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Enhanced index tracking optimal portfolio selection

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ABSTRACT

In this paper we present an analytical solution for an uni-period enhanced index tracking problem with limited number of assets held in the tracking portfolio. We consider an approach in which the tracking portfolio is composed of a given subset of assets, and the value function is written as the trade-off between the tracking error and excess return, balanced by an appropriate choice of a risk aversion parameter. This formulation allows an analytical comparison of the betas and value functions of the optimal portfolios with and without tracking. Our approach provides readily implementable formulae, being consequently easier for numerical implementation.

1. Introduction

Since the classical mean-variance model proposed by Markowitz (Markowitz, 1952), the asset allocation problem has been widely studied in the finance literature. More recently the portfolio optimization problem has been extended to cover several different situations such as robust portfolio optimization, multi-period mean-variance models and portfolio selection problems in the presence of regime switching (see, for example, Costa and Paiva, 2002; Li and Ng, 2000; Kim et al., 2013 and Bae et al., 2014).

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Another type of portfolio optimization problem consists of establishing an optimal allocation so that the portfolio's return replicates the return of a market index (benchmark). This problem is the so-called index tracking problem, where the utility tracking error function is based on the difference between the portfolio's return and the benchmark's return (problems of this nature are addressed in Roll (1992); Alexander and Baptista (2010) and Bajeux-Besnainou et al. (2011), for example). In the same spirit, the problem known as enhanced index tracking (or enhanced indexation) aims at obtaining returns above the reference index (excess return), while minimizing the tracking error. This kind of problem is studied in Wu et al. (2007); Canakgoz and Beasley (2008); Li et al. (2011) and Guastaroba and Speranza (2012).

In general, the asset allocation problem aims at minimizing the portfolio variance under specified restrictions such as limitations on the weights of the assets, liquidity constraints and limitation on the number of assets included in the portfolio. Because of its practical relevance, the portfolio optimization model with limited number of assets has been intensively studied in the literature (see, for instance, Cesarone et al., 2013 and Fastrich et al., 2014). Particularly, this kind of problem is inherent in the passive portfolio management (tracking problem) where the goal is to find a portfolio with a small number of assets to track a chosen benchmark. There are many papers in the literature dealing with the problem of limiting the number of assets in the index tracking and enhanced index tracking problems (see Beasley et al., 2003; Canakgoz and Beasley, 2008 and Guastaroba and Speranza, 2012, for example). Usually it is imposed cardinality constraints to restrict the number of assets, so that it is possible to construct a tracking portfolio selecting an optimal subset of assets that are present in the benchmark index.

Another approach, as used in Yao et al. (2006) and Edirisinghe (2013), is to consider the tracking portfolio for a given subset of assets belonging to the market index. In Yao et al. (2006) this kind of problem is considered under a continuous-time infinite horizon discounted stochastic linear quadratic control problem framework while in Edirisinghe (2013) the author focus on the uni-period problem of minimizing the variance of the tracking error subject to a specified mean tracking error return, and provides an analytical solution for the problem as well as a comparison with the classical Markowitz mean-variance model.

Differently from Yao et al. (2006), which considered an infinite horizon continuous-time horizon discounted stochastic linear quadratic control problem, in this paper we focus on obtaining an analytical solution for an uni-period enhanced index tracking problem with limited number of assets held in the tracking portfolio. When compared with Edirisinghe (2013) we should point out that the goals of the papers are different since Edirisinghe (2013) aims at minimizing the variance of the tracking error subject to a specified mean tracking error return while the present paper aims at minimizing an utility tracking error function based on the difference between the portfolio's return and the benchmark's return. As it will be further explained in Section 2.1 the advantage of this formulation is that the trade-off between the tracking error and excess return can be balanced by an appropriate choice of a risk aversion parameter. Moreover, this kind of objective function naturally arises when considering the maximization of the expected value of a negative exponential utility function for the normal case. Furthermore, our formulation also allows, similarly as in Edirisinghe (2013), an analytical comparison between the optimal portfolios with and without tracking. Finally it is worth noting that our approach provides closed and readily implementable formulae, being consequently easier for numerical implementation than the semidefinite programming computational tool used in Yao et al. (2006).

As aforementioned, we compare the solution obtained for the enhanced index tracking problem with the one derived without the tracking requirement in the objective function and we find that (i) the tracking optimal portfolio beta increases with respect to the beta of the optimal portfolio without tracking; (ii) the value function of the problem without tracking for the tracking optimal portfolio increases with respect to the one for the optimal portfolio without tracking; (iii) the value function of the tracking problem for the tracking optimal portfolio decreases with respect to the one for the optimal portfolio without tracking. Furthermore, we can see that the performance of the considered enhanced indexation problem strongly depends on the set of assets selected as the tracking portfolio.

The remainder of the paper is organized as follows. Section 2 includes the formulation to the enhanced index tracking problem with limited number of assets held in the tracking portfolio, an analytical formula for the optimal tracking portfolio composition and a comparison between the tracking optimal portfolio and the optimal portfolio without tracking. In Section 3 a numerical example for an enhanced indexation portfolio using a few stocks is presented. Finally, Section 4 presents some final remarks.

2. Enhanced index tracking problem

This section presents the formulation to the enhanced index tracking problem with limited number of assets (Section 2.1), as well as an analytical formula for the optimal tracking portfolio composition and the conditions for its optimality (Section 2.2). Furthermore, the solution obtained for the enhanced index tracking problem is compared with the one derived without the tracking requirement in the objective function (Section 2.3).

2.1. Problem formulation

In general the enhanced index tracking problem (or enhanced indexation) aims at replicating a market index (or benchmark portfolio) as well as outperform the index by generating excess return (return above the return of the index), without purchasing all of the assets that make up the index. We refer to the set of assets that we choose to track a benchmark index as a tracking portfolio.

First of all consider a tracking portfolio with return P composed of n assets with returns $R_1, \dots, R_n$ and covariance matrix $\Sigma > 0$. We denote by ω_i the weight invested in the asset with return R_i, so that $P = \sum_{i=1}^{n} \omega_i R_i$ and $\sum_{i=1}^{n} \omega_i = 1$. Note that, as usual, in order to get an analytical solution, negative values for ω_i (short positions) are allowed. Consider a benchmark portfolio ω_B composed of n assets, $\omega_B = (\omega_{B1} \dots \omega_{Bn})'$, with return P_B and expected return μ_B. Let P_e be the error between the return obtained from the tracking portfolio and the return obtained from the benchmark portfolio, so that

$$P_e = P - P_B = (\omega - \omega_B)' R, \tag{1}$$

with $\omega = (\ \omega_1 \dots \quad \omega_n \ )'$ and $R = (\ R_1 \dots \quad R_n \ )'$. The main two measures of interest to formulate an enhanced indexation problem are excess return and tracking error. We define the excess return and the tracking error as the expected value and the variance of P_e, respectively, so that

$$\mu_e = E(R_e) = (\omega - \omega_B)' E(R), \tag{2}$$

$$\sigma_e^2 = E((R_e - \mu_e)^2) = (\omega - \omega_B)' \Sigma (\omega - \omega_B). \tag{3}$$

As considered above the tracking portfolio ω consists of all assets belonging to the benchmark portfolio. We introduce now the restriction on the number of assets, so that the tracking portfolio is composed of $p < n$ assets previously specified. Assuming without loss of generality that the first p of the n assets have been selected as the tracking portfolio, we denote by $w = (\ \omega_1 \dots \omega_p \ )'$ the vector with the weights of the assets that compose the tracking portfolio, by $\mathcal{R} = (\ R_1 \dots R_p \ )'$ the random vector with the returns of the assets, by $\Gamma = E((\mathcal{R} - r)(\mathcal{R} - r)') > 0$ the covariance matrix of $\mathcal{R}$, with $r = E(\mathcal{R})$. Furthermore, we define the $p \times n$ covariance matrix by $\tilde{\Sigma} = E((\mathcal{R} - r)(R - E(R))')$. Using the above notation we have from (1) that $P_e = w'\mathcal{R} - \omega_B' R$ and $\mu_e = E(P_e) = w'r - \mu_B$. Then, rewriting (2) and (3), the excess return and the tracking error considering limited number of assets are given by

$$\mu_e = w'r - \mu_B, \tag{4}$$

$$\sigma_e^2 = w'\Gamma w - 2w'\tilde{\Sigma}\omega_B + \sigma_B^2, \tag{5}$$

where σ_B^2 represents the variance of the benchmark portfolio.

Beasley et al. (2003) and Dose and Cincotti (2005) consider a single generalized objective function for index tracking problem that represents an implicit trade-off between tracking error and excess return. Based on this approach we consider an objective function given by $J = \varrho\sigma_e^2 - \xi\mu_e$, where ϱ and ξ are positive real numbers. Considering (4) and (5), the enhanced index tracking optimization problem with limited number of assets is written in the form:

$$\begin{aligned} &\text{Minimize } \ \varrho\left(w'\Gamma w - 2w'\tilde{\Sigma}\omega_B\right) - \xi\left(w'r - \mu_B\right) \\ &\text{subject to} \\ &\qquad w'e = 1, \\ &\qquad w \in \mathbb{R}^p, \end{aligned} \tag{6}$$

where e is a vector of ones of suitable dimension.

Note that the trade-off between the tracking error and excess return is balanced by the weights $\varrho > 0$ and $\xi \geq 0$. As we are going to see in (12), the optimal solution for the problem (6) depends only on the ratio ξ/ϱ. Thus, by an appropriate choice of this ratio a manager could define one of the three investment strategies: (i) achieve an average return higher than a reference index (active management), (ii) track a reference index with a positive excess return (enhanced index tracking) or (iii) replicate the return of a reference index (index tracking). We highlight that there is no specific value for the ratio ξ/ϱ related to the each type of investment strategy but, comparatively speaking, we should have for strategy (i) a higher value for the ratio (giving more weight to the excess return), for strategy (ii) an intermediate value (giving more weight to the tracking error), and for strategy (iii) a smaller value close or equal to zero (giving almost all or all the weight to the tracking error).

Another motivation for analysing problem (6) is when considering the maximization of the expected value of a negative exponential utility function for the normal case. Indeed, suppose that the return R is a normal random vector. Then we have that P_e is also a normal random variable, with mean μ_e and variance σ_e^2. Consider the negative exponential utility function $U(P_e) = -e^{-\eta P_e}$, where $\eta > 0$. The excess of return optimization problem for this case would be:

$$\text{Maximize } E(U(P_e)) \quad \text{subject to } w'e = 1, \; w \in \mathbb{R}^p. \tag{7}$$

Since $U(P_e)$ is a lognormal random variable, it follows that $E(U(P_e)) = -e^{-\eta(\mu_e - \frac{1}{2}\eta\sigma_e^2)}$ and, by monotonicity, problem (7) is equivalent to maximizing the function $\mu_e - \frac{1}{2}\eta\sigma_e^2$ subject to $w'e = 1$ which, in turn, is equivalent to solving problem (6) for $\varrho = \frac{\eta}{2}$ and $\xi = 1$.

Usually the research related to the enhanced index tracking problem impose cardinality constraints to restrict the number of assets (see for instance Beasley et al., 2003; Canakgoz and Beasley, 2008; Guastaroba and Speranza, 2012). Note that in the problem (6) the tracking portfolio consists of a given subset of assets, so that an investor has to define beforehand the number of assets that compose the tracking portfolio. In other words, this approach does not allow to select an optimal subset of p assets (with $p < n$) that are present in the benchmark portfolio (similar approach for index tracking problem can be seen in Yao et al. (2006) and Edirisinghe (2013)).

2.2. Optimality composition

This subsection presents the analytical solution for the enhanced index tracking problem with limited number of assets presented in Section 2.1, using the method of Lagrange multipliers. Consider the following Lagrangian of the problem (6)

$$\mathcal{L}(w, \lambda) = \varrho\left(w'\Gamma w - 2w'\widetilde{\Sigma}\omega_B\right) - \xi\left(w'r - \mu_B\right) + \lambda(w'e - 1), \tag{8}$$

where λ is the Lagrange multiplier.

Taking the derivative and equalling to zero, and denoting by $w*$ the vector w that satisfies these conditions, we get from the first-order conditions that

$$\begin{cases} \varrho\left(2\Gamma w^* - 2\widetilde{\Sigma}\omega_B\right) - \xi r + \lambda e = 0 \\ e'w^* = 1 \end{cases}. \tag{9}$$

It follows that

$$w^* = \Gamma^{-1}\left(\widetilde{\Sigma}\omega_B + \frac{\xi r}{2\varrho} - \frac{\lambda e}{2\varrho}\right), \tag{10}$$

and therefore, from (9) and (10) we have $e'\Gamma^{-1}\left(\widetilde{\Sigma}\omega_B + \frac{\xi r}{2\varrho} - \frac{\lambda e}{2\varrho}\right) = 1$, so that the Lagrange multiplier λ is given by

$$\lambda = -\frac{2\varrho}{\alpha}\mathcal{T}, \tag{11}$$

with $\mathcal{T} = 1 - e'\Gamma^{-1}\widetilde{\Sigma}\omega_B - \xi\frac{e'\Gamma^{-1}r}{2\varrho}$ and $\alpha = e'\Gamma^{-1}e$. Replacing (11) into (10), we have that the optimal tracking portfolio composition is given by

$$w^* = \Gamma^{-1}\left(\widetilde{\Sigma}\omega_B + \frac{\xi}{2\varrho}r + \frac{\mathcal{T}}{\alpha}e\right). \tag{12}$$

Applying the second-order conditions, we get that the second derivative of the Lagrangian (8) with respect to w, denoted by $L(w)$, is $L(w) = \Gamma > 0$ and, thus, the solution found is indeed a local minimum. Since the objective function is convex, and the feasible set is a convex set, the local minimum is in fact the global minimum, showing the optimality of (12).

Remark 1. For the case in which ω_B is not available we can rewrite the error in equation (1) as $R_e = w'\mathcal{R} - R_M$, where R_M is the return of the market index. Then, the tracking error in (5) is given by $\sigma_e^2 = w'\Gamma w - 2\sigma_M^2 w'\beta + \sigma_M^2$, where β is the p-vector of asset betas and σ_M^2 is the variance of the market index. Following the same steps applied above, we can see that the optimal solution for the problem (6) is given by (12) replacing $\widetilde{\Sigma}\omega_B$ by $\sigma_M^2\beta$.

Remark 2. If the assets in the portfolio are the same as in the benchmark, that is, $p = n$ and $\mathcal{R} = R$, we have that $\Gamma = \widetilde{\Sigma}$ and therefore for this case the solution in (12) reduces to $w^* = \omega_B + (\frac{\xi}{2\varrho})\Gamma^{-1}r - (\frac{\xi}{2\varrho\alpha}e'\Gamma^{-1}r)\Gamma^{-1}e$. It is easy to see that as the tracking risk aversion parameter ϱ goes to infinity, the optimal solution $w*$ tends to ω_B, which is a possible solution since $\mathcal{R} = R$, and has no tracking error risk.

2.3. Comparison with the optimal portfolio

The goal of this subsection is to compare the solution obtained in Section 2.2 for the enhanced index tracking problem with the one derived without the tracking requirement in the objective function (optimal problem), that is, the one obtained from the following problem:

$$\text{Minimize } \varrho\left(w'\Gamma w\right) - \xi\left(w'r\right) \quad \text{subject to } w'e = 1, \;\; w \in \mathbb{R}^p. \tag{13}$$

Following the same procedure as in Section 2.2 we get that the optimal solution for the problem (13) is given by

$$\widetilde{w} = \Gamma^{-1}\left(\frac{\xi}{2\varrho}r + \frac{\widetilde{\mathcal{T}}}{\alpha}e\right), \tag{14}$$

with $\widetilde{\mathcal{T}} = 1 - \xi\frac{e'\Gamma^{-1}r}{2\varrho}$. Comparing with (12) and from Remark 1, we get that

$$w^* = \widetilde{w} + \sigma_M^2\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right). \tag{15}$$

Set $\beta^* = \beta'w^*$, $\widetilde{\beta} = \beta'\widetilde{w}$ (the betas for the portfolios $w*$ and $\widetilde{w}$ respectively), $\mathrm{J}^* = \varrho\left(w^{*\prime}\Gamma w^* - 2\sigma_M^2 w^{*\prime}\beta\right) - \xi\left(w^{*\prime}r - \mu_B\right)$, $\widetilde{\mathrm{J}} = \varrho\left(\widetilde{w}'\Gamma\widetilde{w} - 2\sigma_M^2\widetilde{w}'\beta\right) - \xi\left(\widetilde{w}'r - \mu_B\right)$ (the value functions of problem (6) for the portfolios $w*$ and $\widetilde{w}$ respectively, using the notation as in Remark 1), and $\mathrm{H}^* = \varrho\left(w^{*\prime}\Gamma w^*\right) - \xi\left(w^{*\prime}r\right)$, $\widetilde{\mathrm{H}} = \varrho\left(\widetilde{w}'\Gamma\widetilde{w}\right) - \xi\left(\widetilde{w}'r\right)$ (the value functions of problem (13) for the portfolios $w*$ and $\widetilde{w}$ respectively). In what follows it will be convenient to define

$$\mathrm{C} = \left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)'\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right). \tag{16}$$

Notice that $\mathrm{C} \geq 0$ since $\Gamma > 0$ (a positive definite matrix). Moreover, recalling that $\alpha = e'\Gamma^{-1}e$ we get from (16) that

$$\mathrm{C} = \beta'\Gamma^{-1}\beta - \frac{(e'\Gamma^{-1}\beta)^2}{\alpha}. \tag{17}$$

We have the following result, similar to Proposition 3.2 in Edirisinghe (2013) for the traditional mean-variance optimal portfolio, but written in a more direct way in terms of the parameters of the model and for the mean variance trade off objective functions considered in problems (6) and (13).

Proposition 1. *We have that: (a) The tracking optimal portfolio beta increases $\sigma_M^2\mathrm{C}$ with respect to the optimal portfolio beta, that is, $\beta^* - \widetilde{\beta} = \sigma_M^2\mathrm{C}$; (b) The value function of the problem (13) for the tracking optimal portfolio increases $\varrho\sigma_M^4\mathrm{C}$ with respect to the one for optimal portfolio, that is, $\mathrm{H}^* - \widetilde{\mathrm{H}} = \varrho\sigma_M^4\mathrm{C}$; (c) The value function of the problem (6) for the tracking optimal portfolio decreases $\varrho\sigma_M^4\mathrm{C}$ with respect to the one for optimal portfolio, that is, $\mathrm{J}^* - \widetilde{\mathrm{J}} = -\varrho\sigma_M^4\mathrm{C}$.*

Proof. For (a) we have from (15) and (17) that $\beta^* = \beta' w^* = \beta'\widetilde{w} + \sigma_M^2\left(\beta'\Gamma^{-1}\beta - \frac{(e'\Gamma^{-1}\beta)^2}{\alpha}\right) = \widetilde{\beta} + \sigma_M^2 \mathrm{C}$, showing the desired result. For (b), from (15) we have that

$$\begin{aligned} w^{*\prime}\Gamma w^* &= \left\{\widetilde{w} + \sigma_M^2\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)\right\}'\Gamma\left\{\widetilde{w} + \sigma_M^2\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)\right\} \\ &= \widetilde{w}'\Gamma\widetilde{w} + 2\sigma_M^2\widetilde{w}'\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right) + \sigma_M^4\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)'\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right). \end{aligned} \tag{18}$$

Since $\widetilde{w}'e = 1$ and $\widetilde{w} - \frac{\Gamma^{-1}e}{\alpha} = \frac{\xi}{2\varrho}\Gamma^{-1}\left(r - \frac{e'\Gamma^{-1}r}{\alpha}e\right)$, we get that

$$\widetilde{w}'\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right) = \frac{\xi}{2\varrho}\beta'\Gamma^{-1}\left(r - \frac{e'\Gamma^{-1}r}{\alpha}e\right). \tag{19}$$

Similarly from (15), we have that

$$r'w^* = r'\left\{\widetilde{w} + \sigma_M^2\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)\right\} = r'\widetilde{w} + \sigma_M^2\beta'\Gamma^{-1}\left\{r - \frac{e'\Gamma^{-1}r}{\alpha}e\right\}. \tag{20}$$

Combining (18), (19), and (20) we conclude that $\mathrm{H}^* - \widetilde{\mathrm{H}} = \left(\varrho w^{*\prime}\Gamma w^* - \xi r'w^*\right) - \left(\varrho\widetilde{w}'\Gamma\widetilde{w} - \xi r'\widetilde{w}\right) = \varrho\sigma_M^4\mathrm{C}$, showing the desired result. For (c) we have from (b) that $\mathrm{J}^* - \widetilde{\mathrm{J}} = \left(\varrho(w^{*\prime}\Gamma w^* - 2\sigma_M^2 w^{*\prime}\beta) - \xi r'w^*\right) - \left(\varrho(\widetilde{w}'\Gamma\widetilde{w} - 2\sigma_M^2\widetilde{w}'\beta) - \xi r'\widetilde{w}\right) = \mathrm{H}^* - \widetilde{\mathrm{H}} - 2\varrho\sigma_M^2(w^* - \widetilde{w})'\beta = \varrho\sigma_M^4\mathrm{C} - 2\varrho\sigma_M^2(w^* - \widetilde{w})'\beta$. But from (15), $w^* - \widetilde{w} = \sigma_M^2\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right)$. Therefore, from (17), $(w^* - \widetilde{w})'\beta = \beta'(w^* - \widetilde{w}) = \sigma_M^2\beta'\Gamma^{-1}\left(\beta - \frac{e'\Gamma^{-1}\beta}{\alpha}e\right) = \sigma_M^2\left(\beta'\Gamma^{-1}\beta - \frac{(e'\Gamma^{-1}\beta)^2}{\alpha}\right) = \sigma_M^2\mathrm{C}$. Combining the results we conclude that $\mathrm{J}^* - \widetilde{\mathrm{J}} = \varrho\sigma_M^4\mathrm{C} - 2\varrho\sigma_M^4\mathrm{C} = -\varrho\sigma_M^4\mathrm{C}$ completing the proof. □

Remark 3. Notice that C does not depend on the parameters ϱ and ξ. From Proposition 1 and considering ϱ fixed we notice that the values $\beta^* - \widetilde{\beta} = \sigma_M^2\mathrm{C}$, $\mathrm{J}^* - \widetilde{\mathrm{J}} = -\varrho\sigma_M^4\mathrm{C}$ and $\mathrm{H}^* - \widetilde{\mathrm{H}} = \varrho\sigma_M^4\mathrm{C}$ hold for all values of $\xi > 0$.

3. Numerical example

The purpose of this section is to verify the results presented in Proposition 1 and to compare the performance of the optimal portfolio with tracking ($w*$) and without tracking ($\widetilde{w}$) using the methodology presented in Section 2. We consider an enhanced index tracking strategy whose tracking portfolio is constructed using the stocks negotiated in the Brazilian stock exchange. We have chosen as benchmark index the called Bovespa index, since it is the most representative index of the Brazilian stock market (its composition can be seen on http://www.bmfbovespa.com.br/indices/ResumoCarteiraTeorica.aspx?Indice=Ibovespa&idioma=en-us).

It is important to note that the approach considered in this paper assumes a given set of assets (belonging to the market index) as a tracking portfolio. That is, the number of assets (as well as the kind of assets) is not defined using an optimal procedure. Thus, a manager can select a set of assets based on his own experience and expectation about some aspect of the market (such as liquidity, credit risk, sector, weight of the assets in the index or any other criteria). We consider a tracking portfolio composed of nine stocks belonging to the materials sector (chosen arbitrarily) included in the Bovespa index (considering the composition on 07/20/2015), as presented in Table 1.

Taking a historical series of monthly prices in the period from December 2009 to April 2015 (a sample monthly return with size $T = 64$) the variance of the Bovespa index is $\sigma_M^2 = 0,301\%$ (with $\mu_B = -0,31\%$)

Table 1
Composition of the tracking portfolio selected from the Bovespa index (on 07/20/2015). The columns "Code", "Name" and "Weight" show the ticker symbol, the trading name and the weights of each stock in the index, respectively.

N	Code	Name	Weight (%)
1	VALE5	VALE	3.178
2	GGBR4	GERDAU	0.633
3	VALE3	VALE	2.553
4	USIM5	USIMINAS	0.223
5	CSNA3	SID NACIONAL	0.366
6	FIBR3	FIBRIA	1.031
7	GOAU4	GERDAU MET	0.137
8	SUZB5	SUZANO PAPEL	0.748
9	BRKM5	BRASKEM	0.396

and the parameters Γ, r and β are given by

$$\Gamma = \begin{pmatrix} 0.0058 & 0.0031 & 0.0060 & 0.0044 & 0.0076 & 0.0011 & 0.0034 & 0.0015 & 0.0033 \\ 0.0031 & 0.0067 & 0.0032 & 0.0062 & 0.0063 & 0.0033 & 0.0065 & 0.0029 & 0.0033 \\ 0.0060 & 0.0032 & 0.0064 & 0.0048 & 0.0081 & 0.0013 & 0.0034 & 0.0019 & 0.0036 \\ 0.0044 & 0.0062 & 0.0048 & 0.0193 & 0.0137 & 0.0051 & 0.0066 & 0.0082 & 0.0053 \\ 0.0076 & 0.0063 & 0.0081 & 0.0137 & 0.0182 & 0.0038 & 0.0065 & 0.0063 & 0.0052 \\ 0.0011 & 0.0033 & 0.0013 & 0.0051 & 0.0038 & 0.0088 & 0.0033 & 0.0075 & 0.0027 \\ 0.0034 & 0.0065 & 0.0034 & 0.0066 & 0.0065 & 0.0033 & 0.0070 & 0.0031 & 0.0037 \\ 0.0015 & 0.0029 & 0.0019 & 0.0082 & 0.0063 & 0.0075 & 0.0031 & 0.0120 & 0.0023 \\ 0.0033 & 0.0033 & 0.0036 & 0.0053 & 0.0052 & 0.0027 & 0.0037 & 0.0023 & 0.0122 \end{pmatrix},$$

$$r = \begin{pmatrix} -0.0083 & -0.0152 & -0.0077 & -0.0215 & -0.0138 & 0.0016 & -0.0182 & 0.0004 & 0.0009 \end{pmatrix}',$$

$$\beta = \begin{pmatrix} 0.8223 & 0.9198 & 0.8661 & 1.5794 & 1.5445 & 0.6133 & 0.9555 & 0.7409 & 0.8905 \end{pmatrix}'.$$

Using the notation as in Remark 1, for an enhanced index tracking strategy with $\varrho = 0.8$ and $\xi = 0.15$, it follows that

$$w^* = \begin{pmatrix} 0.816 & 0.718 & -0.278 & 0.013 & -0.092 & 0.174 & -0.729 & 0.170 & 0.208 \end{pmatrix}',$$
$$\widetilde{w} = \begin{pmatrix} 1.401 & 0.875 & -0.662 & -0.091 & -0.268 & 0.214 & -0.935 & 0.270 & 0.195 \end{pmatrix}'.$$

Thus, we have that $\beta^* - \widetilde{\beta} = 0.2516 = \sigma_M^2 \mathrm{C}$ (e.g. the tracking optimal portfolio beta increases with respect to the optimal portfolio beta), $\mathrm{H}^* - \widetilde{\mathrm{H}} = 0.00061 = \varrho\sigma_M^4 \mathrm{C}$ (e.g. the value function of the optimal problem for the tracking optimal portfolio increases with respect to the one for optimal portfolio) and $\mathrm{J}^* - \widetilde{\mathrm{J}} = -0.00061 = -\varrho\sigma_M^4 \mathrm{C}$ (e.g. the value function of tracking problem for the tracking optimal portfolio decreases with respect to the one for optimal portfolio), confirming the results presented in Proposition 1.

In the following, we compare the performance of the optimal portfolio with and without tracking over the time period [0; 64]. Let $X^*(t)$ and $\widetilde{X}(t)$ be respectively the value of the tracking portfolio related to the optimal composition $w*$ and the value of the portfolio without tracking related to the optimal composition $\widetilde{w}$, as calculated above. Then, the values for both optimal portfolios with and without tracking are written as

$$X^*(t+1) = \left(1 + \mathcal{R}'(t) w^*\right) X^*(t), \tag{21}$$

$$\widetilde{X}(t+1) = \left(1 + \mathcal{R}'(t) \widetilde{w}\right) \widetilde{X}(t), \tag{22}$$

Table 2
Tracking portfolios whose composition is defined based on arbitrary selection criteria. The column "ξ" shows the weight assigned to the excess return term of the model (6) that maximizes the positive excess return rate (column "PER") for each tracking portfolio.

Portfolio	Selection criteria	ξ	PER (%)
P1	the $p = 9$ stocks belonging to the materials sector	0.2013	67
P2	the random subset of $p = 10$ stocks that does not belong to the tracking portfolio P1	0.0343	97
P3	the random subset of $p = 7$ stocks present in the Bovespa index	0.0555	80
P4	the subset of $p = 20$ stocks with higher beta in the Bovespa index	0.0011	75

where $\mathcal{R}(t) = (R_1(t)\dots R_p(t))'$, with $t = 0, \dots, T-1$, and $X^*(0) = \widetilde{X}(0) = X_0$. Similarly, let $Y(t)$ be the value of the reference portfolio (associated with the market index) so that

$$Y(t+1) = \big(1 + R_M(t)\big)Y(t), \tag{23}$$

where $R_M(t)$ is the return of the market index and $Y(0) = Y_0$.

Taking $X_0 = 100$ and $Y_0 = 100$, Fig. 1 shows the cumulative monthly portfolio values obtained from equations (21), (22) and (23). Since the goal is to track the market index with positive excess return, we observe that (on average) the enhanced indexation portfolio ($w*$) performs better than the optimal portfolio ($\widetilde{w}$). This is supported by the fact that optimal tracking portfolio $w*$ has a positive excess return rate (PER) of 61% and the optimal portfolio $\widetilde{w}$ has a positive excess return rate of 78%, over the time period [0, 64], while their root mean square are given by $RMS_{w^*} = 0.12$ and $RMS_{\widetilde{w}} = 0.28$. Using the ratio $IR = PER/RMS$, we can note that $IR_{w^*} = 5.55 > IR_{\widetilde{w}} = 2.82$, showing that the portfolio $w*$ performs better than the portfolio $\widetilde{w}$ (comparatively speaking).

As the value of the tracking portfolio tracks the market index more closely than the value of the portfolio without tracking (optimal portfolio), the correlation between the tracking portfolio and market index ($\rho^* = 0.63$) is higher than the correlation between the optimal portfolio and market index ($\widetilde{\rho} = 0.39$) so that the tracking portfolio beta (β^*) is higher than the optimal portfolio beta ($\widetilde{\beta}$). Consequently, the systematic risk of the tracking portfolio is higher than the systematic risk of the optimal portfolio as already seen above ($\beta^* > \widetilde{\beta}$). In fact, as seen in Proposition 1 and Remark 3, we have that $\beta^* \geq \widetilde{\beta}$ for all values of $\xi > 0$, since $C \geq 0$, so that the systematic risk of the tracking portfolio will be always higher than the systematic risk of the portfolio without tracking.

Finally, it is important to note that the performance of a tracking portfolio using the solution (12) can be improved by the balance between the weights ϱ and ξ, but it strongly depends on the set of p assets selected as the tracking portfolio. For example, we can see that the minimum RMS of the tracking portfolio $w*$ is 0.09 (with $\xi \to 0$), so that it is not possible to track the market index more closely. On the other hand, if we select a tracking portfolio composed of all stocks that belong to the market index ($p = n$) we are able to track the index as closely as we want. To emphasize the impact of the choice of a set of assets on the performance of a tracking portfolio, we present in the Table 2 the positive excess return rates (column "PER") for four tracking portfolios chosen from arbitrary selection criteria (column "Selection criteria"). In this case, taking $\varrho = 0.8$ for all tracking portfolios, we determined the weight ξ that maximizes the positive excess return rate so that $RMS = 0.16$ (e.g. all the tracking portfolios have the same level of tracking error with maximum positive excess return rate).

The main result that we can draw from Table 2 is that the selection criteria has significant impact on the performance of a tracking portfolio (as we can see from column "PER"). This is an important factor that we have to take into account when applying the model (6). Note that in this particular study the tracking portfolio P2 presents better performance compared to the other portfolios (note that $IR_2 > IR_3 > IR_4 > IR_1$), showing that increasing the number of assets in the tracking portfolio does not lead to an increase in the performance (see for example the tracking portfolios P2 and P4, respectively). Finally, we have to highlight that from the approach considered in this paper it is not possible to select the best subset of stocks to hold and a combination of the choice of the stocks and optimal tracking portfolio would be an interesting continuation for the issues raised above.

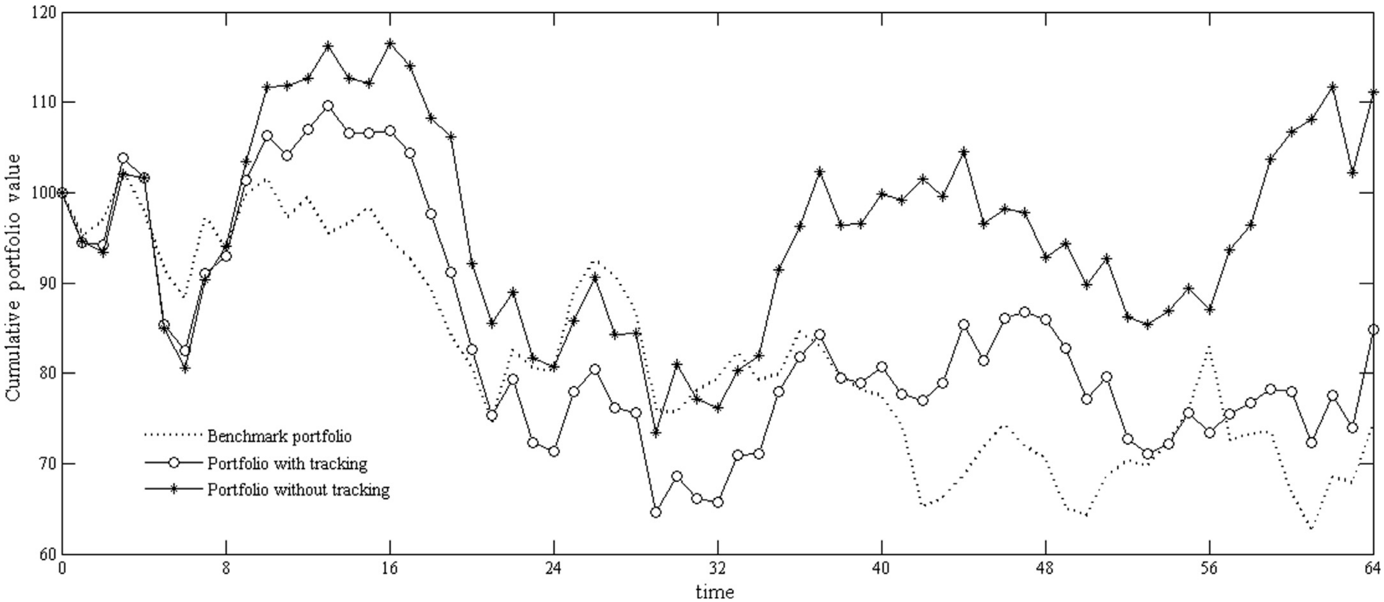

Fig. 1. Cumulative monthly portfolio values, obtained by application of the optimal enhanced indexation composition (12) and the optimal portfolio without tracking (14).

4. Concluding remarks

This paper deals with the enhanced index tracking problem, which aims at replicating a market index as well as generating positive excess return without purchasing all of the assets that constitute the index. We consider an approach where the tracking portfolio is composed of a given subset of assets belonging to a market index. Using the method of Lagrange multipliers, we present an analytical solution for an uni-period enhanced indexation problem with limited number of assets held in the tracking portfolio and a comparison with the optimal portfolio without tracking. This approach yields to a framework with less technical and computational complexity, and consequently easier to be implemented in real applications, when compared to other methods used in the literature. On the other hand, it is not possible to select an optimal subset of assets to compose the tracking portfolio. Finally, we highlight that the selection criteria used to define the subset of assets has significant impact on the performance of the tracking portfolio. This is an important factor that we have to take into account when using this approach in practical applications.

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