PPSP

Reformulations for Project Portfolio Selection Problem

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2021/06/22

Operations Research Applications and Implementation

Imagine that you were a boss...

Project A

3 employees / $10 profit

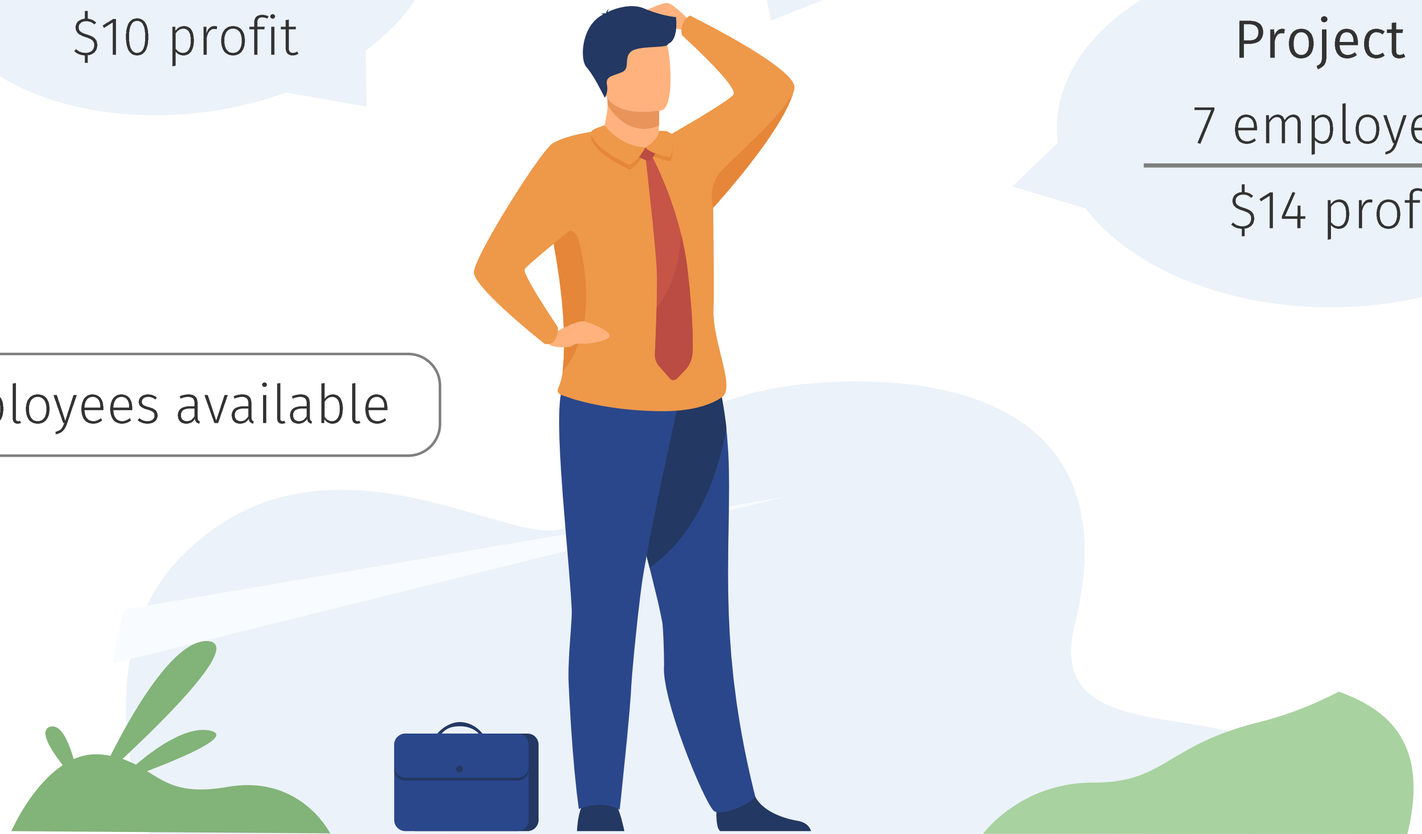
Project B

4 employees / $12 profit

Project C

7 employees / $14 profit

10 employees available



Think and make a decision!

- If we launch project A and project B, then…
- What if we launch project B and project C…
- What if we launch project A, B and C…
- Ok! Let's…

Good!

But better?

Even OPTIMAL?

(in acceptable time)

Agenda

1. Problem Statement
2. Model Formulation
   - Conventional Model
   - Proposed Model
3. Model Comparison
4. Experiment
5. Conclusion

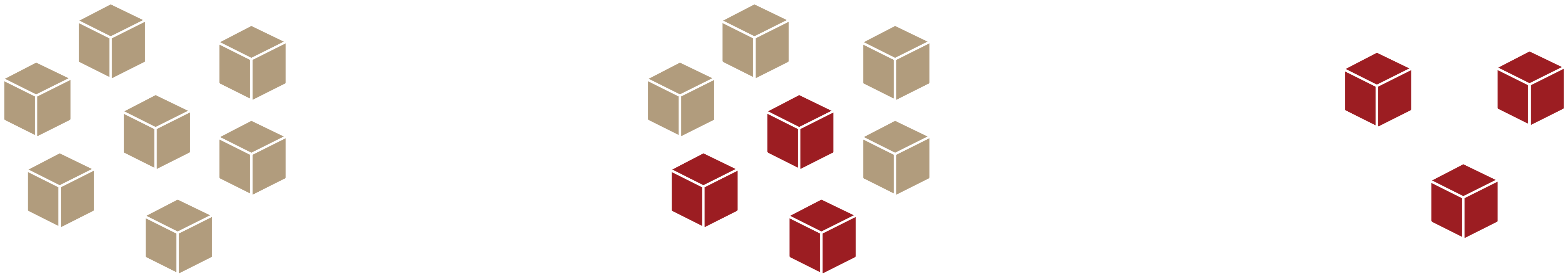
/01 Problem Statement

Project interdependency

Cardinality constraint

Project Portfolio Selection Problem (PPSP)

Select a set of m projects from a whole of N projects



Project candidates Selected projects Maximized benefit

with considering

- Project interdependency
- Cardinality constraint
- Precedence relationship, Employee competence, Divisibility …

Project Interdependency

- Benefit Interdependency
  - Synergistic benefit: when two or more projects are selected and executed
- Resource Interdependency
  - Sharing resources among various projects

Cardinality constraint

- Limit the total number of selected projects
- Equality or inequality constraint

Notations

- Parameters (All below are nonnegative.)
  - N: number of projects to be selected
  - m: limit of cardinality
  - $r_i, r_{i,j}, r_{i,j,k}$: benefit derived from implementing project i, ij and ijk , respectively
  - $d_i, d_{i,j}, d_{i,j,k}$: required amount of resource by project i, ij and ijk, respectively
  - b: the available amount of resource
- Decision variables
  - $x_i = 1$, if project i is selected; otherwise 0, for $i = 1, \ldots, \mathrm{N}$

Nonlinear model

$$\underbrace{Max \sum_{i=1}^{N} r_i x_i + \sum_{i=1}^{N-1} \sum_{j=1+1}^{N} r_{i,j} \underline{x_i x_j} + \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} r_{i,j,k} \underline{x_i x_j x_k}}_{\text{total benefit}}$$

$$s.t. \quad \underbrace{\sum_{i=1}^{N} x_i = m}_{\text{cardinality constraint}}$$

$$\underbrace{\sum_{i=1}^{N} d_i x_i - \sum_{i=1}^{N-1} \sum_{j=1+1}^{N} d_{i,j} \underline{x_i x_j} + \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} d_{i,j,k} \underline{x_i x_j x_k} \leq b}_{\text{resource constraint}}$$

$$x_i \in \{0, 1\}, \; i = 1, \ldots, N$$

Polynomial Integer Programming

→ Linearize

/02 Model Formulation

Conventional Model

Proposed Model

2-1 Conventional Model

- Introduce $y_{i,j} := x_i x_j$ and $z_{i,j,k} := x_i x_j x_k$,

  where $y = (y_{1,2}, \ldots, y_{1,N}, y_{2,3}, \ldots, y_{N-1,N}) \in \mathbb{R}^{\frac{N(N-1)}{2}} = \binom{N}{2}$

  $z = (z_{1,2,3}, \ldots, z_{1,N-1,N}, z_{2,3,4}, \ldots, z_{N-2,N-1,N}) \in \mathbb{R}^{\frac{N(N-1)(N-2)}{6}} = \binom{N}{3}$

$$Max \sum_{i=1}^{N} r_i x_i + \sum_{i=1}^{N} \sum_{j>i}^{N} r_{i,j} y_{i,j} + \sum_{i=1}^{N} \sum_{j>i}^{N} \sum_{k>j}^{N} r_{i,j,k} z_{i,j,k}$$

$$s.t. \sum_{i=1}^{N} x_i = m$$

$$\sum_{i=1}^{N} d_i x_i - \sum_{i=1}^{N} \sum_{j>i}^{N} d_{i,j} y_{i,j} + \sum_{i=1}^{N} \sum_{j>i}^{N} \sum_{k>j}^{N} d_{i,j,k} z_{i,j,k} \leq b$$

$$\underbrace{\sum_{j>i}^{N} y_{i,j} + \sum_{j<i}^{N} y_{j,i} + \sum_{j>i}^{N} \sum_{k>j}^{N} z_{i,j,k} + \sum_{i>j}^{N} \sum_{k>i}^{N} z_{j,i,k} + \sum_{k>j}^{N} \sum_{i>k}^{N} z_{j,k,i} = \frac{m(m-1)}{2} x_i, \text{ for } i = 1, \ldots, N}_{\Leftrightarrow\ x_i x_j = y_{i,j} \text{ and } x_i x_j x_k = z_{i,j,k}}$$

$$x_i \in \{0,1\} \text{ and } 0 \leq y_{i,j}, z_{i,j,k} \leq 1, \text{ for } i,j,k = 1, \ldots, N, \text{ and } i < j < k$$

2-2 Proposed Model

- Eliminate y to reduce the number of continuous variables

$$Max \ \sum_{i=1}^{N} r_i x_i + \underbrace{\frac{1}{m-2}\sum_{i=1}^{N}\sum_{j>i}^{N}\sum_{k>j}^{N}(r_{i,j}+r_{i,k}+r_{j,k})z_{i,j,k}}_{=\sum_{i=1}^{N}\sum_{j>i}^{N} r_{i,j}y_{i,j}} + \sum_{i=1}^{N}\sum_{j>i}^{N}\sum_{k>j}^{N} r_{i,j,k}z_{i,j,k}$$

$$s.t. \ \sum_{i=1}^{N} x_i = m$$

$$\sum_{i=1}^{N} d_i x_i - \underbrace{\frac{1}{m-2}\sum_{i=1}^{N}\sum_{j>i}^{N}\sum_{k>j}^{N}(d_{i,j}+d_{i,k}+d_{j,k})z_{i,j,k}}_{=\sum_{i=1}^{N}\sum_{j>i}^{N} d_{i,j}y_{i,j}} + \sum_{i=1}^{N}\sum_{j>i}^{N}\sum_{k>j}^{N} d_{i,j,k}z_{i,j,k} \le b$$

$$\underbrace{\sum_{j>i}^{N}\sum_{k>j}^{N} z_{i,j,k} + \sum_{i>j}^{N}\sum_{k>i}^{N} z_{j,i,k} + \sum_{k>j}^{N}\sum_{i>k}^{N} z_{j,k,i} = \frac{(m-1)(m-2)}{2}x_i, \ \text{for } i = 1,\ldots,N}_{\Leftrightarrow x_i x_j x_k = z_{i,j,k}}$$

$$x_i \in \{0,1\} \text{ and } 0 \le z_{i,j,k} \le 1, \ \text{for } i,j,k = 1,\ldots,N, \ \text{and } i<j<k$$

/03 Model Comparison

Complexity Analysis

Model Comparison

Models	# of 0-1 variables	# of continuous variables	# of equality constraints	# of inequality constraints
Conventional	N	$\frac{N(N-1)}{2} + \frac{N(N-1)(N-2)}{6}$	$1+N$	1
Proposed	N	$\frac{N(N-1)(N-2)}{6}$	$1+N$	1

- N = the # of candidate projects
- In the equality cardinality constraint cases, the number of continuous variables drops.

/04 Experiment

Data Generating Process

Instance Type	# of instances	(N, m)
Small-PPSP	50	(16, 3)
Medium-PPSP	50	(32, 5)
Large-PPSP	10	(64, 7)

Parameters

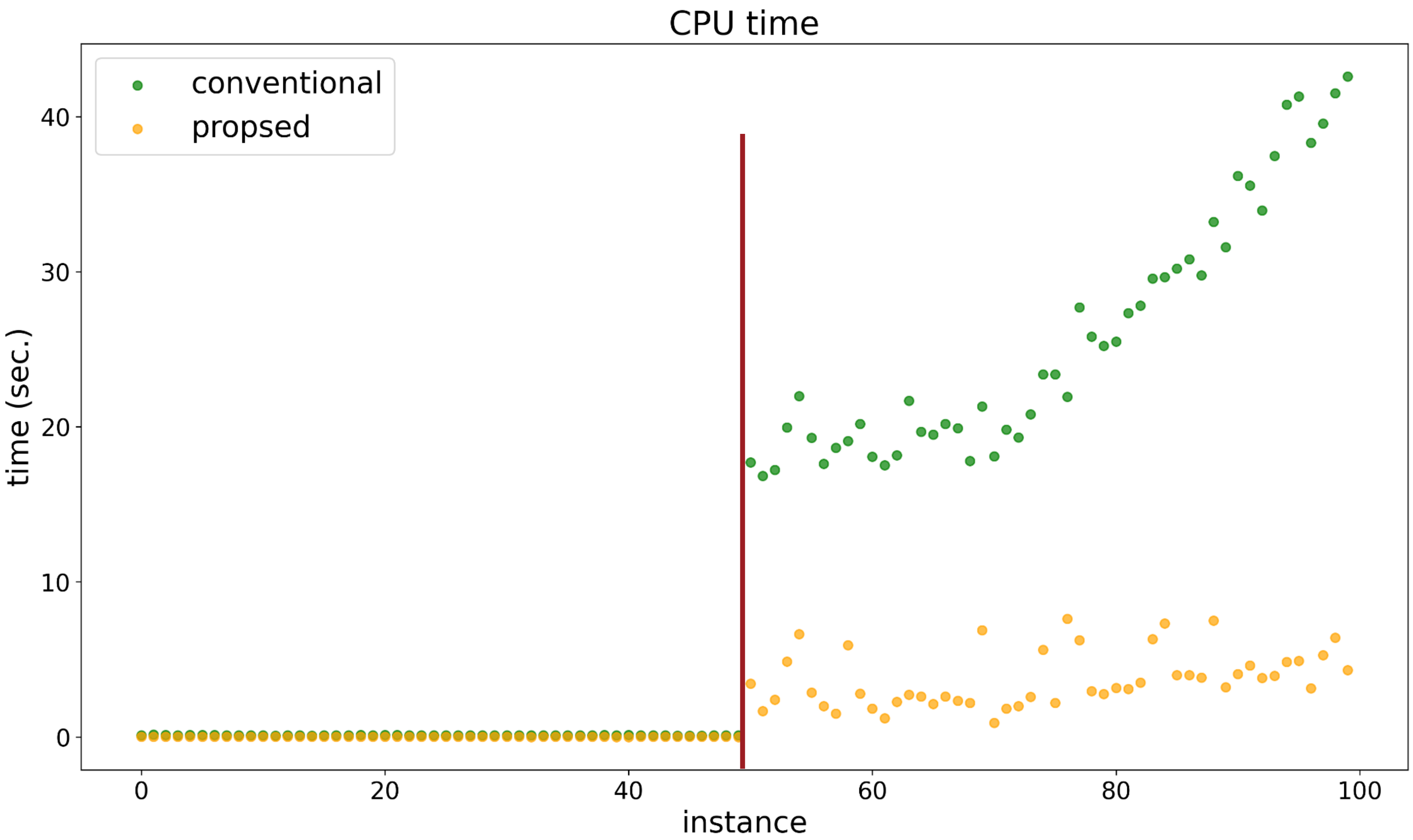
- Resource
  - $b \sim \text{Uniform}(0.05G, 0.1G), G = \sum_i d_i - \sum_i \sum_{j>i} d_{i,j} + \sum_i \sum_{j>i} \sum_{k>j} d_{i,j,k}$
  - $d_i \sim \text{Uniform}(1{,}10)$ / $d_{i,j} \sim \text{Uniform}(5{,}20)$ / $d_{i,j,k} \sim \text{Uniform}(10{,}50)$
- Benefit
  - $r_i \sim \text{Uniform}(10{,}100)$ / $r_{i,j} \sim \text{Uniform}(50{,}200)$ / $r_{i,j,k} \sim \text{Uniform}(100{,}500)$

Computational Result

Instance type (N, m)	model	# of 0-1 variables	# of continuous variables	# of linear Constraints	Avg. CPU time (sec.)	Std. dev. CPU time (sec.)
Small (16, 3)	Conventional	16	680	18	0.1098	0.0112
	Proposed	16	560	18	0.0030	0.0004
Medium (32, 5)	Conventional	32	5456	34	25.8070	7.8189
	Proposed	32	4960	34	3.7361	1.7651
Large (64, 7)	Conventional	64	43680	66	-	-
	Proposed	64	41664	66	-	-

'-' indicates the running time for GUROBI exceeds 1 CPU hour for all instances.

Computational Result



Small scale

Medium scale

/05 Conclusion

Conclusion

- Project Portfolio Selection Problem with considering cardinality constraint and interdependency
- Linearization technique, which can be applied to similar problems (even higher orders)
- Reducing the number of binary variables is a critical issue

References

- Xingmei Li, Yao-Huei Huang, Shu-Cherng Fang and Zhibin Deng (2016), Reformulations for project portfolio selection problem considering interdependence and cardinality, *Pacific Journal of Optimization, 12(2)*, 355-366.
- Xingmei Li, Yao-Huei Huang , Shu-Cherng Fang , Youzhong Zhang (2020), An alternative efficient representation for the project portfolio selection problem, *European Journal of Operational Research, 281(1)*, 100-113.

Thank you for listening.

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