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Two new mean–variance enhanced index tracking models based on uncertainty theory

Tingting Yang, Xiaoxia Huang *

School of Economics and Management, University of Science and Technology Beijing, Beijing 100083, China

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ABSTRACT

The enhanced index tracking (EIT) problem is concerned with selecting a tracking portfolio that achieves an excess return over a given benchmark with a minimum tracking error. This paper explores the EIT problem by providing two new mean–variance EIT models based on uncertainty theory where stock returns are treated as uncertain variables instead of random variables and stock return distributions are estimated by experts instead of from historical data. First, this paper formulates an uncertain enhanced index tracking (UEIT) model and analyzes the characteristic of the UEIT frontier. Then to reduce the tracking portfolio's risk, this paper adds a risk index (RI) constraint to the UEIT model and proposes a UEIT-RI model. Next, by comparing the UEIT and UEIT-RI models this paper gives the advantages of the two models. Investors can choose the model according to their preferences. Finally, this paper conducts numerical examples to illustrate the application of the two models and the analysis results.

1. Introduction

Passive management and active management are two investment strategies in the industry. Passive investors copy the market and obtain the average returns (Sant'Anna et al., 2020), while active investors use quantitative tools such as mean–variance model (Markowitz, 1952) to try to get the excess returns. However, evidences have shown that markable number of actively managed funds do not outperform its benchmark over the long term (Caporin & Lisi, 2013; Gruber, 1996). As a consequence, investors often prefer to follow hybrid strategies using both passive and active managements. The enhanced index tracking (EIT) is such a hybrid strategy, which is designed to track the benchmark closely, yet providing some value added. Since the EIT strategy is favored in the industry, the EIT problems have been paid more attention in financial academia. The EIT problem is concerned with selecting a tracking portfolio that beats a given benchmark on return while having a minimum tracking error relative to this benchmark. Nowadays, many researchers and fund managers use optimization models to solve the EIT problems. Then various EIT optimization models are proposed to help investors build their portfolios (Canakgoz & Beasley, 2009; Guastaroba et al., 2016; Sant'Anna et al., 2017). Existing EIT models can be classified according to the tracking error functions, such as mean–variance EIT model (Paulo et al., 2016), mean-absolute deviation EIT model (Rudolf et al., 1999) and mean-absolute downside deviation EIT model (Koshizuka et al., 2009), etc. Studies have shown that tracking error variance is preferable to minimizing other tracking error functions (Gngi & Strub, 2020). So the mean–variance EIT models are favored by investors.

Under the mean–variance EIT framework, an important work is given by Roll (1992) who formulates a criterion of minimizing the tracking error variance for a given expected excess return and then combines the mean–variance model with the factor model by adding a constraint on the beta of the tracking portfolio. Then scholars extend the mean–variance EIT model. Clarke et al. (1994)

* Corresponding author.
E-mail addresses: oyanglin@163.com (T. Yang), hxiaoxia@manage.ustb.edu.cn (X. Huang).

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introduce the aversion to regret into the tracking error variance analysis and show that an investor with higher averse to regret prefers the portfolio with a lower tracking error variance and expected excess return. Paulo et al. (2016) propose a mean–variance EIT model with limited number of assets held in the tracking portfolio and give the analytical solution of it. Gngi and Strub (2020) add real-life constraints such as transaction costs, investment budget and Min./max. weights to the mean–variance EIT model. Since adding real-life constraints will make the model difficult to find analytical solutions, Mutunge and Haugland (2018) adopt heuristic approaches to solve their mean–variance EIT model. These models and algorithms help investor well manage their tracking portfolios.

Since returns and risks are what investors concerned most, many studies focus on the above two parameters of tracking portfolios. Hodges (1976) uses a Markowitz model and compares the tradeoff curve relating variance to return in excess of the index with the tradeoff curve for the Markowitz portfolio optimization model. Roll (1992) shows the tracking portfolio is risky for investors compared with standard Markowitz mean–variance portfolio and uses a constraint on the beta of the tracking portfolio to alleviate the overly risk problem. Then scholars begin to focus on how to reduce the risk of tracking portfolio. Their approaches are to impose a limit on the amount of risk that investors can take, but differ on the measure of risk used. For example, Alexander and Baptista (2008), Jorion (2003) and Alexander and Baptista (2010) add tracking portfolio's variance constraint, value-at-risk constraint and ex-ante alpha constraint, respectively, to reduce the tracking portfolio's risk.

The above studies are of great help to investors in building tracking portfolios, but their conclusions are all based on the assumption that securities returns are random variables. Though probability theory is a powerful tool for dealing with indeterminate parameters, its application requires sufficient and valid historical data so that the obtained distribution is close enough to the long-run frequency. However, there are some situations in reality that people have few historical data or historical data are not valid due to unexpected events in the market and society. For example, the outbreak of COVID-19 has caused a series of unexpected events, which leads to great fluctuations in the security markets so that historical data cannot reflect the future effectively. When there are no data or data are invalid, people have to rely on experts' estimations. However, findings of Tversky and Kahneman (1986) reveal that human usually overweight unlikely events, which means people's estimations usually contain much wider range of values than the real case. If people's estimations still be treated as probability distributions and probability theory is used to solve problem, counterintuitive results may occur (Liu, 2012). To handle the problem in such kind of situation, some researches use robust portfolio selection models. For example, Pflug and Wozabal (2007) apply a maximin approach which uses a "confidence set" for the probability distribution to solve the optimal portfolio problems in cases when the underlying probability model is not perfectly known, and show the tradeoff between return, risk and robustness in view of the model ambiguity. Kang et al. (2019) adopt a nonparametric bootstrap approach to calibrate the levels of ambiguity and present a computationally tractable optimization method for a robust mean-CVaR portfolio selection model under the condition of distribution ambiguity. Though robust portfolio selection model is a meaningful exploration to deal with indeterminate parameters, it is still based on the framework of probability theory.

In order to model humans' estimations which are different from randomness, Liu (2007) proposes a new theory, i.e., uncertainty theory. By using uncertainty theory no counterintuitive results occur. Nowadays, uncertainty theory has been applied in solving various optimization problems (Gao, 2011; Liu, 2010). Particularly, Huang (2010) is the first to use the uncertainty theory to study portfolio selection systematically and has established a spectrum of uncertain portfolio models, such as uncertain mean-variance model and mean-semivariance model. Gradually uncertain portfolio selection problems have attracted the attention of scholars. Huang and Yang (2020) consider background risk in uncertain mean–variance model for risky asset portfolio selection and discuss the impacts of background risk on risky asset investment in three aspects: portfolio position, efficient frontier and model efficiency. Huang and Jiang (2021) further establish an uncertain mean–variance utility model and study the effects of changes in mean and standard deviation of uncertain background asset as well as the initial proportion in background asset on capital allocation to risky and risk-free securities. Wang and Huang (2019) study uncertain portfolio with options and show portfolios with options gain higher returns than those without options. Huang and Di (2020) focus on uncertain portfolio with mental accounts. Jin et al. (2019) research multi-period uncertain portfolio selection problem. Besides, scholars further discuss the performances of uncertain portfolios. Studies show that the returns or the expected returns of uncertain portfolios are higher than those of traditional portfolios in some situations. For example, Huang and Yang (2020) present the evidence that the return of portfolio got from the uncertain mean–variance model is greater than that of traditional Markowitz's mean–variance model in volatile markets. Xue et al. (2019) find that the expected return of aggregate portfolio when security return rates are uncertain variables is greater than the expected return of aggregate portfolio when security return rates are random variables in the case of facing invalid data. It can be seen from the above examples that the uncertain portfolio selection models have some advantages in the above mentioned situations.

So far, no work deals with the uncertain EIT problem. How to build the uncertain EIT models? What are the return and risk of such uncertain EIT portfolios? How to reduce the uncertain EIT portfolios' risk? The lack of research on these issues motivates us to do the study. Different from Huang and Yang (2020) which is concerned about the impact of background risk on uncertain portfolio selection and how to manage uncertain portfolios with the changes of background risk, our paper attempts to provide deep theoretical insights into uncertain EIT problems and help investors make decisions with uncertain EIT models. Different from Huang (2012b) which proposes a new risk measurement tool, i.e., risk index (RI) and uses the RI to construct a new uncertain mean-risk index portfolio model, our paper cares about constructing the uncertain mean–variance EIT model, the characteristic of the uncertain EIT portfolio, how to reduce the risk of the uncertain EIT portfolio by using the RI constraint and how is the effect of the RI constraint in our proposed model. Contributions of this paper to the literature are as follows. First, we propose an uncertain enhanced index tracking (UEIT) model, analyze the return and risk of the UEIT portfolio, and further offer the form of the UEIT frontier. Second, we add a RI constraint to the UEIT model and propose a UEIT-RI model which aims to reduce the risk of tracking portfolio. RI is an average loss below zero, which is easier for investors to understand and feel the risk. Third, we compare the UEIT and UEIT-RI models. Each model has its own advantage. Since the focus of the two models are different, investors should choose different models according to their preferences.

The rest of the paper is organized as follows. Section 2 addresses the UEIT model and analyzes the property of the UEIT frontier. Section 3 proposes the UEIT-RI model and compares it with UEIT model. Section 4 conducts numerical examples to help readers better understand and apply our models to manage portfolios. Section 5 concludes the paper. Finally, Appendix A provides necessary knowledge of uncertainty theory for readers to better understand this paper and Appendix B provides the proofs of the theorems in the paper.

2. The UEIT model

In this section, we propose a UEIT model and analyze the characteristic of the UEIT frontier.

2.1. Assumptions and notations

Assume that stocks in the asset universe have different excepted returns and variances. And the bigger excepted return, the bigger variance. In the meanwhile, short-selling is allowed. And our UEIT model employs the following notations:

n The number of stocks in investors' asset universe.

$\boldsymbol{X_I}$ An $n\times 1$ vector represents a benchmark that is known, and x_{Ii} is the ith entry in this vector, representing the proportion of stock i in the benchmark.

$\boldsymbol{X_P}$ An $n\times 1$ vector represents a tracking portfolio, and x_{Pi} is the ith entry in this vector, representing the proportion of stock i in the tracking portfolio.

$\boldsymbol{X}$ An $n\times 1$ vector represents the difference, stock by stock, between $\boldsymbol{X_P}$ and $\boldsymbol{X_I}$. And x_i is the ith entry in this vector, representing the alteration of the stock i. Note that $\boldsymbol{X}$ is self-financing, i.e., $\boldsymbol{X}^T\mathbf{1} = 0$ where $\mathbf{1}$ is an $n\times 1$ vector whose entries are all 1. Note also that x_i can be positive, zero or negative. Positive entry means purchasing stocks, zero entry means no alteration, and negative entry means selling stocks.

ξ_i The uncertain return of stock i.

r_i The uncertain return of portfolio i, e.g., r_I represents benchmark's uncertain return and r_P represents the tracking portfolio's uncertain return.

e_i The expected return of stock or portfolio i.

σ_i The standard deviation of stock or portfolio i.

G The expected excess return over the benchmark's return, $G = e_P - e_I$.

2.2. Uncertain model

In our paper, we use variance to measure the tracking error and tracking error variance is $V[r_P - r_I] = V[\sum_{i=1}^{n}(x_{Pi} - x_{Ii})\xi_i]$. Following the criterion of minimizing the tracking error variance for a given expected excess return over its benchmark, the UEIT model can be stated as follows:

$$\begin{cases} \min\ V\left[\sum_{i=1}^{n}(x_{Pi} - x_{Ii})\xi_i\right] \\ \text{subject to:} \\ \qquad E\left[\sum_{i=1}^{n}(x_{Pi} - x_{Ii})\xi_i\right] = G \\ \qquad \sum_{i=1}^{n}(x_{Pi} - x_{Ii}) = 0,\ i = 1, 2, \dots, n \end{cases} \tag{1}$$

where V and E are the variance and the expected value operators of uncertain variables. Since $x_i = x_{Pi} - x_{Ii}$, model (1) can be transformed into model (2):

$$\begin{cases} \min\ V\left[\sum_{i=1}^{n}x_i\xi_i\right] \\ \text{subject to:} \\ \qquad E\left[\sum_{i=1}^{n}x_i\xi_i\right] = G \\ \qquad \sum_{i=1}^{n}x_i = 0,\ i = 1, 2, \dots, n. \end{cases} \tag{2}$$

Model (2) cannot be calculated directly because it has two operators of uncertain variables. The deterministic form will facilitate solving the model, so we need to offer the deterministic form of model (2). As stock return distributions are usually described by normal distributions, Theorem 1 will give the deterministic form of model (2) when ξ_i take normal uncertainty distributions.

Theorem 1. *If the stock returns take normal uncertainty distributions, i.e., $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, $i = 1, 2, \ldots, n$, respectively, model (2) is equivalent to the following form:*

$$
\begin{cases}
\min \left[\sum_{i=1}^{n} |x_i| \sigma_i \right]^2 \\
\textit{subject to}: \\
\qquad \sum_{i=1}^{n} x_i e_i = G \\
\qquad \sum_{i=1}^{n} x_i = 0, \ i = 1, 2, \ldots, n.
\end{cases}
\tag{3}
$$

Proof. The proof is shown in Appendix B. □

In order to solve model (3), we introduce the non-negative decision variables u_i and v_i. Let $u_i = \dfrac{x_i + |x_i|}{2}$ and $v_i = \dfrac{|x_i| - x_i}{2}$. We can get $x_i = u_i - v_i$ and $|x_i| = u_i + v_i$. Substitute these two equations into model (3), we get

$$
\begin{cases}
\min \left[\sum_{i=1}^{n} (u_i + v_i) \sigma_i \right]^2 \\
\textit{subject to}: \\
\qquad \sum_{i=1}^{n} (u_i - v_i) e_i = G \\
\qquad \sum_{i=1}^{n} (u_i - v_i) = 0 \\
\qquad u_i \geq 0 \\
\qquad v_i \geq 0, \ i = 1, 2, \ldots, n.
\end{cases}
\tag{4}
$$

Inspired by Huang and Yang (2020), we can get the solution of model (4). Then by transformation, we get the optimal solution of model (3) $\boldsymbol{X}^* = [0, 0, \ldots, x_k, \ldots, x_l, \ldots, 0, 0]^T$ where

$$
\begin{bmatrix} x_k \\ x_l \end{bmatrix} = \begin{bmatrix} \dfrac{-G}{e_l - e_k} \\ \\ \dfrac{G}{e_l - e_k} \end{bmatrix}.
\tag{5}
$$

Four points are worth noticing. (i) $\boldsymbol{X}^*$ gives the optimal alteration between the tracking portfolio $\boldsymbol{X_P}$ and the benchmark $\boldsymbol{X_I}$. So we can get the composition of tracking portfolio $\boldsymbol{X_P}$ because of $\boldsymbol{X_P} = \boldsymbol{X_I} + \boldsymbol{X}^*$. Since the tracking portfolio $\boldsymbol{X_P}$ is expected to outperform the benchmark, its expected return is

$$
e_P = e_I + G.
\tag{6}
$$

(ii) In the process of solving model (3), we assume that $e_l > e_k$. So $\boldsymbol{X}^*$ is negative in the kth entry and positive in the lth entry. It means that if investors want to change portfolio $\boldsymbol{X_I}$ to $\boldsymbol{X_P}$, they need to adjust two stocks' weights by selling the kth stock and purchasing the lth stock. This point is consistent with what we have observed in financial markets. When investors want to enhance their return, they usually sell low-yield stocks and buy high-yield stocks. (iii) $\boldsymbol{X}^*$ is completely independent of portfolio $\boldsymbol{X_I}$, implying that $\boldsymbol{X}^*$ is the same no matter how $\boldsymbol{X_I}$ changes. (iv) $\boldsymbol{X}^*$ depends on G, the targeted excess return over benchmark. The weights in $\boldsymbol{X}^*$ are increasing with G, which is consistent with practices of investors. The higher excess return investors pursue, the more high-yield stocks they need to buy and the more low-yield stocks they need to sell.

2.3. The UEIT frontier

In this section, we analyze the characteristic of the UEIT frontier which can give investors a panoramic view of portfolio return and risk. First, we have got the expected return of the tracking portfolio according to Eq. (6). Next, we need know the standard deviation of the tracking portfolio. Keep in mind that the return of the tracking portfolio is $r_P = \sum_{i=1}^n x_{Pi}\xi_i$. As ξ_i take normal uncertainty distributions, we can prove that r_P also takes a normal uncertainty distributions, i.e.,

$$r_P \sim \mathcal{N}\left(\sum_{i=1}^n x_{Pi}e_i,\ \sum_{i=1}^n |x_{Pi}|\sigma_i\right).$$

So we can have $\sigma_P = \sum_{i=1}^n |x_{Pi}|\sigma_i$. Since $x_{Pi} = x_{Ii} + x_i$, we get

$$\sigma_P = \sum_{i=1}^n |x_{Pi}|\sigma_i = \sum_{i=1}^n |(x_{Ii} + x_i)|\sigma_i. \tag{7}$$

Substituting Eq. (5) into Eq. (7) and rearranging, we get $\sigma_P = \sum_{i=1}^n |x_{Ii}|\sigma_i - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_{Ik} + x_k|\sigma_k + |x_{Il} + x_l|\sigma_l$. Since we can prove that $\sigma_I = \sum_{i=1}^n |x_{Ii}|\sigma_i$, we have

$$\sigma_P = \sigma_I - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_{Ik} + x_k|\sigma_k + |x_{Il} + x_l|\sigma_l. \tag{8}$$

We have known that $x_k < 0$ and $x_l > 0$ according to Eq. (5). Since x_{Ik} and x_{Il} are the weights of stocks k and l respectively in the benchmark index in financial market, we consider $x_{Ik} > 0$ and $x_{Il} > 0$. When $|x_k| < |x_{Ik}|$, we rearrange Eq. (8) and have

$$\begin{aligned}\sigma_P &= \sigma_I - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_{Ik} + x_k|\sigma_k + |x_{Il} + x_l|\sigma_l\\ &= \sigma_I - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_{Ik}|\sigma_k - |x_k|\sigma_k + |x_{Il}|\sigma_l + |x_l|\sigma_l\\ &= \sigma_I + |x_l|\sigma_l - |x_k|\sigma_k.\end{aligned} \tag{9}$$

When $|x_k| \geq |x_{Ik}|$, we rearrange Eq. (8) and have

$$\begin{aligned}\sigma_P &= \sigma_I - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_{Ik} + x_k|\sigma_k + |x_{Il} + x_l|\sigma_l\\ &= \sigma_I - |x_{Ik}|\sigma_k - |x_{Il}|\sigma_l + |x_k|\sigma_k - |x_{Ik}|\sigma_k + |x_{Il}|\sigma_l + |x_l|\sigma_l\\ &= \sigma_I + |x_k|\sigma_k + |x_l|\sigma_l - 2|x_{Ik}|\sigma_k.\end{aligned} \tag{10}$$

Since we have got x_k and x_l according to Eq. (5), we can calculate σ_P.

Now we have known e_P and σ_P. Next we analyze the characteristic of the UEIT frontier. The following theorem will give the form of the UEIT frontier.

Theorem 2. *The frontier of portfolio $\boldsymbol{X_P}$ in (σ_P, e_P) space takes the following form:*

$$\sigma_P = \begin{cases}(e_P - e_I) \times \dfrac{\sigma_l - \sigma_k}{e_l - e_k} + \sigma_I, & when \quad e_P < (e_l - e_k) \times x_{Ik} + e_I,\\ (e_P - e_I) \times \dfrac{\sigma_l + \sigma_k}{e_l - e_k} + \sigma_I - 2|x_{Ik}|\sigma_k, & when \quad e_P \geq (e_l - e_k) \times x_{Ik} + e_I.\end{cases} \tag{11}$$

Proof. Substituting Eq. (5) into Eqs. (9) and (10), we can get Eq. (11). □

Three points are worth noticing. (i) The UEIT frontier change with the benchmark return distribution. From Eq. (11), it is seen that σ_P decrease with σ_I. It means that if investors track a low-risk benchmark, the risk of the UEIT portfolio will also be reduced accordingly. An implication is worth noting. All else unchanged, reducing the risk of benchmark helps control the risk of portfolio $\boldsymbol{X_P}$.

(ii) Keep in mind that the tracking error is $r_X = r_P - r_I = \sum_{i=1}^n x_i\xi_i$. According to the optimal solution $\boldsymbol{X}^*$, we know that $r_X = x_k\xi_k + x_l\xi_l$. Similarly, we can prove that r_X takes a normal uncertainty distributions, i.e., $r_X \sim \mathcal{N}(x_k e_k + x_l e_l, |x_k|\sigma_k + x_l\sigma_l)$. So we have the standard deviation of tracking error

$$\sigma_X = |x_k|\sigma_k + x_l\sigma_l. \tag{12}$$

Substituting Eq. (5) into Eq. (12), we can have

$$\sigma_X = \frac{\sigma_l + \sigma_k}{e_l - e_k} \times G. \tag{13}$$

Observing Eq. (13), we find that σ_X is independent of the benchmark and increases with G. Since $e_P = e_I + G$, an implication comes. If investors want to obtain a portfolio $\boldsymbol{X_P}$ with a high expected return e_P, they can set a comparatively small G but track a benchmark with big e_I. In this way, a smaller G guarantees a smaller σ_X. But it is not that the smaller σ_X, the better. There is a trade-off between σ_X and G. When we transform Eq. (13) into the following form

$$\frac{e_l - e_k}{\sigma_l + \sigma_k} = \frac{G}{\sigma_X}, \tag{14}$$

the right side of Eq. (14) represents information ratio which is introduced by Grinold and Kahn (1995). Similar to the suggestion of Da Silva et al. (2009), the best trade-off is to maximize the information ratio.

(iii) If $x_{Ik} = 0$ which means the benchmark does not contain stock k, according to Eqs. (10) and (12), we have $\sigma_P = \sigma_I + \sigma_X$. If $x_{Ik} \neq 0$, we have $\sigma_P < \sigma_I + \sigma_X$.

3. The UEIT model with a constraint on risk index

In Section 2, UEIT model focuses on controlling the relative risk (tracking error variance) and neglect to control the total risk (risk of portfolio $\boldsymbol{X_P}$). In this section, we add a new constraint to control the total risk. Next, we first introduce the new risk control method.

3.1. Risk index

Risk index (RI) is an alternative risk measurement presented in Huang (2012b) first. It is a direct tool for measuring portfolio risk and can give investors a more intuitive sense of loss. Huang and Ying (2013) and Wang and Huang (2019) use the RI as the risk measurement in their researches, respectively. In this paper, we will continue to use this risk measurement method. Here we use the tracking portfolio $\boldsymbol{X_P}$ to illustrate the definition of RI. Remember that r_P represent the return of portfolio $\boldsymbol{X_P}$. RI of $\boldsymbol{X_P}$ is defined as follows:

$$RI(r_P) = E[(0 - r_P)^+]$$

where

$$(0 - r_P)^+ = \begin{cases} 0 - r_P, & if\ r_P \leq 0 \\ 0, & if\ r_P > 0. \end{cases}$$

From the definition we can see that RI is an average return below zero, which directly measures the loss of investors. How to calculate RI? We will introduce a proposition from Huang (2012b) for calculating it.

Proposition 1 (*Huang, 2012b*). *Let ξ be an uncertain security return with continuous uncertainty distribution Φ whose inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$, and r_1 be the risk-free interest rate. Then the risk index of the security return can be calculated via*

$$RI(\xi) = \int_0^{\beta} (r_1 - \Phi^{-1}(\alpha))d\alpha,$$

where β is defined by $\Phi^{-1}(\beta) = r_1$.

3.2. The UEIT-RI model

Now we focus on controlling the total risk of tracking portfolio $\boldsymbol{X_P}$ and use RI to measure the total risk. Adding the RI constraint to model (2), the UEIT-RI model becomes the following form:

$$\begin{cases} \min\ V\left[\sum_{i=1}^{n} x_i\xi_i\right] \\ \text{subject to:} \\ \qquad E\left[\sum_{i=1}^{n} x_i\xi_i\right] = G \\ \qquad RI\left[\sum_{i=1}^{n} (x_{Ii} + x_i)\xi_i\right] \leq C \\ \qquad \sum_{i=1}^{n} x_i = 0,\ i = 1, 2, \ldots, n \end{cases} \tag{15}$$

where C is investors' risk tolerance. To facilitate model solving, Theorem 3 gives the deterministic form of model (15) when $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$.

Theorem 3. *If the stock returns take normal uncertainty distributions, i.e., $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, $i = 1, 2, \ldots, n$, respectively, model (15) is equivalent to the following form:*

$$\begin{cases} \min \left[\sum_{i=1}^{n} |x_i|\sigma_i \right]^2 \\ \textit{subject to:} \\ \quad \sum_{i=1}^{n} x_i e_i = G \\ \quad \beta\Big(0 - \sum_{i=1}^{n} (x_{Ii} + x_i) e_i\Big) - \dfrac{\sqrt{3}\sum_{i=1}^{n} |x_{Ii} + x_i|\sigma_i}{\pi} \Big(\beta \ln \beta + (1-\beta)\ln(1-\beta)\Big) \le C \\ \quad \beta = \dfrac{\exp\Big(\pi(0 - \sum_{i=1}^{n}(x_{Ii}+x_i)e_i)/\sqrt{3}\sum_{i=1}^{n}|x_{Ii}+x_i|\sigma_i\Big)}{1+\exp\Big(\pi(0 - \sum_{i=1}^{n}(x_{Ii}+x_i)e_i)/\sqrt{3}\sum_{i=1}^{n}|x_{Ii}+x_i|\sigma_i\Big)} \\ \quad \sum_{i=1}^{n} x_i = 0, \; i = 1, 2, \ldots, n \end{cases} \tag{16}$$

Proof. The proof is shown in Appendix B. □

It is difficult to get the analytical solution of model (16), but we can calculate the numerical solution by computer. Moreover, we can easily conclude the optimal solution of UEIT-RI model now depends on portfolio $\boldsymbol{X_I}$, in contrast to the optimal solution of UEIT model. If investors adopt the UEIT-RI policy, they would not all make the same trades and their optimal trades depend on their benchmarks.

3.3. Discussions on the UEIT-RI model

Does the RI constraint work well? To discover it, we will compare the UEIT-RI model with the UEIT model. First, we compare the objective values of these two models. For the minimization problem, under other conditions unchanged, the smaller the constraint set is, the larger the objective function is. So we can easily get that the objective value of UEIT-RI model is bigger than or equal to that of UEIT model. The UEIT model is superior to the UEIT-RI model in terms of objective value, meaning that the UEIT model has a smaller tracking error variance. However, the purpose of adding RI constraint is to control the total risk. So what is the result of comparing the two models in terms of total risk control? Next we compare the risks of UEIT-RI and UEIT portfolios. Since we add the RI-constraint to the UEIT model, it is easy to get that the RI value of the UEIT-RI portfolio is smaller than or equal to that of the UEIT portfolio. So the UEIT-RI portfolio dominates the UEIT portfolio from the perspective of controlling total risk. With the same expected return, the UEIT-RI portfolio has a smaller total risk measured by RI. What if total risk is measured by the variance, the result still holds according to Corollary 1.

Corollary 1. *When ξ_i are normal uncertain variables, the variance of the UEIT-RI portfolio is smaller than or equal to that of the UEIT portfolio.*

Proof. The proof is shown in Appendix B. □

In summary, according to the above comparisons, the UEIT-RI portfolio has a higher tracking error variance than the UEIT portfolio with the same expected return, but it has a lower total risk measured either by variance or RI. So the RI constraint do help to control the total risk of tracking portfolio.

Next, we will discuss how the variance of the UEIT-RI portfolio changes with C. If the value of C decreases, we know that the RI value of the UEIT-RI portfolio also decreases. According to the proof process of Corollary 1, we can get the following corollary easily.

Corollary 2. *When ξ_i are normal uncertain variables, the variance of the UEIT-RI portfolio decreases with C.*

Corollary 2 provides a way to reduce the variance of the UEIT-RI portfolio. Variance and RI are two tools to measure the risk. Variance is a popular one while RI is more direct. Corollary 2 makes a connection between them. Whether the tracking portfolio risk is measured by variance or RI, when C decreases, the total risk of the tracking portfolio becomes smaller.

4. Numerical examples

In order to clearly illustrate the modeling idea, we present some numerical examples.

Table 1
Uncertainty annual returns of 12 stocks in the investor's asset universe in year 2020.

Stock i	Stock code	Distribution	Stock i	Stock code	Distribution
1	600929	$\mathcal{N}(0.056, 0.089)$	7	601698	$\mathcal{N}(0.176, 0.267)$
2	603214	$\mathcal{N}(0.089, 0.120)$	8	600009	$\mathcal{N}(0.187, 0.290)$
3	601990	$\mathcal{N}(0.100, 0.150)$	9	601330	$\mathcal{N}(0.199, 0.299)$
4	600104	$\mathcal{N}(0.110, 0.180)$	10	002371	$\mathcal{N}(0.210, 0.300)$
5	000034	$\mathcal{N}(0.120, 0.210)$	11	600547	$\mathcal{N}(0.230, 0.310)$
6	002032	$\mathcal{N}(0.156, 0.240)$	12	603712	$\mathcal{N}(0.250, 0.340)$

Table 2
The stock weights in the benchmark.

Stock i	Stock code	Weight (%)
4	600104	10
5	000034	10
6	002032	15
8	600009	30
10	002371	15
11	600547	20

4.1. The data

An investor selects twelve stocks from Shanghai Stock Exchange and Shenzhen Stock Exchange as his/her asset universe. And the twelve stock codes are showed in Table 1. Due to the outbreak of COVID-19, historical data in 2020 cannot effectively predict the future. So the twelve stock annual return distributions in 2020 are given based on an expert's estimations instead of pure historical data. With the expert's estimations, the return distributions of the twelve candidate stocks are obtained via the method given in Huang (2012a). The interested readers can refer to it. Table 1 shows the return distributions of the twelve stocks. Regarding the benchmark, it can be a stock index or an existing portfolio. Here the investor adopts an existing portfolio that the expert recommends. The benchmark is shown in Table 2. Note that the stocks in the benchmark are in the investor's asset universe. So the benchmark can be expressed as

$$\boldsymbol{X_I} = (0, 0, 0, 0.1, 0.1, 0.15, 0, 0.3, 0, 0.15, 0.2, 0)^T.$$

The expected return and stand deviation of the benchmark are $e_I = 0.180$ and $\sigma_I = 0.269$ respectively.

4.2. Computational results of the UEIT model

This subsection illustrates how to get the UEIT portfolio. A UEIT portfolio and the UEIT frontier will be given, and the sensitivity analysis w.r.t the benchmark will be conducted.

4.2.1. The UEIT portfolio

The UEIT portfolio can be got by solving model (3). If G is set at 0.02 level, by running Matlab 2016, the optimal solution is

$$\boldsymbol{X}^* = (-0.103, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.103)^T.$$

So the tracking portfolio is

$$\begin{aligned}\boldsymbol{X_P} &= \boldsymbol{X_I} + \boldsymbol{X}^* \\ &= (-0.103, 0, 0, 0.1, 0.1, 0.15, 0, 0.3, 0, 0.15, 0.2, 0.103)^T,\end{aligned}$$

and its expected return is $e_P = 0.2$ and the stand deviation is $\sigma_P = 0.313$.

In Table 3, the first column is the benchmark, the second column is the optimal alteration and the third column is the tracking portfolio. Note that the first entry is -0.103 in alteration $\boldsymbol{X}^*$, which means selling the first stock and the proportion is 0.103. Intuitively, selling out low-yield stocks and buy more high-yield stocks can increase the return definitely, and it is true actually.

4.2.2. The UEIT frontier

In order to show the relationship between risk and return of UEIT portfolios, we depict the UEIT frontier in $\sigma_P^2 - e_P$ space when $G \in [-0.1, 0.1]$, which is shown in Fig. 1. As seen in Fig. 1, if G is positive, portfolio $\boldsymbol{X_P}$ lies above the benchmark $\boldsymbol{X_I}$, and σ_P increases with e_P; If G is negative, portfolio $\boldsymbol{X_P}$ lies down the benchmark $\boldsymbol{X_I}$, and σ_P increases with e_P decrease. When G is negative, $\boldsymbol{X_P}$ is an suboptimal portfolio because there exists a portfolio with the same standard deviation but higher return. But it does not influence the applicability of our model. Usually, we manage our portfolio to get a higher return, not a lower return.

Besides, we compare the UEIT frontier with the EV efficient frontier in $\sigma_P^2 - e_P$ space, which is shown in Fig. 2. The dash curve represents UEIT frontier and solid curve represents EV efficient frontier. In Roll's paper (Roll, 1992), he has pointed out the tracking portfolios may have high risk, and our UEIT portfolio also has this defect. From Fig. 2 we can see that variances of UEIT portfolios are larger than those of EV portfolios, indicating the overly risky problem and the reason why we proposed the UEIT-RI model.

Table 3
Positions of initial portfolio, alteration, and tracking portfolio.

Stock i	Benchmark portfolio $\boldsymbol{X}_{\boldsymbol{I}}$	Alteration $\boldsymbol{X}^*$	Tracking portfolio $\boldsymbol{X}_{\boldsymbol{P}}$
1	0	−0.103	−0.103
2	0	0	0
3	0	0	0
4	0.1	0	0.1
5	0.1	0	0.1
6	0.15	0	0.15
7	0	0	0
8	0.3	0	0.3
9	0	0	0
10	0.15	0	0.15
11	0.2	0	0.2
12	0	0.103	0.103

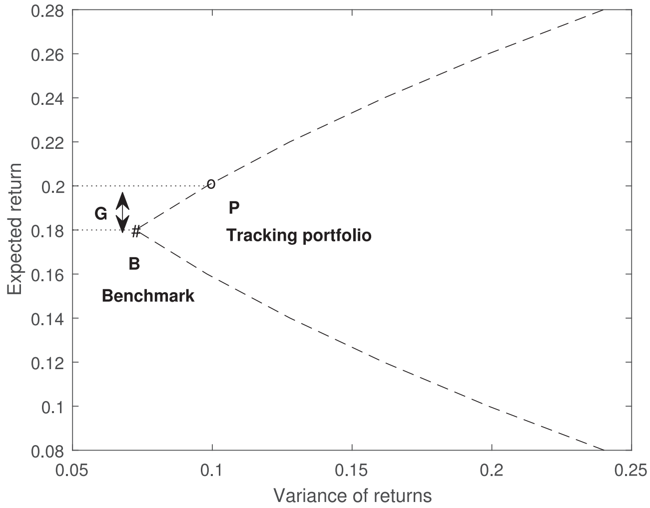

Fig. 1. The UEIT frontier in $\sigma_P^2 - e_P$ space.

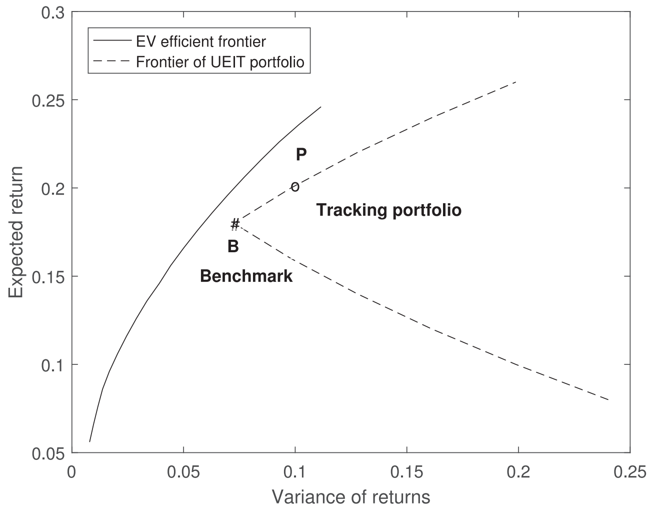

Fig. 2. The UEIT frontier and EV efficient frontier in $\sigma_P^2 - e_P$ space.

4.2.3. Sensitivity analysis w.r.t the benchmark

In order to test the influence of different benchmarks on UEIT portfolios, we adjust the benchmark from $\boldsymbol{X}_{\boldsymbol{I}}$ to $\boldsymbol{X}_{I1}$ and draw the UEIT frontier again, which is shown in Fig. 3. Note that the benchmark $\boldsymbol{X}_{\boldsymbol{I}}$ is not on the EV efficient frontier and the benchmark

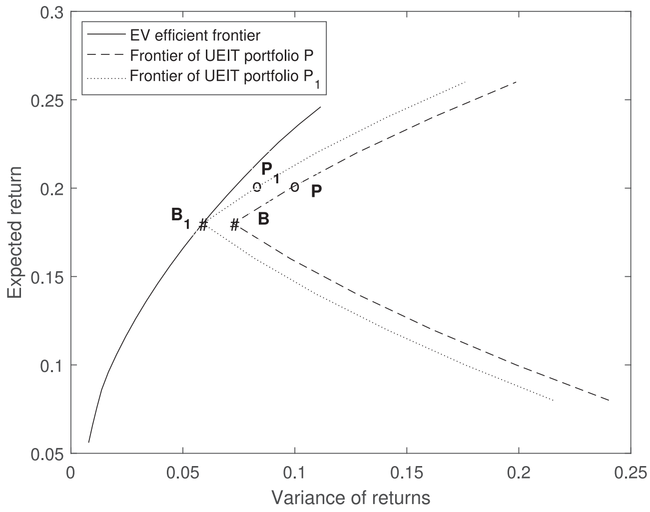

Fig. 3. The UEIT frontiers with two different levels of benchmark variance in $\sigma_P^2 - e_P$ space.

$\boldsymbol{X}_{I1}$ which has the same expected return as $\boldsymbol{X}_{\boldsymbol{I}}$ is on the EV efficient frontier. As seen in Fig. 3, the UEIT frontier moves in the same direction as the benchmark moves. This means that tracking a benchmark with lower variance can reduce the variance of UEIT portfolios. And if the benchmark is on the EV efficient frontier, the UEIT frontier and the EV efficient frontier intersect at the benchmark point.

4.3. Computational results of UEIT-RI model

This subsection illustrates the idea of UEIT-RI model. A UEIT-RI portfolio will be given and the sensitivity analysis will be conducted.

4.3.1. The UEIT-RI portfolio

To ensure that UEIT and UEIT-RI portfolios are comparable, the same parameter settings are used, the benchmark $\boldsymbol{X}_{\boldsymbol{I}} = (0,0,0,0.1,0.1,0.15,0,0.3,0,0.15,0.2,0)^T$, target return $G = 0.02$. Suppose the tolerance of risk is 0.02, i.e., $C = 0.02$. By running Matlab 2016, the optimal solution of UEIT-RI model is

$$\boldsymbol{X}^{RI} = (-0.041,0,0,-0.086,0,0,0,0,0,0,0,0.127)^T.$$

So the tracking portfolio is

$$\begin{aligned}\boldsymbol{X}_{\boldsymbol{P}}^{RI} &= \boldsymbol{X}_{\boldsymbol{I}} + \boldsymbol{X}^{RI}\\ &= (-0.041,0,0,0.014,0.1,0.15,0,0.3,0,0.15,0.2,0.127)^T.\end{aligned}$$

Its expected return is $e_P^{RI} = 0.2$, and its risk index is $RI = 0.02$ and the standard deviation is $\sigma_P^{RI} = 0.3$.

4.3.2. Sensitivity analysis w.r.t the risk tolerance

In order to test the influence of risk tolerance on the decisions-making results of portfolio selection, we adjust the risk tolerable C and conduct experiments again. In Table 4, when $C = 0.015$, the objective value is 0.015 and the standard deviation of portfolio $\boldsymbol{X}_{\boldsymbol{P}}$ is 0.288. When $C = 0.020$, the objective value is 0.004 and the standard deviation of portfolio $\boldsymbol{X}_{\boldsymbol{P}}$ is 0.300; When $C = 0.025$, the objective value is 0.002 and the standard deviation of portfolio $\boldsymbol{X}_{\boldsymbol{P}}$ is 0.313. Evidence shows that (i) objective values decrease as C increases; (ii) risk of portfolio $\boldsymbol{X}_{\boldsymbol{P}}$ increases with C. Besides, one point should be noticed. When $C = 0.005$, the UEIT-RI model has no solution because RI constraint is not met. When $C = 0.03$, the objective value and the standard deviation of portfolio $\boldsymbol{X}_{\boldsymbol{P}}$ are still 0.002 and 0.313 respectively. If C is too small, the UEIT-RI model has no solution; and if C is too big, RI constraint will lose its effect. So investors should choose an appropriate C value.

4.4. Comparison

In this section, two comparisons will be made. First, we compare the UEIT-RI model with the UEIT model to show the effect of RI constraint. Second, we compare the performances of benchmark, UEIT-RI and UEIT portfolios in the in-sample and out-of-sample periods.

Table 4
Optimal tracking portfolios of UEIT-RI model at different tolerance of risk.

C	Obj.	σ_P^{RI}	RI
0.015	0.015	0.288	0.015
0.020	0.004	0.300	0.020
0.025	0.002	0.313	0.025

Note: $G = 0.02$, the benchmark is $\boldsymbol{X}_I$.

Table 5
The objective values of UEIT-RI and UEIT models.

G	Obj. of the UEIT-RI model	Obj. of the UEIT model
0.03	0.004	0.004
0.04	0.011	0.008
0.05	0.020	0.012
0.06	0.035	0.018

Note: $C = 0.03$.

Table 6
The risk of UEIT-RI and UEIT portfolios.

G	The standard deviation of the UEIT-RI portfolio	The standard deviation of the UEIT portfolio
0.03	0.335	0.335
0.04	0.348	0.358
0.05	0.358	0.379
0.06	0.369	0.402

Note: $C = 0.03$.

4.4.1. Comparison of the UEIT-RI model with the UEIT model

We compare models with and without RI constraint. Table 5 provides the objective values of the two models and Table 6 provides the risk of the tracking portfolios determined by the two models.

In Table 5, when $G = 0.03$, the objective values of UEIT-RI and UEIT models are both 0.004. This implies that RI constraint does not work because the risk tolerance is too big relative to the expected return. When $G = 0.04$, the objective values of two models are 0.011 and 0.008; when G gradually increases to 0.06, the two objective values are also increasing, and the objective value of the UEIT-RI model is always greater than that of the UEIT model; when $G = 0.07$, the UEIT-RI model has no solution because RI constraint is not met.

In Table 6, when $G = 0.03$, the standard deviations of UEIT-RI and UEIT portfolios are the same because RI constraint does not work. When G increases from 0.03 to 0.06 gradually, the standard deviations of UEIT-RI portfolios are always smaller than those of UEIT portfolios. When $G = 0.07$, the UEIT-RI model has no solution. Evidence shows that RI constraint helps in controlling the risk.

4.4.2. In-sample performances vs. out-of-sample performances

In this section, we provide the performances of UEIT and UEIT-RI portfolios in in-sample and out-of-sample periods. The in-sample period is from Jan 02, 2020 to Dec 31, 2020 and the out-of-sample period is from Jan 04, 2021 to Aug 11, 2021. Fig. 4 shows the holding period returns (HPRs) of benchmark, UEIT and UEIT-RI portfolios in whole time period. The black dot line represents the HPR of benchmark, the blue dash line represents the HPR of UEIT portfolio and the red solid line represents the HPR of UEIT-RI portfolio. As we seen in Fig. 4, during the in-sample period, the UEIT and UEIT-RI portfolios both track the benchmark closely and provide higher-than-benchmark returns. During the out-of-sample period, the UEIT and UEIT-RI portfolios perform well too. The HPRs of benchmark, UEIT and UEIT-RI portfolios at the end of in-sample period are 18.23%, 20.59% and 21.46% respectively. And the HPRs of benchmark, UEIT and UEIT-RI portfolios at the end of whole time period are 36.04%, 38.12% and 40.45% respectively. Through the above comparison, it is found that our proposed strategies perform well not only during the in-sample period but also during the out-of-sample period.

4.5. Implications

Through the above analysis, we give some implications which we hope will help investors.

(i) When investors want to increase their expect return, they need to sell out low expected return stocks and then buy more high expected return stocks. The higher performance investors pursue, the more low-yield stocks they need to sell and the more high-yield stocks they need to buy. This conclusion is the same as the actual operation of investors we have observed.

(ii) The UEIT and UEIT-RI models provide different criteria for investors to manage their tracking portfolios. UEIT portfolios have a smaller tracking error variance while UEIT-RI portfolios have a smaller total risk. So investors can choose the UEIT model or the UEIT-RI model according to their preference.

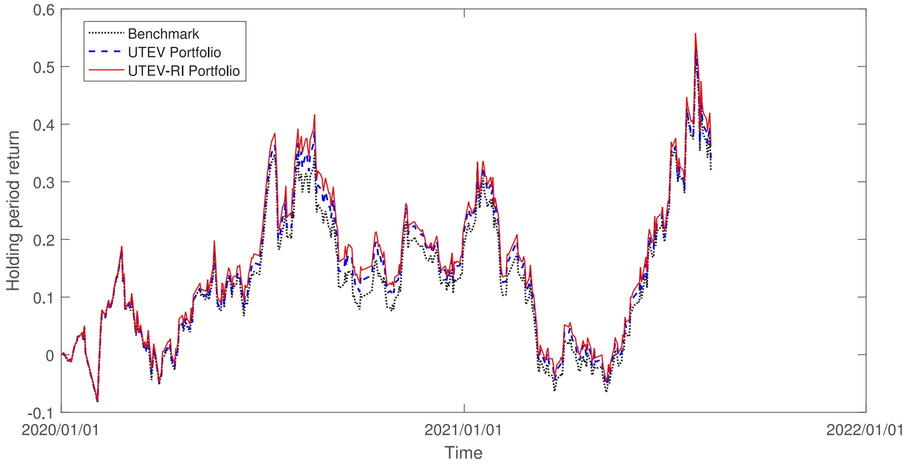

Fig. 4. The holding period returns of benchmark, UEIT and UEIT-RI portfolios in whole time period.

(iii) There are two ways to mitigate the overly risk problem of the tracking portfolio. Tracking a benchmark with smaller variance can decrease the variance of $\boldsymbol{X_P}$. Using the UEIT-RI model can also achieve this goal and the variance of $\boldsymbol{X_P}$ decreases with C. But there is a point to note when applying the UEIT-RI model. If C is too small, the UEIT-RI model has no solution. If C is too large, RI constraint will not work.

5. Conclusion

In real financial market, stock returns are often affected by unexpected events, which can lead to historical data failing to predict the future effectively. In this case, estimating the future based on experts' knowledge and experience is more reasonable than simply applying historical data, so stock returns should be treated as uncertain variables in some situations.

Under uncertainty theory framework, this paper has studied how to manage EIT portfolios when investors have no data or invalid data. We have proposed a UEIT model and have further put forward a UEIT-RI model as the model improvement. Through theoretical and numerical analysis, we have got the following results. (i) The UEIT model can guide investors to manage their portfolios to beat the benchmark on return when they have no data or invalid data. (ii) The UEIT model may be generally overly risky for investors, but the UEIT-RI model can mitigate the inherent flaw. (iii) UEIT portfolios have a smaller tracking error variance, while UEIT-RI portfolios have smaller portfolio risk. So investors need to choose the model according to their preference.

There are many things to do in the future. Considering the impact of transaction costs, in the future research, we will track the index considering a subset of the securities constituting the benchmark.

CRediT authorship contribution statement

Tingting Yang: Conceptualization, Methodology, Software, Data curation, Writing – original draft. **Xiaoxia Huang:** Methodology, Writing – reviewing and editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Let us review some necessary knowledge about the uncertainty theory.

Uncertainty theory is an axiomatic mathematics that models human uncertainty. Uncertain measure, denoted by M, is defined based on four axioms of uncertainty theory (Liu, 2007) and is proved to be monotonous (Liu, 2010). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma,\ \mathrm{L},\ \mathrm{M})$ to the set of real numbers and is characterized by uncertainty distribution.

Normal uncertain variable is the variable that has the following normal uncertainty distribution

$$\Phi(t) = \left(1 + \exp\left(\frac{\pi(e-t)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad t \in \Re,$$

where e and σ are real numbers and $\sigma > 0$. For convenience, it is denoted in the paper by $\xi \sim \mathcal{N}(e,\sigma)$. An uncertainty distribution $\Phi(t)$ is called regular if it is a continuous and strictly increasing function with respect to t at which $0 < \Phi(t) < 1$, and $\lim_{t\to-\infty}\Phi(t) = 0$, $\lim_{t\to+\infty}\Phi(t) = 1$. It is seen that normal uncertainty distribution is regular.

The expected value of an uncertain variable is defined as follows:

Definition 1 (Liu, 2007)**.** Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{\infty} \mathrm{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{M}\{\xi \leq r\}\mathrm{d}r \tag{A.1}$$

provided that at least one of the two integrals is finite.

Theorem 4 (Liu, 2010)**.** *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. If its expected value exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{A.2}$$

The variance of an uncertain variable is defined as follows.

Definition 2 (Liu, 2007)**.** Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E[(\xi - e)^2]. \tag{A.3}$$

Theorem 5 (Yao, 2015)**.** *Let ξ be an uncertain variable with a regular uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha. \tag{A.4}$$

For more expositions on uncertainty theory, the interested readers can consult the book Liu (2010).

Appendix B

Proof of Theorem 1.

First, we give some knowledge about uncertain variables. It has been proved in Huang (2010) that (i) the sum of two independent normal uncertain variables is still a normal uncertain variable, i.e.,

$$\mathcal{N}(e_1,\sigma_1) + \mathcal{N}(e_2,\sigma_2) = \mathcal{N}(e_1+e_2,\sigma_1+\sigma_1).$$

(ii) The product of a normal uncertain variable and a scalar number $k > 0$ is also a normal uncertain variable, and

$$k \cdot \mathcal{N}(e,\sigma) = \mathcal{N}(ke, k\sigma).$$

Then we proof Theorem 1. According to the above knowledge in Huang (2010), we can prove that when ξ_i take normal uncertain distributions, i.e., $\xi_i \sim (e_i,\sigma_i)$, $\sum_{i=1}^n x_i\xi_i$ also takes a normal uncertain distribution. So we get

$$\sum_{i=1}^n x_i\xi_i \sim \mathcal{N}\left(\sum_{i=1}^n x_ie_i, \sum_{i=1}^n |x_i|\sigma_i\right).$$

For a normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$, we can calculated that its expected value $E[\xi] = e$ and variance $V[\xi] = \sigma^2$. So the expected value and variance of $\sum_{i=1}^n x_i\xi_i$ are

$$E\left[\sum_{i=1}^n x_i\xi_i\right] = \sum_{i=1}^n x_ie_i,$$
$$V\left[\sum_{i=1}^n x_i\xi_i\right] = \left[\sum_{i=1}^n |x_i|\sigma_i\right]^2.$$

Therefore Theorem 1 is proved.

Proof of Theorem 3. Let Υ denote the uncertainty distribution of $\sum_{i=1}^n (x_{Ii} + x_i)\xi_i$. According to Proposition 1, we can prove that

$$RI\left[\sum_{i=1}^n (x_{Ii} + x_i)\xi_i\right] = -\int_0^{\beta} \Upsilon^{-1}(\alpha)d\alpha$$

where β is defined by $\Upsilon^{-1}(\beta) = 0$. When $\xi \sim \mathcal{N}(e, \sigma)$, we can prove that

$$RI(\xi) = \beta(0 - e) - \frac{\sqrt{3}\sigma}{\pi}\left(\beta \ln \beta + (1 - \beta)\ln(1 - \beta)\right)$$

where $\beta = \dfrac{\exp\left(\pi(0 - e)/\sqrt{3}\sigma\right)}{1 + \exp\left(\pi(0 - e)/\sqrt{3}\sigma\right)}$. We also know that

$$\sum_{i=1}^{n}(x_{Ii} + x_i)\xi_i \sim \mathcal{N}\left(\sum_{i=1}^{n}(x_{Ii} + x_i)e_i, \sum_{i=1}^{n}|x_{Ii} + x_i|\sigma_i\right).$$

So Theorem 3 is easy to reach.

Proof of Corollary 1. Let RI_1 and RI_2 represent risk index values of UEIT portfolios and UEIT-RI portfolios. We already know that $RI_1 \geq RI_2$. Let Υ_1 and Υ_2 represent the uncertainty distributions of returns of these two portfolios. According to the definition of RI, we have $\beta_1 = \Upsilon_1(0)$ and $\beta_2 = \Upsilon_2(0)$. Let σ_1 and σ_2 represent the standard deviations corresponding to these two portfolios. Since the two portfolios have the same expected return, we use e to represent them.

Assume that $\sigma_1 < \sigma_2$, then we have

$$\left(1 + exp\left(\frac{\pi \times e}{\sqrt{3}\sigma_1}\right)\right)^{-1} < \left(1 + exp\left(\frac{\pi \times e}{\sqrt{3}\sigma_2}\right)\right)^{-1},$$

which means $\Upsilon_1(0) < \Upsilon_2(0)$. So $\beta_1 < \beta_2$. If $\beta \geq 0.5$, the portfolio is too bad for the investor. So $\beta < 0.5$. When $\sigma_1 < \sigma_2$, for any $\alpha \in (0, 0.5)$, we can have

$$\int_0^{\beta_1}\left(0 - (e + \frac{\sqrt{3}\sigma_1}{\pi} ln\frac{\alpha}{1 - \alpha})\right)d\alpha < \int_0^{\beta_2}\left(0 - (e + \frac{\sqrt{3}\sigma_2}{\pi} ln\frac{\alpha}{1 - \alpha})\right)d\alpha.$$

This means $RI_1 < RI_2$, which is contradictory with our result that $RI_1 \geq RI_2$. So $\sigma_1 \geq \sigma_2$. And so $\sigma_1^2 \geq \sigma_2^2$. Therefore Corollary 1 is proved.

References

Alexander, G. J., & Baptista, A. M. (2008). Active portfolio management with benchmarking: adding a value-at-risk constraint. *Journal of Economic Dynamics and Control*, *32*(3), 779–820.

Alexander, G. J., & Baptista, A. M. (2010). Active portfolio management with benchmarking: A frontier based on alpha. *Journal of Banking & Finance*, *34*(9), 2185–2197.

Canakgoz, N. A., & Beasley, J. E. (2009). Mixed-integer programming approaches for index tracking and enhanced indexation. *European Journal of Operational Research*, *196*(1), 384–399.

Caporin, M., & Lisi, F. (2013). A conditional single index model with local covariates for detecting and evaluating active portfolio management. *The North American Journal of Economics and Finance*, *26*, 236–249.

Clarke, R. G., Krase, S., & Statman, M. (1994). Tracking errors, regret, and tactical asset allocation. *The Journal of Portfolio Management*, *20*(3), 16.

Da Silva, A. S., Lee, W., & Pornrojnangkool, B. (2009). The black–litterman model for active portfolio management. *The Journal of Portfolio Management*, *35*(2), 61–70.

Gao, Y. (2011). Shortest path problem with uncertain arc lengths. *Computers and Mathematics with Applications*, *62*(6), 2591–2600.

Gngi, M., & Strub, O. (2020). Tracking and outperforming large stock-market indices. *Omega*, *90*, Article 101999.

Grinold, R. C., & Kahn, R. N. (1995). *Active portfolio management*. Richard D. Irwin, Chicago.

Gruber, M. J. (1996). Another puzzle: The growth in actively managed mutual funds. *The Journal of Finance*, *51*(3), 783–810.

Guastaroba, G., Mansini, R., Ogryczak, W., & Speranza, M. G. (2016). Linear programming models based on omega ratio for the enhanced index tracking problem. *European Journal of Operational Research*, *251*(3), 938–956.

Hodges, S. D. (1976). Problems in the application of portfolio selection models. *Omega*, *4*(6), 699–709.

Huang, X. (2010). *Portfolio analysis: from probabilistic to credibilistic and uncertain approaches*. Berlin: Springer-Verlag.

Huang, X. (2012). Mean-variance models for portfolio selection subject to experts' estimations. *Expert Systems with Applications*, *39*(5), 5887–5893.

Huang, X. (2012). A risk index model for portfolio selection with returns subject to experts' estimations. *Fuzzy Optimization and Decision Making*, *11*(5), 451–463.

Huang, X., & Di, H. (2020). Uncertain portfolio selection with mental accounts. *International Journal of Systems Science*, *51*(12), 2079–2090.

Huang, X., & Jiang, G. (2021). Portfolio management with background risk under uncertain mean-variance utility. *Fuzzy Optimization and Decision Making*, *20*(3), 315–330.

Huang, X., & Yang, T. (2020). How does background risk affect portfolio choice: An analysis based on uncertain mean-variance model with background risk. *Journal of Banking & Finance*, *111*, Article 105726.

Huang, X., & Ying, H. (2013). Risk index based models for portfolio adjusting problem with returns subject to experts' evaluations. *Economic Modelling*, *30*, 61–66.

Jin, X., Chen, N., & Yuan, Y. (2019). Multi-period and tri-objective uncertain portfolio selection model: A behavioral approach. *The North American Journal of Economics and Finance*, *47*, 492–504.

Jorion, P. (2003). Portfolio optimization with tracking-error constraints. *Financial Analysts Journal*, *59*(5), 70–82.

Kang, Z., Li, X., Li, Z., & Zhu, S. (2019). Data-driven robust mean-CVaR portfolio selection under distribution ambiguity. *Quantitative Finance*, *19*(1), 105–121.

Koshizuka, T., Konno, H., & Yamamoto, R. (2009). Index-plus-alpha tracking subject to correlation constraint. *International Journal of Optimization: Theory, Methods and Applications*, *1*(2), 215–224.

Liu, B. (2007). *Uncertainty theory*. Berlin: Springer-Verlag.

Liu, B. (2010). *Uncertainty theory: A branch of mathematics for modeling human uncertainty*. Berlin: Springer-Verlag.

Liu, B. (2012). Why is there a need for uncertainty theory. *Journal of Uncertain Systems*, *6*(1), 3–10.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, *7*(1), 77–91.
Mutunge, P., & Haugland, D. (2018). Minimizing the tracking error of cardinality constrained portfolios. *Computers & Operations Research*, *90*, 33–41.
Paulo, W. L. d., Oliveira, E. M. d., & Valle Costa, O. L. d. (2016). Enhanced index tracking optimal portfolio selection. *Finance Research Letters*, *16*, 93–102.
Pflug, G., & Wozabal, D. (2007). Ambiguity in portfolio selection. *Quantitative Finance*, *7*(4), 435–442.
Roll, R. (1992). A mean/variance analysis of tracking error. *The Journal of Portfolio Management*, *18*(4), 13–22.
Rudolf, M., Wolter, H. J., & Zimmermann, H. (1999). A linear model for tracking error minimization. *Journal of Banking & Finance*, *23*(1), 85–103.
Sant'Anna, L. R., Caldeira, J. a. F., & Filomena, T. P. (2020). Lasso-based index tracking and statistical arbitrage long-short strategies. *The North American Journal of Economics and Finance*, *51*, Article 101055.
Sant'Anna, L. R., Filomena, T. P., & Caldeira, J. a. F. (2017). Index tracking and enhanced indexing using cointegration and correlation with endogenous portfolio selection. *The Quarterly Review of Economics and Finance*, *65*, 146–157.
Tversky, A., & Kahneman, D. (1986). Rational choice and the framing of decisions. *Journal of Business*, *59*, 251–278.
Wang, X., & Huang, X. (2019). A risk index to model uncertain portfolio investment with options. *Economic Modelling*, *80*, 284–293.
Xue, L., Di, H., Zhao, X., & Zhang, Z. (2019). Uncertain portfolio selection with mental accounts and realistic constraints. *Journal of Computational and Applied Mathematics*, *346*, 42–52.
Yao, K. (2015). A formula to calculate the variance of uncertain variable. *Soft Computing*, *19*, 2947–2953.