations** that could alter a word.

Two words $x$ and $y$ are said to be **equivalent** if there is a finite sequence of operations that change $x$ to $y$.

A word $x$ is a **reduced word** if no reducing operation could be preformed on it. That is, $x$ contains no factor $g_i$ which is the identity of a group $G_\alpha$, and contains no consecutive factors $g_i, g_{i+1}$ that belong to the same group.

**Proposition 1.** *The statement above defines a valid equivalent relation $\sim$.*

*Proof.* We check that

- $x \sim x$ clearly from definition.
- $x \sim y \implies y \sim x$ since we have inverse operations.
- $x \sim y \wedge y \sim z \implies x \sim z$ since we could combine the sequence of operations that changes $x$ to $y$ and that changes $y$ to $z$ to obtain a sequence of operations that changes $x$ to $z$.

□

**Proposition 2.** *Free product exists and is unique up to isomorphism.*

*Proof of existence.* Denote the equivalence classes by $[x]$ using the equivalence relation mentioned above. Let $G = W/\sim = \{[x] : x \in W\}$ and define the multiplication as $[x][y] = [xy]$. We shall check that the multiplication is well defined.

Suppose that $x$ and $x'$ differ by only one operations. Then for all $y$, it is easy to see that $xy \sim x'y$ and $yx \sim yx'$ since we could still preform that operation on the $x, x'$ part and turn the former word to the later. Iterating this kind of equivalence shows that $v \sim v'$ and $w \sim w'$ implies $vw \sim v'w'$. Hence the multiplication is well defined.

Then we check that with this multiplication, $G$ is indeed a group.

**associative:** Simply by the associative of word concatenation.

**indentity:** The equivalent class of empty word $[1]$ is the indentity obviously.

**inverse:** For any word $x = g_1g_2 \cdots g_n$. Let $y = g_n^{-1}g_{n-1}^{-1} \cdots g_1^{-1}$. It is easy to see that $[xy] = [yx] = [1]$, hence $[x][y] = [y][x] = [1]$.

Let $i_\alpha(g) = [g]$, then each $i_\alpha$ is a homomorphism since $i_\alpha(g_1g_2) = [g_1g_2] = [g_1][g_2] = i_\alpha(g_1)i_\alpha(g_2)$.

Now, given $H$ and homomorphisms $\varphi_\alpha$, we shall construct $\varphi$. First define $\Phi :: W \to H$ by

$$\Phi(w) = \begin{cases} 1_H & \text{if } w \text{ is the empty word.} \\ \varphi_\alpha(g) & \text{if } w = g \text{ has length 1 and } g \in G_\alpha. \\ \Phi(g)\Phi(w') & \text{otherwise, and let } w = gw'. \end{cases}$$

It is easy to see that $\Phi(w_1w_2) = \Phi(w_1)\Phi(w_2)$. Let $\varphi([w]) = \Phi(w)$ for each $[w] \in G$ and we shall check that it is well defined. It is equivalent to ensure that $w \sim w' \implies \Phi(w) = \Phi(w')$. So we prove that

- $\Phi(w_1w_2) = \Phi(w_1 1_{G_\alpha} w_2)$: Since $\Phi(w_1 1_{G_\alpha} w_2) = \Phi(w_1)\Phi(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\varphi_\alpha(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\Phi(w_2) = \Phi(w_1w_2)$ by the fact that every homomorphism $\varphi_\alpha$ maps identity to identity.
- $\Phi(w_1g^*w_2) = \Phi(w_1g_1g_2w_2)$, if $g^* = g_1g_2$: WLOG assume $g^* \in G_\alpha$, then $\Phi(w_1g^*w_2) = \Phi(w_1)\phi_\alpha(g^*)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1g_2)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1)\phi_\alpha(g_2)\Phi(w_2) = \Phi(w_1g_1g_2w_2)$.

Hence if $w \sim w'$, using the two equalities above and iterating through a sequences of operations that changes $w$ to $w'$, we found that $\Phi(w) = \Phi(w')$, thus $\varphi$ is well defined. Also $\varphi([w_1][w_2]) = \varphi([w_1w_2]) = \Phi(w_1w_2) = \Phi(w_1)\Phi(w_2) = \varphi([w_1])\varphi([w_2])$, so $\varphi$ is indeed a homomorphism.

Finally, we show that $\varphi$ is unique. It is easy to see that each element $[w] \in G$ could be generate by $\{[g] : g \in \bigcup_\alpha G_\alpha\}$, the equivalent classes of words with length 1, and for each $g_\alpha$ in each $G_\alpha$, $\varphi([g_\alpha]) = \varphi(i_\alpha(g_\alpha)) = \varphi_\alpha(g_\alpha)$ is determined, hence $\varphi$ is unique. □

*Proof of uniqueness.* The $G$ constructed above is a coproduct of $\{G_\alpha\}$. By the uniqueness of coproduct, the proof is complete.

Or we could proof it in the old-fashioned way. If $(G', \{i'_\alpha\})$ is another pair that satisfies the universal property. By the univeral property of $G$, exists $\varphi$, $\varphi \circ i_s = i'_s$. Swapping the roles of $G$ and $G'$, we know that exists $\varphi'$, $\varphi' \circ i'_s = i_s$. So $\varphi' \circ \varphi \circ i_s = i_s$, but notice that the identity mapping $1_G$ also satisfies $1_G \circ i_s = i_s$. By the uniqueness of mapping in the definition of universal property, we have $1_G = \varphi' \circ \varphi$, similary $1_{G'} = \varphi \circ \varphi'$ so $\varphi$ 1-1 onto and hence $G \cong G'$. □

Since we construct the free product by equivalence classes, some may wish to find a canonical representative for each equivalence class. So we prove the following proposition.

**Proposition 3.** *Each word is equivalent to exactly one reduced word, and hence each equivalence class has a unique reduced word.*

*Proof.* Let $w = g_1g_2 \cdots g_n$ is a word in $W$. We associate to $w$ a sequence of reduced words $x_0, x_1, \cdots, x_n$ in $W$ define by the following recursive algorithm:

Let $x_0$ be the empty word 1. If $x_{i-1} = h_1 h_2 \cdots h_k$ (when $x_{i-1}$ is an empty word, $k = 0$), define $x_i$ by

$$x_i = \begin{cases} h_1 \cdots h_k & \text{if } g_i = 1_{\mathfrak{G}(g_i)} \\ g_i & \text{otherwise, if } k = 0 \\ h_1 \cdots h_k g_i & \text{otherwise, if } \mathfrak{G}(h_k) \neq \mathfrak{G}(g_i) \\ h_1 \cdots h_{k-1} & \text{otherwise, if } h_k g_i = 1_{\mathfrak{G}(g_i)} \\ h_1 \cdots h_{k-1} g^* & \text{otherwise, and let } h_k g_i = g^* \neq 1_{\mathfrak{G}(g_i)} \end{cases}$$

and let $r(w) = x_n$.

Then we could check inductively such that for each $i$, the following holds:

- $x_i \sim g_1 g_2 \cdots g_i$: By induction, $x_{i-1} \sim g_1 g_2 \cdots g_{i-1}$. Since in each case $x_i$ could be transformed from $x_{i-1} g_i$ by some reducing operations. And we've already prove that $x \sim x', y \sim y' \implies xy \sim x'y'$, hence $x_i \sim x_{i-1} g_i \sim g_1 g_2 \cdots g_i$.
- Each $x_i$ is a reduced word: Since by induction we know that $x_{i-1} = h_1 \cdots h_k$ is reduced. So if $x_i$ is obtain by case $\#1, 2, 4$, $x_i$ is a reduced word. In case #3, because $g_i \neq 1_{\mathfrak{G}(g_i)}$ and $\mathfrak{G}(g_i) \neq \mathfrak{G}(h_k)$, and in case #5, $g^* \neq 1_{\mathfrak{G}(g^*)}$ and $\mathfrak{G}(g^*) = h_k \neq h_{k-1}$, so $x_i$ is a reduced word in both case.

So each word $w$ is equivalent to a reduce word $r(w)$.

If $w = g_1 \cdots g_n$ is reduced, than $x_1$ is obtained by the second case, and $x_i, i > 1$ is obtained by the third case, hence $r(w) = g_1 \cdots g_n = w$.

Now consider the equivalent words

$$w = g_1 \cdots g_j g_{j+1} \cdots g_n \quad \text{and} \quad w' = g_1 \cdots g_j 1_{G_\alpha} g_{j+1} \cdots g_n$$

which induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n+1}$ by the algorithm above. We have $x_j = x'_j$, and $x'_{j+1}$ would be produce by case #1 and hence $x'_{j+1} = x'_j = x_j$. So after appending the same factors $g_{j+1} \cdots g_n$, the output should be the same, hence $x_n = x'_{n+1} \implies r(w) = r(w')$.

Next consider the equivalent words

$$w = g_1 \cdots g_{j-1} g_j g_{j+1} g_{j+2} \cdots g_n \quad \text{and} \quad w' = g_1 \cdots g_{j-1} g^* g_{j+2} \cdots g_n$$

Similarly, they would induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n-1}$, and we have $x_{j-1} = x'_{j-1}$. Let $x_{j-1} = h_1 h_2 \cdots h_k$. Now there are a lot of painful cases to check.

- $g_j = 1_{\mathfrak{G}(g_j)}$ or $g_{j+1} = 1_{\mathfrak{G}(g_{j+1})}$: If $g_j = 1_{\mathfrak{G}(g_j)}$ then $g^* = g_{j+1}$, so this case turn out to be the case above ($w$ is formed by inserting an identity into $w'$). Similar argument holds when $g_{j+1} = 1_{\mathfrak{G}(g_j)}$, hence we could assume $g_j, g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ below.
- otherwise, $k = 0$, or $k \neq 0$ but $\mathfrak{G}(h_k) \neq \mathfrak{G}(g_j)$. There are two cases:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1}$ (case #1), $x_j = x_{j-1} g_j$ (case #2 or #3) and $x_{j+1} = x_{j-1}$ (case #4).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1} g^*$ (case #2, #3), $x_j = x_{j-1} g_j$ (case #2, #3) and $x_{j+1} = x_{j-1} g^*$ (case #3).
- otherwise $\mathfrak{G}(h_k) = \mathfrak{G}(g_j)$, and if $h_k g_j = 1_{\mathfrak{G}(g_j)}$, let $y = h_1 \cdots h_{k-1}$:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $g_j g_{j+1} = g^* = 1 = h_k g_j$, which forces $h_k = g_{j+1} = g_j^{-1} \neq 1_{\mathfrak{G}(g_j)}$. Hence $x'_j = x_{j-1}$ (case #1), $x_j = h_1 \cdots h_{k-1}$ (case #4) and $x_{j+1} = h_1 \cdots h_k = x_{j-1}$ (case #2 or #3, since $x_{j-1}$ reduced, $k - 1 = 0$ or $\mathfrak{G}(h_{k-1}) \neq \mathfrak{G}(h_k) = \mathfrak{G}(g_{j+1})$).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Now $h_k g^* = g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ by assumption. Then $x'_j = y g_{j+1}$ (case #5), $x_j = y$ (case #4) and $x_{j+1} = y g_{j+1}$ (case #2, #3).
- otherwise $h_k g_j = \hat{g} \neq 1_{\mathfrak{G}(g_j)}$, let $y = h_1 \cdots h_{k-1}$:

– $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1} = yh_k$ (case #1), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = yh_k$ (case #5, since $\hat{g}g_{j+1} = h_k g_j g_{j+1} = h_k g^* = h_k \neq 1$).
– $g^* \neq 1_{\mathfrak{G}(g_j)}$: Let $\tilde{g} = h_k g^* = h_k g_j g_{j+1} = \hat{g}g_{j+1}$, then $x'_j = y\tilde{g}$ (case #5), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = y\tilde{g}$ (case #5, since $\hat{g}g_{j+1} = \tilde{g}$).

No matter which case, the result is that $x'_j = x_{j+1}$. After appending the same factors $g_{j+2} \cdots g_n$, the output should be the same, hence $r(w) = r(w')$. Extend the result to an sequence of operations of word, we conclude that $w \sim w' \implies r(w) = r(w')$. If $w$ is equivalent to two reduced word $w', w''$, then $w' \sim w''$ hence $r(w') = r(w'')$. But we proved that $r(x) = x$ if $x$ is a reduced word, hence $w' = w''$.

Finally, notice that if $w$ is not a reduced word, by definition it means that we could preform a reducing operation on $w$. After a reducing operation, the length of the word decrease. Repeat this process. Since the length couldn't decrease below 0, eventually we would get a reduced word $w'$ which is equivalent to $w$.

Combine the result above, we conclude that every word $w$ is equivalent to exactly one reduced word, and hence each equivalent class has exactly one reduced word. □

**Example 1.** Consider $C_2 * C_3$, where $C_2 = \langle a \rangle, C_3 = \langle b \rangle$. By the theorem above, each element of $C_2 * C_3$ correspond to a reduced word in $W(\{a, b\})$.

It is easy to see that all the reduced word in $W(\{a, b\})$ has the form

$$x = a^{n_1} b^{n_2} a^{n_3} b^{n_4} \cdots a^{n_{2k-1}} b^{n_{2k}}$$

such that $n_{2i+1} = 1$ and $n_{2i} = 1$ or 2, with the exception that $n_1, n_{2k}$ could be 0.

Any other word in $W(\{a, b\})$ could be reduced to the form. For example,

$$\begin{aligned} aabbaabaaabba &= a^2 b^2 a^2 b a^3 b^2 a \\ &= 1_{C_2} \cdot b^2 \cdot 1_{C_2} \cdot bab^2 a \\ &= b^2 bab^2 a = b^3 ab^2 a \\ &= 1_{C_3} \cdot ab^2 a \\ &= ab^2 a \end{aligned}$$

**Proposition 4.** *Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a set of groups, and suppose that $\langle S_\alpha \mid R_\alpha \rangle$ is a presentation for each $G_\alpha$, then $\left\langle \bigcup_\alpha S_\alpha \,\middle|\, \bigcup_\alpha R_\alpha \right\rangle$ is a presentation of the free product $\ast_\alpha G_\alpha$.*

Recall that $\langle S \mid R \rangle$ is defined to be $F(S)/N(R)$ where $F(S)$ is the free group of $S$ and $N(R)$ is the smallest normal subgroup containing $R$.

*Proof.* Let $S = \bigcup_\alpha S_\alpha, R = \bigcup_\alpha R_\alpha$ and define $G = \langle S \mid R \rangle$. Now consider the function $\psi$ which carry each element $x \in S_\alpha$ to $F(S)$ by:

1. First carry it from $S_\alpha$ into $S$ by the inclusion map.
2. Then carry it from $S$ into $F(S)$ by the mapping given by the free group.

Now by the universal property of the free group, there exist a map $\tilde{f}_\alpha :: F(S_\alpha) \to F(S)$. It is easy to see that $\tilde{f}_\alpha$ simply maps a word $w \in F(S_\alpha)$ to the same word in $F(S)$. Compose with the quotient map $q$ from $F(S)$ to $F(S)/N(R) = G$, we get a map $f_\alpha$ from $F(S_\alpha)$ to $G$.

Now, let $q_\alpha$ be the quotient map from $F(S_\alpha)$ to $G_\alpha$. For each word $r \in R_\alpha \subseteq F(S_\alpha)$, $\tilde{f}_\alpha$ send $r$ into $R$, so $f_\alpha(r) = 1_G$, hence $\ker q_\alpha = N(R_\alpha) \subseteq \ker f_\alpha$ since $N(R_\alpha)$ is the smallest normal subgroup containing $R_\alpha$ and a kernal is always normal. By factor theorem, $f_\alpha$ descends to a homomorphism $i_\alpha :: G_\alpha \to G$.

If we could prove that $(G, i_\alpha)$ has the universal property of a free product, by the uniqueness of free product we would have $G \cong \ast_\alpha G_\alpha$.

Let $H$ be a group, and let $\varphi_\alpha :: G_\alpha \to H$ be a system of homomorphisms. Consider $\tilde{\varphi}_\alpha :: F(S_\alpha) \to H$ by $\tilde{\varphi}_\alpha = \varphi_\alpha \circ q_\alpha$. Define $\tilde{g} :: S \to H$ which sending $x \in S$ by

1. First carry $x$ to one of the $S_\alpha$ by the inverse of inclusion map $\iota^{-1}$, it could be done since $\{S_\alpha\}$ is consider to be disjoint.
2. Then carry it into $F(S_\alpha)$ by the mapping given by the free group.
3. Finally carry it into $H$ by $\tilde{\varphi}_\alpha$.

Since $F(S)$ is the free group of $S$, by its universal property, exists a homomorphism $g :: F(S) \to H$. Now for each $r \in R \subseteq F(S)$, $r$ lies in a unique $R_\alpha$, so

$$g(r) = \tilde{\varphi}_\alpha(r) = \varphi_\alpha(q_\alpha(r)) = \varphi_\alpha(1_{G_\alpha}) = 1_H.$$

Consequently $\ker q = N(R) \subseteq \ker g$, and by factor theorem, $g$ descends into a homomorphism $\varphi :: G \to H$. This $\varphi$ satisfies:

$$\varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = g \circ \tilde{f}_\alpha = \tilde{\varphi}_\alpha = \varphi_\alpha \circ q_\alpha$$

and notice that $q_\alpha$ onto $G_\alpha$, hence we conclude that $\varphi \circ i_\alpha = \varphi_\alpha$. which proof that $G$ has the desire mapping required by the universal property for any $H$.

$$\begin{tikzcd}
S_\alpha \arrow[r] \arrow[d, shift right, "\iota"'] \arrow[dr, "\psi"] & F(S_\alpha) \arrow[r, "q_\alpha"] \arrow[d, "\tilde{f}_\alpha"'] \arrow[dr, "f_\alpha"] \arrow[rr, bend left, "\tilde{\varphi}_\alpha"] & G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "i_\alpha"] & H \\
S \arrow[u, shift right, "\iota^{-1}"'] \arrow[r] \arrow[urrr, "\tilde{g}"] & F(S) \arrow[r, "q"'] \arrow[urr, "g"] & G \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Figure 2: The commute diagram in the proof.

Finally, for the uniqueness, if $\varphi$ makes $\varphi_\alpha = i_\alpha \circ \varphi$, then consider $h = \varphi \circ q$, since

$$\varphi_\alpha \circ q_\alpha = \varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = h \circ \tilde{f}_\alpha.$$

So $h$ is determined for all $x \in \bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$. But $\bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$ contains all generator of $F(S)$ (i.e., all the words with length 1), hence $h$ is uniquely determined. Since $q$ is onto, we conclude that $\varphi$ is also uniquely determined.

Hence we prove that $G$ satisfies the universal property, and $\left\langle \bigcup_\alpha S_\alpha \,\middle|\, \bigcup_\alpha R_\alpha \right\rangle \cong \ast_\alpha G_\alpha$. $\square$

**Example 2.** Consider again $C_2 * C_3$. Since $C_2 = \langle a \mid a^2 \rangle, C_3 = \langle b \mid b^3 \rangle$, so $C_2 * C_3 = \langle a, b \mid a^2, b^3 \rangle$.

**Definition 3.** Let $\mathcal{C}$ be a category, $X, Y, Z$ be objects, and $f :: Z \to X,\ g :: Z \to Y$ are two morphisms. The **pushout** of the morphisms $f, g$ is a tuple $(P, i_1, i_2)$ such that $P$ is an object, $i_1 :: X \to P,\ i_2 :: Y \to P$ with $i_1 \circ f = i_2 \circ g$, and $(P, i_1, i_2)$ has the following universal property: For any $(Q, j_1, j_2)$ satisfying $j_1 \circ f = j_2 \circ g$, there is an unique morphism $\varphi :: P \to Q$ that makes $\varphi \circ i_1 = j_1, \varphi \circ i_2 = j_2$.

A common notation is $P = X \sqcup_Z Y$.

**Definition 4.** The amalgamated free product $G_1 *_F G_2$ is the pushout $G_1 \sqcup_F G_2$. That is, given $\psi_1 :: F \to G_1, \psi_2 :: F \to G_2$, the amalgamated free product is a tuple $(G, i_1, i_2)$ such that $G$ is a group, $i_1, i_2$ are homomorphisms from $G_1$ to $G$ and $G_2$ to $G$, respectively, with $i_1 \circ \psi_1 = i_2 \circ \psi_2$. It should also satisfied the following universal property: given $(H, j_1, j_2)$ such that $H$ is a group, $j_1, j_2$ are homomorphisms from $G_1$ to $H$ and $G_2$ to $H$, with $j_1 \circ \psi_1 = j_2 \circ \psi_2$, then there exists a unique homomorphism $\varphi$ such that $j_1 = \varphi \circ i_1, j_2 = \varphi \circ i_2$.

$$\begin{tikzcd}
 & G_1 \arrow[d, "i_1"] \arrow[dr, "j_1"] & \\
F \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & G_1 *_F G_2 \arrow[r, "\varphi"] & H \\
 & G_2 \arrow[u, "i_2"'] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Figure 3: The diagram of the amalgamated free product

**Proposition 5.** *Amalgamate free product exists and is unique up to isomorphism.*

*Proof.* Consider the free product $G_1 * G_2$, where $\tilde{i}_1, \tilde{i}_2$ are homomorphisms given by the free product from $G_1$ to $G_1 * G_2$ and $G_2$ to $G_1 * G_2$.

We construct $G_1 *_F G_2 = (G_1 * G_2)/N$, where $N$ is the smallest normal subgroup containing the elements $\left\{ \left[\psi_1(x)\psi_2(x)^{-1}\right] : x \in F \right\}$ in $G_1 * G_2$. Let $q$ be the quotient map $G_1 * G_2 \to G_1 * G_2/N$ and $i_1 = q \circ \tilde{i}_1, i_2 = q \circ \tilde{i}_2$. Then for all $x \in F$,

$$i_1(\psi_1(x))\, i_2(\psi_2(x))^{-1} = q(\tilde{i}_1(\psi_1(x)))\, q\big(\tilde{i}_2\left(\psi_2(x)^{-1}\right)\big) = q\big([\psi_1(x)][\psi_2(x)^{-1}]\big) = q([\psi_1(x)\psi_2(x)^{-1}]) = 1$$

Hence $i_1(\psi_1(x)) = i_2(\psi_2(x)), \forall x \implies i_1 \circ \psi_1 = i_2 \circ \psi_2$.

Now, given $H, j_1, j_2$. By the universal property of free product, exists $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \tilde{i}_\alpha = j_\alpha$ for $\alpha \in \{1, 2\}$.

Next, for all $x \in F$, $[\psi_1(x)\psi_2(x)^{-1}] = [\psi_1(x)][\psi_2(x)^{-1}] = \tilde{i}_1(\psi_1(x))\, \tilde{i}_2\big(\psi_2(x)^{-1}\big)$, so

$$\tilde{\varphi}\big([\psi_1(x)\psi_2(x)^{-1}]\big) = (\tilde{\varphi} \circ \tilde{i}_1)(\psi_1(x))\, (\tilde{\varphi} \circ \tilde{i}_2)\big(\psi_2(x)^{-1}\big) = (j_1 \circ \psi_1)(x)\, \big((j_2 \circ \psi_2)(x)\big)^{-1} = 1$$

since $j_1 \circ \psi_1 = j_2 \circ \psi_2$. By the fact that $N$ is the smallest normal subgroup containing $\left\{ \left[\psi_1(x)\psi_2(x)^{-1}\right] : x \in F \right\}$, we conclude that $N \subseteq \ker \tilde{\varphi}$, and consequently $\tilde{\varphi}$ descends into a homomorphism $\varphi :: G_1 *_F G_2 \to H$. And we have $\varphi \circ i_\alpha = \varphi \circ q \circ \tilde{i}_\alpha = \tilde{i}_\alpha \circ \tilde{\varphi} = j_\alpha$ for $\alpha \in \{1, 2\}$, hence $\varphi$ satisfies the requirements.

Finally, for the uniqueness, suppose $\varphi$ makes $j_\alpha = i_\alpha \circ \varphi$ for $\alpha \in \{1, 2\}$. then consider $h = \varphi \circ q$, since $h \circ \tilde{i}_\alpha = \varphi \circ q \circ \tilde{i}_\alpha = i_\alpha \circ \varphi = j_\alpha$ So $h$ is determined for all $x \in \operatorname{Im} \tilde{i}_1 \cup \operatorname{Im} \tilde{i}_2$. But $\operatorname{Im} \tilde{i}_1 \cup \operatorname{Im} \tilde{i}_2$ contains all the generators of $G_1 * G_2$ (i.e., all the reduced words with length 1 and the empty word), hence $h$ is uniquely determined. Since $q$ is onto, we conclude that $\varphi$ is also uniquely determined.

$$\begin{tikzcd}
 & & G_1 \arrow[dl, "\tilde{i}_1"'] \arrow[d, "i_1"] \arrow[dr, "j_1"] & \\
F \arrow[urr, "\psi_1"] \arrow[drr, "\psi_2"'] & G_1 * G_2 \arrow[r, "q"] \arrow[rr, bend right, "\tilde{\varphi}"'] & G_1 *_F G_2 \arrow[r, "\varphi"] & H \\
 & & G_2 \arrow[ul, "\tilde{i}_2"] \arrow[u, "i_2"'] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Figure 4: The diagram of groups and maps mentioned in the proof

□

For readers interested in applications of free product, the textbook mentioned the **Seifert–van Kampen theorem** in algebraic topology, which states that if $X$ is a topological space which is the union of two open and path connected subspaces $U_1, U_2$, and there intersection is path connected and nonempty. Then the fundamental group of $X$ is the free product of the fundamental groups $U_1, U_2$ amalgamated by the fundamental groups of $U_1 \cup U_2$.