Definition 1. Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty set of groups G_α (We shall assume that these groups are disjoint). The **free product** of $\mathcal{S}$ is a pair $(G, \{i_\alpha\})$ where G is a group and $i_\alpha :: G_\alpha \to G$ are homomorphism which has the following universal property: For any group H and homomorphisms $\varphi_\alpha :: G_\alpha \to H$, there exists a unique group homomorphism $\varphi :: G \to H$ such that $\varphi \circ i_\alpha = \varphi_\alpha$ for all $\alpha \in \mathcal{A}$ (e.g., the diagram below commutes). We write $G = \mathop{\ast}_\alpha G_\alpha$ (some also write $G = \prod^* G_\alpha$ or $G = \coprod G_\alpha$).

$$\begin{tikzcd}
G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "\varphi_\alpha"'] & \mathop{\ast}_\alpha G_\alpha \arrow[d, "\exists\varphi"] \\
& H
\end{tikzcd}$$

Figure 1: The diagram of the free product

Remark. The free product could also be defined as the **coproduct** of $\{G_\alpha\}$ if we consider the morphisms as group homomorphisms in the category of groups.

Definition 2. A **word** (with respect to a free product) is a finite sequence $g_1 g_2 \cdots g_n$ such that every g_i is in one of G_α. Each g_i in it is called a **factor**. We shall denote the empty word by 1. The **length** of a word is the number of factors in it. Define $W(\{G_\alpha\})$ or simply W to be the set of all of the words. For each g, let $\mathfrak{G}(g)$ be the group which g belongs, and g^{-1} be the inverse of g in that group.

Write xy as the **concatenate** of two words x, y.

We define two **reducing operations** on words

1. Drop a factor g_i which is the identity element.
2. Collapse two factors $g_i g_{i+1}$ to a single one g^* if $\mathfrak{G}(g_i) = \mathfrak{G}(g_{i+1})$ and their product in that group is g^*.

Together with the inverse of these two reducing operations, we have four **operations** that could alter a word.

Two words x and y are said to be **equivalent** if there is a finite sequence of operations that change x to y.

A word x is a **reduced word** if no reducing operation could be preformed on it. That is, x contains no factor g_i which is the identity of a group G_α, and contains no consecutive factors g_i, g_{i+1} that belong to the same group.

Proposition 1. *The statement above defines a valid equivalent relation $\sim$.*

Proof. We check that

- $x \sim x$ clearly from definition.
- $x \sim y \implies y \sim x$ since we have inverse operations.
- $x \sim y \wedge y \sim z \implies x \sim z$ since we could combine the sequence of operations that changes x to y and that changes y to z to obtain a sequence of operations that changes x to z.

□

Proposition 2. *Free product exists and is unique up to isomorphism.*

Proof of existence. Denote the equivalence classes by $[x]$ using the equivalence relation mentioned above. Let $G = W/\sim = \big\{[x] : x \in W\big\}$ and define the multiplication as $[x][y] = [xy]$. We shall check that the multiplication is well defined.

Suppose that x and x' differ by only one operations. Then for all y, it is easy to see that $xy \sim x'y$ and $yx \sim yx'$ since we could still preform that operation on the x, x' part and turn the former word to the later. Iterating this kind of equivalence shows that $v \sim v'$ and $w \sim w'$ implies $vw \sim v'w'$. Hence the multiplication is well defined.

Then we check that with this multiplication, G is indeed a group.

associative: Simply by the associative of word concatenation.

indentity: The equivalent class of empty word [1] is the indentity obviously.

inverse: For any word $x = g_1 g_2 \cdots g_n$. Let $y = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$. It is easy to see that $[xy] = [yx] = [1]$, hence $[x][y] = [y][x] = [1]$.

Let $i_\alpha(g) = [g]$, then each i_α is a homomorphism since $i_\alpha(g_1 g_2) = [g_1 g_2] = [g_1][g_2] = i_\alpha(g_1) i_\alpha(g_2)$.

Now, given H and homomorphisms φ_α, we shall construct φ. First define $\Phi :: W \to H$ by

$$\Phi(w) = \begin{cases} 1_H & \text{if } w \text{ is the empty word.} \\ \varphi_\alpha(g) & \text{if } w = g \text{ has length 1 and } g \in G_\alpha. \\ \Phi(g)\Phi(w') & \text{otherwise, and let } w = gw'. \end{cases}$$

It is easy to see that $\Phi(w_1 w_2) = \Phi(w_1)\Phi(w_2)$. Let $\varphi([w]) = \Phi(w)$ for each $[w] \in G$ and we shall check that it is well defined. It is equivalent to ensure that $w \sim w' \implies \Phi(w) = \Phi(w')$. So we prove that

- $\Phi(w_1 w_2) = \Phi(w_1 1_{G_\alpha} w_2)$: Since $\Phi(w_1 1_{G_\alpha} w_2) = \Phi(w_1)\Phi(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\varphi_\alpha(1_{G_\alpha})\Phi(w_2) = \Phi(w_1)\Phi(w_2) = \Phi(w_1 w_2)$ by the fact that every homomorphism φ_α maps identity to identity.
- $\Phi(w_1 g^* w_2) = \Phi(w_1 g_1 g_2 w_2)$, if $g^* = g_1 g_2$: WLOG assume $g^* \in G_\alpha$, then $\Phi(w_1 g^* w_2) = \Phi(w_1)\phi_\alpha(g^*)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1 g_2)\Phi(w_2) = \Phi(w_1)\phi_\alpha(g_1)\phi_\alpha(g_2)\Phi(w_2) = \Phi(w_1 g_1 g_2 w_2)$.

Hence if $w \sim w'$, using the two equalities above and iterating through a sequences of operations that changes w to w', we found that $\Phi(w) = \Phi(w')$, thus φ is well defined. Also $\varphi([w_1][w_2]) = \varphi([w_1 w_2]) = \Phi(w_1 w_2) = \Phi(w_1)\Phi(w_2) = \varphi([w_1])\varphi([w_2])$, so φ is indeed a homomorphism.

Finally, we show that φ is unique. It is easy to see that each element $[w] \in G$ could be generate by $\left\{[g] : g \in \bigcup_\alpha G_\alpha\right\}$, the equivalent classes of words with length 1, and for each g_α in each G_α, $\varphi([g_\alpha]) = \varphi(i_\alpha(g_\alpha)) = \varphi_\alpha(g_\alpha)$ is determined, hence φ is unique. □

Proof of uniqueness. The G constructed above is a coproduct of $\{G_\alpha\}$. By the uniqueness of coproduct, the proof is complete.

Or we could proof it in the old-fashioned way. If $(G', \{i'_\alpha\})$ is another pair that satisfies the universal property. By the univeral property of G, exists φ, $\varphi \circ i_s = i'_s$. Swapping the roles of G and G', we know that exists φ', $\varphi' \circ i'_s = i_s$. So $\varphi' \circ \varphi \circ i_s = i_s$, but notice that the identity mapping 1_G also satisfies $1_G \circ i_s = i_s$. By the uniqueness of mapping in the definition of universal property, we have $1_G = \varphi' \circ \varphi$, similary $1_{G'} = \varphi \circ \varphi'$ so φ 1-1 onto and hence $G \cong G'$. □

Since we construct the free product by equivalence classes, some may wish to find a canonical representative for each equivalence class. So we prove the following proposition.

Proposition 3. *Each word is equivalent to exactly one reduced word, and hence each equivalence class has a unique reduced word.*

Proof. Let $w = g_1 g_2 \cdots g_n$ is a word in W. We associate to w a sequence of reduced words $x_0, x_1, \cdots, x_n$ in W define by the following recursive algorithm:
Let x_0 be the empty word 1. If $x_{i-1} = h_1 h_2 \cdots h_k$ (when x_{i-1} is an empty word, $k = 0$), define x_i

by

$$x_i = \begin{cases} h_1 \cdots h_k & \text{if } g_i = 1_{\mathfrak{G}(g_i)} \\ g_i & \text{otherwise, if } k = 0 \\ h_1 \cdots h_k g_i & \text{otherwise, if } \mathfrak{G}(h_k) \neq \mathfrak{G}(g_i) \\ h_1 \cdots h_{k-1} & \text{otherwise, if } h_k g_i = 1_{\mathfrak{G}(g_i)} \\ h_1 \cdots h_{k-1} g^* & \text{otherwise, and let } h_k g_i = g^* \neq 1_{\mathfrak{G}(g_i)} \end{cases}$$

and let $r(w) = x_n$.

Then we could check inductively such that for each i, the following holds:

- $x_i \sim g_1 g_2 \cdots g_i$: By induction, $x_{i-1} \sim g_1 g_2 \cdots g_{i-1}$. Since in each case x_i could be transformed from $x_{i-1} g_i$ by some reducing operations. And we've already prove that $x \sim x', y \sim y' \implies xy \sim x'y'$, hence $x_i \sim x_{i-1} g_i \sim g_1 g_2 \cdots g_i$.
- Each x_i is a reduced word: Since by induction we know that $x_{i-1} = h_1 \cdots h_k$ is reduced. So if x_i is obtain by case #1, 2, 4, x_i is a reduced word. In case #3, because $g_i \neq 1_{\mathfrak{G}(g_i)}$ and $\mathfrak{G}(g_i) \neq \mathfrak{G}(h_k)$, and in case #5, $g^* \neq 1_{\mathfrak{G}(g^*)}$ and $\mathfrak{G}(g^*) = h_k \neq h_{k-1}$, so x_i is a reduced word in both case.

So each word w is equivalent to a reduce word $r(w)$.

If $w = g_1 \cdots g_n$ is reduced, than x_1 is obtained by the second case, and $x_i, i > 1$ is obtained by the third case, hence $r(w) = g_1 \cdots g_n = w$.

Now consider the equivalent words

$$w = g_1 \cdots g_j g_{j+1} \cdots g_n \quad \text{and} \quad w' = g_1 \cdots g_j 1_{G_\alpha} g_{j+1} \cdots g_n$$

which induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n+1}$ by the algorithm above. We have $x_j = x'_j$, and x'_{j+1} would be produce by case #1 and hence $x'_{j+1} = x'_j = x_j$. So after appending the same factors $g_{j+1} \cdots g_n$, the output should be the same, hence $x_n = x'_{n+1} \implies r(w) = r(w')$.

Next consider the equivalent words

$$w = g_1 \cdots g_{j-1} g_j g_{j+1} g_{j+2} \cdots g_n \quad \text{and} \quad w' = g_1 \cdots g_{j-1} g^* g_{j+2} \cdots g_n$$

Similarly, they would induce $x_0, \cdots, x_n$ and $x'_0, \cdots, x'_{n-1}$, and we have $x_{j-1} = x'_{j-1}$. Let $x_{j-1} = h_1 h_2 \cdots h_k$. Now there are a lot of painful cases to check.

- $g_j = 1_{\mathfrak{G}(g_j)}$ or $g_{j+1} = 1_{\mathfrak{G}(g_{j+1})}$: If $g_j = 1_{\mathfrak{G}(g_j)}$ then $g^* = g_{j+1}$, so this case turn out to be the case above (w is formed by inserting an identity into w'). Similar argument holds when $g_{j+1} = 1_{\mathfrak{G}(g_j)}$, hence we could assume $g_j, g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ below.
- otherwise, $k = 0$, or $k \neq 0$ but $\mathfrak{G}(h_k) \neq \mathfrak{G}(g_j)$. There are two cases:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1}$ (case #1), $x_j = x_{j-1} g_j$ (case #2 or #3) and $x_{j+1} = x_{j-1}$ (case #4).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1} g^*$ (case #2, #3), $x_j = x_{j-1} g_j$ (case #2, #3) and $x_{j+1} = x_{j-1} g^*$ (case #3).
- otherwise $\mathfrak{G}(h_k) = \mathfrak{G}(g_j)$, and if $h_k g_j = 1_{\mathfrak{G}(g_j)}$, let $y = h_1 \cdots h_{k-1}$:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $g_j g_{j+1} = g^* = 1 = h_k g_j$, which forces $h_k = g_{j+1} = g_j^{-1} \neq 1_{\mathfrak{G}(g_j)}$. Hence $x'_j = x_{j-1}$ (case #1), $x_j = h_1 \cdots h_{k-1}$ (case #4) and $x_{j+1} = h_1 \cdots h_k = x_{j-1}$ (case #2 or #3, since x_{j-1} reduced, $k - 1 = 0$ or $\mathfrak{G}(h_{k-1}) \neq \mathfrak{G}(h_k) = \mathfrak{G}(g_{j+1})$).
  - $g^* \neq 1_{\mathfrak{G}(g_j)}$: Now $h_k g^* = g_{j+1} \neq 1_{\mathfrak{G}(g_j)}$ by assumption. Then $x'_j = y g_{j+1}$ (case #5), $x_j = y$ (case #4) and $x_{j+1} = y g_{j+1}$ (case #2, #3).
- otherwise $h_k g_j = \hat{g} \neq 1_{\mathfrak{G}(g_j)}$, let $y = h_1 \cdots h_{k-1}$:
  - $g^* = 1_{\mathfrak{G}(g_j)}$: Then $x'_j = x_{j-1} = y h_k$ (case #1), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = y h_k$ (case #5, since $\hat{g} g_{j+1} = h_k g_j g_{j+1} = h_k g^* = h_k \neq 1$).

- $g^* \neq 1_{\mathfrak{G}(g_j)}$: Let $\tilde{g} = h_k g^* = h_k g_j g_{j+1} = \hat{g} g_{j+1}$, then $x'_j = y\tilde{g}$ (case #5), $x_j = y\hat{g}$ (case #5) and $x_{j+1} = y\tilde{g}$ (case #5, since $\hat{g}g_{j+1} = \tilde{g}$).

No matter which case, the result is that $x'_j = x_{j+1}$. After appending the same factors $g_{j+2} \cdots g_n$, the output should be the same, hence $r(w) = r(w')$. Extend the result to an sequence of operations of word, we conclude that $w \sim w' \implies r(w) = r(w')$. If w is equivalent to two reduced word w', w'', then $w' \sim w''$ hence $r(w') = r(w'')$. But we proved that $r(x) = x$ if x is a reduced word, hence $w' = w''$.

Finally, notice that if w is not a reduced word, by definition it means that we could preform a reducing operation on w. After a reducing operation, the length of the word decrease. Repeat this process. Since the length couldn't decrease below 0, eventually we would get a reduced word w' which is equivalent to w.

Combine the result above, we conclude that every word w is equivalent to exactly one reduced word, and hence each equivalent class has exactly one reduced word. □

Example 1. Consider $C_2 * C_3$, where $C_2 = \langle a \rangle, C_3 = \langle b \rangle$. By the theorem above, each element of $C_2 * C_3$ correspond to a reduced word in $W(\{a, b\})$.

It is easy to see that all the reduced word in $W(\{a, b\})$ has the form

$$x = a^{n_1} b^{n_2} a^{n_3} b^{n_4} \cdots a^{n_{2k-1}} b^{n_{2k}}$$

such that $n_{2i+1} = 1$ and $n_{2i} = 1$ or 2, with the exception that n_1, n_{2k} could be 0.

Any other word in $W(\{a, b\})$ could be reduced to the form. For example,

$$\begin{aligned}
aabbaabaaabba &= a^2 b^2 a^2 b a^3 b^2 a \\
&= 1_{C_2} \cdot b^2 \cdot 1_{C_2} \cdot bab^2 a \\
&= b^2 bab^2 a = b^3 ab^2 a \\
&= 1_{C_3} \cdot ab^2 a \\
&= ab^2 a
\end{aligned}$$

Proposition 4. *Let $\mathcal{S} = \{G_\alpha : \alpha \in \mathcal{A}\}$ be a set of groups, and suppose that $\left\langle S_\alpha \;\middle|\; R_\alpha \right\rangle$ is a presentation for each G_α, then $\left\langle \bigcup_\alpha S_\alpha \;\middle|\; \bigcup_\alpha R_\alpha \right\rangle$ is a presentation of the free product $\mathop{\ast}_\alpha G_\alpha$.*

Recall that $\left\langle S \;\middle|\; R \right\rangle$ is defined to be $F(S)/N(R)$ where $F(S)$ is the free group of S and $N(R)$ is the smallest normal subgroup containing R.

Proof. Let $S = \bigcup_\alpha S_\alpha, R = \bigcup_\alpha R_\alpha$ and define $G = \left\langle S \;\middle|\; R \right\rangle$. Now consider the function ψ which carry each element $x \in S_\alpha$ to $F(S)$ by:

1. First carry it from S_α into S by the inclusion map.
2. Then carry it from S into $F(S)$ by the mapping given by the free group.

Now by the universal property of the free group, there exist a map $\tilde{f}_\alpha :: F(S_\alpha) \to F(S)$. It is easy to see that $\tilde{f}_\alpha$ simply maps a word $w \in F(S_\alpha)$ to the same word in $F(S)$. Compose with the quotient map q from $F(S)$ to $F(S)/N(R) = G$, we get a map f_α from $F(S_\alpha)$ to G.

Now, let q_α be the quotient map from $F(S_\alpha)$ to G_α. For each word $r \in R_\alpha \subseteq F(S_\alpha)$, $\tilde{f}_\alpha$ send r into R, so $f_\alpha(r) = 1_G$, hence $\ker q_\alpha = N(R_\alpha) \subseteq \ker f_\alpha$ since $N(R_\alpha)$ is the smallest normal subgroup containing R_α and a kernal is always normal. By factor theorem, f_α descends to a homomorphism $i_\alpha :: G_\alpha \to G$.

If we could prove that (G, i_α) has the universal property of a free product, by the uniqueness of free product we would have $G \cong \mathop{\ast}_\alpha G_\alpha$.

Let H be a group, and let $\varphi_\alpha :: G_\alpha \to H$ be a system of homomorphisms. Consider $\tilde{\varphi}_\alpha :: F(S_\alpha) \to H$ by $\tilde{\varphi}_\alpha = \varphi_\alpha \circ q_\alpha$. Define $\tilde{g} :: S \to H$ which sending $x \in S$ by

1. First carry x to one of the S_α by the inverse of inclusion map ι^{-1}, it could be done since $\{S_\alpha\}$ is consider to be disjoint.
2. Then carry it into $F(S_\alpha)$ by the mapping given by the free group.
3. Finally carry it into H by $\tilde{\varphi}_\alpha$.

Since $F(S)$ is the free group of S, by its universal property, exists a homomorphism $g :: F(S) \to H$. Now for each $r \in R \subseteq F(S)$, r lies in a unique R_α, so

$$g(r) = \tilde{\varphi}_\alpha(r) = \varphi_\alpha(q_\alpha(r)) = \varphi_\alpha(1_{G_\alpha}) = 1_H.$$

Consequently $\ker q = N(R) \subseteq \ker g$, and by factor theorem, g descends into a homomorphism $\varphi :: G \to H$. This φ satisfies:

$$\varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = g \circ \tilde{f}_\alpha = \tilde{\varphi}_\alpha = \varphi_\alpha \circ q_\alpha$$

and notice that q_α onto G_α, hence we conclude that $\varphi \circ i_\alpha = \varphi_\alpha$. which proof that G has the desire mapping required by the universal property for any H.

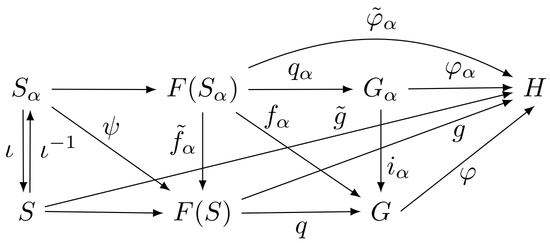

Figure 2: The commute diagram in the proof.

Finally, for the uniqueness, if φ makes $\varphi_\alpha = i_\alpha \circ \varphi$, then consider $h = \varphi \circ q$, since

$$\varphi_\alpha \circ q_\alpha = \varphi \circ i_\alpha \circ q_\alpha = \varphi \circ f_\alpha = \varphi \circ q \circ \tilde{f}_\alpha = h \circ \tilde{f}_\alpha.$$

So h is determined for all $x \in \bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$. But $\bigcup_\alpha \operatorname{Im} \tilde{f}_\alpha$ contains all generator of $F(S)$ (i.e., all the words with length 1), hence h is uniquely determined. Since q is onto, we conclude that φ is also uniquely determined.

Hence we prove that G satisfies the universal property, and $\left\langle \bigcup_\alpha S_\alpha \,\middle|\, \bigcup_\alpha R_\alpha \right\rangle \cong \ast_\alpha G_\alpha$. $\square$

Example 2. Consider again $C_2 * C_3$. Since $C_2 = \langle a \mid a^2 \rangle, C_3 = \langle b \mid b^3 \rangle$, so $C_2 * C_3 = \langle a, b \mid a^2, b^3 \rangle$.

Definition 3. Let $\mathcal{C}$ be a category, X, Y, Z be objects, and $f :: Z \to X$, $g :: Z \to Y$ are two morphisms. The **pushout** of the morphisms f, g is a tuple (P, i_1, i_2) such that P is an object, $i_1 :: X \to P$, $i_2 :: Y \to P$ with $i_1 \circ f = i_2 \circ g$, and (P, i_1, i_2) has the following universal property: For any (Q, j_1, j_2) satisfying $j_1 \circ f = j_2 \circ g$, there is an unique morphism $\varphi :: P \to Q$ that makes $\varphi \circ i_1 = j_1, \varphi \circ i_2 = j_2$.

A common notation is $P = X \sqcup_Z Y$.

Definition 4. The amalgamated free product $G_1 *_F G_2$ is the pushout $G_1 \sqcup_F G_2$. That is, given $\psi_1 :: F \to G_1, \psi_2 :: F \to G_2$, the amalgamated free product is a tuple (G, i_1, i_2) such that G is a group, i_1, i_2 are homomorphisms from G_1 to G and G_2 to G, respectively, with $i_1 \circ \psi_1 = i_2 \circ \psi_2$. It should also satisfied the following universal property: given (H, j_1, j_2) such that H is a group, j_1, j_2 are homomorphisms from G_1 to H and G_2 to H, with $j_1 \circ \psi_1 = j_2 \circ \psi_2$, then there exists a unique homomorphism φ such that $j_1 = \varphi \circ i_1, j_2 = \varphi \circ i_2$.

Proposition 5. *Amalgamate free product exists and is unique up to isomorphism.*

$$\begin{tikzcd}
& G_1 \arrow[d, "i_1"] \arrow[dr, "j_1"] & \\
F \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & G_1 *_F G_2 \arrow[r, "\varphi"] & H \\
& G_2 \arrow[u, "i_2"'] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Figure 3: The diagram of the amalgamated free product

Proof. Consider the free product $G_1 * G_2$, where $\tilde{i}_1, \tilde{i}_2$ are homomorphisms given by the free product from G_1 to $G_1 * G_2$ and G_2 to $G_1 * G_2$.

We construct $G_1 *_F G_2 = (G_1 * G_2)/N$, where N is the smallest normal subgroup containing the elements $\left\{\left[\psi_1(x)\psi_2(x)^{-1}\right] : x \in F\right\}$ in $G_1 * G_2$. Let q be the quotient map $G_1 * G_2 \to G_1 * G_2/N$ and $i_1 = q \circ \tilde{i}_1, i_2 = q \circ \tilde{i}_2$. Then for all $x \in F$,

$$i_1(\psi_1(x))\, i_2(\psi_2(x))^{-1} = q(\tilde{i}_1(\psi_1(x)))\, q\big(\tilde{i}_2\left(\psi_2(x)^{-1}\right)\big) = q\big([\psi_1(x)][\psi_2(x)^{-1}]\big) = q([\psi_1(x)\psi_2(x)^{-1}]) = 1$$

Hence $i_1(\psi_1(x)) = i_2(\psi_2(x)), \forall x \implies i_1 \circ \psi_1 = i_2 \circ \psi_2$.

Now, given H, j_1, j_2. By the universal property of free product, exists $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \tilde{i}_\alpha = j_\alpha$ for $\alpha \in \{1, 2\}$.

Next, for all $x \in F$, $[\psi_1(x)\psi_2(x)^{-1}] = [\psi_1(x)][\psi_2(x)^{-1}] = \tilde{i}_1(\psi_1(x))\, \tilde{i}_2\big(\psi_2(x)^{-1}\big)$, so

$$\tilde{\varphi}\big([\psi_1(x)\psi_2(x)^{-1}]\big) = (\tilde{\varphi} \circ \tilde{i}_1)(\psi_1(x))\, (\tilde{\varphi} \circ \tilde{i}_2)\big(\psi_2(x)^{-1}\big) = (j_1 \circ \psi_1)(x)\, \big((j_2 \circ \psi_2)(x)\big)^{-1} = 1$$

since $j_1 \circ \psi_1 = j_2 \circ \psi_2$. By the fact that N is the smallest normal subgroup containing $\left\{\left[\psi_1(x)\psi_2(x)^{-1}\right] : x \in F\right\}$, we conclude that $N \subseteq \ker \tilde{\varphi}$, and consequently $\tilde{\varphi}$ descends into a homomorphism $\varphi :: G_1 *_F G_2 \to H$. And we have $\varphi \circ i_\alpha = \varphi \circ q \circ \tilde{i}_\alpha = \tilde{i}_\alpha \circ \tilde{\varphi} = j_\alpha$ for $\alpha \in \{1, 2\}$, hence φ satisfies the requirements.

Finally, for the uniqueness, suppose φ makes $j_\alpha = i_\alpha \circ \varphi$ for $\alpha \in \{1, 2\}$. then consider $h = \varphi \circ q$, since $h \circ \tilde{i}_\alpha = \varphi \circ q \circ \tilde{i}_\alpha = i_\alpha \circ \varphi = j_\alpha$ So h is determined for all $x \in \operatorname{Im} \tilde{i}_1 \cup \operatorname{Im} \tilde{i}_2$. But $\operatorname{Im} \tilde{i}_1 \cup \operatorname{Im} \tilde{i}_2$ contains all the generators of $G_1 * G_2$ (i.e., all the reduced words with length 1 and the empty word), hence h is uniquely determined. Since q is onto, we conclude that φ is also uniquely determined.

$$\begin{tikzcd}
& & G_1 \arrow[dl, "\tilde{i}_1"'] \arrow[d, "i_1"] \arrow[dr, "j_1"] & \\
F \arrow[urr, "\psi_1"] \arrow[drr, "\psi_2"'] & G_1 * G_2 \arrow[r, "q"] \arrow[rr, bend right, "\tilde{\varphi}"'] & G_1 *_F G_2 \arrow[r, "\varphi"] & H \\
& & G_2 \arrow[ul, "\tilde{i}_2"] \arrow[u, "i_2"'] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Figure 4: The diagram of groups and maps mentioned in the proof

□

For readers interested in applications of free product, the textbook mentioned the **Seifert–van Kampen theorem** in algebraic topology, which states that if X is a topological space which is the union of two open and path connected subspaces U_1, U_2, and there intersection is path connected and nonempty. Then the fundamental group of X is the free product of the fundamental groups U_1, U_2 amalgamated by the fundamental groups of $U_1 \cup U_2$.