Definition 1. Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty set of groups G_α. The **free product** of $\mathcal{S}$ is a pair $(G, \{i_\alpha\})$ where G is a group and $i_\alpha :: G_\alpha \to G$ are homomorphism which has the following universal property: For any group H and homomorphisms $\varphi_\alpha :: G_\alpha \to H$, there exists a unique group homomorphism $\varphi :: G \to H$ such that $\varphi \circ i_\alpha = \varphi_\alpha$ for all $\alpha \in \mathcal{A}$ (e.g., the diagram below commutes). We write $G = \mathop{\ast}_\alpha G_\alpha$ (some also write $G = \prod^* G_\alpha$ or $G = \coprod G_\alpha$).

$$\begin{tikzcd} G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "\varphi_\alpha"'] & \mathop{\ast}_\alpha G_\alpha \arrow[d, "\exists\varphi"] \\ & H \end{tikzcd}$$

Figure 1: The diagram of the free product

Remark. The free product could also be defined as the **coproduct** of $\{G_\alpha\}$ if we consider the morphisms as group homomorphisms in the category of groups.

Definition 2. A **word** (respect to free product) is a finite sequence $g_1 g_2 \cdots g_n$ such that each g_i is in some G_α, and each g_i in it is called a **factor**. We shall denote the empty word by 1. The **length** of a word is the number of factors in it. Define $W(\{G_\alpha\})$ or simply W to be the set of all the words. For each g, let $\mathfrak{G}(g)$ be the group which g belongs, and g^{-1} be the inverse of g in that group.

Write xy as the **concatenate** of two words x, y.

We define two **reducing** operations on words

1. Drop a factor g_i which is the identity element.
2. Collapse two factors $g_i g_{i+1}$ to a single one g^* if $\mathfrak{G}(g_i) = \mathfrak{G}(g_{i+1})$ and their product in that group is g^*.

Together with the inverse of these two reducing operations, we have four operations that could alter a word.

Two words x and y are said to be **equivalent** if there is a finite sequence of operations that change x to y.

A word x is a **reduced word** if no reducing operation could be preform on it. That is, x contains no factor g_i which is the identity of a group G_α and contains no two consecutive factors g_i, g_{i+1} that lies in the same group.

Proposition 1. *The statement above defines a valid equivalent relation $\sim$.*

Proof. We check that

- $x \sim x$ clearly from definition.
- $x \sim y \implies y \sim x$ since we have inverse operations.
- $x \sim y \wedge y \sim z \implies x \sim z$ since we could combine the sequence of operations that changes x to y and changes y to z to obtain a sequence of operations that changes x to z.

□

Proposition 2. *Free product exists and is unique up to isomorphism.*

Proof of existence. Denote the equivalence classes by $[x]$ using the equivalence relation mentioned above. Let $G = W/\sim = \{[x] : x \in W\}$ and define the multiplication as $[x][y] = [xy]$. We shall check that the multiplication is well defined.

Suppose that x and x' differ by only one operations. Then for all y, it is easy to see that $xy \sim x'y$ and $yx \sim yx'$ since we could still preform the same kind of operation on the x, x' part and turn the former word to the later. Iteration of this kind of relationship show that $v \sim v'$ and $w \sim w'$ implies $vw \sim v'w'$. Hence the multiplication is well defined.

Then we check that with this multiplication, G is indeed a group.

associative: Simply by the associative of word concatenate.

indentity: The equivalent class of empty word [1] is the indentity obviously.

inverse: For any word $x = g_1 g_2 \cdots g_n$. Let $y = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$. It is easy to see that $xy = yx = 1$, hence $[x][y] = [y][x] = [1]$.

Let $i_\alpha(g) = [g]$, then each i_α is a homomorphism since $i_\alpha(g_1 g_2) = [g_1 g_2] = [g_1][g_2] = i_\alpha(g_1) i_\alpha(g_2)$.

Now, given H and homomorphisms φ_α, we shall construct φ. First define $\Phi :: W \to H$ by

$$\Phi(w) = \begin{cases} 1_H & \text{if } w \text{ is the empty word.} \\ \varphi_\alpha(g) & \text{if } w = g \text{ has length 1 and } g \in G_\alpha. \\ \Phi(g)\Phi(w') & \text{otherwise, and let } w = gw'. \end{cases}$$

It is easy to see that $\Phi(w_1 w_2) = \Phi(w_1)\Phi(w_2)$. Let $\varphi([w]) = \Phi(w)$ for each $[w] \in G$ and we shall check that it is well defined. It is equivalent to ensure that $w \sim w' \implies \Phi(w) = \Phi(w')$. So we prove that

- $\Phi(w_1 w_2) = \Phi(w_1 1_{G_\alpha} w_2)$: Since $\Phi(w_1 1_{G_\alpha} w_2) = \Phi(w_1)\Phi(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\varphi_\alpha(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\Phi(w_2) = \Phi(w_1 w_2)$ by the fact that every homomorphism φ_α maps identity to identity.
- $\Phi(w_1 g w_2) = \Phi(w_1 g_1 g_2 w_2)$, if $g = g_1 g_2$: WLOG assume $g \in G_\alpha$, then $\Phi(w_1 g w_2) = \Phi(w_1)\phi_\alpha(g)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1 g_2)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1)\phi_\alpha(g_2)\Phi(w_2) = \Phi(w_1 g_1 g_2 w_2)$.

Hence if $w \sim w'$, using the two equalities above and iterate through a sequences of operator that changes w to w', we found that $\Phi(w) = \Phi(w')$, thus φ is well defined. Also $\varphi([w_1][w_2]) = \varphi([w_1 w_2]) = \Phi(w_1 w_2) = \Phi(w_1)\Phi(w_2) = \varphi([w_1])\varphi([w_2])$, so φ is indeed a homomorphism.

Finally, we show that φ is unique. It is easy to see that each element $[w] \in G$ could be generate by the equivalent classes of words with length 1, and for each g_α in each G_α, $\varphi([g_\alpha]) = \varphi(i_\alpha(g_\alpha)) = \varphi_\alpha(g_\alpha)$ is determined, hence φ is unique. ☐

Proof of uniqueness. The G constructed above is a coproduct. By the uniqueness of coproduct, the proof is complete. ☐

Since we construct free product by equivalence classes, some may wish to find a canonical representative for each equivalence class. So we prove the following proposition.

Proposition 3. *Each word is equivalent to exactly one reduced word, and hence each equivalent class has a unique reduced word.*

Proof. Let $w = g_1 g_2 \cdots g_n$ is a word in W. We associate to w a sequence of reduced words $x_0, x_1, \cdots, x_n$ in W define by the following recursive algorithm:
Let $x_0 = 1$, the empty word. If $x_{i-1} = h_1 \cdots h_k$ (when x_{i-1} is an empty word, $k = 0$), define x_i by

$$x_i = \begin{cases} h_1 \cdots h_k & \text{if } g_i = 1_{\mathfrak{G}(g_i)} \\ g_i & \text{otherwise, if } k = 0 \\ h_1 \cdots h_k g_i & \text{otherwise, if } \mathfrak{G}(h_k) \neq \mathfrak{G}(g_i) \\ h_1 \cdots h_{k-1} & \text{otherwise, if } h_k g_i = 1_{\mathfrak{G}(g_i)} \\ h_1 \cdots h_{k-1} g^* & \text{otherwise, and let } h_k g_i = g^* \neq 1_{\mathfrak{G}(g_i)} \end{cases}$$

and let $r(w) = x_n$.

Then we could check inductively such that for each i, the following holds:

- $x_i \sim g_1g_2\cdots g_i$, since in every cases x_i could be transformed from $x_{i-1}g_i$ by some reducing operations. And we've already prove that $x \sim x', y \sim y' \implies xy \sim x'y'$, hence $x_i \sim x_{i-1}g_i \sim g_1g_2\cdots g_i$.
- Each x_i is a reduced word. Since by induction we know that $x_{i-1} = h_1\cdots h_k$ is reduced. So if x_i is obtain by case #1, 2, 4, x_i is a reduced word. In case #3, because $g_i \neq 1_{\mathfrak{G}(g_i)}$ and $\mathfrak{G}(g_i) \neq \mathfrak{G}(h_k)$, and in case #5, $g^* \neq 1_{\mathfrak{G}(g^*)}$ and $\mathfrak{G}(g^*) = h_k \neq h_{k-1}$, so x_i is a reduced word in both case.

So each word w is equivalent to a reduce word $r(w)$.

If $w = g_1\cdots g_n$ is reduced, than x_1 is obtained by the second case, and $x_i, i > 1$ is obtained by the third case, hence $r(w) = g_1\cdots g_n = w$.

Now consider the equivalent words

$$w = g_1\cdots g_jg_{j+1}\cdots g_n \quad \text{and} \quad w' = g_1\cdots g_j1_{G_\alpha}g_{j+1}\cdots g_n$$

which induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n+1}$ by the algorithm above. We have $x_j = x'_j$, and x'_{j+1} would be produce by case #1 and hence $x'_{j+1} = x'_j = x_j$. So after appending the same factors $g_{j+1}\cdots g_n$, the output should be the same, hence $x_n = x'_{n+1} \implies r(w) = r(w')$.

Next consider the equivalent words

$$w = g_1\cdots g_{j-1}g_jg_{j+1}g_{j+2}\cdots g_n \quad \text{and} \quad w' = g_1\cdots g_{j-1}g^*g_{j+2}\cdots g_n$$

similarly, would induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n-1}$ by the algorithm above and we have $x_{j-1} = x'_{j-1}$. Let $x_{j-1} = h_1h_2\cdots h_k$. Now there are a lot of painful cases to check.

- $g_j = 1_{\mathfrak{G}(g_j)}$: Then $g^* = g_{j+1}$, so this case turn out to be the case above. (w is formed by inserting an identity in w'). Similar argument holds when $g_{j+1} = 1_{\mathfrak{G}(g_j)}$, hence we could assume $g_j, g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ below.
- otherwise, $k = 0$, or $k \neq 0$ but $\mathfrak{G}(h_k) \neq \mathfrak{G}(g_j)$. There are two cases:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1}$ (case #1), $x_j = x_{j-1}g_j$ (case #2 or #3) and $x_{j+1} = x_{j-1}$ (case #4).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1}g^*$ (case #2, #3), $x_j = x_{j-1}g_j$ (case #2, #3) and $x_{j+1} = x_{j-1}g^*$ (case #3).
- otherwise $\mathfrak{G}(h_k) = \mathfrak{G}(g_j)$, and if $h_kg_j = 1_{\mathfrak{G}(g_j)}$, let $y = h_1\cdots h_{k-1}$:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $g_jg_{j+1} = g^* = 1 = h_kg_j$, which forces $h_k = g_{j+1} = g_j^{-1} \neq 1_{\mathfrak{G}(g_j)}$. Hence $x'_j = x_{j-1}$ (case #1), $x_j = h_1\cdots h_{k-1}$ (case #4) and $x_{j+1} = h_1\cdots h_k = x_{j-1}$ (case #3, since x_{j-1} reduced, $\mathfrak{G}(h_{k-1}) \neq \mathfrak{G}(h_k)$).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Now $h_kg^* = g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ by assumption. Then $x'_j = yg_{j+1}$ (case #5), $x_j = y$ (case #4) and $x_{j+1} = yg_{j+1}$ (case #2, #3).
- otherwise $h_kg_j = \hat{g} \neq 1_{\mathfrak{G}(g_j)}$, let $y = h_1\cdots h_{k-1}$:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1} = yh_k$ (case #1), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = yh_k$ (case #5, since $\hat{g}g_{j+1} = h_kg_jg_{j+1} = h_kg^* = h_k \neq 1$).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Let $\tilde{g} = h_kg^* = h_kg_jg_{j+1} = \hat{g}g_{j+1}$, then $x'_j = y\tilde{g}$ (case #5), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = y\tilde{g}$ (case #5, since $\hat{g}g_{j+1} = \tilde{g}$).

No matter which case, the result is that $x'_j = x_{j+1}$. After appending the same factors $g_{j+2}\cdots g_n$, the output should be the same, hence $r(w) = r(w')$. Extend the result to an sequence of operations of word, we conclude that $w \sim w' \implies r(w) = r(w')$. Then if w is equivalent to two reduced word w', w'', then $w' \sim w''$ hence $r(w') = r(w'')$. But we proved that $r(x) = x$ if x is a reduced word, hence $w' = w''$.

Finally, notice that if w is not a reduced word, it means that we could preforming a reducing operation on w. After a reducing operation, the length of the word decrease. Since the length couldn't decrease below 0, eventually we would get a reduced word w' which is equivalent to w.

Combine the result above, we conclude that every word w is equivalent to exactly one reduced word, and hence each equivalent class has exactly one reduced word. $\square$

Example 1. Consider $C_2 * C_3$, where $C_2 = \langle a \rangle, C_3 = \langle b \rangle$. By the theorem above, each element of $C_2 * C_3$ correspond to a reduced word in $W(\{a, b\})$.

It is easy to see that all the reduced word in $W(\{a, b\})$ has the form

$$x = a^{n_1} b^{n_2} a^{n_3} b^{n_4} \cdots a^{n_{2k-1}} b^{n_{2k}}$$

such that $n_{2i+1} = 1$ and $n_{2i} = 1$ or 2, with the exception that n_1, n_{2k} could be 0.

Any other word in $W(\{a, b\})$ could be reduced to the form. For example,

$$\begin{aligned} aabbaabaaabba &= a^2 b^2 a^2 b a^3 b^2 a \\ &= 1_{C_2} \cdot b^2 \cdot 1_{C_2} \cdot bab^2 a \\ &= b^2 bab^2 a = b^3 ab^2 a \\ &= 1_{C_3} \cdot ab^2 a \\ &= ab^2 a \end{aligned}$$

Proposition 4. *Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a set of groups, and suppose that $\langle S_\alpha \mid R_\alpha \rangle$ is a presentation, then $\left\langle \bigcup_\alpha S_\alpha \;\middle|\; \bigcup_\alpha R_\alpha \right\rangle$ is a presentation of the free product $\mathop{\ast}_\alpha G_\alpha$.*

Recall that $\langle S \mid R \rangle$ is defined to be $F(S)/N(R)$ where $F(S)$ is the free group of S and $N(R)$ is the smallest normal subgroup containing R.

Proof. Let $S = \bigcup_\alpha S_\alpha, R = \bigcup_\alpha R_\alpha$ and define $G = \langle S \mid R \rangle$. Now consider the function ψ which carry each element $x \in S_\alpha$ to $F(S)$ by:

1. First carry it from S_α into S by the inclusion map.
2. Then carry it from S into $F(S)$ by the mapping given by the free group.

Now by the universal property of the free group, there exist a map $\tilde{f}_\alpha :: F(S_\alpha) \to F(S)$. It is easy to see that $\tilde{f}_\alpha$ simply maps a word $w \in F(S_\alpha)$ to the same word in $F(S)$. Compose with the quotient map q from $F(S)$ to $G = F(S)/N(R)$, we get a map f_α from $F(S_\alpha)$ to G.

Now, let q_α be the quotient map from $F(S_\alpha)$ to G_α. For each word $r \in R_\alpha \subseteq F(S_\alpha)$, $\tilde{f}_\alpha$ send r into R, so $f_\alpha(r) = 1_G$, hence $\ker q_\alpha = N(R_\alpha) \subseteq \ker f$ since $N(R_\alpha)$ is the smallest normal subgroup containing R_α and a kernal is always normal. By factor theorem, f_α descends to a homomorphism $i_\alpha :: G_\alpha \to G$.

If we could prove that (G, i_α) has the universal property of a free product, by the uniqueness of free product we would have $G \cong \mathop{\ast}_\alpha G_\alpha$.

Let H be a group, and let $\varphi_\alpha :: G_\alpha \to H$ be a system of homomorphisms. Consider $\tilde{\varphi}_\alpha :: F(S_\alpha) \to H$ by $\tilde{\varphi}_\alpha = \varphi \circ q_\alpha$. Define $\tilde{g} :: S \to H$ which sending $x \in S$ by

1. First carry x to one of S_α by the inverse of inclusion map ι^{-1}, it could be done since S_α is consider to be disjoint.
2. Then send into $F(S_\alpha)$ by the mapping given by the free group.
3. Finally send into H by $\tilde{\varphi}_\alpha$.

Since $F(S)$ is the free group of S, by its universal property, exists a homomorphism $g :: F(S) \to H$. Now for each $r \in R \subseteq F(S)$, r lies in a unique R_α, so

$$g(r) = \tilde{q}_\alpha(r) = \varphi_\alpha(q_\alpha(r)) = \varphi_\alpha(1_{G_\alpha}) = 1_H.$$

Consequently $\ker q = N(R) \subseteq \ker g$, and by factor theorem, g descends into a homomorphism $\varphi :: G \to H$. This φ satisfies:

$$\varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = g \circ \tilde{f}_\alpha = \tilde{\varphi}_\alpha = \varphi_\alpha \circ q_\alpha$$

and notice that q_α onto G_α, hence we conclude that $\varphi \circ i_\alpha = \varphi_\alpha$. which proof that G has the desire mapping required by the universal property for any H.

$$\begin{tikzcd}
S_\alpha \arrow[r] \arrow[d, shift right, "\iota"'] \arrow[dr, "\psi"] & F(S_\alpha) \arrow[r, "q_\alpha"] \arrow[d, "\tilde{f}"'] \arrow[dr, "f"] & G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "i_\alpha"] & H \\
S \arrow[r] \arrow[u, shift right, "\iota^{-1}"'] \arrow[urrr, "\tilde{g}"] & F(S) \arrow[r, "q"'] \arrow[urr, "g"] & G \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Figure 2: The compute diagram in the proof.

Finally, for the uniqueness of φ, if $\varphi_\alpha = i_\alpha \circ \varphi$, then consider $h = \varphi \circ q$, since

$$\varphi_\alpha \circ q_\alpha = \varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = h \circ \tilde{f}_\alpha.$$

So h is determined for all $x \in \bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$. But $\bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$ contains all generator of $F(S)$ (i.e., all the words with length 1), hence h is uniquely determined. Since q is onto, we conclude that φ is also uniquely determined.

Hence we prove that G satisfies the universal property, and hence $\left\langle \bigcup_\alpha S_\alpha \;\middle|\; \bigcup_\alpha R_\alpha \right\rangle \cong \mathop{\ast}_\alpha G_\alpha$. $\square$

Example 2. Consider again $C_2 * C_3$. Since $C_2 = \langle a \mid a^2 \rangle, C_3 = \langle b \mid b^3 \rangle$, so $C_2 * C_3 = \langle a, b \mid a^2, b^3 \rangle$.

Definition 3. Let $\mathcal{C}$ be a category, X, Y, Z be objects, and $f :: Z \to X, g :: Z \to Y$ are two morphisms. The **pushout** of the morphisms f, g is a tuple (P, i_1, i_2) such that P is an object, $i_1 :: X \to P, i_2 :: Y \to P$ such that $i_1 \circ f = i_2 \circ g$, and (P, i_1, i_2) has the following universal property: For any (Q, j_1, j_2) satisfied $j_1 \circ f = j_2 \circ g$, there is a unique $\varphi :: P \to Q$ that makes $\varphi \circ i_1 = j_1, \varphi \circ i_2 = j_2$.

A common notation is $P = X \sqcup_Z Y$.

Definition 4. The amalgamated free product $G_1 *_F G_2$ is the pushout $G_1 \sqcup_F G_2$. That is, given $\psi_1 :: F \to G_1, \psi_2 :: F \to G_2$, the amalgamated free product is a tuple (G, i_1, i_2) such that G is a group, i_1, i_2 are homomorphisms from G_1, G_2 to G, respectively, and $i_1 \circ \psi_1 = i_2 \circ \psi_2$. It should also satisfied the universal property, given (H, j_1, j_2) such that H is a group, j_1, j_2 are homomorphisms from G_1, G_2 to H, and $j_1 \circ \psi_1 = j_2 \circ \psi_2$, then there exists a unique homomorphism ϕ such that $j_1 = \varphi \circ i_1, j_2 = \varphi i_2$.

Theorem 1. *Amalgamate free product exists and is unique up to isomorphism.*

Remark. The Amalgamate free product $G_1 *_F G_2$ could be constructed by $(G_1 * G_2)/N$, where N is the smallest normal subgroup containing $\{\psi_1(x)\psi_2(x)^{-1} : x \in F\}$ in $G_1 * G_2$.

For readers interested in applications of free product, one could refer to the **Seifert–van Kampen theorem** in algebraic topology, which states that if X is a topological space which is the union of two open and path connected subspaces U_1, U_2, and there intersection is path connected and nonempty. Then the fundamental group of X is the free product of the fundamental groups U_1, U_2 with amalgamated by the fundamental groups of $U_1 \cup U_2$.