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A risk-gain dominance maximization approach to enhanced index tracking

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ABSTRACT

Following the research strands of enhanced index tracking and of portfolio performance measures optimization, we propose to choose, among the feasible asset portfolios of a given market, the one that maximizes the geometric mean of the differences between its risk and gain and those of a suitable reference benchmark, such as the market index. This approach, which has a peculiar geometric interpretation and enjoys remarkable features, provides the efficient portfolio that dominates the largest amount of portfolios dominating the reference benchmark index. Preliminary empirical results highlight good out-of-sample performances of our approach compared with those of the market index.

1. Introduction

Portfolio selection is a typical decision problem under uncertainty, where the drivers of uncertainty are the asset returns. Generally, the aim is to choose the fractions of a given capital invested in each of n assets, such that the resulting portfolio return satisfies specific criteria. Starting with the seminal work of Markowitz (1952, 1959), the key problem in asset allocation is to select a portfolio with appropriate features in terms of gain and risk. From a mathematical viewpoint, the synthetic indices that represent gain and risk are modeled by functions of n real variables to be optimized simultaneously: this results in a bi-objective problem. Solving multiobjective programs usually consists in computing the Pareto optimal solution that best suites the decision maker, generally carried out by scalarization. Just with respect to the latter approach, it is hardly possible here to even summarize the huge amount of solution methods that have been proposed in the literature (see Miettinen, 2012 for both theoretical bases and review of the related literature).

Differently from expected utility maximization approaches (see Blay and Markowitz, 2013, the recent Carleo et al., 2017; Markowitz, 2014 and the references therein), in this paper, following the research strands of enhanced index tracking (see, e.g, Bruni et al., 2012; 2013; 2015; 2017; Canakgoz and Beasley, 2009; Coleman et al., 2006; Corielli and Marcellino, 2006; de Paulo et al., 2016; Focardi and Fabozzi, 2004; Guastaroba and Speranza, 2012; Okay and Akman, 2003) and of portfolio performance measures optimization (see, e.g., Righi and Borenstein, 2017; Stoyanov et al., 2007), we propose a *novel* portfolio selection approach that tries to combine the advantages of the gain-risk analysis with the benefits of the enhanced index tracking strategy, which aims to obtain returns above a reference index, possibly selecting a reduced number of assets. More precisely, our approach consists in finding a Pareto optimal portfolio that maximizes, as performance measure, the (weighted) geometric mean of the differences between its risk

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and gain and those of a suitable benchmark index (see Audet et al., 2008; Lootsma et al., 1995). This results in a no-preference strategy (i.e., a method that provides a Pareto optimal portfolio not depending on the preferences of the decision maker, see Miettinen, 2012) that requires the solution of a nonlinear nonconvex (constrained) single-objective problem.

This method presents distinctive features. First, the Pareto optimal portoflio provided by our method has the significant property that any other feasible portfolio, for which one of the two objectives is "improved" (e.g., entailing a larger gain or a smaller risk than the ones obtained by our strategy) *by a factor*, is such that the corresponding value of the other objective must be "worse" than the one guaranteed by our approach *by at least the same factor*. Thus, an investor who wants to choose a feasible portfolio improving on the one given by our approach w.r.t. risk or gain, must be willing to accept a worsening of the other objective *at least as large*.

Interestingly and similarly to what happens with gain-risk ratio models, also our approach has a nice geometric interpretation, to be described shortly. Somehow related to this feature, in terms of utility, our strategy has the characteristic of leading to a portoflio dominating the largest amount of portfolios that dominate, in turn, the reference index. Hence, in a sense, our approach determines the portfolio that in the risk-gain plane dominates the most a given benchmark index.

Finally, the underlying optimization problem, which is not sensitive to the scaling of the risk and gain measures, can be solved very efficiently in practice. Indeed, even though globally solving a nonconvex program is generally difficult, any (nontrivial) stationary point of the problem at hand turns out to be globally optimal. Therefore, we are able to efficiently calculate a global (Pareto) optimum by means of standard (gradient-like) techniques. We stress that, for all the performed computational tests, the amount of time needed to reach a solution is less than a second.

Furthermore, based on real-world financial datasets, the empirical testing of our method, from both in-sample and out-of-sample perspectives, confirms the theoretical insights and highlights the good practical performance of our proposal.

2. The dominance maximization approach

Let us start by considering the general portfolio selection problem for the gain-risk analysis:

$$\begin{array}{ll} \max\limits_{x} & \gamma_P(x) \\ \min\limits_{x} & \rho_P(x) \\ \text{s.t.} & x \in \Delta, \end{array} \tag{1}$$

where $\gamma_P(x)$: $\mathbb{R}^n \to \mathbb{R}$ is any continuous concave measure of gain and $\rho_P(x)$: $\mathbb{R}^n \to \mathbb{R}$ is any continuous convex measure of risk. Furthermore, we denote the set of feasible portfolios by $\Delta \triangleq \{x \in \mathbb{R}^n : \ x \geq l, \ e^T x = 1\}$, where $e \in \mathbb{R}^n$ is a vector of ones and $l \in \mathbb{R}^n$ is a vector of lower bounds. Then, Δ is a nonempty, convex and compact feasible set. Note that Δ incorporates the budget constraint $e^T x = 1$ and generic lower bounds on the portfolio weights. Therefore, Δ could contain long-only portfolios by setting $l = 0$ (i.e., $0 \leq x_i \leq 1$), but, when $l < 0$, it could also include long-short portfolios, i.e., portfolios that can have both positive and negative weights that sum to 1.

Definition 2.1 (*Global Pareto optimal portfolios*). A feasible portfolio $\bar{x}$ to the bi-objective problem 1 is said to be global Pareto optimal (or *efficient*) if there not exists any feasible portfolio x such that either $\gamma_P(x) \geq \gamma_P(\bar{x})$ and $\rho_P(x) < \rho_P(\bar{x})$, or $\gamma_P(x) > \gamma_P(\bar{x})$ and $\rho_P(x) \leq \rho_P(\bar{x})$.

In other words, a feasible solution to (1) is Pareto optimal if there is no other feasible solution that is equally good w.r.t. each objective function, and better than at least one objective. In what follows we present a strategy to find a Pareto optimum that enjoys significant features.

Let $(\rho_P^{\text{ref}}, \gamma_P^{\text{ref}}) \in \mathbb{R}^2$ be a suitable reference point in the risk-gain plane, corresponding to a given benchmark index. Just to give some examples, one can consider as a reference the so-called nadir point, where ρ_P^{ref} is the risk of the portfolio with maximal gain and γ_P^{ref} is the gain of the minimum risk portfolio; or, in an enhanced indexation perspective, the market index, where ρ_P^{ref} and γ_P^{ref} are the market index risk and gain, respectively. Obviously, any portfolio $x \in \Delta$, for which $\gamma_P(x) \geq \gamma_P^{\text{ref}}$ and $\rho_P(x) \leq \rho_P^{\text{ref}}$, is such that $(\rho_P^{\text{ref}} - \rho_P(x)) \geq 0$ and $(\gamma_P(x) - \gamma_P^{\text{ref}}) \geq 0$.

In the light of the above considerations, we select portfolios by maximizing, as performance measure, the weighted geometric mean of the distances between their values of risk and gain and those of the benchmark index, i.e., $(\gamma_P(x) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x))$ (see Lootsma et al., 1995). Thus, the gain-risk dominance model can be formulated as follows:

$$\begin{array}{ll} \max\limits_{x} & (\gamma_P(x) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x)) \\ \text{s.t.} & \gamma_P(x) \geq \gamma_P^{\text{ref}} \\ & \rho_P(x) \leq \rho_P^{\text{ref}} \\ & x \in \Delta. \end{array} \tag{2}$$

We denote the set of feasible portfolios in (2) by $X \triangleq \{x \in \Delta : \ \gamma_P(x) \geq \gamma_P^{\text{ref}}, \ \rho_P(x) \leq \rho_P^{\text{ref}}\}$, and the nonnegative objective function (on X) by $A(x) \triangleq (\gamma_P(x) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x))$. Clearly, in view of the constraints in 2, γ_P^{ref} and ρ_P^{ref} are the worst values for gain and risk that a feasible portfolio can entail.

Since in the risk-gain plane any feasible portfolio $x \in X$ identifies a rectangle R_x with base $(\rho_P^{\text{ref}} - \rho_P(x)) \geq 0$ and height $(\gamma_P(x) - \gamma_P^{\text{ref}}) \geq 0$, $A(x)$ can also be interpreted geometrically as the area of such a rectangle (see Fig. 1). We also remark that,

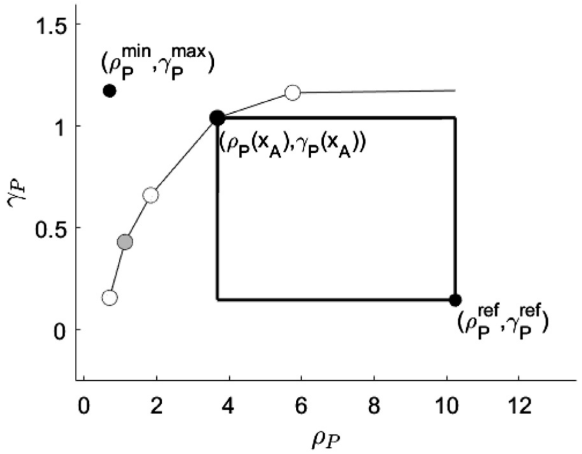

Fig. 1. Efficient frontier for the NASDAQComp dataset. The portfolio entailed by our approach, using the nadir as reference, is marked in black, while that obtained by the Sharpe ratio maximization in grey. The Markowitz portfolios for 3 different target return levels are marked in white.

obviously, x dominates all the portfolios whose objective values belong to the rectangle. Our approach, maximizing the area of the rectangle R_x, provides a portfolio that dominates "the most" the benchmark index in the risk-gain plane, and it is also a Pareto optimal solution of problem (1). Of course, just like enhanced index tracking (see, e.g, Bruni et al., 2015) and gain-risk ratios strategies (see, e.g, Stoyanov et al., 2007), our method is sensitive to the specific choice for the reference index.

Let, from now on, x_A be a global optimal point for problem (2). In a nutshell, choosing among portfolios by addressing (2) has the following prominent advantages and distinctive features:

(i) x_A is a global Pareto efficient portfolio for (1) (see Propositions 2.2 and 2.3);
(ii) if an investor decides to choose a portfolio different from x_A that has an improved risk or gain w.r.t. x_A *by a given factor*, then the price to pay is the corresponding worsening of the other objective *by at least the same factor*. Thus, an investor who wants to choose any other feasible portfolio improving on x_A w.r.t. risk or gain, must be willing to accept a worsening of the other objective *at least as large* (see Proposition 2.4);
(iii) in terms of utility, the portfolio x_A, maximizing the area of the rectangle R_x, dominates the largest amount of portfolios that dominate the reference index. Therefore, in a sense, x_A is the portfolio that dominates the most the benchmark (see Fig. 1 for an illustration of this feature);
(iv) finding a solution of (2) does not depend on the objectives' scales, thus allowing to "robustly" deal with non homogeneous objectives, such as risk and gain (see Remark 2.5);
(v) even though problem (2) is a nonlinear nonconvex optimization problem, x_A turns out to be easily and efficiently computable.

With the following proposition, which is reminiscent of a similar result in Audet et al. (2008), we prove the claim in (i) establishing the link between problem (2) and the original bi-objective optimization problem (1). Essentially, Proposition 2.2 states that one can choose among the risk-gain *efficient* portfolios the one that maximizes A by simply addressing the single-objective problem (2).

Proposition 2.2. *Any global solution x_A for problem (2) such that $A(x_A) > 0$ is a global Pareto optimum for problem (1).*

Proof. Suppose by contradiction that x_A (maximizing the area $A(x) = (\gamma_P(x) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x))$) is not Pareto optimal for (1): hence, there exists $\hat{x} \in X$ such that, without loss of generality, $\gamma_P(\hat{x}) > \gamma_P(x_A)$, $\rho_P(\hat{x}) \leq \rho_P(x_A)$ (see Definition 2.1). Thus, we have that $\gamma_P(\hat{x}) - \gamma_P^{\text{ref}} > \gamma_P(x_A) - \gamma_P^{\text{ref}} > 0$, $\rho_P^{\text{ref}} - \rho_P(\hat{x}) \geq \rho_P^{\text{ref}} - \rho_P(x_A) > 0$ and, therefore, $A(\hat{x}) > A(x_A)$, which contradicts the optimality of x_A for problem (2). □

We remark that the assumption $A(x_A) > 0$ in Proposition 2.2 is not demanding since the optimal value of problem (2) reaches zero only in pathological cases. Specifically, in Proposition 2.3 we show that, if condition $A(x_A) > 0$ is not verified, then at least an objective coincides with its *reference* value for every feasible point, and, thus, can be dropped.

Proposition 2.3. *Let x_A be any global solution for problem (2). Then, $A(x_A) = 0$ if and only if either $\gamma_P(x) = \gamma_P^{\text{ref}}$ for all $x \in X$ or $\rho_P(x) = \rho_P^{\text{ref}}$ for all $x \in X$.*

Proof. We only show the necessity. Let $A(x_A) = (\gamma_P(x_A) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x_A)) = 0$ and suppose by contradiction that $\hat{x}, \check{x} \in X$ exist such that $\gamma_P(\hat{x}) > \gamma_P^{\text{ref}}$ and $\rho_P(\check{x}) < \rho_P^{\text{ref}}$. Clearly, we have $\rho_P(\hat{x}) = \rho_P^{\text{ref}}$ and $\gamma_P(\check{x}) = \gamma_P^{\text{ref}}$ because $A(x) = 0$ for every $x \in X$, due to $A(x_A) = 0$. Setting $\bar{x} = \frac{1}{2}\hat{x} + \frac{1}{2}\check{x}$, we obtain, by the concavity of γ_P, $\gamma_P(\bar{x}) \geq \frac{1}{2}\gamma_P(\hat{x}) + \frac{1}{2}\gamma_P^{\text{ref}} > \gamma_P^{\text{ref}}$ and, thanks to the convexity of ρ_P, $\rho_P(\bar{x}) \leq \frac{1}{2}\rho_P^{\text{ref}} + \frac{1}{2}\rho_P(\check{x}) < \rho_P^{\text{ref}}$. Therefore, $A(\bar{x}) = (\gamma_P(\bar{x}) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(\bar{x})) > 0$, in contradiction with the fact that the optimal value

$A(x_A)$ is zero. □

From now on, in view of the above results, we refer to a generic global solution of problem (2) as a Pareto-optimal portfolio x_A of (1).

The following Proposition 2.4 justifies the claim in (ii). Specifically, an investor may prefer to choose a feasible portfolio $\tilde{x}$ that entails an improvement in terms of either risk or gain w.r.t. their values at x_A. Then, Proposition 2.4 allows one to estimate *how much* worse (w.r.t. its value at x_A) the value of the remaining objective would be. Interestingly, by switching from x_A to a different feasible portfolio $\tilde{x}$, the improvement obtained in terms of an objective is always smaller than the worsening achieved in terms of the other objective.

Proposition 2.4. *An efficient portfolio x_A enjoys the following properties:*

(a) *if a feasible portfolio $\tilde{x} \in X$ is such that $\gamma_P(\tilde{x}) - \gamma_P^{\text{ref}} = \alpha(\gamma_P(x_A) - \gamma_P^{\text{ref}})$ for some $\alpha \geq 1$, then $\alpha(\rho_P^{\text{ref}} - \rho_P(\tilde{x})) \leq \rho_P^{\text{ref}} - \rho_P(x_A)$;*
(b) *if a feasible portfolio $\tilde{x} \in X$ is such that $\rho_P^{\text{ref}} - \rho_P(\tilde{x}) = \beta(\rho_P^{\text{ref}} - \rho_P(x_A))$ for some $\beta \geq 1$, then $\beta(\gamma_P(\tilde{x}) - \gamma_P^{\text{ref}}) \leq \gamma_P(x_A) - \gamma_P^{\text{ref}}$.*

Proof. The claims are due to the following relations:

$$\alpha = \frac{\gamma_P(\tilde{x}) - \gamma_P^{\text{ref}}}{\gamma_P(x_A) - \gamma_P^{\text{ref}}} \leq \frac{\rho_P^{\text{ref}} - \rho_P(x_A)}{\rho_P^{\text{ref}} - \rho_P(\tilde{x})} = \frac{1}{\beta},$$

where the inequality holds since $A(x_A) \geq A(x)$ for every $x \in X$. □

As previously mentioned, due to the constraints in 2, γ_P^{ref} and ρ_P^{ref} are the worst values of gain and risk that a feasible portfolio can entail. Hence, the larger $\gamma_P(x) - \gamma_P^{\text{ref}}$ and $\rho_P^{\text{ref}} - \rho_P(x)$, the better. In the light of this consideration, the above result shows that if one wants to improve risk or gain (w.r.t. their values at x_A) by a factor, then the price to pay is a corresponding worsening of the other objective (w.r.t. its value at x_A) *by at least the same factor*. To be more explicit, let us assume that the difference in gain between a feasible portfolio $\tilde{x}$ and the benchmark is twice as big as the difference in gain between x_A and the benchmark. Then, relative to the benchmark, the risk entailed by $\tilde{x}$ is twice as worse than the risk obtained choosing x_A. For a practical illustration of this feature see the column '**impr** wors' in Table 4 and the related description.

As detailed in Remark 2.5, another notheworthy feature of the approach is that finding a solution of (2) does not depend on the objectives' scales. Hence, just like gain-risk ratio maximization approaches, our method seems particularly fit to handle non homogeneous quantities, such as risk and gain.

Remark 2.5. If the decision maker prefers to scale, for example, the risk measure ρ_P by a positive factor ζ, then the reference value becomes $\zeta\rho_P^{\text{ref}}$ and, $\arg\max\{(\gamma_P(x) - \gamma_P^{\text{ref}})(\rho_P^{\text{ref}} - \rho_P(x)) : x \in X\} = \arg\max\{(\gamma_P(x) - \gamma_P^{\text{ref}})\zeta(\rho_P^{\text{ref}} - \rho_P(x)) : x \in X\}$, for every $\zeta > 0$. □

Some comments are in order. Like many other methods (think, e.g., about the *tangency* portfolio) depend on the specific gain and risk measures that have been used (see, e.g., Stoyanov et al., 2007), also our strategy generally selects different portfolios for different gain and risk measures. This aspect may be interpreted as an evidence of the flexibility of the approach. It is also worth noticing that if we introduce a risk-free asset in our analysis, all the above considerations and the properties of the approach remain valid. As further remark, we also observe that, given a standard reference point such as the nadir, x_A cannot consist in investing the whole budget on the risk-free asset.

While the simple ideas underlying our approach seem appealing, the question about how one can solve problem (2) still remains open. Indeed, even with a concave γ_P and a convex ρ_P, the objective function in (2) is nonconvex in general. In fact, it is only semistrictly quasiconvex, see Theorem 5.15 in Avriel et al. (2010). However, if we assume that γ_P and ρ_P are continuously differentiable on an open set containing $\{x \in X : A(x) > 0\}$, then we can exploit Theorem 5.17(a) in Avriel et al. (2010), which shows that any stationary point $\bar{x} \in X$ for which $A(\bar{x}) \neq 0$ turns out to be a global optimum for (2). Therefore, thanks to Proposition 2.2, $\bar{x}$ is also a Pareto optimal portfolio that maximizes $A(x)$. Now, we are left to resort to a procedure that allows one to efficiently compute a stationary point for problem (2) with a corresponding positive value of the area $A(x)$. For this we can rely on standard nonlinear programming techniques such as the projected gradient algorithm with a sufficiently small constant stepsize, having the forethought to choose a starting point $x^0 \in X$ such that $A(x^0) > 0$. We recall that the projected gradient algorithm is a sequential convex approximation method (for further details cf. Facchinei et al., 2014; 2015; 2017; Scutari et al., 2014; 2017a; 2017b) whose core step consists in the calculation of the projection of a gradient iteration on the feasible set. For further technical details regarding algorithmic aspects, the reader can refer to Cesarone et al. (2018).

3. Empirical and computational results

In this section, we test the risk-gain dominance maximization method on some datasets provided in Bruni et al. (2016). Data consist in weekly returns time series of assets belonging to several major stock markets (see Table 1).

For the practical testing of our approach, we employ the portfolio expected return and volatility as gain and risk measures, respectively. We use linear returns and assume that n assets are available in an investment universe. We denote by $\gamma_P(x) = 100\mu^T x$ the mean of the portfolio return (in percentage) and by $\rho_P(x) = 100\sqrt{x^T \Sigma x}$ the portfolio volatility (in percentage), where μ is the vector whose components μ_i are the expected returns of n assets, and Σ is their covariance matrix, whose generic element σ_{ij} is the covariance of the returns of asset i and asset j with $i, j = 1, \ldots, n$.

Table 1
Datasets synthetic description.

Index	#assets	#weeks	From-To
DowJones	28	1363	2/1990 - 4/2016
NASDAQ100	82	596	11/2004 - 4/2016
FTSE100	83	717	7/2002 - 4/2016
SP500	442	595	11/2004 - 4/2016
NASDAQComp	1203	685	2/2003 - 4/2016

We first perform an in-sample analysis by comparing the properties of the portfolios obtained by our approach with those of the *tangency* and of the Markowitz portfolios (Section 3.1). Then, in the context of Enhanced Indexation, using a rolling time window scheme of evaluation, we examine the out-of-sample performance of the portfolios maximizing the weighted geometric mean of the distances between their values of risk and gain and those of the market index (Section 3.2).

All the experiments have been implemented in Matlab R2017b and executed on an Intel Core i7-4702MQ CPU@ 2.20GHz × 8 with Ubuntu 14.04 LTS 64-bit.

3.1. In-sample analysis

To study the properties of the portoflios obtained by our approach and to compare it with other classical methods, we consider problem (2) without short sales and as reference point the nadir vector, namely $\rho_P^{\mathbf{ref}}$ is the risk (in percentage) of the portfolio with maximal gain and $\gamma_P^{\mathbf{ref}}$ is the gain (in percentage) of the minimum risk portfolio. We also denote the ideal values (in percentage) of risk and gain by $\rho_P^{\min} \triangleq 100\min\{\sqrt{x^T\Sigma x} : x \in X\}$ and $\gamma_P^{\max} \triangleq 100\max_{i\in\{1,\dots,n\}}\{\mu_i\}$, respectively (to support intuition, see Fig. 1). All these data are collected in Table 2.

For each market we compare the portfolio obtained by our approach (denoted by 'area max') with other Pareto optimal portfolios in the risk-gain plane (see Fig. 1). More precisely, we consider

• the portfolio obtained by maximizing the Sharpe ratio (denoted by 'Sharpe')

$$\begin{aligned} \max_x \quad & \frac{\mu^T x - r_f}{\sqrt{x^T\Sigma x}} \\ \text{s.t.} \quad & x \in \Delta, \end{aligned} \tag{3}$$

where we set $r_f = 0$ (see, e.g., Cornuejols and Tütüncü, 2006, pp. 155–158);

• three *efficient* portfolios obtained by the classical Mean-Variance (MV) approach

$$\begin{aligned} \min_x \quad & x^T\Sigma x \\ \text{s.t.} \quad & \mu^T x \geq \eta \\ & x \in \Delta. \end{aligned} \tag{4}$$

where η is a fixed level of the portfolio expected return (Markowitz, 1959). More precisely, we solve problem (4) for the following three different levels of target returns: $100\eta = \alpha(\gamma_P^{\max} - \gamma_P^{\min}) + \gamma_P^{\min}$ with $\alpha = 0.01$, 0.5 and 0.99 (corresponding to 'low', 'medium', and 'high' risk strategies, respectively, as also shown in Table 4).

When addressing the nonconvex problems (2) and (3), we resort to a projected gradient-like algorithm, employing as stopping criterion the distance (in infinity norm) between two successive iterates, which is required to be smaller than 1e-5. The quadratic problem (4) is solved by using `quadprog` with `interior-point-convex` algorithm. In Table 3, referring to problem (2), we report the number of iterations **#iter** and the elapsed time (in seconds) **time** needed to satisfy the stopping criterion. We stress that, for each stock market considered, the amount of time to solve problem (2) is less than a second. The preliminary results of the in-sample analysis are summarized in Table 4.

For each method we report: the computed gain $\gamma_P \triangleq 100\mu^T\tilde{x}$, risk $\rho_P \triangleq 100\sqrt{\tilde{x}^T\Sigma\tilde{x}}$, and area $A(\tilde{x}) \triangleq (\gamma_P(\tilde{x}) - \gamma_P^{\mathbf{ref}})(\rho_P^{\mathbf{ref}} - \rho_P(\tilde{x}))$, and the number **#assets$_P$** of assets that have been selected, where $\tilde{x}$ is the optimal portfolio obtained from each model. Furthermore, we also provide the normalized distance of the computed values of risk and gain from the ideal ones ($\rho_P^{\min}$, $\gamma_P^{\max}$), namely

Table 2
Ideal and nadir values for each considered dataset.

Index	$\gamma_P^{\mathbf{ref}}$	$\gamma_P^{\max}$	$\rho_P^{\min}$	$\rho_P^{\mathbf{ref}}$
DowJones	0.214	0.605	2.000	5.891
NASDAQ100	0.242	1.030	1.975	8.219
FTSE100	0.254	0.802	1.727	8.056
SP500	0.190	1.032	1.471	8.226
NASDAQComp	0.148	1.174	0.698	10.240

Table 3
Details regarding the numerical solution of problem (2).

Index	$\gamma_P(x^0)$	$\rho_P(x^0)$	$A(x^0)$	#iter	time
DowJones	0.587	5.277	0.229	416	0.015
NASDAQ100	0.833	5.648	1.519	233	0.016
FTSE100	0.564	4.483	1.108	353	0.019
SP500	0.946	6.389	1.389	164	0.143
NASDAQComp	1.162	6.785	3.501	99	0.645

Table 4
In-sample performances.

	Method	γ_P	ρ_P	A	#assets$_P$	$\|\beta\|_2$	impr/wors γ_P	wors/impr ρ_P
DowJones	area max	0.523	3.439	0.758	8	0.425	1.000	1.000
	Sharpe	0.436	2.816	0.684	11	0.481	1.392	**1.254**
	MV: 'low' risk	0.218	2.000	0.015	15	0.990	77.25	**1.587**
	'medium' risk	0.410	2.651	0.634	11	0.526	1.576	**1.321**
	'high' risk	0.602	5.044	0.329	2	0.782	**1.256**	2.895
NASDAQ100	area max	0.880	3.872	2.772	8	0.358	1.000	1.000
	Sharpe	0.724	3.071	2.479	14	0.427	1.324	**1.184**
	MV: 'low' risk	0.250	1.976	0.049	11	0.990	79.75	**1.436**
	'medium' risk	0.636	2.733	2.160	13	0.515	1.619	**1.262**
	'high' risk	1.022	7.601	0.482	2	0.901	**1.222**	7.034
FTSE100	area max	0.654	3.624	1.774	4	0.404	1.000	1.000
	Sharpe	0.456	2.159	1.193	12	0.636	1.980	**1.330**
	MV: 'low' risk	0.259	1.727	0.035	23	0.991	80.00	**1.428**
	'medium' risk	0.528	2.579	1.501	9	0.518	1.460	**1.236**
	'high' risk	0.796	7.719	0.183	2	0.947	**1.355**	13.15
SP500	area max	0.866	3.759	3.020	9	0.392	1.000	1.000
	Sharpe	0.568	2.258	2.256	17	0.563	1.788	**1.336**
	MV: 'low' risk	0.198	1.473	0.057	25	0.990	84.50	**1.512**
	'medium' risk	0.611	2.439	2.439	16	0.520	1.606	**1.295**
	'high' risk	1.024	7.553	0.562	2	0.900	**1.234**	6.637
NASDAQComp	area max	1.041	3.672	5.861	23	0.338	1.000	1.000
	Sharpe	0.432	1.124	2.588	83	0.724	3.144	**1.388**
	MV: 'low' risk	0.159	0.698	0.098	79	0.989	81.18	**1.453**
	'medium' risk	0.661	1.834	4.312	67	0.514	1.929	**1.280**
	'high' risk	1.164	5.750	4.561	3	0.529	**1.138**	1.463

$$\|\boldsymbol{\beta}\|_2 \triangleq \sqrt{\left(\frac{\gamma_P^{\max} - \gamma_P(\tilde{x})}{\gamma_P^{\max} - \gamma_P^{\mathrm{ref}}}\right)^2 + \left(\frac{\rho_P(\tilde{x}) - \rho_P^{\min}}{\rho_P^{\mathrm{ref}} - \rho_P^{\min}}\right)^2}.$$

Clearly, the smaller the distance $\|\boldsymbol{\beta}\|_2$, the more risk and gain of the selected portfolio tend to those of the ideal portfolio. Finally, in columns '**impr** wors', we report the numerical evidences for the peculiar feature underlined in Proposition 2.4. Specifically, we distinguish two cases: if the portfolio $\tilde{x}$ entails an improvement w.r.t. x_A in terms of γ_P, then we indicate in the table the quantities

$$\textbf{impr}\ \gamma_P = \frac{\boldsymbol{\gamma_P(\tilde{x}) - \gamma_P^{\mathrm{ref}}}}{\boldsymbol{\gamma_P(x_A) - \gamma_P^{\mathrm{ref}}}} \leq \frac{\rho_P^{\mathrm{ref}} - \rho_P(x_A)}{\rho_P^{\mathrm{ref}} - \rho_P(\tilde{x})} = \text{wors}\ \rho_P.$$

Whereas, if the improvement is obtained in terms of ρ_P, then we indicate in the table the quantities

$$\text{wors}\ \gamma_P = \frac{\gamma_P(x_A) - \gamma_P^{\mathrm{ref}}}{\gamma_P(\tilde{x}) - \gamma_P^{\mathrm{ref}}} \geq \frac{\boldsymbol{\rho_P^{\mathrm{ref}} - \rho_P(\tilde{x})}}{\boldsymbol{\rho_P^{\mathrm{ref}} - \rho_P(x_A)}} = \textbf{impr}\ \rho_P.$$

The computational tests confirm the theoretical insights of previous sections (see Table 4). In particular, our approach actually provides a solution that maximizes the corresponding area. Also, by moving from x_A to another computed $\tilde{x}$, the worsening in terms of an objective is always greater than the corresponding improvement in terms of the other objective. Employing A as performance measure, while enjoying the distinctive features highlighted in Section 2, also gives values of risk and gain that turn out to be close (in Euclidean norm) to the ideal ones.

3.2. Out-of-sample analysis: Enhanced indexation

For the out-of-sample analysis, we adopt a rolling time window approach: we allow for the possibility of rebalancing the portfolio

Table 5
Out-of-sample (annualized) performances.

	Method	Mean	Std	Sharpe ratio	Info ratio	#assets
Dow Jones	area max	**11.492**	**15.085**	**0.762**	0.137	11.5
	MI	8.476	16.939	0.500		–
NASDAQ100	area max	**13.104**	**15.893**	**0.824**	0.036	14.2
	MI	12.636	22.037	0.573		–
FTSE100	area max	**12.844**	**15.251**	**0.842**	0.339	16.6
	MI	4.368	18.150	0.241		–
SP500	area max	**8.060**	**13.369**	**0.603**	0.072	21.8
	MI	6.344	20.227	0.314		–

composition during the holding period, at fixed intervals. We set a period of 100 weeks (around 2 years) for the in-sample window and of 4 weeks (around 1 month) for the out-of-sample window, with rebalancing allowed every 4 weeks. We examine the out-of-sample performance of the portfolios obtained by solving the optimization problem 2 without short sales and using the Market Index (MI) as reference point: $\rho_P^{\mathbf{ref}}$ is the MI volatility (in percentage) and $\gamma_P^{\mathbf{ref}}$ is the MI expected return (in percentage). For these preliminary results, we compare the performance of the portfolios obtained by our approach with that of the MI in terms of annualized mean (**Mean**), volatility (**Std**) and Sharpe Ratio (**Sharpe ratio**) (Sharpe, 1966; 1994) of the out-of-sample portfolio returns and those of the benchmark index. We also compute the Information Ratio (**Info ratio**), that is the ratio of the expected value to the standard deviation of the difference between the out-of-sample portfolio returns and those of the benchmark index (Goodwin, 1998). Finally, we report the average number of stocks selected by our approach (**#assets**).

In Table 5, we provide the out-of-sample performance results for each dataset, with the best results marked in bold. In these preliminary tests, we do not consider the NASDAQComp dataset; in fact, we observed that, due to the large number (1203) of assets as opposed to the relatively short training period (100 weeks), more sophisticated numerical procedures are required to compute, for each training window, suitable starting points. We leave this issue to future research.

We observe that for all the performance measures considered, there is a clear dominance of the portfolio obtained by our approach w.r.t. the MI.

4. Conclusions and further research

We propose a *novel* approach to select a risk-gain efficient portfolio that dominates the largest amount of portfolios dominating a reference benchmark such as the market index. Although our method requires the solution of a nonlinear nonconvex optimization problem, we show how to efficiently compute an optimal portfolio in practice. Our empirical analysis on real-world financial datasets witnesses that our approach, even when specialized to Enhanced Indexation, is also attractive from a practical viewpoint.

Our future research is focused on a more thorough analysis of the out-of-sample performance of our strategy with respect to different asset allocation models. Furthermore, from a theoretical viewpoint, we wish to compare our method with other classical techniques aimed at optimizing performance measures such as the gain-risk ratios (cf., e.g., Sharpe, STARR and Rachev ratios). In addition, referring to the context of Nash equilibrium problems (see, e.g., Aussel and Sagratella, 2017; Dreves et al., 2011; Facchinei and Lampariello, 2011; Facchinei and Sagratella, 2011; Sagratella, 2016; 2017a; 2017b; Scutari et al., 2012), we envisage that the peculiar nature of our approach makes it fit particularly well into the noncooperative scenario of multi-portfolio selection.

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